4 Quantum Field Theory II – Exercise sheet 3 (2025-05-28)

4.1 Exercise 1 - Tree-level= Classical Field Theory

We want to understand the claim that tree-level diagrams correspond to classical field theory (which is equivalently stated by saying that $\hbar$ is the loop counting parameter). We consider ϕ^3 theory with action:

$$S[\phi] = \int d^4x \left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{3!}\phi^3 + J\phi \right), \tag{1}$$

where J is a fixed (non-dynamical) external source. Our main goal is to solve the field equations as a perturbation theory in λ, making the power series ansatz:

$$\phi = \phi_0 + \lambda\phi_1 + \lambda^2\phi_2 + \lambda^3\phi_3 + \cdots .$$

1) Determine the Euler-Lagrange equations of (1) and use this to write the field equations for ϕ_0, ϕ_1, and ϕ_2.
2) Solve the above equations for ϕ_0, ϕ_1, and ϕ_2 in terms of J by use of a suitable Greens function like the Feynman propagator $D_F(x-y)$.
3) Find a graphical notation to represent the above solutions and use these Feynman diagrams to determine the solution to order λ^3, i.e., to determine ϕ_3. Convince yourself directly that the equations hold.

   Hint: Nobody can stop you from reading section 3.5 of the book by Schwartz.
4) Consider now the 'on-shell action' obtained by substituting the solution $\phi(J)$ into (1):

$$S_{\text{on-shell}}[J] := S[\phi(J)]. \tag{2}$$

   The claim is that the n-point tree-level amplitude can be obtained from the on-shell action via:

$$(2\pi)^4\delta^{(4)}(k_1+\cdots+k_n)M_n^{\text{tree}}(k_1,\ldots,k_n) = (-1)^n\prod_{i=1}^{n}\int d^4x_i\, e^{ik_i\cdot x_i}\left(\Box_{x_i}-m^2\right)\frac{\delta}{\delta J(x_i)}S_{\text{on-shell}}[J]\bigg|_{J=0}. \tag{3}$$

   Check this claim by looking at the $2\to 2$ scattering.

1) From $\delta S = 0$ we obtain the Euler-Lagrange equations - so we calculate the terms

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)} &= -\frac{1}{2}\frac{\partial g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi}{\partial(\partial_\alpha\phi)} && (115)\\
&= -\frac{1}{2}(g^{\mu\nu}\delta^\alpha_\mu\partial_\nu\phi + g^{\mu\nu}\partial_\mu\phi\delta^\alpha_\nu) && (116)\\
&= -\partial^\alpha\phi && (117)\\
\frac{\partial\mathcal{L}}{\partial\phi} &= -m^2\phi - \frac{\lambda}{2}\phi^2 + J && (118)\\
&\to -\partial_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)} + \frac{\partial\mathcal{L}}{\partial\phi} = 0 && (119)\\
&\to \partial^\alpha\partial_\alpha\phi - m^2\phi - \frac{\lambda}{2}\phi^2 + J = 0 && (120)\\
&\to (\Box - m^2)\phi - \frac{\lambda}{2}\phi^2 + J = 0 && (121)
\end{aligned}$$

As we are working in $g^{\mu\nu} = \text{diag}(-1,1,1,1)$ (meaning $\Box = -\partial_t^2 + \triangle$) the signs of the KG equation are ok. The substituting in the series expansion

$$(\Box - m^2)(\phi_0 + \lambda\phi_1 + \lambda^2\phi_2 + \cdots) - \frac{\lambda}{2}(\phi_0 + \lambda\phi_1 + \lambda^2\phi_2 + \cdots)^2 + J = 0$$

$$(\Box - m^2)(\phi_0 + \lambda\phi_1 + \lambda^2\phi_2 + \cdots) - \frac{\lambda}{2}(\phi_0^2 + 2\phi_0\phi_1\lambda + (\phi_1^2 + 2\phi_0\phi_2)\lambda^2 + (2\phi_1\phi_2 + 2\phi_0\phi_3)\lambda^3 + \cdots) + J = 0$$

we obtain

$$\begin{aligned}
\lambda^0 :&\quad (\Box - m^2)\phi_0 = -J && (122)\\
\lambda^1 :&\quad (\Box - m^2)\phi_1 = \frac{1}{2}\phi_0^2 && (123)\\
\lambda^2 :&\quad (\Box - m^2)\phi_2 = \phi_0\phi_1 && (124)\\
\lambda^3 :&\quad (\Box - m^2)\phi_3 = \frac{1}{2}(\phi_1^2 + 2\phi_0\phi_2) && (125)
\end{aligned}$$

2) With the definition of the Feynman propagator D_F

$$D_F(x,y) = i\int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2+m^2-i\epsilon}e^{ip(x-y)} \tag{126}$$
$$\rightarrow (\Box_x - m^2)D_F(x,y) = -\int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2+m^2-i\epsilon}e^{ip(x-y)}[i^2(-(p_0)^2+\vec{p}^2)-m^2] \tag{127}$$
$$= -i\int \frac{d^4p}{(2\pi)^4}\frac{-(p^2+m^2)}{p^2+m^2-i\epsilon}e^{ip(x-y)} \tag{128}$$
$$= +i\int \frac{d^4p}{(2\pi)^4}e^{ip(x-y)} \tag{129}$$
$$= i\delta^{(4)}(x-y) \tag{130}$$

we find

$$\phi_0(x) = \int d^4y\, D_F(x-y)[iJ(y)] \tag{131}$$
$$= i\int d^4y\, D_F(x-y)J(y) \tag{132}$$

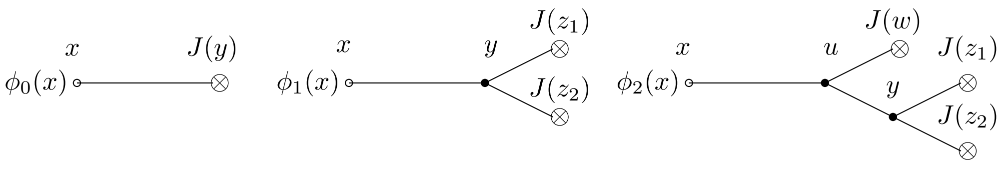
and

$$\phi_1(x) = (-i)\frac{1}{2}\int d^4y\, D_F(x-y)\left(\underbrace{i\int d^4z_1\, D_F(y-z_1)J(z_1)}_{=\phi_0(y)}\cdot\underbrace{i\int d^4z_2\, D_F(y-z_2)J(z_2)}_{=\phi_0(y)}\right) \tag{133}$$
$$= \frac{i}{2}\int d^4y\, D_F(x-y)\left(\int d^4z_1\, D_F(y-z_1)J(z_1)\cdot\int d^4z_2\, D_F(y-z_2)J(z_2)\right) \tag{134}$$
$$= \frac{i}{2}\iiint d^4y\, d^4z_1\, d^4z_2\, D_F(x-y)D_F(y-z_1)J(z_1)D_F(y-z_2)J(z_2) \tag{135}$$

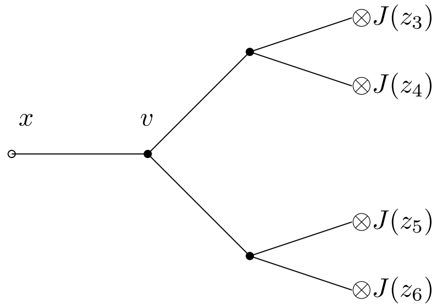
and

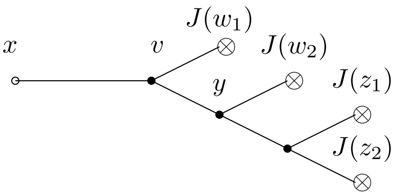
$$\phi_2(x) = (-i)\int d^4u\, D_F(x-u) \tag{136}$$
$$\left(\underbrace{i\int d^4wD_F(u-w)J(w)}_{=\phi_0(u)}\cdot\underbrace{\frac{i}{2}\iiint d^4y\, d^4z_1\, d^4z_2\, D_F(u-y)D_F(y-z_1)J(z_1)D_F(y-z_2)J(z_2)}_{=\phi_1(u)}\right) \tag{137}$$
$$= \frac{i}{2}\iiiint\!\!\int d^4u\, d^4w\, d^4y\, d^4z_1\, d^4z_2\, D_F(x-u)D_F(u-w)J(w)D_F(u-y)D_F(y-z_1)J(z_1)D_F(y-z_2)J(z_2) \tag{138}$$

3) Graphical representation



Now constructing the λ^3 term $\phi_3(x)$ (three black nodes)

$\sim \phi_1^2$ 

$\sim \phi_2\phi_0$ 

We can also calculate - and obtain the same result

$$\phi_3(x) = (-i)\int dv\, D_F(x-v)\left(2\,\underbrace{i\int dw_1\, D_F(v-w_1)J(w_1)}_{=\phi_0(v)}\cdot\right. \tag{139}$$

$$\cdot\underbrace{\frac{i}{2}\iiiint\!\!\int du\, dw_2\, dy\, dz_1\, dz_2\, D_F(v-u)D_F(u-w_2)J(w_2)D_F(u-y)D_F(y-z_1)J(z_1)D_F(y-z_2)J(z_2)}_{=\phi_2(v)}+ \tag{140}$$

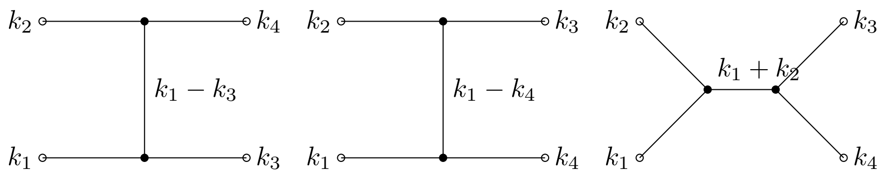
$$+\underbrace{\frac{i}{2}\iiint dy_1\, dz_3\, dz_4\, D_F(v-y_1)D_F(y_1-z_3)J(z_3)D_F(y_1-z_4)J(z_4)}_{\phi_1(v)}\cdot \tag{141}$$

$$\left.\cdot\underbrace{\frac{i}{2}\iiint dy_2\, dz_5\, dz_6\, D_F(v-y_2)D_F(y_2-z_5)J(z_5)D_F(y_2-z_6)J(z_6)}_{\phi_1(v)}\right) \tag{142}$$

$$= i\int dv\, dw_1\, du\, dw_2\, dy\, dz_1\, dz_2\, D_{xv}D_{vw_1}J(w_1)D_{vu}D_{uw_2}J(w_2)D_{uy}D_{yz_1}J(z_1)D_{yz_2}J(z_2) \tag{143}$$

$$+\frac{i}{4}\int dv\, dy_1\, dz_3\, dz_4\, dy_2\, dz_5\, dz_6\, D_{xv}D_{vy_1}D_{y_1z_3}J(z_3)D_{y_1z_4}J(z_4)D_{vy_2}D_{y_2z_5}J(z_5)D_{y_2z_6}J(z_6) \tag{144}$$

4) Intuitively we expect only stuff to happen at λ^2 order because the only relevant (low order) tree level diagrams are. We calculate anyway - so substituting (shortening the notation $D_F(x-y) \equiv D_{xy}$)



$$S_{\text{on-shell}}[J] = S[\phi(J)] \tag{145}$$

$$= S[\phi_0 + \lambda\phi_1 + \lambda^2\phi_2 + ...] \tag{146}$$

$$= \int d^4x \left(-\frac{1}{2}(\Box - m^2)(\phi_0 + \lambda\phi_1 + \lambda^2\phi_2 + ...) - \frac{\lambda}{3!}(\phi_0 + \lambda\phi_1 + \lambda^2\phi_2 + ...)^3 + J(\phi_0 + \lambda\phi_1 + \lambda^2\phi_2 + ...) \right) \tag{147}$$

$$= \int d^4x \left(-\frac{1}{2}(-J + \lambda\frac{1}{2}\phi_0^2 + \lambda^2\phi_0\phi_1 + ...) - \frac{\lambda}{3!}(\phi_0^3 + 3\phi_1\phi_0^2\lambda + 3\phi_0(\phi_1^2 + \phi_0\phi_2)\lambda^2 + ...) + J(\phi_0 + \lambda\phi_1 + \lambda^2\phi_2 + ...) \right) \tag{148}$$

$$= \int d^4x\, J \left(\phi_0 + \frac{1}{4} \right) + \lambda \int d^4x \left(-\frac{1}{6}\phi_0^2 \left[\phi_0 + \frac{3}{2} \right] + J\phi_1 \right) + \lambda^2 \int d^4x \left(-\frac{1}{2}\phi_0\phi_1(\phi_0 + 1) + J\phi_2 \right) + ... \tag{149}$$

Now we look at the individual contributions (shortening notation) - and doing to first functional integral in baby steps

$$S_0 = \int d^4x\, J \left(\phi_0 + \frac{1}{4} \right) \tag{150}$$

$$= \int dx\, J(x) \left(i \int dy\, D_F(x-y)J(y) + \frac{1}{4} \right) \tag{151}$$

$$= \frac{1}{4} \int dx\, J(x) + i \int dx\, dy J(x) D_{xy} J(y) \tag{152}$$

$$\rightarrow \frac{\delta S_0[J]}{\delta J(x_i)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[\frac{1}{4} \int dx\, \left[J(x) + \epsilon\delta(x - x_i) \right] - \frac{1}{4} \int dx\, J(x) \right] \tag{153}$$

$$+ i \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[i \int dx\, dy (J(x) + \epsilon\delta(x - x_i)) D_{xy} (J(y) + \epsilon\delta(y - x_i)) - i \int dx\, dy J(x) D_{xy} J(y) \right] \tag{154}$$

$$= \frac{1}{4} \int dx\, \delta(x - x_j) + i \int dx\, dy\, [\delta(x - x_i) D_{xy} J(y) + J(x) D_{xy} \delta(y - x_i)] \tag{155}$$

$$= \frac{1}{4} + i \int dy\, D_{x_i y} J(y) + i \int dx\, D_{x x_i} J(x) \tag{156}$$

this terms contains only one J - performing the other three functional derivatives $\delta/\delta J(x_k)$ will result in a zero.
Now we can calculate a bit faster

$$S_1 = -\frac{1}{6} \int dx\, \phi_0^3 - \frac{1}{4} \int dx\, \phi_0^2 + \int dx\, J\phi_1 \tag{157}$$

$$= -\frac{i^3}{6} \int dx\, dy_1\, dy_2\, dy_3\, D_{xy_3} J(y_3) D_{xy_2} J(y_2) D_{xy_1} J(y_1) - \frac{i^2}{4} \int dx\, dy_1\, dy_2 D_{xy_2} J(y_2) D_{xy_1} J(y_1) \tag{158}$$

$$+ \frac{i}{2} \int dx\, dy\, dz_1\, dz_2\, J(x)\, D_{xy} D_{yz_1} J(z_1) D_{yz_2} J(z_2) \tag{159}$$

$$\rightarrow \frac{\delta S_1[J]}{\delta J(x_i)} = -\frac{i^3}{6} \int dx\, dy_1\, dy_2\, D_{xx_i} D_{xy_2} J(y_2) D_{xy_1} J(y_1) - \frac{i^3}{6} \int dx\, dy_1\, dy_3\, D_{xy_3} J(y_3) D_{xx_i} D_{xy_1} J(y_1) \tag{160}$$

$$- \frac{i^3}{6} \int dx\, dy_2\, dy_3\, D_{xy_3} J(y_3) D_{xy_2} J(y_2) D_{xx_1} \tag{161}$$

$$- \frac{i^2}{4} \int dx\, dy_1\, D_{xx_i} D_{xy_1} J(y_1) - \frac{i^2}{4} \int dx\, dy_2 D_{xy_2} J(y_2) D_{xx_i} \tag{162}$$

$$+ \frac{i}{2} \int dy\, dz_1\, dz_2\, \; D_{x_1 y} D_{yz_1} J(z_1) D_{yz_2} J(z_2) + \frac{i}{2} \int dx\, dy\, dz_2\, J(x)\, D_{xy} D_{yx_1} D_{yz_2} J(z_2) \tag{163}$$

$$+ \frac{i}{2} \int dx\, dy\, dz_1\, J(x)\, D_{xy} D_{yz_1} J(z_1) D_{yx_i} \tag{164}$$

this terms contains only two J - performing the other three functional derivatives $\delta/\delta J(x_k)$ will result in a zero. And

$$S_2 = -\frac{1}{2}\int dx\, \phi_0^2\phi_1 - \frac{1}{2}\int dx\, \phi_0\phi_1 + \int dx\, J(x)\phi_2 \tag{165}$$

$$= -\frac{i^2}{2}\int dx\, dy_1 d\, y_2\, D_{xy_2}J(y_2)D_{xy_1}J(y_1)\frac{i}{2}\iiint d^4y\, d^4z_1\, d^4z_2\, D_{xy}D_{yz_1}J(z_1)D_{yz_2}J(z_2) \tag{166}$$

$$-\frac{i^2}{2}\int dx\, dy_1 d\, D_{xy_2}J(y_2)\frac{i}{2}\iiint d^4y\, d^4z_1\, d^4z_2\, D_{xy}D_{yz_1}J(z_1)D_{yz_2}J(z_2) \tag{167}$$

$$+\int dx\, J(x)\frac{i}{2}\int du\, dw\, dy\, dz_1\, dz_2\, D_{xu}D_{uw}J(w)D_{uy}D_{yz_1}J(z_1)D_{yz_2}J(z_2) \tag{168}$$

$$= -\frac{i^3}{4}\int dx\, dy_1 d\, y_2\, dy\, dz_1\, dz_2\, D_{xy_2}J(y_2)D_{xy_1}J(y_1)D_{xy}D_{yz_1}J(z_1)D_{yz_2}J(z_2) \tag{169}$$

$$+\frac{i}{2}\int dx\, du\, dw\, dy\, dz_1\, dz_2\;\; J(x)D_{xu}D_{uw}J(w)D_{uy}D_{yz_1}J(z_1)D_{yz_2}J(z_2) + \mathcal{O}(J^3) \tag{170}$$

Now first integral - one derivative

$$\frac{\delta}{\delta J(x_1)}\int dx\, dy_1\, dy_2\, dy\, dz_1\, dz_2\, D_{xy_2}J(y_2)D_{xy_1}J(y_1)D_{xy}D_{yz_1}J(z_1)D_{yz_2}J(z_2) \tag{171}$$

$$= \int dx\, dy_1\, dy\, dz_1\, dz_2\, D_{xx_1}D_{xy_1}J(y_1)D_{xy}D_{yz_1}J(z_1)D_{yz_2}J(z_2) \tag{172}$$

$$+\int dx\, dy_2\, dy\, dz_1\, dz_2\, D_{xy_2}J(y_2)D_{xx_1}D_{xy}D_{yz_1}J(z_1)D_{yz_2}J(z_2) \tag{173}$$

$$+\int dx\, dy_1\, dy_2\, dy\, dz_2\, D_{xy_2}J(y_2)D_{xy_1}J(y_1)D_{xy}D_{yx_1}D_{yz_2}J(z_2) \tag{174}$$

$$+\int dx\, dy_1\, dy_2\, dy\, dz_1\, D_{xy_2}J(y_2)D_{xy_1}J(y_1)D_{xy}D_{yz_1}J(z_1)D_{yz_1} \tag{175}$$

two derivatives

$$\frac{\delta}{\delta J(x_2)}\frac{\delta}{\delta J(x_1)}\int dx\, dy_1\, dy_2\, dy\, dz_1\, dz_2\, D_{xy_2}J(y_2)D_{xy_1}J(y_1)D_{xy}D_{yz_1}J(z_1)D_{yz_2}J(z_2) \tag{176}$$

$$= +\int dx\, dy\, dz_1\, dz_2\, D_{xx_1}D_{xx_2}D_{xy}D_{yz_1}J(z_1)D_{yz_2}J(z_2) \tag{177}$$

$$+\int dx\, dy_1\, dy\, dz_2\, D_{xx_1}D_{xy_1}J(y_1)D_{xy}D_{yx_2}D_{yz_2}J(z_2) \tag{178}$$

$$+\int dx\, dy_1\, dy\, dz_1\, D_{xx_1}D_{xy_1}J(y_1)D_{xy}D_{yz_1}J(z_1)D_{yx_2} \tag{179}$$

$$+\int dx\, dy_2\, dy\, dz_2\, D_{xy_2}J(y_2)D_{xx_1}D_{xy}D_{yx_2}D_{yz_2}J(z_2) \tag{180}$$

$$+\int dx\, dy\, dz_1\, dz_2\, D_{xx_2}D_{xx_1}D_{xy}D_{yz_1}J(z_1)D_{yz_2}J(z_2) \tag{181}$$

$$+\int dx\, dy_2\, dy\, dz_1\, D_{xy_2}J(y_2)D_{xx_1}D_{xy}D_{yz_1}J(z_1)D_{yx_2} \tag{182}$$

$$+\int dx\, dy_1\, dy\, dz_2\, D_{xx_2})D_{xy_1}J(y_1)D_{xy}D_{yx_1}D_{yz_2}J(z_2) \tag{183}$$

$$+\int dx\, dy_2\, dy\, dz_2\, D_{xy_2}J(y_2)D_{xx_2}D_{xy}D_{yx_1}D_{yz_2}J(z_2) \tag{184}$$

$$+\int dx\, dy_1\, dy_2\, dy\, D_{xy_2}J(y_2)D_{xy_1}J(y_1)D_{xy}D_{yx_1}D_{yx_2} \tag{185}$$

$$+\int dx\, dy_1\, dy\, dz_1\, D_{xx_2}D_{xy_1}J(y_1)D_{xy}D_{yz_1}J(z_1)D_{yz_1} \tag{186}$$

$$+\int dx\, dy_2\, dy\, dz_1\, D_{xy_2}J(y_2)D_{xx_2}D_{xy}D_{yz_1}J(z_1)D_{yz_1} \tag{187}$$

$$+\int dx\, dy_1\, dy_2\, dy\, D_{xy_2}J(y_2)D_{xy_1}J(y_1)D_{xy}D_{yx_2}D_{yz_1} \tag{188}$$

three derivatives

$$\frac{\delta}{\delta J(x_3)}\frac{\delta}{\delta J(x_2)}\frac{\delta}{\delta J(x_1)}\int dx\,dy_1\,dy_2\,dy\,dz_1\,dz_2\,D_{xy_2}J(y_2)D_{xy_1}J(y_1)D_{xy}D_{yz_1}J(z_1)D_{yz_2}J(z_2) \tag{189}$$

$$= +\int dx\,dy\,dz_2\,D_{xx_1}D_{xx_2}D_{xy}D_{yx_3}D_{yz_2}J(z_2) + \int dx\,dy\,dz_1\,D_{xx_1}D_{xx_2}D_{xy}D_{yz_1}J(z_1)D_{yx_3} \tag{190}$$

$$+\int dx\,dy\,dz_2\,D_{xx_1}D_{xx_3}D_{xy}D_{yx_2}D_{yz_2}J(z_2) + \int dx\,dy_1\,dy\,D_{xx_1}D_{xy_1}J(y_1)D_{xy}D_{yx_2}D_{yx_3} \tag{191}$$

$$+\int dx\,dy\,dz_1\,D_{xx_1}D_{xx_3}D_{xy}D_{yz_1}J(z_1)D_{yx_2} + \int dx\,dy_1\,dy\,D_{xx_1}D_{xy_1}J(y_1)D_{xy}D_{yx_3}D_{yx_2} \tag{192}$$

$$+\int dx\,dy\,dz_2\,D_{xx_3}D_{xx_1}D_{xy}D_{yx_2}D_{yz_2}J(z_2) + \int dx\,dy_2\,dy\,D_{xy_2}J(y_2)D_{xx_1}D_{xy}D_{yx_2}D_{yx_3} \tag{193}$$

$$+\int dx\,dy\,dz_2\,D_{xx_2}D_{xx_1}D_{xy}D_{yx_3}D_{yz_2}J(z_2) + \int dx\,dy\,dz_1\,D_{xx_2}D_{xx_1}D_{xy}D_{yz_1}J(z_1)D_{yx_3} \tag{194}$$

$$+\int dx\,dy\,dz_1\,D_{xx_3}D_{xx_1}D_{xy}D_{yz_1}J(z_1)D_{yx_2} + \int dx\,dy_2\,dy\,D_{xy_2}J(y_2)D_{xx_1}D_{xy}D_{yx_3}D_{yx_2} \tag{195}$$

$$+\int dx\,dy\,dz_2\,D_{xx_2})D_{xx_3}D_{xy}D_{yx_1}D_{yz_2}J(z_2) + \int dx\,dy_1\,dy\,D_{xx_2})D_{xy_1}J(y_1)D_{xy}D_{yx_1}D_{yx_3} \tag{196}$$

$$+\int dx\,dy\,dz_2\,D_{xx_3}D_{xx_2}D_{xy}D_{yx_1}D_{yz_2}J(z_2) + \int dx\,dy_2\,dy\,D_{xy_2}J(y_2)D_{xx_2}D_{xy}D_{yx_1}D_{yx_3} \tag{197}$$

$$+\int dx\,dy_1\,dy\,D_{xx_3}D_{xy_1}J(y_1)D_{xy}D_{yx_1}D_{yx_2} + \int dx\,dy_2\,dy\,D_{xy_2}J(y_2)D_{xx_3}D_{xy}D_{yx_1}D_{yx_2} \tag{198}$$

$$+\int dx\,dy\,dz_1\,D_{xx_2}D_{xx_3}D_{xy}D_{yz_1}J(z_1)D_{yz_1} + \int dx\,dy_1\,dy\,D_{xx_2}D_{xy_1}J(y_1)D_{xy}D_{yx_3}D_{yz_1} \tag{199}$$

$$+\int dx\,dy\,dz_1\,D_{xx_3}D_{xx_2}D_{xy}D_{yz_1}J(z_1)D_{yz_1} + \int dx\,dy_2\,dy\,D_{xy_2}J(y_2)D_{xx_2}D_{xy}D_{yx_3}D_{yz_1} \tag{200}$$

$$+\int dx\,dy_2\,dy\,D_{xx_3}D_{xy_1}J(y_1)D_{xy}D_{yx_2}D_{yz_1} + \int dx\,dy_1\,dy\,D_{xy_2}J(y_2)D_{xx_3}D_{xy}D_{yx_2}D_{yz_1} \tag{201}$$

four derivatives

$$\frac{\delta}{\delta J(x_4)}\frac{\delta}{\delta J(x_3)}\frac{\delta}{\delta J(x_2)}\frac{\delta}{\delta J(x_1)}\int dx\,dy_1\,dy_2\,dy\,dz_1\,dz_2\,D_{xy_2}J(y_2)D_{xy_1}J(y_1)D_{xy}D_{yz_1}J(z_1)D_{yz_2}J(z_2) \tag{202}$$

$$= +\int dx\,dy\,D_{xx_1}D_{xx_2}D_{xy}D_{yx_3}D_{yx_4} + \int dx\,dy\,D_{xx_1}D_{xx_2}D_{xy}D_{yx_4}D_{yx_3} \tag{203}$$

$$+\int dx\,dy\,D_{xx_1}D_{xx_3}D_{xy}D_{yx_2}D_{yx_4} + \int dx\,dy\,D_{xx_1}D_{xx_4}D_{xy}D_{yx_2}D_{yx_3} \tag{204}$$

$$+\int dx\,dy\,D_{xx_1}D_{xx_3}D_{xy}D_{yx_4}D_{yx_2} + \int dx\,dy\,D_{xx_1}D_{xx_4}D_{xy}D_{yx_3}D_{yx_2} \tag{205}$$

$$+\int dx\,dy\,D_{xx_3}D_{xx_1}D_{xy}D_{yx_2}D_{yx_4} + \int dx\,dy\,D_{xx_4}D_{xx_1}D_{xy}D_{yx_2}D_{yx_3} \tag{206}$$

$$+\int dx\,dy\,D_{xx_2}D_{xx_1}D_{xy}D_{yx_3}D_{yx_4} + \int dx\,dy\,D_{xx_2}D_{xx_1}D_{xy}D_{yx_4}D_{yx_3} \tag{207}$$

$$+\int dx\,dy\,D_{xx_3}D_{xx_1}D_{xy}D_{yx_4}D_{yx_2} + \int dx\,dy\,D_{xx_4}D_{xx_1}D_{xy}D_{yx_3}D_{yx_2} \tag{208}$$

$$+\int dx\,dy\,D_{xx_2}D_{xx_3}D_{xy}D_{yx_1}D_{yx_4} + \int dx\,dy\,D_{xx_2}D_{xx_4}D_{xy}D_{yx_1}D_{yx_3} \tag{209}$$

$$+\int dx\,dy\,D_{xx_3}D_{xx_2}D_{xy}D_{yx_1}D_{yx_4} + \int dx\,dy\,D_{xx_4}D_{xx_2}D_{xy}D_{yx_1}D_{yx_3} \tag{210}$$

$$+\int dx\,dy\,D_{xx_3}D_{xx_4}D_{xy}D_{yx_1}D_{yx_2} + \int dx\,dy\,D_{xx_4}D_{xx_3}D_{xy}D_{yx_1}D_{yx_2} \tag{211}$$

$$+\int dx\,dy\,D_{xx_2}D_{xx_3}D_{xy}D_{yx_4}D_{yz_1} + \int dx\,dy\,D_{xx_2}D_{xx_4}D_{xy}D_{yx_3}D_{yz_1} \tag{212}$$

$$+\int dx\,dy\,D_{xx_3}D_{xx_2}D_{xy}D_{yx_4}D_{yz_1} + \int dx\,dy\,D_{xx_4}D_{xx_2}D_{xy}D_{yx_3}D_{yz_1} \tag{213}$$

$$+\int dx\,dy\,D_{xx_3}D_{xx_4}D_{xy}D_{yx_2}D_{yz_1} + \int dx\,dy\,D_{xx_4}D_{xx_3}D_{xy}D_{yx_2}D_{yz_1} \tag{214}$$

$$\frac{\delta}{\delta J(x_4)}\frac{\delta}{\delta J(x_3)}\frac{\delta}{\delta J(x_2)}\frac{\delta}{\delta J(x_1)}\int dx\,dy_1\,dy_2\,dy\,dz_1\,dz_2\,D_{xy_2}J(y_2)D_{xy_1}J(y_1)D_{xy}D_{yz_1}J(z_1)D_{yz_2}J(z_2)$$
$$= 4\int dx\,dy\,D_{xx_1}D_{xx_2}D_{xy}D_{yx_3}D_{yx_4} + 4\int dx\,dy\,D_{xx_1}D_{xx_3}D_{xy}D_{yx_2}D_{yx_4} + 4\int dx\,dy\,D_{xx_1}D_{xx_4}D_{xy}D_{yx_2}D_{yx_3}$$
$$+4\int dx\,dy\,D_{xx_2}D_{xx_3}D_{xy}D_{yx_1}D_{yx_4} + 4\int dx\,dy\,D_{xx_2}D_{xx_4}D_{xy}D_{yx_1}D_{yx_3} + 4\int dx\,dy\,D_{xx_3}D_{xx_4}D_{xy}D_{yx_3}D_{yx_4}$$

Second integral - analog calculation

$$\frac{\delta}{\delta J(x_4)}\frac{\delta}{\delta J(x_3)}\frac{\delta}{\delta J(x_2)}\frac{\delta}{\delta J(x_1)}\int dx\,du\,dw\,dy\,dz_1\,dz_2\;J(x)D_{xu}D_{uw}J(w)D_{uy}D_{yz_1}J(z_1)D_{yz_2}J(z_2) \tag{215}$$
$$= 2\int du\,dy\;D_{x_1u}D_{ux_2}D_{uy}D_{yx_3}D_{yx_4} + 2\int du\,dy\;D_{x_2u}D_{ux_2}D_{uy}D_{yx_3}D_{yx_4} + 2\int du\,dy\;D_{x_1u}D_{ux_3}D_{uy}D_{yx_2}D_{yx_4} \tag{216}$$
$$+2\int du\,dy\;D_{x_3u}D_{ux_1}D_{uy}D_{yx_2}D_{yx_4} + 2\int du\,dy\;D_{x_1u}D_{ux_4}D_{uy}D_{yx_2}D_{yx_3} + 2\int du\,dy\;D_{x_4u}D_{ux_1}D_{uy}D_{yx_2}D_{yx_3} \tag{217}$$
$$+2\int du\,dy\;D_{x_2u}D_{ux_3}D_{uy}D_{yx_1}D_{yx_4} + 2\int du\,dy\;D_{x_3u}D_{ux_2}D_{uy}D_{yx_1}D_{yx_4} + 2\int du\,dy\;D_{x_2u}D_{ux_4}D_{uy}D_{yx_1}D_{yx_3} \tag{218}$$
$$+2\int du\,dy\;D_{x_4u}D_{ux_2}D_{uy}D_{yx_1}D_{yx_3} + 2\int du\,dy\;D_{x_3u}D_{ux_4}D_{uy}D_{yx_1}D_{yx_2} + 2\int du\,dy\;D_{x_4u}D_{ux_3}D_{uy}D_{yx_1}D_{yx_2} \tag{219}$$

This last two expression is all that survives the funtional derivatives and setting $J \to 0$.

Now we can calculate one example term of the first integral using the Klein-Gordon Greens function property of the Feynman propagator $(\Box_x - m^2)D_F(x-y) = i\delta^{(4)}(x-y)$

$$\int d^4x_1\,e^{i(k_1x_1)}(\Box_{x_1} - m^2)\int dx\,dy\,D_{xx_1}D_{xx_2}D_{xy}D_{yx_3}D_{yx_4} = \int dx\,dy\,e^{i(k_1x_1)}\,(-i)\delta^{(4)}(x-x_1)D_{xx_2}D_{xy}D_{yx_3}D_{yx_4} \tag{220}$$
$$= -i\int dx\,dy\,e^{i(k_1x)}\,D_{xx_2}D_{xy}D_{yx_3}D_{yx_4} \tag{221}$$

so we can generalize to the x_1, x_2, x_3, x_4 integration - and do the Fourier trafo of the Feynman propagator (via substitution $u = x - y, v = y$, meaning $x = u + v, y = v, dxdy = dudv$)

$$\int\prod_{i=1}^{4} d^4x_i\,e^{i(k_ix_i)}(\Box_{x_i} - m^2)\int dx\,dy\,D_{xx_1}D_{xx_2}D_{xy}D_{yx_3}D_{yx_4} = (-i)^4\int dx\,dy\,e^{i(k_1x+k_2x+k_3y+k_4y)}D_{xy} \tag{222}$$
$$= \int dx\,dy\,e^{i(k_1+k_2)x+i(k_3+k_4)y}D_F(x-y) \tag{223}$$
$$= \int du\,dv\,e^{i(k_1+k_2)(u+v)+i(k_3+k_4)v}D_F(u) \tag{224}$$
$$= \int dv\,e^{i(k_1+k_2+k_3+k_4)v}\cdot\int du\,e^{i(k_1+k_2)u}D_F(u) \tag{225}$$
$$= (2\pi)^4\delta^{(4)}(k_1+k_2+k_3+k_4)\frac{i}{(k_1+k_2)^2 - m^2 + i\epsilon} \tag{226}$$

we see that all the 6 different terms end up in the similar expression.

Now we can do one term of the second integral

$$\int \prod_{i=1}^{4} d^4x_i\, e^{i(k_i x_i)}(\Box_{x_i} - m^2) \int du\, dy\;\, D_{x_1 u} D_{u x_2} D_{uy} D_{y x_3} D_{y x_4} = (-i)^4 \int du\, dy\, e^{i(k_1 u + k_2 u + k_3 y + k_4 y)} D_F(u-y) \tag{227}$$

$$= (-i)^4 \int du\, dy\, e^{i(k_1+k_2)u} e^{i(k_3+k_4)y} D_F(u-y) \tag{228}$$

$$= (2\pi)^4 \delta^{(4)}(k_1+k_2+k_3+k_4) \frac{i}{(k_1+k_2)^2 - m^2 + i\epsilon} \tag{229}$$

So in total we should get three types of terms for the pairs (s, t, u-channels) k_1, k_2 and k_1, k_3 and k_1, k_4 (I would need some more time to collect all prefactors)

$$\int \prod_{i=1}^{4} d^4x_i\, e^{i(k_i x_i)}(\Box_{x_i} - m^2)\; \frac{\delta}{\delta J(x_i)} S[(J)]\bigg|_{J=0} = \lambda^2 \int \prod_{i=1}^{4} d^4x_i\, e^{i(k_i x_i)}(\Box_{x_i} - m^2)\; \frac{\delta}{\delta J(x_i)} S_2[(J)]\bigg|_{J=0} \tag{230}$$

$$\sim \lambda^2 (2\pi)^4 \delta^{(4)}(k_1+k_2+k_3+k_4) \left(\frac{1}{(k_1+k_2)^2 - m^2} + \frac{1}{(k_1+k_3)^2 - m^2} + \frac{1}{(k_1+k_4)^2 - m^2} \right) \tag{231}$$