Estimation and Decision Systems 2023/24- Activity sheet 1

1- a) Write a MATLAB function that calculates the autocovariance sequence and spectral density of a signal. This function should have the following inputs:

- the signal (e);
- The maximum lag for which the autocovariance sequence will be computed (τ_{max});
- A vector (ω) holding the frequencies used for calculating the signal spectrum;

Outputs:

- Autocvariance sequence $\lambda_e(\tau)$, $-\tau_{max} \leq \tau \leq \tau_{max}$;
- Vector (τ) with the lags for which the autocovariancen is calculated;
- Auto-spectrum $\phi_e(\omega)$ at the specified freqeuncies in ω;

Save the function in the file *AutoCovSpectra.m*.

b) Generate a realization of white noise, $e(t)$, with zero mean, variance $\sigma_e^2 = 4$, and 2000 points (use the variable **e1** to store the white noise realization).

c) Use the function you wrote to determine estimates of the autocovariance sequence $\lambda_{ee}(\tau)$ and the spectral density $\phi_{ee}(\omega)$ with $-20 \leq \tau \leq 20$ and w a vector of 201 equally spaced points between $-\pi$ and π (use the variables **lag1c**, **lambda1c**, **w1c**, and **phi1c** for the lag vector, autocovariance sequence, frequency vector, and spectral density, respectively).

d) Make the autocovariance sequence plot (save it to the file *Lambda1d.fig*).

e) Make a plot comparing the estimated and the theorectical power density spectrum. Set the x-axis limits equal to $-\pi$ and π and the y-axis limits equal to 0 and 6. Save the plot in a file named *Phi1e.fig*.

f) Write a function that performs a whiteness test on a signal $e(t)$ (save the function in the file *whiteness.m* and the test's obtained plot in the file *whiteness.fig*). The *whiteness* function should have the following inputs:

- The signal (e);
- Maximum lag for which to calculate the autocovariance sequence (τ_{max});
- Significance level (α, see notes);

It has to create the plot as described in the end-of-guide notes and have the following outputs:

- The normalized autocovariance sequence $s(\tau)$ (see notes);
- Vector τ with the lags for which the autocovariance sequence points were calculated.
- k_α, the confidence interval limit.

Test if **e1** is white noise. Save the plot generated by the whiteness function in the file *White.fig*

g) Find estimates of the autocovariance sequence and spectral density with $\tau_{max} = 200$ and ω equal to the one used in **1c)** (use the variables **lag1g**, **lambda1g**, and **phi1g**)."

h) Compare the estimates of $\phi_{ee}(\omega)$ obtained in 1c) and 1g) with the theorectical value. You can visualize this through a graph displayingthe three functions. Save this grpah in file *Phi1c1g.fig.* Provide your analysis of the results, indicating which estimate is closer to the true value and attempt to explain why this is the case.

i) Verify if there is any negative value in **phi1c** or **phi1g**. Comment the result.

j) Save all the data and results of this problem in the file emphTPC2-1.mat.

2 - a) Simulate the system

$$\begin{aligned} x(t+1) &= 0.8x(t) + u(t) \\ y(t) &= x(t). \end{aligned}$$

Using the realization of $e(t)$ calculated in the previous problem as input (use the variable **y2** to store $y(t)$).

b) Plot the time-domain graph of the output (save it as *Y2.fig*). Do you think y(t) exhibits characteristics of a weakly stationary stochastic process? Please explain your reasoning.

c) Calculate the autocovariance sequence and spectral density of $y(t)$ with $\tau_{max} = 20$ and ω identical to that in 1c) (use the variable **w2** to store the values of ω, and the variables **lagy2**, **lambday2**, and **phiy2** for the lag vector, autocovariance sequence, and spectral density, respectively).

d) Plot the estimates and theorectical values of $\lambda_{yy}(\tau)$ as a function of τ and the estimates and theorectical values of $\phi_{yy}(\omega)$ as a function of ω (save both plots in the files *Lambda2.fig* and *Phi2.fig*).

e) Save all the data and results of this problem in the file emphTPC2-2.mat

3 Consider the stochastic process described by

$$\begin{aligned} x(t+1) &= x(t) + e(t) \\ y(t) &= x(t) \end{aligned}$$

where $e(t)$ is the stochastic process defined in problem 1.

a) Show that if $e(t) = 0$ for $t \leq 0$, then

$$y(t) = e(1) + e(2) + \cdots + e(t-1)$$

b) Show that $y(t)$ is not a stationary process.

c) Using the realization of $e(t)$ generated in 1a), determine a realization of $y(t)$ (store $y(t)$ in **y3**).

d) Create the plot of $y(t)$ as a function of t (save this plot in the file *Y3.fig*).

e) Do you think that with just this realization, you could calculate a good estimate of $\lambda_{yy}(0) = \mathbf{E}\left\{y^2(t)\right\}$? Why?

f) Generate 10 different realizations of $e(t)$, each with a length of 100, and perform 10 simulations of the system, the $e(t)$ realizations as inputs. Save the system inputs and ouputs in the columns of the matrices **E3** and **Y3**, respectively. Use these realizations to estimate the variance of $y(t)$, $\lambda_{yy}(0,t) = \mathbf{E}\left\{y^2(t)\right\}$ for $t = 1, \ldots, 100$. Save the results in the variable **var_y03**, Plot $\lambda_{yy}(0,t)$ as a function of t, and save it in the file *vary03.fig*.

g) Comment on the plot of the previous item and compare the estimates of $\lambda_{yy}(0,t)$ with their theoretical values.

h) Save all the data and results of this problem in the file emphTPC2-3.mat.

Notes:

- In the function you will develop in problem 1, you can use the **xcov** command from MATLAB's signal processing toolbox to calculate the autocovariance sequence of the signal.
- Use the definition of power spectral density

$$\phi_e(\omega) = \sum_{\tau=-\infty}^{\infty} \lambda_e(\tau) e^{-j\omega\tau}.$$

to calculate the estimates of $\Phi_{ee}(\omega)$. Here, the problem is that we only have $\lambda_e(\tau)$ for a finite interval of τ ($-\tau_{max} \leq \tau \leq \tau_{max}$). Hence, we have to approximate $\Phi_{ee}(\omega)$ by the finite sum

$$\phi_e(\omega) = \sum_{\tau=-\tau_{max}}^{\tau_{max}} \lambda_e(\tau)e^{-j\omega\tau}.$$

This implies that, in addition to the error due to the uncertainty of $e(t)$, there will be another due to the finite length of the sequence used for spectrum calculation. This error is systematic and causes a deviation of the expected value of the estimate from the true value. When this happens, it is said that the estimate is biased. What is the magnitude of this bias, and how can we reduce it? To answer this question, let's consider the pedestal signal

$$p_{\tau_{max}}(t) = \begin{cases} 1, & -\tau_{max} \leq t \leq \tau_{max} \\ 0, & |t| > \tau_{max} \end{cases}$$

Then, our estimate of $\phi_{ee}(\omega)$ is the Fourier transform of

$$\lambda\bar{e}\bar{e}(\tau) = \begin{cases} \lambda_{ee}(\tau), & -\tau_{max} \leq \tau \leq \tau_{max} \\ 0, & |\tau| > \tau_{max} \end{cases} = p_{\tau_{max}}(\tau)\lambda_e(\tau)$$

i.e

$$\hat{\phi}_{ee}(\omega) = \phi_{\bar{e}\bar{e}}(\omega) = \sum_{\tau=-\tau_{max}}^{\tau_{max}} \lambda_e(\tau)e^{-j\omega\tau}.$$

If $\lambda_e(\tau) = 0$ for $|\tau| > \tau_{max}$, there is no bias. Therefore, if $e(t)$ is white noise, $\hat{\phi}_{ee}(\omega)$ is unbiased. Conversely, if $e(t)$ is not white noise, the bias decreases as τmax increases because $\lambda_{ee}(\tau)$ is absolutely summable, which is both a necessary and sufficient condition for the existence of $\phi_{ee}(\omega)$. Consequently, as τ_{max} tends towards infinity, the sum of $\sum_{\tau=\tau_{max}}^{\infty} \lambda_{ee}(\tau)$ approaches zero

SUGGESTION: Plot the theoretical $\phi_{\bar{e}\bar{e}}(\omega)$ for $\tau_{max} = 20$ and $\tau_{max} = 200$ and compare them with the ones you estimated from the realization of $e(t)$.

- **Whiteness Test**: If $e(t)$ is zero-mean Gaussian white noise and its auto-covariance sequence is estimated as

$$\hat{\lambda}_{ee}(\tau) = \frac{1}{N - \tau_{max}} \sum_{t=\tau_{max}+1}^{N} e(t)e(t-\tau)$$

then the distribution of $s(\tau) = \dfrac{\hat{\lambda}ee(\tau)}{\hat{\lambda}ee(0)}$, $\tau \neq 0$, converges to the normal distribution with zero mean and variance $\dfrac{1}{N - \tau_{max}}$. Hence, for a sufficiently large N, we consider that distribution of $s(\tau)$ has reached the assymtotic distribution. Therefore, the whiteness test of $e(t)$ consists of plotting $s(\tau)$ for $\tau = 0, \ldots, \tau = \tau_{max}$ and drawing two lines parallel to the τ axis at values $\frac{k_\alpha}{\sqrt{N - \tau_{max}}}$ and $-\frac{k_\alpha}{\sqrt{N - \tau_{max}}}$, where $[-k_\alpha, k_\alpha]$ is the interval of the standard normal distribution (zero mean and unit variance) with a probability of occurrence α. If $F(z)$ is the cumulative probability function (also known as the probability function) of the standard normal distribution, then

$$k_\alpha = F^{-1}\left(1 - \frac{\alpha}{2}\right) = -F^{-1}\left(\frac{\alpha}{2}\right)$$

The function $F^{-1}(z)$ is implemented in the MATLAB command *norminv*.

VARIABLES

Problem 1

- **e1** - White noise realization of item b).
- **lag1c** - Time lags of the autocovariance function estimated in item 1c).
- **lambda1c** - Estimates of autocovariance sequence calculated in 1c).
- **w1c** - Frequencies the power spectral density estimated in 1c).
- **phi1c**- Power spectral density estimates calculated in 1c).
- **lag1g** - Vector of time lags of the Autocovariance function estimated in 1g).
- **lambda1g** - stimates of autocovariance sequence calculated calculated in 1g).
- **phi1g**- Power spectral density estimates calculated in 1g).

Problem 2

- **y2** - Output of the stable system with input **e1** (item 2a)).
- **w2** - Contains the frequencies the power spectral density calculated in 2c) (equal to **w1c**.
- **lagy2** - Time lags of the autocovariance function estimated in item 2c).
- **lambday2** - Vector with the estimates of autocovariance sequence calculated in 2c).
- **phiy2** - Power spectral density estimates calculated in 2c).

Problema 3

- **y3** - Output of the unstable system (integrator) with input **e1** (item 3c)).
- **E3** - Matrix whose columns hold the input signals required for simulations to estimate of the output variance of the integrator.
- **Y3** - Matrix holding the outputs of the integrator simulation.
- **var_y03** - Vector holding the output variance of the integrator.

GENERATED FILES

Problem 1

- *AutoCovSpectra.m* - MATLAB function that estimates the autocovariance and power spectral density of a stochastic process from a realization (item 1a)).
- *whiteness.m* - MATLAB function for conducting the whiteness test (item 1f))
- *Lambda1d.fig* - Autcovariance plot of item 1d)
- *Phi1e.fig* - Power spectral density plot of item 1e)
- *White.fig* - Whiteness test plot of item 1f)
- *Phi1c1g.fig* - Plot comparing the spectral density estimates calculated in items 1c) and 1g).
- *TPC2-1.mat* - File that stores all the variables of problem 1.

Problema 2

- *Y2.fig* - Time-domain graph of the stable system output, $y_2(t)$, simulated in 2a) (item 2b)).
- *Lambda2.fig* - Plot of the autocvariance sequence of $y_2(t)$ (item 2d)).
- *Phi2.fig* - Plot of the power spectral density of $y_2(t)$ (item 2d)).
- *TPC2-2.mat* - File that stores all the variables of problem 2.

Problema 3

- *Y3.fig* - Time-domain graph of the sintegrator output, $y_3(t)$, simulated in 3c) (item 3d)).
- *vary03.fig* - Plot of the variance of $y_3(t)$ as a function of time.
- *TPC2-3.mat* - ile that stores all the variables of problem 3.