# Quantify the butterfly effect for the Lorenz system


Tiange Chang

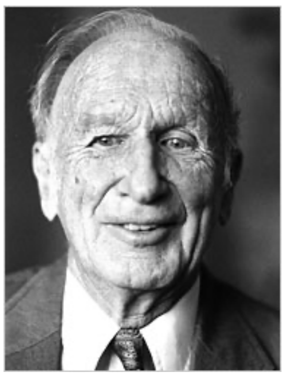
MTH 463_Final Project



*Abstract*

*This paper is an introduction to the Lorenz system, the butterfly effect and the chaos theory. I will use Matlab to illustrate how the change of initial condition in Lorenz equation and the butterfly effect give the chaos. I will also provide some applications in real life.*


## I. Introduction



*Edward N. Lorenz* [2]

*Edward Norton Lorenz* was an American mathematician and meteorologist who established the theoretical basis of weather and climate predictability, as well as the basis for computer-aided atmospheric physics and meteorology. [1]

His discovery of deterministic chaos "profoundly influenced a wide range of basic sciences and brought about one of the most dramatic changes in mankind's view of nature since Sir Isaac Newton," according to the committee that awarded him the 1991 Kyoto Prize for basic sciences in the field of earth and planetary sciences. [2]

**"**In 1961, Lorenz was running a numerical computer model to redo a weather prediction from the middle of the previous run as a shortcut. He entered the initial condition 0.506 from the printout instead of entering the full precision 0.506127 value. The result was a completely different weather scenario."[9]

In 1963, Edward Lorenz developed a simplified mathematical model for atmospheric convection. [4] The model is a system of three ordinary differential equations now known as the *Lorenz equations*:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y - x),$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = x(\rho - z) - y,$$

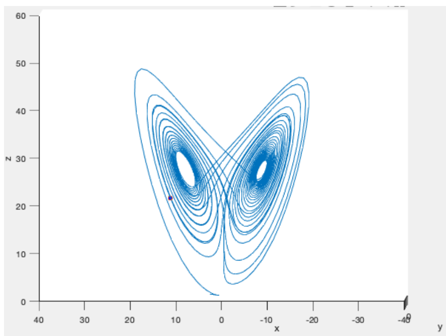
$$\frac{\mathrm{d}z}{\mathrm{d}t} = xy - \beta z.$$

where: x = the spatial average of the hydrodynamic velocity
y = temperature
z = temperature gradient
σ = Prandtl number (dimensionless constant)
$\rho$ = Rayleigh number of the fluid (dimensionless constant)
$\beta$ = a constant related to the aspect ratio of the domain under consideration [3]

*The Chaos theory*, when the present determines the future, but the approximate present does not approximately determine the future said by Lorenz.

The phenomenon of a solution of the system of equations never repeats its past history exactly for certain parameter regimes; moreover, all approximate repetitions have finite duration became known as chaotic dynamics. [3]

*The butterfly effect* is the sensitive dependence on initial conditions in which a small change in one state of a deterministic nonlinear system can result in large differences in a later state. [5]



*A sample solution of the Lorenz attractor when $\rho$ = 28, $\sigma$ = 10, and $\beta$ = 8/3.* *Figure_1*

# II. Results

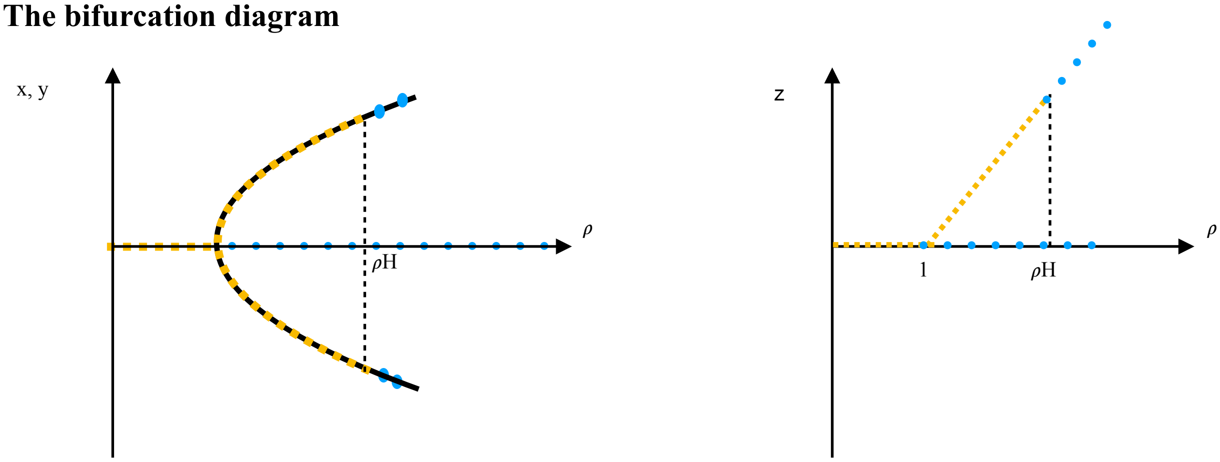
**- The equilibrium solution**

When $0 < \rho < 1$, the Lorenz equations have only equilibrium solution, which is the trivial solution: $x = y = z = 0$. [3]

The nontrivial solution is $x = y = \pm \sqrt{(\beta (\rho - 1))}$, $z = \rho - 1$ which exists for $\rho > 1$. [3]

For $\rho > 1$, the Lorenz equations admit two more equilibrium solutions, corresponding to the critical points $C\pm = (\pm\sqrt{(\beta (\rho - 1))}, \pm\sqrt{(\beta (\rho - 1))}, \rho - 1)$.

**- The bifurcation diagram**




Where ····· are stable and ······ are unstable. [3]

**- Numerically, the equilibrium point $C_+$ is unstable for $\rho > \rho_H$.**

Set σ = 10, and β = 8/3. Since. $\rho_H = \sigma*(1 + 2*(\beta + 2)/(\sigma - \beta - 1)) \approx 24$. I set $\rho = 26$, x0 ≈ [8.1750, 8.1550, 25.0100], Tmax = 40, then xeq = $\sqrt{}(\beta (\rho - 1) = 8.1650$, yeq = $\sqrt{}(\beta (\rho - 1) =$ 8.1650 and zeq = $\rho - 1 = 25$.

As a result, I had figure_2 where x0 is the red dot and t is the green dot.

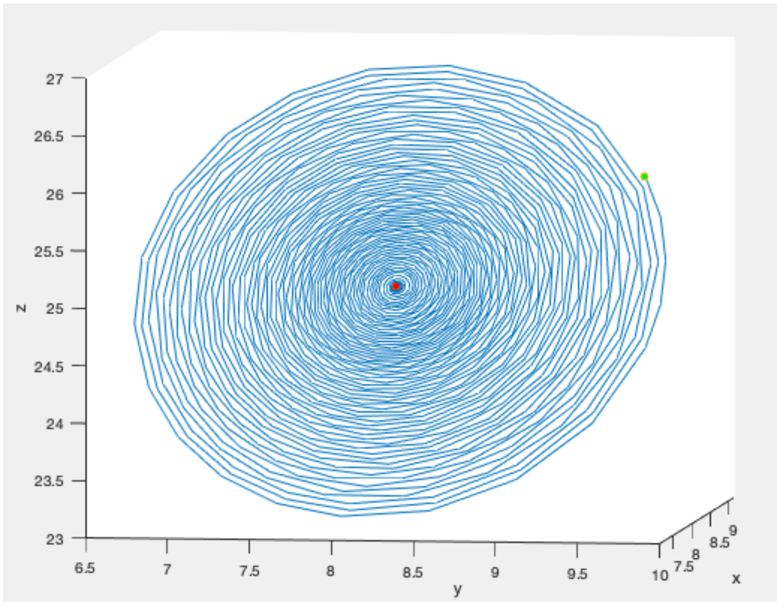



*Figure_2*

Now use the same number but change x0 to [8.2650, 8.0650, 25.1000].
As a result, I had figure_2 where x0 is the red dot and t is the green dot.

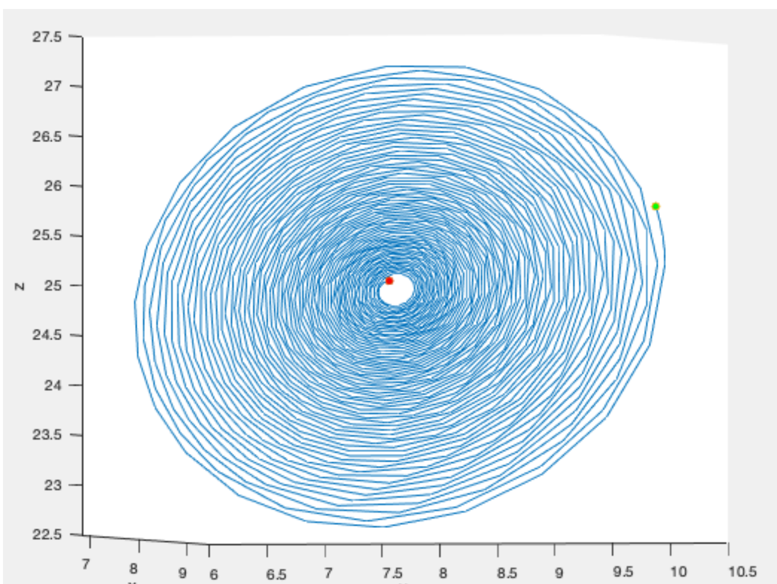
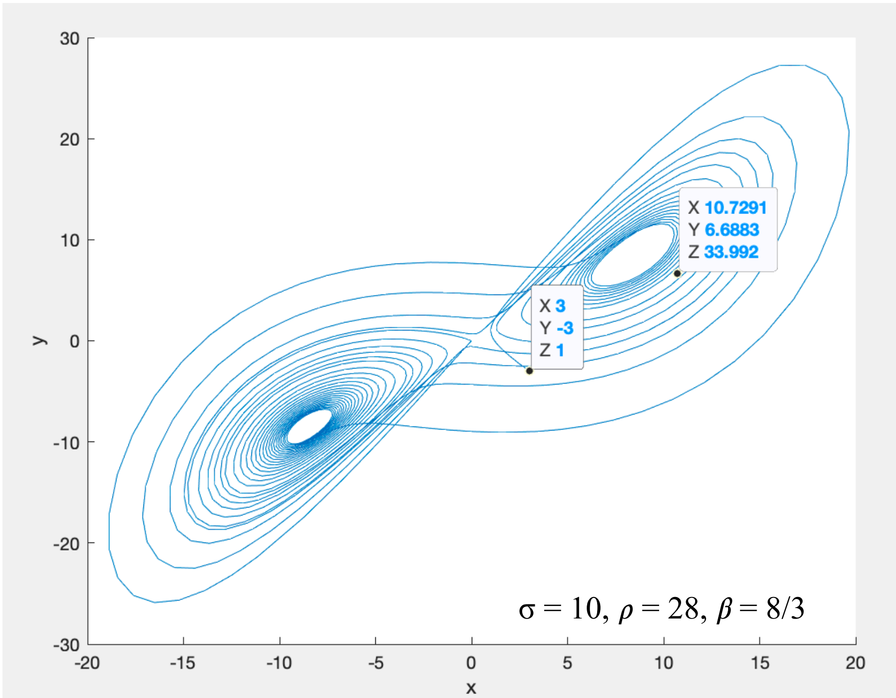
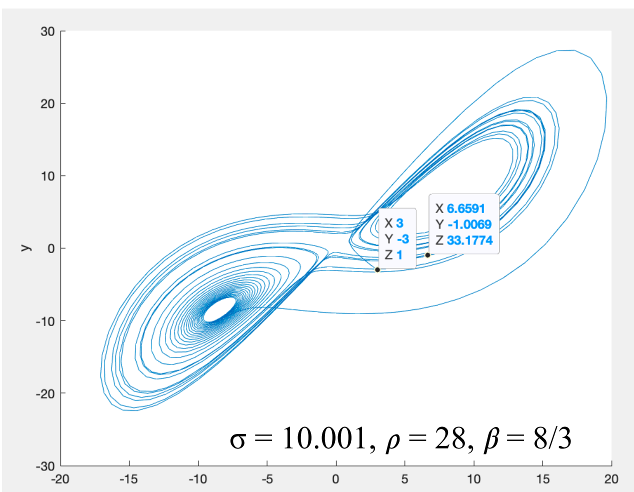
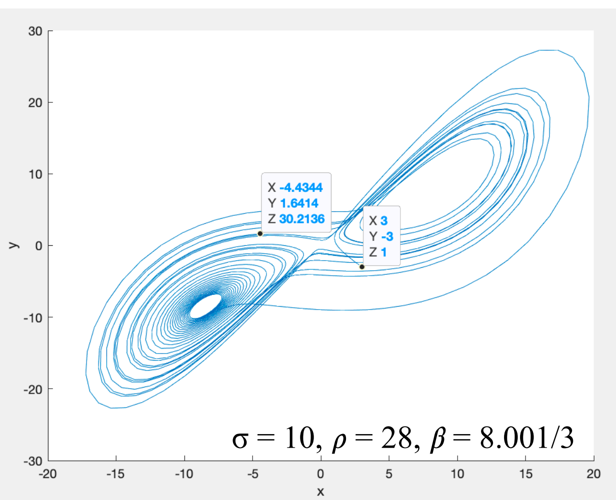


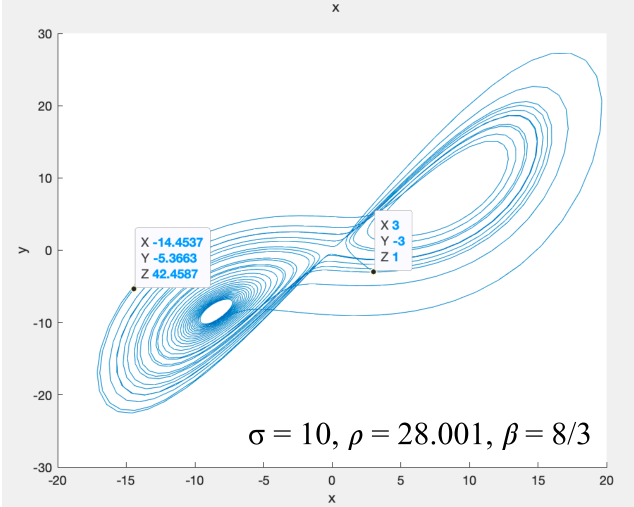

*Figure_3*

**- The change of initial conditions will give chaos**



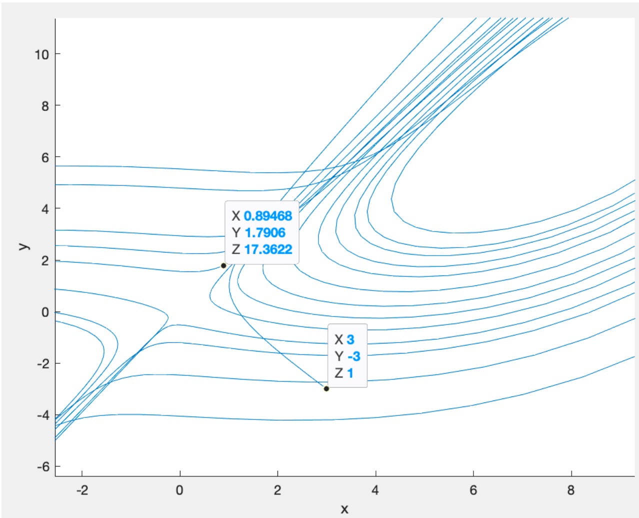



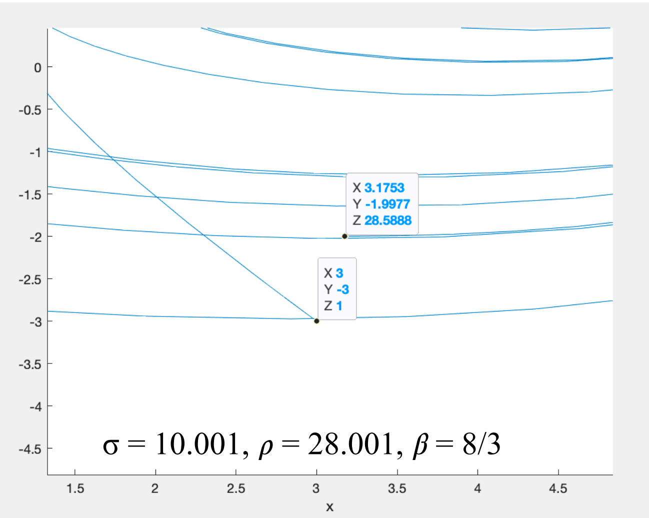



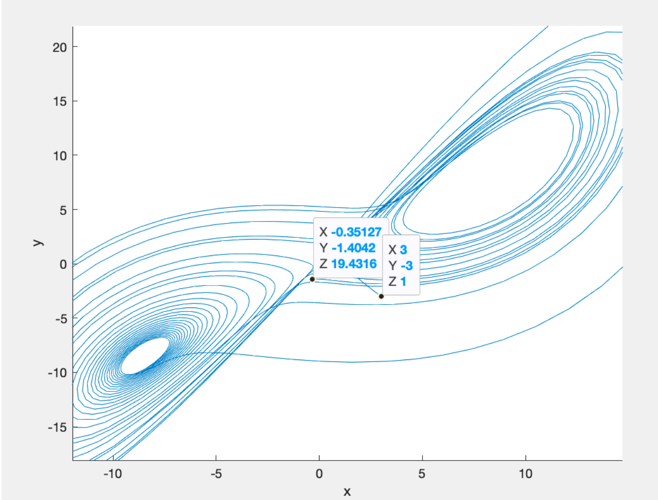

The bigger figure on the top gives the initial conditions $\sigma = 10$, $\rho = 28$, $\beta = 8/3$. Three smaller figures below have only 0.001 difference in only one of the three parameters. These trajectories for the sane period time (T = 30) starting at the same exact point (3,-3,1) but end with three completely different points which give the chaos.

Now I am changing two parameters out of there with only 0.001 difference. Since some of the points are very close to each other, figures below are zoomed.



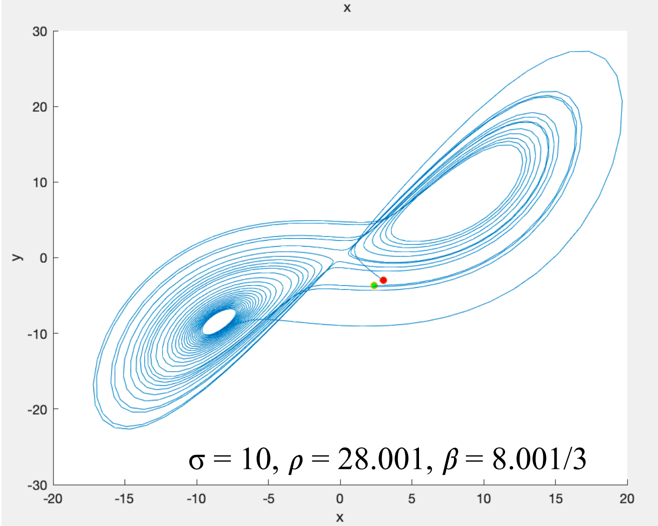




σ = 10.001, $\rho$ = 28.001, $\beta$ = 8/3

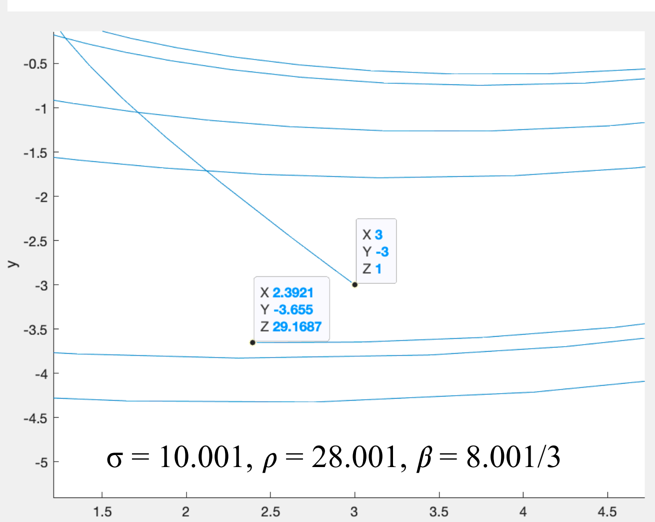



The result remains the same.

Finally, I will change all three parameters to 0.001 different than my original figure(the bigger figure on the top).




σ = 10, $\rho$ = 28.001, $\beta$ = 8.001/3




σ = 10.001, $\rho$ = 28.001, $\beta$ = 8.001/3

## III. Discussion

Just like Lorenz wrote in his paper in 1963: “applied to the atmosphere ... they indicate that prediction of the sufficiently distant future is impossible by any method, unless the present conditions are known exactly. In view of the inevitable inaccuracy and incompleteness of weather observations, precise very-long-range forecasting would seem to be non-existent.”

Since a very little change can lead to a different result, to predict the weather accurately is nearly impossible. However, on October 23th 2019, Google’s quantum computer Sycamore was able to complete a problem in 200 seconds that a supercomputer would need 10,000-years to solve, according to their estimates.[10] My assumption is that if the quantum computer can be widespread used, maybe we can accurately predict the weather.

## IV. Summary

In this paper, I introduced the Lorenz system, the butterfly effect and the chaos theory. I calculated the equilibrium solution, the bifurcation diagram. I used Matlab to calculate and plot the numerical solutions and how the change of initial conditions will give chaos.

By doing this project, I learnt the history and applications of the butterfly effect. I can now relate some daily life cases to the butterfly effect such as why the weather forecast is not accurate. Studying this project has also improved my coding skill in Matlab.

## V. References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