Book of Solutions

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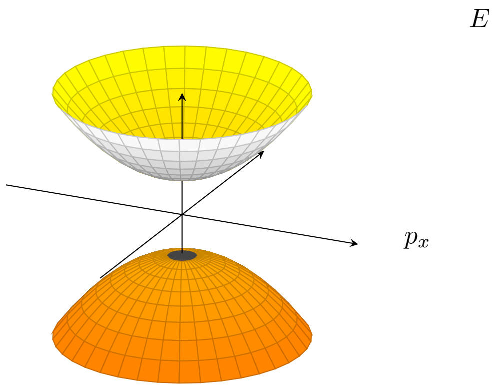
Chapter 1

Introduction

There is a theory which states that if ever anyone discovers exactly what the Universe is for and why it is here, it will instantly disappear and be replaced by something even more bizarre and inexplicable. There is another theory which states that this has already happened.



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Chapter 2

Useful formulas

$$\left(\int_{-\infty}^{\infty} dx e^{-x^2}\right)^2 = \int_{-\infty}^{\infty} dx e^{-x^2} \cdot \int_{-\infty}^{\infty} dy e^{-y^2} \tag{2.1}$$

$$= \int_{\mathbb{R}^2} e^{-(x^2+y^2)} dx\, dy \tag{2.2}$$

$$= \int_0^{2\pi} \int_0^{2\pi} e^{-r^2} r dr \tag{2.3}$$

$$= -2\pi \left. \frac{e^{-r^2}}{2} \right|_0^{\infty} = \pi \tag{2.4}$$

2.1 Common integrals

$$\int_{-\infty}^{\infty} dx e^{-ax^2} = \sqrt{\frac{\pi}{a}} \qquad a > 0, a \in \mathbb{R} \tag{2.5}$$

$$\int_{-\infty}^{\infty} dx e^{-ax^2+bx+c} = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}+c} \qquad a > 0, a, b, c \in \mathbb{R} \tag{2.6}$$

$$\int_{-\infty}^{\infty} dx e^{iax^2} = \sqrt{\frac{\pi}{a}} e^{\frac{i\pi}{4}} \qquad a > 0, a \in \mathbb{R} \tag{2.7}$$

$$\text{modified Bessel} \quad K_0(a\beta) = \int_0^{\infty} dx \frac{\cos(ax)}{\sqrt{\beta^2 + x^2}} \qquad \text{Gradshteyn, Ryzhik 7ed (3.754)} \tag{2.8}$$

$$\text{modified Bessel} \quad K_1(a\beta) = \frac{1}{\beta} \int_0^{\infty} dx \frac{x \sin(ax)}{\sqrt{\beta^2 + x^2}} \tag{2.9}$$

2.2 Common Fourier integrals

$$\int_{-\infty}^{\infty} dy e^{-ay^2} e^{-iby} = \sqrt{\frac{\pi}{a}} e^{-\frac{b^2}{4a}} \qquad a > 0, a, b \in \mathbb{R} \tag{2.10}$$

$$\int_{-\infty}^{\infty} dy e^{iay^2} e^{-iby} = \sqrt{\frac{\pi}{a}} e^{\frac{i}{4}\left(\pi - \frac{b^2}{a}\right)} \qquad a > 0, a, b \in \mathbb{R} \tag{2.11}$$

$$\int_{-\infty}^{\infty} dy e^{-(a+ic)y^2} e^{-iby} = \sqrt{\frac{\pi}{a+ic}} e^{-\frac{b^2}{4(a+ic)}} \qquad a > 0, a, b, c \in \mathbb{R} \tag{2.12}$$

$$= \sqrt{\frac{\pi}{a^2+c^2}} \sqrt{a-ic}\, e^{-\frac{b^2}{4(a^2+c^2)}(a-ic)} \tag{2.13}$$

2.3 Residue theorem

$$\int_\Gamma f = 2\pi i \sum_{a\in D_{\text{Singu}}} \text{ind}_\Gamma(a)\,\text{Res}_a f \tag{2.14}$$

Winding number $\text{ind}_\Gamma(a)$, Residue $\text{Res}_a f = c_{-1}$ from Laurent series at singularity a

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n \tag{2.15}$$

2.4 Common contour integrals

$$G(t-t') = -\frac{1}{2\pi}\int_{-\infty}^{+\infty} dE \frac{e^{-iE(t-t')}}{E^2-\omega^2+i\epsilon} = \frac{i}{2\omega}e^{i\omega|t|} \qquad \text{Sredniki (7.12)} \tag{2.16}$$

$$D(x-y) = \frac{4\pi}{(2\pi)^3}\int_0^\infty dp \frac{p^2 e^{i\sqrt{p^2+m^2}t}}{2\sqrt{p^2+m^2}} = \frac{1}{4\pi^2}\int_m^\infty dE\sqrt{E^2-m^2}e^{iEt} \qquad \text{PS (2.51)} \tag{2.17}$$

$$D(x-y) = \frac{-i}{2(2\pi)^2 r}\int_{-\infty}^{+\infty} dp \frac{pe^{ipr}}{\sqrt{p^2+m^2}} = \frac{1}{4\pi^2 r}\int_m^\infty d\rho \frac{\rho e^{-\rho r}}{\sqrt{\rho^2-m^2}} \qquad \text{PS (2.52)} \tag{2.18}$$

$$V(r) = \frac{1}{(2\pi)^2 ir}\int_{-\infty}^{\infty} dp \frac{pe^{ipr}}{p^2+m^2} = \frac{1}{4\pi r}e^{-mr} \qquad \text{PS (4.126)} \tag{2.19}$$

2.5 Feynman integral tricks

2.5.1 First example

$$\int_0^\infty \frac{e^{-t^2(x^2+1)}}{x^2+1}dx \tag{2.20}$$

2.5.2 Second example

We are trying to evaluate the integral without using contour integrals

$$\int_{-\infty}^{\infty} \frac{\log(x^4+1)}{x^2+1} = 2\int_0^\infty \frac{\log(x^4+1)}{x^2+1}dx \tag{2.21}$$

$$= 2\int_0^\infty \frac{\log[(x^2-i)(x^2+i)]}{x^2+1}dx \tag{2.22}$$

$$= 2\int_0^\infty \frac{\log(x^2-i)}{x^2+1} + \frac{\log(x^2+i)}{x^2+1}dx \tag{2.23}$$

$$= 2\left(I(-i)+I(i)\right) \tag{2.24}$$

Now the trick - come up with a parameter t inside the integral

$$\begin{aligned}
I(t) &= \int_0^\infty \frac{\log(x^2+t)}{x^2+1}dx && (2.25)\\
I(0) &= \int_0^\infty \frac{2\log(x)}{x^2+1}dx && (2.26)\\
&\stackrel{x=1/u}{=} \int_\infty^0 \frac{2\log(1/u)}{1/u^2+1}\frac{-1}{u^2}du && (2.27)\\
&= (-1)^2\int_0^\infty \frac{-2\log(u)}{1+u^2}du && (2.28)\\
&= -I(0) = 0 && (2.29)
\end{aligned}$$

and differentiate with respect to t (without checking if are allowed to switch the integral and the differentiation)

$$\begin{aligned}
\frac{dI}{dt} &= \int_0^\infty \frac{1}{x^2+1}\frac{1}{x^2+t}dx && (2.30)\\
&= \int_0^\infty \frac{1/(t-1)}{x^2+1} + \frac{1/(1-t)}{x^2+t}dx && (2.31)\\
&= \left.\frac{\arctan x}{t-1}\right|_0^\infty - \left.\frac{\arctan\frac{x}{\sqrt{t}}}{(t-1)\sqrt{t}}\right|_0^\infty && (2.32)\\
&= \frac{\pi}{2}\frac{1}{t-1}\frac{\sqrt{t}-1}{\sqrt{t}} && (2.33)\\
&= \frac{\pi}{2}\frac{1}{\sqrt{t}(1+\sqrt{t})} && (2.34)
\end{aligned}$$

and now we can integrate

$$\begin{aligned}
I(t) &= \frac{\pi}{2}\int \frac{1}{\sqrt{t}(1+\sqrt{t})}dt && (2.35)\\
&\stackrel{u=\sqrt{t}}{=} \frac{\pi}{2}\int \frac{1}{u(1+u)}2u\,du \qquad \frac{du}{dt} = \frac{1}{2\sqrt{t}}, \rightarrow dt = 2u\,du && (2.36)\\
&= \pi\log(1+u) + c \qquad \text{with } I(0) = 0 \rightarrow c = 0 && (2.37)\\
&= \pi\log(1+\sqrt{t}) && (2.38)
\end{aligned}$$

then with $i = e^{i\pi/2+2\pi k}$ and $-i = e^{-i\pi/2+2\pi n}$

$$\begin{aligned}
\int_{-\infty}^\infty \frac{\log(x^4+1)}{x^2+1} &= 2\left(I(-i)+I(i)\right) && (2.39)\\
&= 2\pi\left(\log(1+\sqrt{-i}) + \log(1+\sqrt{i})\right) && (2.40)\\
&= 2\pi\log[(1+\sqrt{-i})(1+\sqrt{i})] && (2.41)\\
&= 2\pi\log[1+\sqrt{-i}+\sqrt{i}+\sqrt{-i^2}] && (2.42)\\
&= 2\pi\log[2+\sqrt{-i}+\sqrt{i}] && (2.43)\\
&= 2\pi\log[2+e^{-i\pi/4}e^{i\pi n}+e^{i\pi/4}e^{i\pi k}] && (2.44)\\
&= 2\pi\log[2+\sqrt{2}] \qquad \text{just setting } n,k=0 \text{ to ensure a real solution} && (2.45)\\
&= \pi\log[(2+\sqrt{2})^2] && (2.46)\\
&= \log[(6+4\sqrt{2})^\pi] && (2.47)\\
& && (2.48)
\end{aligned}$$

2.6 Fourier transformation

Starting from the Fourier integral theorem we have some freedom to distribute the 2π between back and forth transformation ($a, b \in \mathbb{R}$)

$$F(k) = \sqrt{\frac{|b|}{(2\pi)^{1-a}}} \int_{-\infty}^{\infty} f(x) e^{ibkx} dx \quad \leftrightarrow \quad f(x) = \sqrt{\frac{|b|}{(2\pi)^{1+a}}} \int_{-\infty}^{\infty} F(t) e^{-ibkx} dk \tag{2.49}$$

2.7 Laplace transformation

$$Y(s) = \int_0^{\infty} f(t) e^{-st} dt \tag{2.50}$$

then

$$\int_0^{\infty} f'(t) e^{-st} dt = f(t) e^{-st}\big|_0^{\infty} - \int_0^{\infty} f(t)(-s) e^{-st} dt \tag{2.51}$$

$$= -f(0) + s \int_0^{\infty} f(t) e^{-st} dt \tag{2.52}$$

$$= sY(s) - f(0) \tag{2.53}$$

$$\int_0^{\infty} f''(t) e^{-st} dt = ... \tag{2.54}$$

$$= s^2 Y(s) - s f'(0) - f(0) \tag{2.55}$$

2.8 Delta distribution

$$x\delta(x) = 0 \tag{2.56}$$

$$\int \delta(x) e^{-ikx} dx = 1 \tag{2.57}$$

$$\int e^{ik(x-y)} dk = 2\pi\delta(x-y) \tag{2.58}$$

$$\int g(x)\delta(f(x))dx = \sum_{x_i : f(x_i)=0} \int_{x_i-\epsilon}^{x_i+\epsilon} g(x)\delta(f(x))dx \tag{2.59}$$

$$= \sum_{x_i} \int_{x_i-\epsilon}^{x_i+\epsilon} g(x)\delta\left(f(x_i) + f'(x_i)(x-x_i) + \frac{1}{2} f''(x_i)(x-x_i)^2 + ...\right) dx \tag{2.60}$$

$$= \sum_{x_i} \int_{x_i-\epsilon}^{x_i+\epsilon} g(x)\delta\left(f'(x_i)(x-x_i)\right) dx \tag{2.61}$$

$$= \sum_{x_i} \int_{(x_i-\epsilon)f'}^{(x_i+\epsilon)f'} g\left(\frac{u}{f'(x_i)}\right) \delta\left(u - f'(x_i)x_i\right) \frac{1}{f'(x_i)} du \tag{2.62}$$

$$= \sum_{x_i} \int_{(x_i-\epsilon)|f'|}^{(x_i+\epsilon)|f'|} g\left(\frac{u}{f'(x_i)}\right) \frac{1}{|f'(x_i)|} \delta\left(u - f'(x_i)x_i\right) du \tag{2.63}$$

$$= \sum_{x_i} g(x_i) \frac{1}{|f'(x_i)|} \tag{2.64}$$

Important restriction: x_i are the **simple** zeros

2.9 Bessel functions

- Bessel ODE $x^2y'' + xy' + (x^2 - \nu^2)y = 0 \qquad \mathrm{Re}\,\nu \geq 0$

$$y = c_1 y_1 + c_2 y_2 \tag{2.65}$$
$$= \begin{cases} c_1 J_\nu + c_2 J_{-\nu} & \nu \notin \mathbb{Z} \\ c_1 J_\nu + c_2 Y_\nu & \nu = 0, 1, 2, \ldots \end{cases} \tag{2.66}$$

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k} \qquad \text{Bessel function} \tag{2.67}$$
$$Y_\nu(x) = \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi} \qquad \text{Neumann/Weber function} \tag{2.68}$$
$$Y_n(x) = \lim_{\alpha\to n} Y_\alpha(x) \tag{2.69}$$
$$H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x) \qquad \text{Hankel function 1. kind} \tag{2.70}$$
$$H_\nu^{(2)}(x) = J_\nu(x) - iY_\nu(x) \qquad \text{Hankel function 2. kind} \tag{2.71}$$

If $\nu = n$ then

$$J_n = \frac{1}{\pi}\int_0^\pi \cos(x\sin\varphi - n\varphi)d\varphi \tag{2.72}$$
$$= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{i(x\sin\varphi - n\varphi)}d\varphi \tag{2.73}$$

- Modified Bessel ODE $x^2y'' + xy' - (x^2 + \nu^2)y = 0$

$$I_\nu(x) = i^{-\nu} J_\nu(ix) \tag{2.74}$$
$$K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin\nu\pi} \tag{2.75}$$
$$K_n(x) = \lim_{\alpha\to n} K_\alpha(x) \tag{2.76}$$
$$\tag{2.77}$$

If $\mathrm{Re}\,x > 0$ then

$$K_n = \int_0^\pi e^{-x\cosh t}\cosh\nu t\,dt \tag{2.78}$$

2.10 Γ, ζ function

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt \tag{2.79}$$
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{2.80}$$

then with $t/n = x$ and $dx = dt/n$

$$\zeta(s)\Gamma(s) = \sum_{n=1}^{\infty} \int_0^{\infty} \frac{1}{n^s} t^{s-1} e^{-t} dt \tag{2.81}$$

$$= \sum_{n=1}^{\infty} \int_0^{\infty} \frac{1}{n^s} t^{s-1} e^{-t} n\, dx \tag{2.82}$$

$$= \sum_{n=1}^{\infty} \int_0^{\infty} \frac{t^{s-1}}{n^{s-1}} e^{-nx} dx \tag{2.83}$$

$$= \int_0^{\infty} x^{s-1} \sum_{n=1}^{\infty} e^{-nx} dx \tag{2.84}$$

$$= \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx \tag{2.85}$$

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx \tag{2.86}$$

2.11 n-dimensional unit spheres

$$\pi^{n/2} = \left(\int_{-\infty}^{\infty} dt e^{-t^2} \right)^n \tag{2.87}$$

$$= \int_{R^n} e^{-|x|^2} dx \tag{2.88}$$

$$= \int_0^{\infty} \int_{\omega_n} e^{-r^2} r^{n-1} dr\, ds \tag{2.89}$$

$$= \int_{\omega_n} ds \cdot \int_0^{\infty} e^{-r^2} r^{n-1} dr \tag{2.90}$$

$$= |\omega_n| \cdot \frac{1}{2} \int_0^{\infty} e^{-\rho} \rho^{\frac{n}{2}-1} d\rho \tag{2.91}$$

$$= |\omega_n| \cdot \frac{1}{2} \Gamma\left(\frac{n}{2}\right) \tag{2.92}$$

Therefore

$$|\omega_n| = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \tag{2.93}$$

$$V_n = |\omega_n| \int_0^1 r^{n-1} dr \tag{2.94}$$

$$= \frac{|\omega_n|}{n} \tag{2.95}$$

2.12 Vector Analysis

Identities

$$\nabla \times \nabla \phi \equiv 0 \tag{2.96}$$

$$\nabla \cdot \nabla \times \mathbf{A} \equiv \mathbf{0} \tag{2.97}$$

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \triangle \mathbf{A} \tag{2.98}$$

Gauss and Stokes Theorem

$$\oint_{\partial V} \mathbf{A} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{A}\, dV \tag{2.99}$$

$$\oint_{\partial S} \mathbf{A} \cdot d\mathbf{r} = \int_V \nabla \times \mathbf{A}\, d\mathbf{S} \tag{2.100}$$

Helmholtz-Hodge decomposition

$$\mathbf{E} = \mathbf{E}_{\parallel} + \mathbf{E}_{\perp} \tag{2.101}$$

$$\nabla \times \mathbf{E}_{\parallel} = 0 \tag{2.102}$$

$$\nabla \cdot \mathbf{E}_{\perp} = 0 \tag{2.103}$$

With

$$-\frac{1}{4\pi} \triangle \frac{1}{|\mathbf{x} - \mathbf{x}'|} = \delta(\mathbf{x} - \mathbf{x}') \tag{2.104}$$

$$\triangle \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla \times \nabla \times \mathbf{E} \tag{2.105}$$

we can construct

$$\mathbf{E}(\mathbf{x}) = \int \mathbf{E}(\mathbf{x}')\delta(\mathbf{x} - \mathbf{x}')dx' \tag{2.106}$$

$$= -\frac{1}{4\pi} \int \mathbf{E}(\mathbf{x}') \triangle \frac{1}{|\mathbf{x} - \mathbf{x}'|} dx' \tag{2.107}$$

$$= -\frac{1}{4\pi} \triangle \int \mathbf{E}(\mathbf{x}') \frac{1}{|\mathbf{x} - \mathbf{x}'|} dx' \tag{2.108}$$

$$= -\frac{1}{4\pi} \nabla \int \mathbf{E}(\mathbf{x}') \nabla \cdot \frac{1}{|\mathbf{x} - \mathbf{x}'|} dx' + \frac{1}{4\pi} \nabla \times \int \mathbf{E}(\mathbf{x}') \nabla \times \frac{1}{|\mathbf{x} - \mathbf{x}'|} dx' \tag{2.109}$$

$$\tag{2.110}$$

2.13 Laplace operator

$$\nabla \cdot X = \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} X^i \right) \tag{2.111}$$

$$(\nabla f)^i = g^{ij} \partial_j f \tag{2.112}$$

$$\triangle f = \nabla \cdot \nabla f \tag{2.113}$$

$$= \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} g^{ij} \partial_j f \right) \tag{2.114}$$

$$= \sum_i \frac{\partial^2}{\partial x_j^2} \tag{2.115}$$

$$\frac{\partial}{\partial x_i} \frac{\partial f}{\partial x_i} = \frac{\partial}{\partial x_i} \left(\frac{\partial y_j}{\partial x_i} \frac{\partial f}{\partial y_j} \right) \tag{2.116}$$

$$= \frac{\partial y_k}{\partial x_i} \frac{\partial}{\partial y_k} \left(\frac{\partial y_j}{\partial x_i} \frac{\partial f}{\partial y_j} \right) \tag{2.117}$$

$$= \frac{\partial^2 y_j}{\partial x_i^2} \frac{\partial f}{\partial y_j} + \frac{\partial y_j}{\partial x_i} \frac{\partial y_k}{\partial x_i} \frac{\partial^2 f}{\partial y_j \partial y_k} \tag{2.118}$$

$$\tag{2.119}$$

With $f = f(r)$ and $r = \sqrt{x_1^2 + ... + x_n^2}$ we have

$$\triangle f(r) = \sum_i \frac{r - x_i \frac{x_i}{r}}{r^2} \frac{\partial f}{\partial r} + \frac{x_i^2}{r^2} \frac{\partial^2 f}{\partial r^2} \tag{2.120}$$

$$= \frac{nr - r}{r^2} \frac{\partial f}{\partial r} + \frac{r^2}{r^2} \frac{\partial^2 f}{\partial r^2} \tag{2.121}$$

$$= \frac{(n-1)}{r} \frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial r^2} \tag{2.122}$$

2.14 ODE solving strategies

2.14.1 Special ODEs

Bernoulli	$y' + p(x)y + q(x)y^n = 0$	
Ricatti	$y' + p(x)y + q(x)y^2 = r(x)$	
d'Alembert	$y = x \cdot g(y') + h(y')$	
Exact	$M(x, y) + N(x, y)y' = 0 \quad (\partial_y M = \partial_x N)$	
Airy	$y'' - xy = 0$	
Bessel	$x^2 y'' + xy' + (x^2 - \nu^2)y = 0$	$\mathrm{Re}\,\nu \geq 0$
modified Bessel	$x^2 y'' + xy' - (x^2 + \nu^2)y = 0$	
Hermite	$y'' - 2xy' + 2ny = 0$	$n = 0, 1, 2, 3, ...$
Laguerre	$xy'' + (1 - x)y' + ny = 0$	$n = 0, 1, 2, 3, ...$
Legendre	$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$	$n = 0, 1, 2, 3, ...$
Weber Hermite	$y'' + \left(\nu + \frac{1}{2} - \frac{1}{4}x^2\right) y = 0$	

2.14.2 1st order ODE

1. Is separable $y' = g(x)h(y) \quad \rightarrow \quad \int \frac{dy}{h(y)} = \int g(x)dx$ - done.

2. Is linear homogen $y' + f(x)y = 0$ go to 1.

$$y(x) = Ce^{-\int f(x)dx} \tag{2.123}$$

3. Is linear inhomogen $y' + f(x)y = g(x)$ general solution

$$y(x) = y_{\text{hom}}(x) + y_{\text{spec}}(x) \tag{2.124}$$
$$y_{\text{hom}}(x) = Ce^{-\int f(x)dx} \tag{2.125}$$
$$y_{\text{spec}}(x) = C(x)e^{-\int f(x)dx} \tag{2.126}$$
$$\rightarrow C(x)'e^{-\int f(x)dx} - f(x)C(x)e^{-\int f(x)dx} + f(x)C(x)e^{-\int f(x)dx} = g(x) \tag{2.127}$$
$$\rightarrow C(x)' = g(x)e^{\int f(x)dx} \tag{2.128}$$
$$\rightarrow C(x) = \int g(x)e^{\int f(x)dx}dx + c_1 \tag{2.129}$$
$$y(x) = Ce^{-\int f(x)dx} + \left(\int g(x)e^{\int f(x)dx}dx + c_1\right) e^{-\int f(x)dx} \tag{2.130}$$

solve homogen (go to 2) then variation of constants.

4. Is linear Bernoulli $y' + f(x)y + g(x)y^n = 0$ divide by y^n and subs $z = \frac{1}{y^{n-1}}$ then go to 3.

5. Is linear Ricatti $y' + f(x)y + g(x)y^2 = r(x)$ substitude with $y = Q\frac{w'}{w}$ to linearize it

$$Q'\frac{w'}{w} + \frac{Qw''}{w} - \frac{Qw'^2}{w^2} + fQ\frac{w'}{w} + gQ^2\frac{w'^2}{w^2} = r \tag{2.131}$$
$$Q'\frac{w'}{w} + \frac{Qw''}{w} + (gQ-1)\,Q\frac{w'^2}{w^2} + fQ\frac{w'}{w} = r \qquad gQ - 1 = 0 \tag{2.132}$$
$$Q'w' + Qw'' + fQw' = rw \tag{2.133}$$
$$w'' + \frac{Q' + fQ}{Q}w' - \frac{r}{Q}w = 0 \quad Q = 1/g, Q' = -1/g^2 \tag{2.134}$$
$$w'' + \left(-\frac{1}{g} + f\right) w' - rgw = 0 \tag{2.135}$$

6. Is exact $M(x,y) + N(x,y)y' = 0$ with $M_y = N_y$ then solution $\Phi(x,y) = C$ because

$$0 = \frac{d\Phi(x,y)}{dx} = \frac{\partial\Phi}{\partial x} + \frac{\partial\Phi}{\partial y}\frac{dy}{dx} \tag{2.136}$$
$$d\Phi = M(x,y)dx + N(x,y)dy \tag{2.137}$$
$$\Phi(x,y) = \int_{y_0}^{y} N(x,v)dv + \int_{x_0}^{x} M(u,y)du \tag{2.138}$$

or easier

$$\frac{\partial\Phi}{\partial x} = M \quad \rightarrow \quad \Phi = \int Mdx + G(y) \quad \rightarrow \quad \frac{\partial\Phi}{\partial y} = N \tag{2.139}$$

7. If nothing works try if of form $y' = f\left(\frac{y}{x}\right)$ and subs $z = y/x$ and go to 1.

8. If still nothing works try $y = u(x) \cdot v(x)$

2.14.3 2nd order ODE

Linear homegeneous equation - we can simplify

$$y'' + a(x)y' + b(x)y = 0 \quad \to \quad y = f(x)u \tag{2.140}$$
$$f''u + 2f'u' + u'' + a(f'u + fu') + bfu = 0 \tag{2.141}$$
$$u'' + (2f' + af)u' + (f'' + af' + bf)u = 0 \quad \to \quad 2f' + af = 0 \tag{2.142}$$
$$u'' + q(x)u = 0 \tag{2.143}$$

Why is it hard to solve 2nd order ODE

$$y'' + a(x)y' + b(x)y = 0 \tag{2.144}$$
$$(D^2 + aD + b)y = 0 \tag{2.145}$$

Lets try to factorize the differential operator

$$(D + A)(D + B)y = 0 \tag{2.146}$$
$$(D^2 + (A + B)D + B' + AB)y = 0 \tag{2.147}$$

Once we know A and B we can solve

$$(D + B)y \equiv w \tag{2.148}$$
$$w' + Aw = 0 \quad \text{(very simple to solve for } w) \tag{2.149}$$
$$y' + by = w \quad \text{(simple to solve for } y) \tag{2.150}$$

But how to find A and B

$$A + B = a, \quad B' = -AB + b \tag{2.151}$$
$$\to B' = -aB + B^2 + b \tag{2.152}$$

which gives the Riccati equation - but linearizing it leads back to the same linear, homogeneous 2nd order equation (which we started with).

1. Is inhomogeneous equation with constant coefficients $ay'' + by' + cy = r(x)$

$$a[s^2Y - sy(0) - y'(0)] + b[sY - y(0)] + cY = \mathcal{L}(r(x)) \tag{2.153}$$
$$(as^2 + bs + c)Y - asy(0) - ay'(0) - by(0) = \mathcal{L}(r(x)) \tag{2.154}$$
$$Y = \frac{\mathcal{L}(r(x)) + (as + b)y(0) + ay'(0)}{as^2 + bs + c} \tag{2.155}$$

... write me ...

2.14.4 n-th order ODE

1. Linear homogen $c_n y^n + ... + c_2 y'' + c_1 y' + c_0 y = 0$ ansatz $y = e^{\alpha x}$ then solve polynom for α, for repeated root α_1 try $y = xe^{\alpha_1 x}, x^2 e^{\alpha_1 x}, ...$

2.15 Greenfunctions and ODEs

2.15.1 Harmonic Oscillator

$$\ddot{G}(t - t') + 2\gamma\dot{G}(t - t') + \omega_0 G(t - t') = \delta(t - t') \tag{2.156}$$

with $G(t-t') = (2\pi)^{-1/2} \int e^{i\omega(t-t')} y(\omega) d\omega$

$$\frac{1}{\sqrt{2\pi}} \int e^{i\omega(t-t')} \left((i\omega)^2 y + 2\gamma(i\omega)y + \omega_0 y \right) d\omega = \frac{1}{2\pi} \int e^{i\omega(t-t')} d\omega \tag{2.157}$$

then

$$y(\omega) = \frac{1}{\sqrt{2\pi}} \frac{1}{\omega_0^2 - \omega^2 + 2i\gamma\omega} \tag{2.158}$$

$$G(t-t') = \frac{1}{2\pi} \int \frac{e^{-i\omega(t-t')}}{\omega_0^2 - \omega^2 + 2i\gamma\omega} d\omega \tag{2.159}$$

and the general solution is given by

$$\ddot{x}(t) + 2\gamma\dot{x}(t) + \omega_0 x(t) = f(t) \quad \rightarrow \quad x(t) = \int G(t-t') f(t) \tag{2.160}$$

2.16 Greenfunctions and PDEs

The Greensfunction $G(x, y)$ for a general PDE $D_x u(x) = f(x)$ is defined by

$$D_x G(x, y) = \delta(x - y). \tag{2.161}$$

This means that general solution of the PDE can be expressed as

$$u(x) = \int G(x, y) f(y) dy \tag{2.162}$$

because

$$\begin{aligned} D_x u(x) &= D_x \int G(x, y) f(y) dy && (2.163) \\ &= \int D_x G(x, y) f(y) dy && (2.164) \\ &= \int \delta(x - y) f(y) dy && (2.165) \\ &= f(x) && (2.166) \end{aligned}$$

2.16.1 Poisson equation $\triangle u(x) = f(x)$

The n-dimensional Fourier transform of $\triangle_x G(x,y) = \delta(x-y)$ and integration by parts gives

$$\frac{1}{(2\pi)^{n/2}} \int d^n x \, \triangle_x G(x,y) e^{-ikx} = \frac{1}{(2\pi)^{n/2}} \underbrace{\int d^n x \, \delta(x-y) e^{-ikx}}_{=e^{-iky}} \tag{2.167}$$

$$\frac{1}{(2\pi)^{n/2}} \int d^n x \, G(x,y)(-ik)^2 e^{-ikx} = \frac{1}{(2\pi)^{n/2}} e^{-iky} \tag{2.168}$$

$$(-ik)^2 g(k) = \frac{1}{(2\pi)^{n/2}} e^{-iky} \tag{2.169}$$

$$\rightarrow g(k) = -\frac{1}{(2\pi)^{n/2}} \frac{1}{k^2} e^{-iky} \tag{2.170}$$

we can now use the Fourier transform of the Greensfunction and transform back.

- Case $n=1$: The function has a pole at $k=0$ and the Laurent series is given by

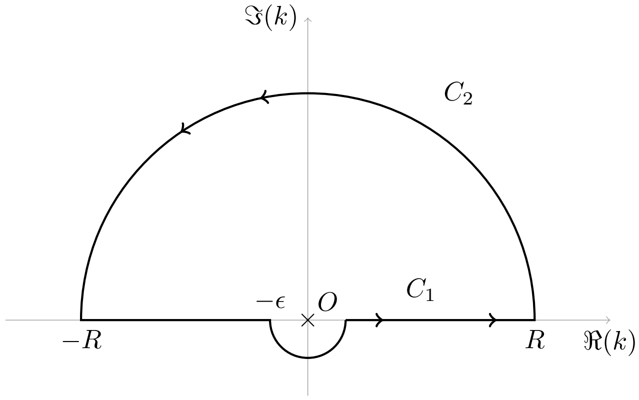
$$\frac{e^{ik(x-y)}}{k^2} = \frac{1}{k^2} + i(x-y)\frac{1}{k} - \frac{(x-y)^2}{2} - \frac{i(x-y)^3}{6} k + ... \tag{2.171}$$

  with $\mathrm{Res} = i(x-y)$. We can now use the residue theorem to evaluate the integral

$$G(x,y) = -\frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \, \frac{e^{ik(x-y)}}{k^2} = -\frac{1}{2\pi} \int_{C_1} dk \, \frac{e^{ik(x-y)}}{k^2} \tag{2.172}$$

$$= -\frac{1}{2\pi} \left(\underbrace{\int_C dk \, \frac{e^{ik(x-y)}}{k^2}}_{=2\pi i \text{ Res}} - \underbrace{\int_{C_2} dk \, \frac{e^{ik(x-y)}}{k^2}}_{=0} \right) \tag{2.173}$$

$$= (x-y) \tag{2.174}$$



- Case $n=2$:

$$G(x,y) = -\frac{1}{2\pi} \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_1 dk_2 \, \frac{e^{i(k_1(x_1-y1)+k_2(x_2-y2))}}{k_1^2 + k_2^2} \tag{2.175}$$

$$= -\frac{1}{4\pi^2} \int_0^{\infty} \int_0^{2\pi} dk \, d\phi \, \frac{e^{ik|x-y|\cos\phi}}{k^2} k \tag{2.176}$$

$$= \frac{1}{2\pi} \int_0^{\infty} dk \frac{1}{k} \frac{1}{2\pi} \int_0^{2\pi} d\phi \, e^{ik|x-y|\cos\phi} \tag{2.177}$$

$$= \frac{1}{2\pi} \int_0^{\infty} dk \, \frac{J_0(k|x-y|)}{k} = -\frac{1}{2\pi} \int_0^{\infty|x-y|} dk' \, \frac{J_0(k')}{k'} \tag{2.178}$$

The last integral diverges but we try a nasty trick (?!?)

$$\frac{dG}{dx} = -\frac{1}{2\pi}\frac{d}{dx}\int_0^\infty dk\,\frac{J_0(k|x-y|)}{k} \tag{2.179}$$

$$= -\frac{1}{2\pi}\int_0^\infty dk\,J_1(k|x-y|) \tag{2.180}$$

$$= -\frac{1}{2\pi}\frac{1}{|x-y|} \tag{2.181}$$

Now simple integration yields

$$G(x,y) = -\frac{1}{2\pi}\log(|x-y|) \tag{2.182}$$

- Case $n = 3$:

$$G(x,y) = \frac{1}{(2\pi)^3}\int d^3k\,\frac{1}{k^2}e^{ik(x-y)} \tag{2.183}$$

$$= \frac{1}{(2\pi)^3}\int dk\underbrace{\int d\phi}_{=|\omega_2|}\int d\theta\,e^{ik|x-y|\cos\theta}\sin\theta \tag{2.184}$$

$$= -\frac{1}{(2\pi)^2}\int dk\int_{-1}^{+1} e^{ik|x-y|\cos\theta}d\cos\theta \tag{2.185}$$

$$= -\frac{1}{(2\pi)^2}\int dk\frac{e^{ik|x|}-e^{-ik|x-y|}}{ik|x-y|} \tag{2.186}$$

$$= -\frac{1}{2\pi^2}\int_0^\infty dk\frac{\sin k|x-y|}{k|x-y|} \tag{2.187}$$

$$= -\frac{1}{2\pi^2}\frac{1}{|x-y|}\int_0^\infty dk'\frac{\sin k'}{k'} \tag{2.188}$$

$$= -\frac{1}{4\pi}\frac{1}{|x-y|} \tag{2.189}$$

- Case $n > 3$: ...

Alternatively we can use the Gauss theorem with $\vec{F} = \nabla_x G(x,y)$

$$\int_V \nabla\cdot\vec{F}dx = \int_{\partial V}\vec{F}\cdot d\vec{S} \tag{2.190}$$

$$\int_{K_r(y)}\triangle_x G(x,y)dx = \int_{\partial_{K_r(y)}}\nabla G\cdot d\vec{S} \tag{2.191}$$

$$1 = \frac{\partial G(r,0)}{\partial r}|\omega_n|r^{n-1} \tag{2.192}$$

$$\frac{\partial G(r,0)}{\partial r} = \frac{r^{-n+1}}{|\omega_n|} \tag{2.193}$$

$$G(x,y) = \begin{cases} \frac{1}{|\omega_2|}\log|x-y| & n=2 \\ -\frac{1}{|\omega_n|(n-2)}\frac{1}{|x-y|^{n-2}} & n\geq 3 \end{cases} \tag{2.194}$$

2.16.2 Wave equation $\left(\frac{1}{c^2}\partial_{tt} - \triangle\right) u(x,t) = j(x,t)$

- The free fundamental solution (no source with $j(x,t) = 0$)

$$u(\vec{x},t) = e^{-i(k_0 t - \vec{k}\vec{x})} \tag{2.195}$$
$$\rightarrow \quad -\frac{k_0^2}{c^2} + \vec{k}^2 + \mu^2 = 0 \tag{2.196}$$
$$\rightarrow \quad k_0 = \pm c\sqrt{\vec{k}^2} \tag{2.197}$$

- The free solution (no source with $f(x,t) = 0$) with initial conditions

$$u(\vec{x},0) = u_0(\vec{x}), \qquad \left.\frac{\partial u}{\partial x}\right|_{t=0} = u_1(\vec{x}) \tag{2.198}$$

Then we find by applying the differential operator to the Fourier transformation

$$u(\vec{x},t) = \frac{1}{(2\pi)^4}\int d^3k \int d\omega\, \tilde{u}(\vec{k},\omega) e^{i(\vec{k}\vec{x}-\omega t)} \tag{2.199}$$
$$\left(\frac{1}{c^2}\partial_{tt} - \triangle\right) u(\vec{x},t) = \frac{1}{(2\pi)^4}\int d^3k \int d\omega\, \tilde{u}(\vec{k},\omega) \left(\frac{1}{c^2}\partial_{tt} - \triangle\right) e^{i(\vec{k}\vec{x}-\omega t)} = 0 \tag{2.200}$$
$$\left(\frac{\omega^2}{c^2} - \vec{k}^2\right)\tilde{u}(\vec{k},\omega) = 0 \qquad \rightarrow \omega = \pm ck \tag{2.201}$$

This leads to the ansatz which we can transform back

$$\tilde{u}(\vec{k},\omega) = a_+(\vec{k})\delta(\omega + ck) + a_-(\vec{k})\delta(\omega - ck) \tag{2.202}$$
$$u(\vec{x},t) = \frac{1}{(2\pi)^4}\int d^3k \int d\omega \left(a_+(\vec{k})\delta(\omega + ck) + a_-(\vec{k})\delta(\omega - ck)\right) e^{i(\vec{k}\vec{r}-\omega t)} \tag{2.203}$$
$$= \frac{1}{(2\pi)^4}\int d^3k \left(a_+(\vec{k})e^{i(\vec{k}\vec{x}+ckt)} + a_-(\vec{k})e^{i(\vec{k}\vec{x}-ckt)}\right) \tag{2.204}$$

Obeying the initial conditions

$$u_0(\vec{x}) = \frac{1}{(2\pi)^4}\int d^3k\, e^{i\vec{k}\vec{x}} \left(a_+(\vec{k}) + a_-(\vec{k})\right) \tag{2.205}$$
$$u_1(\vec{x}) = \frac{i}{(2\pi)^4}\int d^3k\, cke^{i\vec{k}\vec{x}} \left(a_+(\vec{k}) - a_-(\vec{k})\right) \tag{2.206}$$

then leads to expressions for $a_\pm$

$$\int d\vec{x} e^{-i\vec{q}\vec{x}} u_0(\vec{x}) = \frac{1}{(2\pi)^4}\int d^3k \int d\vec{x}\, e^{i(\vec{k}-\vec{q})\vec{x}} \left(a_+(\vec{k}) + a_-(\vec{k})\right) \tag{2.207}$$
$$= \frac{1}{2\pi}\int d^3k \delta(\vec{k}-\vec{q}) \left(a_+(\vec{k}) + a_-(\vec{k})\right) \tag{2.208}$$
$$= \frac{1}{2\pi}\left(a_+(\vec{q}) + a_-(\vec{q})\right) \tag{2.209}$$
$$\int d\vec{x} e^{-i\vec{q}\vec{x}} u_1(\vec{x}) = \frac{icq}{2\pi}\left(a_+(\vec{q}) - a_-(\vec{q})\right) \tag{2.210}$$
$$\rightarrow a_\pm(\vec{q}) = \pi \int d\vec{x} e^{-i\vec{q}\vec{x}} \left(u_0(\vec{x}) \mp \frac{i}{cq} u_1(\vec{x})\right) \tag{2.211}$$

Inserting $a_\pm$ into the original Fourier transform (and renaming the integration variable x by y)

$$u(\vec{x},t) = \frac{1}{2(2\pi)^3}\int d^3y \int d^3k e^{i\vec{k}(\vec{x}-\vec{y})}\left[\left(u_0(\vec{y}) - \frac{i}{ck}u_1(\vec{y})\right)e^{ickt} + \left(u_0(\vec{y}) + \frac{i}{cq}u_1(\vec{y})\right)e^{-ickt}\right] \tag{2.212}$$

$$= \frac{1}{2(2\pi)^3}\int d^3y \int d^3k e^{i\vec{k}(\vec{x}-\vec{y})}\left[\left(e^{ickt}+e^{-ickt}\right)u_0(\vec{y}) - \frac{i}{ck}\left(e^{ickt}-e^{-ickt}\right)u_1(\vec{y})\right] \tag{2.213}$$

$$= \int d^3y\left[\partial_t D(\vec{x}-\vec{y},t)u_0(\vec{y}) + D(\vec{x}-\vec{y},t)u_1(\vec{y})\right] \tag{2.214}$$

with

$$D(\vec{z},t) = -\frac{i}{2(2\pi)^3}\int d^3k \frac{e^{i\vec{k}\vec{z}}}{ck}(e^{ickt}-e^{-ickt}) \tag{2.215}$$

The above calculation is basically valid in any dimension so we will get explicit expressions for $n = 1, 2, 3$

1. $D(\vec{z},t)$ can be simplified (in one dimensions)

$$D(z,t) = -\frac{i}{2(2\pi)}\int dk \frac{e^{ikz}}{ck}(e^{ickt}-e^{-ickt}) \tag{2.216}$$

$$= -\frac{i}{4\pi c}\int_{-\infty}^{\infty} dk \frac{1}{k}\left(e^{-ik(-z-ct)} - e^{-ik(-z+ct)}\right) \tag{2.217}$$

$$= -\frac{i}{4\pi c}\left[-i\pi \mathrm{sgn}(-z-ct) + i\pi \mathrm{sgn}(-z+ct)\right] \tag{2.218}$$

$$= \frac{1}{4c}\left[\mathrm{sgn}(z+ct) + \mathrm{sgn}(-z+ct)\right] \tag{2.219}$$

$$= \frac{1}{4c}\left[\mathrm{sgn}(z+ct) - \mathrm{sgn}(z-ct)\right] \tag{2.220}$$

$$= \begin{cases} +\frac{1}{2c} & |z| < ct, \quad t > 0 \\ 0 & |z| > ct, \\ -\frac{1}{2c} & |z| < ct, \quad t < 0 \end{cases} \tag{2.221}$$

Which vanishes outside the light cone but NOT inside. The explicit solution (for $t > 0$) is then given as

$$\partial_t D(x-\xi,t) = \frac{1}{4c}\left[2\delta(x-\xi+ct)c + 2\delta(-(x-\xi)+ct)c\right] \tag{2.222}$$

$$u(x,t) = \int d\xi \frac{1}{4c}\left[2\delta(x-\xi+ct)c + 2\delta(-(x-\xi)+ct)c\right]u_0(\xi) \tag{2.223}$$

$$+ \frac{1}{4c}\int_{-\infty}^{+\infty}\left[\mathrm{sgn}((x-\xi)+ct) + \mathrm{sgn}(-(x-\xi)+ct)\right]u_1(\xi)d\xi \tag{2.224}$$

$$= \frac{1}{2}\left[u_0(x+ct) + u_0(x-ct)\right] + \frac{1}{2c}\int_{K(x)_{ct}} u_1(\xi)d\xi \tag{2.225}$$

where $K(x)_{ct}$ is a 1-dimensional sphere of radius ct around x - meaning the interval $[x-ct, x+ct]$.

2. $D(\vec{z},t)$ can be simplified (in two dimensions)

$$D(\vec{z},t) = -\frac{i}{2(2\pi)^2}\int d^2k \frac{e^{i\vec{k}\vec{z}}}{ck}(e^{ickt} - e^{-ickt}) \tag{2.226}$$

$$= -\frac{i}{2(2\pi)^2}\int_0^\infty dk\; k\frac{1}{ck}(e^{ickt} - e^{-ickt})\int d\phi\, e^{ikz\cos\phi} \tag{2.227}$$

$$= -\frac{i}{2(2\pi)^2 c}\int_0^\infty dk\;(e^{ickt} - e^{-ickt})\cdot 2\pi J_0(kz) \tag{2.228}$$

$$= -\frac{i(2\pi)2i}{2(2\pi)^2 c}\int_0^\infty dk\;\cdot J_0(kz)\sin(ctk) \tag{2.229}$$

$$= \frac{1}{2\pi c}\begin{cases} 0 & 0 < ct < z \\ \frac{1}{\sqrt{c^2t^2 - z^2}} & 0 < z < ct \end{cases} \tag{2.230}$$

where we used 6.671-7 of Gradshteyn, Ryzhik - Table of integrals, series and products 7ed. The explicit solution is then given by

$$u(\vec{x},t) = \frac{1}{2\pi c}\partial_t\left(\int_{K(\vec{x})_{ct}} d^2\xi \frac{u_0(\vec{\xi})}{\sqrt{c^2t^2 - |\vec{x}-\vec{\xi}|^2}}\right) + \frac{1}{2\pi c}\int_{K(\vec{x})_{ct}} d^2\xi \frac{u_1(\vec{\xi})}{\sqrt{c^2t^2 - |\vec{x}-\vec{\xi}|^2}} \tag{2.231}$$

where $K(\vec{x})_{ct}$ is a disc of radius ct at $\vec{x}$.

3. $D(\vec{z},t)$ can be simplified (in three dimensions)

$$D(\vec{z},t) = -\frac{i}{2(2\pi)^3}\int d^3k \frac{e^{i\vec{k}\vec{z}}}{ck}(e^{ickt} - e^{-ickt}) \tag{2.232}$$

$$= -\frac{i}{2(2\pi)^3}\int_0^\infty dk\; k^2\int d\phi\int d\theta \sin\theta \frac{e^{ikz\cos\theta}}{ck}2i\sin(ckt) \tag{2.233}$$

$$= -\frac{(2\pi)(2i)i}{2(2\pi)^3 c}\int_0^\infty dk\; k\sin(ckt)\int d\theta\sin\theta e^{ikz\cos\theta} \tag{2.234}$$

$$= \frac{1}{(2\pi)^2 c}\int_0^\infty dk\; k\sin(ckt)\frac{i}{kz}e^{ikz\cos\theta}|_0^\pi \tag{2.235}$$

$$= \frac{i}{(2\pi)^2 cz}\int_0^\infty dk\sin(ckt)\left(e^{-ikz} - e^{ikz}\right) \tag{2.236}$$

$$= \frac{i}{(2\pi)^2 cz}\int_0^\infty dk\frac{i}{2}(e^{-ickt} - e^{ickt})\left(e^{-ikz} - e^{ikz}\right) \tag{2.237}$$

$$= \frac{-1}{2(2\pi)^2 cz}\int_0^\infty dk(e^{-ik(ct+z)} - e^{ik(-ct+z)} - e^{-ik(-ct+z)} + e^{ik(ct+z)}) \tag{2.238}$$

$$= \frac{-1}{2(2\pi)^2 cz}\int_{-\infty}^\infty dk(-e^{ik(-ct+z)} + e^{ik(ct+z)}) \tag{2.239}$$

$$= \frac{-1}{4\pi cz}\left[\delta(z+ct) - \delta(z-ct)\right] \tag{2.240}$$

as $z, c > 0$ we have

$$D(\vec{z},t) = \frac{1}{4\pi zc}\begin{cases} -\delta(|\vec{z}| + ct) & (t < 0) \\ 0 & (t = 0) \\ +\delta(|\vec{z}| - ct) & (t > 0) \end{cases} \tag{2.241}$$

Vanishes outside and inside the light cone but NOT on the light cone. The explicit solution is then given as

$$u(\vec{x},t) = \int d^3\xi \left[\partial_t D(\vec{x}-\vec{\xi},t)u_0(\vec{\xi}) + D(\vec{x}-\vec{\xi},t)u_1(\vec{\xi})\right] \tag{2.242}$$

$$= \frac{1}{4\pi c}\partial_t \int d^3\xi \,\frac{\delta(|\vec{x}-\vec{\xi}|-ct)}{|\vec{x}-\vec{\xi}|}u_0(\vec{\xi}) + \frac{1}{4\pi c}\int d^3\xi \,\frac{\delta(|\vec{x}-\vec{\xi}|-ct)}{|\vec{x}-\vec{\xi}|}u_1(\vec{\xi}) \tag{2.243}$$

$$= \frac{1}{4\pi c}\partial_t \int d^3\xi' \,\frac{\delta(|\vec{\xi'}|-ct)}{|\vec{\xi'}|}u_0(\vec{x}-\vec{\xi'}) + \frac{1}{4\pi c}\int d^3\xi' \,\frac{\delta(|\vec{\xi'}|-ct)}{|\vec{\xi'}|}u_1(\vec{x}-\vec{\xi'}) \tag{2.244}$$

$$= \frac{1}{4\pi c}\partial_t \int d\Omega_{\vec{\chi}} \int d\xi'\,\xi'^2 \frac{\delta(|\vec{\xi'}|-ct)}{|\vec{\xi'}|}u_0(\vec{x}-\vec{\xi'}) + \frac{1}{4\pi c}\int d\Omega_{\vec{\chi}} \int d\xi'\,\xi'^2 \frac{\delta(|\vec{\xi'}|-ct)}{|\vec{\xi'}|}u_1(\vec{x}-\vec{\xi'}) \tag{2.245}$$

$$= \frac{1}{4\pi}\partial_t \left(t\int d\Omega_{\vec{\chi}}\, u_0(\vec{x}-ct\vec{\chi})\right) + \frac{t}{4\pi}\int d\Omega_{\vec{\chi}}\, u_1(\vec{x}-ct\vec{\chi}) \tag{2.246}$$

$$= \dots \tag{2.247}$$

$$= \frac{t}{4\pi(ct)^2}\partial_t \left(t\int_{\partial K(\vec{x})_{ct}} u_0(\xi)dA_\xi\right) + \frac{t}{4\pi(ct)^2}\int_{\partial K(\vec{x})_{ct}} u_1(\xi)dA_\xi \tag{2.248}$$

- Sourced solution

  Knowing the Greens function defined by

$$\left(\frac{1}{c^2}\partial_{tt} - \triangle\right) G(\vec{x}-\vec{x}',t-t') = \delta(\vec{x}-\vec{x}')\delta(t-t') \tag{2.249}$$

  allows us to write the solutions of $\left(\frac{1}{c^2}\partial_{tt} - \triangle\right)u(x,t) = j(x,t)$ as

$$u(\vec{x},t) = \int d^nx'\,dt'\;G(\vec{x}-\vec{x}',t-t')j(\vec{x}',t') \tag{2.250}$$

  because

$$\left(\frac{1}{c^2}\partial_{tt} - \triangle\right)u(\vec{x},t) = \int d^nx'\,dt'\;\left(\frac{1}{c^2}\partial_{tt} - \triangle\right)G(\vec{x}-\vec{x}',t-t')j(\vec{x}',t') \tag{2.251}$$

$$= \int d^nx'\,dt'\;\delta(\vec{x}-\vec{x}')\delta(t-t')j(\vec{x}',t') \tag{2.252}$$

$$= j(\vec{x},t) \tag{2.253}$$

$$\left(\frac{1}{c^2}\partial_{tt} - \triangle\right) G(\vec{x}-\vec{x}',t-t') = \delta(\vec{x}-\vec{x}')\delta(t-t') \tag{2.254}$$

$$G(\vec{x}-\vec{x}',t-t') = G(\vec{r},\tau) \tag{2.255}$$

$$= \frac{1}{(2\pi)^{n+1}}\int d^nk d\omega\, \tilde{G}(\vec{k},\omega)e^{i(\vec{k}\vec{r}-\omega\tau)} \tag{2.256}$$

  then

$$\left(-\frac{\omega^2}{c^2}+k^2\right)\tilde{G}(k,\omega) = 1 \tag{2.257}$$

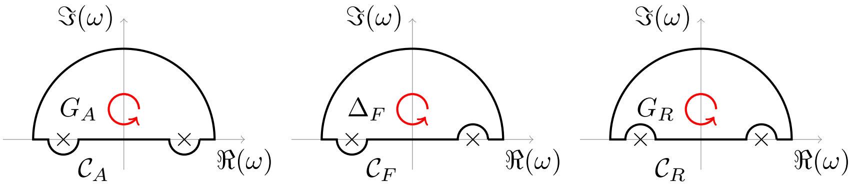
$$\tilde{G}(k,\omega) = \frac{c^2}{-\omega^2+c^2k^2} = \frac{c}{2k}\left(\frac{1}{\omega+ck} - \frac{1}{\omega-ck}\right) \tag{2.258}$$

and therefore

$$G(\vec{r},\tau) = \frac{1}{(2\pi)^{n+1}} \int d^n k d\omega\, \tilde{G}(\vec{k},\omega) e^{i(\vec{k}\vec{r}-\omega\tau)} \tag{2.259}$$
$$= \frac{1}{(2\pi)^{n+1}} \int d^n k\, e^{i\vec{k}\vec{r}} \int d\omega\, \frac{c^2}{-\omega^2+c^2k^2} e^{-i\omega\tau} \tag{2.260}$$
$$= \frac{c}{2(2\pi)^{n+1}} \int d^n k\, \frac{1}{k} e^{i\vec{k}\vec{r}} \int d\omega \left(\frac{1}{\omega+ck} - \frac{1}{\omega-ck}\right) e^{i\omega\tau} \tag{2.261}$$

Now we need to transform back - but the result depends on the number of space dimensions

$\tau < 0$



$\tau > 0$

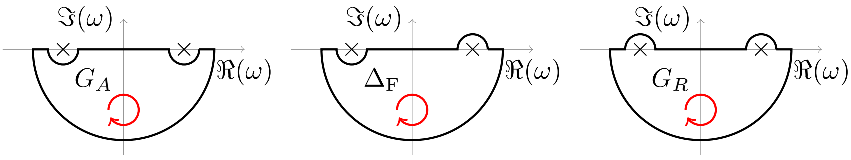


Figure 2.1: Possible contours for the Fourier back transformation of the Greens functions of the wave equation caused by the poles $\pm ck$

1. Case $n = 1$

$$G(r,\tau) = \frac{1}{(2\pi)^2} \int dk d\omega\, \frac{c^2}{-\omega^2+c^2k^2} e^{i(kr-\omega\tau)} \tag{2.262}$$
$$= \frac{1}{(2\pi)^2} \int dk\, e^{ikr} \int d\omega\, \frac{c^2}{-\omega^2+c^2k^2} e^{-i\omega\tau} \tag{2.263}$$
$$= \frac{c}{2(2\pi)^2} \int dk\, \frac{e^{ikr}}{k} \int d\omega \left(\frac{1}{\omega+ck} - \frac{1}{\omega-ck}\right) e^{-i\omega\tau} \tag{2.264}$$
$$\tag{2.265}$$

Now we can evaluate using the residue theorem (additional factor -1 if contour closes in mathematical negative direction)

$$G_R(r,\tau>0) = \frac{c}{2(2\pi)^2}\int dk\,\frac{e^{ikr}}{k}(-1)2\pi i\left(e^{ick\tau}-e^{-ick\tau}\right) \tag{2.266}$$

$$= -\frac{ic}{4\pi}\int_{-\infty}^{\infty} dk\,\frac{1}{k}\left(e^{ik(r+c\tau)}-e^{ik(r-c\tau)}\right) \tag{2.267}$$

$$= -\frac{ic}{4\pi}\left(i\sqrt{\frac{\pi}{2}}\mathrm{sign}(r+c\tau)-i\sqrt{\frac{\pi}{2}}\mathrm{sign}(r-c\tau)\right) \tag{2.268}$$

$$= \frac{c}{4\sqrt{2\pi}}\left(\mathrm{sgn}(r+c\tau)-\mathrm{sgn}(r-c\tau)\right) \tag{2.269}$$

$$= \begin{cases} +\frac{c}{4\sqrt{2\pi}} & |z|<c\tau, \quad \tau>0 \\ 0 & |z|>ct, \\ -\frac{c}{4\sqrt{2\pi}} & |z|<c\tau, \quad \tau<0 \end{cases} \tag{2.270}$$

$$G_R(r,\tau<0) = 0 \tag{2.271}$$

$$G_A(r,\tau>0) = 0 \tag{2.272}$$

$$G_A(r,\tau<0) = ... \tag{2.273}$$

2. Case $n=2$???

3. Case $n=3$

$$G(\vec{r},\tau) = \frac{1}{(2\pi)^4}\int d^3k\, e^{i\vec{k}\vec{r}}\int d\omega\,\frac{c^2}{-\omega^2+c^2k^2}e^{-i\omega\tau} \tag{2.274}$$

$$= \frac{2\pi}{(2\pi)^4}\int dk\,k^2\,d\theta\,\sin\theta\,e^{ikr\cos\theta}\int d\omega\,\frac{c^2}{-\omega^2+c^2k^2}e^{-i\omega\tau} \tag{2.275}$$

$$= \frac{2\pi}{(2\pi)^4 ir}\int dk\,k\,(e^{ikr}-e^{-ikr})\int d\omega\,\frac{c^2}{-\omega^2+c^2k^2}e^{-i\omega\tau} \tag{2.276}$$

The poles at $\omega=\pm ck$ make the value of the integral not unique. Using the residue theorem we can evaluate the integral but the value will depend on the chosen contour - which means the Greens function is NOT unique!

Applying the wave operator to the solution we obtain

$$G(\vec{r},\tau) = \frac{1}{(2\pi)^4}\int d^3k\int d\omega\,\frac{c^2}{-\omega^2+c^2k^2}e^{i(\vec{k}\vec{r}-\omega\tau)} \tag{2.277}$$

$$\Box G(\vec{r},\tau) = \frac{1}{(2\pi)^4}\int d^3k\int d\omega\,e^{i(\vec{k}\vec{r}-\omega\tau)} = \delta(\vec{r})\delta(\tau) \tag{2.278}$$

where we now can integrate along any ω-contour (even along the ω axis) as the poles are gone. This means the all contours give (potentially different) but valid Greens functions. The physical interpretation is that the different Greens function depend on the boundary conditions.

Now we can evaluate using the residue theorem (additional factor -1 if contour closes in mathematical negative direction)

$$G_R(\vec{r}, \tau > 0) = \frac{c^2}{(2\pi)^3 ir} \int_0^\infty dk\, k\, (e^{ikr} - e^{-ikr}) \int d\omega\, \frac{1}{-\omega^2 + c^2k^2} e^{-i\omega\tau} \tag{2.279}$$

$$= \frac{c}{2(2\pi)^3 ir} \int_0^\infty dk\, (e^{ikr} - e^{-ikr}) \int d\omega \left(\frac{1}{\omega + ck} - \frac{1}{\omega - ck} \right) e^{-i\omega\tau} \tag{2.280}$$

$$= \frac{c}{2(2\pi)^3 ir} \int_0^\infty dk\, (e^{ikr} - e^{-ikr})(-1)2\pi i \left(e^{ick\tau} - e^{-ick\tau} \right) \tag{2.281}$$

$$= \frac{c}{2(2\pi)^2 r} \int_0^\infty dk\, (e^{ikr} - e^{-ikr}) \left(e^{-ick\tau} - e^{ick\tau} \right) \tag{2.282}$$

$$= \frac{c}{2(2\pi)^2 r} \int_{-\infty}^\infty dk\, (e^{ik(r-c\tau)} - e^{-ik(r+c\tau)}) \tag{2.283}$$

$$= \frac{c}{4\pi r} \left(\delta(r - c\tau) - \delta(r + c\tau) \right) \tag{2.284}$$

$$= \frac{c}{4\pi r} \delta(r - c\tau) \qquad (r, \tau > 0) \tag{2.285}$$

$$G_R(\vec{x} - \vec{x}', t - t' > 0) = \frac{c}{4\pi} \frac{\delta(|\vec{x} - \vec{x}'| - c(t - t'))}{|\vec{x} - \vec{x}'|} \tag{2.286}$$

$$G_R(\vec{x} - \vec{x}', t - t' < 0) = 0 \tag{2.287}$$

$$G_A(\vec{x} - \vec{x}', t - t' > 0) = 0 \tag{2.288}$$

$$G_A(\vec{x} - \vec{x}', t - t' < 0) = \ldots \tag{2.289}$$

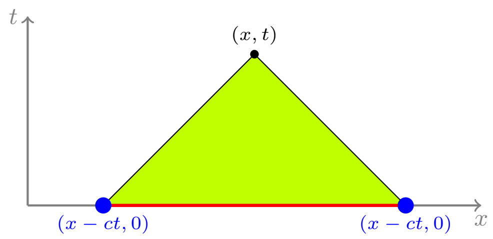
- Summary

$$u(x,t)_R^{1\mathrm{D}} = \frac{1}{2} \left[u_0(x + ct) + u_0(x - ct) \right] + \frac{1}{2c} \int_{K^{1\mathrm{D}}(x)_{ct}} u_1(\xi) d\xi + \frac{c}{2\sqrt{2\pi}} \int_{\mathrm{pLC}} j(\xi, \tau)\, d\xi d\tau \tag{2.290}$$

$$u(\vec{x},t)_R^{2\mathrm{D}} = \frac{1}{2\pi c} \partial_t \left(\int_{K^{2\mathrm{D}}(\vec{x})_{ct}} d^2\xi \frac{u_0(\vec{\xi})}{\sqrt{c^2t^2 - |\vec{x} - \vec{\xi}|^2}} \right) + \frac{1}{2\pi c} \int_{K^{2\mathrm{D}}(\vec{x})_{ct}} d^2\xi \frac{u_1(\vec{\xi})}{\sqrt{c^2t^2 - |\vec{x} - \vec{\xi}|^2}} + ??? \tag{2.291}$$

$$u(\vec{x},t)_R^{3\mathrm{D}} = \frac{t}{4\pi(ct)^2} \partial_t \left(t \int_{\partial K^{3\mathrm{D}}(\vec{x})_{ct}} u_0(\xi) dA_\xi \right) + \frac{t}{4\pi(ct)^2} \int_{\partial K^{3\mathrm{D}}(\vec{x})_{ct}} u_1(\xi) dA_\xi + \frac{c}{4\pi} \int_{\partial\mathrm{pLC}} \frac{j(\vec{\xi}, \tau)}{|\vec{r} - \vec{\xi}|} d^3\xi d\tau \tag{2.292}$$

$1d$



2.16.3 Klein-Gordon equation $\left(\frac{1}{c^2}\partial_{tt} - \triangle + \mu^2\right) u(x,t) = j(x,t)$

- The free fundamental solution (no source with $j(x,t) = 0$)

$$u(\vec{x},t) = e^{-i(k_0 t - \vec{k}\vec{x})} \tag{2.293}$$
$$\rightarrow \quad \frac{(-ik_0)^2}{c^2} - (i\vec{k})^2 + \mu^2 = 0 \tag{2.294}$$
$$\rightarrow \quad -\frac{k_0^2}{c^2} + \vec{k}^2 + \mu^2 = 0 \tag{2.295}$$
$$\rightarrow \quad k_0 = \omega = \pm c\sqrt{\vec{k}^2 + \mu^2} \tag{2.296}$$

- Free solutions of

$$\left(\frac{1}{c^2}\partial_{tt} - \triangle + \mu^2\right) u(\vec{x},t) = 0 \tag{2.297}$$

$$\mathit{\Delta}(x) = -\int_C \frac{d^4k}{(2\pi)^4} \frac{e^{-ikx}}{k^2 - \mu^2} = \frac{1}{2i} \int \frac{d^3k}{(2\pi)^3} \frac{e^{-ikx} - e^{ikx}}{\sqrt{\vec{k}^2 + \mu^2}} \tag{2.298}$$
$$\mathit{\Delta}^{\pm}(x) = -\int_{C^{\pm}} \frac{d^4k}{(2\pi)^4} \frac{e^{-ikx}}{k^2 - \mu^2} = \mp\frac{i}{2} \int \frac{d^3k}{(2\pi)^3} \frac{e^{\mp ikx}}{\sqrt{\vec{k}^2 + \mu^2}} \tag{2.299}$$

- Sourced solution

  Using

$$\delta(\vec{x}) = \frac{1}{2\pi} \int dk e^{i\vec{k}\vec{x}} \tag{2.300}$$
$$G(\vec{x},t) = \frac{1}{(2\pi)^d} \int d\vec{k}\, d\omega\, g(\vec{k},\omega) e^{i\vec{k}\vec{x}} e^{-i\omega t} \tag{2.301}$$

  note the sign change of the frequency/time transform.

  1. Case $n = 1$

Name	Symbol	Contour
Feynman propagator	Δ_F	$\mathcal{C}_F$
Dyson propagator	Δ_D	$\mathcal{C}_D$
Retarded propagator	Δ_R	$\mathcal{C}_R$
Advanced propagator	Δ_A	$\mathcal{C}_A$
Principle-part propagator	$\bar{\Delta}$	$\bar{\mathcal{C}}$

Table 2.1: Greens functions

To find the Green function perform a 2d Fourier transform

$$\left(\frac{1}{c^2}\partial_{tt} - \triangle + \mu^2\right) G(x - x_0, t - t_0) = \delta(x - x_0)\delta(t - t_0) \tag{2.302}$$

$$\left(-\frac{\omega^2}{c^2} + k^2 + \mu^2\right) \tilde{G}(k, \omega) = 1 \tag{2.303}$$

$$\tilde{G}(k, \omega) = \frac{c^2}{-\omega^2 + c^2(k^2 + \mu^2)} \tag{2.304}$$

$$= \frac{c}{2\sqrt{k^2 + \mu^2}} \left(\frac{1}{\omega + c\sqrt{k^2 + \mu^2}} - \frac{1}{\omega - c\sqrt{k^2 + \mu^2}}\right) \tag{2.305}$$

First Fourier back transformation of ω to t

$$w(k, t - t_0) = \frac{1}{2\pi}\int d\omega\, v(\vec{k}, \omega) e^{-i\omega(t - t_0)} \tag{2.306}$$

$$= \frac{c}{(2\pi)2\sqrt{k^2 + \mu^2}} \int_{-\infty}^{+\infty} d\omega\, e^{-i\omega(t-t_0)} \left(\frac{1}{\omega + c\sqrt{k^2 + \mu^2}} - \frac{1}{\omega - c\sqrt{k^2 + \mu^2}}\right) \tag{2.307}$$

we recognize the two poles at $\pm c\sqrt{k^2 + \mu^2}$ on the real axis. Using the residue theorem we can decide pick four (five) contours which subsequently result in different Green functions

$t - t_0 < 0$

$t - t_0 > 0$

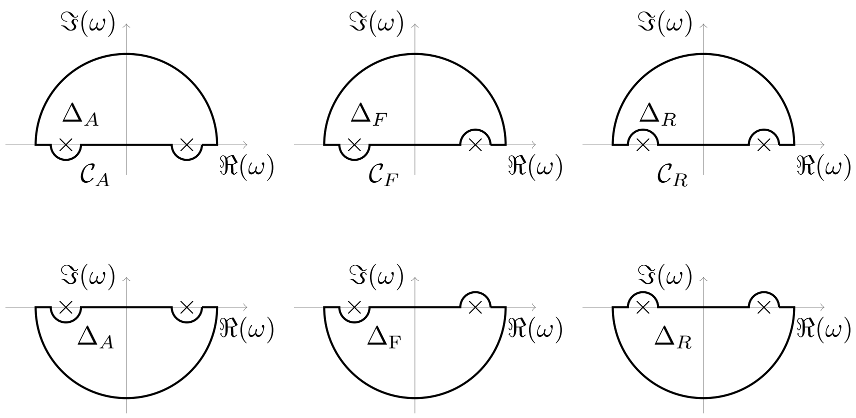


Figure 2.2: Possible contours for the Fourier back transformation of the one dimensional Klein-Gordon Greens functions caused by the poles $\pm c\sqrt{k^2 + \mu^2}$

$$\int_{-\infty}^{\infty} f\, d\omega + \int_{\text{half circ}} f\, d\omega = 2\pi i \text{Res} f \tag{2.308}$$

$t - t_0 < 0:$

$$w_A(k, t-t_0) = \frac{2\pi i c}{4\pi\sqrt{k^2+\mu^2}} \left[e^{ic(t-t_0)\sqrt{k^2+\mu^2}} - e^{-ic(t-t_0)\sqrt{k^2+\mu^2}} \right] \tag{2.309}$$

$$= \frac{-c}{\sqrt{k^2+\mu^2}} \sin\left(c(t-t_0)\sqrt{k^2+\mu^2} \right) \tag{2.310}$$

$$w_F(k, t-t_0) = \frac{ic}{2\sqrt{k^2+\mu^2}} \left[-e^{-ic(t-t_0)\sqrt{k^2+\mu^2}} \right] \tag{2.311}$$

$$w_R(k, t-t_0) = 0 \tag{2.312}$$

$t - t_0 > 0:$

$$w_A(k, t-t_0) = 0 \tag{2.313}$$

$$w_F(k, t-t_0) = \frac{2\pi i c}{4\pi\sqrt{k^2+\mu^2}} \left[-e^{-ic(t-t_0)\sqrt{k^2+\mu^2}} \right] \tag{2.314}$$

$$w_R(k, t-t_0) = \frac{2\pi i c}{4\pi\sqrt{k^2+\mu^2}} \left[e^{ic(t-t_0)\sqrt{k^2+\mu^2}} - e^{-ic(t-t_0)\sqrt{k^2+\mu^2}} \right] \tag{2.315}$$

$$= \frac{-ic}{\sqrt{k^2+\mu^2}} \sin\left(c(t-t_0)\sqrt{k^2+\mu^2} \right) \tag{2.316}$$

Second Fourier back transformation

$$u(x,t) = \frac{1}{2\pi} \int dk\, e^{ikx} w(k,t) \tag{2.317}$$

$$= \frac{c}{4\pi^2} \int dk \frac{e^{ikx}}{\mu\sqrt{k^2/\mu^2+1}} \left[e^{-ict\mu\sqrt{k^2/\mu^2+1}} - e^{ict\mu\sqrt{k^2/\mu^2+1}} \right] \tag{2.318}$$

Now substitude $k/\mu = \sinh s$ and $1 + \sinh^2 s = \cosh^2 s$

$$u(x,t) = \frac{c}{4\pi^2} \int \mu \cosh s\, ds \frac{e^{ix\mu \sinh s}}{\mu \cosh s} \left[e^{-ict\mu \cosh s} - e^{ict\mu \cosh s} \right] \tag{2.319}$$

$$= \frac{c}{4\pi^2} \int ds\, e^{ix\mu \sinh s} \left[e^{-ict\mu \cosh s} - e^{ict\mu \cosh s} \right] \tag{2.320}$$

as well as

$$x = \frac{1}{\mu} z \cosh y \tag{2.321}$$

$$ct = \frac{1}{\mu} z \sinh y \tag{2.322}$$

$$\to x^2 - c^2 t^2 = \frac{1}{\mu^2} z^2 \tag{2.323}$$

which gives

$$u(x,t) = \frac{c}{4\pi^2} \int ds\, e^{iz \cosh y \sinh s} \left[e^{-iz \sinh y \cosh s} - e^{iz \sinh y \cosh s} \right] \tag{2.324}$$

$$= \frac{c}{4\pi^2} \int ds \left(e^{iz(\cosh y \sinh s - \sinh y \cosh s)} - e^{iz(\cosh y \sinh s + \sinh y \cosh s)} \right) \tag{2.325}$$

$$= \frac{c}{4\pi^2} \int ds \left(e^{iz \sinh(s-y)} - e^{iz \sinh(s+y)} \right) \tag{2.326}$$

$$= \frac{c}{4\pi^2} \int ds\ \left[\cos(z \sinh(s-y)) + i \sin(z \sinh(s-y)) - e^{iz \sinh(s+y)} \right] \tag{2.327}$$

$$z^2 = \mu^2(x^2 - c^2t^2) \tag{2.328}$$

$$\psi_0(x,t) = \frac{i}{\pi c}\partial_t \int_0^\infty dy\ \cos(z \sinh y) \quad \text{for} \quad \psi_0(x,0) = \delta(x) \tag{2.329}$$

$$\psi(x,t) = \int dy\, f(y)\psi_0(x-y,t) \quad \text{for} \quad \psi(x,0) = f(x) \tag{2.330}$$

2. Case $n = 3$

2.16.4 Helmholtz equation $(\triangle + k^2)u(x) = f(x)$

The Greens function is given by $(\triangle_x + k^2)G(x,y) = \delta(x-y)$

2.16.5 Feynman propagator $(\triangle - k^2)\, u(x) = f(x)$

2.16.6 Heat equation $(\partial_t - k\triangle)\, u(x) = f(x)$

Homogenous case $(\partial_t - k\triangle)\, G(x,t) = 0$

$$G(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}} \tag{2.331}$$

2.16.7 Relativistic Heat equation $(\partial_{tt} + 2\gamma\partial_t - c^2\triangle)\, u(x) = f(x)$

2.16.8 Sine-Gordon equation equation $\left(\frac{1}{c^2}\partial_{tt} - \triangle +\right) u(x,t) + \sin u(x,t) = 0$

2.16.9 Kortegweg-De Vries equation equation $\partial_t u + 6u \cdot \partial_x u + \partial_{xxx} u = 0$

2.17 Probability

- Hypothesis H: Steve is a librarian
- Evidence E: Steve likes reading books

Question: Whats the probability of the hypothesis is true given the evidence is true $P(H|E)$

$$P(H|E) \equiv \frac{P(E \cap H)}{P(E)} \qquad P(E|H) \equiv \frac{P(E \cap H)}{P(H)} \tag{2.332}$$

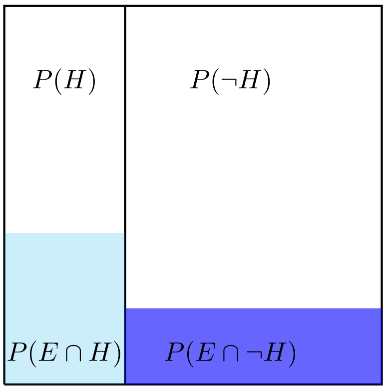
$$\rightarrow \qquad P(H|E) = \frac{P(E|H) \cdot P(H)}{P(E)} = \frac{P(H) \cdot P(E|H)}{P(H) \cdot P(E|H) + P(\neg H) \cdot P(E|\neg H)} \tag{2.333}$$

alternatively

$$P(H|E) = \frac{\#\text{allPeople} \cdot P(H) \cdot P(E|H)}{\#\text{allPeople} \cdot P(H) \cdot P(E|H) + \#\text{allPeople} \cdot P(\neg H) \cdot P(E|\neg H)} \tag{2.334}$$

$$= \frac{P(H) \cdot P(E|H)}{P(H) \cdot P(E|H) + P(\neg H) \cdot P(E|\neg H)} \tag{2.335}$$

$$= \frac{P(H) \cdot P(E|H)}{P(E)} \tag{2.336}$$



2.18 Matrices

1. inverse $A^{-1}A = \mathbb{I}$
   - therefore $\mathbb{I} = (AB)(B^{-1}A^{-1}) \quad \rightarrow \quad (AB)^{-1} = B^{-1}A^{-1}$
2. Hermitian transpose $A^\dagger = (\overline{A})^T = \overline{A^T}$
   - $(AB)^\dagger = B^\dagger A^\dagger$ therefore $\mathbb{I} = (AA^{-1})^\dagger = (A^{-1})^\dagger A^\dagger \quad \rightarrow \quad (A^\dagger)^{-1} = (A^{-1})^\dagger$

$$\langle x|Ay\rangle = \sum_k x_k^* (\vec{A}_{\text{row}k} \cdot \vec{y}) = \sum_{k,l} x_k^* A_{kl} y_l \tag{2.337}$$

$$\langle Bx|y\rangle = \sum_k (\vec{B}_{\text{row}k} \cdot \vec{x})^* y_k = \sum_{k,l} B_{kl}^* x_l^* y_k \tag{2.338}$$

3. Orthorgonal $A^T = A^{-1}$
4. Unitary $A^\dagger = A^{-1}$
5. Hermitian $A^\dagger = A$

2.19 Diagonalization

Any matrix A is called diagonalizable if there exists an invertible matrix S such that

$$D_A = S^{-1}AS \tag{2.339}$$

is a diagonal matrix. The diagonalizability of A is equivalent to the fact that the $\{\vec{v}_i\}$ are all linearly independent.

To find S and D_A one has to find the eigensystem $\{\lambda_i, \vec{v}_i\}$ with $A\vec{v}_i = \lambda_i \vec{v}_i$. Then $D_A S$ and S can be written as $S = (\vec{v}_1, ..., \vec{v}_n)$ and $D_A = \text{diag}(\lambda_1, ..., \lambda_n)$ because $AS = (A\vec{v}_1, ..., A\vec{v}_n) = (\lambda_1\vec{v}_1, ..., \lambda_n\vec{v}_n) = SD_A$.

2.20 Functional derivatives

Let $F[\phi]$ a functional, i.e. a mapping from a Banach space $\mathcal{M}$ to the field of real or complex numbers. The functional (Frechet) derivative $\delta F[\phi]/\delta\phi$ is defined by

$$\begin{aligned}
\delta F &= \int dx \frac{\delta F[\phi]}{\delta\phi(x)} \cdot \delta\phi(x) && (2.340)\\
&= \int dx \frac{\delta F[\phi]}{\delta\phi(x)} \cdot \epsilon\delta(x-y) && (2.341)\\
&= \epsilon \frac{\delta F[\phi]}{\delta\phi(y)} && (2.342)\\
&= F[\phi + \epsilon\delta(x-y)] - F[\phi] && (2.343)
\end{aligned}$$

which means

$$\begin{aligned}
\frac{\delta F[\phi]}{\delta\phi[y]} &= \lim_{\epsilon\to 0} \frac{F[\phi + \epsilon\delta(x-y)] - F[\phi]}{\epsilon} && (2.344)\\
F[\phi + \epsilon\delta(x-y)] &= F[\phi] + \epsilon \frac{\delta F[\phi]}{\delta\phi(y)} && (2.345)\\
&= F[\phi] + \epsilon \int dx \frac{\delta F[\phi]}{\delta\phi(x)} \cdot \delta(x-y) && (2.346)
\end{aligned}$$

- Product rule $F[\phi] = G[\phi]H[\phi]$

$$\begin{aligned}
\frac{\delta F[\phi]}{\delta\phi(x)} &= \frac{\delta(G[\phi]H[\phi])}{\delta\phi} && (2.347)\\
&= \lim_{\epsilon\to 0} \frac{G[\phi + \epsilon\delta(x-y)]H[\phi + \epsilon\delta(x-y)] - G[\phi]H[\phi]}{\epsilon} && (2.348)\\
&= \lim_{\epsilon\to 0} \frac{\left(G[\phi] + \epsilon\frac{\delta G}{\delta\phi}\right)\left(H[\phi] + \epsilon\frac{\delta H}{\delta\phi}\right) - G[\phi]H[\phi]}{\epsilon} && (2.349)\\
&= \lim_{\epsilon\to 0} \frac{G[\phi]H[\phi] + \epsilon G[\phi]\frac{\delta H}{\delta\phi} + \frac{\delta G}{\delta\phi}H[\phi] + \epsilon^2 \frac{\delta G}{\delta\phi}\frac{\delta H}{\delta\phi} - G[\phi]H[\phi]}{\epsilon} && (2.350)\\
&= G[\phi]\frac{\delta H[\phi]}{\delta\phi(x)} + \frac{\delta G[\phi]}{\delta\phi(x)}H[\phi] && (2.351)
\end{aligned}$$

- Chain rule $F[G[\phi]]$

$$\delta F = \int dx \frac{\delta F[G[\phi]]}{\delta \phi(x)} \delta \phi(x) \tag{2.352}$$

$$\delta G = \int dy \frac{\delta G[\phi]}{\delta \phi(y)} \delta \phi(y) \tag{2.353}$$

$$\delta F = \int dz \frac{\delta F[G]}{\delta G(z)} \delta G(z) \tag{2.354}$$

$$= \int dz \frac{\delta F[G]}{\delta G(z)} \int dy \frac{\delta G[\phi]}{\delta \phi(y)} \delta \phi(y) \tag{2.355}$$

$$= \int dy \underbrace{\int dz \frac{\delta F[G]}{\delta G(z)} \frac{\delta G[\phi]}{\delta \phi(y)}}_{= \frac{\delta F[G[\phi]]}{\delta \phi(y)}} \delta \phi(y) \tag{2.356}$$

$$\frac{\delta F[G[\phi]]}{\delta \phi(y)} = \int dz \frac{\delta F[G]}{\delta G(z)} \frac{\delta G[\phi]}{\delta \phi(y)} \tag{2.357}$$

- Chain rule (special case) $F[g[\phi]]$

$$\frac{\delta F[g[\phi]]}{\delta \phi(y)} = ... \tag{2.358}$$

$$= \frac{\delta F}{\delta g(\phi(y))} \frac{dg(\phi)}{d\phi(y)} \tag{2.359}$$

Some examples

1. $F[\phi] = \int dx \phi(x) \delta(x)$

$$\frac{\delta F[\phi]}{\delta \phi(y)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\int dx (\phi(x) + \epsilon \delta(x-y)) \delta(x)) - \int dx\, \phi(x) \delta(x) \right) \tag{2.360}$$

$$= \int dx\, \delta(x-y)) \delta(x) \tag{2.361}$$

$$= \delta(y) \tag{2.362}$$

2. $F[\phi] = \int dx \phi(x)$

$$\frac{\delta F[\phi]}{\delta \phi(y)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\int dx (\phi(x) + \epsilon \delta(x-y))) - \int dx\, \phi(x) \right) \tag{2.363}$$

$$= \int dx\, \delta(x-y) \tag{2.364}$$

$$= 1 \tag{2.365}$$

3. $F_x[\phi] = \phi(x)$

$$\frac{\delta \phi(x)}{\delta \phi(y)} = \frac{\delta F_x[\phi]}{\delta \phi(y)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left((\phi(x) + \epsilon \delta(x-y)) - \phi(x) \right) \tag{2.366}$$

$$= \delta(x-y) \tag{2.367}$$

4. $F[\phi] = \int dx \phi(x)^n$

$$\frac{\delta F[\phi]}{\delta \phi(y)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\int dx (\phi(x) + \epsilon \delta(x-y)))^n - \int dx\, \phi(x)^n \right) \tag{2.368}$$

$$= \int dx\, n \phi(x)^{n-1} \delta(x-y) \tag{2.369}$$

$$= n \phi(y)^{n-1} \tag{2.370}$$

5. $F[\phi] = \int dx \left(\frac{\phi(x)}{dx}\right)^n$
6. $F_y[\phi] = \int dz K(y,z)\phi(z)$

$$\frac{\delta F_y[\phi]}{\delta\phi(x)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int dz(K(y,z)(\phi(z)+\epsilon\delta(z-x)) - \int dz K(y,z)\phi(z)\right) \tag{2.371}$$
$$= \int dz\, K(y,z)\delta(z-x) \tag{2.372}$$
$$= K(y,x) \tag{2.373}$$

7. $F_x[\phi] = \nabla\phi(x)$

$$\frac{\delta F[\phi]}{\delta\phi(y)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\nabla_x(\phi(x)+\epsilon\delta(x-y)) - \nabla_x\phi(x)\right) \tag{2.374}$$
$$= \nabla_x\delta(x-y) \tag{2.375}$$

8. $F[\phi] = g\left(G[\phi(x)]\right)$

$$\frac{\delta F[\phi]}{\delta\phi(y)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}g(G[\phi(x)+\epsilon\delta(x-y)]) - g(G[\phi(x)]) \tag{2.376}$$
$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}g(G[\phi(x)] + \epsilon\frac{\delta G}{\delta\phi}) - g(G[\phi(x)]) \tag{2.377}$$
$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}g(G[\phi(x)]) + g'\epsilon\frac{\delta G}{\delta\phi} - g(G[\phi(x)]) \tag{2.378}$$
$$= \frac{\delta G}{\delta\phi}g'(G[\phi(x)]) \tag{2.379}$$

2.21 Space hierarchy

1. K-Vector space $(K, \oplus, \odot)$
   - set V, field K with $(K, +, \cdot)$
   - vector addition $\oplus : V \times V \to V$
   - scalar multiplication $\odot : K \times V \to V$
2. Topological vector space
   - K-vector space
   - continuous (smooth) vector addition and scalar multiplication
3. Metric (vector) space (M, d)
   - set M, metric $d : M \times M \to \mathbb{R}$
   - $d(x,y) = 0 \Leftrightarrow x = y$
   - $d(x,y) = d(y,x)$
   - $d(x,y) + d(y,z) \geq d(x,z)$
   - from the requirements above follows $d(x,y) \geq 0$
4. Normed vector space $(V, \|\cdot\|)$
   - K-vector space V, norm $\|\cdot\| : V \to \mathbb{R}$
   - Typically $K \in (\mathbb{R}, \mathbb{C})$ to have a definition of $|\lambda|$

- $\|x\| \geq 0$
- $\|x\| = 0 \Leftrightarrow x = 0$
- $\|\lambda x\| = |\lambda|\|x\|$ with $\lambda \in K$
- $\|x\| + \|y\| \geq \|x + y\|$
- with $d(x, y) := \|x - y\|$ every normed vector space has also a metric
- a metric does NOT induce a always norm as the linearity/homogeneity of the norm is not guaranteed

5. Banach space (complete normed vector space)
   - normed K-vector space $(V, \|\cdot\|)$ with $K \in (\mathbb{R}, \mathbb{C})$
   - completeness: every Cauchy sequence converges (with the metric induced by the norm) to a well defined limit
   - if the space is just a metric space (without a norm) the space is called Cauchy space
6. Hilbert space (complete vector space with a scalar product)
   - K-vector space V with $K \in (\mathbb{R}, \mathbb{C})$
   - scalar product $\langle \cdot, \cdot \rangle : V \times V \to K$
   - $\langle \lambda x_1 + x_2, y \rangle = \langle \lambda x_1, y \rangle + \langle \lambda x_2, y \rangle$
   - $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$ for $\lambda \in K$
   - $\langle x, y \rangle = \overline{\langle y, x \rangle}$ which implies $\langle x, x \rangle \in \mathbb{R}$
   - $\langle x, x \rangle > 0$
   - $\langle x, x \rangle = 0 \Leftrightarrow x = 0$
   - completeness: every Cauchy sequence converges (with the metric induced by the norm which is itself induced by the scalar product) to a well defined limit
   - without completeness the space is called Pre-Hilbert space

2.22 Tensors

- For a vector $\mathbf{A}$ the expression $\mathbf{A}^2$ is the squared distance between tip and tail.
- The inner product of two vectors can then be defined by the parallelogram law

$$\mathbf{A} \cdot \mathbf{B} \equiv \frac{1}{4}\left[(\mathbf{A} + \mathbf{B})^2 - (\mathbf{A} - \mathbf{B})^2\right] \tag{2.380}$$

- A rank-n tensor $\mathbf{T} = \mathbf{T}(\_, \_, \_)$ is real-valued linear function of n vectors.

$$\mathbf{T}(\alpha\mathbf{A} + \mu\mathbf{B}, \mathbf{C}, \mathbf{D}) = \alpha\mathbf{T}(\mathbf{A}, \mathbf{C}, \mathbf{D}) + \beta\mathbf{T}(\mathbf{B}, \mathbf{C}, \mathbf{D}) \tag{2.381}$$

- Metric tensor

$$\mathbf{g}(\mathbf{A}, \mathbf{B}) \equiv \mathbf{A} \cdot \mathbf{B} \tag{2.382}$$

- A vector is a tensor of rank one

$$\mathbf{A}(\mathbf{C}) \equiv \mathbf{A} \cdot \mathbf{C} \tag{2.383}$$

- Tensor product

$$\mathbf{A} \otimes \mathbf{B} \otimes \mathbf{C}(\mathbf{E}, \mathbf{F}, \mathbf{G}) \equiv \mathbf{A}(\mathbf{E})\mathbf{B}(\mathbf{F})\mathbf{C}(\mathbf{G}) = (\mathbf{A} \cdot \mathbf{E})(\mathbf{B} \cdot \mathbf{F})(\mathbf{C} \cdot \mathbf{G}) \tag{2.384}$$

- Contraction

$$1\&3 \text{ contraction}(\mathbf{A}\otimes\mathbf{B}\otimes\mathbf{C}\otimes\mathbf{D}) \equiv (\mathbf{A}\cdot\mathbf{C})\mathbf{B}\otimes\mathbf{D} \tag{2.385}$$

- Orthogonal basis

$$\mathbf{e}_j\cdot\mathbf{e}_k = \delta_{jk} \tag{2.386}$$

- Component expansion

$$\begin{aligned}
\mathbf{A} = A_j\mathbf{e}_j \quad &\rightarrow \quad A_j = \mathbf{A}(\mathbf{e}_j) = \mathbf{A}\cdot\mathbf{e}_j && (2.387)\\
\mathbf{T} = T_{abc}\mathbf{e}_a\otimes\mathbf{e}_b\otimes\mathbf{e}_c \quad &\rightarrow \quad T_{ijk} = \mathbf{T}(\mathbf{e}_i,\mathbf{e}_j,\mathbf{e}_k) && (2.388)\\
1\&3 \text{ contraction}(\mathbf{R}) \quad &\rightarrow \quad R_{ijik} && (2.389)\\
\mathbf{g} \quad &\rightarrow \quad g_{jk} = \mathbf{g}(\mathbf{e}_j,\mathbf{e}_k) = \mathbf{e}_j\cdot\mathbf{e}_k = \delta_{jk} && (2.390)
\end{aligned}$$

2.23 Tensors Index rules

A_{ij} - i-th row and j-th column

$$C = AB \tag{2.391}$$

$$C_{ij} = \sum_k A_{ik}C_{kj} \tag{2.392}$$

Matrix - Vector

$$\Lambda\mathbf{a} \rightarrow \Lambda^i{}_j a^j \tag{2.393}$$

Vector - Vector

$$\begin{aligned}
\mathbf{a}\cdot\mathbf{b} &\equiv G(\mathbf{a},\mathbf{b}) && (2.394)\\
&= G(\sum_i a^i\mathbf{e}_i, \sum_j b^j\mathbf{e}_j) && (2.395)\\
&= \sum_{ij} a^i b^j G(\mathbf{e}_i,\mathbf{e}_j) && (2.396)\\
&= \sum_{ij} a^i b^j g_{ij} && (2.397)\\
&= a^i g_{ij} b^j \qquad = \mathbf{a}^T\, G\, \mathbf{b} && (2.398)\\
&= a_i b^i && (2.399)
\end{aligned}$$

Matrix - Matrix

$$\begin{aligned}
\eta_{\alpha\beta}dx^\alpha dx^\beta &= \eta_{\mu\nu}(\Lambda^\mu{}_\alpha dx^\alpha)(\Lambda^\nu{}_\beta dx^\beta) && (2.400)\\
\mathbf{dx}^T\eta\mathbf{dx} &= (\Lambda\mathbf{dx})^T\eta\Lambda\mathbf{dx} = \mathbf{dx}^T(\Lambda^T\eta\Lambda)\mathbf{dx} && (2.401)\\
\eta &= \Lambda^T\eta\Lambda && (2.402)\\
\eta_{\alpha\beta} &= \Lambda^\mu{}_\alpha\eta_{\mu\nu}\Lambda^\nu{}_\beta && (2.403)
\end{aligned}$$

$$\begin{aligned}
F^{ab} &= \Lambda^a_c\Lambda^b_d F^{cd} \qquad \rightarrow \Lambda F\Lambda^T && (2.404)\\
F_{ab} &= \Lambda^c_a\Lambda^d_b F_{cd} \qquad \rightarrow \Lambda^T F\Lambda && (2.405)
\end{aligned}$$

$$\begin{aligned}
F^a_b &= \eta^{ac}F_{cb} \qquad \rightarrow \eta F && (2.406)\\
F^{ad} &= \eta^{db}\eta^{ac}F_{cb} = \eta^{ac}F_{cb}\eta^{bd} \qquad \rightarrow \eta F\eta^T && (2.407)\\
F_{ab}F^{ab} &= -F_{ba}F^{ab} \qquad \rightarrow -\mathrm{tr}(FF) && (2.408)
\end{aligned}$$

2.24 Tensorproduct

Given two Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ with complete orthonormal basis $\{\phi_{i\in I}\}$ and $\{\eta_{k\in K}\}$. Combining two basis vectors from each of the two Hilbert spaces gives

$$\psi_l \equiv \phi_i \otimes \eta_k := (\phi_i, \eta_k) \tag{2.409}$$

with $l = (i,k) \in \Lambda = I \times K$ being an element of the cartesian product of the index sets I and K. The scalar product of the Hilbert space are defined by

$$\langle \phi_{i_1}, \phi_{i_2} \rangle_1 = \delta_{i_1,i_2} \tag{2.410}$$

$$\langle \eta_{k_1}, \eta_{k_2} \rangle_2 = \delta_{k_1,k_2} \tag{2.411}$$

The $\{\psi_{l\in\Lambda}\}$ build a basis of the Product Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$

$$\psi = \sum_{l\in\Lambda} c_l \psi_l \tag{2.412}$$

$$= \sum_{i\in I} \sum_{k\in K} c_{(i,k)}\, \phi_i \otimes \eta_k \tag{2.413}$$

with the rules

$$(\phi + \phi') \otimes \eta = \phi \otimes \eta + \phi' \otimes \eta \tag{2.414}$$

$$\phi \otimes (\eta + \eta') = \phi \otimes \eta + \phi \otimes \eta' \tag{2.415}$$

$$(c\phi) \otimes \eta = c(\phi \otimes \eta) = \phi \otimes (c\eta) \tag{2.416}$$

The scalar product of the Product Hilbert space can be defined by

$$\langle \phi \otimes \eta, \phi' \otimes \eta' \rangle \equiv \langle \phi, \phi' \rangle_1 \langle \eta, \eta' \rangle_2 \tag{2.417}$$

resulting in

$$\langle \phi_i \otimes \eta_k, \phi_{i'} \otimes \eta_{k'} \rangle \equiv \langle \phi_i, \phi_{i'} \rangle_1 \langle \eta_k, \eta_{k'} \rangle_2 = \delta_{ii'} \delta_{kk'} \tag{2.418}$$

and therefore

$$\langle \psi, \psi' \rangle = \sum_{i,i'\in I} \sum_{k,k'\in K} c_{(i,k)} c_{(i',k')} \, \langle \phi_i \otimes \eta_k, \phi_{i'} \otimes \eta_{k'} \rangle = \sum_{i\in I} \sum_{k\in K} c_{(i,k)} c_{(i,k)} \tag{2.419}$$

Operators A and B defined on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ can be promoted to operators on $\mathcal{H}_1 \otimes \mathcal{H}_2$ by

$$A \to A \otimes 1_B \tag{2.420}$$

$$B \to 1_A \otimes B \tag{2.421}$$

$$A + B \equiv A \otimes 1_B + 1_A \otimes B \tag{2.422}$$

then

$$(A \otimes B)(|\phi_A\rangle \otimes |\phi_B\rangle) := A|\phi_A\rangle \otimes B|\phi_B\rangle \tag{2.423}$$

with eigenvectors $A|\phi_a\rangle = a|\phi_a\rangle$ and $B|\phi_b\rangle = b|\phi_b\rangle$

$$(A + B)(|\phi_a\rangle \otimes |\phi_b\rangle) = ... = (a + b)|\phi_a\rangle \otimes |\phi_b\rangle \tag{2.424}$$

Examples

$$\mathbb{R}^3 \otimes \mathbb{R}^3 \simeq \mathbb{R}^9 \tag{2.425}$$

$$\mathbb{R} \otimes \mathbb{R} \otimes \mathbb{R} \simeq \mathbb{R} \tag{2.426}$$

$$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \simeq \mathbb{R}^3 \tag{2.427}$$

2.25 Pauli Matrices

Properties of the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \tag{2.428}$$

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = i\epsilon_{ijk}\frac{\sigma_k}{2} \qquad \{\sigma_i, \sigma_j\} = 2\delta_{ij} \tag{2.429}$$

$$\mathrm{Tr}\sigma_i = 0 \qquad \mathrm{Tr}(\sigma_i\sigma_j) = 2\delta_{ij} \tag{2.430}$$

$$\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k \qquad \sigma_i^2 = 1 \tag{2.431}$$

$$\sum_i (\sigma_i)_{ab}(\sigma_i)_{cd} = 2(\delta_{bc}\delta_{ad} - \frac{1}{2}\delta_{ab}\delta_{cd}) \tag{2.432}$$

The fundamental representation of SU(2) is given by 2×2 matrices U (with $U^\dagger U = 1$ and $\det U = 1$) which operate on two-component column vectors (fundamental doublet or Pauli spinor) $\xi' = U\xi$. A general matrix U can be expressed as

$$U = e^{\frac{i}{2}\theta_i\sigma_i}. \tag{2.433}$$

2.26 Clifford Algebras

Cliff$(1, d-1)$ is defined as set of d matrices of shape $n \times n$

$$\{(\gamma^\mu)^A_{\ B}\}_{\mu\in\{0,1,\ldots,d-1\}} \qquad A, B = 1, \ldots, n \tag{2.434}$$

which obey

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbf{1}_{n\times n} \tag{2.435}$$

$$(\gamma^\mu)^A_{\ B}(\gamma^\nu)^B_{\ C} + (\gamma^\nu)^A_{\ B}(\gamma^\mu)^B_{\ C} = 2\eta^{\mu\nu}(\mathbf{1}_{n\times n})^A_{\ C} \tag{2.436}$$

with spinor indices A, B and space-time indices μ, ν.

Some properties of Clifford algebras

- With $\mathrm{diag}\,\eta^{\mu\nu} = (1, -1, -1, -1)$

$$(\gamma^0)^2 = \mathbf{1}_{n\times n} \qquad (\gamma^i)^2 = -\mathbf{1}_{n\times n} \tag{2.437}$$

- The irreducible representations of Cliff$(1, d-1)$ have dimensions
  - d even: $n = 2^{d/2}$
  - d odd: $n = 2^{(d-1)/2}$

d	1	2	3	4	5, 6, 7, 8
algebra	Cliff$(1,0)$	Cliff$(1,1)$	Cliff$(1,2)$	Cliff$(1,3)$	Cliff$(1,d-1)$
n	1	2	2	4	4, 8, 8, 16

Table 2.2: •

- The d matrices $\{\gamma^\mu\}$ of the Clifford algebra Cliff$(1, d-1)$ induce $d(d-1)/2$ matrices $S^{\rho\sigma}$

$$(S^{\rho\sigma})^A_{\ B} = \frac{i}{4}[\gamma^\rho, \gamma^\sigma]^A_{\ B} \tag{2.438}$$

  which form a representation of the Lie algebra (of the Lorentz group) $\mathfrak{so}(1, d-1)$.

- For $d = 4$ we have Cliff(1,3) which contains $1 + (d-1) = 4$ matrices of shape 4×4

$$\gamma^0, \gamma^1, \gamma^2, \gamma^3 \tag{2.439}$$

which induces 6 matrices $S^{01}, S^{12}, S^{23}, S^{02}, S^{03}, S^{13}$ which are the generators of $\mathfrak{so}(1,3)$.

- Chiral/Dirac representation of Cliff(1,3)

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1}_{2\times2} \\ \mathbf{1}_{2\times2} & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{2.440}$$

The γ's act on a complex vector space - the space of Dirac spinors $(\gamma^\mu)^A_B \psi^B$. A Lorentz trafo look like

$$\psi^A(x) \to \left[e^{-i\omega_{\rho\sigma}S^{\rho\sigma}}\right]^A_B \psi^B(x) \tag{2.441}$$

$$= \left[e^{-i(\omega_{01}S^{01}+\ldots+\omega_{23}S^{23})}\right]^A_B \psi^B(x) \tag{2.442}$$

$$= \left[e^{\frac{1}{4}(\omega_{01}[\gamma^0,\gamma^1]+\ldots+\omega_{23}[\gamma^2,\gamma^3])}\right]^A_B \psi^B(x) \tag{2.443}$$

2.27 Spinors

3D vector ($x, y, z \in \mathbb{R}$) can be written as a Pauli vector (via the Pauli matrices) which can be written as a product of Pauli spinors ($\xi_1, \xi_2 \in \mathbb{C}$)

$$\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Longleftrightarrow x\sigma_x + y\sigma_y + z\sigma_z = \vec{v} \cdot \vec{\sigma} \tag{2.444}$$

$$= x\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + y\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + z\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \tag{2.445}$$

$$= \begin{pmatrix} z & x - yi \\ x + yi & -z \end{pmatrix} \tag{2.446}$$

$$= \begin{pmatrix} -\xi_1\xi_2 & \xi_1^2 \\ -\xi_2^2 & \xi_1\xi_2 \end{pmatrix} \tag{2.447}$$

$$= \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \begin{pmatrix} -\xi_2 & \xi_1 \end{pmatrix} \tag{2.448}$$

Rotating a vector

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{2.449}$$

Rotating the associated Pauli vector

$$\begin{pmatrix} \cos\theta/2 & i\sin\theta/2 \\ i\sin\theta/2 & \cos\theta/2 \end{pmatrix} \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} \begin{pmatrix} \cos\theta/2 & i\sin\theta/2 \\ i\sin\theta/2 & \cos\theta/2 \end{pmatrix}^\dagger \tag{2.450}$$

$$= \underbrace{\begin{pmatrix} \cos\theta/2 & i\sin\theta/2 \\ i\sin\theta/2 & \cos\theta/2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}}_{\ldots} \underbrace{\begin{pmatrix} -\xi_2 & \xi_1 \end{pmatrix} \begin{pmatrix} \cos\theta/2 & i\sin\theta/2 \\ i\sin\theta/2 & \cos\theta/2 \end{pmatrix}^\dagger}_{\ldots} \tag{2.451}$$

As you need two Pauli spinores to represent a 3-vector and their associated rotations contain only half the angles we can regard a spinor as a rank 1/2 tensor.

3-vectors are represented by two Pauli spinors while 4-vectors are represented by Weyl spinors. Weyl representation of 4-vectors

$$\vec{X} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \Longleftrightarrow ct\mathbb{I} + x\sigma_x + y\sigma_y + z\sigma_z = ct\mathbb{I} + \vec{x}\cdot\vec{\sigma} \tag{2.452}$$

$$= X^\mu \sigma_\mu \tag{2.453}$$

$$= \begin{pmatrix} ct+z & x-yi \\ x+yi & ct-z \end{pmatrix} \tag{2.454}$$

For 4-vectors we replace the σ by the γ-matrices and obtain Weyl spinors

$$\vec{X} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \Longleftrightarrow ct\gamma^0 + x\gamma^1 + y\gamma^2 + z\gamma^3 \tag{2.455}$$

- Grassmann algebra (exterior algebra): contains a wedge product
- Clifford algebra (geometric algebra): contains a wedge product and a scalar product

algebra	signature	equation	object
$Cl_{4,2}$	$+,+,+,+,-,-$	Twistor	twistor
$Cl_{1,3}$	$+,-,-,-$	Dirac	rel spin-1/2
$Cl_{3,0}$	$+,+,+$	Pauli	spin-1/2
$Cl_{0,1}$	$-$	Schroedinger	spin-0

Chapter 3

Primers

3.1 Classical Mechanics

3.1.1 Lagrangian Mechanics

$$L = T - V, \qquad S = \int L(q, \dot{q}, t) dt \tag{3.1}$$

Integration by parts - neglecting boundary terms

$$\delta S = \int \delta L dt = 0 \tag{3.2}$$

$$\delta L = \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} \tag{3.3}$$

$$= \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right]\delta q \tag{3.4}$$

Canonical momentum

$$p = \frac{\partial L}{\partial \dot{q}} \tag{3.5}$$

Cyclic coordinates

$$\frac{\partial L}{\partial q} = 0 \quad \rightarrow \quad \frac{\partial L}{\partial \dot{q}} = p = \text{const} \tag{3.6}$$

3.2 Classical Electrodynamics

$$\eta_{ab} = \eta^{ab} = \text{diag}(1, -1, -1, -1) \tag{3.7}$$

$$\mathbf{A} \rightarrow A^i = \begin{pmatrix} A^0 \\ \vec{A} \end{pmatrix} \qquad A_i = \begin{pmatrix} A^0 \\ -\vec{A} \end{pmatrix} \tag{3.8}$$

$$\mathbf{E} = -\nabla A^0 - \partial_t \mathbf{A} \tag{3.9}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{3.10}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.11}$$

$$F_{10} = \partial_x A_0 - \partial_t A_x = \partial_x A^0 + \partial_t A^x = -E_x \tag{3.12}$$

$$F_{21} = \partial_y A_x - \partial_x A_y = -\partial_y A^x + \partial_x A^y = B_z \tag{3.13}$$

$$F_{31} = \partial_z A_x - \partial_x A_z = -\partial_z A^x + \partial_x A^z = -B_y \tag{3.14}$$

$$F_{\mu\nu} = F_{\downarrow\downarrow} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \qquad F^{\mu\nu} = F_{\uparrow\uparrow} = \eta F_{\downarrow\downarrow}\eta^T = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{3.15}$$

$$F_{\mu\nu}F^{\mu\nu} = -\mathrm{tr}(F_{\downarrow\downarrow}.F_{\uparrow\uparrow}) = 2(\mathbf{B}^2 - \mathbf{E}^2) \qquad F^{\mu\lambda}F_{\lambda\nu} = ... \tag{3.16}$$

3.3 Quantum Mechanics

3.3.1 Pictures

Prelims - at $t = t_0$

$$|\psi_H\rangle = |\psi(t_0)\rangle \tag{3.17}$$

and obviously

$$U(t, t_0) = U^{-1}(t_0, t) \tag{3.18}$$

$$U^\dagger(t, t_0)U(t, t_0) = 1 \tag{3.19}$$

1. Schroedinger - time dependency in the states

$$i|\psi(t)\rangle = H|\psi(t)\rangle \tag{3.20}$$

$$|\psi(t)\rangle = U(t, t_0)|\psi(t_0)\rangle \tag{3.21}$$

$$iU(t, t_0) = HU(t, t_0) \tag{3.22}$$

$$\frac{\partial H}{\partial t} = 0 \quad \rightarrow \quad U(t, t_0) = e^{-iH(t-t_0)} \tag{3.23}$$

Time evolution with $H|E_k\rangle = E_k|E_k\rangle$

$$|\psi(t)\rangle = U(t, t_0)|\psi(t_0)\rangle \tag{3.24}$$

$$= U(t, t_0)\sum_k |E_k\rangle\langle E_k|\psi(t_0)\rangle \tag{3.25}$$

$$= \sum_k e^{-iH(t-t_0)}|E_k\rangle\langle E_k|\psi(t_0)\rangle \tag{3.26}$$

$$= \sum_k e^{-iE_k(t-t_0)}|E_k\rangle\langle E_k|\psi(t_0)\rangle \tag{3.27}$$

Measurement

$$\langle A(t)\rangle = \langle\psi(t)|A_S|\psi(t)\rangle \tag{3.28}$$

2. Heisenberg - time dependency in the operators

$$\langle A(t)\rangle = \langle\psi(t)|A_S|\psi(t)\rangle \tag{3.29}$$

$$= \langle\psi(t_0)|U^\dagger(t, t_0)A_S U(t, t_0)|\psi(t_0)\rangle \tag{3.30}$$

$$= \langle\psi(t_0)|A_H(t)|\psi(t_0)\rangle \tag{3.31}$$

$$\rightarrow A_H(t) = U^\dagger(t, t_0)A_S U(t, t_0) \tag{3.32}$$

Time derivative

$$\frac{d}{dt}A_H(t) = \left(\frac{d}{dt}U^\dagger(t,t_0)\right) A_S U(t,t_0) + U^\dagger(t,t_0)\left(\frac{d}{dt}A_S\right) U(t,t_0) + U^\dagger(t,t_0) A_S \left(\frac{d}{dt}U(t,t_0)\right) \tag{3.33}$$

$$= U^\dagger(t,t_0) i(HA_S - A_S H)U(t,t_0) + U^\dagger(t,t_0)\frac{\partial A_S}{\partial t}U(t,t_0) \tag{3.34}$$

$$= i[H,A_S] + \underbrace{U^\dagger(t,t_0)\frac{\partial A_S}{\partial t}U(t,t_0)}_{\equiv \frac{\partial A_H}{\partial t}} \tag{3.35}$$

$$= i[H,A_S] + \frac{\partial A_H}{\partial t} \tag{3.36}$$

3. Dirac - H_0 time dependency in the operators and H_1 time dependency in the states

$$\langle A(t)\rangle = \langle \psi(t)|A_S|\psi(t)\rangle \tag{3.37}$$

$$= \langle \psi(t_0)|U^\dagger(t,t_0)A_S U(t,t_0)|\psi(t_0)\rangle \tag{3.38}$$

$$= \langle \psi(t_0)|U^\dagger(t,t_0)\underbrace{U_0(t,t_0)U_0^\dagger(t,t_0)}_{=1} A_S \underbrace{U_0(t,t_0)U_0^\dagger(t,t_0)}_{=1} U(t,t_0)|\psi(t_0)\rangle \tag{3.39}$$

$$\to A_D = U_0^\dagger(t,t_0)A_S U_0(t,t_0) \tag{3.40}$$

$$\to |\psi_D(t)\rangle = U_0^\dagger(t,t_0)U(t,t_0)|\psi(t_0)\rangle \tag{3.41}$$

Now calc evolution between the TWO Dirac states $|\psi_D(t_1)\rangle$ and $|\psi_D(t_2)\rangle$

$$|\psi_D(t_1)\rangle = U_0^\dagger(t_1,t_0)U(t_1,t_0)|\psi(t_0)\rangle \tag{3.42}$$

$$|\psi_D(t_2)\rangle = U_0^\dagger(t_2,t_0)U(t_2,t_0)|\psi(t_0)\rangle \tag{3.43}$$

$$= U_0^\dagger(t_2,t_0)U(t_2,t_0)\left(U_0^\dagger(t_1,t_0)U(t_1,t_0)\right)^{-1}|\psi_D(t_1)\rangle \tag{3.44}$$

$$= U_0^\dagger(t_2,t_0)U(t_2,t_0)U^{-1}(t_1,t_0)\left(U_0^\dagger(t_1,t_0)\right)^{-1}|\psi_D(t_1)\rangle \tag{3.45}$$

$$= U_0^\dagger(t_2,t_0)U(t_2,t_0)U(t_0,t_1)U_0^\dagger(t_1,t_0)|\psi_D(t_1)\rangle \tag{3.46}$$

$$= U_0^\dagger(t_2,t_0)U(t_2,t_1)U_0^\dagger(t_1,t_0)|\psi_D(t_1)\rangle \tag{3.47}$$

with $t_0 = 0$ and H_0 time-independent

$$U_D(t_2,t_1) = U_0^\dagger(t_2,0)U(t_2,t_1)U_0^\dagger(t_1,0)|\psi_D(t_1)\rangle \tag{3.48}$$

$$= e^{iH_0t_2}U(t_2,t_1)e^{iH_0t_1} \tag{3.49}$$

3.3.2 3D Spherical well

$$\left\{-\frac{\hbar^2}{2m}\triangle + V(r)\right\}\psi = E\psi \tag{3.50}$$

$$\left\{-\frac{\hbar^2}{2m}\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \frac{1}{r^2}\triangle_{\phi\theta}\right] + V(r)\right\}\psi = E\psi \tag{3.51}$$

$$\left\{\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \frac{1}{r^2}\triangle_{\phi\theta} - \frac{2m[V(r)-E]}{\hbar^2}\right\}\psi = 0 \tag{3.52}$$

Separation $\psi = R(r)Y(\phi,\theta)$

$$\frac{r^2\left\{\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{2m[V(r)-E]}{\hbar^2}\right\}R(r)}{R(r)} = l(l+1) = -\frac{\triangle_{\phi,\theta}Y(\phi,\theta)}{Y(\phi,\theta)} \tag{3.53}$$

$$\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{l(l+1)}{r^2}-\frac{2m[V(r)-E]}{\hbar^2}\right)R(r)=0 \tag{3.54}$$

With the definition of the well potential

$$V(r)=\begin{cases}-V_0 & r<a\\ 0 & r>a\end{cases} \tag{3.55}$$

With $-V_0 < E < 0$

$$k=\frac{\sqrt{2m[E+V_0]}}{\hbar} \tag{3.56}$$

$$\kappa=\frac{\sqrt{2m(-E)}}{\hbar} \tag{3.57}$$

be have with $\rho = kr$ and $\rho = i\kappa r$

$$\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{l(l+1)}{r^2}+\begin{pmatrix}k^2\\-\kappa^2\end{pmatrix}\right]R(r)=0 \tag{3.58}$$

$$\left[\frac{d^2}{d\rho^2}+\frac{2}{\rho}\frac{d}{d\rho}-\frac{l(l+1)}{\rho^2}+1\right]R(\rho)=0 \tag{3.59}$$

$$\left[\rho^2\frac{d^2}{d\rho^2}+2\rho\frac{d}{d\rho}+\rho^2-l(l+1)\right]R(\rho)=0 \tag{3.60}$$

Independent solutions

$$R(\rho)=Aj_l(\rho)+By_l(\rho) \tag{3.61}$$

$$=A\sqrt{\frac{\pi}{2\rho}}J_{l+1/2}(\rho)+B\sqrt{\frac{\pi}{2\rho}}Y_{l+1/2}(\rho) \tag{3.62}$$

Here the requirements

- regular at the origin with $R(r)\sim r^l$
- continous and differentiable at $r=a$
- exponential decay outside to ensure normalizability

and here a quick overview of the two functions and a special linear combination

$j_l(x)=(-x)^l\left(\frac{1}{x}\frac{d}{dx}\right)^l\frac{\sin x}{x}$	$y_l(x)=-(-x)^l\left(\frac{1}{x}\frac{d}{dx}\right)^l\frac{\cos x}{x}$	$h_0^{(1)}(x)=j_l(ix)+iy_l(ix)$
$j_0(x)=\frac{\sin x}{x}$	$y_0(x)=-\frac{\cos x}{x}$	$h_0^{(1)}(x)=-\frac{e^{-x}}{x}$
$j_1(x)=\frac{\sin x}{x^2}-\frac{\cos x}{x}$	$y_1(x)=-\frac{\cos x}{x}-\frac{\sin x}{x}$	$h_1^{(1)}(x)=i(1+x)\frac{e^{-x}}{x^2}$
$J_2(x)=...$	$y_l(x)=...$	$h_2^{(1)}(x)=(x^2+3x+3)\frac{e^{-x}}{x^3}$

We see that j_l is suitable for the inside and $h_l^{(1)}$ for the outside.

$$R(\rho)=\begin{cases}Aj_l(\rho) & r<a\\ Ch_l^{(1)}(\rho) & r>a\end{cases} \tag{3.63}$$

Now $l = 0$

$$Aj_0(\rho = ka) = Ch_0^{(1)}(\rho = \kappa a) \quad \rightarrow \quad A\frac{\sin ka}{ka} = -C\frac{e^{-\kappa a}}{\kappa a} \tag{3.64}$$

$$A\partial_r j_0(\rho = ka) = C\partial_r h_0^{(1)}(\rho = \kappa a) \quad \rightarrow \quad A\frac{\sin ak}{a}\left(\cot ka - \frac{1}{ka}\right) = C\frac{e^{-\kappa a}}{a}\left(1 + \frac{1}{\kappa a}\right) \tag{3.65}$$

By substituting first into the second equation we kick out A and C and obtain

$$\cot ka = -\frac{\kappa}{k} \tag{3.66}$$

$$\cot\sqrt{\frac{2ma^2}{\hbar^2}[E + V_0]} = -\sqrt{\frac{-E}{E + V_0}} \tag{3.67}$$

Now $l = 1$

$$Aj_1(\rho = ka) = Ch_1^{(1)}(\rho = \kappa a) \quad \rightarrow \quad A\left(-\frac{\cos ka}{ka} + \frac{\sin ka}{k^2a^2}\right) = iC\frac{e^{-\kappa a}}{\kappa^2a^2}(1 + \kappa a) \tag{3.68}$$

$$A\partial_r j_1(\rho = ka) = C\partial_r h_1^{(1)}(\rho = \kappa a) \quad \rightarrow \quad A\left(2\frac{\cos ka}{ka^2} + \frac{\sin ka}{k^2a^3}(a^2k^2 - 2)\right) = -iC\frac{e^{-\kappa a}}{\kappa^2a^3}(\kappa^2a^2 + 2\kappa a + 2) \tag{3.69}$$

Then

$$\cot ka = \frac{k^2 + ak^2\kappa + \kappa^2}{ak\kappa^2} \tag{3.70}$$

3.4 Quantum statistics

Quick thermodynamics review

$$\text{1st law} \quad dU = \delta Q + \delta W \tag{3.71}$$

$$\text{2nd law} \quad dS = dS_i + \frac{\delta Q}{T}, \qquad dS_i > 0 \tag{3.72}$$

$$\text{Gibbs Fund.Form} \rightarrow dS = \frac{1}{T}dU - \frac{1}{T}\delta W = \frac{1}{T}dU + \frac{1}{T}\sum_i y_i dX_i \tag{3.73}$$

$$\rightarrow \left.\frac{dS}{dU}\right|_{X_i} = \frac{1}{T} \qquad \rightarrow \qquad U = U(T, X_i) \tag{3.74}$$

$$\rightarrow \left.\frac{dS}{dX_i}\right|_{U,X_j} = \frac{y_i}{T} \qquad \rightarrow \qquad y_i = y_i(T, X_j) \tag{3.75}$$

3.4.1 Density matrix - statistical operator

Using the principle of equal probability

$$\hat{\varrho} = \sum_k p_k |\Psi_k\rangle\langle\Psi_k| \tag{3.76}$$

$$= \frac{1}{\Omega}\sum_k |\Psi_k\rangle\langle\Psi_k| \tag{3.77}$$

$$\mathrm{Tr}\hat{\varrho} = 1 \tag{3.78}$$

$$S = -k\langle\hat{\varrho}\rangle \tag{3.79}$$

$$= -k\mathrm{Tr}(\hat{\varrho}\log\hat{\varrho}) \tag{3.80}$$

$$\tag{3.81}$$

3.4.2 Canonical ensemble

Represents all states of a system in thermodynamic equilibrium. Meaning the temperature T and therefore the mean energy $\bar{E} = U$ is fixed but the total energy can fluctuate

$$Z = \mathrm{Tr}\left[\exp\left(-\frac{\hat{H}}{kT}\right)\right] \tag{3.82}$$

$$\hat{\varrho} = \frac{1}{Z(T)}\exp\left(-\frac{\hat{H}}{kT}\right) = \frac{1}{Z(T)}\sum_k |\Psi_k\rangle \exp\left(-\frac{E_k}{kT}\right)\langle\Psi_k| \tag{3.83}$$

$$F = -kT\log Z \tag{3.84}$$

$$\frac{\partial F}{\partial T} = -S \tag{3.85}$$

$$U = F + TS \tag{3.86}$$

3.4.3 Great Canonical ensemble

Represents all states of a system in thermodynamic equilibrium. Meaning the temperature T and therefore the mean energy $\bar{E} = U$ is fixed but the total energy can fluctuate

$$\mathcal{Z} = \mathrm{Tr}\left[\exp\left(-\frac{\hat{H} - \mu\hat{N}}{kT}\right)\right] \tag{3.87}$$

$$\hat{\varrho} = \frac{1}{\mathcal{Z}(T)}\exp\left(-\frac{\hat{H} - \mu\hat{N}}{kT}\right) \tag{3.88}$$

$$\mathcal{F} = -kT\log\mathcal{Z} \tag{3.89}$$

$$\left(\frac{\partial\mathcal{F}}{\partial T}\right)_\mu = -S \qquad \left(\frac{\partial\mathcal{F}}{\partial\mu}\right)_T = -\bar{N} = -\langle\hat{N}\rangle \tag{3.90}$$

3.5 Special relativity

Definition of line element

$$ds^2 = dx^\mu dx_\nu = \eta_{\mu\nu}dx^\mu dx^\nu \tag{3.91}$$

$$= dx^T\eta dx \tag{3.92}$$

Definition of Lorentz transformation

$$dx^\mu = \Lambda^\mu{}_\nu dx^\nu \tag{3.93}$$

By postulate the line element ds is invariant under Lorentz transformation

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu \tag{3.94}$$

$$\stackrel{!}{=} \eta_{\alpha\beta}\Lambda^\alpha{}_\mu dx^\mu\Lambda^\beta{}_\nu dx^\nu \quad\rightarrow\quad \eta_{\mu\nu} = \eta_{\alpha\beta}\Lambda^\alpha{}_\mu\Lambda^\beta{}_\nu \tag{3.95}$$

or analog

$$ds^2 = dx^T\eta dx \tag{3.96}$$

$$\stackrel{!}{=} (\Lambda dx)^T\eta(\Lambda dx) \tag{3.97}$$

$$= dx^T\Lambda^T\eta\Lambda dx \quad\rightarrow\quad \eta = \Lambda^T\eta\Lambda \tag{3.98}$$

Observation with the eigentime $d\tau = ds/c$ and 3-velocity $dx^i = v^i dt$

$$\frac{ds^2}{d\tau^2} = c^2 = c^2\frac{dt^2}{d\tau^2} - \frac{dx^i}{dt}\frac{dx_i}{dt}\left(\frac{dt}{d\tau}\right)^2 \tag{3.99}$$

$$1 = \frac{dt^2}{d\tau^2}\left(1 - \frac{v^i v_i}{c^2}\right) \quad \to \quad \frac{dt}{d\tau} \equiv \gamma = \left(\sqrt{1 - \frac{v^2}{c^2}}\right)^{-1} \tag{3.100}$$

Definition of 4-velocity with 3-velocity $d\vec{x} = \vec{v}dt$

$$u^\mu \equiv \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt}\frac{dt}{d\tau} = \quad \to \quad u^\mu u_\mu = \eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = \frac{ds^2}{d\tau^2} = c^2 \tag{3.101}$$

$$= (c, \vec{v})\gamma \tag{3.102}$$

Object moving in x direction with v meaning $dx = v \cdot dt$ compared to rest frame $dx' = 0$

$$c^2 dt'^2 = ds^2 = c^2 dt^2 - v^2 dt^2 \tag{3.103}$$

$$= c^2 dt^2\left(1 - \frac{v^2}{c^2}\right) \tag{3.104}$$

$$dt' = \frac{ds}{c} \equiv d\tau = dt\sqrt{1 - \frac{v^2}{c^2}} = \frac{dt}{\gamma} \tag{3.105}$$

Definition 4-momentum (using the 3-momentum $\vec{p} = \gamma m\vec{v}$)

$$p^\mu \equiv m u^\mu = (\gamma mc, \gamma m\vec{v}) = \left(\frac{E_p}{c}, \vec{p}\right) \quad \to \quad p^\mu p_\mu = m^2 u^\mu u_\mu = m^2 c^2 \tag{3.106}$$

$$\to \quad (p^0)^2 - p^i p_i = m^2 c^2 \tag{3.107}$$

$$\to \quad p^0 = \sqrt{m^2c^2 + \vec{p}^2} \tag{3.108}$$

$$\to \quad E_p = \sqrt{m^2c^4 + \vec{p}^2c^2} \tag{3.109}$$

$$= \frac{mc^2}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}} \tag{3.110}$$

First we observe

$$\eta_{\mu\nu} = \eta_{\alpha\beta}\Lambda^\alpha_{\ \mu}\Lambda^\beta_{\ \nu} \tag{3.111}$$

$$\det(\eta) = \det(\Lambda)^2\det(\eta) \tag{3.112}$$

$$1 = \det(\Lambda)^2. \tag{3.113}$$

Now we see

$$\Lambda_\gamma^{\ \nu}\Lambda^\gamma_{\ \mu} = \eta_{\alpha\gamma}\eta^{\nu\beta}\Lambda^\alpha_{\ \beta}\Lambda^\gamma_{\ \mu} \tag{3.114}$$

$$= \eta^{\nu\beta}(\eta_{\alpha\gamma}\Lambda^\alpha_{\ \beta}\Lambda^\gamma_{\ \mu}) \tag{3.115}$$

$$= \eta^{\nu\beta}\eta_{\beta\mu} \tag{3.116}$$

$$= \delta^\nu_{\ \mu} \tag{3.117}$$

which means in matrix notation $\Lambda_\gamma^{\ \nu} = (\Lambda^{-1})^\nu_{\ \gamma}$. General transformation laws for tensors of first order

$$V'^\alpha = \Lambda^\alpha_{\ \beta}V^\beta \tag{3.118}$$

$$\eta_{\alpha\mu}V'^\alpha = \eta_{\alpha\mu}\Lambda^\alpha_{\ \beta}V^\beta = \eta_{\alpha\mu}\Lambda^\alpha_{\ \beta}(\eta^{\nu\beta}V_\nu) \tag{3.119}$$

$$V'_\mu = \Lambda_\mu^{\ \nu}V_\nu \tag{3.120}$$

$$\to \quad \Lambda_\mu^{\ \nu} = \eta_{\alpha\mu}\eta^{\nu\beta}\Lambda^\alpha_{\ \beta} \tag{3.121}$$

and second order

$$T'^{\alpha\beta} = \Lambda^{\alpha}_{\ \mu}\Lambda^{\beta}_{\ \nu}T^{\mu\nu} \tag{3.122}$$
$$\eta_{\alpha\delta}\eta_{\beta\gamma}T'^{\alpha\beta} = \eta_{\alpha\delta}\eta_{\beta\gamma}\Lambda^{\alpha}_{\ \mu}\Lambda^{\beta}_{\ \nu}T^{\mu\nu} = \eta_{\alpha\delta}\eta_{\beta\gamma}\Lambda^{\alpha}_{\ \mu}\Lambda^{\beta}_{\ \nu}(\eta^{\mu\rho}\eta^{\nu\sigma}T_{\rho\sigma}) \tag{3.123}$$
$$T'_{\delta\gamma} = \Lambda_{\delta}^{\ \rho}\Lambda_{\gamma}^{\ \sigma}T_{\rho\sigma}. \tag{3.124}$$

The general transformation is therefore given by

$$T'_{\mu_1\mu_2...}{}^{\nu_1\nu_2...} = \Lambda_{\mu_1}{}^{\rho_1}\Lambda_{\mu_2}{}^{\rho_2}...\Lambda^{\nu_1}{}_{\sigma_1}\Lambda^{\nu_2}{}_{\sigma_2}...T'_{\rho_1\rho_2...}{}^{\sigma_1\sigma_2...} \tag{3.125}$$

There exist two invariant tensors

$$\eta'_{\mu\nu} = \eta_{\alpha\beta}\Lambda^{\alpha}_{\ \mu}\Lambda^{\beta}_{\ \nu} = \Lambda_{\beta\mu}\Lambda^{\beta}_{\ \nu} = \eta_{\mu\sigma}\Lambda_{\beta}^{\ \sigma}\Lambda^{\beta}_{\ \nu} = \eta_{\mu\sigma}\delta^{\sigma}_{\ \nu} = \eta_{\mu\nu} \tag{3.126}$$
$$\epsilon'^{\mu\nu\rho\sigma} = \Lambda^{\mu}_{\ \alpha}\Lambda^{\nu}_{\ \beta}\Lambda^{\rho}_{\ \gamma}\Lambda^{\sigma}_{\ \delta}\epsilon'^{\alpha\beta\gamma\delta} \equiv \epsilon^{\mu\nu\rho\sigma}\det(\Lambda) = \pm\epsilon^{\mu\nu\rho\sigma} \tag{3.127}$$

Due to the possibility of the minus sign the Levi-Civita symbol ϵ is sometimes called pseudo-tensor.

3.6 Perturbation theory

1. Find a hard problem
2. Introduce an ϵ
3. Assume the solution can be expressed as a power series $x_s = \sum_k a_k\epsilon^k$
4. Find all a_k and sum them up
5. Set $\epsilon = 1$

Now consider solving $x^5 + x = 1$

$$x^5 + \epsilon x = 1 \tag{3.128}$$
$$\rightarrow x = 1 - \frac{1}{5}\epsilon - \frac{1}{25}\epsilon^2 - \frac{1}{125}\epsilon^3 + 0\epsilon^4 + \frac{21}{15625}\epsilon^5 + ... \tag{3.129}$$

or

$$\epsilon x^5 + x = 1 \tag{3.130}$$
$$\rightarrow x = 1 - \epsilon + 5\epsilon^2 - 35\epsilon^3 + 285\epsilon^4 - 2530\epsilon^5 + ... \tag{3.131}$$

Method of dominant balance

- Asymptotics $f(x)$ $g(x)$ for $x \to x_0$

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = 1 \tag{3.132}$$

- Neglectable $f(x) \ll g(x)$ for $x \to x_0$

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = 0 \tag{3.133}$$

Chapter 4

Groups

4.0.1 Overview

$\mathbb{F}$	GL$(n,\mathbb{F})$	SL$(n,\mathbb{F})$	U(n)	SU(n)	O(n)	SO(n)
$\mathbb{R}$	n^2	n^2-1	-	-	$n(n-1)/2$	$n(n-1)/2$
$\mathbb{C}$	$2n^2$	$2(n^2-1)$	n^2	n^2-1	$n(n-1)$	$n(n-1)$

Table 4.1: Dimensions of common Lie groups (number of independent real parameters)

Observation: $\dim(\mathrm{SO}(n,\mathbb{F})) = \dim(\mathrm{O}(n,\mathbb{F}))$ - sign that SO(n) is not connected

4.0.2 SO(2)

Group of rotations in two dimensions - therefore rotations are naturally given by a 2×2 matrix R with parameter α (and the generator X)

$$R = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \qquad -iX = \left.\frac{\partial R}{\partial \alpha}\right|_{\alpha=0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{4.1}$$

acting on vectors (x,y). This is therefore also a 2-dimensional (real) representation of SO(2) - it is even an irrep. In a complex space the vector can be written as $z = x+iy$ and the rotation is represented by $e^{i\alpha}$ - which serves as a one dimensional complex representation.

There are actually infinitely many (non-equivalent) 1-dimensional standard irreps

$$D^k(\alpha) = e^{-ik\alpha},\ k = 0, \pm 1, \pm 2, ... \tag{4.2}$$

4.0.3 SO(3) - What we know from quantum mechanics

The angular momentum algebra is given by $[J_i, J_j] = i\hbar\varepsilon_{ijk}J_k$. We know that

$$J^2|jm\rangle = j(j+1)|jm\rangle \qquad j \in \left\{0, \frac{1}{2}, 1, \frac{3}{2}, ...\right\}, \quad m = -j, ..., j \tag{4.3}$$

$$J_z|jm\rangle = m|jm\rangle \tag{4.4}$$

meaning that J^2 and J_3 can be diagonalized at the same time. For each j there is a $2j+1$ dimensional irrep on the Hilbert space. The subspace spanned by the states $\{|jm\rangle\}_{m\in\{-j,...,j\}}$ is called $\mathfrak{h}_j$. The states of two added angular momenta j_1 and j_2 are in the space $\mathfrak{h}_{j_1j_2} = \mathfrak{h}_{j_1}\otimes\mathfrak{h}_{j_2}$ spanned by the tensor product of the eigenstates of $(J^2_{j_1}, J_{j_1,3})$ and $(J^2_{j_2}, J_{j_2,3})$

$$|j_1m_1j_2m_2\rangle = |j_1, m_1\rangle \otimes |j_2, m_2\rangle \tag{4.5}$$

The operators J^2, J_3, $J_{j_1}^2$ and $J_{j_2}^2$ commute which means they share one set of eigenfunctions $|j_1, j_2, j, m\rangle$ which also spans $\mathfrak{h}_{j_1,j_2}$. Both basis set are connected by the Clebsch-Gordon coefficients

$$|j_1, j_2, j, m\rangle = \sum_{m_1,m_2} \langle j_1, m_1, j_2, m_2 | j_1, j_2, j, m\rangle |j_1 m_1, j_2, m_2\rangle \tag{4.6}$$

The dimension of the Product space is given by

$$\dim(\mathfrak{h}_{j_1} \otimes \mathfrak{h}_{j_2}) = (2j_1 + 1)(2j_2 + 1). \tag{4.7}$$

The tensor product representations decomposes as (Clebsch-Gordan decomposition)

$$\mathfrak{h}_{j_1} \otimes \mathfrak{h}_{j_1} \cong \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} \mathfrak{h}_j \tag{4.8}$$

$$= \mathfrak{h}_{j_1+j_2} \oplus \mathfrak{h}_{j_1+j_2-1} \oplus ... \oplus \mathfrak{h}_{j_1-j_2+1} \oplus \mathfrak{h}_{|j_1-j_2|} \tag{4.9}$$

Examples

$$j_1 = \frac{1}{2}, j_2 = \frac{1}{2} \quad \rightarrow \quad 2 \otimes 2 = 1 \oplus 3 \tag{4.10}$$

$$j_1 = 1, j_2 = 1 \quad \rightarrow \quad 3 \otimes 3 = 1 \oplus 3 \oplus 5 \tag{4.11}$$

4.0.4 SO(3)

Group of rotations in three dimensions

$$v^2 = \vec{v}^T \vec{v} \tag{4.12}$$

$$= (R\vec{v})^T (R\vec{v}) \tag{4.13}$$

$$= \vec{v}^T R^T R \vec{v} \tag{4.14}$$

$$\rightarrow R^T R = I \tag{4.15}$$

therefore rotations around the 3 coordinate axis are naturally given by three 3×3 matrices R_i (with the generators X_i)

$$R_3 = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad -iX_3 = \frac{\partial R}{\partial \alpha}\bigg|_{\alpha=0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.16}$$

$$R_2 = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix} \qquad -iX_2 = \frac{\partial R}{\partial \alpha}\bigg|_{\alpha=0} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \tag{4.17}$$

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \qquad -iX_1 = \frac{\partial R}{\partial \alpha}\bigg|_{\alpha=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \tag{4.18}$$

which also a 3-dimensional representation of SO(3). The generators obey the commutation relation

$$[X_i, X_i] = i\varepsilon_{ijk} X_k \tag{4.19}$$

4.0.5 SU(2)

Unitary transformation like a complex rotation - so the condition is

$$U^\dagger U = I \qquad \text{or} \qquad U^\dagger = U^{-1}$$

Construction of a generic SU(2) matrix

$$
\begin{aligned}
U &= \begin{pmatrix} a & c \\ c & d \end{pmatrix} \qquad ad - bc = 1 \\
\rightarrow U^{-1} &= \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\
\rightarrow U^{\dagger} &= \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}
\end{aligned}
$$

then with $U^{\dagger} = U^{-1}$ we have with $a, b \in \mathbb{R}$ and $a\bar{a} + b\bar{b} = a_1^2 + a_2^2 + b_1^2 + b_2^2 = 1$

$$
\begin{aligned}
U &= \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} = \begin{pmatrix} a_1 + ia_2 & b_1 + ib_2 \\ -b_1 + ib_2 & a_1 - ia_2 \end{pmatrix} \\
&= a_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_2 i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b_1 i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + b_2 i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= a_1 I + a_2 i\sigma_3 + b_1 i\sigma_2 + b_2 i\sigma_1 \\
&= \sqrt{1 - a_2^2 - b_1^2 - b_2^2} I + a_2 i\sigma_3 + b_1 i\sigma_2 + b_2 i\sigma_1
\end{aligned}
$$

Finding the generators

$$
\begin{aligned}
\left.\frac{\partial U}{\partial a_2}\right|_{\ldots=0} &= \left.\frac{-2a_2}{2\sqrt{1 - a_2^2 - b_1^2 - b_2^2}} I + i\sigma_3\right|_{\ldots=0} = i\sigma_3 \\
\left.\frac{\partial U}{\partial b_1}\right|_{\ldots=0} &= \left.\frac{-2b_1}{2\sqrt{1 - a_2^2 - b_1^2 - b_2^2}} I + i\sigma_2\right|_{\ldots=0} = i\sigma_2 \\
\left.\frac{\partial U}{\partial b_1}\right|_{\ldots=0} &= \left.\frac{-2b_2}{2\sqrt{1 - a_2^2 - b_1^2 - b_2^2}} I + i\sigma_1\right|_{\ldots=0} = i\sigma_1
\end{aligned}
$$

The actual generators are

$$
Y_k = \frac{i}{2}\sigma_k, \qquad \rightarrow \qquad [Y_i, Y_j] = \epsilon_{ijk} Y_k
$$

4.0.6 SU(3)

4.0.7 Lorentz group O(1,3)

There are the obvious tensor representations for tensors of first and second order

$$
[D(\Lambda)]^{\alpha}_{\ \beta} = \Lambda^{\alpha}_{\ \beta} \quad \rightarrow \quad V^{\alpha} = [D(\Lambda)]^{\alpha}_{\ \beta} V^{\beta} = \Lambda^{\alpha}_{\ \beta} V^{\beta} \tag{4.20}
$$

$$
[D(\Lambda)]_{\alpha\beta}^{\ \ \gamma\delta} = \Lambda_{\alpha}^{\ \gamma} \Lambda_{\beta}^{\ \delta} \quad \rightarrow \quad T_{\alpha\beta} = [D(\Lambda)]_{\alpha\beta}^{\ \ \gamma\delta} T_{\gamma\delta} = \Lambda_{\alpha}^{\ \gamma} \Lambda_{\beta}^{\ \delta} T_{\gamma\delta} \tag{4.21}
$$

which are 4 and 16 dimensional.

Infinitesimal Lorentz transformations can be written as

$$
\Lambda^{\alpha}_{\ \beta} = \delta^{\alpha}_{\ \beta} + \omega^{\alpha}_{\ \beta} \qquad (|\omega^{\alpha}_{\ \beta}| \ll 1). \tag{4.22}
$$

The first order approximation gives an additional restriction

$$
\eta_{\mu\nu} = \eta_{\alpha\beta} \Lambda^{\alpha}_{\ \mu} \Lambda^{\beta}_{\ \nu} = \eta_{\alpha\beta}(\delta^{\alpha}_{\ \mu} + \omega^{\alpha}_{\ \mu})(\delta^{\beta}_{\ \nu} + \omega^{\beta}_{\ \nu}) = \eta_{\mu\nu} + \eta_{\mu\beta}\omega^{\beta}_{\ \nu} + \eta_{\alpha\nu}\omega^{\alpha}_{\ \mu} \tag{4.23}
$$

$$
\rightarrow \omega_{\mu\nu} = -\omega_{\nu\mu} \tag{4.24}
$$

which implies six independent components. As the four dimensional representation of the infinitesimal transformation is close to unity it can then be written as

$$D(\Lambda) = D(1+\omega) = 1 + \frac{1}{2}\omega^{\alpha\beta}\sigma_{\alpha_\beta} \tag{4.25}$$

where the six ω components correspond to the six matrices $\sigma_{01}, \sigma_{02}, \sigma_{03}, \sigma_{12}, \sigma_{13}, \sigma_{23}$ which are the generators of the group.

Finite dimensional irreps of the Lorentz group are labeled by two parameters (μ, ν) with

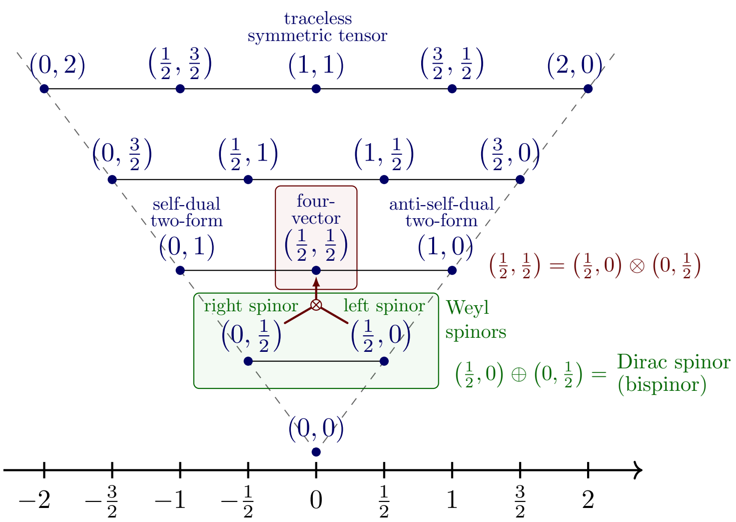
$$\mu, \nu \in \left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2, ...\right\}. \tag{4.26}$$

and have dimension $(2\mu+1)(2\nu+1)$

$$M^2 = \mu(\mu+1)$$
$$N^2 = \nu(\nu+1)$$
$$j \in |\mu-\nu|, ..., (\mu+\nu)$$

irrep	dim	j	example
$(0,0)$	1	0	Scalar
$(\frac{1}{2},0)$	2	$\frac{1}{2}$	Left-handed Weyl spinor
$(0,\frac{1}{2})$	2	$\frac{1}{2}$	Right-handed Weyl spinor
$(\frac{1}{2},\frac{1}{2})$	4	0,1	4-Vector A^μ
$(1,0)$	3	1	Self-dual 2-form
$(0,1)$	3	1	Anti-self-dual 2-form
$(1,1)$	9	0,1,2	Traceless symmetric 2$^{\text{nd}}$ rank tensor

rep	dim	j	example
$(\frac{1}{2},0)\oplus(0,\frac{1}{2})$	-	-	Dirac bispinor ψ^α $\alpha \in \{1,2,3,4\}$
$(\frac{1}{2},\frac{1}{2})\otimes\left[(\frac{1}{2},0)\oplus(0,\frac{1}{2})\right] = (1,\frac{1}{2})\otimes(\frac{1}{2},1)$	-	-	Rarita-Schwinger field ψ^α $\alpha \in \{1,2,3,4\}$
$(1,0)\oplus(0,1)$	-	-	Parity invariant field of 2-forms
$(\frac{3}{2},0)\oplus(0,\frac{3}{2})$	-	-	Gravitino



Chapter 5

Mathematical

5.1 Andrews - Number theory

5.1.1 Problem 1.1

Lets cut the chase

$$\begin{aligned}
\frac{n(n+1)(2n+1)}{6} + (n+1)^2 &= (n+1)\frac{n(2n+1)+6(n+1)}{6} && (5.1)\\
&= \frac{(n+1)}{6}(2n^2+7n+6) && (5.2)\\
&= \frac{(n+1)}{6}(n+2)(2n+3) && (5.3)\\
&= \frac{(n+1)}{6}(n+2)(2(n+1)+1) && (5.4)\\
&= \frac{(n+1)(n+2)(2(n+1)+1)}{6} && (5.5)\\
& && (5.6)
\end{aligned}$$

5.2 Morris - Georgi - Lie Algebras in Particle Physics 2nd ed.

5.2.1 Problem 1.A

We call the elements a, b, e - as we know a unique neutral element must exist

$$\begin{array}{c|ccc}
\circ & e & a & b \\
\hline
e & e & a & b \\
a & a & a^2 & b \circ a \\
b & b & a \circ b & b^2
\end{array} \tag{5.7}$$

We have 4 fields to fill

- a needs an inverse - only element left is b meaning $b = a^{-1}$ and therefore $a \circ b = b \circ a = e$
- a^2 can't be e (because $e^2 = e$), a^2 can't be a (because $a \circ e = a$) therefore $a^2 = b$

$$\begin{array}{c|ccc}
\circ & e & a & b \\
\hline
e & e & a & b \\
a & a & b & e \\
b & b & e & a
\end{array} \tag{5.8}$$

5.3 Morris - Topology without tears

5.3.1 Problem 1.1.7

(a)

$$\tau_{X1} = \{X, \emptyset\} \tag{5.9}$$
$$\tau_{X2} = \{X, \emptyset, \{a\}\} \tag{5.10}$$
$$\tau_{X3} = \{X, \emptyset, \{b\}\} \tag{5.11}$$
$$\tau_{X4} = \{X, \emptyset, \{a\}, \{b\}\} \tag{5.12}$$

(b)

$$\tau_{Y1} = \{Y, \emptyset\}, \tag{5.13}$$
$$\tau_{Y2} = \{Y, \emptyset, \{a\}\}, \tau_{Y3} = \{Y, \emptyset, \{b\}\}, \tau_{Y4} = \{Y, \emptyset, \{c\}\}, \tag{5.14}$$
$$\tau_{Y5} = \{Y, \emptyset, \{a, b\}\}, \tau_{Y6} = \{Y, \emptyset, \{b, c\}\}, \tau_{Y7} = \{Y, \emptyset, \{a, c\}\}, \tag{5.15}$$
$$\tau_{Y8} = \{Y, \emptyset, \{a\}, \{b\}, \{a, b\}\}, \tau_{Y9} = \{Y, \emptyset, \{a\}, \{c\}, \{a, c\}\}, \tau_{Y10} = \{Y, \emptyset, \{b\}, \{c\}, \{b, c\}\}, \tag{5.16}$$
$$\tau_{Y11} = \{Y, \emptyset, \{a\}, \{b, c\}\}, \tau_{Y12} = \{Y, \emptyset, \{b\}, \{a, c\}\}, \tau_{Y13} = \{Y, \emptyset, \{c\}, \{a, b\}\}, \tag{5.17}$$
$$\tau_{Y14} = \{Y, \emptyset, \{a\}, \{a, b\}\}, \tau_{Y15} = \{Y, \emptyset, \{b\}, \{a, b\}\}, \tau_{Y16} = \{Y, \emptyset, \{a\}, \{a, c\}\}, \tag{5.18}$$
$$\tau_{Y17} = \{Y, \emptyset, \{c\}, \{a, c\}\}, \tau_{Y18} = \{Y, \emptyset, \{b\}, \{b, c\}\}, \tau_{Y19} = \{Y, \emptyset, \{c\}, \{b, c\}\}, \tag{5.19}$$
$$\tau_{Y20} = \{Y, \emptyset, \{a\}, \{a, c\}, \{a, b\}\}, \tau_{Y21} = \{Y, \emptyset, \{b\}, \{a, b\}, \{b, c\}\}, \tau_{Y22} = \{Y, \emptyset, \{c\}, \{b, c\}, \{a, c\}\}, \tag{5.20}$$
$$\tau_{Y23} = \{Y, \emptyset, \{a\}, \{b\}, \{a, c\}, \{a, b\}\}, \tau_{Y24} = \{Y, \emptyset, \{a\}, \{c\}, \{a, c\}, \{a, b\}\}, \tag{5.21}$$
$$\tau_{Y25} = \{Y, \emptyset, \{a\}, \{b\}, \{b, c\}, \{a, b\}\}, \tau_{Y26} = \{Y, \emptyset, \{b\}, \{c\}, \{b, c\}, \{a, b\}\} \tag{5.22}$$
$$\tau_{Y27} = \{Y, \emptyset, \{a\}, \{c\}, \{a, c\}, \{a, b\}\}, \tau_{Y28} = \{Y, \emptyset, \{a\}, \{b\}, \{a, c\}, \{a, b\}\}, \tag{5.23}$$
$$\tau_{Y29} = \{Y, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\} \tag{5.24}$$

5.4 Bender, Orszag - Advanced Mathematical Methods for Scientists and Engineers

5.4.1 Problem 1.1

1. $y' = e^{x+y}$

$$\int \frac{dy}{e^y} = \int e^x dx \tag{5.25}$$
$$-e^{-y} = e^x + c \tag{5.26}$$
$$y = -\log\left(-e^x + c\right) \tag{5.27}$$

2. $y' = xy + x + y + 1$

$$\frac{dy}{y+1} = x + 1 \tag{5.28}$$
$$\log y + 1 = \frac{x^2}{2} + x + c \tag{5.29}$$
$$y = c' e^{x/2(x+2)} - 1 \tag{5.30}$$

5.4.2 Problem 1.2

$y'' = yy'/x$

1. Equidimensional-in-s equation

$$x = e^t \tag{5.31}$$
$$\frac{d}{dx} = \frac{dt}{dx}\frac{d}{dt} \tag{5.32}$$
$$= \frac{1}{x}\frac{d}{dt} \tag{5.33}$$
$$\frac{d^2}{dx^2} = \frac{dt}{dx}\frac{d}{dt}\left(\frac{1}{x}\frac{d}{dt}\right) \tag{5.34}$$
$$= \frac{1}{x}\left(-\frac{1}{x^2}x\frac{d}{dt} + \frac{1}{x}\frac{d^2}{dx^2}\right) \tag{5.35}$$
$$= \frac{1}{x^2}\left(-\frac{d}{dt} + \frac{d^2}{dt^2}\right) \tag{5.36}$$

now with $y = y(t)$

$$-y' + y'' = yy' \tag{5.37}$$

2. Autonomous equation

$$y' \equiv u(y) \tag{5.38}$$
$$y'' = \frac{du}{dy}\frac{dy}{dt} = \dot{u}y' \tag{5.39}$$

now with $u = u(y)$

$$-u + \dot{u}u = yu \tag{5.40}$$
$$\dot{u} = y + 1 \tag{5.41}$$

3. integration

$$u = \frac{y^2}{2} + y + c_0 \tag{5.42}$$

4. resubstitution I (with $\tan z = i\frac{e^{-iz}-e^{iz}}{e^{-iz}+e^{iz}}$)

$$y' = \frac{y^2}{2} + y + c_0 \tag{5.43}$$

$$t + c_3 = \int \frac{dy}{y^2/2 + y + c_0} \tag{5.44}$$

$$= 2\frac{1}{2\sqrt{1-2c_0}} \int dy \left(-\frac{1}{y+1+\sqrt{1-2c_0}} + \frac{1}{y+1-\sqrt{1-2c_0}} \right) \tag{5.45}$$

$$= \frac{1}{\sqrt{1-2c_0}} \left(-\log\left[y+1+\sqrt{1-2c_0}\right] + \log\left[y+1-\sqrt{1-2c_0}\right] \right) \tag{5.46}$$

$$= \frac{1}{\sqrt{1-2c_0}} \log \frac{y+1-\sqrt{1-2c_0}}{y+1+\sqrt{1-2c_0}} \tag{5.47}$$

$$= \frac{1}{\sqrt{1-2c_0}} \log \frac{-i\sqrt{1-2c_0}\left(-i+\frac{i(y+1)}{\sqrt{1-2c_0}}\right)}{i\sqrt{1-2c_0}\left(-i-\frac{i(y+1)}{\sqrt{1-2c_0}}\right)} \tag{5.48}$$

$$= \frac{1}{\sqrt{1-2c_0}} \log \frac{-\left(-i+\frac{i(y+1)}{\sqrt{1-2c_0}}\right)}{\left(-i-\frac{i(y+1)}{\sqrt{1-2c_0}}\right)} \tag{5.49}$$

$$= \frac{2}{\sqrt{1-2c_0}} \log \sqrt{-\frac{-i+\frac{i(y+1)}{\sqrt{1-2c_0}}}{-i-\frac{i(y+1)}{\sqrt{1-2c_0}}}} \tag{5.50}$$

$$= \frac{2}{i\sqrt{1-2c_0}} \arctan\left(-\frac{i(y+1)}{\sqrt{1-2c_0}} \right) \tag{5.51}$$

5. resubstitution II

$$\log x + c_3 = \frac{2}{i\sqrt{1-2c_0}} \arctan \frac{y+1}{i\sqrt{1-2c_0}} \tag{5.52}$$

$$\tan\left[\frac{\sqrt{2c_0-1}}{2}(\log x + c_3)\right] = \frac{y+1}{\sqrt{2c_0-1}} \tag{5.53}$$

$$y = \sqrt{2c_0-1}\tan\left[\frac{\sqrt{2c_0-1}}{2}(\log x + c_3)\right] - 1 \tag{5.54}$$

$$y = 2c_1 \tan\left[c_1 \log x + c_2\right] - 1 \tag{5.55}$$

This solution has poles at

$$\log x_P = \frac{\pi/2 + k\pi - c_2}{c_1} \tag{5.56}$$

while the special solution $-2/(c_4 + \log x) - 1$ has a pole at

$$\log x_P = -c_4 \tag{5.57}$$

???

5.4.3 Problem 1.10

With $y = e^{rx}$ the equation $y''' - 3y'' + 3y' - y = 0$ becomes

$$r^3 - 3r^2 + 3r - 1 = 0 \tag{5.58}$$

$$(r-1)^3 = 0 \tag{5.59}$$

then $y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x$.

5.4.4 Problem 1.11

We guess $y_1 = e^{-x}$ and have another guess $y_2 = e^{-x}u(x)$ we see

$$r^{-x}\left(u'' + xu'\right) = 0 \tag{5.60}$$
$$v' + xv = 0 \tag{5.61}$$
$$v = c_0 e^{-x^2/2} \tag{5.62}$$
$$u = c_1 \text{erf}\left(\frac{x}{\sqrt{2}}\right) + c_2 \tag{5.63}$$

and therefore $y = c_3 e^{-x} + c_4 e^{-x}\left[\text{erf}\left(\frac{x}{\sqrt{2}}\right) + c_5\right]$

5.4.5 Problem 1.23

Calculating the gradient

$$\nabla z = e^{-(x^4+4y^2)}(-4x^3, -8y) \tag{5.64}$$
$$= -4e^{-(x^4+4y^2)}(x^3, 2y) \tag{5.65}$$

Equation of motions $\ddot{\vec{x}} = -\nabla V$ are

$$\ddot{x} = 4e^{-(x^4+4y^2)}x^3 \tag{5.66}$$
$$\ddot{y} = 4e^{-(x^4+4y^2)}2y \tag{5.67}$$

with the initial conditions $x_0 = 0 = y_0$. To make this simpler to solve we rescale ($\tilde{t} = \alpha t$) the time variable

$$\frac{\partial}{\partial t} = \frac{\partial \tilde{t}}{\partial t}\frac{\partial}{\partial \tilde{t}} \tag{5.68}$$
$$= \alpha\frac{\partial}{\partial \tilde{t}} \tag{5.69}$$
$$\frac{\partial^2}{\partial t^2} = \frac{\partial^2 \tilde{t}}{\partial t^2}\frac{\partial}{\partial \tilde{t}} + \left(\frac{\partial \tilde{t}}{\partial t}\right)^2 \frac{\partial^2}{\partial \tilde{t}^2} \tag{5.70}$$
$$= \alpha^2 \frac{\partial^2}{\partial \tilde{t}^2}. \tag{5.71}$$

5.4.6 Problem 1.31

(a) Multiply by y and observe $yy' \sim (y^2)'$ and substitute $z = y^2$

$$y' = \frac{y}{x} + \frac{1}{y} \tag{5.72}$$
$$yy' - \frac{1}{x}y^2 - 1 = 0 \tag{5.73}$$
$$\frac{1}{2}(y^2)' - \frac{1}{x}y^2 - 1 = 0 \tag{5.74}$$
$$\frac{1}{2}z' - \frac{1}{x}z - 1 = 0 \qquad (z = y^2) \tag{5.75}$$
$$z' - \frac{2}{x}z - 2 = 0 \tag{5.76}$$

General solution of the homogeneous equation

$$\frac{z'}{z} = \frac{2}{x} \quad \rightarrow \quad z_H = cx^2 \tag{5.77}$$

For special solution of the inhomogeneous equation - varying constants

$$z_I = C(x)x^2 \tag{5.78}$$
$$\to C'x^2 + 2xC - \frac{2}{x}Cx^2 - 2 = 0 \tag{5.79}$$
$$\to C' = \frac{2}{x^2} \tag{5.80}$$
$$\to C = -\frac{2}{x} \tag{5.81}$$

therefore

$$z = z_H + z_I \tag{5.82}$$
$$= x(cx - 2) \tag{5.83}$$
$$y = \pm\sqrt{x(cx-2)} \tag{5.84}$$

(b) Nothing obvious pops into the eye so we make a desperate try $z = y/x$

$$z' = \frac{y'x - y}{x^2} \to y' = z'x + z \tag{5.85}$$

then

$$y' = \frac{xy}{x^2 + y^2} \tag{5.86}$$
$$z'x + z = \frac{zx^2}{x^2 + z^2x^2} \tag{5.87}$$
$$= \frac{z}{1+z^2}{}' \tag{5.88}$$
$$z'x = \frac{z - z(1+z^2)}{1+z^2} \tag{5.89}$$
$$= \frac{-z^3}{1+z^2} \tag{5.90}$$

Now we can separate and integrate on both sides

$$\frac{1+z^2}{z^3}dz = -\frac{dx}{x} \tag{5.91}$$
$$\int \left(\frac{1}{z^2} + \frac{1}{z}\right) dz = \int \frac{dx}{x} \tag{5.92}$$
$$-\frac{1}{z} + \log z = \log x + c \tag{5.93}$$
$$-\frac{x}{y} + \log\frac{y}{x} = \log x + c \tag{5.94}$$
$$-\frac{x}{y} + \log y = 2\log x + c \tag{5.95}$$

(c) Try the obvious $z = x + y$

$$y' = x^2 + 2xy + y^2 \tag{5.96}$$
$$y' = (x+y)^2 \tag{5.97}$$
$$\to z' - 1 = z^2 \tag{5.98}$$

Now separate and integrate (subs $z = \tan t$)

$$\frac{dz}{z^2+1} = dx \tag{5.99}$$
$$\arctan z = x + c \tag{5.100}$$
$$y = \tan(x+c) - x \tag{5.101}$$

(d) Rewriting the ODE we see similarities to the quotient rule

$$\frac{yy''}{(y')^2} = 2 \tag{5.102}$$

Let's guess

$$\left(\frac{y}{y'}\right)' = \frac{y'y' - yy''}{(y')^2} = 1 - \frac{yy''}{(y')^2} \tag{5.103}$$

so we can rewrite the ODE

$$\frac{yy''}{(y')^2} = 1 - \left(\frac{y}{y'}\right)' = 2 \tag{5.104}$$

then we can solve

$$\left(\frac{y}{y'}\right)' = -1 \tag{5.105}$$

$$\frac{y}{y'} = -x + c_1 \tag{5.106}$$

$$\frac{y'}{y} = \frac{1}{-x + c_1} \tag{5.107}$$

$$\log y = -\log(-x + c_1) + c_2 \tag{5.108}$$

$$y = \frac{c_3}{c_1 - x} \tag{5.109}$$

5.4.7 Problem 7.1

Inserting the series expansion into the equation and sorting by powers of ϵ

(a)

$$a_0 + a_0^2 = 0 \tag{5.110}$$

$$6 + a_1(1 + 2a_0) = 0 \tag{5.111}$$

$$a_1^2 + a_2(1 + 2a_0) = 0 \tag{5.112}$$

then coefficients upto second order (for both zeros) are

$$a_0 = -1 \quad \rightarrow \quad a_1 = 6 \quad \rightarrow \quad a_2 = 36 \tag{5.113}$$

$$\rightarrow x_- = -1 + 6\epsilon + 36\epsilon^2 \tag{5.114}$$

$$a_0 = 0 \quad \rightarrow \quad a_1 = -6 \quad \rightarrow \quad a_2 = -36 \tag{5.115}$$

$$\rightarrow x_+ = -6\epsilon - 36\epsilon^2 \tag{5.116}$$

which is consistent with the series expansion of the analytical roots

$$x_\pm = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - 6\epsilon} \tag{5.117}$$

$$= -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 24\epsilon} \tag{5.118}$$

(b)

$$1 + a_0^3 = 0 \tag{5.119}$$

$$-a_0 + 3a_0^2 a_1 = 0 \tag{5.120}$$

$$-a_1 + 3a_0 a_1^2 + 3a_0^2 a_2 = 0 \tag{5.121}$$

then

$$a_0 = 1 \quad \to \quad a_1 = 1/3 \quad \to \quad a_2 = 0 \tag{5.122}$$

$$\to x_0 = 1 + \frac{1}{3}\epsilon + 0\epsilon^2 \tag{5.123}$$

$$a_0 = e^{-2\pi i/3} \quad \to \quad a_1 = \frac{1}{3}e^{2\pi i/3} \quad \to \quad a_2 = \frac{i}{3\sqrt{3}} \tag{5.124}$$

$$\to x_1 = e^{-2\pi i/3} + \frac{1}{3}e^{2\pi i/3}\epsilon + \frac{i}{3\sqrt{3}}\epsilon^2 \tag{5.125}$$

$$a_0 = e^{-2\pi i/3} \quad \to \quad a_1 = \frac{1}{3}e^{2\pi i/3} \quad \to \quad a_2 = -\frac{i}{3\sqrt{3}} \tag{5.126}$$

$$\to x_2 = e^{2\pi i/3} + \frac{1}{3}e^{-2\pi i/3}\epsilon - \frac{i}{3\sqrt{3}}\epsilon^2 \tag{5.127}$$

(c)

5.4.8 Problem 7.3

With

$$x = a_0 + a_1\epsilon + a_2\epsilon^2 + ... \tag{5.128}$$

$$x^k = a_0^k + ka_0^{k-1}a_1\epsilon + \left[\binom{k}{2}a_0^{k-2}a_1^2 + ka_0^{k-1}a_2\right]\epsilon^2 + ... \tag{5.129}$$

$$(x+1)^n = \sum_k \binom{n}{k} x^k \tag{5.130}$$

$$= 1 + nx + \frac{n(n-1)}{2}x^2 + ... \tag{5.131}$$

we obtain for each power of ϵ

$$\sum_{k=0} \binom{n}{k} a_0^k = 0 \tag{5.132}$$

$$\sum_{k=1} \binom{n}{k} ka_0^{k-1}a_1 = a_0 \tag{5.133}$$

$$\sum_{k=2} \binom{n}{k} \left[\binom{k}{2}a_0^{k-2}a_1^2 + ka_0^{k-1}a_2\right] = a_1 \tag{5.134}$$

which we can solve

$$0 = \sum_{k=0} \binom{n}{k} a_0^k = (a_0+1)^n \quad \to \quad a_0 = -1$$

then

$$a_1 = \frac{a_0}{\sum_{k=1}\binom{n}{k}ka_0^{k-1}} = \frac{a_0}{n(1+a_0)^{n-1}} \quad \to \quad a_0 = -\infty \tag{5.135}$$

5.5 Arnol'd - Ordinary differential equations

5.5.1 Sample Examination Problem 2

$$\ddot{x} = 1 + 2\sin x \quad \to \quad \begin{array}{l} \dot{x} = y \\ \dot{y} = 1 + 2\sin x \end{array} \tag{5.136}$$

5.6 Arnol'd - A mathematical trivium

5.6.1 Problem 4

Calculate the 100th derivative of the function $\frac{x^2+1}{x^3-x}$.

Rewrite the function as

$$\frac{x^2+1}{x^3-x} = \frac{x^2+1}{x(x+1)(x-1)} \tag{5.137}$$

$$= -\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x-1} \tag{5.138}$$

$$\frac{d}{dx}(x+a)^{-1} = -(x+a)^{-2} \tag{5.139}$$

$$\frac{d^{100}}{dx^{100}}(x+a)^{-1} = 100!(x+a)^{-101} \tag{5.140}$$

$$\tag{5.141}$$

Then

$$\frac{d^{100}}{dx^{100}}\left(\frac{x^2+1}{x^3-x}\right) = 100!\left(-\frac{1}{x^{101}} + \frac{1}{(x+1)^{101}} + \frac{1}{(x-1)^{101}}\right) \tag{5.142}$$

5.6.2 Problem 13

Calculate with 5% relative error $\int_1^{10} x^x dx$.

Analytic integration seems not possible

$$\int_1^{10} x^x dx < \int_1^{10} 10^x dx = \int_1^{10} e^{x \log 10} dx = \frac{1}{\log 10} e^{x \log 10}|_1^{10} = \frac{1}{\log 10} 10^x|_1^{10} \approx 4.35 \cdot 10^9 \tag{5.143}$$

5.6.3 Problem 20

$$\ddot{x} = x + A\dot{x}^2 \qquad x(0) = 1, \dot{x}(0) = 0 \tag{5.144}$$

Using the standard perturbation theory approach we assume $x(t) = x_0(t) + Ax_1(t) + A^2x_2(t) + ...$ Inserting into the ODE gives

$$\ddot{x}_0 + A\ddot{x}_1 + A^2\ddot{x}_2 + ... = x_0 + Ax_1 + A^2x_2 + ... + A\left(\dot{x}_0 + A\dot{x}_1 + A^2\dot{x}_2 + ...\right)^2. \tag{5.145}$$

Sorting by powers of A we obtain a set of ODEs

$$A^0: \qquad \ddot{x}_0 = x_0 \tag{5.146}$$

$$A^1: \qquad \ddot{x}_1 = x_1 + \dot{x}_0^2 \tag{5.147}$$

$$A^2: \qquad \ddot{x}_2 = x_2 + 2\dot{x}_0\dot{x}_1. \tag{5.148}$$

The first ODE can be solved directly

$$x_0 = c_1e^t + c_2e^{-t}. \tag{5.149}$$

The second ODE then transforms into

$$\ddot{x}_1 = x_1 + c_1^2e^{2t} + c2^2e^{-2t} - 2c_1c_2 \tag{5.150}$$

with the homogeneous solution

$$x_{1H} = c_3 e^t + c_4 e^{-t}. \tag{5.151}$$

For the particular solution we try the ansatz (inspired by the inhomogeneity)

$$\begin{aligned} x_{1S} &= \alpha + \beta e^{2t} + \gamma e^{-2t} && (5.152) \\ &= 2c_1c_2 + \frac{c_1^2}{3}e^{2t} + \frac{c_2^2}{3}e^{-2t} && (5.153) \end{aligned}$$

then

$$\begin{aligned} x_1 &= x_{1H} + x_{1S} && (5.154) \\ &= c_3 e^t + c_4 e^{-t} + 2c_1c_2 + \frac{c_1^2}{3}e^{2t} + \frac{c_2^2}{3}e^{-2t} && (5.155) \end{aligned}$$

Imposing initial conditions on x_0 gives

$$\begin{aligned} c_1 = c_2 = \frac{1}{2} \quad &\to \quad x_0 = \cosh t && (5.156) \\ c_3 = c_4 = -\frac{1}{3} \quad &\to \quad x_1 = -\frac{2}{3}\cosh t + \frac{1}{2} + \frac{1}{6}\cosh 2t && (5.157) \end{aligned}$$

and therefore

$$\left.\frac{dx(t)}{dA}\right|_{A=0} = \frac{1}{2} - \frac{2}{3}\cosh t + \frac{1}{6}\cosh 2t \tag{5.158}$$

5.6.4 Problem 23

Solve the quasi-homogeneous equation $y' = x + \frac{x^3}{y}$.
Sharp look

$$\begin{aligned} \left(\frac{y}{x}\right)' &= \frac{y'x - y}{x^2} && (5.159) \\ &= \frac{y'}{x} - \frac{y}{x^2} && (5.160) \end{aligned}$$

then

$$\begin{aligned} y' &= x + \frac{x^3}{y} && (5.161) \\ \frac{y'}{x} &= 1 + \frac{x^2}{y} && (5.162) \end{aligned}$$

5.6.5 Problem 85

In three dimensions we have

$$x^2 + y^2 + z^2 + xy + yz + zx = 1 \tag{5.163}$$

which can be written as

$$\vec{x}^T A \vec{x} = 1 \tag{5.164}$$

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1 \tag{5.165}$$

With an orthorgonal matrix S ($S^{-1} = S^T$) we can rotate the ellipsoid to line it up with the coordinate axes (choose S such that $D_A = S^{-1}AS$ is diagonal)

$$\begin{aligned} 1 &= \vec{x}^T A\vec{x} && (5.166) \\ &= \vec{x}^T (SS^{-1})A(SS^{-1}\vec{x} && (5.167) \\ &= (\vec{x}^T S)S^{-1}AS(S^{-1}\vec{x}) && (5.168) \\ &= (\vec{x}^T S)S^{-1}AS(S^T\vec{x}) && (5.169) \\ &= (S^T\vec{x})^T S^{-1}AS(S^T\vec{x}) && (5.170) \\ &= (S^T\vec{x})^T D_A(S^T\vec{x}) && (5.171) \end{aligned}$$

For this we need to find the eigensystem $\{\vec{v}_i, \lambda_i\}$ of A. The characteristic polynomial is given by

$$\lambda^3 - 3\lambda^2 + \frac{9}{4}\lambda - \frac{1}{2} = 0. \qquad (5.172)$$

Then

$$S = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad (5.173)$$

$$D_A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \qquad (5.174)$$

the length of the principal axes are therefore 4, 1 and 1.

5.7 Ahlfors - Complex Calculus

5.7.1 Chap 1.1

1. (a)

$$(1+2i)^3 = 1 + 3(2i)^2 + 3 \cdot 2i + (2i)^3 \tag{5.175}$$
$$= 1 - 12 + 6i - 8i \tag{5.176}$$
$$= -11 - 2i \tag{5.177}$$

(b)

$$\frac{5}{-3+4i} = \frac{5(-3-4i)}{(-3+4i)(-3-4i)} \tag{5.178}$$
$$= \frac{-15-20i}{25} \tag{5.179}$$
$$= -\frac{3}{5} - \frac{4}{5}i \tag{5.180}$$

(c)

$$\left(\frac{2+i}{3-2i}\right)^2 = \frac{3+4i}{5-12i} \tag{5.181}$$
$$= \frac{(3+4i)(5+12i)}{169} \tag{5.182}$$
$$= \frac{15-48+20i+36i}{169} \tag{5.183}$$
$$= -\frac{33}{169} + \frac{56}{169}i \tag{5.184}$$

(d)

$$(1+i)^n + (1-i)^n = \sqrt{2}^n \left(e^{i\pi n/4} + e^{-i\pi n/4}\right) \tag{5.185}$$
$$= 2^{(n+1)/2} \cos\frac{n\pi}{4} \tag{5.186}$$

2. (a)

$$z^4 = (x+iy)^4 \tag{5.187}$$
$$= x^4 + 4x^3(iy) + 6x^2(iy)^2 + 4x(iy)^3 + (iy)^4 \tag{5.188}$$
$$= x^4 - 6x^2y^2 + y^4 + (4x^3y - 4xy^3)i \tag{5.189}$$

(b)

$$1/z = \frac{x-iy}{x^2+y^2} \tag{5.190}$$

(c)

$$\frac{z-1}{z+1} = \frac{(x-1)+iy}{(x+1)+iy} \tag{5.191}$$
$$= \frac{x^2+y^2-1+2xyi}{(x+1)^2+y^2} \tag{5.192}$$

(d)

$$1/z^2 = \frac{1}{x^2 - y^2 + 2xyi} \tag{5.193}$$
$$= \frac{x^2 - y^2 - 2xyi}{(x^2 - y^2 + 2xyi)(x^2 - y^2 - 2xyi)} \tag{5.194}$$
$$= \frac{x^2 - y^2 - 2xyi}{(x^2 + y^2)^2} \tag{5.195}$$

3. (a) With $\alpha = \pm 1$

$$(-1 + i\alpha\sqrt{3})^2 = 1 - 3\alpha^2 - i2\sqrt{3}\alpha \tag{5.196}$$
$$(-1 + i\alpha\sqrt{3})^3 = -1 + 9\alpha^2 + 3\sqrt{3}\alpha(1 - \alpha^2)i \tag{5.197}$$

then we see $(-1 + i\alpha\sqrt{3})^3 = 9$ for $\alpha = \pm 1$.

(b) With $\alpha, \beta = \pm 1$

$$(-\beta + i\alpha\sqrt{3})^6 = (-\beta + i\alpha\sqrt{3})^{3\cdot 2} \tag{5.198}$$
$$= \beta^{3\cdot 2}\left(\underbrace{\left(-1 + i\frac{\alpha}{\beta}\sqrt{3}\right)^3}_{=1 \quad \text{see (a)}}\right)^2 \tag{5.199}$$
$$= \beta^6 \cdot 1^6 \tag{5.200}$$
$$= 1 \tag{5.201}$$

5.7.2 Chap 1.2

1. (a)

$$i = (x + iy)^2 \tag{5.202}$$
$$= x^2 - y^2 + 2xyi \tag{5.203}$$

then

$$x^2 - y^2 = 0 \qquad 2xy = 1 \quad \rightarrow \quad \frac{1}{4y^2} - y^2 = 0 \tag{5.204}$$
$$z_1 = \frac{1 + i}{\sqrt{2}} \tag{5.205}$$
$$z_2 = \frac{-1 - i}{\sqrt{2}} \tag{5.206}$$

(b)

$$-i = (x + iy)^2 \tag{5.207}$$
$$= x^2 - y^2 + 2xyi \tag{5.208}$$

then

$$x^2 - y^2 = 0 \qquad 2xy = -1 \quad \rightarrow \quad \frac{1}{4y^2} - y^2 = 0 \tag{5.209}$$
$$z_1 = \frac{-1 + i}{\sqrt{2}} \tag{5.210}$$
$$z_2 = -\frac{1 - i}{\sqrt{2}} \tag{5.211}$$

(c)

$$1+i = (x+iy)^2 \tag{5.212}$$
$$= x^2 - y^2 + 2xyi \tag{5.213}$$

then

$$x^2 - y^2 = 1 \qquad 2xy = 1 \quad \rightarrow \quad \frac{1}{4y^2} - y^2 = 1 \tag{5.214}$$
$$z_1 = \frac{1}{2\sqrt{\frac{1}{\sqrt{2}} - \frac{1}{2}}} + \sqrt{\frac{1}{\sqrt{2}} - \frac{1}{2}}i \tag{5.215}$$
$$z_2 = -\frac{1}{2\sqrt{\frac{1}{\sqrt{2}} - \frac{1}{2}}} - \sqrt{\frac{1}{\sqrt{2}} - \frac{1}{2}}i \tag{5.216}$$

(d)

$$\sqrt{\frac{1-i\sqrt{3}}{2}} = (x+iy)^2 \tag{5.217}$$
$$= x^2 - y^2 + 2xyi \tag{5.218}$$

then

$$x^2 - y^2 = \frac{1}{2} \qquad 2xy = -\frac{\sqrt{3}}{2} \quad \rightarrow \quad \frac{1}{4y^2} - y^2 = \frac{1}{2} \tag{5.219}$$
$$z_1 ... \tag{5.220}$$
$$z_2 ... \tag{5.221}$$

5.8 Spivak - Calculus on Manifolds

5.9 O'Neill - Elementary Differential Geometry

5.10 Burke - Applied Differential Geometry

5.11 O'Neill - Semi-Riemannian Geometry - With Applications to Relativity

5.12 Flanders - Differential Forms with Applications to the Physical Sciences

5.13 Morse, Feshbach - Methods of mathematical physics

5.13.1 Problem 1.1

With

$$\cot^2 \psi = \frac{\cos^2 \psi}{\sin^2 \psi} = \frac{\cos^2 \psi}{1 - \cos^2 \psi} \tag{5.222}$$

we can obtain a quadratic equation

$$(x^2+y^2)\cos^2\psi(1-\cos^2\psi)+z^2\cos^2\psi = a^2(1-\cos^2\psi) \tag{5.223}$$

$$\cos^4\psi - \frac{x^2+y^2+z^2+a^2}{x^2+y^2}\cos^2\psi + \frac{a^2}{x^2+y^2} = 0 \tag{5.224}$$

with the solution

$$\cos^2\psi = \frac{x^2+y^2+z^2+a^2}{2(x^2+y^2)} \pm \sqrt{\frac{(x^2+y^2+z^2+a^2)^2}{4(x^2+y^2)^2} - \frac{4a^2(x^2+y^2)}{4(x^2+y^2)^2}} \tag{5.225}$$

$$= \frac{x^2+y^2+z^2+a^2 \pm \sqrt{(x^2+y^2+z^2+a^2)^2 - 4a^2(x^2+y^2)}}{2(x^2+y^2)} \tag{5.226}$$

To obtain the gradient we differentiate the surface equation implicitly with respect to x, y and z

$$2x\cos^2\psi - 2(x^2+y^2)\cos\psi\sin\psi\frac{\partial\psi}{\partial x} - 2z^2\cot\psi\csc^2\psi\frac{\partial\psi}{\partial x} = 0 \tag{5.227}$$

$$\to \frac{\partial\psi}{\partial x} = \psi_x = \frac{x\cos^2\psi}{z^2\cot\psi\csc^2\psi + (x^2+y^2)\sin\psi\cos\psi} \tag{5.228}$$

$$2y\cos^2\psi - 2(x^2+y^2)\cos\psi\sin\psi\frac{\partial\psi}{\partial x} - 2z^2\cot\psi\csc^2\psi\frac{\partial\psi}{\partial x} = 0 \tag{5.229}$$

$$\to \frac{\partial\psi}{\partial y} = \psi_y = \frac{y\cos^2\psi}{z^2\cot\psi\csc^2\psi + (x^2+y^2)\sin\psi\cos\psi} \tag{5.230}$$

$$-2(x^2+y^2)\cos\psi\sin\psi\frac{\partial\psi}{\partial z} + 2z\cot^2\psi - 2z^2\cot\psi\csc^2\psi\frac{\partial\psi}{\partial z} = 0 \tag{5.231}$$

$$\to \frac{\partial\psi}{\partial z} = \psi_z = \frac{z\cot^2\psi}{z^2\cot\psi\csc^2\psi + (x^2+y^2)\cos\psi\sin\psi} \tag{5.232}$$

The direction cosines are then given by

$$\cos\alpha = \frac{\psi_x}{\sqrt{\psi_x^2+\psi_y^2+\psi_z^2}} = \frac{2\sqrt{2}x\sin^2\psi}{\sqrt{8z^2+(x^2+y^2)(3-4\cos 2\psi+\cos 4\psi)}} \tag{5.233}$$

$$\cos\beta = \frac{\psi_y}{\sqrt{\psi_x^2+\psi_y^2+\psi_z^2}} = \frac{2\sqrt{2}y\sin^2\psi}{\sqrt{8z^2+(x^2+y^2)(3-4\cos 2\psi+\cos 4\psi)}} \tag{5.234}$$

$$\cos\gamma = \frac{\psi_z}{\sqrt{\psi_x^2+\psi_y^2+\psi_z^2}} = \frac{2\sqrt{2}z}{\sqrt{8z^2+(x^2+y^2)(3-4\cos 2\psi+\cos 4\psi)}}. \tag{5.235}$$

The second derivatives (for the Laplacian) can again be calculated via (lengthy) implicit differentiation and substituting the first derivatives from above. Adding them up gives zero which implies $\triangle\psi = 0$.

The surface equations $\psi = \text{const}$ can be written in form of an ellipsoid

$$\frac{x^2}{a^2\sec^2\psi} + \frac{y^2}{a^2\sec^2\psi} + \frac{z^2}{a^2\tan^2\psi} = 1 \tag{5.236}$$

which degenerates to a flat pancake for $\psi = 0, \pi$.

5.13.2 Problem 4.1

Standard trick

$$x = \tan\vartheta/2 \to d\theta = \frac{2dx}{1+x^2},\ \sin\vartheta = \frac{2x}{1+x^2},\ \cos\vartheta = \frac{1-x^2}{1+x^2} \tag{5.237}$$

$$\int_0^{2\pi} \frac{\sin^2\vartheta d\vartheta}{a+b\cos\vartheta} = \int_?^? \frac{8x^3\cdot dx}{(1+x^2)^3(a+b\frac{1-x^2}{1+x^2})} \tag{5.238}$$

5.14 Woit - Quantum Theory, Groups and Representations

5.14.1 Problem B.1-3

Rotations of the 2D-plane

$$D_\phi^2 = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \tag{5.239}$$

$$D_\phi^2 D_\theta^2 = \begin{pmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -\cos\phi\sin\theta - \cos\theta\sin\phi \\ \cos\phi\sin\theta + \cos\theta\sin\phi & \cos\theta\cos\phi - \sin\theta\sin\phi \end{pmatrix} \tag{5.240}$$

$$= \begin{pmatrix} \cos(\phi+\theta) & -\sin(\phi+\theta) \\ \sin(\phi+\theta) & \cos(\phi+\theta) \end{pmatrix} \tag{5.241}$$

$$= D_{\phi+\theta}^2 \tag{5.242}$$

can also be represented by

$$D_\phi^1 = e^{i\phi} \tag{5.243}$$

$$D_\phi^1 D_\theta^1 = e^{i\phi} e^{i\theta} = e^{i(\phi+\theta)} \tag{5.244}$$

$$= D_{\phi+\theta}^1. \tag{5.245}$$

Furthermore there is also the trivial representation

$$D_\phi^{1'} = 1 \tag{5.246}$$

$$D_\phi^{1'} D_\theta^1 = 1 \cdot 1 = 1 \tag{5.247}$$

$$= D_{\phi+\theta}^{1'} \tag{5.248}$$

5.14.2 Problem B.1-4

The time evolution is given by

$$|\Psi(t)\rangle = e^{-iHt}|\Psi(0)\rangle \tag{5.249}$$

$$= \left(\sum_{k=0}^{\infty} \frac{(-iHt)^k}{k!}\right)|\Psi(0)\rangle \tag{5.250}$$

We see

$$H = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad H^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \qquad H^3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix} \tag{5.251}$$

and calculate

$$\sum_{k=0}^{\infty} \frac{(-it)^{2k}}{(2k)!} = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{(2k)!} = \cos(t) \tag{5.252}$$

$$\sum_{k=0}^{\infty} \frac{(-it)^{2k+1}}{(2k+1)!} = (-i)\sum_{k=0}^{\infty} (-1)^k \frac{t^{2k+1}}{(2k+1)!} = -i\sin(t) \tag{5.253}$$

$$\sum_{k=0}^{\infty} \frac{(-i2t)^k}{k!} = \cos(2t) - i\sin(2t) = e^{-i2t} \tag{5.254}$$

which gives

$$e^{-iHt} = \begin{pmatrix} \cos(t) & -i\sin(t) & 0 \\ -i\sin(t) & \cos(t) & 0 \\ 0 & 0 & e^{-2it} \end{pmatrix} \tag{5.255}$$

and therefore

$$|\Psi(t)\rangle = \begin{pmatrix} \psi_1 \cos(t) - \psi_2 i \sin(t) \\ -\psi_1 i \sin(t) + \psi_2 \cos(t) \\ \psi_3 e^{-2it} \end{pmatrix} \tag{5.256}$$

. To check the result one can calculate both sides of $i\partial_t|\Psi(t)\rangle = H|\Psi(t)\rangle$.

5.14.3 Problem B.2-1

1. With $M = PDP^{-1}$ we have $M^2 = PDP^{-1}PDP^{-1} = PDDP^{-1}$ and see

$$e^{tM} = \sum_{k=0}^{\infty} \frac{(tM)^k}{k!} = \sum_{k=0}^{\infty} \frac{(tPDP^{-1})^k}{k!} = \sum_{k=0}^{\infty} \frac{P(tD)^k P^{-1}}{k!} \tag{5.257}$$

$$= P\left(\sum_{k=0}^{\infty} \frac{(tD)^k}{k!}\right) P^{-1} = Pe^{tD}P^{-1}. \tag{5.258}$$

The eigenvalues of M are given by

$$-\lambda^3 - (-\lambda)(-\pi^2) = 0 \quad \rightarrow \quad \lambda_1 = i\pi,\ \lambda_2 = -i\pi,\ \lambda_3 = 0 \tag{5.259}$$

with the eigenvectors

$$\vec{v}_1 = (-i, 1, 0) \tag{5.260}$$

$$\vec{v}_2 = (i, 1, 0) \tag{5.261}$$

$$\vec{v}_3 = (0, 0, 1) \tag{5.262}$$

we obtain

$$M = PDP^{-1} \tag{5.263}$$

$$= \begin{pmatrix} -i & i & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} i\pi & 0 & 0 \\ 0 & -i\pi & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} i/2 & 1/2 & 0 \\ -i/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.264}$$

With

$$\sum_{k=0}^{\infty} \frac{(i\pi)^k}{k!} = e^{i\pi} \tag{5.265}$$

$$\sum_{k=0}^{\infty} \frac{(-i\pi)^k}{k!} = e^{-i\pi} \tag{5.266}$$

we see

$$tD^k = \begin{pmatrix} (i\pi t)^k & 0 & 0 \\ 0 & (-i\pi t)^k & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.267}$$

$$e^{tD} = \sum_{k=0}^{\infty} \frac{(tD)^k}{k!} = \begin{pmatrix} e^{i\pi t} & 0 & 0 \\ 0 & e^{-i\pi t} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.268}$$

and therefore

$$e^{tM} = Pe^{tD}P^{-1} \tag{5.269}$$

$$= \begin{pmatrix} \frac{1}{2}(e^{-i\pi t} + e^{i\pi t}) & -\frac{1}{2}i(e^{i\pi t} - e^{-i\pi t}) & 0 \\ -\frac{1}{2}i(e^{-i\pi t} - e^{i\pi t}) & \frac{1}{2}(e^{-i\pi t} + e^{i\pi t}) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.270}$$

$$= \begin{pmatrix} \cos(\pi t) & \sin(\pi t) & 0 \\ -\sin(\pi t) & \cos(\pi t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.271}$$

2. Brute force calculation of the matrix powers reveals

$$(tM)^2 = \begin{pmatrix} -(t\pi)^2 & 0 & 0 \\ 0 & -(t\pi)^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (tM)^3 = \begin{pmatrix} 0 & -(t\pi)^3 & 0 \\ (t\pi)^3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.272}$$

$$(tM)^4 = \begin{pmatrix} (t\pi)^4 & 0 & 0 \\ 0 & (t\pi)^4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (tM)^5 \begin{pmatrix} 0 & (t\pi)^5 & 0 \\ -(t\pi)^5 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.273}$$

With

$$1 - \frac{1}{2!}(\pi t)^2 + \frac{1}{4!}(\pi t)^4 + ... = \cos(\pi t) \tag{5.274}$$

$$\pi t - \frac{1}{3!}(\pi t)^3 + \frac{1}{5!}(\pi t)^5 + ... = \sin(\pi t) \tag{5.275}$$

$$-\pi t + \frac{1}{3!}(\pi t)^3 - \frac{1}{5!}(\pi t)^5 + ... = (-\pi t) + \frac{1}{3!}(-\pi t)^3 - \frac{1}{5!}(-\pi t)^5 + ... \tag{5.276}$$

$$= \sin(-\pi t) \tag{5.277}$$

$$= -\sin(\pi t) \tag{5.278}$$

we obtain

$$e^{tM} = \begin{pmatrix} \cos(\pi t) & \sin(\pi t) & 0 \\ -\sin(\pi t) & \cos(\pi t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.279}$$

Problem B.2-2

For the Hamiltonian

$$H = -B_x\sigma_1 = \begin{pmatrix} 0 & -B_x \\ -B_x & 0 \end{pmatrix} \tag{5.280}$$

we find the eigensystem

$$E_1 = -B_x \quad |\psi_1\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{5.281}$$

$$E_2 = +B_x \quad |\psi_2\rangle = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{5.282}$$

The Hamiltonian (with full units) is given by

$$H = -g\frac{q\hbar}{2m}\frac{\sigma_1}{2}B_x \tag{5.283}$$

which translates into energies of

$$E_1 = -g\frac{q\hbar}{4m}B_x \tag{5.284}$$

$$E_2 = g\frac{q\hbar}{4m}B_x. \tag{5.285}$$

The time evolution is them given by

$$
\begin{aligned}
|\psi(t)\rangle &= e^{-\frac{i}{\hbar}Ht}|\psi(0)\rangle && (5.286)\\
&= e^{-i\frac{gq}{4m}\sigma_1 t}|\psi(0)\rangle && (5.287)\\
&= \left[\cos\left(\frac{gq}{4m}\sigma_1 t\right) - i\sin\left(\frac{gq}{4m}\sigma_1 t\right)\right]|\psi(0)\rangle && (5.288)\\
&= \left[\cos\left(\frac{gq}{4m}t\right)\mathbb{I}_2 - i\sin\left(\frac{gq}{4m}t\right)\sigma_1\right]|\psi(0)\rangle && (5.289)\\
&= \begin{pmatrix} \cos\left(\frac{gqt}{4m}\right) & -i\sin\left(\frac{gqt}{4m}\right) \\ -i\sin\left(\frac{gqt}{4m}\right) & \cos\left(\frac{gqt}{4m}\right) \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} && (5.290)\\
&= \begin{pmatrix} \cos\left(\frac{gqt}{4m}\right) \\ -i\sin\left(\frac{gqt}{4m}\right) \end{pmatrix} && (5.291)
\end{aligned}
$$

where we used $\sigma_1^{2n} = \mathbb{I}^n = \mathbb{I}$.

5.15 Baez, Muniain - Gauge Fields, Knots and Gravity

5.15.1 Problem I.1 - Plane waves in vacuum

With

$$\vec{\mathcal{E}} = \vec{E}e^{-i(\omega t - \vec{k}\vec{x})} \tag{5.292}$$

we calculate in cartesian coordinates

1. $\nabla \cdot \vec{\mathcal{E}} = 0$

$$
\begin{aligned}
\nabla \cdot \vec{\mathcal{E}} &= \partial_a \mathcal{E}_a && (5.293)\\
&= \partial_a(e^{-i(\omega t - \vec{k}\vec{x})})E_a\vec{e}^a && (5.294)\\
&= \delta_{ab} i k_b E_a e^{-i(\omega t - \vec{k}\vec{x})}\vec{e}^a && (5.295)\\
&= i k_b E_b e^{-i(\omega t - \vec{k}\vec{x})}\vec{e}^a && (5.296)\\
&= 0 && (5.297)
\end{aligned}
$$

where we assumed $E_a = \text{const}$ and used

$$
\begin{aligned}
0 &= \vec{k}\cdot\vec{E} && (5.298)\\
&= k_a\vec{e}^a E_a\vec{e}^a && (5.299)\\
&= k_a E_a && (5.300)
\end{aligned}
$$

2. $\nabla \times \vec{\mathcal{E}} = i\frac{\partial\vec{\mathcal{E}}}{\partial t}$

$$
\begin{aligned}
\nabla \times \vec{\mathcal{E}} &= \epsilon_{abc}\partial_b\mathcal{E}_c\vec{e}_a && (5.301)\\
&= \epsilon_{abc}E_c\vec{e}_a\partial_b(e^{-i(\omega t - \vec{k}\vec{x})}) && (5.302)\\
&= \epsilon_{abc}E_c\vec{e}_a\delta_{bd} i k_d e^{-i(\omega t - \vec{k}\vec{x})} && (5.303)\\
&= i(\epsilon_{abc}k_b E_c\vec{e}_a)e^{-i(\omega t - \vec{k}\vec{x})} && (5.304)\\
&= i(-i\omega E_a\vec{e}^a)e^{-i(\omega t - \vec{k}\vec{x})} && (5.305)\\
&= i(E_a\vec{e}^a)(-i\omega)e^{-i(\omega t - \vec{k}\vec{x})} && (5.306)\\
&= i\vec{E}\frac{\partial}{\partial t}e^{-i(\omega t - \vec{k}\vec{x})} && (5.307)\\
&= i\frac{\partial\vec{\mathcal{E}}}{\partial t} && (5.308)
\end{aligned}
$$

where we used (typo in the book!)

$$\begin{aligned} -i\omega\vec{E} &= \vec{k}\times\vec{E} && (5.309)\\ &= \epsilon_{abc}k_bE_c\vec{e}_a && (5.310)\end{aligned}$$

5.15.2 Problem I.7 - Adding and multiplying vector fields

1. With $(v+w)f \equiv (f) + w(f)$
   (a) $(v+w)(f+g) = v(f+g) + w(f+g) = vf + vg + wf + wg = (v+w)f + (v+w)g$
   (b) $(v+w)(\alpha f) = v(\alpha f) + w(\alpha f) = \alpha vf + \alpha wf = \alpha(v+w)f$
   (c) $(v+w)(fg) = v(fg)+w(fg) = v(f)g+fv(g)+w(f)g+fw(g) = [(v+w)f]g+f[(v+w)g]$
2. With $(gv)(f) \equiv gv(f)$
   (a) $(gv)(f+h) = gv(f+h) = gv(f) + gv(h) = g(v(f)+v(h)) = gv(f) + gv(h)$
   (b) $gv(\alpha f) = gv(\alpha f) = g\alpha v(f) = \alpha gv(f)$
   (c) $(gv)(fh) = gv(fh) = g(v(f)h + fv(h)) = (gv)(f)h + f(gv)(h)$

5.16 Kreyszig - Introduction to functional analysis

5.16.1 Problem 11.3-1 Problem 3

Physicist: Ground state of the harmonic osci. - time-independent Schroedinger equation for harmonic oscillator

$$\begin{aligned} \psi_0' &= -se^{-s^2/2} && (5.311)\\ &= -s\psi_0 && (5.312)\\ \psi_0'' &= -e^{-s^2/2} + s^2e^{-s^2/2} && (5.313)\\ &= -\psi_0 + s^2\psi_0 && (5.314)\\ &\to -\psi_0'' + s^2\psi_0 = \psi_0 && (5.315)\end{aligned}$$

5.17 Garrity et al. - Algebraic Geometry: A Problem Solving Approach

5.18 Garrity, Neumann-Chun - Electricity and magnetism for mathematicians - A guided path from Maxwell's equations to Yang-Mills

5.19 Guidry - Symmetry, Broken Symmetry, and Topology in Modern Physics

5.19.1 Problem 15.1 - Poincaré transformation I

$$\begin{aligned} g(b,\Lambda) \quad &\to \quad x' = \Lambda x + b && (5.316)\\ g(b',\Lambda')\circ g(b,\Lambda) \quad &\to \quad x'' = \Lambda' x' + b' && (5.317)\\ & \qquad\quad = \Lambda'(\Lambda x + b) + b' && (5.318)\\ & \qquad\quad = \Lambda'\Lambda x + \Lambda' b + b' \quad \to \quad g(\Lambda' b + b', \Lambda'\Lambda) && (5.319)\end{aligned}$$

5.19.2 Problem 15.2 - Poincaré transformation II

$$g(b', I') \circ g(0, \Lambda) = g(b', \Lambda) \tag{5.320}$$
$$\Lambda x + b = T(b) \circ \Lambda x \tag{5.321}$$

5.19.3 Problem 15.3 - Poincaré transformation III

5.20 Boltyanskii, Efremovich - Intuitive Combinatorial Topology

5.21 Nakahara - Geometry, Topology and Physics

5.22 Frankel - The Geometry of Physics

5.23 Sexl, Urbantke - Relativity, Groups, Particles

Chapter 6

Many-body physics

6.1 Fetter, Walecka - Quantum Theory of Many-Particle Systems

6.1.1 2.2 Equation of state for an ultrarelativistic ideal gas

$$\begin{aligned}
\epsilon_p &= \lim_{p \gg m_0 c} \sqrt{(pc)^2 + m_0^2 c^4} && (6.1)\\
&= pc\sqrt{1 + \frac{m_0^2 c^2 \cdot c^2}{p^2 \cdot c^2}} && (6.2)\\
&\approx pc && (6.3)
\end{aligned}$$

Quick thermodynamics review

$$\begin{aligned}
\text{1st law} \quad & dU = \delta Q + \delta W && (6.4)\\
\text{2nd law} \quad & dS = dS_i + \frac{\delta Q}{T}, \qquad dS_i > 0 && (6.5)\\
\text{Gibbs Fund.Form} \rightarrow & dS = \frac{1}{T}dU - \frac{1}{T}\delta W = \frac{1}{T}dU + \frac{1}{T}\sum_i y_i dX_i && (6.6)\\
\rightarrow & \left.\frac{dS}{dU}\right|_{X_i} = \frac{1}{T} \qquad \rightarrow \qquad U = U(T, X_i) && (6.7)\\
\rightarrow & \left.\frac{dS}{dX_i}\right|_{U,X_j} = \frac{y_i}{T} \qquad \rightarrow \qquad y_i = y_i(T, X_j) && (6.8)
\end{aligned}$$

6.2 Coleman - Introduction to Many-Body Physics

6.2.1 Problem 2.1 - Specific heat capacity of a solid

Using the Boltzmann statistics and $E_n = \hbar\omega(n + \frac{1}{2})$ the energy E of a system of N_{AV} harmonic oscillators (in 3d!!) is given by

$$E = 3N_{AV} \frac{\sum_n E_n e^{-\frac{E_n}{k_B T}}}{\sum_n e^{-\frac{E_n}{k_B T}}} \tag{6.9}$$

$$= 3N_{AV} \frac{\sum_n \left(\hbar\omega\left[n + \frac{1}{2}\right]\right) e^{-\frac{n\hbar\omega}{k_B T}} e^{-\frac{\hbar\omega}{2k_B T}}}{\sum_n e^{-\frac{n\hbar\omega}{k_B T}} e^{-\frac{\hbar\omega}{2k_B T}}} \tag{6.10}$$

$$= 3N_{AV}\hbar\omega \left(\frac{1}{2} + \frac{\sum_n n e^{-\frac{n\hbar\omega}{k_B T}}}{\sum_n e^{-\frac{n\hbar\omega}{k_B T}}}\right) \tag{6.11}$$

$$= 3N_{AV}\hbar\omega \left(\frac{1}{2} + \frac{1}{e^{\frac{\hbar\omega}{k_B T}} - 1}\right) \tag{6.12}$$

where we used the sum formulas

$$s_1 = \sum_{n=0} q^n = \frac{1}{1-q} \tag{6.13}$$

$$s_2 = \sum_{n=0} n q^n = q\frac{ds_1}{dq} = \frac{q}{(1-q)^2} \tag{6.14}$$

the specific heat can the be calculated as

$$C_V = \frac{dE}{dT} \tag{6.15}$$

$$= 3N_{AV}\hbar\omega \frac{\exp\left[\frac{\hbar\omega}{kT}\right]\frac{\hbar\omega}{kT^2}}{\left[\exp\left[\frac{\hbar\omega}{kT}\right] - 1\right]^2} \tag{6.16}$$

$$= 3N_{AV} k \frac{x^2 \exp(x)}{\left[\exp(x) - 1\right]^2} \tag{6.17}$$

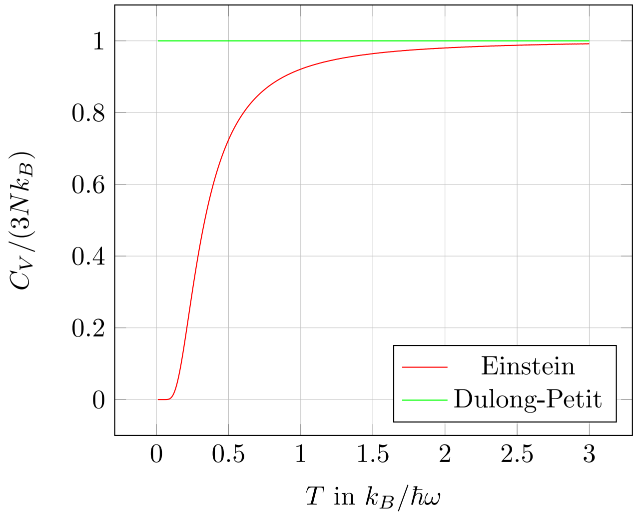
$$= 3N_{AV} k \frac{x^2}{\left[\exp(x/2) - \exp(-x/2)\right]^2} \tag{6.18}$$

$$= 3N_{AV} k \frac{x^2}{\left[\exp(x/2) - \exp(-x/2)\right]^2} \tag{6.19}$$

$$= 3N_{AV} k \left(\frac{x/2}{\sinh(x/2)}\right)^2 \tag{6.20}$$

$$\tag{6.21}$$

The Dulong-Petit rule says $k/2$ per harmonic degree of freedom which means in 3d that $C_V/N = 3k$ (for each harmonic degree there is also a kinetic one - so $f = 6$)


1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
2.5
3
$C_V/(3Nk_B)$
T in $k_B/\hbar\omega$
Einstein
Dulong-Petit

Chapter 7

Nuclear Physics

7.1 Walecka - Theoretical Nuclear And Subnuclear Physics, 2nd Edition (2004)

Chapter 8

Quantum Field Theory

8.1 Lancaster, Blundell - Quantum Field Theory for the gifted amateur

8.1.1 Problem 1.1 - Snell's law via Fermat's principle

The light travels from point A in medium 1 to point B in medium 2. We assume a vertical medium boundary at x_0 and that the light travels within a medium in the straight line. This makes y_0 the free parameter and the the travel time is given by

$$t = \frac{s_{A0}}{c/n_1} + \frac{s_{0B}}{c/n_2} \tag{8.1}$$

$$= \sqrt{\frac{(x_A - x_0)^2 + (y_A - y_0)^2}{c/n_1}} + \sqrt{\frac{(x_0 - x_B)^2 + (y_0 - y_B)^2}{c/n_2}} \tag{8.2}$$

The local extrema of the travel time is given by

$$0 = \frac{dt}{dy_0} \tag{8.3}$$

$$= \frac{y_A - y_0}{s_{A0} c/n_1} + \frac{y_0 - y_B}{s_{0B} c/n_2} \tag{8.4}$$

$$= \frac{\sin\alpha}{c/n_1} - \frac{\sin\beta}{c/n_2} \tag{8.5}$$

and therefore

$$n_1 \sin\alpha = n_2 \sin\beta. \tag{8.6}$$

8.1.2 Problem 1.2 - Functional derivatives I

- $H[f] = \int G(x,y) f(y) dy$

$$\frac{\delta H[f]}{\delta f(z)} = \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left[\int G(x,y)(f(y) + \epsilon\delta(z-y))dy - \int G(x,y)f(y)dy\right] \tag{8.7}$$

$$= \int G(x,y)\delta(z-y))dy \tag{8.8}$$

$$= G(x,z) \tag{8.9}$$

- $I[f] = \int_{-1}^{1} f(x)dx$

$$\frac{\delta^2 I[f^3]}{\delta f(x_0)\delta f(x_1)} = \frac{\delta}{\delta f(x_0)}\frac{\delta I[f^3]}{\delta f(x_1)} \tag{8.10}$$
$$= \frac{\delta}{\delta f(x_0)}\frac{\delta}{\delta f(x_1)}\int_{-1}^{1} f(x)^3 dx \tag{8.11}$$
$$= \frac{\delta}{\delta f(x_0)}\frac{1}{\epsilon}\int_{-1}^{1}(f(x)+\epsilon\delta(x_1-x))^3 - f(x)^3 dx \tag{8.12}$$
$$= \frac{\delta}{\delta f(x_0)}\frac{1}{\epsilon}\int_{-1}^{1}(f(x)^3+3\epsilon f(x)^2\delta(x_1-x)+\mathcal{O}(\epsilon^2) - f(x)^3 dx \tag{8.13}$$
$$= \frac{\delta}{\delta f(x_0)}\begin{cases}3f(x_1)^2 & x_1\in[-1,1]\\ 0 & \text{else}\end{cases} \tag{8.14}$$
$$= \begin{cases}3\frac{1}{\epsilon}[(f(x_1)-\epsilon\delta(x_0-x_1))^2 - f(x_1)^2] & x_1\in[-1,1]\\ 0 & \text{else}\end{cases} \tag{8.15}$$
$$= \begin{cases}6f(x_1)\delta(x_0-x_1) & x_1\in[-1,1]\\ 0 & \text{else}\end{cases} \tag{8.16}$$
$$\tag{8.17}$$

- $J[f] = \int\left(\frac{\partial f}{\partial y}\right)^2 dy$

$$\frac{\delta J[f]}{\delta f(x)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\int\left(\frac{\partial(f+\epsilon\delta(x-y))}{\partial y}\right)^2 dy - \int\left(\frac{\partial f}{\partial y}\right)^2 dy\right] \tag{8.18}$$
$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\int\left(\frac{\partial f}{\partial y}+\epsilon\frac{\partial\delta(x-y)}{\partial y}\right)^2 dy - \int\left(\frac{\partial f}{\partial y}\right)^2 dy\right] \tag{8.19}$$
$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\int\left(\frac{\partial f}{\partial y}\right)^2 + 2\epsilon\frac{\partial f}{\partial y}\frac{\partial\delta(x-y)}{\partial y} + \mathcal{O}(\epsilon^2) - \left(\frac{\partial f}{\partial y}\right)^2 dy\right] \tag{8.20}$$
$$= 2\int\frac{\partial f}{\partial y}\frac{\partial\delta(x-y)}{\partial y}dy \tag{8.21}$$
$$= \text{boundary terms} - 2\int\frac{\partial^2 f}{\partial y^2}\delta(x-y)dy \tag{8.22}$$
$$= -2\int\frac{\partial^2 f}{\partial x^2} \tag{8.23}$$

8.1.3 Problem 1.3 - Functional derivatives II

-

$$\frac{\delta G[f]}{\delta f(x)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int g(y, f+\epsilon\delta(x-y)) - g(y,f)dy \tag{8.24}$$
$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int g(y,f) + \epsilon\frac{\partial g(y,f)}{\partial f}\delta(x-y)) - g(y,f)dy \tag{8.25}$$
$$= \frac{\partial g(x,f)}{\partial f} \tag{8.26}$$

-

$$\frac{\delta H[f]}{\delta f(x)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int g(y, f+\epsilon\delta(x-y), f'+\epsilon\partial_y\delta(x-y)) - g(y,f,f')dy \tag{8.27}$$
$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int g(y,f,f') + \epsilon\frac{\partial g(y,f,f')}{\partial f}\delta(x-y) + \epsilon\frac{\partial g(y,f,f')}{\partial f'}\partial_y\delta(x-y)) - g(y,f,f')dy \tag{8.28}$$
$$= \int \frac{\partial g(y,f,f')}{\partial f}\delta(x-y) + \frac{\partial g(y,f,f')}{\partial f'}\partial_y\delta(x-y))dy \tag{8.29}$$
$$= \frac{\partial g(x,f,f')}{\partial f} + \int \frac{\partial g(y,f,f')}{\partial f'}\partial_y\delta(x-y))dy \tag{8.30}$$
$$= \frac{\partial g(x,f,f')}{\partial f} - \int \partial_y\frac{\partial g(y,f,f')}{\partial f'}\delta(x-y))dy \tag{8.31}$$
$$= \frac{\partial g(x,f,f')}{\partial f} - \partial_x\frac{\partial g(x,f,f')}{\partial f'} \tag{8.32}$$

- Same as above but two times integration by parts is needed. Therefore $(-1)^2 = 1$ giving the term a final + sign.

8.1.4 Problem 1.4 - Functional derivatives III

-

$$\frac{\delta\phi(x)}{\delta\phi(y)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\phi(x) + \epsilon\delta(x-y) - \phi(x)\right) \tag{8.33}$$
$$= \delta(x-y) \tag{8.34}$$

-

$$\frac{\delta\dot\phi(t)}{\delta\phi(t_0)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\dot\phi(t) + \epsilon\partial_t\delta(t-t_0) - \dot\phi(t)\right) \tag{8.35}$$
$$= \frac{d}{dt}\delta(t-t_0) \tag{8.36}$$

8.1.5 Problem 1.5 - Euler-Langrange equations for elastic medium

$$\mathcal{L} = T - V \tag{8.37}$$
$$\frac{\partial\mathcal{L}}{\partial\psi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right) = 0 \tag{8.38}$$

then

$$\frac{\partial\mathcal{L}}{\partial\psi} = 0 \tag{8.39}$$
$$\frac{\partial\mathcal{L}}{\partial(\partial_0\psi)} = \frac{\rho}{2}\int d^3x 2\frac{\partial\psi}{\partial t} \tag{8.40}$$
$$\frac{\partial\mathcal{L}}{\partial(\partial_k\psi)} = -\frac{\mathcal{T}}{2}\int d^3x 2\frac{\partial\psi}{\partial x^k} \tag{8.41}$$
$$\to -\left(\int d^3x[\rho\ddot\psi - \mathcal{T}\nabla^2\psi]\right) = 0 \tag{8.42}$$
$$\to \frac{\rho}{\mathcal{T}}\ddot\psi = \nabla^2\psi \tag{8.43}$$

8.1.6 Problem 1.6 - Functional derivatives IV

$$\frac{\delta Z_0[J]}{\delta J(z_1)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\exp\left(-\frac{1}{2}\int d^4x d^4y (J(x)+\epsilon\delta(x-z_1))\Delta(x-y)(J(y)+\epsilon\delta(y-z_1))\right) \tag{8.44}$$

$$- \exp\left(-\frac{1}{2}\int d^4x d^4y J(x)\Delta(x-y)J(y)\right) \tag{8.45}$$

$$= Z_0[J]\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\exp\left(-\frac{\epsilon}{2}\int d^4x d^4y\, J(x)\Delta(x-y)\delta(y-z_1)+\delta(x-z_1)\Delta(x-y)J(y)\right)-1\right) \tag{8.46}$$

$$= Z_0[J]\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(1-\frac{\epsilon}{2}\int d^4x d^4y\, J(x)\Delta(x-y)\delta(y-z_1)+\delta(x-z_1)\Delta(x-y)J(y)-1\right) \tag{8.47}$$

$$= -\frac{1}{2}Z_0[J]\int d^4x d^4y\, J(x)\Delta(x-y)\delta(y-z_1)+\delta(x-z_1)\Delta(x-y)J(y) \tag{8.48}$$

$$= -\frac{1}{2}Z_0[J]\left(\int d^4x\, J(x)\Delta(x-z_1)+\int d^4y\,\Delta(z_1-y)J(y)\right) \tag{8.49}$$

$$= -Z_0[J]\int d^4y\,\Delta(z_1-y)J(y) \tag{8.50}$$

8.1.7 Problem 2.1 - Commutators of creation and annihilation operators

With $[\hat{x},\hat{p}] = \hat{x}\hat{p}-\hat{p}\hat{x} = i\hbar$

$$[\hat{a},\hat{a}] = \frac{m\omega}{2\hbar}\left(\hat{x}\hat{x}+\frac{i}{m\omega}(\hat{x}\hat{p}+\hat{p}\hat{x})+\frac{i^2}{m^2\omega^2}\hat{p}\hat{p}\right)-\frac{m\omega}{2\hbar}\left(\hat{x}\hat{x}+\frac{i}{m\omega}(\hat{x}\hat{p}+\hat{p}\hat{x})+\frac{i^2}{m^2\omega^2}\hat{p}\hat{p}\right) \tag{8.51}$$

$$= 0 \tag{8.52}$$

$$[\hat{a}^\dagger,\hat{a}^\dagger] = ... = 0 \tag{8.53}$$

$$[\hat{a},\hat{a}^\dagger] = \frac{m\omega}{2\hbar}\left(\hat{x}\hat{x}+\frac{i}{m\omega}(-\hat{x}\hat{p}+\hat{p}\hat{x})-\frac{i^2}{m^2\omega^2}\hat{p}\hat{p}\right)-\frac{m\omega}{2\hbar}\left(\hat{x}\hat{x}+\frac{i}{m\omega}(\hat{x}\hat{p}-\hat{p}\hat{x})-\frac{i^2}{m^2\omega^2}\hat{p}\hat{p}\right) \tag{8.54}$$

$$= \frac{m\omega}{2\hbar}\frac{i}{m\omega}2(-\hat{x}\hat{p}+\hat{p}\hat{x}) \tag{8.55}$$

$$= \frac{i}{\hbar}(-\hat{p}\hat{x}-i\hbar+\hat{p}\hat{x}) \tag{8.56}$$

$$= 1 \tag{8.57}$$

Now the Hamiltonian

$$\hat{a}^\dagger\hat{a} = \frac{m\omega}{2\hbar}\left(\hat{x}\hat{x}+\frac{i}{m\omega}(\hat{x}\hat{p}-\hat{p}\hat{x})-\frac{i^2}{m^2\omega^2}\hat{p}\hat{p}\right) \tag{8.58}$$

$$= \frac{m\omega}{2\hbar}\left(\hat{x}\hat{x}+\frac{i}{m\omega}i\hbar-\frac{i^2}{m^2\omega^2}\hat{p}\hat{p}\right) \tag{8.59}$$

$$= \frac{1}{2m\omega\hbar}\hat{p}^2+\frac{m\omega}{2\hbar}\hat{x}^2-\frac{1}{2} \tag{8.60}$$

$$\hat{a}^\dagger\hat{a}+\frac{1}{2} = \frac{1}{2m\omega\hbar}\hat{p}^2+\frac{m\omega}{2\hbar}\hat{x}^2 \tag{8.61}$$

$$\hbar\omega\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right) = \frac{1}{2m}\hat{p}^2+\frac{m\omega^2}{2}\hat{x}^2 = \hat{H} \tag{8.62}$$

8.1.8 Problem 2.2 - Perturbed harmonic oscillator

We see

$$a + a^\dagger = \sqrt{\frac{2m\omega}{\hbar}} x \tag{8.63}$$
$$(a + a^\dagger)^2 = \frac{2m\omega}{\hbar} x^2 \tag{8.64}$$
$$x^2 = \frac{\hbar}{2m\omega}(a + a^\dagger)^2 \tag{8.65}$$
$$x^4 = (a + a^\dagger)^2 \frac{\hbar}{2m\omega} \cdot \frac{\hbar}{2m\omega}(a + a^\dagger)^2 \tag{8.66}$$

The first order energy perturbation is given by

$$E_n^{(1)} = \langle n|H_1|n\rangle \tag{8.67}$$
$$= \langle n|x^4|n\rangle \tag{8.68}$$
$$= \langle n|x^2 \cdot x^2|n\rangle. \tag{8.69}$$

By splitting H_1 the calculation gets a bit shorter. Using

$$a|n\rangle\sqrt{n}|n\rangle \qquad a^\dagger|n\rangle\sqrt{n+1}|n+1\rangle \tag{8.70}$$

we obtain

$$x^2|n\rangle = \frac{\hbar}{2m\omega}(a + a^\dagger)^2|n\rangle \tag{8.71}$$
$$= \frac{\hbar}{2m\omega}(aa^\dagger + a^\dagger a + (a^\dagger)^2 + a^2)|n\rangle \tag{8.72}$$
$$= \frac{\hbar}{2m\omega}\Big((n+1)|n\rangle + n|n\rangle + \sqrt{n(n-1)}|n-2\rangle + \sqrt{(n+1)(n+2)}|n+2\rangle\Big) \tag{8.73}$$
$$= \frac{\hbar}{2m\omega}\Big((2n+1)|n\rangle + \sqrt{n(n-1)}|n-2\rangle + \sqrt{(n+1)(n+2)}|n+2\rangle\Big) \tag{8.74}$$
$$\langle n|x^2 = (x^2|n\rangle)^\dagger \tag{8.75}$$
$$= \frac{\hbar}{2m\omega}\Big((2n+1)|n\rangle + \sqrt{n(n-1)}|n-2\rangle + \sqrt{(n+1)(n+2)}|n+2\rangle\Big) \tag{8.76}$$

Using the orthogonality of the unperturbed states (eigenstates of the Hamiltonian which is hermitian) we obtain

$$E_n^{(1)} = \langle n|x^2 \cdot x^2|n\rangle \tag{8.77}$$
$$= \frac{\hbar^2}{4m^2\omega^2}\big((2n+1)^2 + n(n-1) + (n+1)(n+2)\big) \tag{8.78}$$
$$= \frac{\hbar^2}{4m^2\omega^2}\big(4n^2 + 4n + 1 + n^2 - n + n^2 + 3n + 2\big) \tag{8.79}$$
$$= \frac{\hbar^2}{4m^2\omega^2}\big(6n^2 + 6n + 3\big) \tag{8.80}$$
$$= \frac{3}{4}\frac{\hbar^2}{m^2\omega^2}\big(2n^2 + 2n + 1\big) \tag{8.81}$$

which gives the desired result using $E_n = E_n^{(0)} + \lambda E_n^{(1)}$

8.1.9 Problem 2.3 - ...

Odd notation $\tilde{x} = \hat{x}$

$$\hat{x}_j = \sqrt{\frac{\hbar}{2\omega_j m}}(\hat{a}_j + \hat{a}^\dagger_{-j}) \tag{8.82}$$

$$x_j = \frac{1}{\sqrt{N}} \sum_k \tilde{x}_k e^{ikja} \tag{8.83}$$

$$= \frac{1}{\sqrt{N}} \sqrt{\frac{\hbar}{m}} \sum_k \frac{1}{\sqrt{2\omega_k}} (\hat{a}_k + \hat{a}^\dagger_{-k}) e^{ikja} \tag{8.84}$$

$$= \frac{1}{\sqrt{N}} \sqrt{\frac{\hbar}{m}} \sum_k \frac{1}{\sqrt{2\omega_k}} (\hat{a}_k e^{ikja} + \hat{a}^\dagger_k e^{-ikja}) \tag{8.85}$$

8.1.10 Problem 2.4 - Wavefunction in space representation

$$\hat{a} = \sqrt{\frac{2\hbar}{m\omega}} \left(\hat{x} + \frac{i}{m\omega}\hat{p}\right), \qquad \hat{a}|0\rangle = 0 \tag{8.86}$$

$$\to \sqrt{\frac{2\hbar}{m\omega}} \langle x| \left(\hat{x} + \frac{i}{m\omega}\hat{p}\right) |0\rangle = 0 \tag{8.87}$$

$$\to \sqrt{\frac{2\hbar}{m\omega}} \left(\langle x|\hat{x}|0\rangle + \frac{i}{m\omega}\langle x|\hat{p}|0\rangle\right) = 0 \tag{8.88}$$

$$\to \sqrt{\frac{2\hbar}{m\omega}} \left(x\langle x|0\rangle + \frac{i}{m\omega}(-i\hbar)\frac{d}{dx}\langle x|0\rangle\right) = 0 \tag{8.89}$$

$$\to \sqrt{\frac{2\hbar}{m\omega}} \left(x + \frac{\hbar}{m\omega}\frac{d}{dx}\right) \langle x|0\rangle = 0 \tag{8.90}$$

Now we can solve the ODE ($\psi_0(x) = \langle x|0\rangle$)

$$\left(x + \frac{\hbar}{m\omega}\frac{d}{dx}\right)\psi_0 = 0 \tag{8.91}$$

$$\int dx\, \psi_0' + \int dx\, \frac{m\omega}{\hbar} x\psi_0 = 0 \tag{8.92}$$

$$\frac{\psi_0'}{\psi_0} = -\frac{m\omega}{\hbar}x \tag{8.93}$$

$$\log\psi_0 = -\frac{m\omega}{2\hbar}x^2 + c \tag{8.94}$$

$$\psi_0 = Ce^{-m\omega x^2/2\hbar} \tag{8.95}$$

Normalization

$$\int dx\, \psi_0^*\, \psi_0 = 1 \tag{8.96}$$

$$C^*C \int dx\, e^{-m\omega x^2/\hbar} = 1 \tag{8.97}$$

$$|C|^2\sqrt{\frac{\pi\hbar}{m\omega}} = 1 \quad \to \quad C = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \tag{8.98}$$

8.1.11 Problem 3.1 - Commutator Fourier Transformation

Bosons - commutator

$$\frac{1}{\mathcal{V}}\sum_{\mathbf{p},\mathbf{q}} e^{i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})}[a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] = \frac{1}{\mathcal{V}}\sum_{\mathbf{p},\mathbf{q}} e^{i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})}\delta_{\mathbf{pq}} \tag{8.99}$$

$$= \frac{1}{\mathcal{V}}\sum_{\mathbf{p}} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \tag{8.100}$$

$$= \frac{1}{L_x L_y L_z}\sum_{n_1=-N/2}^{N/2} e^{i\frac{2\pi n_1}{Na_x}(x_1-y_1)}\cdot\sum_{n_2=-N/2}^{N/2} e^{i\frac{2\pi n_2}{Na_y}(x_2-y_2)}\cdot\sum_{n_3=-N/2}^{N/2} e^{i\frac{2\pi n_3}{Na_z}(x_3-y_3)} \tag{8.101}$$

$$= \left(\frac{1}{L}\sum_{n=-N/2}^{N/2} e^{i\frac{2\pi n}{Na}(x-y)}\right)^3 \qquad \text{with } Na \equiv L \tag{8.102}$$

$$= \left(\frac{1}{L}\frac{Na}{2\pi}\sum_{p_n=-\pi/a}^{\pi/a} e^{ip_n(x-y)}\frac{2\pi}{Na}\right)^3 \qquad \text{with } \sum_{q_n} f(p_n)\Delta p = \int f(p)dp \tag{8.103}$$

$$= \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ip(x-y)}dp\right)^3 \qquad \text{with } N\to\infty,\ a\to 0 \tag{8.104}$$

$$= (\delta(x-y))^3 \tag{8.105}$$

$$= \delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{8.106}$$

with the discretization of the momentum-space $p_j = \left\{\frac{2\pi j}{Na}\right\}_{-N/2}^{N/2}$ and $\Delta p = \frac{2\pi}{Na}$.

Fermions - anticommutator

$$\{c_{\mathbf{p}}, c^\dagger_{\mathbf{q}}\} = \delta_{\mathbf{pq}} \tag{8.107}$$

yields same result.

8.1.12 Problem 3.2 - Harmonic oscillator relations

With

$$[\hat{a}, \hat{a}^\dagger] = 1 \tag{8.108}$$

$$\hat{a}^\dagger\hat{a} = \hat{n} \tag{8.109}$$

$$\frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle = |n\rangle \tag{8.110}$$

Then

(a) $[\hat{a}, (\hat{a}^\dagger)^n]$

$$\begin{aligned}
\hat{a}(\hat{a}^\dagger)^n &= (aa^\dagger)(a^\dagger)^{n-1} && (8.111)\\
&= (a^\dagger a + 1)(a^\dagger)^{n-1} && (8.112)\\
&= a^\dagger a(a^\dagger)^{n-1} + (a^\dagger)^{n-1} && (8.113)\\
&= a^\dagger a a^\dagger (a^\dagger)^{n-2} + (a^\dagger)^{n-1} && (8.114)\\
&= a^\dagger(a^\dagger a + 1)(a^\dagger)^{n-2} + (a^\dagger)^{n-1} && (8.115)\\
&= (a^\dagger)^2 a(a^\dagger)^{n-2} + 2(a^\dagger)^{n-1} && (8.116)\\
&= ... && (8.117)\\
&= (a^\dagger)^n a + n(a^\dagger)^{n-1} && (8.118)\\
\to [\hat{a}, (\hat{a}^\dagger)^n] &= n(a^\dagger)^{n-1} && (8.119)
\end{aligned}$$

(b) $\langle 0|a^n (a^\dagger)^m|0\rangle$

If $n < m$ (similar for $n > m$) we get zero

$$\begin{aligned}
\langle 0|a^n(a^\dagger)^m|0\rangle &\sim \langle 1|a^{n-1}(a^\dagger)^{m-1}|1\rangle && (8.120)\\
&\sim \langle 2|a^{n-2}(a^\dagger)^{m-2}|2\rangle && (8.121)\\
&... && (8.122)\\
&\sim \langle k|(a^\dagger)^{m-k}|k\rangle && (8.123)\\
&= 0 \qquad (\langle k|a^\dagger = 0). && (8.124)
\end{aligned}$$

For $n = m$ we have with the definition

$$\begin{aligned}
\frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle &= |n\rangle && (8.125)\\
(a^\dagger)^n|0\rangle &= \sqrt{n!}|n\rangle && (8.126)\\
\langle 0|a^n(a^\dagger)^m|0\rangle &= \sqrt{n!}^2\langle n|n|\rangle && (8.127)\\
&= n! && (8.128)
\end{aligned}$$

Therefore $\langle 0|a^n(a^\dagger)^m|0\rangle = n!\delta_{nm}$

(c) $\langle m|a^\dagger|n\rangle$

$$\begin{aligned}
\frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle &= |n\rangle && (8.129)\\
a^\dagger\frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle &= a^\dagger|n\rangle && (8.130)\\
\frac{1}{\sqrt{n+1}}a^\dagger\frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle &= \frac{1}{\sqrt{n+1}}a^\dagger|n\rangle = |n+1\rangle && (8.131)
\end{aligned}$$

then

$$\begin{aligned}
\langle m|a^\dagger|n\rangle &= \sqrt{n+1}\langle m|n+1\rangle && (8.132)\\
&= \sqrt{n+1}\delta_{m,n+1} && (8.133)
\end{aligned}$$

(d) $\langle m|a|n\rangle$

$$(\langle m|a)^\dagger = a^\dagger|m\rangle \tag{8.134}$$
$$= \sqrt{m+1}|m+1\rangle \tag{8.135}$$

then

$$\langle m|a|n\rangle = \sqrt{m+1}\delta_{m+1,n} \tag{8.136}$$
$$= \sqrt{n}\delta_{m+1,n} \tag{8.137}$$

8.1.13 Problem 3.2 - 3d Harmonic oscillator

Rewriting the Hamiltonian

$$H = H_1 + H_2 + H_3 \tag{8.138}$$
$$H_i = \frac{p_i^2}{2m} + \frac{1}{2}m\omega^2 x_i^2 \tag{8.139}$$

the we can reutilise the know ladder operators

$$a_i = \sqrt{\frac{m\omega}{2\hbar}}\left(x_i + \frac{i}{m\omega}p_i\right) \tag{8.140}$$
$$a_i^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x_i - \frac{i}{m\omega}p_i\right) \tag{8.141}$$

and the Hamiltonian can be obviously written as the sum

$$H = \hbar\omega\sum_k\left(a_k^\dagger a_k + \frac{1}{2}\right). \tag{8.142}$$

With the classic definition $\vec{L} = \vec{x}\times\vec{p}$ we see (inverting a and $a^\dagger$ to get x and p)

$$L_i = \varepsilon_{ijk}x_j p_k \tag{8.143}$$
$$= -i\varepsilon_{ijk}\sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{\hbar m\omega}{2}}(a_j + a_j^\dagger)(a_k - a_k^\dagger) \tag{8.144}$$
$$= -\frac{i\hbar}{2}\varepsilon_{ijk}(a_j a_k + a_j^\dagger a_k - a_j a_k^\dagger - a_j^\dagger a_k^\dagger) \tag{8.145}$$
$$= -\frac{i\hbar}{2}\varepsilon_{ijk}(a_j^\dagger a_k - \delta_{jk} - a_k^\dagger a_j) \qquad [a_j, a_k^\dagger] = \delta_{jk},\ a_j|0\rangle = 0,\ \langle 0|a_k = 0 \tag{8.146}$$
$$= -\frac{i\hbar}{2}(\varepsilon_{ijk}a_j^\dagger a_k - \varepsilon_{ijk}\delta_{jk} - \varepsilon_{ijk}a_k^\dagger a_j) \tag{8.147}$$
$$= -\frac{i\hbar}{2}(\varepsilon_{ijk}a_j^\dagger a_k - \varepsilon_{ikk} - \varepsilon_{ikj}a_j^\dagger a_k) \qquad \text{reindexing} \tag{8.148}$$
$$= -\frac{i\hbar}{2}(\varepsilon_{ijk}a_j^\dagger a_k + \varepsilon_{ijk}a_j^\dagger a_k) \qquad \varepsilon_{ikk} = 0 \tag{8.149}$$
$$= -i\hbar\varepsilon_{ijk}a_j^\dagger a_k \tag{8.150}$$

Now the new commutation relations

$$
\begin{aligned}
[b_0, b_0^\dagger] &= [a_3, a_3^\dagger] = 1 = \delta_{00} && (8.151)\\
[b_0, b_1^\dagger] &= -\frac{1}{\sqrt{2}}(a_3(a_1^\dagger + ia_2^\dagger) - (a_1^\dagger + ia_2^\dagger)a_3) && (8.152)\\
&= -\frac{1}{\sqrt{2}}(a_3a_1^\dagger + ia_3a_2^\dagger - a_1^\dagger a_3 - ia_2^\dagger a_3) && (8.153)\\
&= -\frac{1}{\sqrt{2}}(\delta_{12} + i\delta_{23}) && (8.154)\\
&= 0 && (8.155)\\
[b_{-1}, b_1^\dagger] &= -\frac{1}{2}((a_1 - ia_2)(a_1^\dagger - ia_2^\dagger) - (a_1^\dagger - ia_2^\dagger)(a_1 - ia_2)) && (8.156)\\
&= -\frac{1}{2}(a_1a_1^\dagger - ia_2a_1^\dagger - ia_1a_2^\dagger - a_2a_2^\dagger - a_1^\dagger a_1 + ia_1^\dagger a_2 + ia_2^\dagger a_1 + a_2^\dagger a_2) && (8.157)\\
&= -\frac{1}{2}(1 - i\cdot 0 - i\cdot 0 - 1) && (8.158)\\
&= 0 && (8.159)\\
&= \delta_{-1,1} && (8.160)\\
... & && (8.161)
\end{aligned}
$$

Now the Hamiltonian with

$$
\begin{aligned}
b_{-1}^\dagger b_{-1} + b_1^\dagger b_1 &= \frac{1}{2}(a_1^\dagger - ia_2^\dagger)(a_1 + ia_2) + \frac{1}{2}(a_1^\dagger + ia_2^\dagger)(a_1 - ia_2) && (8.162)\\
&= \frac{1}{2}(a_1^\dagger a_1 - ia_2^\dagger a_1 + ia_1^\dagger a_2 + a_2^\dagger a_2) + \frac{1}{2}(a_1^\dagger a_1 + ia_2^\dagger a_1 - ia_1^\dagger a_2 + a_2^\dagger a_2) && (8.163)\\
&= a_1^\dagger a_1 + a_2^\dagger a_2 && (8.164)
\end{aligned}
$$

and $b_0^\dagger b_0 = a_3^\dagger a_3$ we have $H = \hbar\omega \sum(1/2 + b_m^\dagger b_m)$. While

$$
\begin{aligned}
-b_{-1}^\dagger b_{-1} + b_1^\dagger b_1 &= -\frac{1}{2}(a_1^\dagger - ia_2^\dagger)(a_1 + ia_2) + \frac{1}{2}(a_1^\dagger + ia_2^\dagger)(a_1 - ia_2) && (8.165)\\
&= -\frac{1}{2}(a_1^\dagger a_1 - ia_2^\dagger a_1 + ia_1^\dagger a_2 + a_2^\dagger a_2) + \frac{1}{2}(a_1^\dagger a_1 + ia_2^\dagger a_1 - ia_1^\dagger a_2 + a_2^\dagger a_2) && (8.166)\\
&= ia_2^\dagger a_1 - ia_1^\dagger a_2 && (8.167)\\
&= -i(-a_2^\dagger a_1 + a_1^\dagger a_2) && (8.168)
\end{aligned}
$$

gives $L^3 = \hbar \sum_m m b_m^\dagger b_m$.

8.2 van Baal - A Course in Field Theory

8.2.1 Problem 1. Violation of causality in 1+1 dimensions

(a) With $H^2 = m^2c^4 + p^2c^2$ and $p = -i\hbar\partial_x$

$$
\begin{aligned}
H\psi(x,t) &= i\hbar\partial_t\psi(x,t) && (8.169)\\
H^2\psi(x,t) &= -\hbar^2\partial_{tt}\psi(x,t) && (8.170)\\
\left(\partial_{xx} - \frac{1}{c^2}\partial_{tt} - \frac{m^2c^2}{\hbar^2}\right)\psi(x,t) &= 0 && (8.171)\\
\left(\Box_x - \frac{m^2c^2}{\hbar^2}\right)\psi(x,t) &= 0 && (8.172)
\end{aligned}
$$

then we try the plane wave ansatz $\psi_k(x,t) = e^{-i(\omega_k t - kx)}$ and see

$$-k^2 + \frac{1}{c^2}\omega_k^2 - \frac{m^2c^2}{\hbar^2} = 0 \tag{8.173}$$

$$\rightarrow \omega_k^2 = k^2c^2 + \frac{m^2c^4}{\hbar^2} \rightarrow \omega_k = \sqrt{k^2c^2 + \frac{m^2c^4}{\hbar^2}}. \tag{8.174}$$

Therefore the general solution is a superposition

$$\psi(x,t) = \int dk f(k)e^{-i(\omega_k t - kx)} + g(k)e^{-i(-\omega_k t - kx)} \tag{8.175}$$

(b) Assume $\psi_0(x,t)$ is a solution then $\psi_0(x-y,t)$ is also a solution

$$\left(\Box_x - \frac{m^2c^2}{\hbar^2}\right)\psi_0(x,t) = 0 \tag{8.176}$$

$$\rightarrow \left(\Box_x - \frac{m^2c^2}{\hbar^2}\right)\psi_0(x-y,t) = 0 \tag{8.177}$$

then with $\psi(x,t) = \int dy\, f(y)\psi_0(x-y,t)$

$$\left(\Box_x - \frac{m^2c^2}{\hbar^2}\right)\psi(x,t) = \int dy\, f(y)\left(\Box_x - \frac{m^2c^2}{\hbar^2}\right)\psi_0(x-y,t) \tag{8.178}$$

$$= 0 \tag{8.179}$$

and

$$\psi(x,0) = \lim_{t\to 0}\int dy\, f(y)\psi_0(x-y,t) \tag{8.180}$$

$$= \int dy\, f(y)\delta(x-y) \tag{8.181}$$

$$= f(x) \tag{8.182}$$

Now we can use the time propagation operator

$$\psi_0(x,t) = e^{-iHt/\hbar}\psi(x,0) \tag{8.183}$$

$$= e^{-it\sqrt{p^2c^2+m^2c^4}/hbar}\delta(x) \tag{8.184}$$

$$= \frac{1}{2\pi\hbar}\int dp\, e^{-it\frac{mc^2}{\hbar}\sqrt{\frac{p^2}{m^2c^2}+1}}e^{ipx/\hbar} \tag{8.185}$$

and use $\cosh^2 u - \sinh^2 u = 1$ and

$$p = mc\sinh u \tag{8.186}$$

$$dp = mc\cosh u\, du \tag{8.187}$$

then

$$\psi_0(x,t) = \frac{mc}{2\pi\hbar}\int du\, e^{-it\frac{mc^2}{\hbar}\sqrt{\sinh^2 u+1}}e^{i\frac{mc}{\hbar}x\sinh u}\cosh u \tag{8.188}$$

$$= \frac{mc}{2\pi\hbar}\int du\, e^{-it\frac{mc^2}{\hbar}\cosh u}e^{i\frac{mc}{\hbar}x\sinh u}\cosh u \tag{8.189}$$

$$= \frac{mc}{2\pi\hbar}\int du\, e^{i\frac{mc}{\hbar}(x\sinh u - ct\cosh u)}\cosh u \tag{8.190}$$

$$= \frac{i}{2\pi c}\partial_t\int du\, e^{i\frac{mc}{\hbar}(x\sinh u - ct\cosh u)}. \tag{8.191}$$

Now we replace x, t by new coordinates v and z

$$x = \frac{\hbar}{mc} z \cosh v \tag{8.192}$$
$$ct = \frac{\hbar}{mc} z \sinh v \tag{8.193}$$
$$\rightarrow x^2 - c^2t^2 = \frac{\hbar^2}{m^2c^2} z^2 \tag{8.194}$$

then we obtain with $y = u - v$

$$\psi_0(x,t) = \frac{i}{2\pi c}\partial_t \int du\, e^{iz(\cosh v \sinh u - \sinh v \cosh u)} \tag{8.195}$$
$$= \frac{i}{2\pi c}\partial_t \int du\, e^{iz\sinh(u-v)} \tag{8.196}$$
$$= \frac{i}{2\pi c}\partial_t \int du\, \left[\cos(z\sinh(u-v)) + i\sin(z\sinh(u-v))\right] \tag{8.197}$$
$$= \frac{i}{2\pi c}\partial_t \int dy\, \left[\cos(z\sinh y) + i\sin(z\sinh y)\right] \tag{8.198}$$
$$= \frac{i}{2\pi c}\partial_t \int_{-\infty}^{\infty} dy\, \cos(z\sinh y) \tag{8.199}$$
$$= \frac{i}{\pi c}\partial_t \int_{0}^{\infty} dy\, \cos(z\sinh y) \tag{8.200}$$

(c)

(d)

8.3 Nastase - Introduction to Quantum Field Theory

8.3.1 Exercise 1.4 Scalar Dirac–Born–Infeld equations of motion

With

$$\frac{\partial(\partial_\mu\phi)^2}{\partial_\nu\phi} = \frac{\partial(\partial_\mu\phi\partial^\mu\phi)}{\partial(\partial_\nu\phi)} \tag{8.201}$$
$$= \frac{\partial(\eta^{\mu\alpha}\partial_\mu\phi\partial_\alpha\phi)}{\partial(\partial_\nu\phi)} \tag{8.202}$$
$$= \eta^{\mu\alpha}\frac{\partial(\partial_\mu\phi\partial_\alpha\phi)}{\partial(\partial_\nu\phi)} \tag{8.203}$$
$$= \eta^{\mu\alpha}(\delta_{\mu\nu}\partial_\alpha\phi + \partial_\mu\phi\delta_{\alpha\nu}) \tag{8.204}$$
$$= \eta^{\mu\alpha}\delta_{\mu\nu}\partial_\alpha\phi + \eta^{\mu\alpha}\delta_{\alpha\nu}\partial_\mu\phi \tag{8.205}$$
$$= \delta^\alpha_\nu\partial_\alpha\phi + \delta^\mu_\nu\partial_\mu\phi \tag{8.206}$$
$$= 2\partial_\nu\phi \tag{8.207}$$

we can calculate the parts for the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \phi} = -\frac{1}{L^4}\frac{L^4\left[\frac{\partial g}{\partial \phi}(\partial_\mu\phi)^2 + 2m^2\phi\right]}{2\sqrt{1+L^4[g(\partial_\mu\phi)^2+m^2\phi^2]}} \tag{8.208}$$

$$= -\frac{\left[\frac{\partial g}{\partial \phi}(\partial_\mu\phi)^2 + 2m^2\phi\right]}{2\sqrt{1+L^4[g(\partial_\mu\phi)^2+m^2\phi^2]}} \tag{8.209}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} = -\frac{1}{L^4}\frac{L^4\left[2g(\partial_\mu\phi)\delta^\mu_\nu\right]}{2\sqrt{1+L^4[g(\partial_\mu\phi)^2+m^2\phi^2]}} \tag{8.210}$$

$$= -\frac{g(\partial_\nu\phi)}{\sqrt{1+L^4[g(\partial_\mu\phi)^2+m^2\phi^2]}} \tag{8.211}$$

$$\partial_\nu\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} = -\frac{g(\partial_\nu\partial_\nu\phi)\sqrt{1+L^4[g(\partial_\mu\phi)^2+m^2\phi^2]} - g(\partial_\nu\phi)\frac{L^4[2g(\partial_\mu\phi)(\partial_\nu\partial_\mu\phi)+2m^2\phi\partial_\nu\phi]}{2\sqrt{1+L^4[g(\partial_\mu\phi)^2+m^2\phi^2]}}}{1+L^4[g(\partial_\mu\phi)^2+m^2\phi^2]} \tag{8.212}$$

Multiplying the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\nu\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} = 0 \tag{8.213}$$

by $\sqrt{1+L^4[g(\partial_\mu\phi)^2+m^2\phi^2]}$ we obtain

$$-\frac{1}{2}\left[\frac{\partial g}{\partial \phi}(\partial_\mu\phi)^2 + 2m^2\phi\right] + g(\partial_\nu\partial_\nu\phi) + \frac{1}{2}g(\partial_\nu\phi)\frac{L^4[2g(\partial_\mu\phi)(\partial_\nu\partial_\mu\phi)+2m^2\phi\partial_\nu\phi]}{1+L^4[g(\partial_\mu\phi)^2+m^2\phi^2]} = 0 \tag{8.214}$$

$$g(\Box\phi - m^2\phi) - \frac{1}{2}\frac{\partial g}{\partial\phi}(\partial_\mu\phi)^2 + gL^4\frac{g(\partial_\nu\phi)(\partial_\mu\phi)(\partial_\nu\partial_\mu\phi)+m^2\phi(\partial_\nu\phi)^2}{1+L^4[g(\partial_\mu\phi)^2+m^2\phi^2]} = 0 \tag{8.215}$$

8.3.2 Exercise 2.1 Equations of motion for an anharmonic

With

$$p = \frac{\partial L}{\partial \dot q} = \dot q \tag{8.216}$$

$$H = p\dot q - L \tag{8.217}$$

$$= p^2 - \frac{p^2}{2} + \frac{\lambda}{4!}q^4 \tag{8.218}$$

$$= \frac{p^2}{2} + \frac{\lambda}{4!}q^4 \tag{8.219}$$

$$\tag{8.220}$$

then

$$\dot p = -\frac{\partial H}{\partial q} = -\frac{\lambda}{3!}q^3 \tag{8.221}$$

$$\dot q = \frac{\partial H}{\partial p} = p \tag{8.222}$$

Phase space path integral

$$M(q',t';q,t) = \mathcal{D}p(t)\mathcal{D}q(t)\exp\left\{i\int_t^{t'} dt[p(t)\dot q(t) - H(p(t),q(t))]\right\} \tag{8.223}$$

$$= \mathcal{D}p(t)\mathcal{D}q(t)\exp\left\{i\int_t^{t'} dt[p(t)\dot q(t) - \frac{p(t)^2}{2} - \frac{\lambda}{4!}q(t)^4]\right\} \tag{8.224}$$

8.4 Straumann - Relativistische Quantentheorie

8.4.1 Problem 1.11.1. Momentum and angular momentum of the radiation field

$$\mathbf{P} = \frac{1}{4\pi c}\int_V \mathbf{E}\times\mathbf{B}\,d^3x \tag{8.225}$$

$$\mathbf{J} = \frac{1}{4\pi c}\int_V [\mathbf{x}\times(\mathbf{E}\times\mathbf{B})]\,d^3x \tag{8.226}$$

In Coulomb gauge we have

$$\mathbf{E} = -\frac{1}{c}\partial_t\mathbf{A} = -\frac{1}{c}\dot{A}_l\mathbf{e}_l \tag{8.227}$$

$$\mathbf{B} = \nabla\times\mathbf{A} = \varepsilon_{ijk}(\partial_j A_k)\mathbf{e}_i \tag{8.228}$$

$$\mathbf{E}\times\mathbf{B} = -\frac{1}{c}\varepsilon_{nli}\mathbf{e}_n(\dot{A}_l\mathbf{e}_l)(\varepsilon_{ijk}(\partial_j A_k)\mathbf{e}_i) \tag{8.229}$$

$$= -\frac{1}{c}\varepsilon_{nli}\mathbf{e}_n\dot{A}_l\varepsilon_{ijk}(\partial_j A_k)\mathbf{e}_i\mathbf{e}_l \tag{8.230}$$

$$= -\frac{1}{c}\varepsilon_{nli}\mathbf{e}_n\dot{A}_l\varepsilon_{ijk}(\partial_j A_k)\delta_{il} \tag{8.231}$$

$$= -\frac{1}{c}\varepsilon_{nli}\varepsilon_{ijk}(\partial_j A_k)\dot{A}_i\mathbf{e}_n \tag{8.232}$$

$$= -\frac{1}{c}(\delta_{nj}\delta_{lk} - \delta_{nk}\delta_{lj})(\partial_j A_k)\dot{A}_l\mathbf{e}_n \tag{8.233}$$

$$= -\frac{1}{c}((\partial_j A_k)\dot{A}_k\mathbf{e}_j - (\partial_j A_k)\dot{A}_j\mathbf{e}_k) \tag{8.234}$$

$$= -\frac{1}{c}((\mathbf{e}_j\partial_j A_k)\dot{A}_k - \dot{A}_j(\partial_j A_k)\mathbf{e}_k) \tag{8.235}$$

$$= -\frac{1}{c}[\nabla(\mathbf{A}\cdot\dot{\mathbf{A}}) - (\dot{\mathbf{A}}\cdot\nabla)\mathbf{A}] \tag{8.236}$$

And from (1.44) and (1.33)

$$\mathbf{A}(x,t) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k},\lambda}\sqrt{\frac{2\pi\hbar c^3}{\omega_k}}\left[a_{\mathbf{k},\lambda}\boldsymbol{\varepsilon}(k,\lambda)e^{i\mathbf{k}\cdot\mathbf{x}} + a^*_{\mathbf{k},\lambda}\boldsymbol{\varepsilon}(k,\lambda)^*e^{-i\mathbf{k}\cdot\mathbf{x}}\right] \tag{8.237}$$

$$= \sum_{\mathbf{k},\lambda}\sqrt{\frac{2\pi\hbar c^3}{\omega_k}}\left[a_{\mathbf{k},\lambda}\mathbf{u}_{\mathbf{k},\lambda}(\mathbf{x}) + a^*_{\mathbf{k},\lambda}\mathbf{u}^*_{\mathbf{k},\lambda}(\mathbf{x})\right] \tag{8.238}$$

$$\tag{8.239}$$

8.4.2 Problem 4.5.1. Approximation for polarization potential

$$\Phi^{\mathrm{Pol}}(\mathbf{x}) = \frac{e}{(2\pi)^3}\int d^3k e^{i\mathbf{k}\cdot\mathbf{x}}\int_{4m^2}^{\infty} d\kappa^2\frac{\Pi(x^2)}{\kappa^2(\kappa^2+\mathbf{k}^2)} \tag{8.240}$$

8.5 RAMOND - Field Theory - A modern primer

8.5.1 Problem 1.1.1 A

With

$$\left(\frac{d(x+\delta x)}{dt}\right)^2 = \left(\frac{dx}{dt}+\delta\frac{dx}{dt}\right)\left(\frac{dx}{dt}+\delta\frac{dx}{dt}\right) \tag{8.241}$$

$$= \left(\frac{dx}{dt}\right)^2 + 2\frac{dx}{dt}\cdot\delta\frac{dx}{dt} + \left(\delta\frac{dx}{dt}\right)^2 \tag{8.242}$$

$$= \left(\frac{dx}{dt}\right)^2 + \frac{d}{dt}\left(\frac{dx}{dt}\delta x\right) - 2\frac{d^2x}{dt^2}\delta x + \left(\delta\frac{dx}{dt}\right)^2 \tag{8.243}$$

where we integrates the second term by parts. Now we can expand the action

$$S = \int dt\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 \tag{8.244}$$

$$S[x+\delta x] = \int dt\frac{1}{2}m\left(\frac{d(x+\delta x)}{dt}\right)^2 \tag{8.245}$$

$$\delta S = -\frac{1}{2}m\int_{t_1}^{t_2} dt\delta x\left(2\frac{d^2x}{dt^2}\right) + \frac{1}{2}m\frac{dx}{dt}\delta x\Big|_{t_1}^{t_2} \tag{8.246}$$

the last (surface) term vanished and we have

$$\frac{d}{dt}\left(\frac{dx}{dt}\right) = 0 \tag{8.247}$$

8.6 MUENSTER - Von der Quantenfeldtheorie zum Standardmodell

8.6.1 Problem 2.1 - 1

(a) The Klein-Gordon equations is given by

$$\left(\partial_\mu\partial^\mu + \frac{m^2c^2}{\hbar^2}\right)\varphi = 0 \tag{8.248}$$

$$\left(c^2\partial_{tt} - \triangle + \frac{m^2c^2}{\hbar^2}\right)\varphi = 0 \tag{8.249}$$

We make the ansatz

$$\varphi = \phi_1 + \phi_2 \tag{8.250}$$

$$\phi_1 = \frac{1}{2}\varphi - \alpha\partial_t\varphi \tag{8.251}$$

$$\phi_2 = \frac{1}{2}\varphi + \alpha\partial_t\varphi \tag{8.252}$$

Then we get expressions for the time derivatives

$$\phi_2 - \phi_1 = 2\alpha\partial_t\varphi \tag{8.253}$$

$$\rightarrow \partial_t\varphi = \frac{1}{2\alpha}(\phi_2 - \phi_1) \tag{8.254}$$

and

$$\partial_{tt}\varphi = c^2\left(\triangle - \frac{m^2c^2}{\hbar^2}\right)\varphi \tag{8.255}$$
$$= c^2\left(\triangle - \frac{m^2c^2}{\hbar^2}\right)(\phi_1+\phi_2) \tag{8.256}$$

Therefore we get for $\phi_{1,2}$

$$\partial_t\phi_1 = \frac{1}{2}\partial_t\varphi - \alpha\partial_{tt}\varphi \tag{8.257}$$
$$= \frac{1}{2\alpha}(\phi_2-\phi_1) - \alpha c^2\left(\triangle - \frac{m^2c^2}{\hbar^2}\right)(\phi_1+\phi_2) \tag{8.258}$$
$$\partial_t\phi_2 = \frac{1}{2}\partial_t\varphi + \alpha\partial_{tt}\varphi \tag{8.259}$$
$$= \frac{1}{2\alpha}(\phi_2-\phi_1) + \alpha c^2\left(\triangle - \frac{m^2c^2}{\hbar^2}\right)(\phi_1+\phi_2) \tag{8.260}$$

which we can write in the form

$$i\hbar\partial_t\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = -i\hbar\alpha c^2\left(\triangle - \frac{m^2c^2}{\hbar^2}\right)\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} + \frac{i\hbar}{2\alpha}\begin{pmatrix}-1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \tag{8.261}$$
$$= i\hbar\begin{pmatrix}-\alpha c^2\left(\triangle - \frac{m^2c^2}{\hbar^2}\right) - \frac{1}{2\alpha} & -\alpha c^2\left(\triangle - \frac{m^2c^2}{\hbar^2}\right) + \frac{1}{2\alpha}\\ \alpha c^2\left(\triangle - \frac{m^2c^2}{\hbar^2}\right) - \frac{1}{2\alpha} & \alpha c^2\left(\triangle - \frac{m^2c^2}{\hbar^2}\right) + \frac{1}{2\alpha}\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \tag{8.262}$$

(b) Diagonalization gives

$$i\hbar\partial_t\phi = \hat{H}\phi \tag{8.263}$$
$$\to i\hbar\partial_t S^{-1}\phi = \underbrace{S^{-1}\hat{H}S}_{=h}S^{-1}\phi \tag{8.264}$$

$$\lambda_\pm = \pm\sqrt{2}c\hbar\sqrt{\Delta - \frac{m^2c^2}{\hbar^2}} \tag{8.265}$$
$$= \mp\sqrt{2}mc^2\sqrt{1 - \frac{\hbar^2}{m^2c^2}\triangle} \tag{8.266}$$

A semi-canonical choice for the parameter α is to make the $\triangle$ look like a momentum operator

$$i\hbar\alpha c^2 = -\frac{\hbar^2}{2m} \quad\to\quad \alpha = \frac{i\hbar}{2mc^2} \tag{8.267}$$

8.7 Peskin, Schroeder - An Introduction to Quantum Field Theory

8.7.1 Problem 2.1 - Maxwell equations

(a)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}F_{\mu\nu}F_{\alpha\beta} \tag{8.268}$$
$$= -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\alpha A_\beta - \partial_\beta A_\alpha) \tag{8.269}$$

With

$$\frac{\partial \mathcal{L}}{\partial A_\gamma} - \partial_\sigma \frac{\mathcal{L}}{\partial(\partial_\sigma A_\gamma)} = 0 \tag{8.270}$$

then

$$\frac{\mathcal{L}}{\partial(\partial_\sigma A_\gamma)} = -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\alpha A_\beta - \partial_\beta A_\alpha) - \frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\alpha A_\beta - \partial_\beta A_\alpha) \tag{8.271}$$

$$= -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}(\delta^\sigma_\mu \delta^\gamma_\nu - \delta^\sigma_\nu \delta^\gamma_\mu)(\partial_\alpha A_\beta - \partial_\beta A_\alpha) - \frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\alpha A_\beta - \partial_\beta A_\alpha) \tag{8.272}$$

$$= -\frac{1}{4}(\delta^{\alpha\sigma}\delta^{\beta\gamma} - \delta^{\beta\sigma}\delta^{\alpha\gamma})(\partial_\alpha A_\beta - \partial_\beta A_\alpha) - ... \tag{8.273}$$

$$= -\frac{1}{4}(\partial^\sigma A^\gamma - \partial^\gamma A^\sigma - \partial^\gamma A^\sigma + \partial^\sigma A^\gamma) - ... \tag{8.274}$$

$$= -\frac{1}{4}2F^{\sigma\gamma} - ... \tag{8.275}$$

$$= -F^{\sigma\gamma} \tag{8.276}$$

and therefore

$$\partial_\sigma F^{\sigma\gamma} = 0 \tag{8.277}$$

Rewriting into the common form

$$\gamma = 0 \quad \rightarrow \quad \partial_0 F^{00} + \sum_i \partial_i F^{i0} = 0 \tag{8.278}$$

$$\rightarrow \quad \sum_i \partial_i(-F^{0i}) = 0 \tag{8.279}$$

$$\rightarrow \quad \sum_i \partial_i E^i = 0 \tag{8.280}$$

$$\rightarrow \quad \nabla \cdot \mathbf{E} = 0 \tag{8.281}$$

$$\gamma = k \quad \rightarrow \quad \partial_0 F^{0k} + \sum_i \partial_i F^{ik} = 0 \tag{8.282}$$

$$\rightarrow \quad \partial_0(-E^k) + \sum_i \partial_i F^{ik} = 0 \tag{8.283}$$

$$\rightarrow \quad \partial_0(-E^k) + \sum_i \partial_i(-\epsilon_{ikm}B^m) = 0 \tag{8.284}$$

$$\rightarrow \quad \dot{\mathbf{E}} = \nabla \times \mathbf{B} \tag{8.285}$$

The other two equations come from

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{8.286}$$

$$\rightarrow \quad \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \tag{8.287}$$

(b) With the definition (2.17)

$$T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\lambda)}\partial_\nu A_\lambda - \mathcal{L}\delta^\mu_\nu \tag{8.288}$$

$$= -F^{\mu\lambda}\partial_\nu A_\lambda + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\delta^\mu_\nu \tag{8.289}$$

we rewrite

$$T^{\mu\nu} = -F^{\mu\lambda}\partial^\nu A_\lambda + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\eta^{\mu\nu} \tag{8.290}$$

$$\widehat{T}^{\mu\nu} = -F^{\mu\lambda}\partial^\nu A_\lambda + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\eta^{\mu\nu} + \partial_\lambda(F^{\mu\lambda}A^\nu) \tag{8.291}$$

$$= -F^{\mu\lambda}\partial^\nu A_\lambda + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\eta^{\mu\nu} + \underbrace{(\partial_\lambda F^{\mu\lambda})}_{=0\text{ (Maxwell)}} A^\nu + F^{\mu\lambda}(\partial_\lambda A^\nu) \tag{8.292}$$

$$= F^{\mu\lambda}F_\lambda^{\ \nu} + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\eta^{\mu\nu} \tag{8.293}$$

$$= F^{\mu\lambda}F_{\lambda\sigma}\eta^{\sigma\nu} + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\eta^{\mu\nu} \tag{8.294}$$

$$= F^{\uparrow\uparrow}F_{\downarrow\downarrow}\eta + \frac{1}{4}\mathrm{tr}(-F^{\uparrow\uparrow}F_{\downarrow\downarrow})\eta \tag{8.295}$$

and with

$$F_{\mu\nu} = F_{\downarrow\downarrow} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \qquad F^{\mu\nu} = F_{\uparrow\uparrow} = \eta F_{\downarrow\downarrow}\eta^T = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{8.296}$$

$$F_{\mu\nu}F^{\mu\nu} = -\mathrm{tr}(F_{\downarrow\downarrow}.F_{\uparrow\uparrow}) = 2(\mathbf{B}^2 - \mathbf{E}^2) \qquad F^{\mu\lambda}F_{\lambda\nu} = ... \tag{8.297}$$

we obtain

$$\widehat{T}^{\mu\nu} = \begin{pmatrix} \mathcal{E} & \mathbf{S} \\ \mathbf{S} & ... \end{pmatrix} \tag{8.298}$$

which looks symmetric.

8.7.2 Problem 2.2 - The complex scalar field

(a) Using $\partial_\mu\phi^*\partial^\mu\phi = \partial_\mu\phi^*\eta^{\mu\nu}\partial_\nu\phi = \partial^\mu\phi^*\partial_\mu\phi$ and $\partial^\mu = \eta^{\mu\nu}\partial_\nu = (\partial_0, -\partial_i)$ we find

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial\dot{\phi})} = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \partial^0\phi^* = \partial_0\phi^* = \dot{\phi}^* \tag{8.299}$$

$$\pi^* = \frac{\partial\mathcal{L}}{\partial(\partial\dot{\phi}^*)} = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi^*)} = \partial^0\phi = \partial_0\phi = \dot{\phi} \tag{8.300}$$

then

$$H = \int d^3x[\pi\dot{\phi} + \pi^*\dot{\phi}^* - \mathcal{L}] \tag{8.301}$$

$$= \int d^3x[\pi\pi^* + \pi^*\pi - \partial_\mu\phi^*\eta^{\mu\nu}\partial_\nu\phi + m^2\phi^*\phi] \tag{8.302}$$

$$= \int d^3x[\pi\pi^* + \pi^*\pi - (\underbrace{\dot{\phi}^*\dot{\phi}}_{=\pi\pi^*} - \nabla\phi^*\cdot\nabla\phi) + m^2\phi^*\phi] \tag{8.303}$$

$$= \int d^3x[\pi^*\pi + \nabla\phi^*\cdot\nabla\phi + m^2\phi^*\phi] \tag{8.304}$$

Let's rewrite the Lagrangian with $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi \tag{8.305}$$

$$= \frac{1}{2}\partial_\mu(\phi_1 - i\phi_2)\partial^\mu(\phi_1 + i\phi_2) - \frac{1}{2}m^2(\phi_1 - i\phi_2)(\phi_1 + i\phi_2) \tag{8.306}$$

$$= \frac{1}{2}(\partial_\mu\phi_1\partial^\mu\phi_1 - m^2\phi_1^2) + i\frac{1}{2}(\partial_\mu\phi_2\partial^\mu\phi_2 - m^2\phi_2^2) \tag{8.307}$$

So we use the results for the scalar field

$$\phi_1(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_\mathbf{p}}} \left(a_\mathbf{p}e^{i\mathbf{p}\cdot\mathbf{x}} + a_\mathbf{p}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right) \tag{8.308}$$

$$\pi_1(\mathbf{x}) = -i\int \frac{d^3p}{(2\pi)^3\sqrt{2}}\sqrt{\omega_\mathbf{p}} \left(a_\mathbf{p}e^{i\mathbf{p}\cdot\mathbf{x}} + a_\mathbf{p}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right) \tag{8.309}$$

$$\phi_2(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_\mathbf{p}}} \left(b_\mathbf{p}e^{i\mathbf{p}\cdot\mathbf{x}} + b_\mathbf{p}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right) \tag{8.310}$$

$$\pi_2(\mathbf{x}) = -i\int \frac{d^3p}{(2\pi)^3\sqrt{2}}\sqrt{\omega_\mathbf{p}} \left(b_\mathbf{p}e^{i\mathbf{p}\cdot\mathbf{x}} + b_\mathbf{p}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right) \tag{8.311}$$

$$[a_\mathbf{p}, a_\mathbf{q}^\dagger] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) \tag{8.312}$$

$$[b_\mathbf{p}, b_\mathbf{q}^\dagger] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) \tag{8.313}$$

then

$$\phi(\mathbf{x}) = \frac{1}{\sqrt{2}}\int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_\mathbf{p}}} \left((a_\mathbf{p}+ib_\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}} + (a_\mathbf{p}^\dagger + ib_\mathbf{p}^\dagger)e^{-i\mathbf{p}\cdot\mathbf{x}}\right) \tag{8.314}$$

$$\equiv \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_\mathbf{p}}} \left(\alpha_\mathbf{p}e^{i\mathbf{p}\cdot\mathbf{x}} + \beta_\mathbf{p}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right) \tag{8.315}$$

$$\phi^\dagger(\mathbf{x}) = \frac{1}{\sqrt{2}\int \frac{d^3p}{(2\pi)^3}\sqrt{2\omega_\mathbf{p}}} \left((a_\mathbf{p}^\dagger - ib_\mathbf{p}^\dagger)e^{-i\mathbf{p}\cdot\mathbf{x}} + (a_\mathbf{p}-ib_\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}\right) \tag{8.316}$$

$$\equiv \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_\mathbf{p}}} \left(\alpha_\mathbf{p}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} + \beta_\mathbf{p} e^{i\mathbf{p}\cdot\mathbf{x}}\right) \tag{8.317}$$

With the new defines creation/annihilation operators

$$\alpha_\mathbf{p} = \frac{1}{\sqrt{2}}(a_\mathbf{p}+ib_\mathbf{p}) \quad\rightarrow\quad \alpha_\mathbf{p}^\dagger = \frac{1}{\sqrt{2}}(a_\mathbf{p}^\dagger - ib_\mathbf{p}^\dagger) \tag{8.318}$$

$$\beta_\mathbf{p} = \frac{1}{\sqrt{2}}(a_\mathbf{p}-ib_\mathbf{p}) \quad\rightarrow\quad \beta_\mathbf{p}^\dagger = \frac{1}{\sqrt{2}}(a_\mathbf{p}^\dagger + ib_\mathbf{p}^\dagger) \tag{8.319}$$

we can calculate their commutation relations (**assuming all the cross commutators between $a, a^\dagger$ and $b, b^\dagger$ are zero**)

$$[\alpha_\mathbf{p}, \alpha_\mathbf{q}] = \frac{1}{2}[a_\mathbf{p}+ib_\mathbf{p}, a_\mathbf{q}+ib_\mathbf{q}] \tag{8.320}$$

$$= \frac{1}{2}([a_\mathbf{p}, a_\mathbf{q}] + i[b_\mathbf{p}, a_\mathbf{q}] + i[a_\mathbf{p}, b_\mathbf{q}] - [b_\mathbf{p}, b_\mathbf{q}]) \tag{8.321}$$

$$= \frac{1}{2}i([b_\mathbf{p}, a_\mathbf{q}] + [a_\mathbf{p}, b_\mathbf{q}]) \tag{8.322}$$

$$= 0 \tag{8.323}$$

$$[\alpha_\mathbf{p}^\dagger, \alpha_\mathbf{q}^\dagger] = \frac{1}{2}([a_\mathbf{p}^\dagger - ib_\mathbf{p}^\dagger, a_\mathbf{q}^\dagger - ib_\mathbf{q}^\dagger]) \tag{8.324}$$

$$= \frac{1}{2}([a_\mathbf{p}^\dagger, a_\mathbf{q}^\dagger] - i[b_\mathbf{p}^\dagger, a_\mathbf{q}^\dagger] - i[a_\mathbf{p}^\dagger, b_\mathbf{q}^\dagger] - [b_\mathbf{p}^\dagger, b_\mathbf{q}^\dagger]) \tag{8.325}$$

$$= \frac{1}{2}(-i[b_\mathbf{p}^\dagger, a_\mathbf{q}^\dagger] - i[a_\mathbf{p}^\dagger, b_\mathbf{q}^\dagger]) \tag{8.326}$$

$$= 0 \tag{8.327}$$

$$
\begin{aligned}
[\alpha_{\mathbf{p}}, \alpha_{\mathbf{q}}^\dagger] &= \frac{1}{2}[a_{\mathbf{p}} + ib_{\mathbf{p}}, a_{\mathbf{q}}^\dagger - ib_{\mathbf{q}}^\dagger] && (8.328)\\
&= \frac{1}{2}([a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] + i[b_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] - i[a_{\mathbf{p}}, b_{\mathbf{q}}^\dagger] + [b_{\mathbf{p}}, b_{\mathbf{q}}^\dagger]) && (8.329)\\
&= (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) + i[b_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] - i[a_{\mathbf{p}}, b_{\mathbf{q}}^\dagger] && (8.330)\\
&= (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) && (8.331)
\end{aligned}
$$

$$
\begin{aligned}
[\beta_{\mathbf{p}}, \beta_{\mathbf{q}}^\dagger] &= \frac{1}{2}[a_{\mathbf{p}} - ib_{\mathbf{p}}, a_{\mathbf{q}}^\dagger + ib_{\mathbf{q}}^\dagger] && (8.332)\\
&= \frac{1}{2}([a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] - i[b_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] + i[a_{\mathbf{p}}, b_{\mathbf{q}}^\dagger] + [b_{\mathbf{p}}, b_{\mathbf{q}}^\dagger]) && (8.333)\\
&= (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) && (8.334)
\end{aligned}
$$

$$
\begin{aligned}
[\alpha_{\mathbf{p}}, \beta_{\mathbf{q}}] &= \frac{1}{2}[a_{\mathbf{p}} + ib_{\mathbf{p}}, a_{\mathbf{q}} - ib_{\mathbf{q}}] && (8.335)\\
&= \frac{1}{2}([a_{\mathbf{p}}, a_{\mathbf{q}}] + i[a_{\mathbf{p}}, b_{\mathbf{q}}] + i[b_{\mathbf{p}}, a_{\mathbf{q}}] - [b_{\mathbf{p}}, b_{\mathbf{q}}]) && (8.336)\\
&= 0 && (8.337)
\end{aligned}
$$

$$
\begin{aligned}
[\alpha_{\mathbf{p}}, \beta_{\mathbf{q}}^\dagger] &= \frac{1}{2}[a_{\mathbf{p}} + ib_{\mathbf{p}}, a_{\mathbf{q}}^\dagger - ib_{\mathbf{q}}^\dagger] && (8.338)\\
&= \frac{1}{2}([a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] + i[a_{\mathbf{p}}, b_{\mathbf{q}}^\dagger] + i[b_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] - [b_{\mathbf{p}}, b_{\mathbf{q}}^\dagger]) && (8.339)\\
&= (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) - (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) && (8.340)\\
&= 0 && (8.341)\\
[\alpha_{\mathbf{p}}^\dagger, \beta_{\mathbf{q}}^\dagger] &= 0 && (8.342)
\end{aligned}
$$

As the $\phi\mathbf{x}$ is in the Schroedinger picture there is not time dependency and we can not calculate $\pi(\mathbf{x})$ - therefore we need to transform to the Heisenberg picture. To make it simple we do this first for ϕ_1 and ϕ_2 using $p \cdot x = E_p t - \mathbf{p}\cdot\mathbf{x}$ and $p^2 = E_{\mathbf{p}}^2 - \mathbf{p}^2 = m^2$ (meaning $p^0 \equiv E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$

$$
\begin{aligned}
\phi_1(x) &= e^{iHt}\phi(\mathbf{x})e^{-iHt} && (8.343)\\
&= \ldots && (8.344)\\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}(a_{\mathbf{p}}e^{-ipx} + a_{\mathbf{p}}^\dagger e^{ipx}) && (8.345)\\
\phi_2(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}(b_{\mathbf{p}}e^{-ipx} + b_{\mathbf{p}}^\dagger e^{ipx}) && (8.346)\\
& && (8.347)
\end{aligned}
$$

Here we cheated a bit - we used the result from the scalar Lagrangian - meaning using the scalar Hamiltonian. Then

$$
\begin{aligned}
\phi(x) &= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}\left(\alpha_{\mathbf{p}}e^{-ipx} + \beta_{\mathbf{p}}^\dagger e^{ipx}\right) && (8.348)\\
\phi^\dagger(x) &= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}\left(\alpha_{\mathbf{p}}^\dagger e^{ipx} + \beta_{\mathbf{p}}e^{-ipx}\right) && (8.349)
\end{aligned}
$$

and

$$\rightarrow \quad \pi^*(x) = \dot{\phi}(x) = i\int \frac{d^3p}{(2\pi)^3\sqrt{2}}\sqrt{E_{\mathbf{p}}}\left(-\alpha_{\mathbf{p}}e^{-ipx} + \beta^\dagger_{\mathbf{p}}e^{ipx}\right) \tag{8.350}$$

$$\rightarrow \quad \pi(x) = \dot{\phi}^\dagger(x) = i\int \frac{d^3p}{(2\pi)^3\sqrt{2}}\sqrt{E_{\mathbf{p}}}\left(\alpha^\dagger_{\mathbf{p}}e^{ipx} - \beta_{\mathbf{p}}e^{-ipx}\right) \tag{8.351}$$

The only non-vanishing commutator relations for field and momentum operators are

$$[\phi(\mathbf{x},t),\pi(\mathbf{y},t)] = i\int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}\int \frac{d^3q}{(2\pi)^3\sqrt{2}}\sqrt{E_{\mathbf{q}}}[\alpha_{\mathbf{p}}e^{-ipx} + \beta^\dagger_{\mathbf{p}}e^{ipx}, \alpha^\dagger_{\mathbf{q}}e^{iqy} - \beta_{\mathbf{q}}e^{-iqy}] \tag{8.352}$$

$$= i\int \frac{d^3p\, d^3q}{(2\pi)^6}\frac{1}{2}\sqrt{\frac{E_{\mathbf{q}}}{E_{\mathbf{p}}}}([\alpha_{\mathbf{p}},\alpha^\dagger_{\mathbf{q}}]e^{-ipx+iqy} - [\beta^\dagger_{\mathbf{p}},\beta_{\mathbf{q}}]e^{ipx-iqy}) \tag{8.353}$$

$$= i\int \frac{d^3p\, d^3q}{(2\pi)^6}\frac{1}{2}\sqrt{\frac{E_{\mathbf{q}}}{E_{\mathbf{p}}}}(e^{-ipx+iqy} + e^{ipx-iqy})(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) \tag{8.354}$$

$$= i\int \frac{d^3p}{(2\pi)^3}\frac{1}{2}(e^{-ip(x-y)} + e^{ip(x-y)}) \tag{8.355}$$

$$= i\delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{8.356}$$

$$[\phi^\dagger(\mathbf{x},t),\pi^\dagger(\mathbf{y},t)] = i\delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{8.357}$$

To calculate the Heisenberg equations of motion we start with

$$\nabla\phi(x) = i\int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}\mathbf{p}\left(\alpha_{\mathbf{p}}e^{-ipx} - \beta^\dagger_{\mathbf{p}}e^{ipx}\right) \tag{8.358}$$

$$\nabla\phi^\dagger(x) = i\int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}\mathbf{p}\left(-\alpha^\dagger_{\mathbf{p}}e^{ipx} + \beta_{\mathbf{p}}e^{-ipx}\right) \tag{8.359}$$

and then

$$i\dot{\phi}(x) = [\phi(x),H] = \left[\phi(x), \int d^3y(\pi^\dagger\pi + \nabla\phi^\dagger\cdot\nabla\phi + m^2\phi^\dagger\phi)\right] \tag{8.360}$$

$$= \int d^3y\pi^\dagger(y)[\phi(x),\pi(y)] \tag{8.361}$$

$$= i\pi^\dagger(x) \tag{8.362}$$

$$i\dot{\phi}^\dagger(x) = [\phi^\dagger(x),H] = \left[\phi(x), \int d^3y(\pi^\dagger\pi + \nabla\phi^\dagger\cdot\nabla\phi + m^2\phi^\dagger\phi)\right] \tag{8.363}$$

$$= \int d^3y[\phi^\dagger(x),\pi^\dagger(y)]\pi(y) \tag{8.364}$$

$$= i\pi(x) \tag{8.365}$$

and

$$i\dot{\pi}(x) = [\pi(x), H] = \left[\pi(x), \int d^3y(\pi^\dagger\pi + \nabla\phi^\dagger \cdot \nabla\phi + m^2\phi^\dagger\phi)\right] \tag{8.366}$$
$$= \left[\pi(x), \int d^3y(\pi^\dagger\pi - \triangle\phi^\dagger \cdot \phi + m^2\phi^\dagger\phi)\right] \tag{8.367}$$
$$= \int d^3y(-\triangle\phi^\dagger + m^2\phi^\dagger)[\pi(x), \phi(y)] \tag{8.368}$$
$$= i(\triangle_x - m^2)\phi^\dagger(x) \tag{8.369}$$
$$i\dot{\pi}^\dagger(x) = [\pi^\dagger(x), H] = \left[\pi^\dagger(x), \int d^3y(\pi^\dagger\pi + \nabla\phi^\dagger \cdot \nabla\phi + m^2\phi^\dagger\phi)\right] \tag{8.370}$$
$$= \left[\pi^\dagger(x), \int d^3y(\pi^\dagger\pi - \phi^\dagger \cdot \triangle\phi + m^2\phi^\dagger\phi)\right] \tag{8.371}$$
$$= \int d^3y[\pi^\dagger(x), \phi^\dagger(y)](-\triangle\phi + m^2\phi) \tag{8.372}$$
$$= i(\triangle_x - m^2)\phi(x) \tag{8.373}$$

resulting in

$$i\dot{\pi}(x) \quad \rightarrow \quad \ddot{\phi}^\dagger = (\triangle - m^2)\phi^\dagger \tag{8.374}$$
$$\rightarrow \quad (\Box + m^2)\phi^\dagger = 0 \tag{8.375}$$
$$i\dot{\pi}^\dagger(x) \quad \rightarrow \quad \ddot{\phi} = (\triangle - m^2)\phi \tag{8.376}$$
$$\rightarrow \quad (\Box + m^2)\phi = 0 \tag{8.377}$$

(b)

(c)

(d)

8.7.3 Problem 2.3 - Calculating $D(x-y)$

As we are calculation the vacuum expectation value we need to get the $a^\dagger$'s to the right and the a's to the left

$$\phi(x)\phi(y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}(a_\mathbf{p}e^{-ipx} + a_\mathbf{p}^\dagger e^{ipx}) \int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{q}}}(a_\mathbf{q}e^{-iqy} + a_\mathbf{q}^\dagger e^{iqy}) \tag{8.378}$$
$$= \iint \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{q}}}\frac{1}{\sqrt{2E_\mathbf{p}}}(a_\mathbf{p}e^{-ipx} + a_\mathbf{p}^\dagger e^{ipx})(a_\mathbf{q}e^{-iqy} + a_\mathbf{q}^\dagger e^{iqy}) \tag{8.379}$$
$$= \iint \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{q}}}\frac{1}{\sqrt{2E_\mathbf{p}}}(a_\mathbf{p}a_\mathbf{q}e^{-ipx-iqy} + a_\mathbf{p}^\dagger a_\mathbf{q}e^{ipx-iqy} + a_\mathbf{p}a_\mathbf{q}^\dagger e^{-ipx+iqy} + a_\mathbf{p}^\dagger a_\mathbf{q}^\dagger e^{ipx+iqy}) \tag{8.380}$$
$$= \iint \frac{d^3p\, d^3q}{(2\pi)^6}\frac{1}{\sqrt{4E_\mathbf{q}E_\mathbf{p}}}(a_\mathbf{p}a_\mathbf{q}e^{-ipx-iqy} + (a_\mathbf{q}a_\mathbf{p}^\dagger - (2\pi)^3\delta(\mathbf{q}-\mathbf{p}))e^{ipx-iqy} + a_\mathbf{p}a_\mathbf{q}^\dagger e^{-ipx+iqy} + a_\mathbf{p}^\dagger a_\mathbf{q}^\dagger e^{ipx+iqy}) \tag{8.381}$$

then with $a^\dagger|0\rangle = 0$ and $\langle 0|a = 0$

$$\langle 0|\phi(x)\phi(y)|\rangle = \iint \frac{d^3p\, d^3q}{(2\pi)^6}\frac{1}{\sqrt{4E_{\mathbf{q}}E_{\mathbf{p}}}}(((\langle 0|a_{\mathbf{q}}a^\dagger_{\mathbf{p}}|0\rangle - \langle 0|0\rangle(2\pi)^3\delta(\mathbf{q}-\mathbf{p}))e^{ipx-iqy} + \langle 0|a_{\mathbf{p}}a^\dagger_{\mathbf{q}}|0\rangle e^{-ipx+iqy}) \tag{8.382}$$

$$= \iint \frac{d^3p\, d^3q}{(2\pi)^6}\frac{1}{\sqrt{4E_{\mathbf{q}}E_{\mathbf{p}}}}\left(\left(\frac{\langle \mathbf{q}|\mathbf{p}\rangle}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} - (2\pi)^3\delta(\mathbf{q}-\mathbf{p})\right)e^{ipx-iqy} + \frac{\langle \mathbf{p}|\mathbf{q}\rangle}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}e^{-ipx+iqy}\right) \tag{8.383}$$

$$= \iint \frac{d^3p\, d^3q}{(2\pi)^6}\frac{1}{\sqrt{4E_{\mathbf{q}}E_{\mathbf{p}}}}\left(\left(\frac{2E_{\mathbf{p}}(2\pi)^3\delta^{(3)}(\mathbf{q}-\mathbf{p})}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} - (2\pi)^3\delta(\mathbf{q}-\mathbf{p})\right)e^{ipx-iqy} + \frac{2E_{\mathbf{p}}(2\pi)^3\delta^{(3)}(\mathbf{q}-\mathbf{p})}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}e^{-i \tag{8.384}$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{4E^2_{\mathbf{p}}}}\left(\underbrace{\left(\frac{2E_{\mathbf{p}}}{\sqrt{4E^2_{\mathbf{p}}}} - 1\right)}_{=0}e^{ipx-ipy} + \frac{2E_{\mathbf{p}}}{\sqrt{4E^2_{\mathbf{p}}}}e^{-ipx+ipy}\right) \tag{8.385}$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ip(x-y)} \tag{8.386}$$

Now we can calculate with $x^0 - y^0 = 0$ and $\mathbf{x}-\mathbf{y} = \mathbf{r}$

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ip(x-y)} \tag{8.387}$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-i(E_p(x^0-y^0)-\mathbf{p}\cdot(\mathbf{x}-\mathbf{y}))} \tag{8.388}$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \tag{8.389}$$

transforming to spherical coordinates

$$D(x-y) = 2\pi\int_0^\infty \frac{p^2dp}{(2\pi)^3}\frac{1}{2\sqrt{p^2+m^2}}\int \sin\theta\, e^{ipr\cos\theta}d\theta \tag{8.390}$$

$$= 2\pi\int_0^\infty \frac{p^2dp}{(2\pi)^3}\frac{1}{2\sqrt{p^2+m^2}}\left[\frac{1}{(-ipr)}e^{ipr\cos\theta}\right]_0^\pi \tag{8.391}$$

$$= 2\pi\int_0^\infty \frac{p^2dp}{(2\pi)^3}\frac{1}{2\sqrt{p^2+m^2}}\frac{1}{(-ipr)}(e^{-ipr} - e^{ipr}) \tag{8.392}$$

$$= \frac{i}{2(2\pi)^2r}\int_0^\infty \frac{p\,dp}{\sqrt{p^2+m^2}}(e^{-ipr} - e^{ipr}) \tag{8.393}$$

$$= \frac{i}{2(2\pi)^2r}\left(\int_0^\infty \frac{p\,dp}{\sqrt{p^2+m^2}}e^{-ipr} - \int_0^\infty \frac{p\,dp}{\sqrt{p^2+m^2}}e^{ipr}\right) \tag{8.394}$$

$$= \frac{i}{2(2\pi)^2r}\left(\int_0^\infty \frac{p\,dp}{\sqrt{p^2+m^2}}e^{-ipr} - \int_0^{-\infty} \frac{(-p)\,(-dp)}{\sqrt{(-p)^2+m^2}}e^{i(-p)r}\right) \tag{8.395}$$

$$= \frac{i}{2(2\pi)^2r}\int_{-\infty}^\infty \frac{p\,dp}{\sqrt{p^2+m^2}}e^{-ipr} \tag{8.396}$$

$$= \frac{-i}{2(2\pi)^2r}\int_{-\infty}^\infty \frac{p\,dp}{\sqrt{p^2+m^2}}e^{ipr} \qquad (r\to -r) \tag{8.397}$$

Let's use contour integration (closing the contour above - $\lim_{p\to i\infty} e^{ipr} = e^{-\infty r} = 0$ so the upper half circle integral vanishes). Furthermore we see that the square root becomes zero at $\pm im$.

8.7.4 Problem 3.1 - Lorentz group

$$J_+^i = \frac{1}{2}\left(\frac{1}{2}\epsilon^{ijk}J^{jk} + iJ^{0i}\right) \tag{8.398}$$

8.8 Schwartz - Quantum Field Theory and the Standard Model

8.8.1 Problem 2.2 Special relativity and colliders

1. Quick special relativity recap

$$p'^\mu = \Lambda^\mu_\nu p^\nu \quad p^\mu p_\mu = m^2c^2 \tag{8.399}$$

At rest

$$p^\mu p_\mu = (p^0)^2 - \vec{p}^2 = (p^0)^2 = m^2c^2 \tag{8.400}$$

After Lorentz trafo in x direction

$$\Lambda = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{8.401}$$

$$\begin{aligned} p'^\mu &= (\gamma p^0, -\beta\gamma p^0, 0, 0) && (8.402)\\ &\equiv \left(\frac{E}{c}, \vec{p}\right) && (8.403) \end{aligned}$$

with $p^\mu p_\mu = m^2c^2$ we have $E^2/c^2 + \vec{p}^2 = m^2c^2$.

Now we can solve the problem

$$\begin{aligned} \frac{E_{cm}}{2} &= \sqrt{m_p^2c^4 + p^2c^2} && (8.404)\\ \to p &= \frac{1}{c}\sqrt{\frac{E_{cm}^2}{4} - m_p^2c^4} \equiv \beta\gamma m_p c && (8.405)\\ \to \frac{E_{cm}^2}{4} &= m_p^2c^4(\beta^2\gamma^2 + 1) && (8.406)\\ \to \gamma &= \frac{E_{cm}}{2m_pc^2} && (8.407)\\ \to \beta &= \sqrt{1 - \left(\frac{2m_pc}{E_{cm}}\right)^2} \approx 1 - \frac{1}{2}\left(\frac{2m_pc^2}{E_{cm}}\right)^2 && (8.408)\\ \to c - v &= 2\left(\frac{m_pc^2}{E_{cm}}\right)^2 c = 2.69\text{m/s} && (8.409) \end{aligned}$$

2. Using the velocity addition formula

$$\Delta v = \frac{2v}{1+\frac{v^2}{c^2}} \approx c\left(1 - 2\left[\frac{m_pc^2}{E_{cm}}\right]^4\right) \tag{8.410}$$

8.8.2 Problem 2.3 GZK bound

1. We are utilizing Plancks law

$$w_\nu d\nu = \frac{8\pi h\nu^3}{c^3}\frac{d\nu}{e^{h\nu/k_BT}-1} \tag{8.411}$$

where the spectral energy density w_ν [J m^{-3} s] gives the spacial energy density per frequency interval $d\nu$. The total radiative energy density is then given by

$$\begin{aligned}
\rho_{\text{rad}} &= \frac{8\pi h}{c^3}\int_0 \infty \frac{\nu^3 d\nu}{e^{h\nu/k_BT}-1} &(8.412)\\
&= \frac{8\pi h}{c^3}\cdot\frac{(\pi k_BT)^4}{15h^4} &(8.413)\\
&= \frac{8\pi^5 k_B^4T^4}{15h^3c^3} = 0.26\text{MeV/m}^3. &(8.414)
\end{aligned}$$

The photon density is given by

$$\begin{aligned}
n_{\text{rad}} &= \int_0^\infty \frac{w_\nu}{h\nu}d\nu &(8.415)\\
&= \frac{8\pi}{c^3}\int\frac{\nu^2 d\nu}{e^{h\nu/k_BT}-1} &(8.416)\\
&= \frac{8\pi}{c^3}\cdot\frac{2\zeta(3)k_B^3T^3}{h^3} &(8.417)\\
&= \frac{16\pi\zeta(3)k_B^3T^3}{h^3c^3} = 416\text{cm}^{-3}. &(8.418)
\end{aligned}$$

The average photon energy is then given by

$$\begin{aligned}
E_{\text{ph}} &= \frac{\rho_{\text{rad}}}{n_{\text{rad}}} = \frac{\pi^4}{30\zeta(3)}k_BT = 0.63\text{meV} &(8.419)\\
\lambda_{\text{ph}} &= \frac{hc}{E_{\text{ph}}} = 1.9\text{mm} &(8.420)
\end{aligned}$$

therefore it is called CM(icrowave)B. One obtains slightly other values if the peak of the Planck spectrum is used as definition of the average photon energy.

2. In the center-of-mass system the total momentum before and after the collision vanishes

$$\vec{p}^{cm}_{p^+} + \vec{p}^{cm}_\gamma = 0 = \vec{\hat{p}}^{cm}_{p^+} + \vec{\hat{p}}^{cm}_{\pi^0}. \tag{8.421}$$

which implies for (Lorentz-invariant) norm the systems 4-momentum $P^{cm} = p^{cm}_{p^+} + p^{cm}_{\pi^0}$

$$\begin{aligned}
(P^{cm})^2 &= (E^{cm}_{p^+} + E^{cm}_\gamma)^2 - c^2(\vec{p}^{cm}_{p^+} + \vec{p}^{cm}_\gamma)^2 &(8.422)\\
&= (E^{cm}_{p^+} + E^{cm}_\gamma)^2 &(8.423)\\
&= (E^{cm})^2 &(8.424)\\
&\stackrel{!}{=} (E_{p^+} + E_\gamma)^2 - c^2(\vec{p}_{p^+} + \vec{p}_\gamma)^2 &(8.425)\\
&\stackrel{!}{=} (\hat{E}_{p^+} + \hat{E}_{\pi^0})^2 - c^2(\vec{\hat{p}}_{p^+} + \vec{\hat{p}}_{\pi^0})^2 &(8.426)
\end{aligned}$$

with $p^i = \hbar k^i = \hbar(\omega, \vec{k}) = \hbar(\omega, \frac{2\pi}{\lambda}\vec{e}_k) = h(\nu, \frac{\nu}{c}\vec{e}_k)$ and the values before

$$
\begin{aligned}
E_{p^+} &= m_{p^+}c^2 + T_{p^+} && (8.427)\\
E_\gamma &= h\nu && (8.428)\\
(\vec{p}_{p^+})^2 &= \frac{1}{c^2}\left[(E_{p^+})^2 - (m_{p^+})^2c^4\right] && (8.429)\\
&= \frac{T_{p^+}}{c^2}\left[T_{p^+} + 2m_{p^+}c^2\right] && (8.430)\\
(\vec{p}_\gamma)^2 &= \frac{h^2\nu^2}{c^2} && (8.431)
\end{aligned}
$$

At the threshold the π^0 is created without any kinetic energy. As the total momentum is vanishing the proton also needs to be at rest

$$
\begin{aligned}
(E_{p^+} + E_\gamma)^2 - c^2(\vec{p}_{p^+} + \vec{p}_\gamma)^2 &= \left(m_{p^+}c^2 + m_{\pi^0}c^2\right)^2 && (8.432)\\
E_{p^+}^2 + 2E_{p^+}E_\gamma + E_\gamma^2 - c^2\left(\vec{p}_{p^+}^{\,2} + \vec{p}_\gamma^{\,2} - 2\vec{p}_{p^+}\cdot\vec{p}_\gamma\right) &= \left(m_{p^+}c^2 + m_{\pi^0}c^2\right)^2 && (8.433)\\
m_{p^+}^2c^4 + 2E_{p^+}E_\gamma + 2c^2\vec{p}_{p^+}\cdot\vec{p}_\gamma &= \left(m_{p^+}c^2 + m_{\pi^0}c^2\right)^2 && (8.434)\\
m_{p^+}^2c^4 + 2E_{p^+}E_\gamma + 2E_\gamma\sqrt{E_{p^+}^2 - m_{p^+}^2c^2}\cos\phi &= \left(m_{p^+}c^2 + m_{\pi^0}c^2\right)^2 && (8.435)\\
E_{p^+}E_\gamma + E_\gamma\sqrt{E_{p^+}^2 - m_{p^+}^2c^2}\cos\phi &= \left(m_{p^+} + \frac{m_{\pi^0}}{2}\right)m_{\pi^0}c^4 && (8.436)
\end{aligned}
$$

Now we can square the equation and solve approximately assuming $E_\gamma \ll m_{p^+}c^2$

$$
\begin{aligned}
E_\gamma\sqrt{E_{p^+}^2 - m_{p^+}^2c^2}\cos\phi &= \left(m_{p^+} + \frac{m_{\pi^0}}{2}\right)m_{\pi^0}c^4 - E_{p^+}E_\gamma && (8.437)\\
E_\gamma^2\left(E_{p^+}^2 - m_{p^+}^2c^2\right)\cos^2\phi &= \left(m_{p^+} + \frac{m_{\pi^0}}{2}\right)^2 m_{\pi^0}^2c^8 + (E_{p^+}E_\gamma)^2 - 2E_{p^+}E_\gamma\left(m_{p^+} + \frac{m_{\pi^0}}{2}\right)m_{\pi^0}c^4 && (8.438)\\
-E_\gamma^2 m_{p^+}^2c^2\cos^2\phi &= \left(m_{p^+} + \frac{m_{\pi^0}}{2}\right)^2 m_{\pi^0}^2c^8 - 2E_{p^+}E_\gamma\left(m_{p^+} + \frac{m_{\pi^0}}{2}\right)m_{\pi^0}c^4 && (8.439)\\
E_{p^+} &\approx \frac{\left(m_{p^+} + m_{\pi^0}/2\right)m_{\pi^0}c^4}{2E_\gamma} && (8.440)\\
&= 10.8\cdot 10^{19}\text{eV} && (8.441)
\end{aligned}
$$

3. By assumption the p^+ and the π^0 would rest in the CM system

$$
\begin{aligned}
(P^\mu)^{cm} &= (p_{p^+}^\mu)^{cm} + (p_{\pi^0}^\mu)^{cm} && (8.442)\\
&= \left([m_{p^+} + m_{\pi^0}]c^2, \vec{0}\right) && (8.443)\\
&= \Lambda^\mu_\alpha\left[\hat{p}_{p^+}^\alpha + \hat{p}_{\pi^0}^\alpha\right] && (8.444)\\
&= \Lambda^\mu_\alpha\left[p_{p^+}^\alpha + p_\gamma^\alpha\right] && (8.445)\\
& && (8.446)
\end{aligned}
$$

We can therefore calculate γ

$$\mu = 1: \quad 0 = \underbrace{\Lambda^1_0}_{-\gamma\beta}(E_{p^+} + E_\gamma) + \underbrace{\Lambda^1_1}_{\gamma} c(p^x_{p^+} + p^x_\gamma) \tag{8.447}$$
$$= -\gamma\beta(E_{p^+} + E_\gamma) + \gamma\left(\sqrt{E^2_{p^+} - m^2_p c^4} + E_\gamma\right) \tag{8.448}$$
$$\to \beta = \frac{\sqrt{E^2_{p^+} - m^2_p c^4} + E_\gamma}{E_{p^+} + E_\gamma} \approx \frac{\sqrt{E^2_{p^+} - m^2_p c^4}}{E_{p^+}} \tag{8.449}$$
$$\to \gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{E_{p^+}}{m_{p^+}c^2} \tag{8.450}$$

which can be used to calculate the pion momentum

$$c\hat{p}_{\pi^0} = \Lambda^0_\mu (p^\mu_{\pi^0})^{cm} \tag{8.451}$$
$$= \Lambda^0_0 (p^0_{\pi^0})^{cm} \tag{8.452}$$
$$= \gamma m_{\pi^0} c^2 \tag{8.453}$$
$$= E_{p^+} \frac{m_{\pi^0}}{m_{p^+}}. \tag{8.454}$$

The $p+$ energy after the collision is then given by

$$E_{p^+} + E_\gamma = \hat{E}_{p^+} + \hat{E}_{\pi^0} \tag{8.455}$$
$$\to \hat{E}_{p^+} = E_{p^+} + E_\gamma - \hat{E}_{\pi^0} \tag{8.456}$$
$$= E_{p^+} + E_\gamma - \sqrt{m^2_{\pi^0} c^4 + \hat{\vec{p}}_{\pi^0} c^2} \tag{8.457}$$
$$= E_{p^+} + E_\gamma - \sqrt{m^2_{\pi^0} c^4 + E^2_{p^+} \frac{m^2_{\pi^0}}{m^2_{p^+}}} \tag{8.458}$$
$$= E_{p^+} + E_\gamma - m_{\pi^0} c^2 \sqrt{1 + \frac{E^2_{p^+}}{m^2_{p^+} c^4}} \tag{8.459}$$
$$\approx E_{p^+} - m_{\pi^0} c^2 \frac{E_{p^+}}{m_{p^+} c^2} \tag{8.460}$$
$$= E_{p^+}\left(1 - \frac{m_{\pi^0}}{m_{p^+}}\right) \tag{8.461}$$
$$\approx 0.85 \cdot E_{p^+}. \tag{8.462}$$

8.8.3 Problem 2.5 Compton scattering

1. the binding energy of outer(!!!) electrons is in the eV range while typical X-rays energies are in the keV range.

2. In the nonrelativistic case we have energy and momentum conservation

$$\frac{hc}{\lambda} = \frac{hc}{\lambda'} + \frac{1}{2} m_e v^2 \tag{8.463}$$
$$\frac{h}{\lambda} = \frac{h}{\lambda'} \cos\theta + m_e v \cos\phi \tag{8.464}$$
$$0 = \frac{h}{\lambda'} \sin\theta + m_e v \sin\phi \tag{8.465}$$

then we see

$$v = \sqrt{\frac{2hc}{m_e}\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)} = \sqrt{\frac{2hc}{m_e}\frac{\lambda' - \lambda}{\lambda\lambda'}} \tag{8.466}$$

and

$$\sin\phi = -\frac{h}{m_e v}\frac{1}{\lambda'}\sin\theta \tag{8.467}$$
$$\cos\phi = \frac{h}{m_e v}\frac{1}{\lambda'}\left(\frac{\lambda'}{\lambda} - \cos\theta\right) \tag{8.468}$$
$$\to 1 = \sin^2\phi + \cos^2\phi \tag{8.469}$$
$$= \frac{h^2}{m_e^2 v^2 \lambda'^2}\left(\sin^2\theta + \frac{\lambda'^2}{\lambda^2} - 2\frac{\lambda'}{\lambda}\cos\theta + \cos^2\theta\right) \tag{8.470}$$
$$= \frac{h^2}{m_e^2 v^2 \lambda'^2}\left(1 + \frac{\lambda'^2}{\lambda^2} - 2\frac{\lambda'}{\lambda}\cos\theta\right) \tag{8.471}$$
$$= \frac{h\lambda}{2m_e c\lambda'(\lambda' - \lambda)}\left(1 + \frac{\lambda'^2}{\lambda^2} - 2\frac{\lambda'}{\lambda}\cos\theta\right) \tag{8.472}$$
$$= \frac{h}{2m_e c(\lambda' - \lambda)}\left(\frac{\lambda}{\lambda'} + \frac{\lambda'}{\lambda} - 2\cos\theta\right) \tag{8.473}$$
$$\lambda' - \lambda \approx \frac{h}{m_e c}(1 - \cos\theta) \tag{8.474}$$

where we used $\lambda \approx \lambda'$.

3.

8.8.4 Problem 2.6 Lorentz invariance

1. With $\omega_k = \sqrt{\vec{k}^2 + m^2}$

$$\int_{-\infty}^{\infty} dk^0 \delta(k^2 - m^2)\theta(k^0) = \int_{-\infty}^{\infty} dk^0 \delta({k^0}^2 - [\vec{k}^2 + m^2])\theta(k^0) \tag{8.475}$$
$$= \frac{\theta(\omega_k)}{2\omega_k} + \frac{\theta(-\omega_k)}{2\omega_k} \tag{8.476}$$
$$= \frac{1}{2\omega_k} \tag{8.477}$$

2. Under Lorentz transformations we have $k^2 - m^2 = 0$. For orthochronous transformation we have k^0...

3. Now we can put it all together

$$\int d^4k\delta(k^2 - m^2)\theta(k^0) = \int d^3k \int dk^0 \delta(k^2 - m^2)\theta(k^0) \tag{8.478}$$
$$= \int \frac{d^3k}{2\omega_k} \tag{8.479}$$

8.8.5 Problem 2.7 Coherent states

1.

$$\partial_z \left(e^{-za^\dagger} a e^{-za^\dagger}\right) = -e^{-za^\dagger} a^\dagger a e^{-za^\dagger} + e^{-za^\dagger} a a^\dagger e^{-za^\dagger} \tag{8.480}$$
$$= e^{-za^\dagger}[a, a^\dagger]e^{-za^\dagger} \tag{8.481}$$
$$= 1 \tag{8.482}$$

2. Rolling the a through the $(a^\dagger)^k$ using the commutator $[a, a^\dagger] = 1$

$$a|z\rangle = ae^{za^\dagger}|0\rangle \tag{8.483}$$
$$= a\sum_{k=0} \frac{1}{k!} z^k (a^\dagger)^k |0\rangle \tag{8.484}$$
$$= a|0\rangle + \sum_{k=1} \frac{k}{k!} z^k (a^\dagger)^{k-1}|0\rangle \tag{8.485}$$
$$= z\sum_{n=0} \frac{1}{n!} z^n (a^\dagger)^n |0\rangle \tag{8.486}$$
$$= z|z\rangle \tag{8.487}$$

3. With $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ and using the $|z\rangle$ is an eigenstate of a we have

$$\langle n|z\rangle = \frac{1}{\sqrt{n!}}\langle 0|a^n|z\rangle = \frac{z^n}{\sqrt{n!}}\langle 0|z\rangle = \frac{z^n}{\sqrt{n!}}\langle 0|e^{za^\dagger}|0\rangle \tag{8.488}$$
$$= \frac{z^n}{\sqrt{n!}}\langle 0|1 + za^\dagger + \frac{1}{2}z^2(a^\dagger)^2 + ...|0\rangle \tag{8.489}$$
$$= \frac{z^n}{\sqrt{n!}}\langle 0|0\rangle = \frac{z^n}{\sqrt{n!}} \tag{8.490}$$

where we used $\langle 0|a^\dagger = 0$.

4. With

$$a + a^\dagger = \sqrt{\frac{m\omega}{2}}2q \quad \rightarrow \quad q = \frac{1}{\sqrt{2m\omega}}(a + a^\dagger) \tag{8.491}$$
$$a - a^\dagger = \sqrt{\frac{m\omega}{2}}2\frac{ip}{m\omega} \quad \rightarrow \quad p = -i\frac{\sqrt{m\omega}}{\sqrt{2}}(a - a^\dagger) \tag{8.492}$$

and $a|z\rangle = z|z\rangle$ and $\langle z|a^\dagger = \bar{z}\langle z|$

$$\langle z|q|z\rangle = \frac{1}{\sqrt{2m\omega}}\langle z|a + a^\dagger|z\rangle = \frac{1}{\sqrt{2m\omega}}\langle z|z\rangle(z + \bar{z}) \tag{8.493}$$
$$\langle z|p|z\rangle = -i\frac{\sqrt{m\omega}}{\sqrt{2}}\langle z|a - a^\dagger|z\rangle = -i\frac{\sqrt{m\omega}}{\sqrt{2}}\langle z|z\rangle(z - \bar{z}) \tag{8.494}$$
$$\langle z|q^2|z\rangle = \frac{1}{2m\omega}\langle z|aa + \underbrace{aa^\dagger}_{=1+a^\dagger a} + a^\dagger a + a^\dagger a^\dagger|z\rangle \tag{8.495}$$
$$= \frac{1}{2m\omega}\langle z|z\rangle\left(z^2 + 1 + 2z\bar{z} + \bar{z}^2\right) \tag{8.496}$$
$$\langle z|p^2|z\rangle = -\frac{m\omega}{2}\langle z|aa - \underbrace{aa^\dagger}_{=1+a^\dagger a} - a^\dagger a + a^\dagger a^\dagger|z\rangle \tag{8.497}$$
$$= -\frac{m\omega}{2}\langle z|z\rangle\left(z^2 - 1 - 2z\bar{z} + \bar{z}^2\right) \tag{8.498}$$

Therefore

$$\Delta q^2 = \langle q^2\rangle - \langle q\rangle^2 \tag{8.499}$$
$$= \frac{1}{2m\omega}\left(z^2 + 1 + 2z\bar{z} + \bar{z}^2\right) - \left(\frac{1}{\sqrt{2m\omega}}(z + \bar{z})\right)^2 \tag{8.500}$$
$$= \frac{1}{2m\omega} \tag{8.501}$$

and

$$\Delta p^2 = \langle p^2 \rangle - \langle p \rangle^2 \tag{8.502}$$
$$= -\frac{m\omega}{2}\left(z^2 - 1 - 2z\bar{z} + \bar{z}^2\right) - \left(-i\frac{\sqrt{m\omega}}{\sqrt{2}}(z - \bar{z})\right)^2 \tag{8.503}$$
$$= \frac{m\omega}{2} \tag{8.504}$$

which means

$$\Delta p \Delta q = \frac{1}{\sqrt{2m\omega}}\frac{\sqrt{m\omega}}{\sqrt{2}} = \frac{1}{2}. \tag{8.505}$$

5. At first let's construct the eigenstate $|w\rangle$ for a manually

$$a|w\rangle = c_w|w\rangle \tag{8.506}$$

Expanding the eigenstate with $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$

$$|w\rangle = \sum_n \alpha_n|n\rangle \tag{8.507}$$
$$a|w\rangle = \sum_n \alpha_n\sqrt{n}|n-1\rangle \stackrel{!}{=} c_w\sum_n \alpha_n|n\rangle = c_w|n\rangle \tag{8.508}$$
$$\to \alpha_n\sqrt{n} = c_w\alpha_{n-1} \tag{8.509}$$
$$\to \alpha_n = \frac{c_w}{\sqrt{n}}\alpha_{n-1} \tag{8.510}$$
$$|w\rangle = \sum_n \alpha_0\frac{c_w^n}{\sqrt{n!}}|n\rangle = \alpha_0\sum_n \frac{c_w^n}{n!}(a^\dagger)^n|0\rangle = \alpha_0 e^{c_w a^\dagger}|0\rangle \tag{8.511}$$

Now we do the same for $a^\dagger$

$$a^\dagger|v\rangle = c_v|v\rangle \tag{8.512}$$

Expanding the eigenstate

$$|v\rangle = \sum_n \beta_n|n\rangle \tag{8.513}$$
$$a^\dagger|v\rangle = \sum_n \beta_n\sqrt{n+1}|n+1\rangle \stackrel{!}{=} c_v\sum_n \beta_n|n\rangle = c_v|n\rangle \tag{8.514}$$
$$\to \beta_n\sqrt{n+1} = c_v\beta_{n+1} \tag{8.515}$$
$$\to \beta_{n+1} = \frac{\sqrt{n+1}}{c_v}\beta_n \tag{8.516}$$
$$|v\rangle = \sum_n \beta_0\frac{\sqrt{n!}}{c_v^n}|n\rangle = \beta_0\sum_n \frac{1}{c_v^n}(a^\dagger)^n|0\rangle \tag{8.517}$$

Now we calculate with $\langle 0|a^\dagger = 0$

$$\langle 0|a^\dagger|v\rangle = \beta_0\sum_n \frac{1}{c_v^n}\langle 0|(a^\dagger)^{n+1}|0\rangle \tag{8.518}$$
$$= \beta_0\frac{1}{c_v^0}\langle 0|a^\dagger|0\rangle \tag{8.519}$$
$$\tag{8.520}$$

8.8.6 Problem 3.1 Higher order Lagrangian

With the principle of least action

$$\delta S = \delta \int \mathcal{L} d^4x = \int \delta \mathcal{L} d^4x \tag{8.521}$$

we calculate

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu\phi) + \frac{\partial \mathcal{L}}{\partial(\partial_\nu\partial_\mu \phi)}\delta(\partial_\nu\partial_\mu\phi) + ... \tag{8.522}$$

Now we can integrate each term

$$\delta\mathcal{L}_0 = \int \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi d^4x \tag{8.523}$$

$$\delta\mathcal{L}_1 = \int \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu\phi) d^4x = \int \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\mu\delta\phi d^4x \tag{8.524}$$

$$= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi\bigg|_{\partial\Omega} - \int \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi d^4x \tag{8.525}$$

$$\delta\mathcal{L}_2 = \int \frac{\partial \mathcal{L}}{\partial(\partial_\nu\partial_\mu \phi)}\delta(\partial_\nu\partial_\mu\phi) d^4x = \int \frac{\partial \mathcal{L}}{\partial(\partial_\nu\partial_\mu \phi)}\partial_\nu\delta\partial_\mu\phi d^4x \tag{8.526}$$

$$= \frac{\partial \mathcal{L}}{\partial(\partial_\nu\partial_\mu \phi)}\delta\partial_\mu\phi\bigg|_{\partial\Omega} - \int \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu\partial_\mu \phi)}\delta\partial_\mu\phi d^4x \tag{8.527}$$

$$= \frac{\partial \mathcal{L}}{\partial(\partial_\nu\partial_\mu \phi)}\delta\partial_\mu\phi\bigg|_{\partial\Omega} - \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu\partial_\mu \phi)}\delta\phi\bigg|_{\partial\Omega} + \int \partial_\mu\partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu\partial_\mu \phi)}\delta\phi d^4x \tag{8.528}$$

Requiring that all derivatives vanish at infinity we obtain

$$\delta S = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} + \partial_\mu\partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu\partial_\mu \phi)} - ... \right) \delta\phi \tag{8.529}$$

and therefore

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} + \partial_\mu\partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu\partial_\mu \phi)} - ... = 0 \tag{8.530}$$

8.8.7 Problem 3.5 Spontaneous symmetry

$$\mathscr{L} = -\frac{1}{2}\phi\Box\phi + \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{8.531}$$

$$\frac{\partial \mathscr{L}}{\partial \phi} - \partial_\beta \frac{\partial \mathscr{L}}{\partial(\partial_\beta \phi)} + \partial_\mu\partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu\partial_\mu \phi)} = 0 \tag{8.532}$$

$$\rightarrow -\Box\phi + m^2\phi - \frac{\lambda}{3!}\phi^3 = 0 \tag{8.533}$$

and the Hamiltonian with $-\phi\Box\phi \sim (\partial_\mu\phi)(\partial^\mu\phi) = \eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot\phi} \tag{8.534}$$

$$= \dot\phi \tag{8.535}$$

$$\mathcal{H} = \pi\dot\phi - \mathcal{L} \tag{8.536}$$

$$= (\dot\phi)^2 - \mathcal{L} \tag{8.537}$$

$$= \frac{1}{2}\dot\phi^2 + \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{8.538}$$

(a)

$$m^2\phi - \frac{\lambda}{3!}\phi^3 = 0 \tag{8.539}$$
$$(m^2 - \frac{\lambda}{3!}\phi^2)\phi = 0 \tag{8.540}$$
$$\phi_0 = 0 \quad \to \quad \mathcal{H}[\phi] = 0 \tag{8.541}$$
$$\phi_{1,2} = \pm\sqrt{\frac{3!}{\lambda}}m \quad \to \quad \mathcal{H}[\phi] = -\frac{3m^4}{2\lambda} \tag{8.542}$$

(b)

(c)

8.8.8 Problem 3.6 Yukawa potential

(a) We slit the Lagranian in three parts

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}m^2A_\mu^2 - A_\mu J_\mu \tag{8.543}$$
$$= \mathscr{L}_F + \mathscr{L}_m + \mathscr{L}_J \tag{8.544}$$

with the Euler Lagrange equations

$$\frac{\partial\mathscr{L}}{\partial A_\alpha} - \partial_\beta\frac{\partial\mathscr{L}}{\partial(\partial_\beta A_\alpha)} = 0 \tag{8.545}$$

with

$$\frac{\partial(\partial_\mu A_\nu)}{\partial(\partial_\beta A_\alpha)} = \delta_{\mu\beta}\delta_{\nu\alpha} \tag{8.546}$$

we can calculate

$$\frac{\partial\mathscr{L}_m}{\partial A_\alpha} - \partial_\beta\frac{\partial\mathscr{L}_m}{\partial(\partial_\beta A_\alpha)} = m^2A_\alpha \tag{8.547}$$
$$\frac{\partial\mathscr{L}_J}{\partial A_\alpha} - \partial_\beta\frac{\partial\mathscr{L}_J}{\partial(\partial_\beta A_\alpha)} = -J_\alpha \tag{8.548}$$
$$\frac{\partial\mathscr{L}_F}{\partial A_\alpha} - \partial_\beta\frac{\partial\mathscr{L}_F}{\partial(\partial_\beta A_\alpha)} = -\frac{1}{4}\partial_\beta\left(-2F_{\mu\nu}(\delta_{\mu\beta}\delta_{\nu\alpha} - \delta_{\nu\beta}\delta_{\mu\alpha})\right) \tag{8.549}$$
$$= \frac{1}{4}\partial_\beta\left(2(F_{\beta\alpha} - F_{\alpha\beta})\right) \tag{8.550}$$
$$= \partial_\beta F_{\beta\alpha} \tag{8.551}$$
$$= \partial_\beta\partial_\beta A_\alpha - \partial_\beta\partial_\alpha A_\beta \tag{8.552}$$

to obtain (the Proca equation)

$$\Box A_\alpha - \partial_\beta\partial_\alpha A_\beta + m^2A_\alpha - J_\alpha = 0. \tag{8.553}$$

Now we can calculate the divergence of the equations

$$\partial_\alpha\left(\Box A_\alpha - \partial_\beta\partial_\alpha A_\beta + m^2A_\alpha - J_\alpha\right) = 0. \tag{8.554}$$
$$\Box\partial_\alpha A_\alpha - \partial_\alpha\partial_\alpha\partial_\beta A_\beta + m^2\partial_\alpha A_\alpha - \underbrace{\partial_\alpha J_\alpha}_{=0} = 0 \tag{8.555}$$

which implies $\partial_\alpha A_\alpha = 0$ and therefore

$$\Box A_\alpha + m^2A_\alpha - J_\alpha = 0. \tag{8.556}$$

(b) For A_0 we have for a static potential

$$(\partial_{tt} - \triangle)A_0 + m^2 A_0 - e\delta(x) = 0 \tag{8.557}$$
$$-\triangle A_0 + m^2 A_0 - e\delta(x) = 0. \tag{8.558}$$

A Fourier transformation of the equation of motion yields

$$-(ik)^2 A_0(k) + m^2 A_0(k) - e = 0 \tag{8.559}$$
$$\rightarrow A_0(k) = \frac{e}{k^2 + m^2} \tag{8.560}$$

which we can now transform back

$$A_0 = \frac{e}{(2\pi)^3} \int d^3k \frac{e^{ikx}}{k^2 + m^2} \tag{8.561}$$
$$= \frac{e}{4\pi r} e^{-mr} \tag{8.562}$$

where we used the integral evaluation from Kachelriess Problem 3.5.

(c)

$$\lim_{m \to 0} \frac{e}{4\pi r} e^{-mr} = \frac{e}{4\pi r} \tag{8.563}$$

(d) Scaling down the Coulomb potential exponentially with a characteristic length of $1/m$.

(e)

(f) We can expand and the integrate each term by parts to move over the partial derivatives

$$\mathscr{L}_F = -\frac{1}{4} F_{\mu\nu}^2 \tag{8.564}$$
$$= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) \tag{8.565}$$
$$= -\frac{1}{4}\left(\partial_\mu A_\nu \partial_\mu A_\nu - \partial_\mu A_\nu \partial_\nu A_\mu - \partial_\nu A_\mu \partial_\mu A_\nu + \partial_\nu A_\mu \partial_\nu A_\mu\right) \tag{8.566}$$
$$= -\frac{1}{2}\left(\partial_\mu A_\nu \partial_\mu A_\nu - \partial_\mu A_\nu \partial_\nu A_\mu\right) \tag{8.567}$$
$$= -\frac{1}{2}\left(-A_\nu \partial_\mu \partial_\mu A_\nu + A_\nu \partial_\nu \partial_\mu A_\mu\right) \tag{8.568}$$
$$= \frac{1}{2}\left(A_\mu \Box A_\mu - A_\nu \partial_\nu \underbrace{\partial_\mu A_\mu}_{=0}\right) \tag{8.569}$$
$$= \frac{1}{2} A_\mu \Box A_\mu \tag{8.570}$$

We can plug this into the full Lagrangian (renaming the summation index)

$$\mathscr{L} = \frac{1}{2} A_\mu \Box A_\mu + \frac{1}{2} m^2 A_\mu^2 - A_\mu J_\mu \tag{8.571}$$
$$= \frac{1}{2} A_\mu \left(\Box + m^2\right) A_\mu - A_\mu J_\mu \tag{8.572}$$

then we calculate the derivatives for the Euler-Lagrange equations up to second order (see problem 3.1)

$$\frac{\partial\mathcal{L}}{\partial A_\mu} = \frac{1}{2}\Box A_\mu + m^2 A_\mu - J_\mu \tag{8.573}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\alpha A_\mu)} = 0 \tag{8.574}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\partial_\alpha A_\mu)} = \frac{1}{2}A_\mu \tag{8.575}$$

and get

$$(\Box + m^2)A_\mu = J_\mu \tag{8.576}$$

8.8.9 Problem 3.7 Perihelion shift of Mercury by dimensional analysis

(a) Lets summarize the rules of dimensional analysis

varible	SI unit	equation	natural unit
c	m/s	-	1
$\hbar$	Js	-	1
Velocity	m/s	-	1
mass	kg	$E = mc^2$	E
frequency	1/s	$E = \hbar\omega$	E
time	s	$t = 2\pi/\omega$	E^{-1}
length	m	$s = ct$	E^{-1}
∂_μ	1/m	-	E
momentum	kg m/s	$E = p^2/2m$	E
action	Js	$S = Et$	1
$\mathcal{L}$	J/m^3	$S = \int d^4x\mathcal{L}$	E^4
energy density	J/m^3	$\rho = E/V$	E^4
$T^{\mu\nu}$	J/m^3	$\rho = E/V$	E^4

Now we can do a dimensions count for each term

$$\underbrace{\mathcal{L}}_{=4} = -\frac{1}{2}\underbrace{h\Box h}_{2\cdot[h]+2} + \underbrace{M_{\text{Pl}}^a h^2\Box h}_{=a+3\cdot[h]+2} - \underbrace{M_{\text{Pl}}^b hT}_{b+[h]+4} \tag{8.577}$$

$$\rightarrow \quad [h] = 1 \tag{8.578}$$

$$\rightarrow \quad a = -1 \tag{8.579}$$

$$\rightarrow \quad b = -1 \tag{8.580}$$

(b) Deriving the equations of motions: keeping in mind that the Lagrangian contains second order derivatives with implies and extra term in the Euler-Lagrange equations (see problem 3.1)

$$\mathcal{L} = -\frac{1}{2}h\Box h + \frac{1}{M_{\text{Pl}}}h^2\Box h - \frac{1}{M_{\text{Pl}}}hT \tag{8.581}$$

$$\frac{\partial\mathcal{L}}{\partial h} = -\frac{1}{2}\cdot\Box h + 2\frac{1}{M_{\text{Pl}}}h\Box h - \frac{1}{M_{\text{Pl}}}T \tag{8.582}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial h)} = 0 \tag{8.583}$$

$$\frac{\partial\mathcal{L}}{\partial(\Box h)} = -\frac{1}{2}h + \frac{1}{M_{\text{Pl}}}h^2 \tag{8.584}$$

$$\rightarrow \Box h = \frac{1}{M_{\text{Pl}}}\Box(h^2) + \frac{2}{M_{\text{Pl}}}h\Box h - \frac{1}{M_{\text{Pl}}}T \tag{8.585}$$

which show an extra term. Alternatively we can integrate the Lagrangian by parts (neglecting the boundary terms) and get

$$\mathcal{L} = \frac{1}{2}\partial h \partial h - \frac{1}{M_{\text{Pl}}}\partial(h^2)\partial h - \frac{1}{M_{\text{Pl}}}hT \tag{8.586}$$

$$\frac{\partial \mathcal{L}}{\partial h} = -\frac{1}{M_{\text{Pl}}}T \tag{8.587}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial h)} = \Box h - \frac{1}{M_{\text{Pl}}}\Box(h^2) \tag{8.588}$$

$$\rightarrow \Box h = \frac{1}{M_{\text{Pl}}}\Box(h^2) - \frac{1}{M_{\text{Pl}}}T \tag{8.589}$$

We now assume a solution of the form

$$h = h_0 + \frac{1}{M_{\text{Pl}}}h_1 + \frac{1}{M_{\text{Pl}}^2}h_2 + ... \tag{8.590}$$

$$\rightarrow h^2 = h_0^2 + \frac{1}{M_{\text{Pl}}}2h_0h_1 + \frac{1}{M_{\text{Pl}}^2}(2h_0h_2 + h_1^2) + \frac{1}{M_{\text{Pl}}^3}(2h_1h_2 + 2h_0h_3) + ... \tag{8.591}$$

and obtain (with the Coulomb solution 3.61 and 3.61)

$$k = 0: \quad \Box h_0 = 0 \quad \rightarrow \quad h_0 = 0 \tag{8.592}$$

$$k = 1: \quad \Box h_1 = \Box h_0^2 - m\delta^{(3)} \tag{8.593}$$

$$\Box h_1 = -m\delta^{(3)} \quad \rightarrow \quad h_1 = -\frac{m}{\Box}\delta^{(3)} = \frac{m}{\triangle}\delta^{(3)} = -\frac{m}{4\pi r} \tag{8.594}$$

$$k = 2: \quad \Box h_2 = 2\Box h_0 h_1 \quad \rightarrow \quad h_2 = 0 \tag{8.595}$$

$$k = 3: \quad \Box h_3 = \Box(2h_0h_2 + h_1^2) \tag{8.596}$$

$$\Box h_3 = \Box(h_1^2) \quad \rightarrow \quad h_3 = h_1^2 = \frac{m^2}{16\pi^2 r^2} \tag{8.597}$$

and therefore

$$h = -\frac{m}{4\pi r}\frac{1}{M_{\text{Pl}}} + \frac{m^2}{16\pi^2 r^2}\frac{1}{M_{\text{Pl}}^3} \tag{8.598}$$

$$= -\frac{m}{4\pi r}\sqrt{G_N} + \frac{m^2}{16\pi^2 r^2}\sqrt{G_N^3} \tag{8.599}$$

(c) The Newton potential is actually given by (and additional power of M_{Pl} is missing and we are dropping the 4π)

$$V_N = h_1 \frac{1}{M_{\text{Pl}}} \cdot \frac{1}{M_{\text{Pl}}} = -\frac{G m_{\text{Sun}}}{r} \tag{8.600}$$

the virial theorem implies $E_{\text{kin}} \simeq E_{\text{pot}}$ and therefore

$$\frac{1}{2}J\omega^2 \simeq \frac{G_N m_{\text{Sun}} m_{\text{Mercury}}}{R} \tag{8.601}$$

$$\frac{1}{2}m_{\text{Mercury}}R^2\omega^2 \simeq \frac{G_N m_{\text{Sun}} m_{\text{Mercury}}}{R} \tag{8.602}$$

$$\omega^2 \simeq \frac{G_N m_{\text{Sun}}}{R^3} \tag{8.603}$$

(d)

(e)

(f)

(g)

8.8.10 Problem 3.9 - Photon polarizations

(a) Then using the results from problem 3.6 and the corrected sign in the Lagranian we get

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 - J_\mu A_\mu \tag{8.604}$$
$$= \frac{1}{2}A_\mu \Box A_\mu - J_\mu A_\mu \tag{8.605}$$
$$= \frac{1}{2}A_\mu \Box A_\mu - (\Box A_\mu)A_\mu \tag{8.606}$$
$$= -\frac{1}{2}A_\mu \Box A_\mu \tag{8.607}$$

The equations of motion are $\Box A_\mu = J_\mu$ which can be written in momentum space as $k^2 A_\mu(k) = J_\mu(k)$. Now let's write the Lagranian in momentum space as well

$$\mathcal{L} = \int d^4k e^{ikx} A_\mu(k) k^2 A_\mu(k) \tag{8.608}$$
$$= \int d^4k e^{ikx} \frac{J_\mu(k)}{k^2} k^2 \frac{J_\mu(k)}{k^2} \tag{8.609}$$
$$= \int d^4k e^{ikx} J_\mu(k) \frac{1}{k^2} J_\mu(k) \tag{8.610}$$

(b) In momentum space charge conservation is given by

$$k_\mu J_\mu = 0 \tag{8.611}$$
$$\omega J_0 - \kappa J_1 = 0 \tag{8.612}$$
$$\rightarrow \quad J_1 = \frac{\omega}{\kappa} J_0 \tag{8.613}$$

(c)

$$\mathcal{L} = \int d^4k e^{ikx} J_\mu(k) \frac{1}{k^2} J_\mu(k) \tag{8.614}$$
$$\simeq \frac{J_0^2 - J_1^2 - J_3^2 - J_4^2}{\omega^2 - \kappa^2} \tag{8.615}$$
$$\simeq \frac{J_0^2(1 - \omega^2/\kappa^2)}{\omega^2 - \kappa^2} - \frac{J_3^2 + J_4^2}{\omega^2 - \kappa^2} \tag{8.616}$$
$$\simeq -\frac{J_0^2}{\kappa^2} - \frac{J_3^2 + J_4^2}{\omega^2 - \kappa^2} \tag{8.617}$$
$$\simeq \triangle J_0^2 - \Box(J_3^2 + J_4^2) \tag{8.618}$$

(d)

8.8.11 Problem 3.10 - Graviton polarizations

(a)

(b)

(c)

(d)

8.9 Srednicki - Quantum Field Theory

8.9.1 Problem 1.2 - Schroedinger equation

$$H = \int d^3x a^\dagger(x) \left(-\frac{\hbar^2}{2m} \triangle_x + V(x) \right) a(x) + \frac{1}{2} \int d^3x d^3y V(x-y) a^\dagger(x) a^\dagger(y) a(x) a(y) \quad (8.619)$$

$$|\psi, t\rangle = \int d^3x_1 ... d^3x_n \psi(x_1, ..., x_n; t) a^\dagger(x_1) ... a^\dagger(x_n)|0\rangle \quad (8.620)$$

1. Bosons: With the commutations relation and $a|0\rangle = 0$

$$a(x) a^\dagger(x_1) ... a^\dagger(x_n)|0\rangle = \left(\delta^3(x - x_1) - a^\dagger(x_1) a(x)\right) ... a^\dagger(x_n)|0\rangle \quad (8.621)$$

$$= \sum_{k=1}^{n} (-1)^{k-1} \delta^3(x - x_k) \underbrace{a^\dagger(x_1) ... a^\dagger(x_n)}_{(n-1)\times a^\dagger} |0\rangle \quad (8.622)$$

and similar

$$a(y) a(x) a^\dagger(x_1) ... a^\dagger(x_n)|0\rangle = \sum_{j \neq k}^{n} \delta^3(x - x_k) \delta^3(y - x_j) \underbrace{a^\dagger(x_1) ... a^\dagger(x_n)}_{(n-2)\times a^\dagger} |0\rangle \quad (8.623)$$

we obtain

$$i\hbar \frac{\partial}{\partial t} |\psi, t\rangle = \int d^3x_1 ... d^3x_n \frac{\partial}{\partial t} \psi(x_1, ..., x_n; t) a^\dagger(x_1) ... a^\dagger(x_n)|0\rangle \quad (8.624)$$

and

$$H|\psi, t\rangle = \sum_{k=1}^{n} a^\dagger(x_k) \left(-\frac{\hbar^2}{2m} \triangle_{x_k} + V(x_k) \right) \psi(x_1, ..., x_n; t) \underbrace{a^\dagger(x_1) ... a^\dagger(x_n)}_{(n-1)\times a^\dagger} |0\rangle \quad (8.625)$$

$$+ \frac{1}{2} \sum_{j \neq k}^{n} V(x_k - x_j) \psi(x_1, ..., x_n; t) a^\dagger(x_k) a^\dagger(x_j) \underbrace{a^\dagger(x_1) ... a^\dagger(x_n)}_{(n-2)\times a^\dagger} |0\rangle \quad (8.626)$$

2. Fermions:

8.9.2 Problem 1.3 - Commutator of the number operator

Preliminary calculations (we use the boson commutation relations)

$$a^\dagger(z) a(z) a^\dagger(x) = a^\dagger(z)(\delta(x - z) + a^\dagger(x) a(z)) \quad (8.627)$$

$$= a^\dagger(z) \delta^3(x - z) + a^\dagger(z) a^\dagger(x) a(z) \quad (8.628)$$

$$= a^\dagger(z) \delta^3(x - z) + a^\dagger(x) a^\dagger(z) a(z) \quad (8.629)$$

and

$$a(x) a^\dagger(z) a(z) = (\delta(x - z) + a^\dagger(z) a(x)) a(z) \quad (8.630)$$

$$= \delta^3(x - z) a(z) + a^\dagger(z) a(x) a(z) \quad (8.631)$$

$$= \delta^3(x - z) a(z) + a^\dagger(z) a(z) a(x) \quad (8.632)$$

With

$$N = \int d^3z\; a^\dagger(z)a(z) \tag{8.633}$$

$$H = H_1 + H_{\text{int}} \tag{8.634}$$

$$= \int d^3x\; a^\dagger(x)\left(-\frac{\hbar^2}{2m}\triangle_x + U(x)\right)a(x) + \frac{1}{2}\int d^3xd^3y\; V(x-y)a^\dagger(x)a^\dagger(y)a(y)a(x) \tag{8.635}$$

We are calculating the commutator in two parts. We start with $[N, H_1]$

$$NH_1 = \int d^3xd^3z\left(a^\dagger(z)\delta^3(x-z) + a^\dagger(x)a^\dagger(z)a(z)\right)\left(-\frac{\hbar^2}{2m}\triangle_x + U(x)\right)a(x) \tag{8.636}$$

$$= \int d^3xa^\dagger(x)\left(-\frac{\hbar^2}{2m}\triangle_x + U(x)\right)a(x) + \int d^3xd^3za^\dagger(x)\left(-\frac{\hbar^2}{2m}\triangle_x + U(x)\right)a^\dagger(z)a(z)a(x) \tag{8.637}$$

and

$$H_1N = \int d^3x\; a^\dagger(x)\left(-\frac{\hbar^2}{2m}\triangle_x + U(x)\right)(\delta^3(x-z)a(z) + a^\dagger(z)a(z)a(x)) \tag{8.638}$$

$$= \int d^3xa^\dagger(x)\left(-\frac{\hbar^2}{2m}\triangle_x + U(x)\right)a(x) + \int d^3xd^3za^\dagger(x)\left(-\frac{\hbar^2}{2m}\triangle_x + U(x)\right)a^\dagger(z)a(z)a(x) \tag{8.639}$$

therefore $[N, H_1] = 0$. For the second part $[N, H_{\text{int}}]$ we calculate

$$\begin{aligned}
a_z^\dagger a_z a_x^\dagger a_y^\dagger a_y a_x &= a_z^\dagger(\delta^3_{zx} + a_x^\dagger a_z)a_y^\dagger a_y a_x && (8.640)\\
&= \delta^3_{zx}a_z^\dagger a_y^\dagger a_y a_x + a_z^\dagger a_x^\dagger a_z a_y^\dagger a_y a_x && (8.641)\\
&= \delta^3_{zx}a_y^\dagger a_z^\dagger a_y a_x + a_z^\dagger a_x^\dagger(\delta^3_{zy} + a_y^\dagger a_z)a_y a_x && (8.642)\\
&= \delta^3_{zx}a_y^\dagger a_z^\dagger a_y a_x + \delta^3_{zy}a_z^\dagger a_x^\dagger a_y a_x + a_z^\dagger a_x^\dagger a_y^\dagger a_z a_y a_x && (8.643)\\
&= \delta^3_{zx}a_y^\dagger a_z^\dagger a_y a_x + \delta^3_{zy}a_x^\dagger a_z^\dagger a_y a_x + a_x^\dagger a_y^\dagger a_z^\dagger a_z a_y a_x && (8.644)\\
&\to a_y^\dagger a_x^\dagger a_y a_x + a_x^\dagger a_y^\dagger a_y a_x + a_x^\dagger a_y^\dagger a_z^\dagger a_z a_y a_x && (8.645)
\end{aligned}$$

and

$$\begin{aligned}
a_x^\dagger a_y^\dagger a_y a_x a_z^\dagger a_z &= a_x^\dagger a_y^\dagger a_y(\delta^3_{xz} + a_z^\dagger a_x)a_z && (8.646)\\
&= \delta^3_{xz}a_x^\dagger a_y^\dagger a_y a_z + a_x^\dagger a_y^\dagger a_y a_z^\dagger a_x a_z && (8.647)\\
&= \delta^3_{xz}a_x^\dagger a_y^\dagger a_z a_y + a_x^\dagger a_y^\dagger(\delta^3_{zy} + a_z^\dagger a_y)a_x a_z && (8.648)\\
&= \delta^3_{xz}a_x^\dagger a_y^\dagger a_z a_y + \delta^3_{zy}a_x^\dagger a_y^\dagger a_x a_z + a_x^\dagger a_y^\dagger a_z^\dagger a_y a_x a_z && (8.649)\\
&= \delta^3_{xz}a_x^\dagger a_y^\dagger a_z a_y + \delta^3_{zy}a_x^\dagger a_y^\dagger a_z a_x + a_x^\dagger a_y^\dagger a_z^\dagger a_z a_y a_x && (8.650)\\
&\to a_x^\dagger a_y^\dagger a_x a_y + a_x^\dagger a_y^\dagger a_y a_x + a_x^\dagger a_y^\dagger a_z^\dagger a_z a_y a_x && (8.651)
\end{aligned}$$

We therefore see that the commutator vanishes as well.

8.9.3 Problem 2.1 - Infinitesimal LT

$$\begin{aligned}
g_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma &= g_{\rho\sigma} && (8.652)\\
g_{\mu\nu}\left(\delta^\mu_\rho + \delta\omega^\mu_\rho\right)(\delta^\nu{}_\sigma + \delta\omega^\nu{}_\sigma) &= g_{\rho\sigma} && (8.653)\\
g_{\mu\nu}\left(\delta^\mu_\rho\delta^\nu{}_\sigma + \delta^\nu{}_\sigma\cdot\delta\omega^\mu_\rho + \delta^\mu_\rho\cdot\delta\omega^\nu{}_\sigma + \mathcal{O}(\delta\omega^2)\right) &= g_{\rho\sigma} && (8.654)\\
g_{\rho\sigma} + g_{\mu\sigma}\cdot\delta\omega^\mu_\rho + g_{\rho\nu}\cdot\delta\omega^\nu{}_\sigma &= g_{\rho\sigma} && (8.655)
\end{aligned}$$

which implies

$$\delta\omega_{\sigma\rho} + \delta\omega_{\rho\sigma} = 0 \tag{8.656}$$

8.9.4 Problem 2.2 - Infinitesimal LT II

Important: each $M^{\mu\nu}$ is an operator and $\delta\omega$ is just a coefficient matrix so $\delta\omega_{\mu\nu}M^{\mu\nu}$ ist a weighted sum of operators.

$$U(\Lambda^{-1}\Lambda'\Lambda) = U(\Lambda^{-1})U(\Lambda')U(\Lambda) \tag{8.657}$$

$$U(\Lambda^{-1}(I+\delta\omega')\Lambda) = U(\Lambda^{-1})\left(I + \frac{i}{2\hbar}\delta\omega'_{\mu\nu}M^{\mu\nu}\right)U(\Lambda) \tag{8.658}$$

$$U(I+\Lambda^{-1}\delta\omega'\Lambda) = I + \frac{i}{2\hbar}\delta\omega'_{\mu\nu}U(\Lambda^{-1})M^{\mu\nu}U(\Lambda) \tag{8.659}$$

now we calculate recalling successive LT's $(\Lambda^{-1})^{\varepsilon}{}_{\gamma}\delta\omega'^{\gamma}{}_{\beta}\Lambda^{\beta}{}_{\alpha}x^{\alpha}$

$$(\Lambda^{-1}\delta\omega'\Lambda)_{\rho\sigma} = g_{\varepsilon\rho}(\Lambda^{-1})^{\varepsilon}{}_{\mu}\delta\omega'^{\mu}{}_{\nu}\Lambda^{\nu}{}_{\sigma} \tag{8.660}$$

$$= g_{\varepsilon\rho}\Lambda_{\mu}{}^{\varepsilon}\delta\omega'^{\mu}{}_{\nu}\Lambda^{\nu}{}_{\sigma} \tag{8.661}$$

$$= \delta\omega'_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} \tag{8.662}$$

now we can rewrite $U(I+\Lambda^{-1}\delta\omega'\Lambda)$ and therefore

$$\delta\omega'_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}M^{\rho\sigma} = \delta\omega'_{\mu\nu}U(\Lambda^{-1})M^{\mu\nu}U(\Lambda) \tag{8.663}$$

As all $\delta\omega'$ components are basically independent the equation must hold for each pair μ,ν.

8.9.5 Problem 2.3 - Commutators of LT generators I

LHS:

$$U(\Lambda)^{-1}M^{\mu\nu}U(\Lambda) \simeq \left(I - \frac{i}{2\hbar}\delta\omega_{\alpha\beta}M^{\alpha\beta}\right)M^{\mu\nu}\left(I + \frac{i}{2\hbar}\delta\omega_{\rho\sigma}M^{\rho\sigma}\right) \tag{8.664}$$

$$\simeq M^{\mu\nu} - \frac{i}{2\hbar}\delta\omega_{\rho\sigma}(M^{\rho\sigma}M^{\mu\nu} - M^{\mu\nu}M^{\rho\sigma}) + \mathcal{O}(\delta\omega^2) \tag{8.665}$$

$$= M^{\mu\nu} - \frac{i}{2\hbar}\delta\omega_{\rho\sigma}[M^{\rho\sigma}, M^{\mu\nu}] \tag{8.666}$$

$$= M^{\mu\nu} + \frac{i}{2\hbar}\delta\omega_{\rho\sigma}[M^{\mu\nu}, M^{\rho\sigma}] \tag{8.667}$$

RHS:

$$\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}M^{\rho\sigma} \simeq \left(\delta^{\mu}{}_{\rho} + \delta\omega^{\mu}{}_{\rho}\right)\left(\delta^{\nu}{}_{\sigma} + \delta\omega^{\nu}{}_{\sigma}\right)M^{\rho\sigma} \tag{8.668}$$

$$\simeq M^{\mu\nu} + \delta^{\mu}{}_{\rho}\delta\omega^{\nu}{}_{\sigma}M^{\rho\sigma} + \delta^{\nu}{}_{\sigma}\delta\omega^{\mu}{}_{\rho}M^{\rho\sigma} \tag{8.669}$$

$$\simeq M^{\mu\nu} + \delta\omega^{\nu}{}_{\sigma}M^{\mu\sigma} + \delta\omega^{\mu}{}_{\rho}M^{\rho\nu} \tag{8.670}$$

$$\simeq M^{\mu\nu} + \delta\omega_{\alpha\sigma}g^{\alpha\nu}M^{\mu\sigma} + \delta\omega_{\alpha\rho}g^{\alpha\mu}M^{\rho\nu} \tag{8.671}$$

$$\simeq M^{\mu\nu} + \delta\omega_{\alpha\sigma}(g^{\alpha\nu}M^{\mu\sigma} + g^{\alpha\mu}M^{\sigma\nu}) \tag{8.672}$$

$$\simeq M^{\mu\nu} + \delta\omega_{\rho\sigma}(g^{\rho\nu}M^{\mu\sigma} + g^{\rho\mu}M^{\sigma\nu}) \tag{8.673}$$

$$\simeq M^{\mu\nu} + \frac{1}{2}\delta\omega_{\rho\sigma}\left(g^{\rho\nu}(M^{\mu\sigma} - M^{\sigma\mu}) + g^{\rho\mu}(M^{\sigma\nu} - M^{\nu\sigma})\right) \tag{8.674}$$

$$\simeq M^{\mu\nu} + \frac{1}{2}\delta\omega_{\rho\sigma}\left(g^{\rho\nu}M^{\mu\sigma} - g^{\nu\rho}M^{\sigma\mu} + g^{\rho\mu}M^{\sigma\nu} - g^{\mu\rho}M^{\nu\sigma}\right) \tag{8.675}$$

Now we use the antisymmetry of M

$$\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}M^{\rho\sigma} \simeq M^{\mu\nu} + \frac{1}{2}\delta\omega_{\rho\sigma}\left(g^{\nu\rho}M^{\mu\sigma} - g^{\nu\rho}M^{\sigma\mu} + g^{\rho\mu}M^{\sigma\nu} - g^{\mu\rho}M^{\nu\sigma}\right) \tag{8.676}$$

$$\simeq M^{\mu\nu} - \frac{1}{2}\delta\omega_{\rho\sigma}\left(-g^{\nu\rho}M^{\mu\sigma} + g^{\nu\rho}M^{\sigma\mu} - g^{\rho\mu}M^{\sigma\nu} + g^{\mu\rho}M^{\nu\sigma}\right) \tag{8.677}$$

$$\simeq M^{\mu\nu} - \frac{1}{2}\delta\omega_{\rho\sigma}\left(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma} - g^{\rho\mu}M^{\sigma\nu} + g^{\nu\rho}M^{\sigma\mu}\right) \tag{8.678}$$

$$\simeq M^{\mu\nu} - \frac{1}{2}\delta\omega_{\rho\sigma}\left(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma}\right) - \frac{1}{2}\underbrace{\delta\omega_{\rho\sigma}\left(-g^{\rho\mu}M^{\sigma\nu} + g^{\nu\rho}M^{\sigma\mu}\right)}_{=\underbrace{\delta\omega_{\sigma\rho}\left(-g^{\sigma\mu}M^{\rho\nu} + g^{\nu\sigma}M^{\rho\mu}\right)}_{=-\delta\omega_{\rho\sigma}\left(-g^{\mu\sigma}(-M^{\nu\rho})+g^{\nu\sigma}(-M^{\mu\rho})\right)}} \tag{8.679}$$

$$\simeq M^{\mu\nu} - \frac{1}{2}\delta\omega_{\rho\sigma}\left(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma}\right) - \frac{1}{2}\delta\omega_{\rho\sigma}\left(-g^{\mu\sigma}M^{\nu\rho} + g^{\nu\sigma}M^{\mu\rho}\right) \tag{8.680}$$

$$\simeq M^{\mu\nu} - \frac{1}{2}\delta\omega_{\rho\sigma}\left(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma} - g^{\mu\sigma}M^{\nu\rho} + g^{\nu\sigma}M^{\mu\rho}\right) \tag{8.681}$$

As the components of $\delta\omega$ (besides the antisymmetry) are independent we get

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\hbar\left(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma} - g^{\mu\sigma}M^{\nu\rho} + g^{\nu\sigma}M^{\mu\rho}\right) \tag{8.682}$$

8.9.6 Problem 2.4 - Commutators of LT generators II

Preliminary calculations

$$\epsilon_{ijk}J_k = \varepsilon_{ijk}\frac{1}{2}\varepsilon_{kab}M^{ab} \tag{8.683}$$

$$= -\frac{1}{2}\varepsilon_{kij}\varepsilon_{kab}M^{ab} \tag{8.684}$$

$$= -\frac{1}{2}\left(\delta_{ia}\delta_{jb} - \delta_{ja}\delta_{ib}\right)M^{ab} \tag{8.685}$$

$$= -\frac{1}{2}\left(M^{ij} - M^{ji}\right) \tag{8.686}$$

$$= -M^{ij} \tag{8.687}$$

- With

$$J_1 = \frac{1}{2}(\varepsilon_{123}M^{23} + \varepsilon_{132}M^{32}) \tag{8.688}$$

$$= \varepsilon_{123}M^{23} \tag{8.689}$$

$$= M^{23} \tag{8.690}$$

  then

$$[J_1, J_3] = [M^{23}, M^{12}] \tag{8.691}$$

$$= i\hbar\left(g^{21}M^{32} - g^{31}M^{22} - g^{22}M^{31} + g^{32}M^{21}\right) \tag{8.692}$$

$$= -i\hbar g^{22}M^{31} \tag{8.693}$$

$$= -i\hbar M^{31} \tag{8.694}$$

$$= -i\hbar J_2 \tag{8.695}$$

- analog ...

•

$$
\begin{aligned}
[K^i, K^j] &= [M^{i0}, M^{j0}] &(8.696)\\
&= i\hbar \left(g^{ij} M^{00} - g^{0j} M^{i0} - g^{i0} M^{0j} + g^{00} M^{ij}\right) &(8.697)\\
&= i\hbar \left(-\delta^{ij} M^{00} + M^{ij}\right) &(8.698)\\
&= \begin{cases} i\hbar M^{ij} = -i\hbar \epsilon_{ijk} J_k & (i = j) \\ 0 & (i \neq j) \end{cases} &(8.699)
\end{aligned}
$$

where we used the result from the preliminary calculation in the last step.

8.9.7 Problem 2.7 - Translation operator

The obvious property $T(a)T(b) = T(a+b)$. Then

$$
\begin{aligned}
T(\delta a + \delta b) &= T(\delta a)T(\delta b) &(8.700)\\
&= \left(1 - \frac{i}{\hbar}\delta a_\mu P^\mu\right)\left(1 - \frac{i}{\hbar}\delta b_\nu P^\nu\right) &(8.701)\\
&\simeq 1 - \frac{i}{\hbar}(\delta a_\mu + \delta b_\mu)P^\mu + \frac{1}{\hbar^2}\delta a_\mu \delta b_\mu P^\mu P^\nu &(8.702)
\end{aligned}
$$

and

$$
\begin{aligned}
T(\delta a + \delta b) &= T(\delta b)T(\delta a) &(8.703)\\
&= \left(1 - \frac{i}{\hbar}\delta b_\nu P^\nu\right)\left(1 - \frac{i}{\hbar}\delta a_\mu P^\mu\right) &(8.704)\\
&\simeq 1 - \frac{i}{\hbar}(\delta a_\mu + \delta b_\mu)P^\mu + \frac{1}{\hbar^2}\delta a_\mu \delta b_\mu P^\nu P^\mu &(8.705)
\end{aligned}
$$

which implies $P^\mu P^\nu = P^\nu P^\mu$.

8.9.8 Problem 2.8 - Transformation of scalar field

(a) We start with

$$
\begin{aligned}
U(\Lambda)^{-1}\varphi(x)U(\Lambda) &= \varphi(\Lambda^{-1}x) &(8.706)\\
\left(1 - \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right)\varphi(x)\left(1 + \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right) &= \varphi([\delta^\mu_{\ \nu} - \delta\omega^\mu_{\ \nu}]x^\nu) &(8.707)\\
\varphi(x) - \frac{i}{2\hbar}\delta\omega_{\mu\nu}[M^{\mu\nu}, \varphi(x)] &= \varphi(x) - \delta\omega^\mu_{\ \nu} x^\nu \frac{\partial\varphi}{\partial x^\mu} &(8.708)\\
&= \varphi(x) - \delta\omega^\mu_{\ \nu}\frac{1}{2}\left(x^\nu \frac{\partial\varphi}{\partial x^\mu} - x^\mu \frac{\partial\varphi}{\partial x^\nu}\right) &(8.709)\\
&= \varphi(x) - \delta\omega_{\mu\nu}\frac{1}{2}\left(x^\nu\partial^\mu - x^\mu\partial^\nu\right)\varphi &(8.710)
\end{aligned}
$$

and therefore

$$
[\varphi, M^{\mu\nu}] = \frac{\hbar}{i}(x^\mu\partial^\nu - x^\nu\partial^\mu)\varphi \tag{8.711}
$$

(b) (c) (d) (e) (f)

8.9.9 Problem 3.2 - Multiparticle eigenstates of the hamiltonian

With

$$|k_1...k_n\rangle = a^\dagger_{k_1}...a^\dagger_{k_n}|0\rangle \tag{8.712}$$
$$H = \int \widetilde{dk}\, \omega_k a^\dagger_k a_k \tag{8.713}$$
$$[a_k, a^\dagger_q] = \underbrace{(2\pi)^3 2\omega_k \delta^3(\vec{k}-\vec{q})}_{\delta_{kq}} \tag{8.714}$$

we see that the expression which needs calculating is the creation and annihilation operators. The idea is to use the commutation relations to move the a_k to the right end to use $a_k|0\rangle$

$$\begin{aligned} a^\dagger_k a_k a^\dagger_{k_1}...a^\dagger_{k_n}|0\rangle &= a^\dagger_k(a^\dagger_{k_1}a_k + \delta_{kk_1})a^\dagger_{k_2}...a^\dagger_{k_n}|0\rangle && (8.715)\\ &= \delta_{kk_1}a^\dagger_k a^\dagger_{k_2}...a^\dagger_{k_n}|0\rangle + a^\dagger_k a^\dagger_{k_1}a_k a^\dagger_{k_2}...a^\dagger_{k_n}|0\rangle && (8.716)\\ &= ... && (8.717)\\ &= \sum_j \delta_{kk_j} a^\dagger_k \underbrace{a^\dagger_{k_2}...a^\dagger_{k_n}}_{(n-1)\text{ times with } a_{k_j} \text{ missing}} |0\rangle + a^\dagger_k a^\dagger_{k_1}...a^\dagger_{k_n}\underbrace{a_k|0\rangle}_{=0}. && (8.718) \end{aligned}$$

Therefore we obtain

$$\begin{aligned} H|k_1...k_n\rangle &= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\omega_k \sum_j \delta_{kk_j} a^\dagger_k a^\dagger_{k_2}...a^\dagger_{k_n}|0\rangle && (8.719)\\ &= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\omega_k \sum_j (2\pi)^3 2\omega_k \delta^3(\vec{k}-\vec{k}_j) a^\dagger_k a^\dagger_{k_2}...a^\dagger_{k_n}|0\rangle && (8.720)\\ &= \int d^3k \omega_k \sum_j \delta^3(\vec{k}-\vec{k}_j) a^\dagger_k a^\dagger_{k_2}...a^\dagger_{k_n}|0\rangle && (8.721) \end{aligned}$$

which we can integrate obtaining the desired result

$$\begin{aligned} H|k_1...k_n\rangle &= \sum_j \omega_{k_j} a^\dagger_{k_j} a^\dagger_{k_2}...a^\dagger_{k_n}|0\rangle && (8.722)\\ &= \left(\sum_j \omega_{k_j}\right) a^\dagger_{k_1} a^\dagger_{k_2}...a^\dagger_{k_n}|0\rangle && (8.723)\\ &= \left(\sum_j \omega_{k_j}\right) |k_1...k_n\rangle. && (8.724) \end{aligned}$$

8.9.10 Problem 3.4 - Heisenberg equations of motion for free field

(a) For the translation operator $T(a) = e^{-iP^\mu a_\mu}$ we expand in first order

$$\begin{aligned} T(a)^{-1}\varphi(a)T(a) &= \left(1-(-i)P^\mu a_\mu + \mathcal{O}(a^2)\right)\varphi(x)\left(1+(-i)P^\mu a_\mu + \mathcal{O}(a^2)\right) && (8.725)\\ &= \left(1+iP^\mu a_\mu + \mathcal{O}(a^2)\right)\varphi(x)\left(1-iP^\mu a_\mu + \mathcal{O}(a^2)\right) && (8.726)\\ &\simeq \varphi(x) + ia_\mu P^\mu \varphi(x) - ia_\mu \varphi(x) P^\mu && (8.727)\\ &\simeq \varphi(x) + ia_\mu [P^\mu, \varphi(x)] && (8.728) \end{aligned}$$

for the right hand right we get

$$\varphi(x-a) \simeq \varphi(x) - \partial^\mu \varphi(x) a_\mu \tag{8.729}$$

and therefore

$$i[P^\mu, \varphi(x)] = -\partial^\mu \varphi(x) \tag{8.730}$$

(b) With $\mu = 0$ and $\partial^0 = g_{0\nu}\partial_\nu = -\partial_0$ we have

$$\begin{aligned} i[H, \varphi(x)] &= -\partial^0 \varphi(x) = +\partial_0 \varphi(x) &(8.731)\\ \rightarrow \quad \dot{\varphi}(x) &= i[H, \varphi(x)] &(8.732)\end{aligned}$$

(c) We start with the hamiltonian (3.25)

$$H = \int d^3y \frac{1}{2}\Pi^2(y) + \frac{1}{2}(\nabla_y \varphi(y))^2 + \frac{1}{2}m^2\varphi(y)^2 - \Omega_0 \tag{8.733}$$

- Obtaining $\dot{\varphi}(x) = i[H, \varphi(x)]$

  We need to calculate (setting $x^0 = y^0$ - why can we?)

$$\begin{aligned}
[\Pi^2(y), \varphi(x)] &= \Pi(y)\Pi(y)\varphi(x) - \varphi(x)\Pi(y)\Pi(y) &(8.734)\\
&= \Pi(y)\Pi(y)\varphi(x) - \Pi(y)\varphi(x)\Pi(y) + \Pi(y)\varphi(x)\Pi(y) - \varphi(x)\Pi(y)\Pi(y) &(8.735)\\
&= \Pi(y)[\Pi(y), \varphi(x)] + [\Pi(y), \varphi(x)]\Pi(y) &(8.736)\\
&= 2\Pi(y)(-1)i\delta^3(\vec{y} - \vec{x}) &(8.737)\\
[(\nabla_y\varphi(y))^2, \varphi(x)] &= \nabla_y\varphi(y)\nabla_y\varphi(y)\varphi(x) - \varphi(x)\nabla_y\varphi(y)\nabla_y\varphi(y) &(8.738)\\
&= \nabla_y\varphi(y)[\nabla_y\varphi(y), \varphi(x)] + [\nabla_y\varphi(y), \varphi(x)]\nabla_y\varphi(y) &(8.739)\\
&= \nabla_y\varphi(y)\nabla_y[\varphi(y), \varphi(x)] + \nabla_y[\varphi(y), \varphi(x)]\nabla_y\varphi(y) &(8.740)\\
&= 0 &(8.741)\\
[\varphi(y)^2, \varphi(x)] &= \varphi(y)\varphi(y)\varphi(x) - \varphi(x)\varphi(y)\varphi(y) &(8.742)\\
&= \varphi(y)\varphi(y)\varphi(x) - \varphi(y)\varphi(x)\varphi(y) + \varphi(y)\varphi(x)\varphi(y) - \varphi(x)\varphi(y)\varphi(y) &(8.743)\\
&= \varphi(y)[\varphi(y), \varphi(x)] + [\varphi(y), \varphi(x)]\varphi(y) &(8.744)\\
&= 0 &(8.745)
\end{aligned}$$

  then

$$\begin{aligned}
\int d^3y[\Pi^2(y), \varphi(x)] &= -2i\Pi(x) &(8.746)\\
\int d^3y[(\nabla_y\varphi(y))^2, \varphi(x)] &= \int d^3y \nabla_y\varphi(y)[\nabla_y\varphi(y), \varphi(x)] + [\nabla_y\varphi(y), \varphi(x)]\nabla_y\varphi(y) &(8.747)\\
&= 0 &(8.748)\\
\int d^3y[\varphi(y)^2, \varphi(x)] &= 0 &(8.749)
\end{aligned}$$

  and therefore

$$\begin{aligned}
\dot{\varphi}(x) &= i[H, \varphi(x)] &(8.750)\\
&= i\frac{1}{2}(-2i)\Pi(x) &(8.751)\\
&= \Pi(x) &(8.752)
\end{aligned}$$

- Obtaining $\dot{\Pi}(x) = -i[H, \Pi(x)]$ (sign!?!)

Now we need to calculate - by using the results from above we can now shortcut a bit

$$[\Pi^2(y), \Pi(x)] = 0 \tag{8.753}$$
$$[(\nabla_y\varphi(y))^2, \Pi(x)] = (\nabla_y\varphi(y))(\nabla_y\varphi(y))\Pi(x) - \Pi(x)(\nabla_y\varphi(y))(\nabla_y\varphi(y)) \tag{8.754}$$
$$= (\nabla_y\varphi(y))[(\nabla_y\varphi(y)), \Pi(x)] - [\Pi(x), (\nabla_y\varphi(y))](\nabla_y\varphi(y)) \tag{8.755}$$
$$= (\nabla_y\varphi(y))\nabla_y[\varphi(y), \Pi(x)] - (\nabla_y[\Pi(x), \varphi(y)])(\nabla_y\varphi(y)) \tag{8.756}$$
$$= (\nabla_y\varphi(y))\nabla_y i\delta^3(\vec{x} - \vec{y}) - (\nabla_y(-i)\delta^3(\vec{x} - \vec{y}))(\nabla_y\varphi(y)) \tag{8.757}$$
$$= 2i(\nabla_y\delta^3(\vec{x} - \vec{y}))(\nabla_y\varphi(y)) \tag{8.758}$$
$$[\varphi(y)^2, \Pi(x)] = \varphi(y)\varphi(y)\Pi(x) - \Pi(x)\varphi(y)\varphi(y) \tag{8.759}$$
$$= \varphi(y)\varphi(y)\Pi(x) - \varphi(y)\Pi(x)\varphi(y) + \varphi(y)\Pi(x)\varphi(y) - \Pi(x)\varphi(y)\varphi(y) \tag{8.760}$$
$$= \varphi(y)[\varphi(y), \Pi(x)] + [\varphi(y), \Pi(x)]\varphi(y) \tag{8.761}$$
$$= 2i\varphi(y)\delta^3(\vec{x} - \vec{y}) \tag{8.762}$$

then

$$\int d^3y[\Pi^2(y), \Pi(x)] = 0 \tag{8.763}$$
$$\int d^3y[(\nabla_y\varphi(y))^2, \Pi(x)] = 2i\int d^3y(\nabla_y\delta^3(\vec{x} - \vec{y}))(\nabla_y\varphi(y)) \tag{8.764}$$
$$= -2i\int d^3y\delta^3(\vec{x} - \vec{y})(\nabla_y\nabla_y\varphi(y)) \tag{8.765}$$
$$= -2i\triangle_x\varphi(x) \tag{8.766}$$
$$\int d^3y[\varphi(y)^2, \Pi(x)] = 2i\varphi(x) \tag{8.767}$$

and therefore

$$\dot{\Pi}(x) = -i[H, \Pi(x)] \tag{8.768}$$
$$= -i\left(\frac{1}{2}(-2i)\triangle_x\varphi(x) + \frac{1}{2}m^2 2i\varphi(x)\right) \tag{8.769}$$
$$= -i\left(-i\triangle_x\varphi(x) + m^2 i\varphi(x)\right) \tag{8.770}$$
$$= -\triangle_x\varphi(x) + m^2\varphi(x) \tag{8.771}$$

which finally leads to (with $\Box = \partial_{tt} - \triangle$)

$$\partial^0\partial_0\varphi(x) = \partial^0\Pi(x) \tag{8.772}$$
$$= -\partial_0\Pi(x) \tag{8.773}$$
$$= -(-\triangle_x\varphi(x) + m^2\varphi(x)) \tag{8.774}$$
$$\to (\Box_x + m^2)\varphi(x) = 0 \tag{8.775}$$

(d) With

$$\vec{P} \equiv -\int d^3x\Pi(x)\nabla_x\varphi(x) \tag{8.776}$$

we have to calculate

$$[\vec{P}, \varphi(y)] = -\int d^3x[\Pi(x)\nabla_x\varphi(x), \varphi(y)]. \tag{8.777}$$

Let's start with

$$[\Pi(x)\nabla_x\varphi(x),\varphi(y)] = \Pi(x)\nabla_x\varphi(x)\varphi(y) - \varphi(y)\Pi(x)\nabla_x\varphi(x) \tag{8.778}$$
$$= \Pi(x)\nabla_x\varphi(x)\varphi(y) - (\Pi(x)\varphi(y) + i\delta^3(\vec{x}-\vec{y}))\nabla_x\varphi(x) \tag{8.779}$$
$$= \Pi(x)\nabla_x\varphi(x)\varphi(y) - \Pi(x)\varphi(y)\nabla_x\varphi(x) + i\delta^3(\vec{x}-\vec{y})\nabla_x\varphi(x) \tag{8.780}$$
$$= \Pi(x)\nabla_x(\varphi(x)\varphi(y)) - \Pi(x)\nabla_x(\varphi(y)\varphi(x)) + i\delta^3(\vec{x}-\vec{y})\nabla_x\varphi(x) \tag{8.781}$$
$$= \Pi(x)\nabla_x[\varphi(x),\varphi(y)] + i\delta^3(\vec{x}-\vec{y})\nabla_x\varphi(x) \tag{8.782}$$
$$= i\delta^3(\vec{x}-\vec{y})\nabla_x\varphi(x) \tag{8.783}$$

and then

$$[\vec{P},\varphi(y)] = -i\int d^3x\delta^3(\vec{x}-\vec{y})\nabla_x\varphi(x) \tag{8.784}$$
$$= -i\nabla_y\varphi(y) \tag{8.785}$$

(e) With

$$\Pi(x) = \dot{\varphi}(x) \tag{8.786}$$
$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k}(-i\omega_k)(a_k e^{ikx} - a_k^\dagger e^{-ikx}) \tag{8.787}$$
$$\nabla\varphi(x) = \int \frac{d^3q}{(2\pi)^3 2\omega_k}(i\vec{q})(a_q e^{iqx} - a_q^\dagger e^{-iqx}) \tag{8.788}$$
$$\tag{8.789}$$

then

$$\vec{P} = -\int d^3x\Pi(x)\nabla_x\varphi(x) \tag{8.790}$$
$$= -\iiint d^3x\frac{d^3k}{(2\pi)^3 2\omega_k}\frac{d^3q}{(2\pi)^3 2\omega_k}(-i\omega_k)(i\vec{q})(a_k e^{ikx} - a_k^\dagger e^{-ikx})(a_q e^{iqx} - a_q^\dagger e^{-iqx}) \tag{8.791}$$
$$= -\iiint d^3x\frac{d^3k}{(2\pi)^3 2}\frac{d^3q}{(2\pi)^3 2\omega_k}\vec{q}(a_k a_q e^{i(k+q)x} - a_k^\dagger a_q e^{-i(k-q)x} - a_k a_q^\dagger e^{i(k-q)x} + a_k^\dagger a_q^\dagger e^{-i(k+q)x}) \tag{8.792}$$
$$\tag{8.793}$$

now we can use the commutation relations and reindex

$$= -\iiint d^3x\frac{d^3kd^3q}{4\omega_k(2\pi)^6}\vec{q}(a_k a_q e^{i(k+q)x} - a_k^\dagger a_q e^{-i(k-q)x} - (a_q^\dagger a_k + (2\pi)^3 2\omega_k\delta^3(\vec{k}-\vec{q}))e^{i(k-q)x} + a_k^\dagger a_q^\dagger e^{-i(k+q)x}) \tag{8.794}$$
$$= -\iiint d^3x\frac{d^3kd^3q}{4\omega_k(2\pi)^6}\vec{q}(a_k a_q e^{i(k+q)x} + a_k^\dagger a_q^\dagger e^{-i(k+q)x}) + \iiint d^3x\frac{d^3kd^3q}{4\omega_k(2\pi)^6}\vec{q}2a_k^\dagger a_q e^{-i(k-q)x} \tag{8.795}$$
$$+ \iiint d^3x\frac{d^3kd^3q}{4\omega_k(2\pi)^6}\vec{q}(2\pi)^3 2\omega_k\delta^3(\vec{k}-\vec{q})e^{i(k-q)x} \tag{8.796}$$

Now we can look at the integrals individually and use the asymmetry. The first

$$-\iiint d^3x\frac{d^3kd^3q}{4\omega_k(2\pi)^6}\vec{q}(a_k a_q e^{i(k+q)x} + a_k^\dagger a_q^\dagger e^{-i(k+q)x}) = ... \tag{8.797}$$
$$= 0 \tag{8.798}$$

second

$$\iiint d^3x \frac{d^3kd^3q}{4\omega_k(2\pi)^6}\vec{q}(2\pi)^3 2\omega_k \delta^3(\vec{k}-\vec{q})e^{i(k-q)x} = \iiint d^3x \frac{d^3kd^3q}{2(2\pi)^3}\vec{q}\delta^3(\vec{k}-\vec{q})e^{i(k-q)x} \tag{8.799}$$

$$= \iiint d^3x \frac{d^3k}{2(2\pi)^3}\vec{k} \tag{8.800}$$

$$= 0 \tag{8.801}$$

and third

$$\iiint d^3x \frac{d^3kd^3q}{4\omega_k(2\pi)^6}\vec{q}2a_k^\dagger a_q e^{-i(k-q)x} = \iint \frac{d^3kd^3q}{4\omega_k(2\pi)^6}\vec{q}2a_k^\dagger a_q \int d^3x\ e^{-i(k-q)x} \tag{8.802}$$

$$= \iint \frac{d^3kd^3q}{4\omega_k(2\pi)^6}\vec{q}2a_k^\dagger a_q e^{-i(k-q)x} e^{-i(k^0-q^0)x^0} \int d^3x\ e^{-i(\vec{k}-\vec{q})\vec{x}} \tag{8.803}$$

$$= \iint \frac{d^3kd^3q}{4\omega_k(2\pi)^6}\vec{q}2a_k^\dagger a_q e^{-i(k-q)x} e^{-i(k^0-q^0)x^0} (2\pi)^3\delta^3(\vec{k}-\vec{q}) \tag{8.804}$$

$$= \int \frac{d^3k}{2\omega_k(2\pi)^3}\vec{k}a_k^\dagger a_k \tag{8.805}$$

$$= \int \widetilde{d^3k}\ \vec{k}a_k^\dagger a_k \tag{8.806}$$

Therefore we obtain

$$\vec{P} = \int \frac{d^3k}{2\omega_k(2\pi)^3}\vec{k}a_k^\dagger a_k \tag{8.807}$$

$$= \int \widetilde{d^3k}\ \vec{k}a_k^\dagger a_k \tag{8.808}$$

8.9.11 Problem 3.5 - Complex scalar field

(a) Sloppy way - Calculating the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \varphi} = -m^2\varphi^\dagger \tag{8.809}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = -\partial^\mu \varphi^\dagger \tag{8.810}$$

$$\to -m^2\varphi^\dagger + \partial_\mu\partial^\mu\varphi^\dagger = 0 \tag{8.811}$$

$$\to (\partial_\mu\partial^\mu - m^2)\varphi^\dagger = 0 \tag{8.812}$$

Bit more rigorous with

$$\frac{\delta\phi(x_1,t_1)}{\delta\phi(x_2,t_2)} = \delta(x_1-x_2)\times\delta(t_1-t_2) \tag{8.813}$$

$$\frac{\delta\partial_\mu\phi(x)}{\delta\phi(y)} = \frac{\delta}{\delta\phi(y)}\lim_{\epsilon\to 0}\frac{\phi(x_1,x_\mu+\epsilon,...,x_4)-\phi(x_1,x_2,x_3,x_4)}{\epsilon} \tag{8.814}$$

$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\delta(x_\mu+\epsilon-y_\mu)-\delta(x_\mu-y_\mu)\right)\times\delta(x_1-y_1)\times ...\times\delta(x_4-y_4) \tag{8.815}$$

$$= \frac{\partial}{\partial x^\mu}\delta^4(x-y) \tag{8.816}$$

we get

$$S[\varphi] = \int d^4x \left(-\partial^\mu \varphi^\dagger(x)\partial_\mu\varphi(x) - m^2\varphi^\dagger(x)\varphi(x)\right) \tag{8.817}$$

$$\frac{\delta S[\varphi]}{\delta\varphi(y)} = \int d^4x \left(-\partial^\mu \varphi^\dagger(x)\partial_\mu\delta^4(x-y) - m^2\varphi^\dagger(y)\delta^4(x-y)\right) \tag{8.818}$$

$$= \int d^4x \left(\partial_\mu\partial^\mu \varphi^\dagger(x)\delta^4(x-y) - m^2\varphi^\dagger(x)\delta^4(x-y)\right) \tag{8.819}$$

$$= (\Box_y - m^2)\varphi^\dagger(y) \tag{8.820}$$

(b) With

$$\mathcal{L} = -\partial^0\varphi^\dagger\partial_0\varphi - \partial^a\varphi^\dagger\partial_a\varphi - m^2\varphi^\dagger\varphi + \Omega_0 \tag{8.821}$$

$$= \partial_0\varphi^\dagger\partial_0\varphi - \partial^a\varphi^\dagger\partial_a\varphi - m^2\varphi^\dagger\varphi + \Omega_0 \tag{8.822}$$

$$\Pi = \frac{\partial\mathcal{L}}{\partial\dot\varphi} = \dot\varphi^\dagger \tag{8.823}$$

$$\Pi^\dagger = \frac{\partial\mathcal{L}}{\partial\dot\varphi^\dagger} = \dot\varphi \tag{8.824}$$

$$\to \mathcal{H} = \Pi\dot\varphi + \Pi^\dagger\dot\varphi^\dagger - \mathcal{L} \tag{8.825}$$

$$= \dot\varphi^\dagger\dot\varphi + \dot\varphi\dot\varphi^\dagger - \dot\varphi^\dagger\dot\varphi + (\nabla^a\varphi^\dagger)(\nabla_a\varphi) + m^2\varphi^\dagger\varphi - \Omega_0 \tag{8.826}$$

$$= \Pi^\dagger\Pi + (\nabla^a\varphi^\dagger)(\nabla_a\varphi) + m^2\varphi^\dagger\varphi - \Omega_0 \tag{8.827}$$

(c) Considering the plane wave solutions $e^{i\vec{k}\vec{x}\pm i\omega_k t}$ with

$$kx = g_{\mu\nu}k^\mu x^\nu = g_{00}k^0x^0 + g_{ik}k^ix^k = -\omega_k t + \vec{k}\vec{x} \tag{8.828}$$

we have

$$\varphi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\; a_k e^{ikx} + b_k^\dagger e^{-ikx} \tag{8.829}$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\; a_k e^{i\vec{k}\vec{x}-i\omega_k t} + b_k^\dagger e^{-i\vec{k}\vec{x}+i\omega_k t} \tag{8.830}$$

$$e^{-iqx}\varphi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\; a_k e^{i(k-q)x} + b_k^\dagger e^{-i\vec{k}\vec{x}+i\omega_k t}e^{-iqx} \tag{8.831}$$

$$\int d^3x e^{-iqx}\varphi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\; a_k \underbrace{\int d^3x e^{i(k-q)x}}_{(2\pi)^3\delta^3(\vec{k}-\vec{q})e^{-i(\omega_k-\omega_q)t}} + b_{-k}\underbrace{\int d^3x e^{i(\vec{k}-\vec{q})\vec{x}}}_{(2\pi)^3\delta^3(\vec{k}-\vec{q})} e^{i(\omega_k+\omega_q)t} \tag{8.832}$$

$$= \frac{1}{2\omega_q}\left(a_q + b_{-q}^\dagger e^{2i\omega_q t}\right) \tag{8.833}$$

and

$$\partial_0\varphi(\vec{x},t) = \int \frac{d^3k\,(-i\omega_k)}{(2\pi)^3 2\omega_k}\; a_k e^{i\vec{k}\vec{x}-i\omega_k t} - b_k^\dagger e^{-i\vec{k}\vec{x}+i\omega_k t} \tag{8.834}$$

$$\int d^3x e^{-iqx}\partial_0\varphi(\vec{x},t) = -\frac{i}{2}\left(a_q - b_{-q}^\dagger e^{2i\omega_q t}\right) \tag{8.835}$$

adding both equations gives with $\partial_0 e^{-iqx} = \partial_0 e^{-i(-\omega_k t+\vec{k}\vec{x})} = -i\omega_q e^{-iqx}$ and $f \overset{\leftrightarrow}{\partial_\mu} g = f(\partial_\mu g) - (\partial_\mu f)g$

$$a_q = \omega_q \int d^3x e^{-iqx}\varphi(\vec{x},t) + i\int d^3x e^{-iqx}\partial_0\varphi(\vec{x},t) \tag{8.836}$$
$$= i\int d^3x e^{-iqx}(-i\omega_q + \partial_0)\varphi(\vec{x},t) \tag{8.837}$$
$$= i\int d^3x e^{-iqx} \overset{\leftrightarrow}{\partial_0} \varphi(\vec{x},t) \tag{8.838}$$

To get b_q we solve a second set of equations for $\varphi^\dagger$

$$\varphi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\ a_k e^{ikx} + b_k^\dagger e^{-ikx} \tag{8.839}$$
$$\to \varphi^\dagger(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\ a_k^\dagger e^{-ikx} + b_k e^{ikx} \tag{8.840}$$
$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\ b_k e^{ikx} + a_k^\dagger e^{-ikx} \tag{8.841}$$

Now b_k takes the role of a_k and we can just copy the solution

$$b_q = \omega_q \int d^3x e^{-iqx}\varphi^\dagger(\vec{x},t) + i\int d^3x e^{-iqx}\partial_0\varphi^\dagger(\vec{x},t) \tag{8.842}$$
$$= i\int d^3x e^{-iqx}(-i\omega_q + \partial_0)\varphi^\dagger(\vec{x},t) \tag{8.843}$$
$$= i\int d^3x e^{-iqx} \overset{\leftrightarrow}{\partial_0} \varphi^\dagger(\vec{x},t) \tag{8.844}$$

(d) Starting with the observation

$$[A,B]^\dagger = (AB)^\dagger - (BA)^\dagger \tag{8.845}$$
$$= B^\dagger A^\dagger - A^\dagger B^\dagger \tag{8.846}$$
$$= [B^\dagger, A^\dagger] \tag{8.847}$$
$$= -[A^\dagger, B^\dagger] \tag{8.848}$$

therefore the relevant commutation relations for the fields are

$$[\varphi(\vec{x},t),\varphi(\vec{y},t)] = 0 \qquad \to \qquad [\varphi^\dagger(\vec{x},t),\varphi^\dagger(\vec{y},t)] = 0 \tag{8.849}$$
$$[\varphi^\dagger(\vec{x},t),\varphi(\vec{y},t)] = 0 \tag{8.850}$$
$$[\Pi(\vec{x},t),\Pi(\vec{y},t)] = 0 \qquad \to \qquad [\Pi^\dagger(\vec{x},t),\Pi^\dagger(\vec{y},t)] = 0 \tag{8.851}$$
$$[\Pi^\dagger(\vec{x},t),\Pi(\vec{y},t)] = 0 \tag{8.852}$$
$$[\varphi(\vec{x},t),\Pi(\vec{y},t)] = i\delta^3(\vec{x}-\vec{y}) \qquad \to \qquad [\varphi^\dagger(\vec{x},t),\Pi^\dagger(\vec{y},t)] = i\delta^3(\vec{x}-\vec{y}) \tag{8.853}$$
$$[\varphi^\dagger(\vec{x},t),\Pi(\vec{y},t)] = 0 \qquad \to \qquad [\varphi(\vec{x},t),\Pi^\dagger(\vec{y},t)] = 0 \tag{8.854}$$

with the previous results

$$a_q = i\int d^3x e^{-iqx}(-i\omega_q + \partial_0)\varphi(\vec{x},t) \tag{8.855}$$

$$= i\int d^3x e^{-iqx}(-i\omega_q\varphi(\vec{x},t) + \Pi^\dagger(\vec{x},t)) \tag{8.856}$$

$$a_q^\dagger = i\int d^3x e^{iqx}(i\omega_q\varphi^\dagger(\vec{x},t) + \Pi(\vec{x},t)) \tag{8.857}$$

$$b_q = i\int d^3x e^{-iqx}(-i\omega_q + \partial_0)\varphi^\dagger(\vec{x},t) \tag{8.858}$$

$$= i\int d^3x e^{-iqx}(-i\omega_q\varphi(\vec{x},t) + \Pi(\vec{x},t)) \tag{8.859}$$

$$b_q^\dagger = i\int d^3x e^{iqx}(i\omega_q\varphi^\dagger(\vec{x},t) + \Pi^\dagger(\vec{x},t)) \tag{8.860}$$

let's calculate each of the commutators

$$[a_k, a_q^\dagger] = \iint d^3x\, d^3y\, e^{-ikx}e^{iqy}\left(\omega_k\omega_q[\varphi_x,\varphi_y^\dagger] - i\omega_q[\varphi_x,\Pi_y] + i\omega_q[\Pi_x^\dagger,\varphi_y^\dagger] + [\Pi_x^\dagger,\Pi_y]\right) \tag{8.861}$$

$$= \iint d^3x\, d^3y\, e^{-i(kx-qy)}\left(-i\omega_q[\varphi_x,\Pi_y] + i\omega_q[\Pi_x^\dagger,\varphi_y^\dagger]\right) \tag{8.862}$$

$$= \iint d^3x\, d^3y\, e^{-i(kx-qy)}\left(-i\omega_q i\delta^3(\vec{x}-\vec{y}) + i\omega_q(-i)\delta^3(\vec{x}-\vec{y})\right) \tag{8.863}$$

$$= (\omega_q + \omega_q)\iint d^3x\, e^{-i(k-q)x} \tag{8.864}$$

$$= (\omega_q + \omega_q)\,(2\pi)^3\delta^3(\vec{k}-\vec{q}) \tag{8.865}$$

$$= 2\omega_q(2\pi)^3\delta^3(\vec{k}-\vec{q}) \tag{8.866}$$

and so on

$$[b_k, b_q^\dagger] = ... = 2\omega_q(2\pi)^3\delta^3(\vec{k}-\vec{q}) \tag{8.867}$$

(e) Now

$$H = \int d^3x\, \Pi^\dagger\Pi + (\nabla^a\varphi^\dagger)(\nabla_a\varphi) + m^2\varphi^\dagger\varphi - \Omega_0 \tag{8.868}$$

$$\Pi^\dagger\Pi = \dot{\varphi}\dot{\varphi}^\dagger \tag{8.869}$$

$$= \int \widetilde{d^3k}\widetilde{d^3q}(i\omega_k)(i\omega_q)\left(a_k e^{ikx} - b_k^\dagger e^{-ikx}\right)\left(a_q^\dagger e^{-iqx} - b_q e^{iqx}\right) \tag{8.870}$$

$$= \int \widetilde{d^3k}\widetilde{d^3q}(-\omega_k\omega_q)\left(a_k a_q^\dagger e^{-iqx}e^{ikx} - b_k^\dagger a_q^\dagger e^{-iqx}e^{-ikx} - a_k b_q e^{iqx}e^{ikx} + b_k^\dagger b_q e^{iqx}e^{-ikx}\right) \tag{8.871}$$

$$= \int \widetilde{d^3k}\widetilde{d^3q}(-\omega_k\omega_q)\left([a_q^\dagger a_k - 2\omega_k(2\pi)^3\delta^3(\vec{k}-\vec{q})]e^{-i(q-k)x} - b_k^\dagger a_q^\dagger e^{-i(q+k)x} - a_k b_q e^{i(q+k)x} + b_k^\dagger b_q e^{i(q-k)x}\right) \tag{8.872}$$

$$(\nabla^a\varphi^\dagger)(\nabla_a\varphi) = \int \widetilde{d^3k}\widetilde{d^3q}(k^a q_a)\left(-a_k^\dagger e^{-ikx} + b_k e^{ikx}\right)\left(a_q e^{iqx} - b_q^\dagger e^{-iqx}\right) \tag{8.873}$$

$$= \int \widetilde{d^3k}\widetilde{d^3q}(k^a q_a)\left(-a_k^\dagger a_q e^{iqx}e^{-ikx} + b_k a_q e^{iqx}e^{ikx} + a_k^\dagger b_q^\dagger e^{-iqx}e^{-ikx} - b_k b_q^\dagger e^{-iqx}e^{ikx}\right) \tag{8.874}$$

$$= \int \widetilde{d^3k}\widetilde{d^3q}(k^a q_a)\left(-a_k^\dagger a_q e^{i(q-k)x} + a_q b_k e^{i(q+k)x} + a_k^\dagger b_q^\dagger e^{-i(q+k)x} - [b_q^\dagger b_k - 2\omega_k(2\pi)^3\delta^3(\vec{k}-\vec{q})]e^{-i(q-k)} \right. \tag{8.875}$$

$$\varphi^\dagger\varphi = \int \widetilde{d^3k}\widetilde{d^3q}\left(a_k^\dagger e^{-ikx} + b_k e^{ikx}\right)\left(a_q e^{iqx} + b_q^\dagger e^{-iqx}\right) \tag{8.876}$$
$$= \int \widetilde{d^3k}\widetilde{d^3q}\left(a_k^\dagger a_q e^{iqx}e^{-ikx} + b_k a_q e^{iqx}e^{ikx} + a_k^\dagger b_q^\dagger e^{-iqx}e^{-ikx} + b_k b_q^\dagger e^{-iqx}e^{ikx}\right) \tag{8.877}$$
$$= \int \widetilde{d^3k}\widetilde{d^3q}\left(a_k^\dagger a_q e^{i(q-k)x} + a_q b_k e^{i(q+k)x} + a_k^\dagger b_q^\dagger e^{-i(q+k)x} + [b_q^\dagger b_k - 2\omega_k(2\pi)^3\delta^3(\vec{k}-\vec{q})]e^{-i(q-k)x}\right) \tag{8.878}$$

then

$$H_{a^\dagger a} = \int \widetilde{d^3k}\widetilde{d^3q}\int d^3x\left[(-\omega_k\omega_q)[a_q^\dagger a_k - 2\omega_k(2\pi)^3\delta^3(\vec{k}-\vec{q})]e^{-i(q-k)x}\right] \tag{8.879}$$
$$+\int \widetilde{d^3k}\widetilde{d^3q}\int d^3x(k^a q_a)\left[-a_k^\dagger a_q e^{i(q-k)x}\right] + m^2 a_k^\dagger a_q e^{i(q-k)x} \tag{8.880}$$
$$= \int \widetilde{d^3k}\widetilde{d^3q}\; a_k^\dagger a_q\left[-\omega_k\omega_q - k^a q_a + m^2\right]\int d^3x e^{i(q-k)x} \tag{8.881}$$
$$-\int \widetilde{d^3k}\widetilde{d^3q}\;(-\omega_k\omega_q)2\omega_q(2\pi)^3\delta^3(\vec{q}-\vec{k})\int d^3x\; e^{i(q-k)x} \tag{8.882}$$
$$= \int \widetilde{d^3k}\frac{d^3q}{(2\pi)^3 2\omega_q}\; a_k^\dagger a_q\left[-\omega_k\omega_q - k^a q_a + m^2\right](2\pi)^3\delta^3(\vec{q}-\vec{k})e^{-i(\omega_q-\omega_k)t} \tag{8.883}$$
$$-\int \frac{d^3k}{(2\pi)^3 2\omega_k}\frac{1}{(2\pi)^3 2\omega_k}\;(-\omega_k^2)2\omega_k(2\pi)^3 e^{-i(\omega_k-\omega_k)t}\int d^3x \tag{8.884}$$
$$= \int \widetilde{d^3k}\frac{1}{2\omega_k}\; a_k^\dagger a_k \underbrace{\left[-\omega_k^2 - \vec{k}^2 + m^2\right]}_{2\omega_k^2!?!?!} + \frac{V}{2(2\pi)^3}\int d^3k\;\omega_k \tag{8.885}$$
$$= \int \widetilde{d^3k}\omega_k\; a_k^\dagger a_k + \frac{V}{2(2\pi)^3}\int d^3k\;\omega_k \tag{8.886}$$

and similar for $H_{b^\dagger b}, H_{ab}, H_{a^\dagger b^\dagger}$.

$$H = \int \widetilde{d^3k}\omega_k\;(a_k^\dagger a_k + b_k^\dagger b_k) + \frac{V}{2(2\pi)^3}\int d^3k\;\omega_k \tag{8.887}$$

8.9.12 Problem 4.1 - Commutator non-hermitian field

With $t = t'$ and $|\vec{x} - \vec{x}'| = r$ we have

$$[\varphi^+(x), \varphi^-(x')]_\pm = \int \widetilde{dk} e^{ik(x-x')} \tag{8.888}$$

$$= \int d^3k \frac{1}{(2\pi)^3 2\omega_k} e^{ik(x-x')} \tag{8.889}$$

$$= \frac{1}{2 \cdot 8\pi^3} \int d^3k \frac{1}{\sqrt{|k|^2 + m^2}} e^{i[\vec{k}(\vec{x}-\vec{x}')]} \tag{8.890}$$

$$= \frac{1}{16\pi^3} \int |k|^2 dk d\phi d\theta \sin\theta \frac{1}{\sqrt{|k|^2 + m^2}} e^{i|k|r\cos\theta} \tag{8.891}$$

$$= \frac{2\pi}{16\pi^3} \int |k|^2 dk \underbrace{d\theta \sin\theta}_{-d\cos\theta} \frac{1}{\sqrt{|k|^2 + m^2}} e^{i|k|r\cos\theta} \tag{8.892}$$

$$= \frac{2\pi}{16\pi^3} \int |k|^2 dk \frac{1}{\sqrt{|k|^2 + m^2}} \int_{-1}^{1} d\cos\theta e^{i|k|r\cos\theta} \tag{8.893}$$

$$= \frac{2\pi}{16\pi^3} \int |k|^2 dk \frac{1}{\sqrt{|k|^2 + m^2}} 2\frac{\sin(|k|r)}{|k|r} \tag{8.894}$$

$$= \frac{1}{4\pi^2 r} \int_0^\infty dk \frac{|k|\sin(|k|r)}{\sqrt{|k|^2 + m^2}} \tag{8.895}$$

With Gradshteyn, Ryzhik 7ed (8.486) - we find for the definition of the modified Bessel function K_1

$$\frac{d}{dz} K_0(z) = -K_1(z) \tag{8.896}$$

and Gradshteyn, Ryzhik 7ed (3.754)

$$\int_0^\infty dx \frac{\cos(ax)}{\sqrt{\beta^2 + x^2}} = K_0(a\beta) \tag{8.897}$$

therefore

$$\frac{d}{da} K_0(a\beta) = \int_0^\infty dx \frac{-x\sin(ax)}{\sqrt{\beta^2 + x^2}} \tag{8.898}$$

$$= \beta K_0'(a\beta) \tag{8.899}$$

$$= -\beta K_1(a\beta) \tag{8.900}$$

$$\to K_1(a\beta) = \frac{1}{\beta} \int_0^\infty dx \frac{x\sin(ax)}{\sqrt{\beta^2 + x^2}} \tag{8.901}$$

which we can use to finish the calculation

$$[\varphi^+(x), \varphi^-(x')]_\pm = \frac{1}{4\pi^2 r} m K_1(mr) \tag{8.902}$$

From `https://dlmf.nist.gov/10.30` we get

$$\lim_{z\to 0} K_\nu(z) \sim \frac{1}{2}\Gamma(\nu) \left(\frac{1}{2}z\right)^{-\nu} \tag{8.903}$$

$$\to \lim_{z\to 0} K_1(z) \sim \frac{1}{2} \left(\frac{1}{2}z\right)^{-1} = 1/z \tag{8.904}$$

and therefore

$$[\varphi^+(x), \varphi^-(x')]_\pm = \frac{1}{4\pi^2 r^2}. \tag{8.905}$$

8.9.13 Problem 5.1 - LSZ reduction for complex scalar field

From Exercise 3.5 we have

$$a_q = i\int d^3x e^{-iqx} \overleftrightarrow{\partial_0} \varphi(\vec{x},t) \tag{8.906}$$

$$a_q^\dagger = -i\int d^3x e^{iqx} \overleftrightarrow{\partial_0} \varphi^\dagger(\vec{x},t) \tag{8.907}$$

$$b_q = i\int d^3x e^{-iqx} \overleftrightarrow{\partial_0} \varphi^\dagger(\vec{x},t) \tag{8.908}$$

$$b_q^\dagger = -i\int d^3x e^{iqx} \overleftrightarrow{\partial_0} \varphi(\vec{x},t) \tag{8.909}$$

then

$$a_1^\dagger(+\infty) - a_1^\dagger(-\infty) = -i\int d^3k f_1(\vec{k})\int d^4x e^{ikx}(-\Box_x + m^2)\varphi^\dagger(x) \tag{8.910}$$

rearranging leads to

$$a_1^\dagger(-\infty) = a_1^\dagger(+\infty) + i\int d^3k f_1(\vec{k})\int d^4x e^{ikx}(-\Box_x + m^2)\varphi^\dagger(x) \tag{8.911}$$

$$a_1(+\infty) = a_1(-\infty) + i\int d^3k f_1(\vec{k})\int d^4x e^{-ikx}(-\Box_x + m^2)\varphi(x) \tag{8.912}$$

$$b_1^\dagger(-\infty) = b_1^\dagger(+\infty) + i\int d^3k f_1(\vec{k})\int d^4x e^{ikx}(-\Box_x + m^2)\varphi^\dagger(x) \tag{8.913}$$

$$b_1(+\infty) = b_1(-\infty) + i\int d^3k f_1(\vec{k})\int d^4x e^{-ikx}(-\Box_x + m^2)\varphi(x) \tag{8.914}$$

then we get for a, b particle scattering with the time ordering operator T (Later time to the Left)

$$\begin{aligned}
\langle f|i\rangle &= \langle 0|a_{1'}(+\infty)b_{2'}(+\infty)a_1^\dagger(-\infty)b_2^\dagger(-\infty)|0\rangle && (8.915)\\
&= \langle 0|Ta_{1'}(+\infty)b_{2'}(+\infty)a_1^\dagger(-\infty)b_2^\dagger(-\infty)|0\rangle && (8.916)\\
&= \langle 0|T(a_{1'}(-\infty) + i\int)(b_{2'}(-\infty) + i\int)(a_1^\dagger(+\infty) + i\int)(b_2^\dagger(+\infty) + i\int)|0\rangle && (8.917)\\
&= i^4\int d^4x_1' e^{-ik_1'x_1'}(-\Box_{x_1'} + m_a^2)\int d^4x_2' e^{-ik_2'x_2'}(-\Box_{x_2'} + m_b^2)\times && (8.918)\\
&\quad\times\int d^4x_1 e^{-ik_1x_1}(-\Box_{x_1} + m_a^2)\int d^4x_2 e^{-ik_2x_2}(-\Box_{x_2} + m_b^2)\langle 0|\phi_{x_1'}\phi_{x_2'}\phi_{x_1}^\dagger\phi_{x_2}^\dagger|0\rangle && (8.919)
\end{aligned}$$

8.9.14 Problem 6.1 - Path integral in quantum mechanics

(a) The transition amplitude $\langle q''|e^{-iH(t''-t')}|q'\rangle$ (particle to start at q', t' and ends at position q'' at time t'') can be written in the Heisenberg picture as

$$\begin{aligned}
\langle q''|e^{-iH(t''-t')}|q'\rangle &= \langle q''|e^{-iHt''}e^{iHt''}e^{-iH(t''-t')}e^{-iHt'}e^{iHt'}|q'\rangle && (8.920)\\
&= \langle q'',t''|e^{iHt''}e^{iH(t''-t')}e^{-iHt'}|q',t'\rangle && (8.921)\\
&= \langle q'',t''|q',t'\rangle. && (8.922)
\end{aligned}$$

Now we can do the standard path integral derivation

$$\langle q'',t''|q',t'\rangle = \int \left(\prod_{j=1}^{N} dq_j\right) \langle q''|e^{-iH\delta t}|q_N\rangle\langle q_N|e^{-iH\delta t}|q_{N-1}\rangle \dots \langle q_1|e^{-iH\delta t}|q'\rangle \tag{8.923}$$

$$= \int \left(\prod_{j=1}^{N} dq_j\right) \int \frac{dp_N}{2\pi} e^{-iH(p_N,q_N)\delta t} e^{ip_N(q'-q_N)} \dots \int \frac{dp'}{2\pi} e^{-iH(p',q')\delta t} e^{ip'(q_1-q')} \tag{8.924}$$

$$= \int \left(\prod_{j=1}^{N} dq_j\right) \left(\prod_{k=0}^{N} \frac{dp_k}{2\pi} e^{ip_k(q_{k+1}-q_k)} e^{-iH(p_k,q_k)\delta t}\right) \quad (q_0 = q', q_{N+1} = q'') \tag{8.925}$$

which under Weyl ordering (see Greiner, Reinhard - field quantization) has to be replaced by

$$\langle q'',t''|q',t'\rangle = \int \left(\prod_{j=1}^{N} dq_j\right) \left(\prod_{k=0}^{N} \frac{dp_k}{2\pi} e^{ip_k(q_{k+1}-q_k)} e^{-iH(p_k,\bar{q}_k)\delta t}\right) \quad \bar{q}_k = (q_{k+1}+q_k)/2 \tag{8.926}$$

$$= \int \left(\prod_{j=1}^{N} dq_j\right) \left(\prod_{k=0}^{N} \frac{dp_k}{2\pi} e^{i[p_k\dot{q}_k - H(p_k,\bar{q}_k)]\delta t}\right) \quad \dot{q}_k = (q_{k+1}-q_k)/\delta t \tag{8.927}$$

$$= \int \left(\prod_{j=1}^{N} dq_j\right) \left(\prod_{k=0}^{N} \frac{dp_k}{2\pi}\right) \left(e^{i\sum_{n=0}^{N}[p_n\dot{q}_n - H(p_n,\bar{q}_n)]\delta t}\right) \tag{8.928}$$

$$= \int \mathcal{D}q\mathcal{D}p \exp\left[i\int_{t'}^{t''} dt\, (p(t)\dot{q}(t) - H(p(t),q(t)))\right] \tag{8.929}$$

Let's now assume $H(p,q)$ has only a quadratic term in p which is independent of q meaning

$$H(p,q) = \frac{p^2}{2m} + V(q) \tag{8.930}$$

then

$$\langle q'',t''|q',t'\rangle = \int \left(\prod_{j=1}^{N} dq_j\right) \left(\prod_{k=0}^{N} \frac{dp_k}{2\pi}\right) \left(e^{i\sum_{n=0}^{N}[p_n\dot{q}_n - \frac{1}{2m}p_n^2 - V(\bar{q}_n)]\delta t}\right) \tag{8.931}$$

We can evaluate a single p-integral using

$$\int_{-\infty}^{\infty} dx e^{-ax^2+bx+c} = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}+c} \tag{8.932}$$

and obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dp_k \left(e^{i[p_k\dot{q}_k - \frac{1}{2m}p_k^2 - V(\bar{q}_k)]\delta t}\right) = \frac{1}{2\pi} e^{-iV(\bar{q}_k)\delta t} \int dp_k \left(e^{i[p_k\dot{q}_k - \frac{1}{2m}p_k^2]\delta t}\right) \tag{8.933}$$

$$= \frac{1}{2\pi} e^{-iV(\bar{q}_k)\delta t} \sqrt{\frac{\pi}{i\frac{\delta t}{2m}}} e^{\frac{-\dot{q}_k^2\delta t^2}{4\frac{i\delta t}{2m}}} \tag{8.934}$$

$$= \frac{1}{2\pi}\sqrt{\frac{2\pi m}{i\delta t}} e^{i\left(\frac{m\dot{q}_k^2}{2} - V(\bar{q}_k)\right)\delta t} \tag{8.935}$$

$$= \sqrt{\frac{m}{2\pi i\delta t}} e^{iL(\bar{q}_k,\dot{q}_k)\delta t}. \tag{8.936}$$

As there are $N+1$ p-integrals we have

$$\mathcal{D}q = \left(\frac{m}{2\pi i\delta t}\right)^{(N+1)/2} \prod_{j=1}^{N} dq_j \tag{8.937}$$

(b) We now assume $V(q) = 0$

$$\langle q'', t''|q', t'\rangle = \int \mathcal{D}q\, e^{i\int_{t'}^{t''} dt \frac{\dot{q}^2}{2m}} \tag{8.938}$$

$$= \lim_{N\to\infty} \left(\frac{m}{2\pi i\delta t}\right)^{\frac{N+1}{2}} \left(\prod_{j=1}^{N} \int_{-\infty}^{\infty} dq_j\, e^{im\frac{(q_j-q_{j+1})^2}{2\delta t^2}\delta t}\right) e^{im\frac{(q'-q_1)^2}{2\delta t}} e^{im\frac{(q_N-q'')^2}{2\delta t}} \tag{8.939}$$

$$= \lim_{N\to\infty} \left(\frac{m}{2\pi i\delta t}\right)^{\frac{N+1}{2}} \left(\prod_{j=3}^{N} \int_{-\infty}^{\infty} dq_j\, e^{im\frac{(q_j-q_{j+1})^2}{2\delta t}}\right) \int dq_2 e^{im\frac{(q_2-q_3)^2}{2\delta t}} \int dq_1 e^{im\frac{(q_1-q_2)^2}{2\delta t}} e^{im\frac{(q_0-q_1)^2}{2\delta t}} \tag{8.940}$$

now we can simplify the q_1-integral

$$\int_{-\infty}^{\infty} dq_1\, e^{im\frac{(q_1-q_2)^2}{2\delta t}} e^{im\frac{(q_0-q_1)^2}{2\delta t}} = \int_{-\infty}^{\infty} dq_1\, e^{\frac{im}{2\delta t}(q_0^2-2q_0q_1+q_1^2+q_1^2-2q_1q_2+q_2^2)} \tag{8.941}$$

$$= e^{\frac{im}{2\delta t}(q_0^2+q_2^2)} \int_{-\infty}^{\infty} dq_1\, e^{\frac{im}{\delta t}(q_1^2-q_1(q_2+q_0))} \tag{8.942}$$

$$= e^{\frac{im}{2\delta t}(q_0^2+q_2^2)} \sqrt{\frac{\pi\delta t}{m}} e^{\frac{i}{4}\left(\pi - \frac{(q_2+q_0)^2 m}{\delta t}\right)} \tag{8.943}$$

$$= e^{\frac{im}{4\delta t}(q_0-q_2)^2} \sqrt{\frac{\pi\delta t}{m}} \sqrt{i} \tag{8.944}$$

$$= e^{\frac{im}{4\delta t}(q_0-q_2)^2} \sqrt{\frac{i\pi\delta t}{m}} \tag{8.945}$$

now simplify the q_2-integral

$$\sqrt{\frac{i\pi\delta t}{m}} \int_{-\infty}^{\infty} dq_2 e^{\frac{im}{2\delta t}(q_2-q_3)^2} e^{\frac{im}{4\delta t}(q_0-q_2)^2} = \sqrt{\frac{i\pi\delta t}{m}} \int_{-\infty}^{\infty} dq_2 e^{\frac{im}{4\delta t}(2q_2^2-4q_3q_2+2q_3^2+q_0^2-2q_0q_2+q_2^2)} \tag{8.946}$$

$$= \sqrt{\frac{i\pi\delta t}{m}} \int_{-\infty}^{\infty} dq_2 e^{\frac{im}{4\delta t}(3q_2^2-(4q_3+2q_0)q_2+2q_3^2+q_0^2)} \tag{8.947}$$

$$= \sqrt{\frac{i\pi\delta t}{m}} e^{\frac{im}{4\delta t}(2q_3^2+q_0^2)} \int_{-\infty}^{\infty} dq_2 e^{\frac{im}{4\delta t}(3q_2^2-(4q_3+2q_0)q_2)} \tag{8.948}$$

$$= \sqrt{\frac{i\pi\delta t}{m}} e^{\frac{im}{4\delta t}(2q_3^2+q_0^2)} \sqrt{\frac{\pi 4\delta t}{3m}} e^{\frac{i}{4}\left(\pi - \frac{(4q_3+2q_0)^2 m}{12\delta t}\right)} \tag{8.949}$$

$$= \sqrt{\frac{i\pi\delta t}{m}} \sqrt{\frac{4i\pi\delta t}{3m}} e^{\frac{im}{6\delta t}(q_3-q_0)^2} \tag{8.950}$$

then we can extend the results (without explicitly proving)

$$\langle q'', t''|q', t'\rangle = \lim_{N\to\infty} \left(\frac{m}{2\pi i\delta t}\right)^{\frac{N+1}{2}} \prod_{j=1}^{N} \sqrt{\frac{2i\pi\delta t}{m}\frac{j}{j+1}} \cdot e^{\frac{im}{2(j+1)\delta t}(q''-q')^2} \tag{8.951}$$

$$= \lim_{N\to\infty} \sqrt{\frac{m}{2\pi i\delta t}} \sqrt{\frac{1}{N+1}} \cdot e^{\frac{im}{2(N+1)\delta t}(q_{N+1}-q_0)^2} \tag{8.952}$$

$$= \sqrt{\frac{m}{2\pi i(t''-t')}} \cdot e^{\frac{im(q''-q')^2}{2(t''-t')}}. \tag{8.953}$$

The exponent has the dimension $\mathrm{kg} \cdot \mathrm{m}^2/s$ which is the same as Js. So we just insert an $\hbar$

$$\langle q'', t''|q', t'\rangle = \sqrt{\frac{m}{2\pi i\hbar(t''-t')}} \cdot e^{\frac{im(q''-q')^2}{2\hbar(t''-t')}}. \tag{8.954}$$

(c) Simple - with $H|k\rangle = \frac{k^2}{2m}|k\rangle$ we get

$$\begin{aligned}
\langle q'', t''|q', t'\rangle &= \langle q''|\exp(-iH(t''-t'))|q'\rangle &(8.955)\\
&= \int dp \int dk \langle q''|p\rangle\langle p|\exp(-iH(t''-t'))|k\rangle\langle k|q'\rangle &(8.956)\\
&= \int dp \int dk \frac{1}{\sqrt{2\pi}} e^{ipq'} \langle p|k\rangle \exp(-i\frac{k^2}{2m}(t''-t'))\frac{1}{\sqrt{2\pi}} e^{-ikq''} &(8.957)\\
&= \int dp \int dk \frac{1}{\sqrt{2\pi}} e^{ipq'} \exp(-i\frac{k^2}{2m}(t''-t'))\delta(k-p)\frac{1}{\sqrt{2\pi}} e^{-ikq''} &(8.958)\\
&= \frac{1}{2\pi}\int dp e^{ip(q'-q'')} \exp(-i\frac{p^2}{2m}(t''-t')) &(8.959)\\
&= \frac{1}{2\pi}\sqrt{-\frac{2m\pi}{t''-t'}} e^{\frac{i}{4}\left(\pi - \frac{-2m(q''-q')^2}{t''-t'}\right)} &(8.960)\\
&= \sqrt{-\frac{im}{2\pi(t''-t')}} e^{-\frac{i}{4}\frac{-2m(q''-q')^2}{t''-t'}} &(8.961)\\
&= \sqrt{\frac{m}{2\pi i(t''-t')}} e^{\frac{-im(q''-q')^2}{2(t''-t')}} &(8.962)
\end{aligned}$$

which is the same as in (b).

8.9.15 Problem 7.1 - Oscillator Green's function I

$$\begin{aligned}
G(t-t') &= \int_{-\infty}^{+\infty} \frac{dE}{2\pi} \frac{e^{-iE(t-t')}}{-E^2+\omega^2-i\epsilon} &(8.963)\\
&= -\frac{1}{2\pi}\int_{-\infty}^{+\infty} dE \frac{e^{-iE(t-t')}}{E^2-\omega^2+i\epsilon} &(8.964)
\end{aligned}$$

with

$$\begin{aligned}
E^2-\omega^2+i\epsilon &= (E+\sqrt{\omega^2-i\epsilon})(E-\sqrt{\omega^2-i\epsilon}) &(8.965)\\
&= \left(E+\omega\sqrt{1-\frac{i\epsilon}{\omega^2}}\right)\left(E-\omega\sqrt{1-\frac{i\epsilon}{\omega^2}}\right) &(8.966)\\
&\simeq \left(E+\omega-\frac{i\epsilon}{2\omega}\right)\left(E-\omega+\frac{i\epsilon}{2\omega^2}\right) &(8.967)
\end{aligned}$$

we can simplify

$$\begin{aligned}
G(\Delta t) &= -\frac{1}{2\pi}\int_{-\infty}^{+\infty} dE e^{-iE\Delta t}\left(\frac{1}{E+\omega-\frac{i\epsilon}{2\omega}} + \frac{1}{E-\omega+\frac{i\epsilon}{2\omega}}\right) &(8.968)\\
&= -\frac{1}{2\pi}\frac{1}{2\left(\omega-\frac{i\epsilon}{2\omega}\right)}\int_{-\infty}^{+\infty} dE e^{-iE\Delta t}\left(-\frac{1}{E+\omega-\frac{i\epsilon}{2\omega}} + \frac{1}{E-\omega+\frac{i\epsilon}{2\omega}}\right) &(8.969)
\end{aligned}$$

Integrating along the closed contour along the lower half plane (seeing that the exponential function makes the arc part vanish - for $\Delta t > 0$) and using the residual theorem (only one pole is inside) we get (with $\epsilon \to 0$)

$$G(\Delta t) = +\frac{1}{2\pi}\frac{1}{2\left(\omega - \frac{i\epsilon}{2\omega}\right)}(2\pi i)e^{-i(\omega - \frac{i\epsilon}{2\omega})\Delta t} \tag{8.970}$$

$$= \frac{i}{2\omega}e^{-i\omega\Delta t} \tag{8.971}$$

For $\Delta t < 0$ we integrate along the contour of the upper plane - combining both results we get

$$G(t) = \frac{i}{2\omega}e^{-i\omega|t|} \tag{8.972}$$

8.9.16 Problem 7.2 - Oscillator Green's function II

We can rewrite the Greens function using the Heaviside theta function

$$|t| = (2\theta(t) - 1)t \tag{8.973}$$

$$\frac{d}{dt}|t| = 2\theta'(t)t + (2\theta(t) - 1) \tag{8.974}$$

$$= 2\underbrace{\delta(t)t}_{=0} + 2\theta(t) - 1 \tag{8.975}$$

$$= 2\theta(t) - 1 \tag{8.976}$$

and then differentiate and use $\theta'(t) = \delta(t)$

$$G(t) = \frac{i}{2\omega}e^{-i\omega(2\theta(t)-1)t} \tag{8.977}$$

$$\partial_t G(t) = \frac{i}{2\omega}e^{-i\omega(2\theta(t)-1)t}(-i\omega)(2\theta(t) - 1)) \tag{8.978}$$

$$= (-i\omega)G(t)\,(2\theta(t) - 1) \tag{8.979}$$

$$\partial_{tt}G(t) = (-i\omega)\partial_t G(t)\,(2\theta(t) - 1) + (-2i\omega)G(t)\delta(t) \tag{8.980}$$

$$= (-i\omega)^2 G(t)\,(2\theta(t) - 1)^2 + (-2i\omega)G(t)\delta(t) \tag{8.981}$$

$$= -\omega^2 G(t) + e^{-i\omega|t|}\delta(t) \tag{8.982}$$

where we used $(2\theta(t) - 1)^2 \equiv 1$

$$\left(\partial_{tt} + \omega^2\right)G(t) = \left(-\omega^2 + \omega^2\right)G(t) + \delta(t) = \delta(t) \tag{8.983}$$

8.9.17 Problem 7.3 - Harmonic Oscillator - Heisenberg and Schroedinger picture

(a) With $\hbar = 1$ and

$$H = \frac{1}{2}P^2 + \frac{1}{2}m\omega^2 Q^2 \tag{8.984}$$

$$[Q, P] = QP - PQ = i \tag{8.985}$$

$$[Q, Q] = [P, P] = 0 \tag{8.986}$$

we obtain for the commutators

$$
\begin{aligned}
[P^2, Q] &= P(PQ) - QP^2 && (8.987)\\
&= P(QP - i) - QP^2 && (8.988)\\
&= (PQ)P - Pi - QP^2 && (8.989)\\
&= (QP - i)P - Pi - QP^2 && (8.990)\\
&= -2Pi && (8.991)\\
[Q^2, P] &= Q(QP) - PQ^2 && (8.992)\\
&= Q(PQ + i) - PQ^2 && (8.993)\\
&= (QP)Q + iQ - PQ^2 && (8.994)\\
&= (PQ + i)Q + iQ - PQ^2 && (8.995)\\
&= 2Qi && (8.996)
\end{aligned}
$$

Then the Heisenberg equations are

$$
\begin{aligned}
\dot{Q}(t) &= i[H, Q(t)] = i\frac{1}{2m}[P^2(t), Q(t)] = \frac{1}{m}P(t) && (8.997)\\
\dot{P}(t) &= i[H, P(t)] = i\frac{1}{2}m\omega^2[Q^2(t), P(t)] = -m\omega^2 Q(t) && (8.998)\\
\to \ddot{Q}(t) &= \frac{1}{m}\dot{P}(t) = -\omega^2 Q(t) && (8.999)
\end{aligned}
$$

with the solutions (initial conditions $Q(0) = Q, P(0) = P$)

$$
\begin{aligned}
Q(t) &= A\cos\omega t + B\sin\omega t \qquad \to A = Q, \quad \omega B = \frac{1}{m}P && (8.1000)\\
&= Q\cos\omega t + \frac{1}{\omega m}P\sin\omega t && (8.1001)\\
P(t) &= m\dot{Q}(t) && (8.1002)\\
&= -m\omega Q\sin\omega t + P\cos\omega t && (8.1003)
\end{aligned}
$$

(b) Using Diracs trick from QM (rewriting H in terms of a and $a^\dagger$)

$$
\begin{aligned}
a &= \sqrt{\frac{m\omega}{2}}(Q + \frac{i}{m\omega}P) && (8.1004)\\
a^\dagger &= \sqrt{\frac{m\omega}{2}}(Q - \frac{i}{m\omega}P) && (8.1005)
\end{aligned}
$$

we can invert the relation

$$
\begin{aligned}
Q &= \frac{1}{\sqrt{2m\omega}}(a^\dagger + a) && (8.1006)\\
P &= i\sqrt{\frac{m\omega}{2}}(a^\dagger - a) && (8.1007)
\end{aligned}
$$

and

$$\begin{aligned}
Q(t) &= Q\cos\omega t + \frac{1}{\omega m}P\sin\omega t && (8.1008)\\
&= \frac{1}{\sqrt{2m\omega}}(a^\dagger + a)\cos\omega t + \frac{1}{\omega m}i\sqrt{\frac{m\omega}{2}}(a^\dagger - a)\sin\omega t && (8.1009)\\
&= \frac{1}{\sqrt{2m\omega}}\left((a^\dagger + a)\cos\omega t + i(a^\dagger - a)\sin\omega t\right) && (8.1010)\\
&= \frac{1}{\sqrt{2m\omega}}\left(a^\dagger(\cos\omega t + i\sin\omega t) + a(\cos\omega t - i\sin\omega t)\right) && (8.1011)\\
&= \frac{1}{\sqrt{2m\omega}}\left(a^\dagger e^{i\omega t} + a e^{-i\omega t}\right) && (8.1012)\\
P(t) &= i\sqrt{\frac{m\omega}{2}}\left(a^\dagger e^{i\omega t} - a e^{-i\omega t}\right) && (8.1013)\\
& && (8.1014)
\end{aligned}$$

(c) Now with $t_1 < t_2$ and the time ordering operator (larger time to the left)

$$\begin{aligned}
\langle 0|TQ(t_1)Q(t_2)|0\rangle &= \frac{1}{2m\omega}\langle 0|T\left(a^\dagger e^{i\omega t_1} + a e^{-i\omega t_1}\right)\left(a^\dagger e^{i\omega t_2} + a e^{-i\omega t_2}\right)|0\rangle && (8.1015)\\
&= \frac{1}{2m\omega}\langle 0|\left(a^\dagger e^{i\omega t_2} + a e^{-i\omega t_2}\right)\left(a^\dagger e^{i\omega t_1} + a e^{-i\omega t_1}\right)|0\rangle && (8.1016)\\
&= \frac{1}{2m\omega}\langle 0|a e^{-i\omega t_2} a^\dagger e^{i\omega t_1}|0\rangle && (8.1017)
\end{aligned}$$

all other terms are vanishing because of $a|0\rangle = 0$ and $\langle 0|a^\dagger = 0$. Then

$$\begin{aligned}
\langle 0|TQ(t_1)Q(t_2)|0\rangle &= \frac{1}{2m\omega}e^{-i\omega(t_2-t_1)}\underbrace{\langle 0|aa^\dagger|0\rangle}_{=1} && (8.1018)\\
&= \frac{1}{2m\omega}e^{-i\omega(t_2-t_1)} && (8.1019)\\
&\equiv \frac{1}{i}G(t_2 - t_1) && (8.1020)
\end{aligned}$$

And now the next case with $t_1 > t_2 > t_3 > t_4$

$$\langle 0|TQ(t_1)Q(t_2)Q(t_3)Q(t_4)|0\rangle = \frac{1}{(2m\omega)^2}\cdots \qquad (8.1021)$$

8.9.18 Problem 7.4 - Harmonic Oscillator with perturbation

As $f(t)$ is a real function we have $\tilde{f}(-E) = (\tilde{f}(E))^*$ then with (7.10)

$$\begin{aligned}
\langle 0|0\rangle_f &= \exp\left[\frac{i}{2}\int_{-\infty}^{+\infty}\frac{dE}{2\pi}\frac{\tilde{f}(E)\tilde{f}(-E)}{-E^2 + \omega^2 - i\epsilon}\right] && (8.1022)\\
&= \exp\left[\frac{i}{2}\int_{-\infty}^{+\infty}\frac{dE}{2\pi}\frac{\tilde{f}(E)\tilde{f}(E)^*}{-E^2 + \omega^2 - i\epsilon}\right] && (8.1023)
\end{aligned}$$

But we actually need to calculate $|\langle 0|0\rangle_f|^2$ therefore we observe with

$$\begin{aligned}
e^{iz} &= e^{i(x+iy)} = e^{-y}e^{ix} = e^{-y}(\cos x + i\sin x) && (8.1024)\\
\rightarrow \quad (e^{iz})^* &= e^{-y}(\cos x - i\sin x) = e^{-y-ix}e^{-i(x-iy)} = e^{-iz^*} && (8.1025)\\
\langle 0|0\rangle_f = e^{iA} \quad \rightarrow \quad |\langle 0|0\rangle_f|^2 &= e^{iA}(e^{iA})^* = e^{iA}e^{-iA^*} = e^{i(A-A^*)} = e^{-2\Im A} && (8.1026)
\end{aligned}$$

Now we calculate the imaginary part of the integral

$$\Im\frac{1}{4\pi}\int_{-\infty}^{+\infty}dE\frac{\tilde{f}(E)\tilde{f}(E)^*}{-E^2+\omega^2-i\epsilon} = \frac{1}{4\pi}\int_{-\infty}^{+\infty}dE\Im\frac{\tilde{f}(E)\tilde{f}(E)^*}{-E^2+\omega^2-i\epsilon} \tag{8.1027}$$

$$= \frac{1}{4\pi}\int_{-\infty}^{+\infty}dE\tilde{f}(E)\tilde{f}(E)^*\Im\frac{1}{-E^2+\omega^2-i\epsilon} \tag{8.1028}$$

$$= \frac{1}{4\pi}\int_{-\infty}^{+\infty}dE\tilde{f}(E)\tilde{f}(E)^*\Im\frac{-E^2+\omega^2+i\epsilon}{(-E^2+\omega^2)^2+\epsilon^2} \tag{8.1029}$$

$$= \frac{1}{4\pi}\int_{-\infty}^{+\infty}dE\tilde{f}(E)\tilde{f}(E)^*\frac{\epsilon}{(-E^2+\omega^2)^2+\epsilon^2} \tag{8.1030}$$

$$\simeq \frac{1}{4\pi}\int_{-\infty}^{+\infty}dE\tilde{f}(E)\tilde{f}(E)^*\pi\delta(-E^2+\omega^2) \tag{8.1031}$$

$$\simeq \frac{1}{4\pi}\int_{-\infty}^{+\infty}dE\tilde{f}(E)\tilde{f}(E)^*\pi\delta((\omega+E)(\omega-E)) \tag{8.1032}$$

$$\simeq \frac{1}{4\cdot 2\omega}(\tilde{f}(\omega)\tilde{f}(\omega)^*+\tilde{f}(-\omega)\tilde{f}(-\omega)^*) \tag{8.1033}$$

$$\simeq \frac{1}{8\omega}(\tilde{f}(\omega)\tilde{f}(\omega)^*+\tilde{f}(\omega)^*\tilde{f}(\omega)) \tag{8.1034}$$

$$\simeq \frac{1}{4\omega}\tilde{f}(\omega)\tilde{f}(\omega)^* \tag{8.1035}$$

then

$$|\langle 0|0\rangle_f|^2 = e^{-2\left(\frac{1}{4\omega}\right)\tilde{f}(\omega)\tilde{f}(\omega)^*} \tag{8.1036}$$

$$= e^{-\frac{1}{2\omega}\tilde{f}(\omega)\tilde{f}(\omega)^*} \tag{8.1037}$$

$$\tag{8.1038}$$

8.9.19 Problem 8.1 - Feynman propagator is Greens function Klein-Gordon equation

With

$$\Delta(x-x') = \frac{1}{(2\pi)^4}\int d^4k\frac{e^{ik(x-x')}}{k^2+m^2-i\epsilon} \tag{8.1039}$$

we have

$$(-\partial_x^2+m^2)\Delta(x-x') = \frac{1}{(2\pi)^4}\int d^4k(-i^2k^2+m^2)\frac{e^{ik(x-x')}}{k^2+m^2-i\epsilon} \tag{8.1040}$$

$$= \frac{1}{(2\pi)^4}\int d^4k\frac{k^2+m^2}{k^2+m^2-i\epsilon}e^{ik(x-x')} \tag{8.1041}$$

$$\simeq \frac{1}{(2\pi)^4}\int d^4k e^{ik(x-x')} \tag{8.1042}$$

$$= \delta^4(x-x') \tag{8.1043}$$

8.9.20 Problem 8.2 - Feynman propagator II

With $\widetilde{dk} = d^3k/((2\pi)^3 2\omega_k)$ and $\omega_k = \sqrt{\vec{k}^2 + m^2}$

$$\Delta(x - x') = \frac{1}{(2\pi)^4} \int d^4k \frac{e^{ik(x-x')}}{k^2 + m^2 - i\epsilon} \tag{8.1044}$$

$$= \frac{1}{(2\pi)^4} \int d^3k \int dk^0 e^{-ik^0(t-t')} \frac{e^{i\vec{k}(\vec{x}-\vec{x}')}}{-(k^0)^2 + \vec{k}^2 + m^2 - i\epsilon} \tag{8.1045}$$

$$= \frac{1}{(2\pi)^4} \int d^3k\, e^{i\vec{k}(\vec{x}-\vec{x}')} \int dE \frac{e^{-iE(t-t')}}{-E^2 + \vec{k}^2 + m^2 - i\epsilon} \tag{8.1046}$$

$$= \frac{1}{(2\pi)^4} \int d^3k\, e^{i\vec{k}(\vec{x}-\vec{x}')} 2\pi \frac{i}{2(\vec{k}^2 + m^2)} e^{-i(\vec{k}^2+m^2)|t-t'|} \tag{8.1047}$$

where we used exercise (7.1). Then

$$\Delta(x - x') = \frac{i}{(2\pi)^3} \int d^3k\, e^{i\vec{k}(\vec{x}-\vec{x}')} \frac{i}{2\omega_k} e^{-i\omega_k|t-t'|} \tag{8.1048}$$

$$= i \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{i\vec{k}(\vec{x}-\vec{x}')} e^{-i\omega_k|t-t'|} \tag{8.1049}$$

$$= i \int \widetilde{dk}\, e^{i\vec{k}(\vec{x}-\vec{x}')-i\omega_k|t-t'|} \tag{8.1050}$$

$$= i\theta(t - t') \int \widetilde{dk}\, e^{i\vec{k}(\vec{x}-\vec{x}')-i\omega_k(t-t')} + i\theta(t' - t) \int \widetilde{dk}\, e^{i\vec{k}(\vec{x}-\vec{x}')+i\omega_k(t-t')} \tag{8.1051}$$

$$= i\theta(t - t') \int \widetilde{dk}\, e^{ik(x-x')} + i\theta(t' - t) \int \widetilde{dk}\, e^{-i\vec{k}(\vec{x}-\vec{x}')+i\omega_k(t-t')} \tag{8.1052}$$

$$= i\theta(t - t') \int \widetilde{dk}\, e^{ik(x-x')} + i\theta(t' - t) \int \widetilde{dk}\, e^{-ik(x-x')} \tag{8.1053}$$

$$\tag{8.1054}$$

8.10 Kachelriess - Quantum Fields - From the Hubble to the Planck scale

8.10.1 Problem 1.1 - Units

1. The fundamental constants are given by

$$k = 1.381 \cdot 10^{-23}\mathrm{m}^2\mathrm{s}^{-2}\mathrm{kg}^1\mathrm{K}^{-1} \tag{8.1055}$$

$$G = 6.674 \cdot 10^{-11}\mathrm{m}^3\mathrm{s}^{-2}\mathrm{kg}^{-1} \tag{8.1056}$$

$$\hbar = 1.054 \cdot 10^{-34}\mathrm{m}^2\mathrm{s}^{-1}\mathrm{kg}^1 \tag{8.1057}$$

$$c = 2.998 \cdot 10^{-8}\mathrm{m}^1\mathrm{s}^{-1} \tag{8.1058}$$

A newly constructed Planck constant has the general form

$$X_P = c^{\alpha_c} \cdot G^{\alpha_G} \cdot \hbar^{\alpha_\hbar} \cdot k^{\alpha_k} \tag{8.1059}$$

and the dimension of X_P is given by $\mathrm{m}^{\beta_m}\mathrm{s}^{\beta_s}\mathrm{kg}^{\beta_{kg}}\mathrm{K}^{\beta_K}$ are determined by

$$\text{Meter} \quad \beta_m = 2\alpha_k + 3\alpha_G + 2\alpha_h + \alpha_c \tag{8.1060}$$

$$\text{Second} \quad \beta_s = -2\alpha_k - 2\alpha_G - \alpha_c - \alpha_h \tag{8.1061}$$

$$\text{Kilogram} \quad \beta_{kg} = \alpha_k - \alpha_G + \alpha_h \tag{8.1062}$$

$$\text{Kelvin} \quad \beta_K = -\alpha_k \tag{8.1063}$$

Solving the linear system gives

$$l_P = \sqrt{\frac{\hbar G}{c^3}} = 1.616 \cdot 10^{-35}\text{m} \tag{8.1064}$$
$$m_P = \sqrt{\frac{\hbar c}{G}} = 2.176 \cdot 10^{-8}\text{kg} \tag{8.1065}$$
$$t_P = \sqrt{\frac{\hbar G}{c^5}} = 5.391 \cdot 10^{-44}\text{s} \tag{8.1066}$$
$$T_P = \sqrt{\frac{\hbar c^5}{Gk^2}} = 1.417 \cdot 10^{-32}\text{K} \tag{8.1067}$$
$$\tag{8.1068}$$

As the constants are made up from QM, SR and GR constants they indicate magnitudes at which a quantum theory of gravity is needed to make a sensible predictions.

2. We use the definition $1\text{barn} = 10^{-28}\text{m}^2$

$$1\text{cm}^2 = 10^{-4}\text{m}^2 \tag{8.1069}$$
$$1\text{mbarn} = 10^{-31}\text{m}^2 \tag{8.1070}$$
$$= 10^{-27}\text{cm}^2 \tag{8.1071}$$

We also have $1\text{eV} = 1.602 \cdot 10^{-19}\text{As} \cdot 1\text{V} = 1.602 \cdot 10^{-19}\text{J}$

$$E = mc^2 \quad \to \quad 1\text{kg} \cdot c^2 = 8.987 \cdot 10^{16}\text{J} = 5.609 \cdot 10^{35}\text{eV} \tag{8.1072}$$
$$\to \quad 1\text{GeV} = 1.782 \cdot 10^{-27}\text{kg} \tag{8.1073}$$
$$E = \hbar\omega \quad \to \quad \frac{1}{1\text{s}} \cdot \hbar = 1.054 \cdot 10^{-34}\text{J} = 6.582 \cdot 10^{-16}\text{eV} \tag{8.1074}$$
$$\to \quad 1\text{GeV}^{-1} = 6.582 \cdot 10^{-25}\text{s} \tag{8.1075}$$
$$E = \frac{\hbar c}{\lambda} \quad \to \quad \frac{1}{1\text{m}} \cdot \hbar c = 3.161 \cdot 10^{-26}\text{J} = 1.973 \cdot 10^{-7}\text{eV} \tag{8.1076}$$
$$\to \quad 1\text{GeV}^{-1} = 1.973 \cdot 10^{-16}\text{m} \tag{8.1077}$$
$$E \sim pc \quad \to \quad 1\text{kgms}^{-1} \cdot c = 2.998 \cdot 10^{8}\text{J} = 1.871 \cdot 10^{27}\text{eV} \tag{8.1078}$$
$$\to \quad 1\text{GeV} = 5.344 \cdot 10^{-19}\text{kgms}^{-1} \tag{8.1079}$$

therefore

$$1\text{GeV}^{-2} = (1.973 \cdot 10^{-16}\text{m})^2 \tag{8.1080}$$
$$= 3.893 \cdot 10^{-32}\text{m}^2 \tag{8.1081}$$
$$= 0.389\text{mbarn} \tag{8.1082}$$

8.10.2 Problem 3.2 - Maxwell Lagrangian

1. First we observe that

$$F_{\mu\nu}F^{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \tag{8.1083}$$
$$= (\partial_\mu A_\nu)(\partial^\mu A^\nu) - (\partial_\mu A_\nu)(\partial^\nu A^\mu) - \underbrace{(\partial_\nu A_\mu)(\partial^\mu A^\nu)}_{=(\partial_\mu A_\nu)(\partial^\nu A^\mu)} + \underbrace{(\partial_\nu A_\mu)(\partial^\nu A^\mu)}_{=(\partial_\mu A_\nu)(\partial^\mu A^\nu)} \tag{8.1084}$$
$$= 2\left((\partial_\mu A_\nu)(\partial^\mu A^\nu) - (\partial_\mu A_\nu)(\partial^\nu A^\mu)\right) \tag{8.1085}$$
$$= 2(\partial_\mu A_\nu)F^{\mu\nu}. \tag{8.1086}$$

The variation is then given by

$$\delta\left(F_{\mu\nu}F^{\mu\nu}\right) = 2\delta\left((\partial_\mu A_\nu)F^{\mu\nu}\right) \tag{8.1087}$$
$$= 2\left[\delta\left(\partial_\mu A_\nu\right)F^{\mu\nu} + (\partial_\mu A_\nu)\delta F^{\mu\nu}\right] \tag{8.1088}$$
$$= 2[\delta\left(\partial_\mu A_\nu\right)\underbrace{(\partial^\mu A^\nu - \partial^\nu A^\mu)}_{=F^{\mu\nu}} + (\partial_\mu A_\nu)\underbrace{(\delta(\partial^\mu A^\nu - \partial^\nu A^\mu))}_{\delta F^{\mu\nu}}] \tag{8.1089}$$
$$= 2\left[\delta\left(\partial_\mu A_\nu\right)\partial^\mu A^\nu - \delta\left(\partial_\mu A_\nu\right)\partial^\nu A^\mu + (\partial_\mu A_\nu)\delta(\partial^\mu A^\nu) - (\partial_\mu A_\nu)\delta(\partial^\nu A^\mu)\right] \tag{8.1090}$$
$$= 4\left[\delta\left(\partial_\mu A_\nu\right)\partial^\mu A^\nu - \delta\left(\partial_\mu A_\nu\right)\partial^\nu A^\mu\right] \tag{8.1091}$$
$$= 4(\partial^\mu A^\nu - \partial^\nu A^\mu)\ \delta(\partial_\mu A_\nu) \tag{8.1092}$$
$$= 4F^{\mu\nu}\ \delta(\partial_\mu A_\nu) \tag{8.1093}$$
$$= 4F^{\mu\nu}\ \partial_\mu(\delta A_\nu) \tag{8.1094}$$

We start with the source free Maxwell equations $\partial_\mu F^{\mu\nu} = 0$

$$0 = \int_\Omega d^4x\ (\delta A_\nu)\partial_\mu F^{\mu\nu} \tag{8.1095}$$
$$= F^{\mu\nu}(\delta A_\nu)|_{\partial\Omega} - \int_\Omega d^4x\ \underbrace{\partial_\mu(\delta A_\nu)F^{\mu\nu}}_{=\frac{1}{4}\delta(F_{\mu\nu}F^{\mu\nu})} \tag{8.1096}$$
$$= \int_\Omega d^4x\ \delta\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) \tag{8.1097}$$

and therefore $\mathscr{L}_{\text{ph}} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$.

2. So we see that the Lagrangian $\mathscr{L}_{\text{ph}} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = 2(\partial_\mu A_\nu)F^{\mu\nu}$ yields the inhomogeneous Maxwell equations

$$\frac{\partial \mathscr{L}_{\text{ph}}}{\partial A_\alpha} - \partial_\beta \frac{\partial \mathscr{L}_{\text{ph}}}{\partial(\partial_\beta A_\alpha)} = 0 \tag{8.1098}$$
$$-\partial_\beta\left[(2\delta_{\alpha\mu}\delta_{\beta\nu}F^{\mu\nu} + 2(\partial_\mu A_\nu)(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\nu_\alpha\delta^\mu_\beta)\right] = 0 \tag{8.1099}$$
$$-\partial_\beta\left[(2F^{\alpha\beta} + 2(\partial^\alpha A^\beta - \partial^\beta A^\alpha)\right] = 0 \tag{8.1100}$$
$$\partial_\beta(F^{\alpha\beta}) = 0 \tag{8.1101}$$

but not the homogeneous ones. They are fulfilled trivially - by construction of $F^{\mu\nu}$.

3. The conjugated momentum is given by

$$\pi_\mu = \frac{\partial \mathscr{L}_{\text{ph}}}{\partial \dot{A}^\mu} \tag{8.1102}$$
$$= F_{0\mu} \tag{8.1103}$$

8.10.3 Problem 3.3 - Dimension of ϕ

1. With $c = 1 = \hbar$ we see

$$E = mc^2 \to E \sim M \tag{8.1104}$$
$$E = \hbar\omega \to T \sim E^{-1} \sim M^{-1} \tag{8.1105}$$
$$s = ct \to L \sim T \sim M^{-1} \tag{8.1106}$$

As $\mathscr{L}$ is an action density we have

$$\mathscr{L} \sim \frac{E \cdot T}{TL^3} \sim M \cdot M^{d-1} = M^d \tag{8.1107}$$

From the explicit form of the scalar Lagrangian we derive

$$\mathscr{L} \sim \frac{[\phi^2]}{M^{-2}} = [\phi^2]M^{-2} \tag{8.1108}$$

and therefore $[\phi] = M^{(d-2)/2}$

2. Using the previous result we see

$$\lambda\phi^3 : \qquad M^d \sim [\lambda]M^{3(d-2)/2} \to d = 6 \tag{8.1109}$$

$$\lambda\phi^4 : \qquad M^d \sim [\lambda]M^{4(d-2)/2} \to d = 4 \tag{8.1110}$$

3. With

$$\mathscr{L} = \frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - \frac{1}{2}m^2\phi^2 + \lambda\phi^4 \tag{8.1111}$$

$$= \frac{1}{2}\eta^{\mu\nu}\left(\partial_\mu\frac{\tilde{\phi}}{\sqrt{\lambda}}\right)\left(\partial_\nu\frac{\tilde{\phi}}{\sqrt{\lambda}}\right) - \frac{1}{2}m^2\frac{\tilde{\phi}^2}{\lambda} + \lambda\frac{\tilde{\phi}^4}{\lambda^2} \tag{8.1112}$$

$$= \frac{1}{\lambda}\left[\frac{1}{2}\eta^{\mu\nu}(\partial_\mu\tilde{\phi})(\partial_\nu\tilde{\phi}) - \frac{1}{2}m^2\tilde{\phi}^2 + \tilde{\phi}^4\right] \tag{8.1113}$$

8.10.4 Problem 3.5 - Yukawa potential

Integration in spherical coordinates yields (with $x = kr$)

$$\int d^3k\frac{e^{-ik\cdot r}}{k^2+m^2} = 2\pi\int\frac{e^{-ikr\cos\theta}}{k^2+m^2}k^2\sin\theta d\theta\, dk \tag{8.1114}$$

$$= -2\pi\int\frac{e^{-ikr\cos\theta}}{k^2+m^2}k^2\ d(\cos\theta)\, dk \tag{8.1115}$$

$$= -2\pi\int\frac{k^2}{ikr}\frac{e^{-ikr\cos\theta}}{k^2+m^2}\bigg|_{-1}^{+1}\ dk \tag{8.1116}$$

$$= -2\pi\int\frac{k}{ir}\frac{e^{-ikr}-e^{+ikr}}{k^2+m^2}\ dk \tag{8.1117}$$

$$= \frac{4\pi}{r}\int_0^\infty\frac{k\sin kr}{k^2+m^2}\ dk \tag{8.1118}$$

$$= \frac{4\pi}{r^2}\int_0^\infty\frac{\frac{x}{r}\sin x}{\frac{x^2}{r^2}+m^2}\ dx \tag{8.1119}$$

$$= \frac{4\pi}{r}\int_0^\infty\frac{x\sin x}{x^2+m^2r^2}\ dx \tag{8.1120}$$

$$\tag{8.1121}$$

Now we use a small trick

$$
\begin{aligned}
&= \frac{2\pi}{ir} \int_0^\infty \frac{x(e^{ix} - e^{-ix})}{x^2 + m^2 r^2}\, dx && (8.1122)\\
&= \frac{2\pi}{ir} \left[\int_0^\infty \frac{x e^{ix}}{x^2 + m^2 r^2}\, dx - \int_0^\infty \frac{x e^{-ix}}{x^2 + m^2 r^2}\, dx \right] && (8.1123)\\
&= \frac{2\pi}{ir} \left[\int_0^\infty \frac{x e^{ix}}{x^2 + m^2 r^2}\, dx - (-1)^3 \int_{-\infty}^0 \frac{y e^{iy}}{y^2 + m^2 r^2}\, dy \right] && (8.1124)\\
&= \frac{2\pi}{ir} \int_{-\infty}^\infty \frac{x e^{ix}}{x^2 + m^2 r^2}\, dx && (8.1125)\\
&= \frac{2\pi}{ir} \int_{-\infty}^\infty \frac{x e^{ix}}{(x + imr)(x - imr)}\, dx && (8.1126)\\
&= \frac{2\pi}{ir} \left(2\pi i \cdot \underbrace{\mathrm{Res}_{x=imr}}_{= \frac{imr \exp(i^2 mr)}{2imr}} - \int_{\text{upper half circle}} \ldots \right) && (8.1127)\\
&= \frac{2\pi^2}{r} e^{-mr} && (8.1128)
\end{aligned}
$$

Therefore

$$
\frac{1}{(2\pi)^3} \int d^3k \frac{e^{-ik\cdot r}}{k^2 + m^2} = \frac{1}{4\pi r} e^{-mr} \qquad (8.1129)
$$

8.10.5 Problem 3.9 - ζ function regularization

1. Calculation the Taylor expansion (using L'Hopital's rule for the limits) we obtain

$$
\begin{aligned}
f(t) &= \frac{t}{e^t - 1} && (8.1130)\\
&= \sum_k \left. \frac{d^k f}{dt^k} \right|_{t=0} t^k && (8.1131)\\
&= 1 - \frac{1}{2} t + \frac{1}{12} t^2 - \frac{1}{12} t^4 + \ldots && (8.1132)\\
&\stackrel{!}{=} B_0 + B_1 t + \frac{B_2}{2} t^2 + \frac{B_3}{6} t^2 + \ldots && (8.1133)\\
\rightarrow B_n &= \{1, -\frac{1}{2}, \frac{1}{6}, 0, \ldots\} && (8.1134)
\end{aligned}
$$

2. Avoiding mathematical rigor we see after playing around for a while

$$\begin{aligned}
\sum_{n=1}^{\infty} n e^{-an} &= -\frac{d}{da}\sum_{n=1}^{\infty} e^{-an} && (8.1135)\\
&= -\frac{d}{da}\sum_{n=1}^{\infty} \left(e^{-a}\right)^n && (8.1136)\\
&= -\frac{d}{da}\frac{1}{1-e^{-a}} && (8.1137)\\
&= -\frac{d}{da}\left(\frac{1}{a}\frac{a}{1-e^{-a}}\right) && (8.1138)\\
&= -\frac{d}{da}\left(\frac{1}{a}f(t)\right) && (8.1139)\\
&= -\frac{d}{da}\left(\frac{1}{a}\sum_{n=0}^{\infty}\frac{B_n}{n!}a^n\right) && (8.1140)\\
&= -\frac{d}{da}\left(\frac{1}{a}\left[1-\frac{a}{2}+\frac{a^2}{12}-\frac{a^4}{720}+...\right]\right) && (8.1141)\\
&= -\frac{d}{da}\left(\frac{1}{a}-\frac{1}{2}+\frac{a}{12}-\frac{a^3}{720}...\right) && (8.1142)\\
&= \frac{1}{a^2}-\frac{1}{12}+\frac{a}{240}-... && (8.1143)\\
&\stackrel{a\to 0}{\to} \frac{1}{a^2}-\frac{1}{12} && (8.1144)
\end{aligned}$$

3. Using the definition of the Riemann ζ function

$$\zeta(s) = \sum_{k=1}^{\infty}\frac{1}{k^s} \tag{8.1145}$$

8.10.6 Problem 4.1 - $Z[J]$ at order λ in ϕ^4 theory

Lets start at (4.6a) with $\mathscr{L}_I = -\lambda/4!\phi^4$

$$\begin{aligned}
Z[J] &= \exp\left[\mathrm{i}\int d^4x\mathscr{L}_I\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}\right)\right]\int \mathcal{D}\phi\exp\left[\mathrm{i}\int d^4x(\mathscr{L}_0+J\phi)\right] && (8.1146)\\
&= \exp\left[\mathrm{i}\int d^4x\mathscr{L}_I\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}\right)\right]Z_0[J] && (8.1147)\\
&= \exp\left[-\frac{\mathrm{i}\lambda}{4!}\int d^4x\left(\frac{\delta^4}{\delta J(x)^4}\right)\right]Z_0[J] && (8.1148)\\
&= Z_0[J]-\frac{\mathrm{i}\lambda}{4!}\int d^4x\left(\frac{\delta^4 Z_0[J]}{\delta J(x)^4}\right)+\dots. && (8.1149)
\end{aligned}$$

Using (4.7)

$$\begin{aligned}
Z_0[J] &= Z_0[0]\exp\left[-\frac{\mathrm{i}}{2}\int d^4yd^4zJ(y)\Delta_F(y-z)J(z)\right] = Z_0[0]e^{iW_0[J]} && (8.1150)\\
W_0[J] &= -\frac{1}{2}\int d^4yd^4zJ(y)\Delta_F(y-z)J(z) && (8.1151)
\end{aligned}$$

we derive (4.10) in various steps

1. Calculating $\frac{\delta W_0[J]}{\delta J(x)}$

$$\frac{\delta W_0[J]}{\delta J(x)} = -\frac{1}{2}\lim_{\epsilon\to 0}\int d^4y d^4z \frac{\left(J(y)+\epsilon\delta^{(4)}(y-x)\right)\Delta_F(y-z)\left(J(z)+\epsilon\delta^{(4)}(z-x)\right)-W_0[J]}{\epsilon} \tag{8.1152}$$

$$= -\frac{1}{2}\int d^4y d^4z \left[\delta^{(4)}(y-x)\Delta_F(y-z)J(z) + J(y)\Delta_F(y-z)\delta^{(4)}(z-x)\right] \tag{8.1153}$$

$$= -\frac{1}{2}\int d^4z \Delta_F(x-z)J(z) - \frac{1}{2}\int d^4y J(y)\Delta_F(y-x) \tag{8.1154}$$

$$= -\int d^4y \Delta_F(y-x)J(y) \tag{8.1155}$$

where we used $\Delta_F(x) = \Delta_F(-x)$.

2. Calculating $\frac{\delta^2 W_0[J]}{\delta J(x)^2}$

$$\frac{\delta^2 W_0[J]}{\delta J(x)^2} = -\int d^4y \Delta_F(y-x)\frac{\delta J(y)}{\delta J(x)} \tag{8.1156}$$

$$= -\int d^4y \Delta_F(y-x)\delta(y-x) \tag{8.1157}$$

$$= -\Delta_F(0) \tag{8.1158}$$

3. Calculating $\delta F[J]/\delta J(x)$ for $F[J] = f\left(W_0[J]\right)$

$$\frac{\delta F[J]}{\delta J(x)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}f(W_0[\phi(x)+\epsilon\delta(x-y)]) - f(W_0[\phi(x)]) \tag{8.1159}$$

$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}f(W_0[\phi(x)] + \epsilon\frac{\delta W_0}{\delta\phi}) - f(W_0[\phi(x)]) \tag{8.1160}$$

$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}f(W_0[\phi(x)]) + g'\epsilon\frac{\delta W_0}{\delta\phi} - f(W_0[\phi(x)]) \tag{8.1161}$$

$$= f'(W_0[J])\frac{\delta W_0}{\delta J} \tag{8.1162}$$

4. Calculating first derivative

$$\frac{\delta}{\mathrm{i}\delta J(x)}\exp\left(\mathrm{i}W_0[J]\right) = \frac{\delta W_0[J]}{\delta J(x)}\exp\left(\mathrm{i}W_0[J]\right) \tag{8.1163}$$

5. Calculating second derivative (using the functional derivative product rule)

$$\left(\frac{\delta}{\mathrm{i}\delta J(x)}\right)^2\exp\left(\mathrm{i}W_0[J]\right) = \left(\left(\frac{\delta W_0[J]}{\delta J(x)}\right)^2 + \frac{1}{i}\frac{\delta^2 W_0[J]}{\delta J(x)^2}\right)\exp\left(\mathrm{i}W_0[J]\right) \tag{8.1164}$$

6. Calculating third derivative

$$\left(\frac{\delta}{\mathrm{i}\delta J(x)}\right)^3\exp\left(\mathrm{i}W_0[J]\right) = \left(\left(\frac{\delta W_0[J]}{\delta J(x)}\right)^3 + \frac{3}{i}\frac{\delta^2 W_0[J]}{\delta J(x)^2}\frac{\delta W_0[J]}{\delta J(x)} + \frac{1}{i^2}\frac{\delta^3 W_0[J]}{\delta J(x)^3}\right)\exp\left(\mathrm{i}W_0[J]\right) \tag{8.1165}$$

7. Calculating fourth derivative

$$\left(\frac{\delta}{\mathrm{i}\delta J(x)}\right)^4 \exp\left(\mathrm{i}W_0[J]\right) = \left(\left(\frac{\delta W_0[J]}{\delta J(x)}\right)^4 + \frac{6}{i}\frac{\delta^2 W_0[J]}{\delta J(x)^2}\left(\frac{\delta W_0[J]}{\delta J(x)}\right)^2 + \frac{3}{i^2}\left(\frac{\delta^2 W_0[J]}{\delta J(x)^2}\right)^2 + \right.$$

$$\left. + \frac{4}{i^2}\frac{\delta W_0[J]}{\delta J(x)}\frac{\delta^3 W_0[J]}{\delta J(x)^3} + \frac{1}{i^3}\frac{\delta^4 W_0[J]}{\delta J(x)^4}\right)\exp\left(\mathrm{i}W_0[J]\right)$$

$$= \left(\left(\frac{\delta W_0[J]}{\delta J(x)}\right)^4 + \frac{6}{i}\frac{\delta^2 W_0[J]}{\delta J(x)^2}\left(\frac{\delta W_0[J]}{\delta J(x)}\right)^2 + \frac{3}{i^2}\left(\frac{\delta^2 W_0[J]}{\delta J(x)^2}\right)^2\right)\exp\left(\mathrm{i}W_0[J]\right)$$

8. Substituting the functional derivatives

$$\left(\frac{\delta}{\mathrm{i}\delta J(x)}\right)^4 \exp\left(\mathrm{i}W_0[J]\right) = \left[\left(\int d^4y \Delta_F(y-x)J(y)\right)^4 + 6i\Delta_F(0)\left(\int d^4y\Delta_F(y-x)J(y)\right)^2\right.$$

$$\left. +3\left(i\Delta_F(0)\right)^2\right]\exp\left(\mathrm{i}W_0[J]\right)$$

8.10.7 Problem 19.1 - Dynamical stress tensor

Preliminaries

- The Laplace expansion of the determinate by row or column is given by

$$|g| = \sum_\kappa g_{\kappa\mu}G_{\kappa\mu} \quad \text{(no sum over } \mu!) \tag{8.1166}$$

  with the cofactor matrix $G_{\kappa\mu}$ (matrix of determinants of minors of g).

- The inverse matrix is given by

$$g^{\alpha\beta} = \frac{1}{|g|}G_{\alpha\beta} \tag{8.1167}$$

- Therefore we have

$$\frac{\partial|g|}{\delta g_{\alpha\beta}} = \frac{\partial\left(\sum_\kappa g_{\kappa\beta}G_{\kappa\beta}\right)}{\delta g_{\alpha\beta}} \tag{8.1168}$$

$$= \delta_{\kappa\alpha}G_{\kappa\beta} \tag{8.1169}$$

$$= G_{\alpha\beta} \tag{8.1170}$$

$$= |g|g^{\alpha\beta} \tag{8.1171}$$

Now we can calculate

$$\delta\sqrt{|g|} = \frac{\partial\sqrt{|g|}}{\delta g_{\mu\nu}}\delta g_{\mu\nu} = \frac{1}{2\sqrt{|g|}}\frac{\partial|g|}{\delta g_{\mu\nu}}\delta g_{\mu\nu} = \frac{1}{2}\sqrt{|g|}g^{\mu\nu}\delta g_{\mu\nu} \tag{8.1172}$$

$$\frac{\delta\sqrt{|g(x)|}}{\delta g_{\mu\nu}(y)} = \frac{1}{2}\sqrt{|g|}\delta(x-y) \tag{8.1173}$$

We now use the action and definition (7.49)

$$S_\mathrm{m} = \int d^4x\sqrt{|g|}\mathscr{L}_\mathrm{m} \tag{8.1174}$$

$$T^{\mu\nu} = \frac{2}{\sqrt{|g|}}\frac{\delta S_\mathrm{m}}{\delta g^{\mu\nu}} \tag{8.1175}$$

$$= \frac{2}{\sqrt{|g|}}\int d^4x\left[\frac{1}{2}\sqrt{|g|}g^{\mu\nu}\mathscr{L}_\mathrm{m} + \sqrt{|g|}\frac{\delta\mathscr{L}_\mathrm{m}}{\delta g_{\mu\nu}}\right] \tag{8.1176}$$

8.10.8 Problem 19.6 - Dirac-Schwarzschild

1. (19.13) - adding the bi-spinor index might be helpful for some readers, see (B.27)
2. (19.13) vs (B.27) naming of generators $J^{\mu\nu}$ vs $\sigma_{\mu\nu}/2$

The Dirac equation in curved space is obtained (from the covariance principle) by replacing all derivatives ∂_k with covariant tetrad derivatives $\mathscr{D}_k$

$$(i\hbar\gamma^k \mathscr{D}_k + mc)\psi = 0 \tag{8.1177}$$

Lets start with the Schwarzschild line element

$$\begin{aligned} ds^2 &= \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2(d\vartheta^2 + \sin^2\vartheta\, d\phi^2) && (8.1178)\\ &= \eta_{mn} d\xi^m d\xi^n && (8.1179)\end{aligned}$$

with

$$d\xi^0 = \left(1 - \frac{2M}{r}\right)^{1/2} dt, \quad d\xi^1 = \left(1 - \frac{2M}{r}\right)^{-1/2} dr, \quad d\xi^2 = r d\vartheta, \quad d\xi^3 = r\sin\vartheta d\phi. \tag{8.1180}$$

and the tetrad fields e^m_μ can then be derived via $d\xi^m = e^m_\mu(x)dx^\mu$.

8.10.9 Problem 23.1 - Conformal transformation

For a change of coordinates we find in general

$$\begin{aligned} x^\mu &\mapsto \tilde{x}^\mu && (8.1181)\\ g_{\mu\nu}(x) \mapsto \tilde{g}_{\mu\nu}(\tilde{x}) &= \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(x) && (8.1182)\end{aligned}$$

which for $x \mapsto \tilde{x} = e^\omega x$ results in (there might be a sign error in (18.1))

$$g_{\mu\nu}(x) \mapsto \tilde{g}_{\mu\nu}(\tilde{x}) = e^{-2\omega} g_{\alpha\beta}(x) \tag{8.1183}$$

while for a conformal transformation we have

$$\begin{aligned} g_{\mu\nu}(x) \mapsto & \tilde{g}_{\mu\nu}(x) = \Omega^2 g_{\alpha\beta}(x) && (8.1184)\\ & \tilde{g}_{\mu\nu}(\tilde{x}) = \Omega^2 g_{\alpha\beta}(e^\omega x) && (8.1185)\end{aligned}$$

8.10.10 Problem 23.2 - Conformal transformation properties

- Christoffel symbol:

$$\begin{aligned} \tilde{g}_{\mu\nu}(x) &= \Omega^2(x) g_{\mu\nu}(x) = e^{2\omega(x)} g_{\mu\nu}(x) && (8.1186)\\ \tilde{g}_{\mu\nu,\alpha} &= 2\Omega\Omega_{,\alpha} g_{\mu\nu} + \Omega^2 g_{\mu\nu,\alpha} && (8.1187)\\ &= \Omega(2g_{\mu\nu}\Omega_{,\alpha} + \Omega g_{\mu\nu,\alpha}) && (8.1188)\end{aligned}$$

and

$$\begin{aligned} \delta^\mu_{\ \nu} &= \tilde{g}^{\mu\alpha}\tilde{g}_{\alpha\nu} = \tilde{g}^{\mu\alpha} g_{\alpha\nu}\Omega^2 && (8.1189)\\ \delta^\mu_{\ \nu} g^{\nu\beta} &= \tilde{g}^{\mu\alpha} g_{\alpha\nu} g^{\nu\beta}\Omega^2 && (8.1190)\\ g^{\mu\beta} &= \tilde{g}^{\mu\alpha}\delta^\beta_\alpha \Omega^2 && (8.1191)\\ \rightarrow \tilde{g}^{\mu\beta} &= \Omega^{-2} g^{\mu\beta} && (8.1192)\end{aligned}$$

we find by using $\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(g_{\alpha\mu,\beta} + g_{\beta\mu,\alpha} - g_{\alpha\beta,\mu} \right)$

$$\tilde{\Gamma}^{\mu}{}_{\alpha\beta} = \frac{1}{2} \tilde{g}^{\mu\nu} \left(\tilde{g}_{\alpha\nu,\beta} + \tilde{g}_{\beta\nu,\alpha} - \tilde{g}_{\alpha\beta,\nu} \right) \tag{8.1193}$$

$$= \frac{1}{2} \Omega^{-2} g^{\mu\nu} \left[\Omega(2g_{\alpha\nu}\Omega_{,\beta} + \Omega g_{\alpha\nu,\beta}) + \Omega(2g_{\beta\nu}\Omega_{,\alpha} + \Omega g_{\beta\nu,\alpha}) - \Omega(2g_{\alpha\beta}\Omega_{,\nu} + \Omega g_{\alpha\beta,\nu}) \right] \tag{8.1194}$$

$$= \Gamma^{\mu}{}_{\alpha\beta} + \Omega^{-1} g^{\mu\nu} \left[g_{\alpha\nu}\Omega_{,\beta} + g_{\beta\nu}\Omega_{,\alpha} - g_{\alpha\beta}\Omega_{,\nu} \right] \tag{8.1195}$$

$$= \Gamma^{\mu}{}_{\alpha\beta} + \Omega^{-1} \left[\delta^{\mu}_{\alpha}\Omega_{,\beta} + \delta^{\mu}_{\beta}\Omega_{,\alpha} - g^{\mu\nu} g_{\alpha\beta}\Omega_{,\nu} \right] \tag{8.1196}$$

- Ricci tensor: with

$$\Omega = e^{2\omega} \tag{8.1197}$$

$$\Omega^{-2}\Omega_{,\lambda} = e^{-4\omega} e^{2\omega} 2\omega_{,\lambda} \tag{8.1198}$$

$$= 2e^{-2\omega}\omega_{,\lambda} \tag{8.1199}$$

$$\Omega_{,\lambda\alpha} = \left(2e^{2\omega}\omega_{,\lambda} \right)_{,\alpha} \tag{8.1200}$$

$$= 4e^{2\omega}\omega_{,\lambda}\omega_{,\alpha} + 2e^{2\omega}\omega_{,\lambda\alpha} \tag{8.1201}$$

$$= 2e^{2\omega} \left(2\omega_{,\lambda}\omega_{,\alpha} + \omega_{,\lambda\alpha} \right) \tag{8.1202}$$

and

$$\partial_\lambda \tilde{\Gamma}^{\mu}{}_{\alpha\beta} = \partial_\lambda \Gamma^{\mu}{}_{\alpha\beta} - \Omega^{-2}\Omega_{,\lambda} \left[\delta^{\mu}_{\alpha}\Omega_{,\beta} + \delta^{\mu}_{\beta}\Omega_{,\alpha} - g^{\mu\nu} g_{\alpha\beta}\Omega_{,\nu} \right] + \Omega^{-1} \left[\delta^{\mu}_{\alpha}\Omega_{,\beta\lambda} + \delta^{\mu}_{\beta}\Omega_{,\alpha\lambda} - (g^{\mu\nu} g_{\alpha\beta}\Omega_{,\nu})_{,\lambda} \right] \tag{8.1203}$$

$$= \partial_\lambda \Gamma^{\mu}{}_{\alpha\beta} - 4\omega_{,\lambda} \left[\delta^{\mu}_{\alpha}\omega_{,\beta} + \delta^{\mu}_{\beta}\omega_{,\alpha} - g^{\mu\nu} g_{\alpha\beta}\omega_{,\nu} \right] + 2 \left[\delta^{\mu}_{\alpha} \left(2\omega_{,\beta}\omega_{,\lambda} + \omega_{,\beta\lambda} \right) + \delta^{\mu}_{\beta} \left(2\omega_{,\alpha}\omega_{,\lambda} + \omega_{,\alpha\lambda} \right) \right] \tag{8.1204}$$

$$- 2 \left[g^{\mu\nu}{}_{,\lambda} g_{\alpha\beta}\omega_{,\nu} + g^{\mu\nu} g_{\alpha\beta,\lambda}\omega_{,\nu} + g^{\mu\nu} g_{\alpha\beta}(2\omega_{,\nu}\omega_{,\lambda} + \omega_{,\nu\lambda}) \right] \tag{8.1205}$$

$$\tag{8.1206}$$

$$\partial_\rho \tilde{\Gamma}^{\rho}{}_{\mu\nu} = \partial_\rho \Gamma^{\rho}{}_{\mu\nu} - 4\omega_{,\rho} \left[\delta^{\rho}_{\mu}\omega_{,\nu} + \delta^{\rho}_{\nu}\omega_{,\mu} - g^{\rho\nu} g_{\mu\nu}\omega_{,\lambda} \right] + 2 \left[\delta^{\rho}_{\mu} \left(2\omega_{,\nu}\omega_{,\rho} + \omega_{,\nu\rho} \right) + \delta^{\rho}_{\nu} \left(2\omega_{,\mu}\omega_{,\rho} + \omega_{,\mu\rho} \right) \right] \tag{8.1207}$$

$$- 2 \left[g^{\rho\lambda}{}_{,\rho} g_{\mu\nu}\omega_{,\lambda} + g^{\rho\lambda} g_{\mu\nu,\rho}\omega_{,\lambda} + g^{\rho\lambda} g_{\mu\nu}(2\omega_{,\lambda}\omega_{,\rho} + \omega_{,\lambda\rho}) \right] \tag{8.1208}$$

$$= \partial_\rho \Gamma^{\rho}{}_{\mu\nu} - 4 \left[2\omega_{,\mu}\omega_{,\nu} - \omega_{,\rho} g^{\rho\nu} g_{\mu\nu}\omega_{,\lambda} \right] + 4 \left(2\omega_{,\nu}\omega_{,\mu} + \omega_{,\nu\mu} \right) \tag{8.1209}$$

$$- 2 \left[g^{\rho\lambda}{}_{,\rho} g_{\mu\nu}\omega_{,\lambda} + g^{\rho\lambda} g_{\mu\nu,\rho}\omega_{,\lambda} + g^{\rho\lambda} g_{\mu\nu}(2\omega_{,\lambda}\omega_{,\rho} + \omega_{,\lambda\rho}) \right] \tag{8.1210}$$

$$= \partial_\rho \Gamma^{\rho}{}_{\mu\nu} + 4g^{\rho\nu} g_{\mu\nu}\omega_{,\lambda}\omega_{,\rho} + 4\omega_{,\nu\mu} - 2 \left[g^{\rho\lambda}{}_{,\rho} g_{\mu\nu}\omega_{,\lambda} + g^{\rho\lambda} g_{\mu\nu,\rho}\omega_{,\lambda} + (2g^{\rho\lambda} g_{\mu\nu}\omega_{,\lambda}\omega_{,\rho} + g^{\rho\lambda} g_{\mu\nu}\omega_{,\lambda\rho}) \right] \tag{8.1211}$$

$$= \partial_\rho \Gamma^{\rho}{}_{\mu\nu} + 4\omega_{,\lambda}\omega_{,\mu} + 4\omega_{,\nu\mu} - 2 \left[g^{\rho\lambda}{}_{,\rho} g_{\mu\nu}\omega_{,\lambda} + g_{\mu\nu,\rho}\omega^{,\rho} + 2g_{\mu\nu}\omega^{,\rho}\omega_{,\rho} + g_{\mu\nu}\omega^{,\rho}{}_{\rho} \right] \tag{8.1212}$$

$$\partial_\nu \tilde{\Gamma}^{\rho}{}_{\mu\rho} = \partial_\nu \Gamma^{\rho}{}_{\mu\rho} - 4\omega_{,\nu} \left[\delta^{\rho}_{\mu}\omega_{,\rho} + \delta^{\rho}_{\rho}\omega_{,\mu} - g^{\rho\kappa} g_{\mu\rho}\omega_{,\kappa} \right] + 2 \left[\delta^{\rho}_{\mu} \left(2\omega_{,\rho}\omega_{,\nu} + \omega_{,\rho\nu} \right) + \delta^{\rho}_{\rho} \left(2\omega_{,\mu}\omega_{,\nu} + \omega_{,\mu\nu} \right) \right] \tag{8.1213}$$

$$- 2 \left[g^{\rho\kappa}{}_{,\nu} g_{\mu\rho}\omega_{,\kappa} + g^{\rho\kappa} g_{\mu\rho,\nu}\omega_{,\kappa} + g^{\rho\kappa} g_{\mu\rho}(2\omega_{,\kappa}\omega_{,\nu} + \omega_{,\kappa\nu}) \right] \tag{8.1214}$$

$$= \partial_\nu \Gamma^{\rho}{}_{\mu\rho} - 4 \left[(d+1)\omega_{,\mu}\omega_{,\nu} - \omega_{,\mu}\omega_{,\nu} \right] + 2(d+1) \left(2\omega_{,\mu}\omega_{,\nu} + \omega_{,\mu\nu} \right) \tag{8.1215}$$

$$- 2 \left[g^{\rho\kappa}{}_{,\nu} g_{\mu\rho}\omega_{,\kappa} + g^{\rho\kappa} g_{\mu\rho,\nu}\omega_{,\kappa} + \delta^{\kappa}_{\mu}(2\omega_{,\kappa}\omega_{,\nu} + \omega_{,\kappa\nu}) \right] \tag{8.1216}$$

$$= \partial_\nu \Gamma^{\rho}{}_{\mu\rho} + 4\omega_{,\mu}\omega_{,\nu} + 2(d+1)\omega_{,\mu\nu} - 2 \left[g^{\rho\kappa}{}_{,\nu} g_{\mu\rho}\omega_{,\kappa} + g^{\rho\kappa} g_{\mu\rho,\nu}\omega_{,\kappa} + (2\omega_{,\mu}\omega_{,\nu} + \omega_{,\mu\nu}) \right] \tag{8.1217}$$

$$= \partial_\nu \Gamma^{\rho}{}_{\mu\rho} + 2d \cdot \omega_{,\mu\nu} - 2 \left[g^{\rho\kappa}{}_{,\nu} g_{\mu\rho}\omega_{,\kappa} + g_{\mu\rho,\nu}\omega^{,\rho} \right] \tag{8.1218}$$

$$\tilde{\Gamma}^{\mu}{}_{\alpha\beta} = \Gamma^{\mu}{}_{\alpha\beta} + \Omega^{-1}\left[\delta^{\mu}_{\alpha}\Omega_{,\beta} + \delta^{\mu}_{\beta}\Omega_{,\alpha} - g^{\mu\nu}g_{\alpha\beta}\Omega_{,\nu}\right] \tag{8.1219}$$

$$\tag{8.1220}$$

$$\begin{aligned}\tilde{\Gamma}^{\rho}{}_{\mu\nu}\tilde{\Gamma}^{\sigma}{}_{\rho\sigma} &= \left(\Gamma^{\rho}{}_{\mu\nu} + \Omega^{-1}\left[\delta^{\rho}_{\mu}\Omega_{,\nu} + \delta^{\rho}_{\nu}\Omega_{,\mu} - g^{\rho\lambda}g_{\mu\nu}\Omega_{,\lambda}\right]\right)\left(\Gamma^{\sigma}{}_{\rho\sigma} + d\cdot\Omega^{-1}\Omega_{,\rho}\right) && (8.1221)\\ &= \Gamma^{\rho}{}_{\mu\nu}\Gamma^{\sigma}{}_{\rho\sigma} + \Gamma^{\rho}{}_{\mu\nu}d\cdot\Omega^{-1}\Omega_{,\rho} + \Gamma^{\sigma}{}_{\rho\sigma}\Omega^{-1}\left[\delta^{\rho}_{\mu}\Omega_{,\nu} + \delta^{\rho}_{\nu}\Omega_{,\mu} - g^{\rho\lambda}g_{\mu\nu}\Omega_{,\lambda}\right] && (8.1222)\\ &\quad + d\cdot\Omega^{-2}\left[\delta^{\rho}_{\mu}\Omega_{,\nu} + \delta^{\rho}_{\nu}\Omega_{,\mu} - g^{\rho\lambda}g_{\mu\nu}\Omega_{,\lambda}\right]\Omega_{,\rho} && (8.1223)\end{aligned}$$

$$\begin{aligned}\tilde{R}_{\mu\nu} &= \tilde{R}^{\rho}{}_{\mu\rho\nu} && (8.1224)\\ &= \partial_\rho\tilde{\Gamma}^{\rho}{}_{\mu\nu} - \partial_\nu\tilde{\Gamma}^{\rho}{}_{\mu\rho} + \tilde{\Gamma}^{\rho}{}_{\mu\nu}\tilde{\Gamma}^{\sigma}{}_{\rho\sigma} - \tilde{\Gamma}^{\sigma}{}_{\nu\rho}\tilde{\Gamma}^{\rho}{}_{\mu\sigma} && (8.1225)\end{aligned}$$

- Curvature scalar

$$\begin{aligned}\tilde{R} &= \tilde{g}^{\mu\nu}\tilde{R}_{\mu\nu} && (8.1226)\\ &= \tilde{g}^{\mu\nu}\left[R_{\mu\nu} - g_{\mu\nu}\Box\omega - (d-2)\nabla_\mu\nabla_\nu\omega + (d-2)\nabla_\mu\omega\nabla_\nu\omega - (d-2)g_{\mu\nu}\nabla^\lambda\omega\nabla_\lambda\omega\right] && (8.1227)\\ &= \Omega^{-2}\left[R - d\Box\omega - (d-2)\Box\omega + (d-2)\nabla^\mu\omega\nabla_\mu\omega - (d-2)d\nabla^\lambda\omega\nabla_\lambda\omega\right] && (8.1228)\\ &= \Omega^{-2}\left[R - 2(d-1)\Box\omega - (d-2)(d-1)\nabla^\lambda\omega\nabla_\lambda\omega\right] && (8.1229)\\ & && (8.1230)\end{aligned}$$

8.10.11 Problem 23.6 - Reflection formula

$$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx \tag{8.1231}$$

8.10.12 Problem 23.7 - Unruh temperature

8.10.13 Problem 24.14 - Jeans length and the speed of sound

We start with the Euler equations

$$\begin{aligned}\frac{D\rho}{Dt} &= -\rho\nabla\cdot\vec{u} \quad\rightarrow\quad \frac{\partial\rho}{\partial t} + \vec{u}\cdot(\nabla\rho) + \rho(\nabla\cdot\vec{u}) = 0 && (8.1232)\\ \frac{D\vec{u}}{Dt} &= -\nabla\left(\frac{P}{\rho}\right) + \vec{g} \quad\rightarrow\quad \frac{\partial\vec{u}}{\partial t} + \vec{u}\cdot(\nabla\vec{u}) + \frac{\nabla P}{\rho} = \vec{g}. && (8.1233)\end{aligned}$$

With the perturbation ansatz (small perturbation in a resting fluid)

$$\begin{aligned}\rho &= \rho_0 + \varepsilon\rho_1(x,t) && (8.1234)\\ P &= P_0 + \varepsilon P_1(x,t) && (8.1235)\\ \vec{u} &= \varepsilon\vec{u}_1(x,t) && (8.1236)\end{aligned}$$

and the Newton equation

$$\triangle\phi = 4\pi G\rho \quad\rightarrow\quad \nabla\cdot\vec{g}_1 = -4\pi G\rho_1 \tag{8.1237}$$

we obtain (with the EoS $P = w\rho$) in order ε

$$\frac{\partial \rho_1}{\partial t} + \rho_0(\nabla \cdot \vec{u}_1) = 0 \tag{8.1238}$$

$$\frac{\partial \vec{u}_1}{\partial t} + \underbrace{\frac{1}{\rho_0}\nabla P_1}_{=\frac{w}{\rho_0}\nabla\rho_1} = \vec{g}_1. \tag{8.1239}$$

Differentiating both (with respect to space and time) we obtain a wave equation

$$\frac{\partial^2 \rho_1}{\partial t^2} - w\triangle\rho_1 = 4\pi G\rho_0\rho_1 \tag{8.1240}$$

with the speed of sound $c_s^2 = w$. Inserting the wave ansatz $\rho_1 \sim \exp[i(\vec{k}\cdot\vec{x} - \omega t)]$ yields the dispersion relation

$$\omega^2 = c_s^2 k^2 - 4\pi G\rho_0. \tag{8.1241}$$

For wave numbers $k_J < \sqrt{4\pi G/c_s^2}$ the ω becomes complex which gives rise to exponentially growing modes. Therefore the Jeans length is given by

$$\lambda_J = \frac{2\pi}{k_J} = c_s\sqrt{\frac{\pi}{G\rho_0}} = \sqrt{\frac{\pi w}{G\rho_0}}. \tag{8.1242}$$

8.10.14 Problem 25.1 - Schwarzschild metric

The simplified vacuum Einstein equations are given by

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0 \tag{8.1243}$$

$$\rightarrow \quad R - \frac{1}{2}R\cdot 4 = 0 \quad \rightarrow \quad R = 0 \tag{8.1244}$$

$$R_{\mu\nu} = 0 \tag{8.1245}$$

Lets start with the metric ansatz (25.4)

$$g_{\mu\nu} = \mathrm{diag}(A(r), -B(r), -r^2, -r^2\sin^2\theta) \tag{8.1246}$$

$$g^{\mu\nu} = \mathrm{diag}(1/A(r), -1/B(r), -1/r^2, -1/r^2\sin^2\theta) \tag{8.1247}$$

The non-vanishing Chistoffel symbols are then

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2}g^{\mu\kappa}(g_{\kappa\lambda,\nu} + g_{\nu\kappa,\lambda} - g_{\nu\lambda,\kappa}) \tag{8.1248}$$

$$\Gamma^0_{01} = \frac{A'}{2A}, \quad \Gamma^1_{00} = \frac{A'}{2B} \quad \Gamma^1_{11} = \frac{B'}{2B} \quad \Gamma^1_{22} = -\frac{r}{B} \quad \Gamma^1_{33} = \frac{r\sin^2\theta}{B} \tag{8.1249}$$

$$\Gamma^2_{12} = 1/r \quad \Gamma^2_{12} = -\cos\theta\sin\theta \quad \Gamma^2_{12} = 1/r \quad \Gamma^2_{12} = \cot\theta \tag{8.1250}$$

The non-vanishing components of the Ricci tensor are

$$R_{00} = \frac{A'}{rB} - \frac{A'^2}{4AB} - \frac{A'B'}{4B^2} + \frac{A''}{2B} \tag{8.1251}$$

$$R_{11} = \frac{A'^2}{4A^2} + \frac{B'}{rB} + \frac{A'B'}{4AB} - \frac{A''}{2A} \tag{8.1252}$$

$$R_{22} = -\frac{1}{B} + 1 - \frac{rA'}{2AB} + \frac{rB''}{2B^2} \tag{8.1253}$$

$$R_{33} = R_{22}\sin^2\theta \tag{8.1254}$$

As there are only the two unknown functions A, B we only need two vacuum equations $R_{00} = 0$ and $R_{11} = 0$. Multiplying the first by B/A and leaving the second one untouched we obtain the system

$$\frac{A'}{rA} - \frac{A'^2}{4A^2} - \frac{A'B'}{4AB} + \frac{A''}{2A} = 0 \tag{8.1255}$$

$$\frac{B'}{rB} + \frac{A'^2}{4A^2} + \frac{A'B'}{4AB} - \frac{A''}{2A} = 0 \tag{8.1256}$$

Adding bot we get $B'/B = -A'/A$ which we can substitude into the first one obtaining

$$\frac{A'}{rA} + \frac{A''}{2A} = 0 \tag{8.1257}$$

$$\to A'(r) = \frac{c_1}{r^2} \tag{8.1258}$$

$$\to A(r) = c_2 - \frac{c_1}{r} \tag{8.1259}$$

now we can solve for $B(r)$

$$\frac{B'}{B} = -\frac{A'}{A} \tag{8.1260}$$

$$\to B(r) = \frac{c_3 r}{c_1 - r c_2} = \frac{-c_3}{c_2 - \frac{c_1}{r}} \tag{8.1261}$$

8.10.15 Problem 26.4 - Fixed points of (26.18)

We start with

$$\text{(F1)} \qquad H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot\phi^2 + V + \rho\right) \tag{8.1262}$$

$$\text{(F2)} \qquad \dot H = -4\pi G\left[\dot\phi^2 + (1+w_m)\rho\right] \tag{8.1263}$$

$$\text{(KG)} \qquad \ddot\phi = -3H\dot\phi - V_{,\phi}. \tag{8.1264}$$

Using $H = \dot a/a$, $N = \ln(a)$ and $\lambda = -V_{,\phi}/(\sqrt{8\pi G}V)$ we obtain for the time derivatives of x and y

$$\dot V = \frac{dV}{d\phi}\frac{d\phi}{dt} = V_{,\phi}\dot\phi \tag{8.1265}$$

$$x = \sqrt{\frac{4}{3}\pi G}\frac{\dot\phi}{H} \quad \to \quad \frac{dx}{dt} = \frac{dx}{dN}\frac{d\ln(a)}{dt} = \frac{dx}{dN}H = \sqrt{\frac{4}{3}\pi G}\frac{\ddot\phi H - \dot\phi\dot H}{H^2} \tag{8.1266}$$

$$y = \sqrt{\frac{8}{3}\pi G}\frac{\sqrt{V}}{H} \quad \to \quad \frac{dy}{dt} = \frac{dy}{dN}\frac{d\ln(a)}{dt} = \frac{dy}{dN}H = \sqrt{\frac{8}{3}\pi G}\frac{\frac{V_{,\phi}\dot\phi}{2\sqrt{V}} - \sqrt{V}\dot H}{H^2}. \tag{8.1267}$$

With the substitutions

$$\dot H = -4\pi G\left[\dot\phi^2 + (1+w_m)\rho\right] \tag{8.1268}$$

$$\ddot\phi = -3H\dot\phi - V_{,\phi} \tag{8.1269}$$

$$V_{,\phi} = -\sqrt{8\pi G}\lambda V \tag{8.1270}$$

$$\rho = \frac{3H^2}{8\pi G} - \frac{1}{2}\dot\phi^2 - V \tag{8.1271}$$

$$\dot\phi = xH/\sqrt{\frac{4}{3}\pi G} \tag{8.1272}$$

$$\sqrt{V} = yH/\sqrt{\frac{8}{3}\pi G} \tag{8.1273}$$

we obtain

$$\frac{dx}{dN} = -3x + \frac{\sqrt{6}}{2}\lambda y^2 + \frac{3}{2}x[(1-w_m)x^2 + (1+w_m)(1-y^2)] \tag{8.1274}$$

$$\frac{dy}{dN} = -\frac{\sqrt{6}}{2}\lambda xy + \frac{3}{2}y[(1-w_m)x^2 + (1+w_m)(1-y^2)]. \tag{8.1275}$$

To find the fix points of (26.17) we need to solve

$$-3x + \frac{\sqrt{6}}{2}\lambda y^2 + \frac{3}{2}x[(1-w_m)x^2 + (1+w_m)(1-y^2)] = 0 \tag{8.1276}$$

$$-\frac{\sqrt{6}}{2}\lambda xy + \frac{3}{2}y[(1-w_m)x^2 + (1+w_m)(1-y^2)] = 0. \tag{8.1277}$$

- An obvious solution is

$$x_0 = 0, y_0 = 0. \tag{8.1278}$$

- Two semi-obvious solutions can be found for $y = 0$ which solves the second equation and transforms the first to the quadratic equation $x^2 - 1 = 0$ which gives

$$x_1 = +1, y_1 = 0 \tag{8.1279}$$

$$x_2 = -1, y_2 = 0. \tag{8.1280}$$

- Substituting the square bracket of the second equation into the first and simplifying the second gives

$$-3x + \frac{\sqrt{6}}{2}\lambda(x^2 + y^2) = 0 \tag{8.1281}$$

$$-\frac{\sqrt{6}}{2}\lambda x + \frac{3}{2}[1 + 2x^2 - (x^2 + y^2) - w_m((x^2 + x^2) - 1)] = 0. \tag{8.1282}$$

  Now we can eliminate $x^2 + y^2$ and obtain a single quadratic equation in x

$$-\frac{\sqrt{6}}{2}\lambda x + \frac{3}{2}\left[1 + 2x^2 - \frac{\sqrt{6}}{\lambda}x - w_m\left(\frac{\sqrt{6}}{\lambda}x - 1\right)\right] = 0 \tag{8.1283}$$

  which can be simplified to

$$x^2 - \frac{3(1+w_m) + \lambda^2}{\sqrt{6}\lambda}x + \frac{1+w_m}{2} = 0. \tag{8.1284}$$

  This gives us two more solutions

$$x_3 = \frac{\lambda}{\sqrt{6}}, y_3 = \sqrt{1 - \frac{\lambda^2}{6}} \qquad (\lambda^2 < 6) \tag{8.1285}$$

$$x_4 = \sqrt{\frac{3}{2}}\frac{1+w_m}{\lambda}, y_4 = \sqrt{\frac{3}{2}}\frac{\sqrt{1-w_m^2}}{\lambda} \qquad (w_m^2 < 1). \tag{8.1286}$$

- Let's quickly check the stability of the fix points. The characteristic equation for the fix points of a 2d system is given by

$$\alpha^2 + a_1(x_i, y_i)\alpha + a_2(x_i, y_i) = 0 \tag{8.1287}$$

$$a_1(x_i, y_i) = -\left(\frac{df_x}{dx} + \frac{df_y}{dy}\right)_{x=x_i, y=y_i} \tag{8.1288}$$

$$a_2(x_i, y_i) = \left.\frac{df_x}{dx}\frac{df_y}{dy} - \frac{df_x}{dy}\frac{df_y}{dx}\right|_{x=x_i, y=y_i} \tag{8.1289}$$

  with the stability classification (assuming for EoS parameter $w_m^2 < 1$)

type	condition	fix point 0	fix point 1	fix point 2
saddle node	$a_2 < 0$	$-1 < w_m < 1$	$\lambda > \sqrt{6}$	$\lambda < -\sqrt{6}$
unstable node	$0 < a_2 < a_1^2/4$	-	$\lambda < \sqrt{6}$	$\lambda > -\sqrt{6}$
unstable spiral	$a_1^2/4 < a_2, a_1 < 0$	-	-	-
center	$0 < a_2, a_1 = 0$	-	-	-
stable spiral	$a_1^2/4 < a_2, a_1 > 0$	-	-	-
stable node	$0 < a_2 < a_1^2/4$	-	-	-

type	fix point 3	fix point 4
saddle node	$3(1+w_m) < \lambda^2 < 6$	-
unstable node	-	-
unstable spiral	-	-
center	-	-
stable spiral	-	$\lambda^2 > \frac{24(1+w_m)^2}{7+9w_m}$
stable node	$\lambda^2 < 3(1+w_m)$	$\lambda^2 < \frac{24(1+w_m)^2}{7+9w_m}$

8.10.16 Problem 26.5 - Tracker solution

Inserting the ansatz

$$\phi(t) = C(\alpha, n) M^{1+\nu} t^{\nu} \tag{8.1290}$$

into the ODE

$$\ddot{\phi} + \frac{3\alpha}{t}\dot{\phi} - \frac{M^{4+n}}{\phi^{n+1}} = 0 \tag{8.1291}$$

gives

$$CM^{1+\nu}\nu(\nu-1)t^{\nu-2} + CM^{1+\nu}\frac{3\alpha}{t}t^{\nu-1} - \frac{M^{4+n}}{C^{n+1}M^{(n+1)(1+\nu)}t^{\nu(n+1)}} = 0 \tag{8.1292}$$

$$CM^{1+\nu}\left[\nu(\nu-1)+3\alpha\right]t^{\nu-2} - \frac{M^{3-\nu(n+1)}}{C^{n+1}}t^{-\nu(n+1)} = 0 \tag{8.1293}$$

From equating coefficients and powers (in t) we obtain

$$\nu = \frac{2}{2+n} \tag{8.1294}$$

$$C(\alpha, n) = \left(\frac{(2+n)^2}{6\alpha(2+n) - 2n}\right)^{\frac{1}{2+n}}. \tag{8.1295}$$

8.11 Banks - Quantum Field Theory

8.11.1 Problem 2.2 - Time evolution operator in the Dirac picture

With the definitions

$$i\partial_t U_S = (H_0 + V)U_S \tag{8.1296}$$

$$U_D(t,t_0) = e^{iH_0t}U_S(t,t_0)e^{-iH_0t_0} \tag{8.1297}$$

we can start rewriting

$$\begin{aligned}
i\partial_t U_D(t,t_0) &= i\partial_t \left(e^{iH_0t}U_S(t,t_0)e^{-iH_0t_0}\right) &(8.1298)\\
&= i^2 H_0 \underbrace{e^{iH_0t}U_S(t,t_0)e^{-iH_0t_0}}_{=U_D} + e^{iH_0t} i[\partial_t U_S(t,t_0)]e^{-iH_0t_0} &(8.1299)\\
&= -H_0U_D(t,t_0) + e^{iH_0t} i[\partial_t U_S(t,t_0)]e^{-iH_0t_0} &(8.1300)\\
&= -H_0U_D(t,t_0) + e^{iH_0t}(H_0+V)U_S(t,t_0)e^{-iH_0t_0} &(8.1301)\\
&= -H_0U_D(t,t_0) + H_0 \underbrace{e^{iH_0t}U_S(t,t_0)e^{-iH_0t_0}}_{=U_D} + e^{iH_0t}VU_S(t,t_0)e^{-iH_0t_0} &(8.1302)\\
&= e^{iH_0t}VU_S(t,t_0)e^{-iH_0t_0} &(8.1303)\\
&= e^{iH_0t}V \underbrace{e^{-iH_0t}e^{iH_0t}}_{=1} U_S(t,t_0)e^{-iH_0t_0} &(8.1304)\\
&= e^{iH_0t}Ve^{-iH_0t}U_D(t,t_0) &(8.1305)
\end{aligned}$$

8.12 Kugo - Eichtheorie

8.12.1 Problem 1.1

With $\Lambda^{\alpha}_{\ \mu} \approx \delta^{\alpha}_{\mu} + \epsilon^{\alpha}_{\ \mu}$ we obtain

$$\begin{aligned} g_{\mu\nu} &= \Lambda^{\alpha}_{\ \mu}\Lambda^{\beta}_{\ \nu} g_{\alpha\beta} && (8.1306)\\ &\simeq \left(\delta^{\alpha}_{\mu} + \epsilon^{\alpha}_{\ \mu}\right)\left(\delta^{\beta}_{\nu} + \epsilon^{\beta}_{\ \nu}\right) g_{\alpha\beta} && (8.1307)\\ &\simeq g_{\mu\nu} + \epsilon^{\alpha}_{\ \mu}\delta^{\beta}_{\nu} g_{\alpha\beta} + \epsilon^{\beta}_{\ \nu}\delta^{\alpha}_{\mu} g_{\alpha\beta} + \mathcal{O}(\epsilon^2) && (8.1308)\\ &\simeq g_{\mu\nu} + \epsilon_{\nu\mu} + \epsilon_{\mu\nu} + \mathcal{O}(\epsilon^2) && (8.1309) \end{aligned}$$

which means that ϵ is antisymmetric $\epsilon_{\nu\mu} = -\epsilon_{\mu\nu}$ and we can write

$$\epsilon_{\nu\mu} = \frac{1}{2}\left(\epsilon_{\nu\mu} - \epsilon_{\mu\nu}\right). \qquad (8.1310)$$

The infinitesimal Poincare transformation can then be written as

$$\begin{aligned} x'^{\mu} &= \Lambda^{\mu}_{\ \alpha} x^{\alpha} + a^{\mu} && (8.1311)\\ &\simeq \left(\delta^{\mu}_{\alpha} + \epsilon^{\mu}_{\ \alpha}\right) x^{\alpha} + a^{\mu} && (8.1312)\\ &\simeq x^{\mu} + \epsilon^{\mu}_{\ \alpha} x^{\alpha} + a^{\mu}. && (8.1313) \end{aligned}$$

The inverted PT is then given by

$$\begin{aligned} x &= \Lambda^{-1}(x' - a) && (8.1314)\\ &= \Lambda^{-1}x' - \Lambda^{-1}a && (8.1315)\\ x^{\mu} &\simeq \left(\delta^{\mu}_{\alpha} - \epsilon^{\mu}_{\ \alpha}\right) x'^{\alpha} - \left(\delta^{\mu}_{\alpha} - \epsilon^{\mu}_{\ \alpha}\right) a^{\alpha} && (8.1316)\\ &\simeq x'^{\mu} - \epsilon^{\mu}_{\ \alpha} x'^{\alpha} - a^{\mu} + \mathcal{O}(\epsilon \cdot a) && (8.1317) \end{aligned}$$

Because of

$$\begin{aligned} \phi'(x') = \phi(x) \quad &\Leftrightarrow \quad \phi'(\Lambda x + a) = \phi(x) && (8.1318)\\ &\Leftrightarrow \quad \phi'(x) = \phi(\Lambda^{-1}(x-a)) && (8.1319) \end{aligned}$$

we can now calculate

$$\begin{aligned} \delta\phi(x) &\equiv \phi'(x) - \phi(x) && (8.1320)\\ &= \phi(\Lambda^{-1}(x-a)) - \phi(x) && (8.1321)\\ &\simeq \phi(x^{\mu} - \epsilon^{\mu}_{\ \alpha} x^{\alpha} - a^{\mu}) - \phi(x) && (8.1322)\\ &\simeq \phi(x) + \partial_{\mu}\phi(x) \cdot (-\epsilon^{\mu}_{\ \alpha} x^{\alpha} - a^{\mu}) - \phi(x) && (8.1323)\\ &\simeq -(a^{\mu} + \epsilon^{\mu}_{\ \alpha} x^{\alpha})\partial_{\mu}\phi(x) && (8.1324)\\ &\simeq -(a^{\mu} + \epsilon^{\mu\alpha} x_{\alpha})\partial_{\mu}\phi(x) && (8.1325)\\ &\simeq -\left(a^{\mu} + \frac{1}{2}\left(\epsilon^{\mu\alpha} - \epsilon^{\alpha\mu}\right) x_{\alpha}\right)\partial_{\mu}\phi(x) && (8.1326)\\ &\simeq -\left(a^{\mu}\partial_{\mu} + \frac{1}{2}\left(\epsilon^{\mu\alpha} x_{\alpha}\partial_{\mu} - \epsilon^{\alpha\mu} x_{\alpha}\partial_{\mu}\right)\right)\phi(x) && (8.1327)\\ &\simeq -\left(a^{\mu}\partial_{\mu} + \frac{1}{2}\left(\epsilon^{\mu\alpha} x_{\alpha}\partial_{\mu} - \epsilon^{\mu\alpha} x_{\mu}\partial_{\alpha}\right)\right)\phi(x) && (8.1328)\\ &\simeq i\left(a^{\mu} i\partial_{\mu} + \frac{1}{2}\epsilon^{\mu\alpha} i\left(x_{\alpha}\partial_{\mu} - x_{\mu}\partial_{\alpha}\right)\right)\phi(x) && (8.1329)\\ &\simeq i\left(a^{\mu} i\partial_{\mu} - \frac{1}{2}\epsilon^{\mu\alpha} i\left(x_{\mu}\partial_{\alpha} - x_{\alpha}\partial_{\mu}\right)\right)\phi(x) && (8.1330)\\ &\simeq i\left(a^{\mu} P_{\mu} - \frac{1}{2}\epsilon^{\mu\alpha} M_{\mu\alpha}\right)\phi(x) && (8.1331) \end{aligned}$$

Calculating the commutators

$$[P_\mu, P_\nu] = 0 \tag{8.1332}$$

$$\begin{aligned}
[M_{\mu\nu}, P_\rho] &= i^2(x_\mu\partial_\nu - x_\nu\partial_\mu)\partial_\rho - i^2\partial_\rho(x_\mu\partial_\nu - x_\nu\partial_\mu) && (8.1333)\\
&= -(x_\mu\partial_\nu - x_\nu\partial_\mu)\partial_\rho + \partial_\rho(x_\mu\partial_\nu - x_\nu\partial_\mu) && (8.1334)\\
&= -x_\mu\partial_\nu\partial_\rho + x_\nu\partial_\mu\partial_\rho + (\partial_\rho g_{\mu\alpha}x^\alpha)\partial_\nu + x_\mu\partial_\rho\partial_\nu - (\partial_\rho g_{\nu\alpha}x^\alpha)\partial_\mu - x_\nu\partial_\rho\partial_\mu && (8.1335)\\
&= (\partial_\rho g_{\mu\alpha}x^\alpha)\partial_\nu - (\partial_\rho g_{\nu\alpha}x^\alpha)\partial_\mu && (8.1336)\\
&= (g_{\mu\alpha}\partial_\rho x^\alpha)\partial_\nu - (g_{\nu\alpha}\partial_\rho x^\alpha)\partial_\mu && (8.1337)\\
&= (g_{\mu\alpha}\delta^\alpha_\rho)\partial_\nu - (g_{\nu\alpha}\delta^\alpha_\rho)\partial_\mu && (8.1338)\\
&= g_{\mu\rho}\partial_\nu - g_{\nu\rho}\partial_\mu && (8.1339)\\
&= -i(g_{\mu\rho}i\partial_\nu - g_{\nu\rho}i\partial_\mu) && (8.1340)\\
&= -i(g_{\mu\rho}P_\nu - g_{\nu\rho}P_\mu) && (8.1341)
\end{aligned}$$

$$[M_{\mu\nu}, M_{\rho,\sigma}] = \text{...painful} \tag{8.1342}$$

8.13 LeBellac - Quantum and Statistical Field Theory

8.13.1 Problem 1.1

Some simple geometry

$$\begin{aligned}
l &= 2a\cos\theta && (8.1343)\\
x &= l\sin\theta && (8.1344)\\
&= 2a\cos\theta\sin\theta && (8.1345)\\
h &= x\tan\theta && (8.1346)\\
&= 2a\sin^2\theta && (8.1347)
\end{aligned}$$

Then the potential is given by

$$\begin{aligned}
V(\phi) &= 2mga\sin^2\theta + \frac{1}{2}Ca^2(2\cos\theta - 1)^2 && (8.1348)\\
\frac{\partial V}{\partial\theta} &= 4mga\sin\theta\cos\theta - 2Ca^2(2\cos\theta - 1)\sin\theta && (8.1349)\\
&= 2a\sin\theta\,(2mg\cos\theta - Ca(2\cos\theta - 1)) && (8.1350)\\
&= 2a\sin\theta\,(2(mg - Ca)\cos\theta + Ca) && (8.1351)\\
&\to \theta_0 = 0 && (8.1352)\\
&\to \theta_{1,2} = \arccos\frac{Ca}{2(Ca - mg)} && (8.1353)
\end{aligned}$$

Stability

$$\begin{aligned}
\frac{\partial^2 V}{\partial\theta^2}(\theta_{1,2}) &= 2a(2mg - Ca) && (8.1354)\\
\frac{\partial^2 V}{\partial\theta^2}(\theta_0) &= 2a(2mg - Ca) && (8.1355)
\end{aligned}$$

Chapter 9

Particle Physics

9.1 Nagashima - Elementary Particle Physics Volume 1: Quantum Field Theory and Particles

9.1.1 Problem 2.1

1. Simple calculation

$$\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c} = \frac{1.6^2 \cdot 10^{-38}}{12 \cdot 9 \cdot 10^{-12} \cdot 10^{-34} \cdot 3 \cdot 10^8} \tag{9.1}$$
$$= \frac{1}{108} 10^{-38+34} \frac{1}{10^{-12+8}} \tag{9.2}$$
$$\approx \frac{1}{137} \tag{9.3}$$

$$\alpha_G = G_N \frac{m_e m_p}{\hbar c} = 7 \cdot 10^{-11} \frac{10^{-30} \cdot 2 \cdot 10^{-27}}{10^{-34} \cdot 3 \cdot 10^8} \tag{9.4}$$
$$= 4 \cdot 10^{-11-30-27} \frac{1}{10^{-34+8}} \tag{9.5}$$
$$= 4 \cdot 10^{-42} \tag{9.6}$$

2. Another simple one

$$1 = G \frac{m_P^2}{\hbar c} \quad \to \quad m_P = \sqrt{\frac{\hbar c}{G}} = 2 \cdot 10^{-8} \text{kg} \tag{9.7}$$
$$E_P = m_P c^2 = \sqrt{\frac{\hbar c^5}{G}} = 2 \cdot 10^9 \text{J} = 1.2 \cdot 10^{19} \text{eV} \tag{9.8}$$

9.1.2 Problem 2.3

Basic approximation with $\Delta m = m$

$$\Delta E \cdot \Delta t \approx \frac{\hbar}{2} \tag{9.9}$$
$$\Delta t \approx \frac{\hbar}{2\Delta m \cdot c^2} \tag{9.10}$$
$$\Delta x = c \cdot \Delta t \approx \frac{\hbar}{2\Delta m \cdot c} \tag{9.11}$$
$$\tag{9.12}$$

Forgetting the factor 2 and knowing $1\text{GeV}^{-1} = 0.197 \cdot 10^{-15}\text{s}$

$$\Delta x_W = \frac{1}{80}\frac{1}{\text{GeV}} = 2.4 \cdot 10^{-18} \tag{9.13}$$

$$\Delta x_Z = \frac{1}{91}\frac{1}{\text{GeV}} = 2.16 \cdot 10^{-18} \tag{9.14}$$

9.1.3 Problem 2.4

$$\Delta m = \frac{\hbar}{\Delta x c} \tag{9.15}$$

$$\Delta E = \Delta m c^2 = \frac{\hbar c}{\Delta x} \tag{9.16}$$

$$\Delta E_{\text{crab}} = 2 \cdot 10^{-25}\text{eV} \tag{9.17}$$

$$\Delta E_{\text{galactic}} = 2 \cdot 10^{-29}\text{eV} \tag{9.18}$$

Chapter 10

General Relativity

10.1 Coleman - Sidney Coleman's Lectures On Relativity

10.1.1 Problem 1.1

Lets simplify

$$\tau(b) - \tau(a) = \int_a^b \sqrt{c^2dt^2 - dx^2 - dy^2 - dz^2} \tag{10.1}$$

$$= \int_a^b cdt\sqrt{1 - \frac{v^2}{c^2}} \tag{10.2}$$

If we LT into the inertial system of Alice her proper time is simply (because $v = 0$)

$$\Delta\tau_A = \int_a^b cdt = ct \tag{10.3}$$

For Bob we obtain

$$\Delta\tau_B = \int_a^b cdt\sqrt{1 - \frac{v(t)^2}{c^2}} \tag{10.4}$$

where the square root is smaller than one as soon the the observer is moving. Therefore it clear that $\Delta\tau_A < \Delta\tau_B$.

10.2 Carroll - Spacetime an Geometry

10.2.1 Problem 1.7

1. Because the metric is symmetric

$$X^{\mu}{}_{\nu} = \eta_{\nu\alpha}X^{\mu\alpha} = X^{\mu\alpha}\eta_{\alpha\nu} \equiv X\eta \tag{10.5}$$

$$= \begin{pmatrix} -2 & 0 & 1 & -1 \\ 1 & 0 & 3 & 2 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & -2 \end{pmatrix} \tag{10.6}$$

2.

$$X_\mu{}^\nu = \eta_{\nu\alpha} X^{\alpha\nu} \equiv \eta X \tag{10.7}$$

$$= \begin{pmatrix} -2 & 0 & -1 & 1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix} \tag{10.8}$$

3.

$$X^{(\mu\nu)} = \frac{1}{2}(X^{\mu\nu} + X^{\nu\mu}) \tag{10.9}$$

$$= \begin{pmatrix} 2 & -1/2 & 0. & -3/2 \\ -1/2 & 0 & 2 & 3/2 \\ 0 & 2 & 0 & 1/2 \\ -3/2 & 3/2 & 1/2 & -2 \end{pmatrix} \tag{10.10}$$

4.

$$X_{\mu\nu} = \eta_{\mu\alpha}\eta_{\nu\beta} X^{\alpha\beta} \equiv \eta X \eta \tag{10.11}$$

$$= \begin{pmatrix} 2 & 0 & -1 & 1 \\ 1 & 0 & 3 & 2 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & -2 \end{pmatrix} \tag{10.12}$$

$$X_{[\mu\nu]} = \frac{1}{2}(X_{\mu\nu} - X_{\nu\mu}) \tag{10.13}$$

$$= \begin{pmatrix} 0 & -1/2 & -1 & -1/2 \\ 1/2 & 0 & 1 & 1/2 \\ 1 & -1 & 0 & -1/2 \\ 1/2 & -1/2 & 1/2 & 0 \end{pmatrix} \tag{10.14}$$

5.

$$X^\lambda{}_\lambda = \eta_{\lambda\alpha} X^{\lambda\alpha} = X^{\lambda_\alpha}\eta_{\alpha\lambda} \equiv \mathrm{Tr}(X\eta) = -4 \tag{10.15}$$

$$\tag{10.16}$$

6.

$$V^\mu V_\mu = V^\mu \eta_{\mu\nu} V^\nu = 7 \tag{10.17}$$

7.

$$V_\mu X^{\mu\nu} = V^\alpha \eta_{\alpha\mu} X^{\mu\nu} \equiv V\eta X = (4, -2, 5, 7) \tag{10.18}$$

10.2.2 Problem 3.3 - Christoffel symbols for diagonal metric

With $g_{\mu\nu} = \mathrm{diag}(g_{11}, g_{22}, g_{33}, g_{44})$ the inverse is given by $g^{\mu\nu} = \mathrm{diag}(1/g_{11}, 1/g_{22}, 1/g_{33}, 1/g_{44})$. Now for $\mu \neq \mu \neq \lambda$ we obtain

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \tag{10.19}$$

$$= \frac{1}{2} g^{\lambda\lambda}(\partial_\mu \underbrace{g_{\nu\lambda}}_{=0} + \partial_\nu \underbrace{g_{\mu\lambda}}_{=0} - \partial_\lambda \underbrace{g_{\mu\nu}}_{=0}) \tag{10.20}$$

$$= 0 \tag{10.21}$$

$$\Gamma^\lambda_{\mu\mu} = \frac{1}{2}g^{\lambda\sigma}(\partial_\mu g_{\mu\sigma} + \partial_\mu g_{\mu\sigma} - \partial_\sigma g_{\mu\mu}) \tag{10.22}$$
$$= \frac{1}{2}g^{\lambda\lambda}(\partial_\mu g_{\mu\lambda} + \partial_\mu g_{\mu\lambda} - \partial_\lambda g_{\mu\mu}) \tag{10.23}$$
$$= -\frac{1}{2}\frac{1}{g_{\lambda\lambda}}\partial_\lambda g_{\mu\mu} \tag{10.24}$$

$$\Gamma^\lambda_{\mu\lambda} = \frac{1}{2}g^{\lambda\sigma}(\partial_\lambda g_{\mu\sigma} + \partial_\mu g_{\lambda\sigma} - \partial_\sigma g_{\lambda\mu}) \tag{10.25}$$
$$= \frac{1}{2}g^{\lambda\lambda}(\partial_\lambda g_{\mu\lambda} + \partial_\mu g_{\lambda\lambda} - \partial_\lambda g_{\lambda\mu}) \tag{10.26}$$
$$= \frac{1}{2}\frac{1}{g^{\lambda\lambda}}\partial_\mu g_{\lambda\lambda} \tag{10.27}$$
$$= \frac{1}{2}\frac{1}{\mathrm{sgn}\cdot|g^{\lambda\lambda}|}\partial_\mu(\mathrm{sgn}\cdot|g_{\lambda\lambda}|) \tag{10.28}$$
$$= \frac{1}{2}\frac{1}{|g^{\lambda\lambda}|}\partial_\mu(|g_{\lambda\lambda}|) \tag{10.29}$$
$$= \frac{1}{2}\partial_\mu \log|g_{\lambda\lambda}| \tag{10.30}$$
$$= \partial_\mu \log\sqrt{|g_{\lambda\lambda}|} \tag{10.31}$$

$$\Gamma^\lambda_{\lambda\lambda} = \frac{1}{2}g^{\lambda\sigma}(\partial_\lambda g_{\lambda\sigma} + \partial_\lambda g_{\lambda\sigma} - \partial_\sigma g_{\lambda\lambda}) \tag{10.32}$$
$$= \frac{1}{2}g^{\lambda\lambda}(\partial_\lambda g_{\lambda\lambda} + \partial_\lambda g_{\lambda\lambda} - \partial_\lambda g_{\lambda\lambda}) \tag{10.33}$$
$$= \frac{1}{2}\frac{\partial_\lambda g_{\lambda\lambda}}{g_{\lambda\lambda}} \tag{10.34}$$
$$= \partial_\lambda \log\sqrt{|g_{\lambda\lambda}|} \tag{10.35}$$

10.3 Poisson - A relativists toolkit

10.3.1 Problem 1.1 - Parallel transport on cone

1. We find the metric by using elementary geometry

$$ds^2 = dr^2 + (r\sin\alpha)^2 d\phi^2 \tag{10.36}$$

2. Trying around a bit - we find

$$x = r\cos(\phi\sin\alpha) \tag{10.37}$$
$$y = r\sin(\phi\sin\alpha) \tag{10.38}$$

$$x = f(r,\phi) \quad \rightarrow \quad dx = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \phi}d\phi \tag{10.39}$$
$$dx = \cos(\phi\sin\alpha)dr - r\sin(\phi\sin\alpha)\sin\alpha\; d\phi \tag{10.40}$$
$$y = g(r,\phi) \quad \rightarrow \quad dy = \frac{\partial g}{\partial r}dr + \frac{\partial g}{\partial \phi}d\phi \tag{10.41}$$
$$dy = \sin(\phi\sin\alpha)dr + r\cos(\phi\sin\alpha)\sin\alpha\; d\phi \tag{10.42}$$

We can then simply check $ds^2 = dx^2 + dy^2$

3. The parallel transport equation for a vector A^λ along curve $x^\mu(s)$ is given by

$$\dot{x}^\mu \nabla_\mu A^\lambda = 0 \tag{10.43}$$
$$\frac{dx^\mu}{ds}\partial_\mu A^\lambda + \Gamma^\lambda_{\mu\nu}\dot{x}^\mu A^\nu = 0 \tag{10.44}$$
$$\frac{\partial A^\lambda}{\partial s} + \Gamma^\lambda_{\mu\nu}\dot{x}^\mu A^\nu = 0 \tag{10.45}$$

We are moving the vector $\vec{A} = (A_r, A_\phi)$ along $\vec{x}(s) = (r, s)$ with $s \in [0, 2\pi]$ and $\dot{\vec{x}}(s) = (0, 1)$. Calculating the Christoffel symbols gives

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \tag{10.46}$$
$$\Gamma^r_{\phi\phi} = -r\sin^2\alpha \tag{10.47}$$
$$\Gamma^\phi_{r\phi} = 1/r \tag{10.48}$$
$$\Gamma^\phi_{\phi r} = 1/r \tag{10.49}$$

and the parallel transport equations simplify to

$$\dot{A}_r + \Gamma^r_{\phi\phi}\dot{x}_\phi A_\phi = 0 \quad \rightarrow \quad \dot{A}_r - r\sin^2\alpha A_\phi = 0 \tag{10.50}$$
$$\dot{A}_\phi + \Gamma^\phi_{\phi r}\dot{x}_\phi A_r = 0 \quad \rightarrow \quad \dot{A}_\phi + \frac{1}{r}A_r = 0. \tag{10.51}$$

which can be solved by Mathematica. To obtain the angle β we calculate first the norm of the vector at a given t

$$|\vec{A}(t)| = g_{\mu\nu}A^\mu(t)A^\nu(t) \tag{10.52}$$
$$= A_r(0)^2 + A_\phi(0)^2 r^2 \sin^2\alpha. \tag{10.53}$$

Now we can calculate the inner product

$$\vec{A}(t = 2\pi) \cdot \vec{A}(0) = g_{\mu\nu}A^\mu(t)A^\nu(0) \tag{10.54}$$
$$= (A_r(0)^2 + A_\phi(0)^2 r^2 \sin^2\alpha)\cos(2\pi\sin\alpha) \tag{10.55}$$

so $\cos\beta = \cos(2\pi\sin\alpha)$.

10.3.2 Problem 1.5 - Killing vectors of a spherical surface

Definition of the Lie derivative

$$\mathcal{L}_{\mathbf{a}}T^\mu = T^\mu_{\ ,\alpha}a^\alpha - T^\alpha a^\mu_{\ ,\alpha} \tag{10.56}$$
$$\mathcal{L}_{\mathbf{a}}T_\mu = T_{\mu,\alpha}a^\alpha + T_\alpha a^\alpha_{\ ,\mu} \tag{10.57}$$
$$\mathcal{L}_{\mathbf{a}}T_{\mu\nu} = T_{\mu\nu,\alpha}a^\alpha + T_{\alpha\nu}a^\alpha_{\ ,\mu} + T_{\mu\alpha}a^\alpha_{\ ,\nu} \tag{10.58}$$

Definition Killing vectors

$$\mathcal{L}_\xi g_{\mu\nu} = 0 \tag{10.59}$$
$$\rightarrow \quad g_{\mu\nu,\alpha}\xi^\alpha + g_{\alpha\nu}\xi^\alpha_{\ ,\mu} + g_{\mu\alpha}\xi^\alpha_{\ ,\nu} = 0 \tag{10.60}$$

Using coordinates $x = (\theta, \phi)$ the metric is given by

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2 \tag{10.61}$$

and we can write down the three Killing equations

$$\mu=\nu=1 \qquad g_{11,1}\xi^1+g_{11}\xi^1_{\ ,1}+g_{11}\xi^1_{\ ,1}=0 \tag{10.62}$$
$$\rightarrow \quad \xi^1_{\ ,1}=0 \tag{10.63}$$
$$\mu=1,\nu=2 \qquad g_{22}\xi^2_{\ ,1}+g_{11}\xi^1_{\ ,2}=0 \tag{10.64}$$
$$\rightarrow \quad \xi^2_{\ ,1}\sin^2\theta+\xi^1_{\ ,2}=0 \tag{10.65}$$
$$\mu=\nu=2 \qquad g_{22,1}\xi^1+g_{22}\xi^2_{\ ,2}+g_{22}\xi^2_{\ ,2}=0 \tag{10.66}$$
$$\rightarrow \quad \xi^1\cos\theta+\xi^2_{\ ,2}\sin\theta=0 \tag{10.67}$$

then we see immediately $\xi^1=f(\phi)$ and set/try $\xi^2=g(\phi,\theta)=u(\theta)v(\phi)$

$$vu_{,\theta}\sin^2\theta+f_{,\phi}=0 \tag{10.68}$$
$$f\cos\theta+uv_{,\phi}\sin\theta=0 \tag{10.69}$$

then we can separate

$$u_{,\theta}\sin^2\theta=-\frac{f_{,\phi}}{v}=A \tag{10.70}$$
$$\frac{f}{v_{,\phi}}=-u\tan\theta=B \tag{10.71}$$

and see $u=-\frac{B}{\tan\theta}$ and simplify

$$Bv_{,\phi\phi}=f_{,\phi}=-Av \tag{10.72}$$
$$\rightarrow v_{,\phi\phi}=\frac{A}{B}v \tag{10.73}$$
$$\rightarrow v=\sin(\sqrt{A/B}\phi+c) \tag{10.74}$$
$$\rightarrow f=Bv_{,\phi}=\sqrt{AB}\cos(\sqrt{A/B}\phi+c) \tag{10.75}$$

Now we find two vectors

$$c=-\pi/2, A=1, B=1 \quad \xi^\mu_{(1)}=(\sin\phi,\cos\phi\cot\theta) \tag{10.76}$$
$$c=0, A=1, B=1 \quad \xi^\mu_{(2)}=(\cos\phi,-\sin\phi\cot\theta) \tag{10.77}$$

but there is a third vector which we did not find as it was not caught by the separation ansatz

$$\xi^\mu_{(3)}=(0,1). \tag{10.78}$$

We could have found it also by calculating the Lie bracket of two Killing vectors we already found

$$Y=\xi_{(1)}=\sin\phi\partial_\theta+\cos\phi\cot\theta\partial_\phi \tag{10.79}$$
$$X=\xi_{(2)}=\cos\phi\partial_\theta-\sin\phi\cot\theta\partial_\phi \tag{10.80}$$
$$\rightarrow [X,Y]=XY-YX=\partial_\phi \tag{10.81}$$

we find (by construction) another Killing vector field - which translates into $\xi^\mu_{(3)}=(0,1)$.

10.4 Wald - General Relativity

10.4.1 5.1

Case $k=-1$

$$ds^2_{k=0}=-d\tau^2+a^2(\tau)\left[dx^2+dy^2+dz^2\right] \tag{10.82}$$
$$=-d\tau^2+a^2(\tau)\left[dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2)\right] \tag{10.83}$$

Case $k = 0$

$$r = \sinh\psi \tag{10.84}$$

$$\rightarrow \quad \frac{dr}{d\psi} = \cosh\psi \tag{10.85}$$

$$\rightarrow \quad d\psi = \frac{dr}{\cosh\psi} = \frac{dr}{\sqrt{1+\sinh^2\psi}} = \frac{dr}{\sqrt{1+r^2}} \tag{10.86}$$

$$d_{k=-1}s^2 = -d\tau^2 + a^2(\tau)\left[d\psi^2 + \sinh^2\psi(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{10.87}$$

$$= -d\tau^2 + a^2(\tau)\left[\frac{dr^2}{1+r^2} + r^2(d\theta^2 + \sin^2\theta d\phi)^2\right] \tag{10.88}$$

Case $k = 1$

$$r = \sin\psi \tag{10.89}$$

$$\rightarrow \quad \frac{dr}{d\psi} = \cos\psi \tag{10.90}$$

$$\rightarrow \quad d\psi = \frac{dr}{\cos\psi} = \frac{dr}{\sqrt{1-\sin^2\psi}} = \frac{dr}{\sqrt{1-r^2}} \tag{10.91}$$

$$ds^2_{k=1} = -d\tau^2 + a^2(\tau)\left[d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\phi)^2\right] \tag{10.92}$$

$$= -d\tau^2 + a^2(\tau)\left[\frac{dr^2}{1-r^2} + r^2(d\theta^2 + \sin^2\theta d\phi)^2\right] \tag{10.93}$$

where $\psi \in (0, \pi)$, $\theta \in (0, \pi)$ and $\phi \in (0, 2\pi)$. The 3-volume is given by

$$V = \int \sqrt{\gamma}\; d\psi d\theta d\phi \tag{10.94}$$

$$= a^3 \int \sin\theta \sin^2\psi \; d\psi d\theta d\phi \tag{10.95}$$

$$= 2\pi^2 a^3 \tag{10.96}$$

Chapter 11

Quantum Gravity

11.1 Hartman - Lectures on Quantum Gravity and Black Holes

11.2 Ammon, Erdmenger - Gauge/Gravity Duality - Foundations and Applications

The authors use $d-1$ spacial dimension and the sign convention

$$\eta_{\mu\nu} = diag(-1, 1, ..., 1) \tag{11.1}$$

which implies

$$\Box = \partial^\mu \partial_\mu = -\partial_t^2 + \triangle \tag{11.2}$$

$$kx = -k^0 x^0 + \vec{k}\vec{x} \tag{11.3}$$

and results in a minus sign in the KG equation.

11.2.1 Problem 1.1.1 - Fourier representation of free scalar field

Ansatz (because KG equation looks quite similar to wave equation) $\phi(x) = a \cdot e^{ikx}$ with $x^\mu = (t, \vec{x})$, $k^\mu = (\omega, \vec{k})$ and $a \in \mathbb{C}$ meaning

$$e^{ikx} \equiv e^{ik^\mu x_\mu} = e^{i\eta_{\mu\nu} k^\mu x^\nu} = e^{i(-k^0 x^0 + \vec{k}\vec{x})} \tag{11.4}$$

Inserting into the equation of motion

$$(\Box - m^2)\phi(x) = (\partial^t \partial_t + \triangle - m^2)\phi(x) \tag{11.5}$$

$$= a(-\partial_t^2 + \triangle - m^2)e^{i(-\omega t + \vec{k}\vec{x})} \tag{11.6}$$

$$= a\left(\omega^2 + i^2 \vec{k}^2 - m^2\right) e^{i(-\omega t + \vec{k}\vec{x})} = 0 \tag{11.7}$$

This implies $\omega^2 - \vec{k}^2 - m^2 = 0$ and therefore $\omega_k \equiv \omega = \sqrt{\vec{k}^2 + m^2}$. One particular solution is therefore $\phi(x) = a \cdot e^{ikx}|_{k^0 = \omega_k}$. The general solution is then given by a superposition

$$\phi(x) = \int d^{d-1}\vec{k} \left[a(\vec{k})e^{ikx}\right] \tag{11.8}$$

to ensure a real valued ϕx we add the conjugate complex solution

$$\phi(x) = \int d^{d-1}\vec{k} \left[a(\vec{k})e^{ikx} + a^*(\vec{k})e^{-ikx}\right]. \tag{11.9}$$

The factor $(2\pi)^{1-d}/2\omega_k$ can be absorbed into $a(k)$.

11.2.2 Problem 1.1.2 - Lagrangian of self-interacting scalar field

The Lagrangian is then

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}} && (11.10)\\ &= -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi(x)\partial_\nu\phi(x) - \frac{1}{2}m^2\phi(x)^2 - \frac{g}{4!}\phi(x)^4. && (11.11)\end{aligned}$$

with the Euler-Lagrange equations

$$\partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0. \tag{11.12}$$

Therefore

$$\begin{aligned}\partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)}\right) &= \partial_\alpha\left(-\frac{1}{2}\eta^{\mu\nu}[\delta_{\mu\alpha}\partial_\nu\phi + \partial_\mu\phi\delta_{\nu\alpha}]\right) && (11.13)\\ &= \partial_\alpha\left(-\frac{1}{2}\eta^{\alpha\nu}\partial_\nu\phi - \frac{1}{2}\eta^{\mu\alpha}\partial_\mu\phi\right) && (11.14)\\ &= -\partial_\alpha\left(\eta^{\alpha\beta}\partial_\beta\phi\right) && (11.15)\\ &= -\partial^\beta\partial_\beta\phi && (11.16)\\ &= -\Box\phi && (11.17)\end{aligned}$$

and

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi - \frac{g}{3!}\phi^3. \tag{11.18}$$

The relevant term in the Euler-Lagrange equations is $\partial\mathcal{L}_{\text{int}}/\partial\phi = -g\phi^3/3!$. The modified equation of motion is therefore

$$(\Box - m^2)\phi(x) - \frac{g}{3!}\phi(x)^3 = 0 \tag{11.19}$$

11.2.3 Problem 1.1.3 - Complex scalar field

$$\begin{aligned}\mathcal{L}_{\text{free}} &= -\partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi && (11.20)\\ &= -\eta^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi - m^2\phi^*\phi && (11.21)\\ &= -\frac{1}{2}\eta^{\mu\nu}\partial_\mu(\phi_1 - i\phi_2)\partial_\nu(\phi_1 + i\phi_2) - \frac{1}{2}m^2(\phi_1^2 + \phi_2^2) && (11.22)\\ &= -\frac{1}{2}\eta^{\mu\nu}\left(\partial_\mu\phi_1\partial_\nu\phi_1 + i\partial_\mu\phi_1\partial_\nu\phi_2 - i\partial_\mu\phi_2\partial_\nu\phi_1 + \partial_\mu\phi_2\partial_\nu\phi_2\right) - \frac{1}{2}m^2(\phi_1^2 + \phi_2^2) && (11.23)\\ &= -\frac{1}{2}\eta^{\mu\nu}\left(\partial_\mu\phi_1\partial_\nu\phi_1 + \partial_\mu\phi_2\partial_\nu\phi_2\right) - \frac{1}{2}m^2(\phi_1^2 + \phi_2^2) && (11.24)\\ &= -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi_1\partial_\nu\phi_1 - \frac{1}{2}m^2\phi_1^2 - \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi_2\partial_\nu\phi_2 - \frac{1}{2}m^2\phi_2^2 && (11.25)\\ &= \mathcal{L}_{\text{free1}} + \mathcal{L}_{\text{free2}} && (11.26)\end{aligned}$$

Equations of motion for ϕ and ϕ^* are given by

$$\begin{aligned}\partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi^*)}\right) - \frac{\partial\mathcal{L}}{\partial\phi^*} &= 0 && (11.27)\\ -\partial_\mu\partial^\mu\phi + m^2\phi &= 0 && (11.28)\\ (\Box - m^2)\phi &= 0 && (11.29)\end{aligned}$$

and

$$\partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^*)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \tag{11.30}$$
$$-\partial_\mu \partial^\mu \phi + m^2 \phi^* = 0 \tag{11.31}$$
$$(\Box - m^2)\phi^* = 0 \tag{11.32}$$

11.2.4 Problem 1.2.1 - Time-independence of Noether charge

The conserved current is

$$\partial_\mu \mathcal{J}^\mu \equiv -\partial_0 \mathcal{J}^0 + \partial_i \mathcal{J}^i = 0. \tag{11.33}$$

Spacial integration using Gauss law on the right hand side gives

$$\int_{\mathbb{R}^{d-1}} d^{d-1}\vec{x}\, \partial_0 \mathcal{J}^0 = \int_{\mathbb{R}^{d-1}} d^{d-1}\vec{x}\, \partial_i \mathcal{J}^i \tag{11.34}$$
$$\partial_0 \int_{\mathbb{R}^{d-1}} d^{d-1}\vec{x}\, \mathcal{J}^0 = \int_{\partial\mathbb{R}^{d-1}} dS\, \mathcal{J}^i \tag{11.35}$$
$$\partial_0 \mathcal{Q} = 0 \tag{11.36}$$

where we used that $\mathcal{J}^i$ is vanishing at infinity.

11.2.5 Problem 1.2.2 - Hamiltonian of scalar field

The Lagrangian of the real free scalar field is given by

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu \phi(x)\partial_\nu \phi(x) - \frac{1}{2}m^2\phi(x)^2. \tag{11.37}$$

The canonical momentum is therefore

$$\Pi = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} \tag{11.38}$$
$$= -\frac{1}{2}2\eta^{ti}\partial_i \phi - \frac{1}{2}2\eta^{tt}\partial_t \phi \tag{11.39}$$
$$= \partial_t \phi. \tag{11.40}$$

Using $\eta_{\mu\nu} = diag(-1, 1, ..., 1)$ the Hamiltonian $\mathcal{H} = \Theta^{tt} = \eta^{t\nu}\Theta^t{}_\nu = -\Theta^t{}_t$ is

$$\Theta^t{}_t = -\frac{\partial \mathcal{L}}{\partial(\partial_t \phi)}\partial_t \phi + \mathcal{L} \tag{11.41}$$
$$= -\Pi \cdot \partial_t \phi + \mathcal{L} \tag{11.42}$$

and therefore

$$\mathcal{H} = \Pi \partial_t \phi - \mathcal{L} \tag{11.43}$$
$$= \Pi^2 - \left(-\frac{1}{2}\eta^{\mu\nu}\partial_\mu \phi(x)\partial_\nu \phi(x) - \frac{1}{2}m^2\phi(x)^2 \right) \tag{11.44}$$
$$= \Pi^2 - \left(\frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\nabla \phi)^2 - \frac{1}{2}m^2\phi(x)^2 \right) \tag{11.45}$$
$$= \frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla \phi)^2 + \frac{1}{2}m^2\phi(x)^2 \tag{11.46}$$

11.2.6 Problem 1.2.3 - Symmetric energy-momentum tensor

The Lorentz transformation

$$\Lambda^{\mu}_{\ \nu} = \delta^{\mu}_{\nu} + \omega^{\mu}_{\ \nu} \tag{11.47}$$

implies the field transformation

$$\begin{aligned} \phi(x^{\mu}) \to \tilde{\phi}(x^{\mu}) &= \phi(x^{\mu} - \omega^{\mu}_{\ \rho} x^{\rho}) && (11.48) \\ &= \phi(x^{\mu}) - \omega^{\mu}_{\ \rho} x^{\rho} \partial_{\mu} \phi && (11.49) \end{aligned}$$

under which the Lagrangian transforms as

$$\begin{aligned} \mathcal{L} \to \tilde{\mathcal{L}} &= \mathcal{L} + \frac{\partial \mathcal{L}}{\partial x^{\mu}} dx^{\mu} && (11.50) \\ &= \mathcal{L} - \omega^{\nu}_{\ \rho} x^{\rho} \partial_{\mu} (\delta^{\mu}_{\nu} \mathcal{L}) && (11.51) \\ &= \mathcal{L} + \partial_{\mu}(\omega^{\nu}_{\ \rho} x^{\rho}) \cdot (\delta^{\mu}_{\nu} \mathcal{L}) - \partial_{\mu}(\omega^{\nu}_{\ \rho} x^{\rho} \delta^{\mu}_{\nu} \mathcal{L}) && (11.52) \\ &= \mathcal{L} + \omega^{\nu}_{\ \rho} \delta^{\rho}_{\mu} \cdot (\delta^{\mu}_{\nu} \mathcal{L}) - \partial_{\mu}(\omega^{\nu}_{\ \rho} x^{\rho} \delta^{\mu}_{\nu} \mathcal{L}) && (11.53) \\ &= \mathcal{L} + \omega^{\rho}_{\ \rho} \mathcal{L} - \partial_{\mu}(\omega^{\nu}_{\ \rho} x^{\rho} \delta^{\mu}_{\nu} \mathcal{L}) && (11.54) \\ &= \mathcal{L} - \partial_{\mu}(\omega^{\nu}_{\ \rho} x^{\rho} \delta^{\mu}_{\nu} \mathcal{L}) && (11.55) \end{aligned}$$

where we used $\omega_{\mu\nu} = -\omega_{\nu\mu}$ meaning

$$\begin{aligned} \omega^{\rho}_{\ \rho} &= \eta^{\alpha\rho} \omega_{\alpha\rho} && (11.56) \\ &= \sum_{\rho} \eta^{0\rho} \omega_{0\rho} + \eta^{1\rho} \omega_{1\rho} + \eta^{2\rho} \omega_{2\rho} + \eta^{3\rho} \omega_{3\rho} && (11.57) \\ &= 0 && (11.58) \end{aligned}$$

in the last step (as η has only diagonal elements and the diagonal elements of ω are zero). With $\delta\phi = -\omega^{\mu}_{\ \rho} x^{\rho} \partial_{\mu} \phi$ and $X^{\mu} = -\omega^{\nu}_{\ \rho} x^{\rho} \delta^{\mu}_{\nu} \mathcal{L}$ we obtain for the conserved current

$$\begin{aligned} \mathcal{J}^{\mu} &= -\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)} \delta\phi + X^{\mu} && (11.59) \\ &= -\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)} (-\omega^{\nu}_{\ \rho} x^{\rho} \partial_{\nu} \phi) + (-\omega^{\nu}_{\ \rho} x^{\rho} \delta^{\mu}_{\nu} \mathcal{L}) && (11.60) \\ &= (-\omega^{\nu}_{\ \rho} x^{\rho}) \left(-\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)} \partial_{\nu}\phi + (\delta^{\mu}_{\nu} \mathcal{L}) \right) && (11.61) \\ &= (-\omega^{\nu}_{\ \rho} x^{\rho}) \Theta^{\mu}_{\ \nu} && (11.62) \\ &= (-\eta^{\nu\alpha} \omega_{\alpha\rho} x^{\rho}) \Theta^{\mu}_{\ \nu} && (11.63) \\ &= -\omega_{\alpha\rho} x^{\rho} \Theta^{\mu\alpha} && (11.64) \\ &= -\frac{1}{2} \omega_{\alpha\rho} (x^{\rho} \Theta^{\mu\alpha} - x^{\alpha} \Theta^{\mu\rho}) && (11.65) \\ &= -\frac{1}{2} \omega_{\alpha\rho} N^{\mu\rho\alpha} && (11.66) \end{aligned}$$

With $\partial_{\mu} \Theta^{\mu}_{\ \nu} = 0$ and $\partial_{\mu} N^{\mu\nu\rho} = 0$ we see

$$\begin{aligned} 0 &= \partial_{\mu} N^{\mu\nu\rho} && (11.67) \\ &= \partial_{\mu} \left(x^{\nu} \Theta^{\mu\rho} - x^{\rho} \Theta^{\mu\nu} \right) && (11.68) \\ &= (\partial_{\mu} x^{\nu}) \Theta^{\mu\rho} + x^{\nu} (\partial_{\mu} \Theta^{\mu\rho}) - (\partial_{\mu} x^{\rho}) \Theta^{\mu\nu} - x^{\rho} (\partial_{\mu} \Theta^{\mu\nu}) && (11.69) \\ &= \delta^{\nu}_{\mu} \Theta^{\mu\rho} + x^{\nu} (\partial_{\mu} \Theta^{\mu\rho}) - \delta^{\rho}_{\mu} \Theta^{\mu\nu} - x^{\rho} (\partial_{\mu} \Theta^{\mu\nu}) && (11.70) \\ &= \Theta^{\nu\rho} - \Theta^{\rho\nu}. && (11.71) \end{aligned}$$

which means that the (canonical) energy-momentum tensor for Poincare invariant field theories is symmetric $\Theta^{\nu\rho} = \Theta^{\rho\nu}$.

11.2.7 Problem 1.2.4 - Callan-Coleman-Jackiw energy-momentum tensor

For the scalar field we have with $\mathcal{L} = -\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}m^2\phi^2$

$$\Theta^\mu_{\ \nu} = -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi + (\delta^\mu_{\ \nu}\mathcal{L}) \tag{11.72}$$

$$= -\left(-\frac{1}{2}\eta^{\alpha\beta}\delta^\mu_\alpha\partial_\beta\phi - \frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\phi\delta^\mu_\beta\right)\partial_\nu\phi + \delta^\mu_{\ \nu}\left(-\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}m^2\phi^2\right) \tag{11.73}$$

$$= \partial^\mu\phi\partial_\nu\phi - \frac{1}{2}\delta^\mu_{\ \nu}(\partial^\beta\phi\partial_\beta\phi + m^2\phi^2) \tag{11.74}$$

which gives in the massless case

$$\Theta^\mu_{\ \nu,\ \text{massless}} = \partial^\mu\phi\partial_\nu\phi - \frac{1}{2}\delta^\mu_{\ \nu}\partial^\beta\phi\partial_\beta\phi \tag{11.75}$$

$$\Theta_{\mu\nu,\ \text{massless}} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\partial^\beta\phi\partial_\beta\phi \tag{11.76}$$

The new improved or Callan–Coleman–Jackiw energy-momentum tensor for a single, real, massless scalar field in d-dimensional Minkowski space is obtained by adding a term proportional to $(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box)\phi^2$ where the proportionality constant is chosen to make the tensor traceless

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\partial_\rho\phi\partial^\rho\phi - \frac{d-2}{4(d-1)}\left(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box\right)\phi^2 \tag{11.77}$$

Let us now check the properties

1. symmetric: obvious

2. conserved: we use the equation of motion $\partial^\mu\partial_\mu\phi = \Box\phi = 0$

$$\partial_\mu T^{\mu\nu} = (\partial_\mu\partial^\mu\phi)\partial^\nu\phi + \partial^\mu\phi(\partial_\mu\partial^\nu\phi) \tag{11.78}$$

$$- \frac{1}{2}\eta^{\mu\nu}\left[(\partial_\mu\partial_\rho\phi)\partial^\rho\phi + \partial_\rho\phi(\partial_\mu\partial^\rho\phi)\right] \tag{11.79}$$

$$- \frac{d-2}{4(d-1)}\Box\partial^\nu\phi^2 + \frac{d-2}{4(d-1)}\eta^{\mu\nu}\partial_\mu\Box\phi^2 \tag{11.80}$$

$$= \partial^\mu\phi(\partial_\mu\partial^\nu\phi) - \frac{1}{2}\left[(\partial^\nu\partial_\rho\phi)\partial^\rho\phi + \partial_\rho\phi(\partial^\nu\partial^\rho\phi)\right] \tag{11.81}$$

$$= 0 \tag{11.82}$$

3. traceless:

$$T^\mu_{\ \mu} = \partial^\mu\phi\partial_\mu\phi - \frac{1}{2}\eta^\mu_{\ \mu}\partial_\rho\phi\partial^\rho\phi - \frac{d-2}{4(d-1)}\left(\partial^\mu\partial_\mu - \eta^\mu_{\ \mu}\Box\right)\phi^2 \tag{11.83}$$

$$= \partial^\mu\phi\partial_\mu\phi - \frac{d}{2}\partial_\rho\phi\partial^\rho\phi - \frac{d-2}{4(d-1)}\left(\partial^\mu\partial_\mu - d\cdot\partial^\mu\partial_\mu\right)\phi^2 \tag{11.84}$$

$$= \frac{2-d}{2}\partial_\rho\phi\partial^\rho\phi - \frac{d-2}{4(d-1)}(1-d)\partial^\mu\partial_\mu\phi^2 \tag{11.85}$$

$$= \frac{2-d}{2}\partial_\rho\phi\partial^\rho\phi + \frac{d-2}{4}\partial^\mu\partial_\mu\phi^2 \tag{11.86}$$

$$= \frac{2-d}{2}\partial_\rho\phi\partial^\rho\phi + \frac{d-2}{4}\partial^\mu(2\phi\partial_\mu\phi) \tag{11.87}$$

$$= \frac{2-d}{2}\left[\partial_\rho\phi\partial^\rho\phi - \partial^\mu\phi\partial_\mu\phi\right] + \frac{d-2}{2}\phi\cdot\Box\phi \tag{11.88}$$

$$= 0. \tag{11.89}$$

11.2.8 Problem 1.2.5 - Noether currents of complex scalar field

$$\mathcal{L}_{\text{free}} = -\partial^\mu \phi^* \partial_\mu \phi - m^2 \phi^* \phi \tag{11.90}$$
$$= -\eta^{\mu\nu} \partial_\nu \phi^* \partial_\nu \phi - m^2 \phi^* \phi \tag{11.91}$$

with the field transformations

$$\phi \to \phi' = e^{i\alpha}\phi = \phi + i\alpha\phi \tag{11.92}$$
$$\phi^* \to \phi^{*'} = e^{-i\alpha}\phi^* = \phi^* - i\alpha\phi^* \tag{11.93}$$
$$\mathcal{L} \to \mathcal{L}' = \mathcal{L} \tag{11.94}$$

we have $\delta\phi = i\alpha\phi$ and $\delta\phi^* = -i\alpha\phi^*$ and $X^\mu = 0$. With

$$\mathcal{J}^\sigma = -\frac{\partial \mathcal{L}}{\partial(\partial_\sigma \phi)} \delta\phi + X^\sigma \tag{11.95}$$

we obtain the the two fields

$$\mathcal{J}^\sigma = -\frac{\partial \mathcal{L}}{\partial(\partial_\sigma \phi)} \delta\phi - \frac{\partial \mathcal{L}}{\partial(\partial_\sigma \phi^*)} \delta\phi^* \tag{11.96}$$
$$= -(\eta^{\sigma\nu}\partial_\nu \phi^*) i\alpha\phi + (\eta^{\sigma\nu}\partial_\nu \phi) i\alpha\phi^* \tag{11.97}$$
$$= i\alpha \left[\phi^*(\partial^\sigma \phi) - \phi(\partial^\sigma \phi^*)\right] \tag{11.98}$$

11.2.9 Problem 1.2.6 - $O(n)$ invariance of action of n free scalar fields

For the n real scalar fields with equal mass m we have

$$\mathcal{L} = -\frac{1}{2}\sum_{j=1}^{n} \left[\eta^{\alpha\beta}(\partial_\alpha \phi_j)(\partial_\beta \phi^j) + m^2 (\phi^j)^2\right] \tag{11.99}$$

the action functional is then

$$S = \int d^d x \mathcal{L} \tag{11.100}$$
$$= -\frac{1}{2}\sum_{j=1}^{n} \int d^d x \left[\eta^{\alpha\beta}(\partial_\alpha \phi_j)(\partial_\beta \phi^j) + m^2 (\phi_j \phi^j)\right] \tag{11.101}$$

With $\phi'^j = R^j{}_k \phi^k$ and the definition of an orthogonal matrix R (inner product is invariant under rotation)

$$x^i x_i = x^i \delta_{ij} x^j \tag{11.102}$$
$$\stackrel{!}{=} R^i{}_a x^a \delta_{ij} R^j{}_b x^b \tag{11.103}$$
$$= \delta_{ij} R^j{}_b R^i{}_a x^a x^b \tag{11.104}$$
$$= R_{ib} R^i{}_a x^a x^b \tag{11.105}$$

we require $R_{ib}R^i{}_a = \delta_{ba}$. Then we can recalculate the action

$$S' = -\frac{1}{2}\sum_{j=1}^{n}\int d^dx\left[\eta^{\alpha\beta}(\partial_\alpha R_{ja}\phi^a)(\partial_\beta R^j{}_b\phi^b) + m^2(R_{ja}\phi^a \cdot R^j{}_b\phi^b)\right] \tag{11.106}$$

$$= -\frac{1}{2}\sum_{j=1}^{n}\int d^dx\left[\eta^{\alpha\beta}R_{ja}R^j{}_b(\partial_\alpha\phi^a)(\partial_\beta\phi^b) + m^2 R_{ja}R^j{}_b(\phi^a \cdot \phi^b)\right] \tag{11.107}$$

$$= -\frac{1}{2}\sum_{b=1}^{n}\int d^dx\left[\eta^{\alpha\beta}\delta_{ab}(\partial_\alpha\phi^a)(\partial_\beta\phi^b) + m^2\delta_{ab}(\phi^a \cdot \phi^b)\right] \tag{11.108}$$

$$= -\frac{1}{2}\sum_{b=1}^{n}\int d^dx\left[\eta^{\alpha\beta}(\partial_\alpha\phi_b)(\partial_\beta\phi^b) + m^2(\phi_b \cdot \phi^b)\right] \tag{11.109}$$

Analog for the complex case.

11.2.10 Problem 1.3.1 - Field commutators of scalar field

From the field

$$\hat\phi(x) = \frac{1}{(2\pi)^{d-1}}\int\frac{d^{d-1}\vec k}{2\omega_k}\left[\hat a(\vec k)e^{ikx} + \hat a^\dagger(\vec k)e^{-ikx}\right]_{k^0=\omega_k} \tag{11.110}$$

we can derive the conjugated momentum

$$\hat\Pi(x) = \partial_t\hat\phi \tag{11.111}$$

$$= \frac{1}{(2\pi)^{d-1}}\int\frac{d^{d-1}\vec k}{2\omega_k}\partial_t\left[\hat a(\vec k)e^{-i\omega_k t}e^{i\vec k\vec x} + \hat a^\dagger(\vec k)e^{i\omega_k t}e^{-i\vec k\vec x}\right] \tag{11.112}$$

$$= \frac{1}{(2\pi)^{d-1}}\int\frac{d^{d-1}\vec k}{2\omega_k}\left[\hat a(\vec k)(-i\omega_k)e^{ikx} + \hat a^\dagger(\vec k)(i\omega_k)e^{-ikx}\right]_{k^0=\omega_k} \tag{11.113}$$

$$= \frac{i}{2(2\pi)^{d-1}}\int d^{d-1}\vec k\left[-\hat a(\vec k)e^{ikx} + \hat a^\dagger(\vec k)e^{-ikx}\right]_{k^0=\omega_k}. \tag{11.114}$$

Now calculating the three commutation relations

- $[\hat\phi(t,\vec x), \hat\phi(t,\vec y)]$

$$= \frac{1}{(2\pi)^{2(d-1)}}\int\frac{d^{d-1}\vec k d^{d-1}\vec q}{4\omega_k\omega_q}\Big((\hat a(\vec k)e^{ikx} + \hat a^\dagger(\vec k)e^{-ikx})(\hat a(\vec q)e^{iqy} + \hat a^\dagger(\vec q)e^{-iqy}) - \tag{11.115}$$

$$(\hat a(\vec q)e^{iqy} + \hat a^\dagger(\vec q)e^{-iqy})(\hat a(\vec k)e^{ikx} + \hat a^\dagger(\vec k)e^{-ikx})\Big) \tag{11.116}$$

the bracket can then be simplified

$$(\hat a(\vec k)e^{ikx} + \hat a^\dagger(\vec k)e^{-ikx})(\hat a(\vec q)e^{iqy} + \hat a^\dagger(\vec q)e^{-iqy}) - (\hat a(\vec q)e^{iqy} + \hat a^\dagger(\vec q)e^{-iqy})(\hat a(\vec k)e^{ikx} + \hat a^\dagger(\vec k)e^{-ikx}) \tag{11.117}$$

$$= [\hat a(\vec k), \hat a(\vec q)]e^{i(kx+qy)} + [\hat a(\vec k), \hat a^\dagger(\vec q)]e^{i(kx-qy)} + [\hat a^\dagger(\vec k), \hat a(\vec q)]e^{i(-kx+qy)} + [\hat a^\dagger(\vec k), \hat a^\dagger(\vec q)]e^{i(-kx-qy)} \tag{11.118}$$

$$= [\hat a(\vec k), \hat a^\dagger(\vec q)]e^{i(kx-qy)} - [\hat a(\vec q), \hat a^\dagger(\vec k)]e^{i(-kx+qy)} \tag{11.119}$$

$$= 2\omega_k(2\pi)^{d-1}\left(\delta^{d-1}(\vec k - \vec q)e^{i(kx-qy)} - \delta^{d-1}(\vec q - \vec k)e^{i(-kx+qy)}\right) \tag{11.120}$$

where we used the given commutation relations for $\hat{a}(\vec{k})$.

$$[\hat{\phi}(t,\vec{x}),\hat{\phi}(t,\vec{y})] = \frac{1}{(2\pi)^{2(d-1)}}\int \frac{d^{d-1}\vec{k}d^{d-1}\vec{q}}{4\omega_k\omega_q}2\omega_k(2\pi)^{d-1}\left(\delta^{d-1}(\vec{k}-\vec{q})e^{i(kx-qy)} - \delta^{d-1}(\vec{q}-\vec{k})e^{i(-kx+qy)}\right) \tag{11.121}$$

$$= \frac{1}{(2\pi)^{d-1}}\int \frac{d^{d-1}\vec{k}d^{d-1}\vec{q}}{2\omega_q}\left(\delta^{d-1}(\vec{k}-\vec{q})e^{i(kx-qy)} - \delta^{d-1}(\vec{q}-\vec{k})e^{i(-kx+qy)}\right) \tag{11.122}$$

$$= \frac{1}{(2\pi)^{d-1}}\int \frac{d^{d-1}\vec{k}d^{d-1}\vec{q}}{2\omega_q}\left(\delta^{d-1}(\vec{k}-\vec{q})e^{i(-\omega_k t+\vec{k}\vec{x}-[-\omega_q t+\vec{q}\vec{y}])}\right. \tag{11.123}$$

$$\left. -\delta^{d-1}(\vec{q}-\vec{k})e^{-i(-\omega_k t+\vec{k}\vec{x}-[-\omega_q t+\vec{q}\vec{y}])}\right) \tag{11.124}$$

$$= \frac{1}{(2\pi)^{d-1}}\int \frac{d^{d-1}\vec{k}d^{d-1}\vec{q}}{2\omega_q}\left(\delta^{d-1}(\vec{k}-\vec{q})e^{i(-[\omega_k-\omega_q]t+\vec{k}\vec{x}-\vec{q}\vec{y})}\right. \tag{11.125}$$

$$\left. -\delta^{d-1}(\vec{q}-\vec{k})e^{-i(-[\omega_k-\omega_q]t+\vec{k}\vec{x}-\vec{q}\vec{y})}\right) \tag{11.126}$$

$$= \frac{1}{(2\pi)^{d-1}}\int \frac{d^{d-1}\vec{k}}{2\omega_k}\left(e^{i\vec{k}(\vec{x}-\vec{y})} - e^{-i\vec{k}(\vec{x}-\vec{y})}\right) \tag{11.127}$$

$$= \frac{1}{2\omega_k}\left(\delta^{d-1}(\vec{y}-\vec{x}) - \delta^{d-1}(\vec{x}-\vec{y})\right) \tag{11.128}$$

$$= 0 \tag{11.129}$$

where we used $\delta(x) = \int dk e^{-2\pi i k x}$ or $\delta^d(x) = \int \frac{d^d k}{(2\pi)^d} e^{-ikx}$.

- $[\hat{\Pi}(t,\vec{x}),\hat{\Pi}(t,\vec{y})]$ Not done yet

- $[\hat{\phi}(t,\vec{x}),\hat{\Pi}(t,\vec{y})]$ Not done yet

11.2.11 Problem 1.3.2 - Lorentz invariant integration measure

We use the property of the δ-function $\delta(f(x)) = \sum_i \frac{\delta(x-a_i)}{]f'(a_i)|}$ where a_i are the zeros of $f(x)$ and $\omega_k = \sqrt{\vec{k}^2+m^2}$. With $\int d^d k$ being manifestly Lorentz invariant

$$dk'^\mu = \Lambda^\mu_\nu dk^\nu \quad \rightarrow \quad \frac{dk'^\mu}{dk^\nu} = \Lambda^\mu_\nu \quad \rightarrow \quad \int d^d k' = |\det(\Lambda^\mu_\nu)|\int d^d k = \int d^d k \tag{11.130}$$

$\delta^d[k^2+m^2]$ being invariant and with $k^0 = \sqrt{\vec{k}^2+m^2}$ we see that k is inside the forward light cone and remains there under orthochrone transformation ($\Theta(k^0)$ is invariant for relevant k) we are convinced that the starting expression is Lorentz invariant (integration over the upper mass shell)

$$\int d^d\vec{k}\delta^d[k^2+m^2]\Theta(k^0) = \int d^{d-1}\vec{k}\int dk^0\delta^d[k^2+m^2]\Theta(k^0) \tag{11.131}$$
$$= \int d^{d-1}\vec{k}\int dk^0\delta^d[-(k^0)^2+\vec{k}^2+m^2]\Theta(k^0) \tag{11.132}$$
$$= \int d^{d-1}\vec{k}\int dk^0\delta^d[\omega_k^2-(k^0)^2]\Theta(k^0) \tag{11.133}$$
$$= \int d^{d-1}\vec{k}\int dk^0\left(\frac{\delta(k^0-\omega_k)}{2\omega_k}+\frac{\delta(k^0+\omega_k)}{2\omega_k}\right)\Theta(k^0) \tag{11.134}$$
$$= \int \frac{d^{d-1}\vec{k}}{2\omega_k}\int dk^0\delta(k^0-\omega_k) \tag{11.135}$$
$$= \int \frac{d^{d-1}\vec{k}}{2\omega_k}. \tag{11.136}$$

As we started with a Lorentz invariant expression the derived measure is also invariant.

11.2.12 Problem 1.3.3 - Retarded Green function

$$\Delta_\mathrm{F} = \int \frac{d^dk}{(2\pi)^d}\frac{e^{ik(x-y)}}{k^2+m^2-i\epsilon} \tag{11.137}$$
$$G_\mathrm{R} = \int \frac{d^dk}{(2\pi)^d}\frac{e^{ik(x-y)}}{-(k^0+i\epsilon)^2+\vec{k}^2+m^2} \tag{11.138}$$

For the poles of G_R we have

$$-(k^0+i\epsilon)^2+\vec{k}^2+m^2 = 0 \tag{11.139}$$
$$k^0 = -i\epsilon \pm \sqrt{\vec{k}^2+m^2} \tag{11.140}$$
$$= -i\epsilon \pm \omega_k \tag{11.141}$$

while we the poles of Δ_F are given by

$$-(k^0)^2+\vec{k^2}+m^2-i\epsilon = 0 \tag{11.142}$$
$$k^0 = \pm\sqrt{\vec{k^2}+m^2-i\epsilon} \tag{11.143}$$
$$= \pm\sqrt{\omega_k^2-i\epsilon} \tag{11.144}$$

Figure 11.1: Poles of G_R (circle) and Δ_F (triangle)

With $|\vec{k}\rangle = a^\dagger(\vec{k})|0\rangle$ and

$$\hat{\phi}(x) = \frac{1}{(2\pi)^{d-1}}\int\frac{d^{d-1}\vec{k}}{2\omega_k}\left[\hat{a}(\vec{k})e^{ikx}+\hat{a}^\dagger(\vec{k})e^{-ikx}\right]_{k^0=\omega_k} \tag{11.145}$$

we obtain

$$\hat{\phi}(x)\hat{\phi}(y) \sim \left(\hat{a}(\vec{k})e^{ikx} + \hat{a}^\dagger(\vec{k})e^{-ikx}\right)\left(\hat{a}(\vec{q})e^{iqy} + \hat{a}^\dagger(\vec{q})e^{-iqy}\right) \tag{11.146}$$

$$= \hat{a}(\vec{k})\hat{a}(\vec{q})e^{i(kx+qy)} + \hat{a}(\vec{k})\hat{a}^\dagger(\vec{q})e^{-i(-kx+qy)} + \hat{a}^\dagger(\vec{k})\hat{a}(\vec{q})e^{i(-kx+qy)} + \hat{a}^\dagger(\vec{k})\hat{a}^\dagger(\vec{q})e^{-i(kx+qy)} \tag{11.147}$$

$$= \hat{a}(\vec{k})\hat{a}(\vec{q})e^{i(kx+qy)} + \hat{a}(\vec{k})\hat{a}^\dagger(\vec{q})e^{-i(-kx+qy)} + \hat{a}^\dagger(\vec{k})\hat{a}^\dagger(\vec{q})e^{-i(kx+qy)} \tag{11.148}$$

$$+ \left(\hat{a}(\vec{q})\hat{a}^\dagger(\vec{k}) - 2\omega_k(2\pi)^{d-1}\delta^{d-1}(\vec{q}-\vec{k})\right)e^{i(-kx+qy)} \tag{11.149}$$

and therefore

$$\langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle = \frac{1}{(2\pi)^{2(d-1)}}\int \frac{d^{d-1}\vec{k}}{2\omega_k}\frac{d^{d-1}\vec{q}}{2\omega_q}\langle 0|\hat{a}(\vec{k})\hat{a}(\vec{q})|0\rangle e^{i(kx+qy)} + \langle 0|\hat{a}(\vec{k})\hat{a}^\dagger(\vec{q})|0\rangle e^{-i(-kx+qy)} \tag{11.150}$$

$$+ \langle 0|\hat{a}^\dagger(\vec{k})\hat{a}^\dagger(\vec{q})|0\rangle e^{-i(kx+qy)} + \left(\langle 0|\hat{a}(\vec{q})\hat{a}^\dagger(\vec{k})|0\rangle - 2\omega_k(2\pi)^{d-1}\delta^{d-1}(\vec{q}-\vec{k})\right)e^{i(-kx+qy)} \tag{11.151}$$

$$= \frac{1}{(2\pi)^{2(d-1)}}\int \frac{d^{d-1}\vec{k}}{2\omega_k}\frac{d^{d-1}\vec{q}}{2\omega_q}\langle\vec{k}|\vec{q}\rangle e^{-i(-kx+qy)} + \left(\langle\vec{q}|\vec{k}\rangle - 2\omega_k(2\pi)^{d-1}\delta^{d-1}(\vec{q}-\vec{k})\right)e^{i(-kx+qy)} \tag{11.152}$$

$$\tag{11.153}$$

Not done yet

11.2.13 Problem 1.3.4 - Feynman rules of ϕ^4 theory

Not done yet

11.2.14 Problem 1.3.5 - Convergence of perturbative expansion

Not done yet

11.2.15 Problem 1.3.6

Not done yet

11.2.16 Problem 1.3.7

Not done yet

11.2.17 Problem 1.3.8

Not done yet

Chapter 12

String Theory

12.1 Zwiebach - A First Course in String Theory

12.1.1 Problem 2.1 - Exercise with units

(a) In esu (electrostatic units) $k_c = 1$ and the mechanical units are cgs (cm, g, sec) which means force is measured in dyn=g·m/s^2=10^{-5}N then via the Coulomb law we have

$$F = k_c \frac{q_1 q_2}{r^2} \tag{12.1}$$

(b)

(c)

12.2 Becker, Becker, Schwarz - String Theory and M-Theroy

12.3 Polchinski - String Theory Volumes 1 and 2

12.3.1 Problem 1.1 - Non-relativistic action limits

(a) We start with (1.2.2) and use $dt = \gamma d\tau$ and $u^\mu = \gamma(c, \vec{v})$ as well as $v \ll c$

$$\begin{aligned}
S_{\mathrm{pp}} &= -mc \int d\tau \sqrt{-\dot{X}^\mu \dot{X}_\mu} && (12.2)\\
&= -mc \int d\tau \sqrt{(c^2 - v^2)\gamma^2} && (12.3)\\
&= -\int mc^2 \cdot dt \sqrt{1 - \frac{v^2}{c^2}} && (12.4)\\
&\approx -\int dt \cdot mc^2 \left(1 - \frac{1}{2}\frac{v^2}{c^2}\right) && (12.5)\\
&= -\int dt \left(mc^2 - \frac{1}{2}mv^2\right) && (12.6)
\end{aligned}$$

(b) We start with (1.2.9) and $X^\mu = X^\mu(\tau, \sigma)$

$$S_{\mathrm{NG}} = \int_M d\tau d\sigma \mathcal{L}_{\mathrm{NG}} \tag{12.7}$$

$$= -\frac{1}{2\pi\alpha'} \int_M d\tau d\sigma \sqrt{-\det h_{ab}} \tag{12.8}$$

$$= -\frac{1}{2\pi\alpha'} \int_M d\tau d\sigma \sqrt{-\det \partial_a X^\mu \partial_b X_\mu} \tag{12.9}$$

$$\tag{12.10}$$

Not done yet

Chapter 13

Astrophysics

13.1 Carroll, Ostlie - An Introduction to Modern Astrophysics

13.2 Weinberg - Lecture on Astrophysics

13.2.1 Problem 1 - Hydrostatics of spherical star

Gravitational force on a mass element must be balanced by the top and bottom pressure (buoyancy)

$$\begin{aligned}
F_p^{\text{top}} - F_p^{\text{bottom}} &= F_g && (13.1)\\
dA \cdot p\left(r + \frac{dr}{2}\right) - dA \cdot p\left(r - \frac{dr}{2}\right) &= -g(r)\rho(r) \cdot dA \cdot dr && (13.2)\\
\frac{dp}{dr} &= -g(r)\rho(r) && (13.3)\\
&= -G\frac{\mathcal{M}(r)}{r^2}\rho(r) && (13.4)
\end{aligned}$$

and therefore

$$\rho(r)\mathcal{M}(r) = -\frac{dp}{dr}\frac{r^2}{G} \tag{13.5}$$

where

$$g(r) = G\frac{\mathcal{M}(r)}{r^2} = \frac{G}{r^2}\int_0^r 4\pi\rho(r')r'^2 dr'. \tag{13.6}$$

The gravitational binding energy Ω is given by

$$
\begin{aligned}
d\Omega &= -G\frac{m_{\text{shell}}\mathcal{M}}{r} && (13.7)\\
\Omega &= -G\int_0^R \frac{4\pi\rho(r)\mathcal{M}(r)}{r}r^2 dr && (13.8)\\
&= -4\pi G\int_0^R r\rho(r)\mathcal{M}(r)dr && (13.9)\\
&= 4\pi\int_0^R \frac{dp}{dr}r^3 dr && (13.10)\\
&= 4\pi p r^3|_0^R - 3\cdot 4\pi\int_0^R p(r)r^2 dr && (13.11)\\
&= 4\pi p_0 R^3 - 3\left(4\pi\int_0^R p(r)r^2 dr\right) && (13.12)\\
&= 4\pi p_0 R^3 - 3\int_{K_R} p(\vec{r})d^3 r. && (13.13)
\end{aligned}
$$

13.2.2 Problem 2 - CNO cycle

$$
\begin{aligned}
&\Gamma(ii) = \Gamma(iii) = \Gamma(iv) = \Gamma(v) = \Gamma(i) && (13.14)\\
&\Gamma(vi) = P\cdot\Gamma(i) && (13.15)\\
&\Gamma(vii) = \Gamma(viii) = \Gamma(ix) = \Gamma(x) = (1-P)\cdot\Gamma(i) && (13.16)
\end{aligned}
$$

Check result!

13.2.3 Problem 3

Not done yet

13.2.4 Problem 4

Not done yet

13.2.5 Problem 5 - Radial density expansion for a polytrope

For the polytrope equation

$$p = K\rho^\Gamma \tag{13.17}$$

we obtain

$$
\begin{aligned}
\frac{dp}{d\rho} &= K\Gamma\rho^{\Gamma-1} && (13.18)\\
&= \Gamma\frac{p}{\rho} && (13.19)
\end{aligned}
$$

With equations (1.1.4/5)

$$
\begin{aligned}
\frac{dp}{dr} &= -\frac{G\mathcal{M}(r)\rho(r)}{r^2} \quad \rightarrow \quad \mathcal{M}(r) = -\frac{p' r^2}{G\rho} && (13.20)\\
\frac{d\mathcal{M}(r)}{dr} &= 4\pi r^2\rho(r) && (13.21)
\end{aligned}
$$

we can obtain a second order ODE by differentiating the first one and substituting $\mathcal{M}'$

$$\mathcal{M}' = -\frac{1}{G}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{d}{dr}p\right) \tag{13.22}$$

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{d}{dr}p\right) + G\mathcal{M}' = 0 \tag{13.23}$$

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{d}{dr}p\right) + 4\pi G r^2\rho = 0 \tag{13.24}$$

now we can substitute the $p = K\rho^\Gamma$ and obtain

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{d}{dr}\rho^\Gamma\right) + \frac{4\pi G}{K}r^2\rho = 0. \tag{13.25}$$

The Taylor expansion

$$\rho(r) = \rho(0)\left[1 + ar^2 + br^4 + ...\right] \tag{13.26}$$

$$\rho(r)^\Gamma = \rho(0)^\Gamma\left[1 + ar^2 + br^4 + ...\right]^\Gamma \tag{13.27}$$

$$= \rho(0)^\Gamma\left[1 + a\Gamma r^2 + \left(b\Gamma + \frac{1}{2}a^2\Gamma(\Gamma-1)\right)r^4 + ...\right] \tag{13.28}$$

$$\frac{1}{\rho} = \frac{1}{\rho(0)}\left[1 - ar^2 + (a^2 - b)r^4 + ...\right] \tag{13.29}$$

can be substituted into the ODE

$$\rho(0)^{\Gamma-1}\frac{d}{dr}\left(r^2\left[1 - ar^2 + (a^2-b)r^4 + ...\right]\left[a\Gamma 2r + \left(b\Gamma + \frac{1}{2}a^2\Gamma(\Gamma-1)\right)4r^3 + ...\right]\right) \tag{13.30}$$

$$+\frac{4\pi G}{K}\rho(0)\left[r^2 + ar^4 + br^6 + ...\right] = 0. \tag{13.31}$$

and sort by powers of r

$$\rho(0)^{\Gamma-1}\frac{d}{dr}\left(2\Gamma ar^3 + \left[-2\Gamma a^2 + 4\left(b\Gamma + \frac{1}{2}a^2\Gamma(\Gamma-1)\right)\right]r^5 + ...\right) + \frac{4\pi G}{K}\rho(0)\left[r^2 + ar^4 + br^6 + ...\right] = 0. \tag{13.32}$$

In second order of r we obtain

$$\rho(0)^{\Gamma-1}2\Gamma a3 + \frac{4\pi G}{K}\rho(0) = 0 \tag{13.33}$$

which results in

$$a = -\frac{2\pi G}{3\Gamma K\rho(0)^{\Gamma-2}} \tag{13.34}$$

13.2.6 Problem 6

Not done yet

13.2.7 Problem 7

Not done yet

13.2.8 Problem 8

Not done yet

13.2.9 Problem 9

Not done yet

13.2.10 Problem 10

Not done yet

13.2.11 Problem 11 - Modified Newtonian gravity

The modified Poisson equation is given by

$$\left(\triangle + \mathcal{R}^{-2}\right)\phi = 4\pi G\rho \tag{13.35}$$

with the Greens function

$$\left(\triangle + \mathcal{R}^{-2}\right)G(\vec{r}) = -\delta^3(\vec{r}). \tag{13.36}$$

The Fourier transform of the Greens function

$$G(\vec{k}) = \int d^3\vec{r}\, G(\vec{r})e^{-i\vec{k}\vec{r}} \tag{13.37}$$

and the field equations are given by

$$\begin{aligned}
\left[k^2 + \mathcal{R}^{-2}\right]G(\vec{k}) &= -1 && (13.38)\\
G(\vec{k}) &= \frac{1}{k^2+\mathcal{R}^{-2}} && (13.39)\\
G(\vec{x}) &= \frac{1}{(2\pi)^3}\int d^3\vec{k}\frac{e^{i\vec{k}\vec{r}}}{k^2+\mathcal{R}^{-2}} && (13.40)\\
&= \frac{1}{(2\pi)^3}2\pi\int_0^\infty\int_0^\pi \frac{e^{ik_r\cdot r\cos\theta}}{k_r^2+\mathcal{R}^{-2}}k_r^2\sin\theta\; d\theta dk_r && (13.41)\\
&= \frac{1}{(2\pi)^3}2\pi\int_0^\infty\left[-\frac{e^{ik_r r\cos\theta}}{ik_r r}\right]_0^\pi \frac{1}{k_r^2+\mathcal{R}^{-2}}k_r^2 dk_r && (13.42)\\
&= \frac{1}{2\pi^2 r}\int_0^\infty \frac{k_r\sin(k_r r)}{k_r^2+\mathcal{R}^{-2}}dk_r && (13.43)\\
& && (13.44)
\end{aligned}$$

The integral can be can be calculated using the residual theorem

$$\begin{aligned}
\int_0^\infty \frac{k_r\sin(k_r r)}{k_r^2+\mathcal{R}^{-2}}dk_r &= \frac{1}{2}\int_{-\infty}^\infty \frac{k_r\sin(k_r r)}{k_r^2+\mathcal{R}^{-2}}dk_r && (13.45)\\
&= \frac{1}{2}\int_{-\infty}^\infty \frac{k_r\sin(k_r r)}{(k_r+i\mathcal{R}^{-1})(k_r-i\mathcal{R}^{-1})}dk_r && (13.46)\\
&= \frac{1}{2}\int_{-\infty}^\infty \frac{k_r\sin(k_r r)}{2k_r}\left(\frac{1}{k_r+i\mathcal{R}^{-1}}+\frac{1}{k_r-i\mathcal{R}^{-1}}\right)dk_r && (13.47)\\
&= \frac{1}{4}\int_{-\infty}^\infty \frac{\sin(k_r r)}{k_r+i\mathcal{R}^{-1}}dk_r + \frac{1}{4}\int_{-\infty}^\infty \frac{\sin(k_r r)}{k_r-i\mathcal{R}^{-1}}dk_r && (13.48)
\end{aligned}$$

Not done yet

13.2.12 Problem 12

Not done yet

Chapter 14

Cosmology

14.1 Ryden - Introduction to Cosmology, 2016

14.1.1 Exercise 2.1

The power emitted by a surface dA under the angle θ ($\theta = 0$ is perpendicular to the surface dA) into the solid angle $d\Omega$ is $B_\nu \cos\theta \, dA \, d\Omega \, d\nu$ with

$$B_\nu(\nu, T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1} \tag{14.1}$$

this is related to the spectral energy density (eqn 2.27) of the photon field by

$$\varepsilon(\nu) = u_\nu = \frac{4\pi}{c} B_\nu. \tag{14.2}$$

The angular integration (for dA in the xy plane) gives

$$\int \cos\theta \, d\Omega = \int_0^{2\pi} \int_0^{\pi/2} \cos\theta \, \sin\theta \, d\theta \, d\phi \tag{14.3}$$

and therefore with $\rho = m/V$ and $V = 4/3\pi R^3$ we obtain

$$\begin{aligned}
P &= \int B_\nu \cos\theta \, dA \, d\Omega \, d\nu && (14.4)\\
&= \pi \int B_\nu \, dA \, d\nu && (14.5)\\
&= \frac{2\pi^5 k^4}{15 c^2 h^3} T^4 \int dA && (14.6)\\
&= \frac{2\pi^5 k^4}{15 c^2 h^3} T^4 \cdot 4\pi R^2 && (14.7)\\
&= \frac{2\pi^5 k^4}{15 c^2 h^3} T^4 \cdot \left(\frac{6\sqrt{\pi} m}{\rho} \right)^{2/3} && (14.8)
\end{aligned}$$

which gives for $m = 75$kg a power of 440W. The net power is obviously smaller $P_{\text{net}} = \sigma A (T_{\text{body}}^4 - T_{\text{ambient}}^4)$.

14.1.2 Exercise 2.2

The photon number density is given by (eqn 2.30)

$$n(\nu) = \frac{\varepsilon(\nu)}{h\nu} = \frac{8\pi\nu^2}{c^3} \frac{1}{e^{h\nu/kT} - 1} \tag{14.9}$$

then the flux across the projected surface of the sphere πR^2 from each direction is given by

$$\begin{aligned}
N &= \int d\Omega \int d\nu \, \pi R^2 c n(\nu) && (14.10)\\
&= 4\pi^2 c R^2 \int d\nu \, n(\nu) && (14.11)\\
&= \zeta(3) \frac{64\pi^3 R^2}{c^2 h^3} (kT)^3 && (14.12)\\
&= 3.2 \cdot 10^{17} && (14.13)
\end{aligned}$$

assuming $m = \rho V = 75\text{kg}$. Analog

$$\begin{aligned}
P &= \int d\Omega \int d\nu \, \pi R^2 c \varepsilon(\nu) && (14.14)\\
&= 4\pi^2 c R^2 \int d\nu \, varepsilon(\nu) && (14.15)\\
&= \frac{32\pi^7 R^2}{15 c^2 h^3} (kT)^4 && (14.16)\\
&= 3.3 \cdot 10^{-5}\text{W} && (14.17)
\end{aligned}$$

14.1.3 Exercise 2.3

Combining the results

$$\begin{aligned}
P_{\text{rad}} &= -440\text{W} && (14.18)\\
P_{\text{absCMB}} &= 3.3 \cdot 10^{-5}\text{W} && (14.19)\\
P_{\text{tot}} &= -440W && (14.20)
\end{aligned}$$

So the astronaut is loosing heat and can not overheat. The astronauts energy loss is given by

$$\begin{aligned}
\Delta E_{\text{heat}} &= mC\Delta T \equiv P_{\text{tot}} \Delta t && (14.21)\\
\rightarrow \frac{\Delta t}{\Delta T} &= \frac{mC}{P_{\text{tot}}} = 716\text{s/K} = 12\text{min/K} && (14.22)
\end{aligned}$$

therefore so the lack of oxygen seems to be most likely.

14.1.4 Exercise 2.4

$$\begin{aligned}
z(r) &= \frac{\lambda(r) - \lambda_{\text{em}}}{\lambda_{\text{em}}} \quad \rightarrow \quad \lambda(r) = [1 + z(r)]\lambda_{\text{em}} && (14.23)\\
E(r) &= hc \frac{1}{\lambda(r)} = \frac{hc}{[1 + z(r)]\lambda_{\text{em}}} && (14.24)\\
\frac{dE}{dr} &= -kE \quad \rightarrow \quad z' - k(1 + z) = 0 && (14.25)\\
&\rightarrow \quad z(r) = ce^{kr} - 1 \quad (z(0) = 0) && (14.26)\\
&\rightarrow \quad z(r) = e^{kr} - 1 && (14.27)\\
&\rightarrow \quad z(r) \approx kr && (14.28)
\end{aligned}$$

with $k = H_0/c$

14.1.5 Exercise 2.5

With

$$n(\nu) = \frac{\varepsilon(\nu)}{h\nu} = \frac{8\pi\nu^2}{c^3}\frac{1}{e^{h\nu/kT}-1} \tag{14.29}$$

$$n_\gamma = \int d\nu\, n(\nu) \tag{14.30}$$

$$= \frac{16\zeta(3)\pi}{c^3h^3}(kT)^3 = \frac{2\zeta(3)}{\pi^2c^3\hbar^3}(kT)^3 \tag{14.31}$$

then

$$n(h\nu > E_0) = \frac{8\pi}{c^3}\int_{E_0/h}^{\infty}\frac{\nu^2}{e^{h\nu/kT}-1}d\nu \tag{14.32}$$

$$\overset{h\nu>E_0\gg kT}{\simeq} \frac{8\pi}{c^3}\int_{E_0/h}^{\infty} e^{-h\nu/kT}\nu^2 d\nu \tag{14.33}$$

$$= \frac{8\pi kT}{c^3h^3}e^{-E_0/kT}(E_0^2 + 2E_0kT + 2(kT)^2) \tag{14.34}$$

$$\simeq \frac{8\pi kT}{c^3h^3}e^{-E_0/kT}E_0^2 \tag{14.35}$$

then

$$\frac{n(h\nu > E_0)}{n_\gamma} = \frac{1}{2\zeta(3)}e^{-E_0/kT}\left(\frac{E_0^2}{kT}\right)^2 \tag{14.36}$$

Using this result we obtain 5.8% of infrared photons. Exact numerical integration gives $6\cdot 10^{-4}$% radio waves (or longer), 91.6% microwaves, 8.4% infrared, 0% optical (and shorter).

14.1.6 Exercise 2.6

Now

$$n(h\nu < E_0) = \frac{8\pi}{c^3}\int_0^{E_0/h}\frac{\nu^2}{e^{h\nu/kT}-1}d\nu \tag{14.37}$$

$$\overset{h\nu<E_0\ll kT}{\simeq} \frac{8\pi}{c^3}\int_{E_0/h}^{\infty}\frac{kT}{h\nu}\nu^2 d\nu \tag{14.38}$$

$$= \frac{4E_0^2\pi kT}{c^3h^3} \tag{14.39}$$

then

$$\frac{n(h\nu < E_0)}{n_\gamma} = \frac{E_0^2}{4\zeta(3)k^2T^2} \tag{14.40}$$

For $\lambda > 3$cm ($hc/\lambda = h\nu < E_0 = hc/\lambda_0$) we obtain 0.6%.

14.1.7 Exercise 3.2

We replace $d\theta$ by $d\varphi$! Calculating the size of the object as distance on the sphere of radius R

$$dl^2 = R^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{14.41}$$

$$= R^2\sin^2\theta d\varphi^2 \tag{14.42}$$

with $\theta = r/R$. Then

$$dl = R \sin \frac{r}{R} \cdot d\varphi \tag{14.43}$$

$$\rightarrow d\varphi = \frac{dl}{R \sin \frac{r}{R}} \tag{14.44}$$

For $r \rightarrow \pi R$ $d\varphi$ increases to 2π

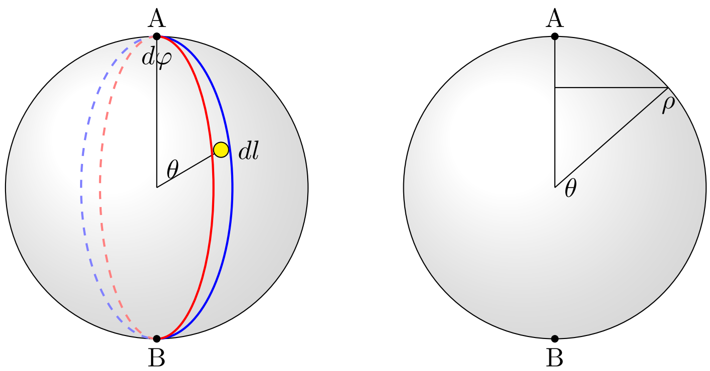


Figure 14.1: (left) Ex 3.2. Spherical universe, Observer at A, object of size dl at distance r, (right) Ex 3.3. Spherical universe, Observer at A

14.1.8 Exercise 3.3

Simple geometry

$$\theta = 2\pi \frac{r}{2\pi R} = \frac{r}{R} \tag{14.45}$$

$$\sin \theta = \frac{\rho}{R} \tag{14.46}$$

$$C = 2\pi\rho \tag{14.47}$$

gives

$$C = 2\pi\rho \tag{14.48}$$

$$= 2\pi R \sin \theta \tag{14.49}$$

$$= 2\pi R \sin \frac{r}{R}. \tag{14.50}$$

For the euclidean case we get of course $C_{\text{Euclid}} = 2\pi r$. Then

$$\Delta s = C_{\text{Euclid}} - C \tag{14.51}$$

$$= 2\pi(r - R \sin \frac{r}{R}) \tag{14.52}$$

$$\simeq \frac{\pi r^3}{3R^2} - \frac{\pi r^5}{60R^2} \tag{14.53}$$

$$\simeq 33.8 \text{ km} \tag{14.54}$$

14.1.9 Exercise 3.4

1. $\kappa = +1$ With

$$\alpha + \beta + \gamma = \frac{A}{R^2} + \pi \tag{14.55}$$

we see that each angle can be maximally π. So

$$A_{\max} = (3\pi - \pi)R^2 = 2\pi R^2. \tag{14.56}$$

It is easy to see that such a (degenerated) triangle (half sphere) can be realized.

A bit more formal - integrating over a triangle-shape slice

$$\begin{aligned} A &= \int_0^\alpha \int_0^\alpha R^2 \sin\theta \, d\theta \, d\phi & (14.57) \\ &= R^2(\alpha - \alpha\cos\alpha) & (14.58) \\ A_{\max} &= A(\alpha = \pi) = 2\pi R^2 & (14.59) \end{aligned}$$

2. $\kappa = 0$ There is no limited to the triangle size.
3. $\kappa = -1$ With

$$A = (\pi - \alpha - \beta - \gamma)R^2 \tag{14.60}$$

we see that the potential maximum is $A_{\max} = \pi R^2$. Now we need to show that such a triangle exists.

14.1.10 Exercise 3.5

With

$$\begin{aligned} dx &= \frac{x}{r}dr + \frac{x}{\sin\theta}\cos\theta \, d\theta + \frac{x}{\cos\phi}(-\sin\phi) \, d\phi & (14.61) \\ dy &= \frac{y}{r}dr + \frac{y}{\sin\theta}\cos\theta \, d\theta + \frac{y}{\sin\phi}\cos\phi \, d\phi & (14.62) \\ dz &= \frac{z}{r}dr + \frac{z}{\cos\theta}(-\sin\theta)d\theta & (14.63) \end{aligned}$$

we obtain

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 & (14.64) \\ &= dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) & (14.65) \end{aligned}$$

14.1.11 Exercise 4.1

$$\begin{aligned} E_{\text{sun}} &= M_{\text{sun}}c^2 = 1.79 \cdot 10^{47}\,\text{J} & (14.66) \\ E_\Lambda &= \varepsilon_\Lambda \frac{4}{3}\pi R^3 = 1.1 \cdot 10^{25}\,\text{J} & (14.67) \end{aligned}$$

14.2 Boerner - The Early Universe - Facts and Fiction (4th edition)

14.2.1 1.1 Friedman equations

1. The Friedman equations in book contain a small typo ($\rho = \varrho$)

$$\begin{aligned} (A) \qquad \ddot{R} &= -\frac{4\pi}{3}(\varrho + 3p)GR + \frac{1}{3}\Lambda R & (14.68) \\ (B) \qquad \dot{R}^2 &= \frac{8\pi}{3}G\varrho R^2 + \frac{1}{3}\Lambda R^2 - K & (14.69) \\ (C) \qquad 0 &= (\varrho R^3)^{\cdot} + p(R^3)^{\cdot} & (14.70) \end{aligned}$$

Calculating the time derivative of (B)

$$2\dot{R}\ddot{R} = \frac{8\pi}{3}G(\dot{\varrho}R^2 + 2\varrho R\dot{R}) + \frac{2}{3}\Lambda R\dot{R} \tag{14.71}$$

$$\ddot{R} = \frac{R}{3}\left(4\pi G\dot{\varrho}\frac{R}{\dot{R}} + 8\pi G\varrho + \Lambda\right) \tag{14.72}$$

and simplifying (A)

$$\ddot{R} = \frac{R}{3}\left(-4\pi G(\varrho + 3p) + \Lambda\right) \tag{14.73}$$

Combining both yields

$$\dot{\varrho}\frac{R}{\dot{R}} + 2\varrho = -(\varrho + 3p) \tag{14.74}$$

$$\dot{\varrho}R = -3(\varrho + p)\dot{R} \tag{14.75}$$

which is (C). Rearranging the order of the steps gives the other two cases.

2. From (C) we have

$$\dot{\varrho} = -3(\varrho + p)\frac{\dot{R}}{R} \tag{14.76}$$

$$= -3\varrho\left(1 + k\varrho^{\gamma-1}\right)\frac{\dot{R}}{R} \tag{14.77}$$

which can be rearranged and integrated

$$\frac{\dot{R}}{R} = \frac{\dot{\varrho}}{-3\varrho\left(1 + k\varrho^{\gamma-1}\right)} \tag{14.78}$$

$$\to -\frac{1}{3(1-\gamma)}\log(k + \varrho^{1-\gamma}) = \log R + c \tag{14.79}$$

$$\to \log(k + \varrho^{1-\gamma}) = -3(1-\gamma)\log R + c' \tag{14.80}$$

$$\to k + \varrho^{1-\gamma} = e^{-3(1-\gamma)\log R + c'} \tag{14.81}$$

$$\to k + \varrho^{1-\gamma} = c'' R^{-3(1-\gamma)} \tag{14.82}$$

$$\to \varrho = \left(c'' R^{3(\gamma-1)} - k\right)^{1/(1-\gamma)} \tag{14.83}$$

with

$$c'' = \frac{k + \varrho_0^{1-\gamma}}{R_0^{3(\gamma-1)}} \tag{14.84}$$

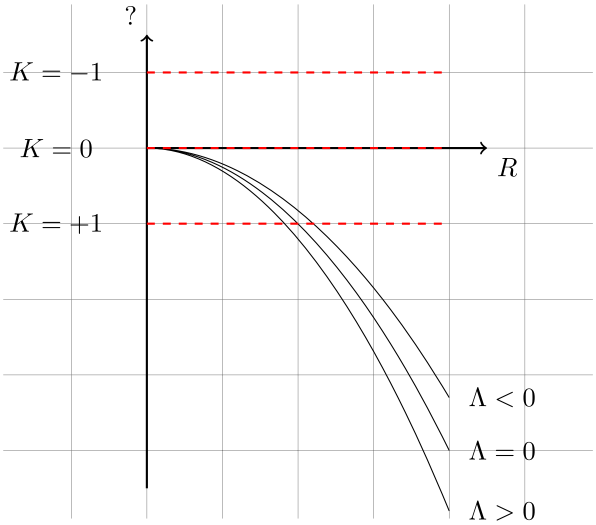
$$\to \quad \varrho = \left([k + \varrho_0^{1-\gamma}]\frac{R}{R_0}^{3(\gamma-1)} - k\right)^{1/(1-\gamma)} \tag{14.85}$$

$$\to \quad \varrho = \left(k\left[\frac{R}{R_0}^{3(\gamma-1)} - 1\right] + \left[\frac{R^3}{\varrho_0 R_0^3}\right]^{\gamma-1}\right)^{1/(1-\gamma)} \tag{14.86}$$

We obtain from (B)

$$\dot{R}^2 - \left(\frac{8\pi}{3}G\varrho + \frac{1}{3}\Lambda\right)R^2 = -K \tag{14.87}$$

which we can interpret as motion of a particle in a changing $-R^2$ potential.



14.3 Baumann - Cosmology (1nd edition)

14.3.1 Problem 1.1 Length scales

object	size	size in m
pepper corn	5mm	0.005m
basketball size 7 (75 cm circumference)	24cm	0.24m
basketball court	30.62yds	28m

1. $R_{\text{Moon}} = 6.5\text{cm}$, d_{ME}=11.3m
2. $R_{\text{Sun}} = 55\text{cm}$, $r_{\text{Earth orbit}}$=118m, $r_{\text{Neptune orbit}}$=3544m
3. $d_{\text{Solar system}}$=0.2mm
4. $R_{\text{Solar neigh}} = 18\text{mm}$
5. $R_{\text{Galaxy}} = 28\text{cm}$
6. $R_{\text{Local Group}} = 56\text{cm}$
7. $R_{\text{Super Cluster}} = 30\text{cm}$

14.3.2 Problem 1.2 Hubble constant

1. $t_{H_0} = 14 \cdot 10^9\text{a}$
2. $d_{H_0} = 140 \cdot 10^9\text{ly}$
3. $\rho_0 = 9 \cdot 10^{-27}\text{kg/m}^3$
4. $n_{\text{H universe}} = \frac{\rho_0 d^3}{m_{\text{H}}} = 10^{79}$, $n_{\text{H brain}} = \frac{m_{\text{brain}}}{m_{H_2O}} = 10^{26}$

14.3.3 Exercise 2.1

Using the Euler Lagrange equations we obtain

$$\frac{\partial L}{\partial r} = mr\dot{\phi}^2, \quad \frac{\partial L}{\partial \dot{r}} = mr\dot{r} \quad \rightarrow \quad \ddot{r} = r\dot{\phi}^2 \tag{14.88}$$

$$\frac{\partial L}{\partial \phi} = 0, \quad \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} \quad \rightarrow \quad \ddot{\phi} = -2\frac{\dot{r}}{r}\dot{\phi} \tag{14.89}$$

14.3.4 Exercise 2.2

Calculating

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\lambda}(g_{\beta\lambda,\alpha} + g_{\alpha\lambda,\beta} - g_{\alpha\beta,\lambda}) \tag{14.90}$$

we need the FRW metric which is given by

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & a^2\gamma \end{pmatrix} \qquad g^{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & \frac{1}{a^2}\gamma^{-1} \end{pmatrix} \tag{14.91}$$

then

$$\begin{aligned}
\Gamma^{i}_{0j} &= \frac{1}{2} g^{i\lambda}(g_{j\lambda,0} + g_{0\lambda,j} - g_{0j,\lambda}) && (14.92)\\
&= \frac{1}{2} g^{il}(g_{jl,0} + g_{0l,j} - g_{0j,l}) && (14.93)\\
&= \frac{1}{2} g^{il} g_{jl,0} = \frac{1}{2} g^{il} \frac{1}{c} \partial_t g_{jl} && (14.94)\\
&= \frac{1}{2}\frac{1}{a^2}\gamma^{il}\frac{1}{c}\partial_t(a^2\gamma_{jl}) = \frac{1}{2}\frac{1}{a^2}\gamma^{il}\frac{1}{c}2a\dot{a}\gamma_{jl} && (14.95)\\
&= \frac{\dot{a}}{a}\frac{1}{c}\gamma^{il}\gamma_{jl} = \frac{\dot{a}}{a}\frac{1}{c}\delta^{i}_{j} && (14.96)
\end{aligned}$$

and

$$\begin{aligned}
\Gamma^{i}_{jk} &= \frac{1}{2} g^{i\lambda}(g_{k\lambda,j} + g_{j\lambda,k} - g_{jk,\lambda}) && (14.97)\\
&= \frac{1}{2} g^{il}(g_{kl,j} + g_{j\lambda,k} - g_{jk,l}) && (14.98)
\end{aligned}$$

14.3.5 Exercise 2.3

With $P^{\mu} = (E/c, P^i)$

$$\begin{aligned}
-m^2c^2 &= g_{\mu\nu}P^{\mu}P^{\nu} && (14.99)\\
&= g_{00}(P^0)^2 + g_{ij}P^iP^j && (14.100)\\
&= -\frac{E^2}{c^2} + a^2\gamma_{ij}P^iP^j && (14.101)\\
&\rightarrow \vec{p}^2 = a^2\gamma_{ij}P^iP^j = \left(\frac{E^2}{c^2} - m^2c^2\right) && (14.102)
\end{aligned}$$

then

$$\begin{aligned}
\frac{E}{c^3}\frac{dE}{dt} &= -\frac{1}{c}a\dot{a}\gamma_{ij}P^iP^j && (14.103)\\
&= -\frac{1}{c}\frac{\dot{a}}{a}\left(\frac{E^2}{c^2} - m^2c^2\right) && (14.104)\\
\frac{E}{E^2 - m^2c^4}dE &= -\frac{da}{a} && (14.105)
\end{aligned}$$

Integrating on both sides

$$\begin{aligned}
\frac{1}{2}\log(E^2 - m^2c^4) &= -\log a + k_1 && (14.106)\\
\sqrt{E^2 - m^2c^4} &= \frac{k_2}{a} && (14.107)\\
pc &= \frac{k_2}{a} && (14.108)
\end{aligned}$$

meaning $p \sim a^{-1}$.

14.3.6 Exercise 2.4

With

$$\frac{dU}{dt} = (c^2\dot{\rho})V + (\rho c^2)\dot{V} \tag{14.109}$$
$$= c^2 k a^3 \dot{\rho} + 3ka^2(\rho c^2)\dot{a} \tag{14.110}$$
$$-P\frac{dV}{dt} = -P \cdot 3ka^2\dot{a} \tag{14.111}$$

then

$$c^2ka^3\dot{\rho} + 3ka^2(\rho c^2)\dot{a} + P \cdot 3ka^2\dot{a} = 0 \tag{14.112}$$
$$\dot{\rho} + 3\rho\frac{\dot{a}}{a} + \frac{P}{c^2} \cdot 3\frac{\dot{a}}{a} = 0 \tag{14.113}$$
$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) = 0 \tag{14.114}$$

14.3.7 Problem 2.1 - Robertson-Walker metric

1. - t proper time measured along the world lines of the galaxies or fluid elements: $g_{00} = \text{const} = -1$
   - Spacial part - isometry at every point means none of the γ_{ij} have preferred time dependency - which can be ultimately factored out

   $$\gamma_{ij} = \gamma_{ij}(t, x^k) = a(t)^2\gamma_{ij}(x^k) \tag{14.115}$$

   - Weyl postulate: The world lines of the fluid elements, that model the universe's matter content, are orthogonal to hypersurfaces of constant time: $g_{0i} \equiv \mathbf{g}(\mathbf{e}_0, \mathbf{e}_1) = 0$

   Therefore

   $$ds^2 = -dt^2 + a(t)^2\gamma_{ij}(x^k)dx^i dx^j \tag{14.116}$$

2. - Spherical symmetry around a point means the proper distance between two points does not change under rotations this means the angular part is $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$
   - θ and ϕ mirror symmetry implies $g_{\hat{r}\phi} = 0$ and $g_{\hat{r}\theta} = 0$ so we are left with

   $$ds^2 = -dt^2 + a(t)^2\left[C(\hat{r})d\hat{r}^2 + D(\hat{r})d\Omega^2\right] \tag{14.117}$$

   at this moment $\hat{r}$ is an arbitrary radial coordinate with $D(\hat{r}) > 0$
   - Defining new radial coordinate $r = \sqrt{D(\hat{r})}$ then

   $$ds^2 = -dt^2 + a(t)^2\left[\tilde{C}(r)dr^2 + r^2d\Omega^2\right] \tag{14.118}$$

   - Now we just rewrite $\tilde{C}(r) > 0$ in a more convenient way

   $$ds^2 = -dt^2 + a(t)^2\left[e^{2\alpha(r)}dr^2 + r^2d\Omega^2\right] \tag{14.119}$$

   Now we calculate the connection coefficients - the non-vanishing ones are

   $$\Gamma^r_{rr} = \alpha' \quad \Gamma^r_{\theta\theta} = -re^{-2\alpha} \quad \Gamma^r_{\phi\phi} = -re^{-2\alpha}\sin^2\theta \tag{14.120}$$
   $$\Gamma^\theta_{\theta r} = 1/r \quad \Gamma^\theta_{r\theta} = 1/r \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \tag{14.121}$$
   $$\Gamma^\phi_{\phi r} = 1/r \quad \Gamma^\phi_{\phi\theta=\cot\theta} \quad \Gamma^\phi_{r\phi} = 1/r \quad \Gamma^\phi_{\theta\phi} = \cot\theta \tag{14.122}$$

then

$$R_{ij} = \begin{pmatrix} \frac{2\alpha'}{r} & 0 & 0 \\ 0 & e^{-2\alpha}(-1+e^{2\alpha}+r\alpha') & 0 \\ 0 & 0 & e^{-2\alpha}\sin^2\theta(-1+e^{2\alpha}+r\alpha') \end{pmatrix} \tag{14.123}$$

$$R_{(3)} = R_{ij}\gamma^{ij} \tag{14.124}$$

$$= \frac{2e^{-2\alpha}(-1+e^{2\alpha}+2r\alpha')}{r^2} \tag{14.125}$$

$$= \frac{2}{r^2}(1-e^{-2\alpha}+2r\alpha' e^{-2\alpha}) \tag{14.126}$$

$$= \frac{2}{r^2}(1-\partial_r[re^{-2\alpha}]) \tag{14.127}$$

3. Solving the differential equation for the constant curvature $\hat{K}$

$$\frac{2}{r^2}(1-\partial_r[re^{-2\alpha}]) = K \tag{14.128}$$

$$\partial_r[re^{-2\alpha}] = 1-\frac{\hat{K}r^2}{2} \tag{14.129}$$

$$re^{-2\alpha} = r-\frac{Kr^3}{6}-b \tag{14.130}$$

$$\alpha = -\frac{1}{2}\log\left(1-\frac{\hat{K}r^2}{6}-\frac{b}{r}\right) \tag{14.131}$$

$$\alpha = \frac{1}{2}\log\left(1-\frac{\hat{K}r^2}{6}-\frac{b}{r}\right)^{-1} \tag{14.132}$$

$$e^{2\alpha} = \frac{1}{1-Kr^2-br^{-1}} \quad (K=\hat{K}/6) \tag{14.133}$$

Locally flat means

$$e^{2\alpha}|_{r=0} = 1 \quad \rightarrow \quad b=0. \tag{14.134}$$

Now we rewrite

$$\frac{1}{1-Kr^2} = \frac{1}{1-k\frac{r^2}{R_0^2}} \tag{14.135}$$

where R_0 is a scaling parameter and k determines the sign of the constant 3-curvature $R_{(3)}$.

4. Using the coordinate transformation

$$d\rho = \dot{a}r\,dt + a\,dr \tag{14.136}$$

$$dT = dt+\frac{1}{2}(\ddot{a}a+\dot{a}^2)r^2\,dt+\dot{a}ar\,dt \tag{14.137}$$

we see

$$dt \simeq \left(1+\frac{\dot{a}^2-a\ddot{a}}{2a^2}\rho^2\right)dT-\frac{\dot{a}}{a}\rho\,d\rho \tag{14.138}$$

$$dr \simeq -\frac{\dot{a}}{a^2}\rho\,dT+\frac{1}{a}\left(1+\frac{\dot{a}^2}{a^2}\rho^2\right)d\rho \tag{14.139}$$

then with $\frac{1}{1-Kr^2} \simeq 1+Kr^2 = 1+k\frac{r^2}{R_0^2} = 1+k\frac{\rho^2}{a^2R_0^2}$ we obtain

$$ds^2 = -dt^2+a(t)^2\left[\frac{1}{1-Kr^2}dr^2+r^2d\Omega^2\right] \tag{14.140}$$

$$= -dT^2\left(1-\frac{\ddot{a}}{a}\rho^2\right)+\left(\frac{\dot{a}^2}{a^2}\rho^2+1+\frac{k}{a^2R_0^2}\rho^2\right)d\rho^2+\rho^2d\Omega^2 \tag{14.141}$$

14.3.8 Problem 2.2 - Geodesics from a simple Lagrangian

1. Calculating every term individually

$$\frac{\partial \mathcal{L}}{\partial x^\alpha} = -\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\dot{x}^\mu \dot{x}^\nu \tag{14.142}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} = -g_{\mu\nu}\left(\frac{\partial \dot{x}^\mu}{\partial \dot{x}^\alpha}\dot{x}^\nu + \dot{x}^\mu \frac{\partial \dot{x}^\nu}{\partial \dot{x}^\alpha}\right) \tag{14.143}$$

$$= -g_{\mu\nu}\left(\delta^\mu_\alpha \dot{x}^\nu + \dot{x}^\mu \delta^\nu_\alpha\right) \tag{14.144}$$

$$= -(g_{\alpha\nu}\dot{x}^\nu + g_{\mu\alpha}\dot{x}^\mu) = -2g_{\alpha\nu}\dot{x}^\nu \tag{14.145}$$

$$\frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} = -\left(\frac{\partial g_{\alpha\nu}}{\partial x^\beta}\dot{x}^\beta \dot{x}^\nu + g_{\alpha\nu}\ddot{x}^\alpha + \frac{\partial g_{\mu\alpha}}{\partial x^\beta}\dot{x}^\beta \dot{x}^\mu + g_{\mu\alpha}\ddot{x}^\mu\right) \tag{14.146}$$

$$= -\left(\frac{\partial g_{\alpha\nu}}{\partial x^\mu}\dot{x}^\mu \dot{x}^\nu + \frac{\partial g_{\mu\alpha}}{\partial x^\nu}\dot{x}^\nu \dot{x}^\mu + 2g_{\mu\alpha}\ddot{x}^\mu\right) \tag{14.147}$$

then the equations of motion are

$$g_{\mu\alpha}\ddot{x}^\mu + \frac{1}{2}\left(\frac{\partial g_{\alpha\nu}}{\partial x^\mu}\dot{x}^\mu \dot{x}^\nu + \frac{\partial g_{\mu\alpha}}{\partial x^\nu}\dot{x}^\nu \dot{x}^\mu - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\dot{x}^\mu \dot{x}^\nu\right) = 0 \tag{14.148}$$

$$g_{\mu\alpha}\ddot{x}^\mu + \frac{1}{2}\left(\frac{\partial g_{\alpha\nu}}{\partial x^\mu} + \frac{\partial g_{\mu\alpha}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right)\dot{x}^\mu \dot{x}^\nu = 0 \tag{14.149}$$

Now we multiply with $g^{\alpha\beta}$ and use $g_{\mu\alpha}g^{\alpha\beta}\ddot{x}^\mu = \delta^\beta_\mu \ddot{x}^\mu = \ddot{x}^\beta$

$$\ddot{x}^\beta + \frac{1}{2}g^{\alpha\beta}\left(\frac{\partial g_{\alpha\nu}}{\partial x^\mu} + \frac{\partial g_{\mu\alpha}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right)\dot{x}^\mu \dot{x}^\nu = 0 \tag{14.150}$$

$$\ddot{x}^\beta + \Gamma^\beta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0 \tag{14.151}$$

2. Calculating the λ derivative of $\mathcal{H}$ along the geodesic (substituting)

$$\frac{\partial \mathcal{H}}{\partial \lambda} = \frac{\partial \mathcal{L}}{\partial \lambda} - \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\gamma}\dot{x}^\gamma\right) \tag{14.152}$$

$$= \frac{\partial \mathcal{L}}{\partial \lambda} - \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\gamma}\right)\dot{x}^\gamma - \frac{\partial \mathcal{L}}{\partial \dot{x}^\gamma}\ddot{x}^\gamma \tag{14.153}$$

$$= \frac{\partial \mathcal{L}}{\partial \lambda} - \underbrace{\frac{\partial \mathcal{L}}{\partial x^\gamma}}_{-\frac{\partial g_{\mu\nu}}{\partial x^\gamma}\dot{x}^\mu \dot{x}^\nu}\dot{x}^\gamma - \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{x}^\gamma}}_{-2g_{\gamma\varepsilon}\dot{x}^\varepsilon}\underbrace{\ddot{x}^\gamma}_{-\Gamma^\gamma_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} \tag{14.154}$$

$$= \frac{\partial \mathcal{L}}{\partial \lambda} + \frac{\partial g_{\mu\nu}}{\partial x^\gamma}\dot{x}^\mu \dot{x}^\nu \dot{x}^\gamma - 2g_{\gamma\varepsilon}\dot{x}^\varepsilon \Gamma^\gamma_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \tag{14.155}$$

$$= \frac{\partial \mathcal{L}}{\partial \lambda} + g_{\mu\nu,\varepsilon}\dot{x}^\mu \dot{x}^\nu \dot{x}^\varepsilon - g_{\gamma\varepsilon}g^{\gamma\sigma}(g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma})\dot{x}^\varepsilon \dot{x}^\mu \dot{x}^\nu \tag{14.156}$$

$$= \frac{\partial \mathcal{L}}{\partial \lambda} + g_{\mu\nu,\varepsilon}\dot{x}^\mu \dot{x}^\nu \dot{x}^\varepsilon - \delta^\sigma_\varepsilon(g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma})\dot{x}^\varepsilon \dot{x}^\mu \dot{x}^\nu \tag{14.157}$$

$$= \frac{\partial \mathcal{L}}{\partial \lambda} - (-g_{\mu\nu,\varepsilon} + g_{\mu\varepsilon,\nu} + g_{\nu\varepsilon,\mu} - g_{\mu\nu,\varepsilon})\dot{x}^\varepsilon \dot{x}^\mu \dot{x}^\nu \quad \text{(reindex)} \tag{14.158}$$

$$= \frac{\partial \mathcal{L}}{\partial \lambda} \tag{14.159}$$

$$= 0 \tag{14.160}$$

then

$$\mathcal{H} = \mathcal{L} - \frac{\partial \mathcal{L}}{\partial \dot{x}^\gamma}\dot{x}^\gamma \tag{14.161}$$

$$= -g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu - (-2g_{\alpha\gamma}\dot{x}^\alpha)\dot{x}^\gamma \tag{14.162}$$

$$= g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \tag{14.163}$$

14.3.9 Problem 2.3 - Christoffel symbols from a Lagrangian

$$\mathcal{L} = -g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \tag{14.164}$$
$$= \dot{t}^2 - a(t)^2(\dot{x}^2+\dot{y}^2+\dot{z}^2) \tag{14.165}$$

Now $\mu = 0, x^\mu = t$

$$\frac{d}{d\lambda}\frac{\partial\mathcal{L}}{\partial\dot{t}} = \frac{\partial\mathcal{L}}{\partial t} \tag{14.166}$$
$$2\ddot{t} = -2a\dot{a}(\dot{x}^2+\dot{y}^2+\dot{z}^2) \tag{14.167}$$
$$\rightarrow\ddot{t} + a\dot{a}(\dot{x}^2+\dot{y}^2+\dot{z}^2) = 0 \tag{14.168}$$
$$\rightarrow\Gamma^0_{11} = \Gamma^0_{22} = \Gamma^0_{33} = a\dot{a} \tag{14.169}$$

Now $\mu = 1, x^\mu = x$

$$\frac{d}{d\lambda}\frac{\partial\mathcal{L}}{\partial\dot{x}} = \frac{\partial\mathcal{L}}{\partial x} \tag{14.170}$$
$$-2\left(\ddot{x}a^2 + \dot{x}2a\frac{\partial a}{\partial t}\frac{\partial t}{\partial\lambda}\right) = 0 \tag{14.171}$$
$$\rightarrow\ddot{x} + 2\frac{\dot{a}}{a}\dot{x}\dot{t} = 0 \tag{14.172}$$
$$\rightarrow\Gamma^1_{01} = \Gamma^1_{10} = \frac{\dot{a}}{a} \tag{14.173}$$

Analog for $\mu = 2, 3$

$$\Gamma^2_{02} = \Gamma^1_{20} = \frac{\dot{a}}{a} \tag{14.174}$$
$$\Gamma^3_{03} = \Gamma^1_{30} = \frac{\dot{a}}{a} \tag{14.175}$$

14.3.10 Problem 2.4 - Geodesics in de Sitter

1.) To derive the conserved quantities we need to find the Killing vectors ξ^α defined by

$$\mathcal{L}_\xi g_{\mu\nu} = 0 \tag{14.176}$$
$$\rightarrow \quad g_{\mu\nu,\alpha}\xi^\alpha + g_{\alpha\nu}\xi^\alpha_{\ ,\mu} + g_{\mu\alpha}\xi^\alpha_{\ ,\nu} = 0 \tag{14.177}$$

which for $ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2d\Omega^2$ is a system of 10 coupled PDEs

$$-\frac{\partial A}{\partial r}\xi^r - 2A\xi^t_{,t} = 0 \qquad (\mu = 1, \nu = 1) \tag{14.178}$$
$$-A\xi^t_{,r} + B\xi^r_{,t} = 0 \qquad (\mu = 1, \nu = 2) \tag{14.179}$$
$$-A\xi^t_{,\theta} + r^2\xi^\theta_{,t} = 0 \qquad (\mu = 1, \nu = 3) \tag{14.180}$$
$$-A\xi^t_{,\phi} + r^2\sin^2\theta\xi^\phi_{,t} = 0 \qquad (\mu = 1, \nu = 4) \tag{14.181}$$
$$\xi^r B' + 2B\xi^r_{,r} = 0 \qquad (\mu = 2, \nu = 2) \tag{14.182}$$
$$B\xi^r_{,\theta} + r^2\xi^\theta_{,r} = 0 \qquad (\mu = 2, \nu = 3) \tag{14.183}$$
$$B\xi^r_{,\phi} + r^2\sin^2\theta\xi^\phi_{,r} = 0 \qquad (\mu = 2, \nu = 4) \tag{14.184}$$
$$\frac{1}{r}\xi^r + \xi^\theta_{,\theta} = 0 \qquad (\mu = 3, \nu = 3) \tag{14.185}$$
$$\xi^\theta_{,\phi} + \sin^2\theta\xi^\phi_{,\theta} = 0 \qquad (\mu = 3, \nu = 4) \tag{14.186}$$
$$\frac{1}{r}\xi^r + \cot\theta\xi^\theta + \xi^\phi_{,\phi} = 0 \qquad (\mu = 4, \nu = 4) \tag{14.187}$$

We guess some solutions

$$\xi^\alpha_{(t)} = (1,0,0,0) \to \partial_t \tag{14.188}$$
$$\xi^\alpha_{(\phi)} = (0,0,0,1) \to \partial_\phi \tag{14.189}$$
$$\xi^\alpha_{(1)} = (0,0,\sin\phi, \cos\phi\cot\theta) \to \sin\phi\partial_\theta + \cos\phi\cot\theta\partial_\phi \tag{14.190}$$
$$\xi^\alpha_{(2)} = (0,0,\cos\phi, -\sin\phi\cot\theta) \to \cos\phi\partial_\theta - \sin\phi\cot\theta\partial_\phi \tag{14.191}$$

With the geodesic equation

$$u^\alpha_{;\beta}u^\beta = 0 \tag{14.192}$$
$$\to \quad (u^\alpha_{,\beta} + \Gamma^\alpha_{\beta\gamma}u^\gamma)u^\beta = 0 \tag{14.193}$$
$$\to \quad \underbrace{u^\alpha_{,\beta}u^\beta}_{\frac{\partial u^\alpha}{\partial x^\beta}\frac{\partial x^\beta}{\partial\lambda}} + \Gamma^\alpha_{\beta\gamma}u^\gamma u^\beta = 0 \tag{14.194}$$
$$\to \quad \frac{du^\alpha}{d\lambda} + \Gamma^\alpha_{\beta\gamma}u^\gamma u^\beta = 0 \tag{14.195}$$

we see with the Killing equation $\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0$

$$\xi_\alpha(u^\alpha_{;\beta}u^\beta) = 0 \tag{14.196}$$
$$(\xi_\alpha u^\alpha)_{;\beta}u^\beta - \xi_{\alpha;\beta}u^\alpha u^\beta = 0 \tag{14.197}$$
$$(\xi_\alpha u^\alpha)_{,\beta}u^\beta - \xi_{\alpha;\beta}u^\alpha u^\beta = 0 \qquad (\xi_\alpha u^\alpha \text{ is a scalar}) \tag{14.198}$$
$$\frac{\partial(\xi_\alpha u^\alpha)}{\partial x^\beta}\frac{dx^\beta}{d\lambda} - \xi_{\alpha;\beta}u^\alpha u^\beta = 0 \qquad \text{symmetry of Killing equation} \tag{14.199}$$
$$\frac{d}{d\lambda}(\xi_\alpha u^\alpha) = 0 \tag{14.200}$$

which means $\xi_\alpha u^\alpha$ is constant along the geodesic. Therefore we find

$$L_1 = g_{\theta\theta}\xi^\theta_{(1)}u^\theta + g_{\phi\phi}\xi^\phi_{(1)}u^\phi \tag{14.201}$$
$$= r^2\sin\phi\cdot\dot\theta + r^2\sin^2\theta\cos\phi\cot\theta\cdot\dot\phi \tag{14.202}$$
$$= \text{const} \tag{14.203}$$
$$L_2 = r^2\cos\phi\cdot\dot\theta - r^2\sin^2\theta\sin\phi\cot\theta\cdot\dot\phi \tag{14.204}$$
$$= \text{const} \tag{14.205}$$
$$\to L_1\sin\phi + L_2\cos\phi = r^2\dot\theta \tag{14.206}$$
$$\to \dot\theta = \frac{1}{r^2}(L_1\sin\phi + L_2\cos\phi) \tag{14.207}$$
$$\to \dot\theta = \frac{\dot\phi\sin^2\theta}{L}(L_1\sin\phi + L_2\cos\phi) \tag{14.208}$$

From here we should?!? to conclude $\dot\theta = 0$ ... and therefore $\theta = \pi/2$=const

$$L = g_{\alpha\beta}\xi^\beta_{(\phi)}u^\alpha \tag{14.209}$$
$$= g_{\phi\phi}u^\phi \tag{14.210}$$
$$= r^2\sin^2\theta\cdot\dot\phi \tag{14.211}$$
$$= r^2\dot\phi \qquad (\theta = \pi/2 = \text{const}) \tag{14.212}$$

and

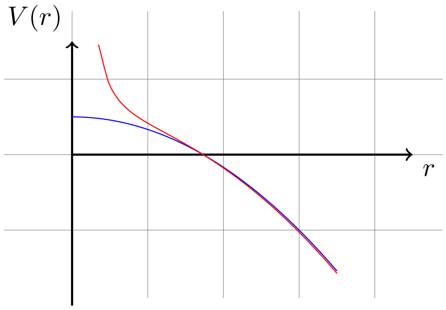
$$E = g_{\alpha\beta}\xi^{\beta}_{(t)}u^{\alpha} \tag{14.213}$$
$$= g_{tt}u^t \tag{14.214}$$
$$= -\left(1-\frac{r^2}{R^2}\right)\dot{t} \tag{14.215}$$

The conserved Hamiltonian is given by

$$\mathcal{H} = g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} \tag{14.216}$$
$$-1 = -\left(1-\frac{r^2}{R^2}\right)\dot{t}^2 + \left(1-\frac{r^2}{R^2}\right)^{-1}\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) \tag{14.217}$$
$$-1 = -\left(1-\frac{r^2}{R^2}\right)\dot{t}^2 + \left(1-\frac{r^2}{R^2}\right)^{-1}\dot{r}^2 + r^2\dot{\phi}^2 \tag{14.218}$$
$$-1 = -\left(1-\frac{r^2}{R^2}\right)^{-1}E^2 + \left(1-\frac{r^2}{R^2}\right)^{-1}\dot{r}^2 + \frac{L^2}{r^2} \tag{14.219}$$

which gives an ODE for $\dot{r}$

$$\left(1-\frac{r^2}{R^2}\right)^{-1}(\dot{r}^2 - E^2) + \left(1+\frac{L^2}{r^2}\right) = 0 \tag{14.220}$$
$$\dot{r}^2 = E^2 - \left(1-\frac{r^2}{R^2}\right)\left(1+\frac{L^2}{r^2}\right) \tag{14.221}$$
$$\dot{r}^2 = E^2 - \left(1-\frac{L^2}{R^2}+\frac{L^2}{r^2}-\frac{r^2}{R^2}\right) \tag{14.222}$$



3.) Small radial velocity means $E \approx 1$ and $L = 0$

$$\dot{r} = \sqrt{E^2 - 1 + \frac{r^2}{R^2}} \tag{14.223}$$
$$= \sqrt{E^2-1}\sqrt{1+\frac{r^2}{R^2(E^2-1)}} \tag{14.224}$$
$$\frac{\dot{r}}{\sqrt{E^2-1}} = \sqrt{1+\frac{r^2}{R^2(E^2-1)}} \tag{14.225}$$
$$\dot{y} = \sqrt{1+\frac{y^2}{R^2}} \qquad (y = r/\sqrt{E^2-1}) \tag{14.226}$$

Now set $\lambda = \tau/R$ then

$$\frac{\partial y}{\partial \lambda} = \frac{\partial y}{\partial \tau}\frac{\partial \tau}{\partial \lambda} = \frac{1}{R}\frac{\partial y}{\partial \tau} \tag{14.227}$$

and with $z = y/R$

$$\frac{\dot{y}}{R} = \sqrt{1 + \frac{y^2}{R^2}} \tag{14.228}$$
$$z' = \sqrt{1 + z^2} \tag{14.229}$$

with the solutions $z = \sinh(\lambda + c)$ and resubstitution we obtain

$$r(\lambda) = R\sqrt{E^2 - 1}\sinh\lambda/R \tag{14.230}$$

and

$$\Delta\lambda = R \cdot \text{arcsinh}\frac{1}{\sqrt{E^2 - 1}} \tag{14.231}$$

$$\frac{dr}{d\lambda} = \sqrt{E^2 - 1}\cosh\lambda/R \tag{14.232}$$
$$= \sqrt{E^2 - 1}\sqrt{1 + \sinh^2\lambda/R} \tag{14.233}$$
$$= \sqrt{E^2 - 1}\sqrt{1 + \frac{r^2}{R^2\sqrt{E^2 - 1}}} \tag{14.234}$$
$$\to t = \int_0^R \frac{dt}{d\lambda} d\lambda = \int dr \frac{-E}{1 - \frac{r^2}{R^2}} \frac{1}{\sqrt{E^2 - 1 + \frac{r^2\sqrt{E^2-1}}{R^2}}} \tag{14.235}$$

14.3.11 Problem 2.5 - Distances

Metric distance d_M, luminosity distance d_L

$$d_M = S_k(\chi) \tag{14.236}$$
$$d_L(z) = (1 + z)d_M(z) \tag{14.237}$$
$$d_A(z) = \frac{d_M(z)}{1 + z} \tag{14.238}$$

14.3.12 Problem 2.6 - Friedmann universes

14.3.13 Problem 2.7 - Einsteins biggest blunder

14.3.14 Problem 2.8 - The accelerating universe

14.3.15 Problem 2.9 - Phantom Dark energy

14.3.16 Exercise 3.1

Let's first rewrite the Zeta function as an integral starting with the common definitions

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt \tag{14.239}$$
$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} \tag{14.240}$$

Then with $t/n = x$ and $dx = dt/n$

$$\zeta(s)\Gamma(s) = \sum_{n=1}^{\infty}\int_0^{\infty}\frac{1}{n^s}t^{s-1}e^{-t}dt \tag{14.241}$$
$$= \sum_{n=1}^{\infty}\int_0^{\infty}\frac{1}{n^s}t^{s-1}e^{-t}n\,dx \tag{14.242}$$
$$= \sum_{n=1}^{\infty}\int_0^{\infty}\frac{t^{s-1}}{n^{s-1}}e^{-nx}dx \tag{14.243}$$
$$= \int_0^{\infty}x^{s-1}\sum_{n=1}^{\infty}e^{-nx}dx \tag{14.244}$$
$$= \int_0^{\infty}\frac{x^{s-1}}{e^x-1}dx \tag{14.245}$$

we obtain

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{x^{s-1}}{e^x-1}dx. \tag{14.246}$$

Now

$$J_-(0) = \int_0^{\infty}\frac{\xi^3}{e^{\xi}-1}d\xi \tag{14.247}$$
$$= \Gamma(4)\zeta(4) \tag{14.248}$$
$$= 3!\zeta(4) \tag{14.249}$$

Furthermore we can write

$$J_-(0) = \int_0^{\infty}\frac{\xi^3}{e^{\xi}-1}d\xi \tag{14.250}$$
$$= \int_0^{\infty}\frac{\xi^3}{(e^{\xi/2}-1)(e^{\xi/2}+1)}d\xi \tag{14.251}$$
$$= \frac{1}{2}\int_0^{\infty}\frac{\xi^3}{e^{\xi/2}-1}d\xi - \frac{1}{2}\int_0^{\infty}\frac{\xi^3}{e^{\xi/2}+1}d\xi \tag{14.252}$$
$$= 8\int_0^{\infty}\frac{x^3}{e^x-1}dx - 8\int_0^{\infty}\frac{x^3}{e^x+1}dx \tag{14.253}$$
$$= 8J_-(0) - 8J_+(0) \tag{14.254}$$
$$\to J_+ = \frac{7}{8}J_-(0) \tag{14.255}$$

14.4 Dodelson, Schmidt - Cosmology (2nd edition)

14.4.1 1.2

We start with

$$\rho_{\rm cr} = \frac{3H_0^2}{8\pi G} \tag{14.256}$$

$$H(t) = \frac{1}{a}\frac{da}{dt} \tag{14.257}$$

$$H(t)^2 = \frac{8\pi G}{3}\left[\varrho(t) + \frac{\Lambda}{3} - \frac{k}{a^2}\right] \tag{14.258}$$

$$= \frac{8\pi G}{3}\left[\rho(t) + \frac{\rho_{\rm cr} - \rho(t_0)}{a^2}\right] \tag{14.259}$$

$$= \frac{8\pi G}{3}\left[\Omega_m\left(\frac{a_0}{a}\right)^3\rho_{\rm cr} + \Omega_\Lambda\rho_{\rm cr} + \frac{\rho_{\rm cr} - \rho(t_0)}{a^2}\right] \tag{14.260}$$

$$= H_0^2\left[\Omega_m\left(\frac{a_0}{a}\right)^3 + \Omega_\Lambda + \frac{\rho_{\rm cr} - \rho(t_0)}{\rho_{\rm cr}a^2}\right] \tag{14.261}$$

$$\tag{14.262}$$

and assume $\rho_{\rm cr} = \rho(t_0)$ (same as Euclidean $k = 0$?!?) and $\Omega_\Lambda + \Omega_m = 1$ and $a_0 = 1$

$$dt = \frac{da}{a}\frac{1}{H(t)} \tag{14.263}$$

$$= \frac{da}{a}\frac{1}{H_0\sqrt{\Omega_m\left(\frac{a_0}{a}\right)^3 + \Omega_\Lambda}} \tag{14.264}$$

$$= \frac{1}{H_0}\frac{da}{a}\left[\frac{1-\Omega_\Lambda}{a^3} + \Omega_\Lambda\right]^{-1/2} \tag{14.265}$$

(a) Now with $\Omega_\Lambda = 0$

$$dt = \frac{1}{H_0}\frac{da}{a}a^{3/2} = \frac{1}{H_0}da\, a^{1/2} \tag{14.266}$$

$$\to t - t_i = \frac{2}{3H_0}(a^{3/2} - a_i^{3/2}) \tag{14.267}$$

$$\to a(t) = \left(\frac{3H_0}{2}(t - t_i) + a_i^{3/2}\right)^{2/3} \tag{14.268}$$

with $a(t = 0) = 0$

$$a(t) = \left(\frac{3H_0}{2}t\right)^{2/3} \tag{14.269}$$

$$\to T = \frac{2}{3H_0} \tag{14.270}$$

(b) ...

14.4.2 1.3 Lyman-α splitting in hydrogen isotopes

The energy eigenvalues are

$$E_n = -\frac{1}{2}\mu c^2\frac{\alpha^2}{n^2} \tag{14.271}$$

$$= -\frac{1}{2}\frac{m_e M_{\rm nuc}}{m_e + M_{\rm nuc}}c^2\frac{\alpha^2}{n^2} \tag{14.272}$$

then

$$\Delta E_{2\to 1} = -\frac{1}{2}\frac{m_e M_{\rm nuc}}{m_e + M_{\rm nuc}} c^2\alpha^2 \left(\frac{1}{2^2} - \frac{1}{1^2}\right) \tag{14.273}$$
$$= \frac{3}{8}\frac{m_e M_{\rm nuc}}{m_e + M_{\rm nuc}} c^2\alpha^2 \tag{14.274}$$
$$= \frac{3}{8}\frac{m_e M_{\rm nuc}}{M_{\rm nuc}(1 + m_e/M_{\rm nuc})} c^2\alpha^2 \tag{14.275}$$
$$= \frac{3}{8}\frac{m_e}{1 + m_e/M_{\rm nuc}} c^2\alpha^2 \tag{14.276}$$

and

$$\Delta E^{\rm D}_{2\to 1} = \frac{3}{8}\frac{m_e}{1 + m_e/2m_p} c^2\alpha^2 \tag{14.277}$$
$$\Delta E^{\rm H}_{2\to 1} = \frac{3}{8}\frac{m_e}{1 + m_e/m_p} c^2\alpha^2 \tag{14.278}$$
$$\to \Delta E^{\rm D}_{2\to 1} = \Delta E^{\rm H}_{2\to 1}\frac{1 + m_e/m_p}{1 + m_e/2m_p} \tag{14.279}$$

and with $E = hc/\lambda$

$$\lambda^{\rm D}_{2\to 1} = \frac{hc}{\Delta E^{\rm D}_{2\to 1}} \tag{14.280}$$
$$= \frac{hc}{\Delta E^{\rm H}_{2\to 1}}\frac{1 + m_e/2m_p}{1 + m_e/m_p} \tag{14.281}$$
$$= \lambda^{\rm H}_{2\to 1}\frac{1 + m_e/2m_p}{1 + m_e/m_p} \tag{14.282}$$
$$= \lambda^{\rm H}_{2\to 1}\left(1 + \frac{m_e}{2m_p}\right)\left(1 - \frac{m_e}{m_p}\right) \tag{14.283}$$
$$\simeq \lambda^{\rm H}_{2\to 1}\left(1 - \frac{1}{2}\frac{m_e}{m_p}\right) \tag{14.284}$$
$$= 1215.67\text{Å} \tag{14.285}$$

furthermore

$$c\frac{\Delta\lambda}{\lambda} = c\frac{\lambda^{\rm D}_{2\to 1} - \lambda^{\rm H}_{2\to 1}}{\lambda^{\rm H}_{2\to 1}} \tag{14.286}$$
$$= \left(1 - \frac{1}{2}\frac{m_e}{m_p}\right) \tag{14.287}$$
$$= 0.999727c \tag{14.288}$$

14.4.3 1.4 Planck law for CMB

Insider hint $1\text{MJy} = 10^6\text{Jansky} = 10^6 \cdot 10^{-26}\text{J}\cdot\text{s}^{-1}\cdot\text{Hz}^{-1}\cdot\text{m}^{-2}$. We start with $c = \lambda\nu = 2\pi\nu/k$

$$I_\nu = \frac{4\pi\hbar\nu^3}{c^2}\frac{1}{e^{2\pi\hbar\nu/k_{\rm B}T} - 1} \tag{14.289}$$

which has the unit energy per area (per frequency per time are cancelling)

$$\frac{\text{Js}\cdot\text{s}^{-3}}{\text{m}^2/\text{s}^2} = \text{J}\cdot\text{m}^{-2} \tag{14.290}$$

then

$$\frac{I_\nu d\nu}{d\Omega} \tag{14.291}$$

14.5 MUKHANOV - Physical foundations of cosmology, 2005

Chapter 15

Quantum Mechanics

15.1 Feynman, Hibbs - Quantum mechanics and path integrals 2ed

15.1.1 2.1

With $\dot{x} = 0$ and $\dot{x}$ =const we see

$$
\begin{aligned}
S &= \int_{t_a}^{t_b} L dt && (15.1)\\
&= \frac{m}{2}\int_{t_a}^{t_b} \dot{x}^2 dt && (15.2)\\
&= \frac{m}{2}\left[\dot{x}x|_{t_a}^{t_b} - \int_{t_a}^{t_b} x\ddot{x}dt\right] && (15.3)\\
&= \frac{m}{2}\frac{x_b - x_a}{t_b - t_b}(x_b - x_a) && (15.4)\\
&= \frac{m}{2}\frac{(x_b - x_a)^2}{t_b - t_b} && (15.5)
\end{aligned}
$$

15.1.2 2.1

With the solution of the equation of motion

$$
\begin{aligned}
\ddot{x} + \omega^2 x = 0 \quad &\rightarrow \quad x = x_0 \sin(\omega t + \varphi_0) = (x_0 \cos\varphi_0)\sin\omega t + (x_0 \sin\varphi_0)\cos\omega t && (15.6)\\
&\rightarrow \quad \dot{x} = (x_0\omega\cos\varphi_0)\cos\omega t - (x_0\omega\sin\varphi_0)\sin\omega t && (15.7)
\end{aligned}
$$

then with (x_a, x_b, t_a, t_b) we can solve for x_0 and φ_0

$$
\begin{aligned}
x_0\cos\varphi_0 &= \frac{x_a\cos\omega t_b - x_b\cos\omega t_a}{\cos\omega t_b \sin\omega t_a - \cos\omega t_a \sin\omega t_b} && (15.8)\\
&= \frac{x_a\cos\omega t_b - x_b\cos\omega t_a}{\sin\omega(t_a - t_b)} && (15.9)\\
x_0\sin\varphi_0 &= -\frac{x_a\frac{\sin\omega t_b}{\sin\omega t_a} - x_b\tan\omega t_a}{-\sin\omega t_b + \cos\omega t_b\tan\omega t_a} && (15.10)\\
&= \frac{x_b\sin\omega t_a - x_a\sin\omega t_b}{\sin\omega(t_a - t_b)} && (15.11)
\end{aligned}
$$

and therefore

$$
\begin{aligned}
v_a &= \frac{x_a\cos\omega t_b - x_b\cos\omega t_a}{\sin\omega(t_a-t_b)}\sin\omega t_a + \frac{x_b\sin\omega t_a - x_a\sin\omega t_b}{\sin\omega(t_a-t_b)}\sin\omega t_a && (15.12)\\
&= -\frac{1}{\sin\omega T}\left[(x_a\cos\omega t_b - x_b\cos\omega t_a)\sin\omega t_a + (x_b\sin\omega t_a - x_a\sin\omega t_b)\sin\omega t_a\right] && (15.13)\\
&= -\frac{1}{\sin\omega T}\left[x_a(\cos\omega t_b\sin\omega t_a - \sin\omega t_a\sin\omega t_b) + x_b(\sin^2\omega t_a - \cos\omega t_a\sin\omega t_a)\right] && (15.14)\\
v_b &= \frac{x_a\cos\omega t_b - x_b\cos\omega t_a}{\sin\omega(t_a-t_b)}\sin\omega t_b + \frac{x_b\sin\omega t_a - x_a\sin\omega t_b}{\sin\omega(t_a-t_b)}\sin\omega t_b && (15.15)\\
&= -\frac{1}{\sin\omega T}\left[x_a(\cos\omega t_b\sin\omega t_b - \sin^2\omega t_b) + x_b(\sin\omega t_a\sin\omega t_b - \cos\omega t_a\sin\omega t_b)\right] && (15.16)
\end{aligned}
$$

Now we can write

$$
\begin{aligned}
S &= \int_{t_a}^{t_b} L dt && (15.17)\\
&= \frac{m}{2}\int_{t_a}^{t_b}(\dot{x}^2 - \omega^2 x^2)dt && (15.18)\\
&= \frac{m}{2}x_0^2\omega^2\int_{t_a}^{t_b} dt\left(\cos^2(\omega t+\varphi) - \sin^2(\omega t+\varphi)\right) && (15.19)\\
&= \frac{m}{2}x_0^2\omega^2\int_{t_a}^{t_b} dt\cos(2[\omega t+\varphi]) && (15.20)\\
&= \frac{m}{4}x_0^2\omega\sin(2[\omega t+\varphi])|_{t_a}^{t_b} && (15.21)\\
&= \frac{m}{2}x_0^2\omega\sin(\omega t+\varphi)\cos(\omega t+\varphi)|_{t_a}^{t_b} && (15.22)\\
&= \frac{m}{2}x\dot{x}|_{t_a}^{t_b} && (15.23)\\
&= \frac{m}{2}(x_b v_b - x_a v_a) && (15.24)\\
&= \frac{m\omega}{2\sin\omega T}\left[(x_a^2 + x_b^2)\cos\omega T - 2x_a x_b\right] && (15.25)
\end{aligned}
$$

15.2 Straumann - Quantenmechanik 2ed

15.2.1 2.1 - Spectral oscillator density

The vanishing electrical field in the surface requires for each standing wave

$$k_i = \frac{\pi}{L}n_i. \tag{15.26}$$

and

$$k^2 = k_x^2 + k_y^2 + k_z^2 \tag{15.27}$$

$$\Delta V = \frac{\pi^3}{L^3}. \tag{15.28}$$

With $k = 2\pi/\lambda = \omega/c$ we have $dk = \frac{d\omega}{c}$ and the volume of a sphere in k-space is given by

$$V(k) = \frac{4}{3}\pi k^3 \tag{15.29}$$

$$dV = 4\pi k^2 dk = 4\pi\frac{\omega^2}{c^2}\frac{d\omega}{c} = 4\pi(2\pi)^3\frac{\nu^2}{c^3}d\nu \tag{15.30}$$

The number of oscillator are then given by the number of points in the positive quadrant (all k_i positive) time two (polarization)

$$dN(\nu) = 2\frac{V(\nu)/8}{\Delta V} = L^3 \frac{8\pi}{c^3}\nu^2 d\nu \tag{15.31}$$

15.2.2 2.2 - Energy variance of the harmonic oscillator

First we obtain an expression for T

$$E = \frac{h\nu}{e^{h\nu/kT} - 1} \quad \rightarrow \quad \frac{h\nu}{kT} = \ln\left(\frac{h\nu}{E} + 1\right) \tag{15.32}$$

which we can use in

$$\frac{dS}{dE} = \frac{1}{T} = \frac{k}{h\nu}\ln\left(\frac{h\nu}{E} + 1\right) \tag{15.33}$$

and take one more derivative

$$\frac{d^2S}{dE^2} = -\frac{k}{h\nu}\frac{\frac{h\nu}{E^2}}{\frac{h\nu}{E} + 1} \tag{15.34}$$

$$= -k\frac{1}{h\nu E + E^2}. \tag{15.35}$$

Now we see

$$\langle(\Delta E)^2\rangle = E^2 + Eh\nu. \tag{15.36}$$

15.2.3 3.6 - 1D molecular potential

With the given coordinate transformation we get for the single terms

$$e^{-\alpha x} = \frac{\alpha\hbar\xi}{2\sqrt{2mA}} \tag{15.37}$$

$$e^{-2\alpha x} = \frac{(\alpha\hbar\xi)^2}{8mA} \tag{15.38}$$

$$\frac{\partial}{\partial x} = \frac{\partial\xi}{\partial x}\frac{\partial}{\partial\xi} \tag{15.39}$$

$$= -\alpha\xi\frac{\partial}{\partial\xi} \tag{15.40}$$

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2\xi}{\partial x^2}\frac{\partial}{\partial\xi} + \left(\frac{\partial\xi}{\partial x}\right)^2\frac{\partial^2}{\partial\xi^2} \tag{15.41}$$

$$= \alpha^2\xi\frac{\partial}{\partial\xi} + (\alpha\xi)^2\frac{\partial^2}{\partial\xi^2} \tag{15.42}$$

and combined

$$-\frac{\hbar^2}{2m}\partial_{xx}\psi + A(e^{-2\alpha x} - 2e^{-\alpha x})\psi = E\psi \tag{15.43}$$

$$-\frac{\hbar^2}{2m}\left(\alpha^2\xi\frac{\partial}{\partial\xi} + (\alpha\xi)^2\frac{\partial^2}{\partial\xi^2}\right)\psi + A\left(\frac{(\alpha\hbar\xi)^2}{8mA} - 2\frac{\alpha\hbar\xi}{2\sqrt{2mA}}\right)\psi = E\psi \tag{15.44}$$

$$\left(\alpha^2\xi\frac{\partial}{\partial\xi} + (\alpha\xi)^2\frac{\partial^2}{\partial\xi^2}\right)\psi - \frac{2mA}{\hbar^2}\left(\frac{(\alpha\hbar\xi)^2}{8mA} - 2\frac{\alpha\hbar\xi}{2\sqrt{2mA}}\right)\psi = -\frac{2mE}{\hbar^2}\psi \tag{15.45}$$

$$\left(\frac{1}{\xi}\frac{\partial}{\partial\xi} + \frac{\partial^2}{\partial\xi^2}\right)\psi - \frac{2mA}{\alpha^2\xi^2\hbar^2}\left(\frac{(\alpha\hbar\xi)^2}{8mA} - 2\frac{\alpha\hbar\xi}{2\sqrt{2mA}}\right)\psi = -\frac{2mE}{\hbar^2\alpha^2\xi^2}\psi \tag{15.46}$$

$$\left(\frac{1}{\xi}\frac{\partial}{\partial\xi} + \frac{\partial^2}{\partial\xi^2}\right)\psi + \left(-\frac{1}{4} + \frac{\sqrt{2mA}}{\alpha\hbar\xi}\right)\psi = -\frac{2mE}{\hbar^2\alpha^2\xi^2}\psi \tag{15.47}$$

$$\left(\frac{1}{\xi}\frac{\partial}{\partial\xi} + \frac{\partial^2}{\partial\xi^2}\right)\psi + \left(-\frac{1}{4} + \frac{n+s+\frac{1}{2}}{\xi}\right)\psi = \frac{s^2}{\xi^2}\psi \tag{15.48}$$

$$\left(\frac{\partial^2}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial}{\partial\xi}\right)\psi + \left(-\frac{1}{4} + \frac{n+s+\frac{1}{2}}{\xi} - \frac{s^2}{\xi^2}\right)\psi = 0. \tag{15.49}$$

The units of ξ is $\sqrt{\mathrm{kg}\cdot\mathrm{J}}/\mathrm{m}^{-1}\mathrm{Js} = 1$ so ξ in dimensionless.

1. Case $\xi \gg 1$ ($x \to -\infty$) Dropping all $1/\xi$ terms

$$\psi'' - \frac{1}{4}\psi = 0 \quad \to \quad \psi = c_1 e^{\xi/2} + c_2 e^{-\xi/2} \tag{15.50}$$

2. Case $0 < \xi \ll 1$ ($x \to +\infty$) Ansatz $\psi \sim \xi^m$

$$m(m-1)\xi^{m-2} + m\xi^{m-2} - \frac{1}{4}\xi^m + \left(n+s+\frac{1}{2}\right)\xi^{m-1} - s^2\xi^{m-2} = 0 \tag{15.51}$$

$$\left[(m^2 - s^2) - \frac{1}{4}\xi^2 + \left(n+s+\frac{1}{2}\right)\xi\right]\xi^{m-2} = 0 \tag{15.52}$$

which for small ξ becomes

$$(m^2 - s^2)\xi^{m-2} = 0 \quad \to \quad \psi = \xi^{\pm s} \tag{15.53}$$

With the two asymptotics we can make a physically sensible ansatz for a full solutions $\psi = \xi^s e^{-\xi/2}u(\xi)$ which leads to

$$\xi u'' + (2s + 1 - \xi)u' + nu = 0 \tag{15.54}$$

To solve this equation we use the Sommerfeld polynomial method

$$u = \sum_k a_k\xi^k \quad \to \quad \sum_k k(k-1)a_k\xi^{k-1} + (2s+1)ka_k\xi^{k-1} - ka_k\xi^k + na_k\xi^k = 0 \tag{15.55}$$

$$\sum_k (k+1)ka_{k+1}\xi^k + (2s+1)(k+1)a_{k+1}\xi^k - ka_k\xi^k + na_k\xi^k = 0 \tag{15.56}$$

$$a_{k+1} = \frac{k-n}{(k+1)(2s+1+k)}a_k. \tag{15.57}$$

The requirement for the series to cut off (making u a finite order polynomial) is $n_k = k$. The energies of the bound states are therefore

$$E_k = -\frac{\alpha^2\hbar^2}{2m}s_k^2 \tag{15.58}$$

$$= -\frac{\alpha^2\hbar^2}{2m}\left[\frac{\sqrt{2mA}}{\alpha\hbar} - (k+1/2)\right]^2 \tag{15.59}$$

$$= -A\left[1 - \frac{\alpha\hbar}{\sqrt{2mA}}(k+1/2)\right]^2 \tag{15.60}$$

where the only valid k are the ones where E_k is in $[-A, 0]$.

15.3 Schwinger - Quantum Mechanics Symbolism of Atomic Measurements

15.3.1 2.1

Observe

$$\int_{-\infty}^{\infty} \left(\theta(x+a) + \theta(a-x)\right) e^{ikx} dx = \int_{-a}^{a} e^{ikx} dx \tag{15.61}$$

$$= \frac{1}{ik}\left(e^{ika} - e^{-ika}\right) \tag{15.62}$$

$$= 2a\frac{\sin ka}{ka} \tag{15.63}$$

$$\lim_{P\to\infty}\int_{-\infty}^{\infty} \frac{d\chi}{\pi}\frac{\sin\chi}{\chi}e^{ik\left(q'+\frac{\chi}{P}\right)} = \frac{1}{\pi}e^{ikq'}\lim_{P\to\infty}\int_{-\infty}^{\infty} d\chi\frac{\sin\chi}{\chi}e^{i\frac{k}{P}\chi} \tag{15.64}$$

15.4 Weinberg - Quantum Mechanics 2nd edition

15.4.1 1.1

- The solution of for a free particle in the interval $-a < x < a$ is given by

$$\left[-\frac{\hbar^2}{2M}\frac{d^2}{dx^2} - E\right]\phi = 0 \tag{15.65}$$

$$\left[\frac{d^2}{dx^2} + \frac{2ME}{\hbar^2}\right]\phi = 0 \tag{15.66}$$

$$\to \phi = A\sin\left(\frac{\sqrt{2ME}}{\hbar}x\right) + B\cos\left(\frac{\sqrt{2ME}}{\hbar}x\right) \tag{15.67}$$

  with the two boundary conditions

$$A\sin\left(\frac{\sqrt{2ME}}{\hbar}(-a)\right) + B\cos\left(\frac{\sqrt{2ME}}{\hbar}(-a)\right) = 0 \tag{15.68}$$

$$A\sin\left(\frac{\sqrt{2ME}}{\hbar}a\right) + B\cos\left(\frac{\sqrt{2ME}}{\hbar}a\right) = 0. \tag{15.69}$$

The possible energy eigenvalues are therefore

$$A = 0, \quad \frac{\sqrt{2ME_{2n+1}}}{\hbar}a = (2n+1)\frac{\pi}{2} \quad \to \quad E_{2n+1} = \frac{\pi^2\hbar^2}{8Ma^2}(2n+1)^2 \tag{15.70}$$

$$\to \quad \phi = \frac{1}{\sqrt{a}}\cos\left(x\frac{\pi}{2a}(2n+1)\right) \tag{15.71}$$

$$B = 0, \quad \frac{\sqrt{2ME_{2n}}}{\hbar}a = 2n\frac{\pi}{2} \quad \to \quad E_{2n} = \frac{\pi^2\hbar^2}{8Ma^2}(2n)^2 \tag{15.72}$$

$$\to \quad \phi = \frac{1}{\sqrt{a}}\sin\left(x\frac{\pi}{2a}(2n)\right) \tag{15.73}$$

where we calculated the normalization via

$$\int_{-a}^{a}\sin^2(kx)dx = \int_{-a}^{a}(1-\cos^2(kx))dx \tag{15.74}$$

$$= 2a - \int_{-a}^{a}\cos^2(kx)dx \quad \to \int_{-a}^{a}\sin^2(kx)dx = a. \tag{15.75}$$

- Lets first calculate the normalization

$$\int_{-a}^{a}(a^2-x^2)^2dx = a^4x - 2a^2\frac{x^3}{3} + \frac{x^5}{5}\Big|_{-a}^{a} \tag{15.76}$$

$$= a^4(2a) - \frac{2}{3}a^2(16a^3) + \frac{1}{5}(64a^5) \tag{15.77}$$

$$= \left(2 - \frac{4}{3} + \frac{2}{5}\right)a^5 = \frac{16}{15}a^5 \tag{15.78}$$

and then obtain

$$\int_{-a}^{a}\frac{1}{\sqrt{\frac{16a^5}{15}}}\left(a^2-x^2\right)\frac{1}{\sqrt{a}}\cos\left(\frac{\pi x}{2a}\right)dx = \frac{8\sqrt{15}}{\pi^3} \tag{15.79}$$

15.4.2 1.2

- We can write the Hamiltonian as

$$H = \frac{\vec{P}^2}{2M} + \frac{M\omega_0^2}{2}\vec{X}^2 \tag{15.80}$$

$$= \sum_{k=1}^{3}\frac{p_k^2}{2M} + \frac{M\omega_0^2}{2}x_k^2 \tag{15.81}$$

the energy is therefore given by

$$E_{n_1,n_2,n_3} = \hbar\omega_0\left(n_1+n_2+n_3+\frac{3}{2}\right) \tag{15.82}$$

$$N_{n=n_1+n_2+n_3} = \sum_{k=0}^{n}(k+1) \tag{15.83}$$

$$= \frac{n(n+1)}{2} + n + 1 \tag{15.84}$$

$$= \frac{(n+1)(n+2)}{2} \tag{15.85}$$

- With (1.4.5), (1.4.15) and $\omega_{01} = \omega_0$ we have

$$\vec{x}]_{01} = e^{i\omega_0 t}\sqrt{\frac{\hbar}{2M\omega_0}} \tag{15.86}$$

$$A^{n=0}_{n=1} = \frac{4e^2\omega_0^3}{3c^3\hbar}\,|[\vec{x}]_{01}|^2 \tag{15.87}$$

$$= \frac{2e^2\omega_0^2}{3c^3 M} \tag{15.88}$$

where with (1.4.15).

15.5 Schwabl - Quantum Mechanics 4th ed

15.5.1 Problem 17.1 - 3d Harmonic oscillator

(a) Represent the 3d oscillator by three 1d oscillators

$$H = \frac{\mathbf{p}^2}{2m} + \frac{m\omega^2}{2}\mathbf{x}^2 \tag{15.89}$$

$$= \frac{p_x^2 + p_y^2 + p_z^2}{2m} + \frac{m\omega^2}{2}(x^2 + y^2 + z^2) \tag{15.90}$$

$$= \sum_k^3 \frac{p_k^2}{2m} + \frac{m\omega^2}{2}x_k^2 \tag{15.91}$$

$$= \hbar\omega \sum_k^3 \left(a_k^\dagger a_k + \frac{1}{2}\right) \tag{15.92}$$

$$= \hbar\omega \sum_k^3 \left(n_k + \frac{1}{2}\right) \tag{15.93}$$

$$\to E = \hbar\omega\left(n_x + n_y + n_y + \frac{3}{2}\right) \tag{15.94}$$

level	1	2	3	4	...	N
energy	$3/2$	$5/2$	$7/2$	$9/2$	...	$3/2 + N$
multi	1	3	6	10	...	$N(N+1)/2$

The eigenfunctions are then

$$\psi(\mathbf{x}) = \psi_{n_x}(x)\psi_{n_y}(y)\psi_{n_z}(z) \tag{15.95}$$

(b)

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} - \frac{2m[V(r) - E]}{\hbar^2}\right)R(r) = 0 \tag{15.96}$$

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} + \frac{2mE}{\hbar^2} - \frac{m^2\omega^2}{\hbar^2}r^2\right)R(r) = 0 \tag{15.97}$$

For the asymptotics $r \to 0$ we set $R(r) = u(r)/r$ and obtain

$$\left(\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2}\right)u(r) = 0 \tag{15.98}$$

assuming $E - V(r)$ is small compared to the $1/r^2$. This gives

$$u(r) = Ar^{l+1} + Br^{-l} \tag{15.99}$$
$$\to u(r) = Ar^{l+1} \tag{15.100}$$

We therefore guess the solution as $R(r) \sim r^l e^{-\alpha r^2}(a_0 + a_1 r + a_2 r^2 + ...) = r^l e^{-\alpha r^2} f(r)$ and substitute into the ODE obtaining a system of algebraic equations for the a_i and E. For the lowed energy levels we obtain

$$l = 0 \quad R(r) = e^{-\frac{m\omega}{2\hbar}r^2} \quad \to \quad E = \frac{3}{2}\hbar\omega \tag{15.101}$$
$$R(r) = e^{-\frac{m\omega}{2\hbar}r^2}\left(1 - \frac{2m\omega r^2}{3\hbar}\right) \quad \to \quad E = \frac{7}{2}\hbar\omega \tag{15.102}$$
$$l = 1 \quad R(r) = e^{-\frac{m\omega}{2\hbar}r^2} r \quad \to \quad E = \frac{5}{2}\hbar\omega \tag{15.103}$$
$$l = 2 \quad R(r) = e^{-\frac{m\omega}{2\hbar}r^2} r^2 \quad \to \quad E = \frac{7}{2}\hbar\omega \tag{15.104}$$

Making the calculation more robust we insert a full series expansion $f(r) = \sum_k a_k r^k$ into the radial equation

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} + \frac{2mE}{\hbar^2} - \frac{m^2\omega^2}{\hbar^2}r^2\right)R(r) = 0$$
$$rf'' + 2(1 + l - 2\alpha r^2)f' - r\left(-\frac{2mE}{\hbar^2} + \alpha(3 + 2l - 2\alpha r^2) + \frac{m^2\omega^2}{\hbar^2}\right)f = 0$$
$$f'' + 2\frac{1 + l - 2\alpha r^2}{r}f' - \left(-\frac{2mE}{\hbar^2} + \alpha(3 + 2l - 2\alpha r^2) + \frac{m^2\omega^2}{\hbar^2}\right)f = 0$$
$$\sum_k \left[k(k-1)a_k + 2(1 + l - 2\alpha r^2)ka_k - \left(-\frac{2mE}{\hbar^2} + \alpha(3 + 2l - 2\alpha r^2) + \frac{m^2\omega^2}{\hbar^2}\right)a_k r^2\right] r^{k-2} = 0$$
$$\sum_k \left[k(k-1)a_k + 2(1+l)ka_k - 2\alpha(k-2)a_{k-2} - \frac{m(m\omega^2 - 2E)}{\hbar^2}a_{k-2} + \alpha(3+2l)a_{k-2} - 2\alpha^2 r^2 a_k r^2\right] r^{k-2} = 0$$

15.5.2 Problem 17.2 - Delta-shell potential

With

$$y = r/a \tag{15.105}$$
$$\frac{d}{dr} = \frac{\partial y}{\partial r}\frac{d}{dy} = \frac{1}{a}\frac{d}{dy} \tag{15.106}$$
$$\frac{d^2}{dr^2} = \frac{d}{dr}\left(\frac{1}{a}\frac{d}{dy}\right) = \frac{1}{a^2}\frac{d}{dy} \tag{15.107}$$

we can rewrite

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} - \frac{2m[V(r) - E]}{\hbar^2}\right)R(r) = 0 \tag{15.108}$$
$$\left(\frac{1}{a^2}\frac{d^2}{dy^2} + \frac{2}{ya}\frac{1}{a}\frac{d}{dy} - \frac{l(l+1)}{y^2a^2} - \frac{2m}{\hbar^2}\left[-\lambda\frac{\hbar^2}{2m}\delta(r-a)\right] + \frac{2mE}{\hbar^2}\right)R(r) = 0 \tag{15.109}$$
$$\left(\frac{1}{a^2}\frac{d^2}{dy^2} + \frac{2}{ya}\frac{1}{a}\frac{d}{dy} - \frac{l(l+1)}{y^2a^2} + \lambda\delta(r-a) + \frac{2mE}{\hbar^2}\right)R(r) = 0 \tag{15.110}$$
$$\left(\frac{d^2}{dy^2} + \frac{2}{y}\frac{d}{dy} - \frac{l(l+1)}{y^2} + ga\delta(r-a) + \frac{2ma^2E}{\hbar^2}\right)R(r) = 0 \tag{15.111}$$

and see

$$y \neq 1 \quad \left(\frac{d^2}{dy^2} + \frac{2}{y}\frac{d}{dy} - \frac{l(l+1)}{y^2} + ak^2\right) R(y) = 0 \tag{15.112}$$

$$k^2 = g + \frac{2maE}{\hbar^2} \tag{15.113}$$

Independent solutions

$$R(y) = Aj_l(y\sqrt{ka}) + By_l(y\sqrt{ka}) \tag{15.114}$$

Here the requirements for the wavefunction

- regular at the origin with $R(r) \sim r^l$
- continuous (not differentiable) at $r = a$ (or $y = 1$)
- jump of the first derivative of ga
- exponential decay outside to ensure normalizability

and here a quick overview of the two functions and a special linear combination

$$j_l(x) = (-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\sin x}{x} \qquad y_l(x) = -(-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\cos x}{x} \qquad h_0^{(1)}(x) = j_l(ix) + iy_l(ix)$$

$$j_0(x) = \frac{\sin x}{x} \qquad y_0(x) = -\frac{\cos x}{x} \qquad h_0^{(1)}(x) = -\frac{e^{-x}}{x}$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} \qquad y_1(x) = -\frac{\cos x}{x} - \frac{\sin x}{x} \qquad h_1^{(1)}(x) = i(1+x)\frac{e^{-x}}{x^2}$$

$$J_2(x) = ... \qquad y_l(x) = ... \qquad h_2^{(1)}(x) = (x^2 + 3x + 3)\frac{e^{-x}}{x^3}$$

We see that j_l is suitable for the inside and $h_l^{(1)}$ for the outside.

$$R(\rho) = \begin{cases} Aj_l(\rho) & r < a \\ Ch_l^{(1)}(\rho) & r > a \end{cases} \tag{15.115}$$

15.6 Sakurai, Napolitano - Modern Quantum Mechanics 3rd ed

15.6.1 5.1 - Harmonic oscillator with linear perturbation

The Hamiltonians are given by

$$\hat{H}_0 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega_o^2 x^2 \tag{15.116}$$

$$\hat{H}_1 = bx \tag{15.117}$$

We remember

$$\phi_0(x) = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4} e^{-m\omega_0 x^2/2\hbar} \tag{15.118}$$

$$E_0 = \frac{1}{2}\hbar\omega_0 \tag{15.119}$$

$$\phi_n(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4} e^{-m\omega_0 x^2/2\hbar} H_n\left(\sqrt{\frac{m\omega_0}{\hbar}}x\right) \tag{15.120}$$

$$E_n = \hbar\omega_0\left(n + \frac{1}{2}\right) \tag{15.121}$$

1. Time independent perturbation theory gives

$$\Delta E_n^{(1)} = \langle n^{(0)}|\hat{H}_1|n^{(0)}\rangle \tag{15.122}$$
$$\Delta E_0^{(1)} = \langle 0^{(0)}|\hat{H}_1|0^{(0)}\rangle = 0 \tag{15.123}$$

The first order energy shift vanishes because of the wave function is even and H_1 is odd. For the first order perturbation of the wave function we observe

$$H_1(x) = 2xH_0(x) \quad \rightarrow \quad \hat{H}_1|0^{(0)}\rangle = \frac{b}{2}\sqrt{2}\sqrt{\frac{\hbar}{m\omega_0}}|1^{(0)}\rangle \tag{15.124}$$
$$\langle m^{(0)}|n^{(0)}\rangle = \delta_{nm} \tag{15.125}$$

Now we can calculate

$$|n^{(1)}\rangle = \sum_{k\neq n} \frac{\langle k^{(0)}|\hat{H}_1|n^{(0)}\rangle}{E_n^{(0)} - E_k^{(0)}}|k^{(0)}\rangle \tag{15.126}$$
$$|0^{(1)}\rangle = \frac{\langle 0^{(0)}|\hat{H}_1|1^{(0)}\rangle}{E_0^{(0)} - E_1^{(0)}}|1^{(0)}\rangle \tag{15.127}$$
$$= -\frac{1}{\hbar\omega_0}b\sqrt{\frac{\hbar}{2m\omega_0}}|1^{(0)}\rangle \tag{15.128}$$
$$= -b\sqrt{\frac{1}{2m\hbar\omega_0^3}}|1^{(0)}\rangle \tag{15.129}$$

Second order enegy perturbation

$$\Delta E_n^{(2)} = \langle n^{(0)}|\hat{H}_1|n^{(1)}\rangle = \sum_{k\neq n} \frac{|\langle k^{(0)}|\hat{H}_1|n^{(0)}\rangle|^2}{E_n^{(0)} - E_k^{(0)}} \tag{15.130}$$
$$\Delta E_0^{(2)} = \langle 0^{(0)}|\hat{H}_1|0^{(1)}\rangle \tag{15.131}$$
$$= b\sqrt{\frac{\hbar}{2m\omega_0}}\langle 1^{(0)}|0^{(1)}\rangle \tag{15.132}$$
$$= b\sqrt{\frac{\hbar}{2m\omega_0}}\langle 1^{(0)}|\left(-b\sqrt{\frac{1}{2m\hbar\omega_0^3}}\right)|1^{(0)}\rangle \tag{15.133}$$
$$= -b^2\frac{1}{2m\omega_0^2} \tag{15.134}$$

2. The linear perturbation does not change the shape of the potential - only shifts the minimum

$$V(x) = \frac{m\omega_0^2}{2}x^2 + bx = \frac{m\omega_0^2}{2}\left(x + \frac{b}{m\omega_0^2}\right)^2 - \frac{b^2}{2m\omega_0^2} \tag{15.135}$$
$$\Delta E^{(\infty)} = -\frac{b^2}{2m\omega_0^2} \tag{15.136}$$

So the second order gives the exact result - interesting to see if higher orders would all vanish or give oscillating contributions.

15.6.2 5.2 - Potential well with linear slope

We will treat the slope as a perturbation with

$$\hat{H}_1 = \frac{V}{L}x \tag{15.137}$$

Therefore the unperturbed wave functions are given by

$$\phi_n = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} \qquad E_n = \frac{\pi^2 \hbar^2}{2mL^2} n^2 \tag{15.138}$$

Then

$$\begin{aligned}
\Delta E_n^{(1)} &= \langle n^{(0)} | \hat{H}_1 | n^{(0)} \rangle && (15.139)\\
&= \frac{V}{L} \frac{2}{L} \int_0^L x \sin^2 \frac{n\pi x}{L} dx && (15.140)\\
&= \frac{2V}{L^2} \int_0^L x \sin^2 \frac{n\pi x}{L} dx && (15.141)\\
&= \frac{2V}{L^2} \int_0^L x \left(1 - \cos^2 \frac{n\pi x}{L}\right) dx && (15.142)\\
&= \frac{2V}{L^2} \frac{L^2}{2} - \Delta E_n^{(1)} && (15.143)
\end{aligned}$$

meaning $\Delta E_n^{(1)} = V/2$.

15.6.3 5.3 - Relativistic perturbation

We can approximate the kinetic energy by

$$\begin{aligned}
E &= \sqrt{m^2 c^4 + p^2 c^2} && (15.144)\\
&\approx mc^2 + \frac{p^2}{2m} - \frac{p^4}{8m^3 c^2} + \frac{p^6}{16 m^5 c^4} + \cdots && (15.145)\\
&= mc^2 + \frac{mc^2}{2} \frac{p^2}{m^2 c^2} - \frac{mc^2}{8} \frac{p^4}{m^4 c^4} + \cdots && (15.146)\\
&= mc^2 \left(1 + \frac{1}{2}\beta^2 - \frac{1}{8}\beta^4 + \cdots\right) && (15.147)
\end{aligned}$$

so

$$\begin{aligned}
\hat{H}_0 &= -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m\omega^2 x^2 && (15.148)\\
\hat{H}_1 &= -\frac{1}{8m^3 c^2} p^4 = -\frac{\hbar^4}{8m^3 c^2} \frac{d^4}{dx^4} && (15.149)
\end{aligned}$$

and we remember

$$\begin{aligned}
\phi_0(x) &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar} && (15.150)\\
E_0 &= \frac{1}{2}\hbar\omega_0 && (15.151)
\end{aligned}$$

then

$$\begin{aligned}
\Delta E_0^{(1)} &= \langle 0^{(0)} | \hat{H}_1 | 0^{(0)} \rangle && (15.152)\\
&= -\frac{\hbar^4}{8m^3 c^2} \int_{-\infty}^{\infty} \phi_0(x)^* \frac{d^4}{d^4 x} \phi_0(x) dx && (15.153)\\
&= -\frac{3\hbar^2 \omega^2}{32 mc^2} && (15.154)
\end{aligned}$$

15.6.4 5.4 - Diatomic atomic rotor

Hamiltonian of the problem is given by

$$H = \frac{L^2}{2I} \quad \rightarrow \quad \hat{H} = -\frac{\hbar^2}{2I}\frac{d^2}{d\varphi^2} \tag{15.155}$$

with the unperturbed solutions

$$\phi_n^{(0)} = Ce^{in\phi} \qquad E_n^{(0)} = \frac{\hbar^2 n^2}{2I} \tag{15.156}$$

where only E_0 is non-degenerate (all other are double degenerated). For the perturbation we use the Hamiltonian

$$\hat{H}_1 = Ed\cos\varphi \tag{15.157}$$

Hmmm....

15.6.5 5.6 - Two dimensional potential well

As the problem separates

$$\left(\hat{H}_x + \hat{H}_y\right)\phi_x\phi_y = (E_x + E_y)\phi_x\phi_y \tag{15.158}$$

$$\phi_y\hat{H}_x\phi_x + \phi_x\hat{H}_y\phi_y = (E_x + E_y)\phi_x\phi_y \tag{15.159}$$

$$\frac{\hat{H}_x\phi_x}{\phi_x} + \frac{\hat{H}_y\phi_y}{\phi_y} = (E_x + E_y) \tag{15.160}$$

the wave function can be written as a product of the 1-dimensional wave functions

$$\phi_{n_x,n_y} = \sqrt{\frac{2}{L}}\sqrt{\frac{2}{L}}\sin\left(\frac{n_x\pi}{L}x\right)\sin\left(\frac{n_y\pi}{L}y\right) \tag{15.161}$$

$$E_{n_x,n_y} = \frac{\pi^2\hbar^2}{2mL^2}(n_x^2 + n_y^2) \tag{15.162}$$

So

$$\phi_{1,1} \quad \rightarrow \quad E_{1,1} = 2\frac{\pi^2\hbar^2}{2mL^2} \tag{15.163}$$

$$\phi_{2,1}, \phi_{1,2} \quad \rightarrow \quad E_{2,1} = 5\frac{\pi^2\hbar^2}{2mL^2} \tag{15.164}$$

$$\phi_{2,2} \quad \rightarrow \quad E_{1,1} = 8\frac{\pi^2\hbar^2}{2mL^2} \tag{15.165}$$

for the non-degenerated levels $E_{1,1}$ and $E_{2,2}$ we get

$$\Delta E_{1,1}^{(1)} = \langle 1, 1^{(0)}|\hat{H}_1|1, 1^{(0)}\rangle \tag{15.166}$$

$$= \frac{1}{4}\lambda L^2 \tag{15.167}$$

$$\Delta E_{2,2}^{(1)} = \langle 2, 2^{(0)}|\hat{H}_1|2, 2^{(0)}\rangle \tag{15.168}$$

$$= \frac{1}{4}\lambda L^2 \tag{15.169}$$

and for the degenerated levels $E_{1,2}/E_{2,1}$ we get

$$H = \begin{pmatrix} \langle 1, 2^{(0)}|\hat{H}_1|1, 2^{(0)}\rangle & \langle 1, 2^{(0)}|\hat{H}_1|2, 1^{(0)}\rangle \\ \langle 2, 1^{(0)}|\hat{H}_1|1, 2^{(0)}\rangle & \langle 2, 1^{(0)}|\hat{H}_1|2, 1^{(0)}\rangle \end{pmatrix} \tag{15.170}$$

with

$$H_{aa} = \langle 1, 2^{(0)}|\hat{H}_1|1, 2^{(0)}\rangle = \frac{\lambda L^2}{4} \tag{15.171}$$

$$H_{ab} = \langle 1, 2^{(0)}|\hat{H}_1|2, 1^{(0)}\rangle = \frac{256\lambda L^2}{81\pi^4} \tag{15.172}$$

$$H_{bb} = \langle 2, 1^{(0)}|\hat{H}_1|2, 1^{(0)}\rangle = \frac{\lambda L^2}{4} \tag{15.173}$$

and $\hat{H}_1 = \lambda xy$ Diagonalising the matrix H gives the perturbation

$$\Delta E^{(1)}_{12,21} = \frac{\lambda L^2}{4} - \frac{256\lambda L^2}{81\pi^4} \tag{15.174}$$

$$\Delta E^{(1)}_{12,21} = \frac{\lambda L^2}{4} + \frac{256\lambda L^2}{81\pi^4} \tag{15.175}$$

$$\tag{15.176}$$

15.6.6 5.8 - Quadratically perturbed harmonic oscillator

$$\hat{H}_1 = \epsilon\frac{1}{2}m\omega^2 x^2 \tag{15.177}$$

$$H_0(x) = 1 \tag{15.178}$$

$$H_2(x) = 4x^2 - 2 \quad \rightarrow \quad x^2 = \frac{H_2}{4} + \frac{1}{2} \tag{15.179}$$

15.6.7 8.1 - Natural units

1. Proton Mass

$$E_p = m_p c^2/e = 0.937\text{GeV} \tag{15.180}$$

2. With $\Delta p \cdot \Delta x \geq \hbar/2$ and $E = \sqrt{m^2c^4 + p^2c^2} \approx pc$

$$E = \Delta pc/e = 98.6\text{MeV} \tag{15.181}$$

Alternatively we have $E = \frac{\hbar c}{e \cdot dx}$ meaning $1\text{fm} = \frac{1}{197.3\text{MeV}}$ and therefore

$$E = \frac{\hbar}{2 \cdot \Delta x}c = 197.3/2\text{MeV} \tag{15.182}$$

3. Solving for α, β, γ

$$M_P = G^\alpha c^\beta \hbar^\gamma \tag{15.183}$$

$$= \left(\frac{\text{Nm}^2}{\text{kg}^2}\right)^\alpha \left(\frac{\text{m}}{\text{s}}\right)^\beta (\text{Js})^\gamma \tag{15.184}$$

$$= \sqrt{\frac{\hbar c}{G}} \tag{15.185}$$

$$E_P = \sqrt{\frac{\hbar c}{G}}c^2\frac{1}{e} = 1.22 \cdot 10^{19}\text{GeV} \tag{15.186}$$

15.6.8 8.2 - Minkowski Metric

The definition implies that $\eta_{\lambda\nu}$ is the inverse of $\eta^{\lambda\nu}$ - simple calculation shows that they are identical. Now we can calculate

$$\eta^{\mu\lambda}\eta^{\nu\sigma}\eta_{\lambda\sigma} = \eta^{\nu\sigma}\delta^{\mu}_{\sigma} \quad (15.187)$$
$$= \eta^{\nu\mu} \quad (15.188)$$

and

$$a^{\mu}b_{\mu} = a_{\alpha}\eta^{\alpha\mu}b^{\beta}\eta_{\beta\mu} = a_{\alpha}b^{\beta}\delta^{\alpha}_{\beta} = a_{\alpha}b^{\alpha} \quad (15.189)$$

15.7 Bethe, Jackiw - Intermediate Quantum Mechanics

15.7.1 1.1 - Atomic units

Set $\hbar = e = m_e = 1$ and $a_B = \frac{4\pi\varepsilon_0\hbar^2}{m_e e^2} = 1$ then $4\pi\varepsilon_0 = 1$ and therefore $\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} = 1/c$

1. energy: $E_{1s} = \frac{1}{2}m_e c^2\alpha^2$ therefore 1 a.u. $= 2 \times 13.6$eV
2. momentum: $p = m_e c$ therefore 1 a.u. $= 2 \cdot 10^{-31}\text{kg} \times 3 \cdot 10^{8}\text{m/s}^2 = 2.73 \cdot 10^{-22}$J
3. angular momentum: $L = \hbar$ therefore 1 a.u. $= 1.04 \cdot 10^{-34}$Js

15.7.2 1.7 - Hydrogen atom with finite nucleus

The field of a uniform sphere of charge Q can be found by Gauss law

$$E_r = \frac{1}{4\pi\epsilon_0} \cdot \begin{cases} Q/a^3 \cdot r & r < R \\ Q/r^2 & r > R \end{cases} \quad (15.190)$$

The potential is then given by

$$\phi = \frac{1}{4\pi\epsilon_0} \cdot \begin{cases} Q/2R\left(3 - \frac{r^2}{R^2}\right) & r < R \\ Q/r & r > R \end{cases} \quad (15.191)$$

Treating this as a perturbation problem the energy shift can be calculated via the perturbation Hamiltonian (switching the electrostatic energy within the finite nucleus)

$$H_1 = (q\phi_{\text{finite}} - q\phi_{\text{point}})\theta(R - a) \quad (15.192)$$
$$= -e\left(\phi_{\text{finite}} - \phi_{\text{point}}\right)\theta(R - a) \quad (15.193)$$
$$= -\frac{e}{4\pi\epsilon_0}\left(\frac{Ze}{2R}\left[3 - \frac{r^2}{R^2}\right] - \frac{Ze}{r}\right)\theta(R - a) \quad (15.194)$$
$$= -\frac{Ze^2}{4\pi\epsilon_0}\left(\frac{1}{2R}\left[3 - \frac{r^2}{R^2}\right] - \frac{1}{r}\right)\theta(R - a) \quad (15.195)$$
$$(15.196)$$

15.7.3 1.9 - Exponential potential

The Schroedinger equation is given by

$$-\frac{1}{2}\triangle_r\psi + V\psi = E\psi \quad (15.197)$$
$$-\frac{1}{2r^2}\partial_r(r^2\partial_r\psi) - \frac{a^2}{8}e^{-r/2r_0}\psi = E\psi \quad (15.198)$$
$$-\frac{1}{2}\left(\frac{2}{r}\psi' + \psi''\right) - \frac{a^2}{8}e^{-r/2r_0}\psi = E\psi \quad (15.199)$$

Ansatz $\psi(r) = u(r)/r$

$$-\frac{1}{2}\left(\frac{2}{r}\frac{u'r-u}{r^2}+\frac{(u''r+u'-u')r^2-2r(u'r-u)}{r^4}\right)-\frac{a^2}{8}e^{-r/2r_0}\frac{u}{r}=E\frac{u}{r} \tag{15.200}$$

$$-\frac{u'}{r^2}+\frac{u}{r^3}-\frac{u''}{2r}+\frac{2u'}{r^2}-\frac{u}{r^3}-\frac{a^2}{8}e^{-r/2r_0}\frac{u}{r}=E\frac{u}{r} \tag{15.201}$$

$$-\frac{u''}{2r}+\frac{u'}{r^2}-\frac{a^2}{8}e^{-r/2r_0}\frac{u}{r}=E\frac{u}{r} \tag{15.202}$$

$$\tag{15.203}$$

Stepwise calculation for the verification of the solution

$$r^2\partial_r\psi = u'r-u \tag{15.204}$$

$$=\frac{1}{2}\left[J_{n-1}(.)-J_{n+1}(.)\right]ar_0e^{-\frac{r}{2r_0}}\frac{-1}{2r_0}r-J_n(.) \tag{15.205}$$

$$=-\frac{1}{4}\left[J_{n-1}(.)-J_{n+1}(.)\right]are^{-\frac{r}{2r_0}}-J_n(.) \tag{15.206}$$

$$=-\frac{1}{4}\left[J_{n-1}(.)-\left(\frac{2n}{ar_0e^{-r/2r_0}}J_n(.)-J_{n-1}(.)\right)\right]are^{-\frac{r}{2r_0}}-J_n(.) \tag{15.207}$$

$$=-\frac{1}{4}\left[2J_{n-1}(.)-\frac{2n}{ar_0e^{-r/2r_0}}J_n(.)\right]are^{-\frac{r}{2r_0}}-J_n(.) \tag{15.208}$$

$$=-\frac{1}{2}J_{n-1}(.)are^{-\frac{r}{2r_0}}+\left(\frac{nr}{2r_0}-1\right)J_n(.) \tag{15.209}$$

$$\frac{1}{r^2}\partial_r(r^2\partial_r\psi)=-\frac{1}{2}\left(J_{n-1}-J_{n+1}\right)a^2\frac{r_0}{r}-\frac{1}{2}J_{n-1}(.)\frac{a}{r^2}e^{-\frac{r}{2r_0}}-\frac{1}{2}J_{n-1}(.)\frac{-a}{2rr_0}e^{-\frac{r}{2r_0}} \tag{15.210}$$

$$+\frac{n}{2r_0r^2}J_n(.)+\left(\frac{nr}{2r_0}-1\right)\frac{1}{2r^2}(J_{n-1}(.)-J_{n+1}(.))ar_0e^{-\frac{r}{2r_0}}\frac{-1}{2r_0} \tag{15.211}$$

$$=\left(J_{n-1}-J_{n+1}\right)\left[-\frac{a^2r_0}{2r}-\left(\frac{nr}{2r_0}-1\right)\frac{ar_0}{4r^2r_0}\right]e^{-\frac{r}{2r_0}} \tag{15.212}$$

Chapter 16

General Physics

16.1 Feynman - Feynman Lectures on Physics

16.1.1 Section G1-1 - 1961 Sep 28 (1.16)

16.1.2 Section G1-2 - 1961 Sep 28 (1.15)

(a) We use the Penman equation to estimate the specific evaporation rate

$$\frac{dm}{dAdt} = \frac{mR_n + \rho_{\text{air}} c_p (\delta e) g_a}{\lambda_v (m+\gamma)} \tag{16.1}$$

$$= \frac{mR_n + \rho_{\text{air}} c_p (\delta e) g_a}{\lambda_v (m + \frac{c_p p}{\lambda_v MW_{\text{ratio}}})} \tag{16.2}$$

$$\approx \frac{mR_n}{\lambda_v (m + \frac{c_p p}{\lambda_v MW_{\text{ratio}}})}. \tag{16.3}$$

The total time is then given by

$$t = \frac{M}{\frac{dm}{dAdt} A} \tag{16.4}$$

$$= \frac{M}{\frac{dm}{dAdt} \pi r^2} \tag{16.5}$$

$$= \frac{M \lambda_v (m + \frac{c_p p}{\lambda_v MW_{\text{ratio}}})}{\pi r^2 m R_n} \tag{16.6}$$

with vapor the water vapor pressure

$$p_{\text{vap}} = \frac{101325\text{Pa}}{760} \exp\left[20.386 - \frac{5132K}{T}\right] \tag{16.7}$$

the slope of the saturation vapor pressure

$$m = \frac{\partial p_{\text{vap}}}{\partial T} = ... \tag{16.8}$$

the air heat capacity $c_p = 1.012\text{Jkg}^{-1}\text{K}^{-1}$, the latent heat of vaporization $\lambda_v = 2.26 \cdot 10^6\text{Jkg}^{-1}$, the net irradiance $R_n = 150\text{Wm}^{-2}$ (average day/night partly shade), the ratio molecular weight of water vapor/dry air $MW_{\text{ratio}} = 0.622$, the pressure $p = 10^5\text{Pa}$, the temperature $T = 298\text{K}$, the water weight $M = 0.5\text{kg}$ and the radius of the glass $r = 0.04\text{m}$. This results in $t = 26$ days.

(b) With the molar mass of water $m_{H2O} = 18\,\mathrm{g}\cdot\mathrm{mol}^{-1}$

$$N = \frac{dm}{dAdt}\frac{N_A}{m_{H20}} \tag{16.9}$$
$$= \frac{mR_n}{\lambda_v(m + \frac{c_p p}{\lambda_v MW_{\mathrm{ratio}}})}\frac{N_A}{m_{H20}} \tag{16.10}$$
$$= 1.47\cdot 10^{17}\mathrm{cm}^{-1}\mathrm{s}^{-1} \tag{16.11}$$

(c) The total mass of water vaporizing on earth in one year is

$$M_{\mathrm{1y\ prec}} = \varepsilon_{\mathrm{ocean}} 4\pi R_E^2 \frac{dm}{dAdt} t_{1y}. \tag{16.12}$$

with $\varepsilon_{\mathrm{ocean}} = 0.7$. In equilibrium this must be equal to the total amount of precipitation. So the average rainfall height is

$$h = \frac{M_{\mathrm{1y\ prec}}}{4\pi R_E^2 \rho_{\mathrm{H2O}}} \tag{16.13}$$
$$= \frac{\varepsilon_{\mathrm{ocean}} t_{1y}}{\rho_{\mathrm{H2O}}}\frac{dm}{dAdt} \tag{16.14}$$
$$= 947\mathrm{mm}. \tag{16.15}$$

which seems reasonable (given that the solar constant is $1,361\mathrm{Wm}^{-2}$ the estimate of $R_n = 150\mathrm{Wm}^{-2}$ seems ok).

16.1.3 Section G-1 - 1961 Oct 5 (?.??)

(a) $\sqrt{s/g}$

(b) mL/T^2

(c) $\rho g h$

(d) $\sqrt{p/\rho}$

(e) gT (need to use the period T as c is not a material constant due to strong dispersion)

(f) $\rho g H^2$

(g) $\sqrt{R/g}$ here we assume the hemisphere rests on the table upside down - so it acts like a pendulum

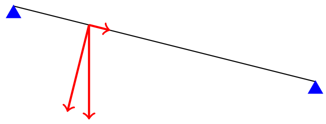
(h) $\sqrt{FL/m}$

16.1.4 Section G-2 - 1961 Oct 5 (?.??)

1. Equilibrium is given by condition

$$m_1 g = m_2 g \sin\alpha \tag{16.16}$$
$$= m_2 g \frac{x}{\sqrt{x^2+a^2}} \tag{16.17}$$
$$\rightarrow \quad m_1^2(x^2+a^2) = m_2^2 x^2 \tag{16.18}$$
$$\rightarrow \quad x = \frac{m_1 a}{\sqrt{m_2^2 - m_1^2}} \tag{16.19}$$
$$\tag{16.20}$$

2. General consideration



3.

4.

5.

16.1.5 Problem Set 3-1 - 1961 Nov 03 (3.16)

Direct measurement can be done for the

- radius of the earth $R_e = 6371\text{km}$
- orbital period of the moon $T_M = 28\text{d}$
- angular diameter of the moon $\delta = 30' = 0.5°$
- earths gravitational acceleration $g = 9.81\text{ms}^2$
- also Sputnik I orbital data can be looked up $a_{\text{satellite}} = R_E + 584\text{km}$ and $T_{\text{satellite}} = 96.2\text{min}$
- height difference between low and high tide $\Delta h = 1\text{m}$

1. We use Keplers 3rd law

$$\frac{a_M^3}{T_M^2} = \frac{a_{\text{satellite}}^3}{T_{\text{satellite}}^2} \tag{16.21}$$

$$a_M = a_{\text{sat}} \left(\frac{T_M}{T_{\text{sat}}}\right)^{2/3} \tag{16.22}$$

then the radius of the moon is given by

$$R_M = \frac{a_M}{2} \tan\delta = \frac{a_{\text{sat}}}{2} \left(\frac{T_M}{T_{\text{sat}}}\right)^{2/3} \tan\delta \tag{16.23}$$

and the mass by

$$m_M = \rho_M V_M = \frac{4}{3}\pi\rho_M R_M^3 \tag{16.24}$$

$$= \frac{4}{3}\pi\rho_M \left(\frac{a_{\text{sat}}}{2} \left(\frac{T_M}{T_{\text{sat}}}\right)^{2/3} \tan\delta\right)^3 \tag{16.25}$$

$$= \frac{1}{6}\pi\rho_M a_{\text{sat}}^3 \left(\frac{T_M}{T_{\text{sat}}}\right)^2 \tan^3\delta \tag{16.26}$$

$$\approx \frac{1}{6}\pi\rho_E a_{\text{sat}}^3 \left(\frac{T_M}{T_{\text{sat}}}\right)^2 \tan^3\delta \tag{16.27}$$

where we approximated the moon by the earth mass density. From the gravitational law we can obtain the earth density by

$$g = \frac{F_g}{m} = \frac{Gm_E}{R_E^2} \quad \rightarrow \quad m_E = \frac{gR_E^2}{G} \tag{16.28}$$

$$\rho_E = \frac{m_E}{V_E} = \frac{m_E}{\frac{4}{3}\pi R_E^3} = \frac{3g}{4\pi G R_E}. \tag{16.29}$$

Therefore the mass of the moon is given by

$$m_M \approx \frac{g}{8GR_E} a_{\text{sat}}^3 \left(\frac{T_M}{T_{\text{sat}}}\right)^2 \tan^3\delta \tag{16.30}$$

$$= 1.16 \cdot 10^{23}\text{kg}. \tag{16.31}$$

2. We use Keplers 3rd law (for the earth-moon system) and the gravitational law for the earth

$$\frac{a_M^3}{T_M^2} = \frac{G(m_E + m_M)}{4\pi^2} \approx \frac{Gm_E}{4\pi^2} = \frac{a_{\text{satellite}}^3}{T_{\text{satellite}}^2} \tag{16.32}$$

$$g = \frac{F_g}{m} = \frac{Gm_E}{R_E^2} \tag{16.33}$$

and obtain

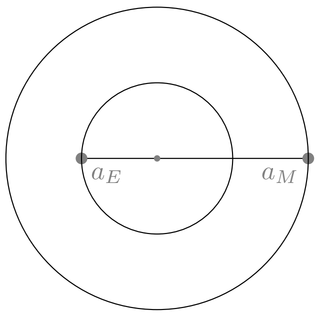
$$\frac{a_{\text{satellite}}^3}{T_{\text{satellite}}^2} = \frac{gR_E^2 + Gm_M}{4\pi^2} \tag{16.34}$$

$$m_M = \frac{4\pi^2}{G}\left(\frac{a_{\text{satellite}}^3}{T_{\text{satellite}}^2} - \frac{gR_E^2}{4\pi^2}\right) \tag{16.35}$$

$$= 7.07 \cdot 10^{21}\text{kg}. \tag{16.36}$$

This result is quite sensitive to the satellite orbital data.

3. We will use the earth tidal data. Lets assume circular orbits with $a_E + a_M = D$ which we can justify by observation (as the moon appears to have constant angular diameter). As reference system we use the center of mass of the system



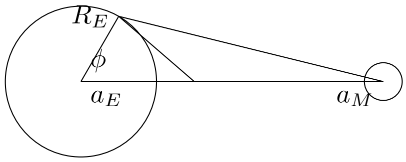
$$m_E\omega^2 a_E = \frac{Gm_E m_M}{D^2} = m_M\omega^2 a_M \tag{16.37}$$

$$\rightarrow a_E = \frac{m_M D}{m_E + m_M} \tag{16.38}$$

$$\rightarrow \omega^2 = \frac{G(m_E + m_M)}{D^3} \tag{16.39}$$

$$\rightarrow \omega^2 a_E^2 = \frac{Gm_M^2}{D(m_E + m_M)} \tag{16.40}$$

$$\rightarrow \frac{R_E}{a_E} = \frac{m_E + m_M}{m_M}\frac{R_E}{D} \tag{16.41}$$



The potential is then given by (gravity of moon and earth as well as the centripetal potential around the center of gravity)

$$
\begin{aligned}
V &= V_{\mathrm{G,moon}} + V_{\mathrm{G,earth}} + V_{\mathrm{cent}} && (16.42)\\
&= -\frac{Gm_M}{\sqrt{R_E^2 + D^2 - 2DR_E\cos\phi}} - \frac{Gm_E}{R_E} - \frac{1}{2}\omega^2(R_E^2 + a_E^2 - 2a_E R_E\cos\phi) && (16.43)\\
&= -\frac{Gm_M}{D\sqrt{\left(\frac{R_E}{D}\right)^2 + 1 - 2\frac{R_E}{D}\cos\phi}} - \frac{Gm_E}{R_E} - \frac{1}{2}\omega^2 a_E^2\left[\left(\frac{R_E}{a_E}\right)^2 + 1 - 2\frac{R_E}{a_E}\cos\phi\right] && (16.44)\\
&\approx -\frac{Gm_M}{D}\left(1 + \frac{R_E}{D}\cos\phi + \frac{3}{2}\left(\frac{R_E}{D}\right)^2\cos^2\phi\right) - \frac{Gm_E}{R_E} && (16.45)\\
&\qquad - \frac{1}{2}\omega^2 a_E^2\left[\left(\frac{R_E}{a_E}\right)^2 + 1 - 2\frac{R_E}{a_E}\cos\phi\right] && (16.46)\\
&\approx -\frac{Gm_M}{D}\left(1 + \frac{R_E}{D}\cos\phi + \frac{3}{2}\left(\frac{R_E}{D}\right)^2\cos^2\phi\right) - \frac{Gm_E}{R_E} && (16.47)\\
&\qquad - \frac{1}{2}\frac{Gm_M^2}{D(m_E+m_M)}\left[\left(\frac{m_E+m_M}{m_M}\frac{R_E}{D}\right)^2 + 1 - 2\frac{m_E+m_M}{m_M}\frac{R_E}{D}\cos\phi\right] && (16.48)\\
&\approx -\frac{Gm_M}{D} - \frac{3Gm_M}{2D}\frac{R_E^2}{D^2}\cos^2\phi - \frac{Gm_E}{R_E} && (16.49)\\
&\qquad - \frac{1}{2}\frac{Gm_M^2}{D(m_E+m_M)}\left[\left(\frac{m_E+m_M}{m_M}\frac{R_E}{D}\right)^2 + \frac{m_M^2 D^2}{m_M^2 D^2}\right] && (16.50)\\
&\approx -\frac{Gm_M}{D} - \frac{3Gm_M}{2D}\frac{R_E^2}{D^2}\cos^2\phi - \frac{Gm_E}{R_E} - \frac{G}{2}\left[(m_E+m_M)\frac{R_E^2}{D^3} + \frac{m_M^2}{m_E+m_M}\frac{1}{D}\right]. && (16.51)
\end{aligned}
$$

with the angular dependent tidal part

$$V_{\mathrm{tidal}} = -\frac{3GR_E^2 m_M}{2D^3}\cos^2\phi. \tag{16.52}$$

The tidal water surface would be formed by the the surface $r_{\mathrm{surf}}(\phi) = R_E + h$ of constant potential. The height difference between low and high tide can then be estimated by

$$
\begin{aligned}
-\frac{3GR_E^2 m_M}{2D^3} &= Gm_E\left(\frac{1}{R_E+h} - \frac{1}{R_E}\right) && (16.53)\\
&\approx Gm_E\left(\frac{1}{R_E\left(1+\frac{h}{R_E}\right)} - \frac{1}{R_E}\right) && (16.54)\\
&\approx \frac{Gm_E}{R_E}\left(\left(1 - \frac{h}{R_E}\right) - 1\right) && (16.55)
\end{aligned}
$$

which gives

$$h = \frac{3R_E^4}{2D^3}\frac{m_M}{m_E}. \tag{16.56}$$

Using the results from above

$$m_E = \frac{gR_E^2}{G} \tag{16.57}$$
$$\omega^2 = \frac{G(m_E + m_M)}{D^3} \tag{16.58}$$
$$\rightarrow \quad D^3 = \frac{G(m_E + m_M)}{\omega^2} = G(m_E + m_M)\frac{T_M^2}{4\pi^2} \tag{16.59}$$

we obtain

$$h = \frac{6\pi^2 R_E^4 T_M^2}{G(m_E + m_M)T_M^2}\frac{m_M}{m_E}. \tag{16.60}$$

and can subsequently solve for m_M

$$m_M = \frac{Ghm_E^2 T^2}{6\pi^2 R_E^4 - Ghm_E T^2} \tag{16.61}$$
$$= \frac{m_E}{\frac{6\pi^2 R_E^4}{Ghm_E T^2} - 1} \tag{16.62}$$
$$= \frac{g^2 h T_M^2 R_E^2}{G(6\pi^2 R_E^2 - ghT_M^2)} \tag{16.63}$$
$$= \frac{gR_E^2}{G\left(\frac{6\pi^2 R_E^2}{ghT_M^2} - 1\right)} \tag{16.64}$$
$$= 1.38 \cdot 10^{23}\text{kg} \tag{16.65}$$

16.1.6 Problem Set 3-3 - 1961 Nov 03 (3.10)

(a) We use Keplers 3rd law for the earth

$$\frac{a_E^3}{T_E^2} = \frac{G(m_S + m_E)}{4\pi^2} \approx \frac{Gm_S}{4\pi^2} \tag{16.66}$$
$$\tag{16.67}$$

and the stars a and b

$$\frac{a^3}{T^2} = \frac{G(m_A + m_B)}{4\pi^2} \tag{16.68}$$
$$\frac{(Ra_E)^3}{(TT_E)^2} = \frac{R^3}{T^2}\frac{a_E^3}{T_E^2} = \frac{R^3}{T^2}\frac{Gm_S}{4\pi^2} = \frac{G(m_A + m_B)}{4\pi^2} \tag{16.69}$$
$$\rightarrow m_A + m_B = \frac{R^3}{T^2}m_S = \frac{729}{25}m_S \tag{16.70}$$

(b) For a the circular orbits we have the stability condition

$$m_A\omega^2 r_A = F_{AB} = m_B\omega^2 r_B \tag{16.71}$$
$$\rightarrow m_A\omega v_A = m_B\omega v_B \tag{16.72}$$
$$\rightarrow \frac{m_A}{m_B} = \frac{v_B}{v_A} = \frac{1}{5} \tag{16.73}$$

with $m_B = 5m_A$ we have

$$m_A = \frac{243}{50}m_S \tag{16.74}$$
$$m_B = \frac{243}{10}m_S. \tag{16.75}$$

16.1.7 Book (2.22)

The center of the spheres build a tetrahedron where each connection has to carry a third of the

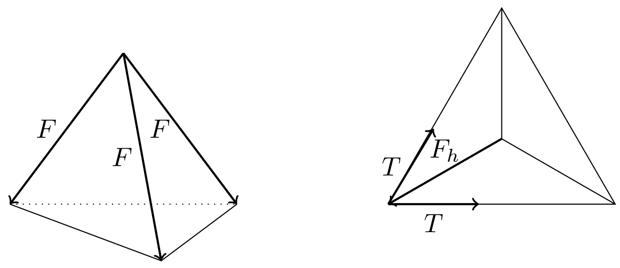


Figure 16.1: Problem (2.22)

weight mg

$$F = \frac{mg}{3\cos\alpha} \tag{16.76}$$

where α is the angle of the edge

$$\cos\alpha = \frac{H}{a} = \frac{\sqrt{a^2 - \left(\frac{2}{3}h\right)^2}}{a} = \frac{\sqrt{a^2 - \left(\frac{2}{3}\frac{\sqrt{3}}{2}a\right)^2}}{a} = \sqrt{2/3}. \tag{16.77}$$

The horizontal projection is then

$$F_h = F\sin\alpha = \frac{mg}{3}\tan\alpha. \tag{16.78}$$

Projecting them in the plane gives

$$F_h = \sqrt{T^2 + T^2 - 2T^2\cos\frac{2\pi}{3}} \tag{16.79}$$

$$\rightarrow T = \frac{mg}{3\sqrt{6}} \tag{16.80}$$

including the safety margin we obtain

$$\widetilde{T} = 3T = \frac{mg}{\sqrt{6}} = 2\text{ton-wt} \tag{16.81}$$

16.1.8 Problem Set 3-4 - 1961 Nov 03 (?.??)

$$g_M = \frac{GM_M}{R_M^2} = \frac{4}{3}G\rho_M R_M = \frac{4}{3}G(0.537\rho_E)(0.716R_E) = 0.384 \cdot g_E \tag{16.82}$$

16.1.9 Problem Set 1a 4-1 - 1961 Nov 10 (11.16)

For the masses we obtain

$$\widetilde{m}_i = \rho_i V_i = \frac{4}{3}\pi(kR_i)^3\rho_i \tag{16.83}$$

$$= k^3 m_i \tag{16.84}$$

The third Kepler law is given by

$$T^2 = a^3\frac{4\pi}{G(M+m)} \tag{16.85}$$

applying the scaling to a we have $\widetilde{a}^3 = (ka)^3$ and therefore

$$\begin{aligned}\widetilde{T}^2 &= \widetilde{a}^3 \frac{4\pi}{G(\widetilde{M} + \widetilde{m})} && (16.86)\\ &= k^3 a^3 \frac{4\pi}{G(k^3 M + k^3 m)} && (16.87)\\ &= a^3 \frac{4\pi}{G(M + m)} && (16.88)\\ &= T^2. && (16.89)\end{aligned}$$

So we conclude that there is no change in T.

16.1.10 Problem Set 1a 4-3 - 1961 Nov 10 (??.??)

With $\vec{a} = (1, 0, 2)$ and $\vec{b} = (1, 4, 0)$

$$\cos\alpha = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|} = 1/\sqrt{85} \tag{16.90}$$

16.1.11 Problem Set 1a 4-6 - 1961 Nov 10 (??.??)

The 3, 4, 5 triangle is rectangular with and incline angle α

$$\begin{aligned}\sin\alpha &= 3/5 && (16.91)\\ \cos\alpha &= 4/5 && (16.92)\end{aligned}$$

and therefore

$$\begin{aligned}F_{A,\|} &= M_A g(\sin\alpha - \mu_A \cos\alpha) && (16.93)\\ F_{B,\|} &= M_B g(\sin\alpha - \mu_B \cos\alpha) && (16.94)\end{aligned}$$

(a) Then

$$\begin{aligned}a &= \frac{F_{A,\|} + F_{B,\|}}{M_A + M_B} && (16.95)\\ &= \frac{M_A(\sin\alpha - \mu_A\cos\alpha) + M_B(\sin\alpha - \mu_B\cos\alpha)}{M_A + M_B} g && (16.96)\\ &= \frac{(M_A + M_B)\sin\alpha - (M_A\mu_A + M_B\mu_B)\cos\alpha}{M_A + M_B} g && (16.97)\\ &= \left(\sin\alpha - \frac{M_A\mu_A + M_B\mu_B}{M_A + M_B}\cos\alpha\right) g && (16.98)\\ &= 4.84\text{m/s}^2 && (16.99)\end{aligned}$$

(b) Newton 3

$$\begin{aligned}F_{A,\|} - M_A a &= T && (16.100)\\ F_{B,\|} - M_B a &= -T && (16.101)\end{aligned}$$

then

$$\begin{aligned}2T &= (F_{A,\|} - F_{B,\|}) - (M_A - M_B)a && (16.102)\\ T &= \frac{M_A M_B}{M_A + M_B}(\mu_B - \mu_A) g\cos\alpha && (16.103)\\ &= 2.09\text{N} && (16.104)\end{aligned}$$

16.1.12 Problem Set 1b 3-3 - 1962 Jan 16 (??.??)

1.

2. For the angular momentum (without precision) we get

$$L = J\omega = \frac{1}{2}MR^2\omega \tag{16.105}$$

3. With a little geometry we see

$$\frac{d\vec{L}}{dt} = \vec{a} \times \vec{M} = M\vec{a} \times \vec{g} \tag{16.106}$$

$$\frac{dL}{L} = \sin d\phi \approx d\phi \tag{16.107}$$

$$\to \Omega_1 = \frac{d\phi}{dt} = \frac{Mag}{L} = \frac{2ag}{R^2\omega} \tag{16.108}$$

$$\to \omega = \frac{2ag}{R^2\Omega_1} \tag{16.109}$$

4.

16.1.13 Problem Set 1b 11-1 - 1962 Feb 16 (20.11)

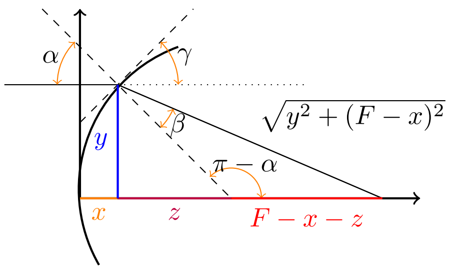


Figure 16.2: Problem (20.11)

We start with Snell's law

$$\sin\alpha = n \sin\beta \tag{16.110}$$

and the sine law

$$\frac{\sin(\pi - \alpha)}{\sqrt{y^2 + (F-x)^2}} = \frac{\sin\beta}{F - x - z} \tag{16.111}$$

we also see

$$\frac{y}{z} = \tan\alpha \tag{16.112}$$

most importantly we have for the slope of the surface

$$\frac{dy}{dx} = \tan\gamma \tag{16.113}$$

$$= \tan(\pi/2 - \alpha) = \cot\alpha = \frac{1}{\tan\alpha} \tag{16.114}$$

Now we can put it all together

$$\frac{\sin(\alpha)}{\sqrt{y^2+(F-x)^2}} = \frac{\sin\alpha}{n(F-x-\frac{y}{\tan\alpha})} \tag{16.115}$$
$$\frac{1}{\sqrt{y^2+(F-x)^2}} = \frac{1}{n[(F-x)-y\frac{dy}{dx}]} \tag{16.116}$$
$$(F-x)-y\frac{dy}{dx} = \frac{1}{n}\sqrt{y^2+(F-x)^2} \tag{16.117}$$

The ODE can be solved by Mathematica which gives two solutions

$$y_1 = \pm\sqrt{2Fx\left(1-\frac{1}{n}\right)-\left(1-\frac{1}{n^2}\right)x^2} \tag{16.118}$$
$$y_2 = \pm\sqrt{2Fx\left(1+\frac{1}{n}\right)-\left(1-\frac{1}{n^2}\right)x^2} \tag{16.119}$$

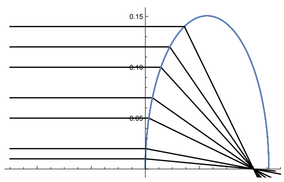


Figure 16.3: Light rays for solution y_1 of Feynman problem (20.11)

16.1.14 Book (20.14)

Using the sine law for the appropriate triangle inside the sphere (cosine law to calculate the length of one side) we obtain

$$\frac{\sin\beta}{R} = \frac{\sin\alpha}{\sqrt{R^2+R^2-2R\cdot R\cos\alpha}} \tag{16.120}$$
$$\frac{\sin\alpha}{nR} = \frac{\sin\alpha}{\sqrt{2}R\sqrt{1-\cdot R\cos\alpha}} \tag{16.121}$$

where we used Snells law. Simplifying further

$$\cos\alpha = 1-\frac{n^2}{2} \tag{16.122}$$
$$\sqrt{1-\sin^2\alpha} = 1-\frac{n^2}{2} \tag{16.123}$$
$$\sin^2\alpha = 1-\left(1-\frac{n^2}{2}\right)^2 \tag{16.124}$$
$$\frac{y^2}{4R^2} = 1-\left(1-\frac{n^2}{2}\right)^2 \tag{16.125}$$
$$\to y = 2nR\sqrt{1-\frac{n^2}{4}} = 1.92R \tag{16.126}$$

16.1.15 Book (20.16)

First lens

$$\frac{1}{f} = \frac{1}{g} + \frac{1}{b} \quad \rightarrow \quad b = \frac{gF}{g-F} \tag{16.127}$$

$$\frac{B}{G} = \frac{b}{g} \quad \rightarrow \quad B = G\frac{b}{g} = \frac{G}{g}\frac{gF}{g-F} \tag{16.128}$$

Distant object means

$$B \simeq \frac{GF}{g} \tag{16.129}$$

The second lens works as a magnifying glass - focussing at infinity means the virtual picture is at infinity and therefore the object (real picture of the first lens) needs to be at the focus of the second lens. The angle is then given by

$$\tan\alpha' = \frac{B}{f} = \frac{GF}{gf} \tag{16.130}$$

without lenses the angle would have been

$$\tan\alpha = \frac{G}{g}. \tag{16.131}$$

Then

$$M = \tan\alpha' / \tan\alpha = F/f. \tag{16.132}$$

16.1.16 Problem Set 1c 13-1 - 1962 May 25 (?.??)

We cheat a little and use the Lagrange formalism

$$L = T - V \tag{16.133}$$

$$= \frac{m_1}{2}\dot{x}^2 + \frac{m_2}{2}\dot{y}^2 - \frac{k_1}{2}x^2 - \frac{k_2}{2}y^2 - \frac{k}{2}(x-y)^2 \tag{16.134}$$

then

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0 \tag{16.135}$$

gives

$$m_1\ddot{x} + k_1 x + k(x-y) = 0 \quad \rightarrow \quad \ddot{x} + \omega_0^2 x + \frac{k}{m_1}(x-y) = 0 \tag{16.136}$$

$$m_2\ddot{y} + k_2 y - k(x-y) = 0 \quad \rightarrow \quad \ddot{y} + \omega_0^2 y + \frac{k}{m_2}(y-x) = 0 \tag{16.137}$$

16.1.17 Problem Set 1c 13-2 - 1962 May 25 (?.??)

We obtain

$$-A\omega^2 + A\omega_0^2 + \frac{k}{m_1}(A-B) = 0 \tag{16.138}$$

$$-B\omega^2 + B\omega_0^2 + \frac{k}{m_2}(B-A) = 0 \tag{16.139}$$

and therefore

$$\omega^2 = \omega_0^2 + \frac{k}{m_1}(1 - B/A) \tag{16.140}$$

$$\omega^2 = \omega_0^2 + \frac{k}{m_2}(1 - A/B) \tag{16.141}$$

both expressions give the same valuers for ω if

$$A/B = 1 \quad \rightarrow \quad \omega = \omega_0 \tag{16.142}$$

$$A/B = -m_2/m_1 \quad \rightarrow \quad \omega = \sqrt{\omega_0 + \frac{k}{m_1} + \frac{k}{m_2}} \tag{16.143}$$

16.2 Jackson - Classical Electrodynamics

16.2.1 Exercise 1.3 Charge densities and the Dirac delta function

$$\rho_a = \frac{Q}{4\pi R^2}\delta(r - R) \quad \rightarrow \quad \int \rho_a d^3r = 4\pi \frac{Q}{4\pi R^2} \int_0^\infty \delta(r - R) r^2 \, dr \tag{16.144}$$

$$= Q \tag{16.145}$$

$$\rho_b = \frac{\lambda}{2\pi b}\delta(r - b) \quad \rightarrow \quad \int \rho_b d^3r = \frac{\lambda}{2\pi b} 2\pi \int_0^L dz \int_0^\infty \delta(r - b) r \, dr \tag{16.146}$$

$$= \lambda L \tag{16.147}$$

$$\rho_c = \frac{Q}{\pi R^2}\theta(R - r)\delta(z) \quad \rightarrow \quad \int \rho_c d^3r = \frac{Q}{\pi R^2} 2\pi \int dz \int_0^\infty \theta(r - R) r \; dr \tag{16.148}$$

$$= \frac{Q}{\pi R^2} 2\pi \int dz \int_0^R r \; dr \tag{16.149}$$

$$= \frac{Q}{\pi R^2} 2\pi \frac{R^2}{2} = Q \tag{16.150}$$

Now we got curvilinear coordinates so we need an additional $1/r$ scaling

$$\rho_d = \frac{Q}{\pi R^2 r}\theta(R - r)\delta(\vartheta - \pi/2) \quad \rightarrow \quad \int \rho_d d^3r = \frac{Q}{\pi R^2} 2\pi \int_0^\infty \frac{r^2}{r}\theta(R - r) \int_0^\pi \delta(\vartheta - \pi/2) \sin\vartheta \; d\vartheta \tag{16.151}$$

$$= \frac{Q}{\pi R^2} 2\pi \int_0^R r \int_0^\pi \delta(\vartheta - \pi/2) \sin\vartheta \; d\vartheta \tag{16.152}$$

$$= \frac{Q}{\pi R^2} 2\pi \frac{R^2}{2} \sin\pi/2 = Q \tag{16.153}$$

16.2.2 Exercise 1.4 Charged spheres

We can utilize the Gauss theorem

$$\oint_S \vec{E} \cdot \vec{n} dA = \frac{1}{\epsilon_0} \int_V \rho(x) d^3x \tag{16.154}$$

$$4\pi r^2 E_r = \frac{q_r}{\epsilon_0} \tag{16.155}$$

$$E_r = \frac{q_r}{4\pi\epsilon_0 r^2} \tag{16.156}$$

assuming a radial electrical field.

- Conducting sphere

$$\rho_{\text{cond}} = Q\delta(r-a) \tag{16.157}$$
$$E_r = \frac{1}{4\pi\epsilon_0} \cdot \begin{cases} 0 & r<a \\ Q/r^2 & r>a \end{cases} \tag{16.158}$$

- Uniform sphere

$$\rho_{\text{hom}} = Q\theta(a-r) \tag{16.159}$$
$$E_r = \frac{1}{4\pi\epsilon_0} \cdot \begin{cases} Q/a^3 \cdot r & r<a \\ Q/r^2 & r>a \end{cases} \tag{16.160}$$

- Nonuniform sphere

$$\rho_{\text{inhom}} = Q\frac{n+3}{a^{n+3}} r^n \quad (r<a) \tag{16.161}$$
$$E_r = \frac{1}{4\pi\epsilon_0} \cdot \begin{cases} Qa^{n+3}r^{n+1} & r<a \\ Q/r^2 & r>a \end{cases} \tag{16.162}$$

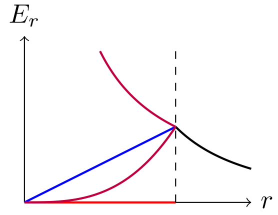


Figure 16.4: Jackson problem (1.4)

16.2.3 Exercise 1.5 Charge density of hydrogen atom

With the potential

$$\Phi = \frac{q}{4\pi\epsilon_0}\frac{e^{-\alpha r}}{r}\left(1+\frac{\alpha r}{2}\right) \tag{16.163}$$

we calculate for $r>0$

$$\begin{aligned}
\rho_1 &= -\epsilon_0 \triangle \Phi && (16.164)\\
&= -\epsilon_0 \frac{1}{r^2}\partial_r(r^2\partial_r\Phi) && (16.165)\\
&= -\frac{q}{4\pi}e^{-\alpha r}\frac{\alpha^3}{2} && (16.166)\\
&= -\frac{q}{\pi a_0^3}e^{-2r/a_0} && (16.167)
\end{aligned}$$

For $r=0$ we have

$$\begin{aligned}
\Phi(r\to 0) &= \frac{q}{4\pi\epsilon_0 r} && (16.168)\\
\to \quad \rho_0 &= q\delta(r) && (16.169)
\end{aligned}$$

Therefore

$$\rho = \rho_0 + \rho_1 \tag{16.170}$$
$$= q\left(\delta^{(3)}(r) - \frac{1}{\pi a_0^3}e^{-2r/a_0}\right) \tag{16.171}$$

Calculating the total charge

$$Q_0 = q\int d^3r\delta(r) = q \tag{16.172}$$
$$Q_1 = 4\pi\int_0^\infty r^2\rho_1 dr \tag{16.173}$$
$$= -\frac{4\pi q}{\pi a_0^3}\int_0^\infty r^2 e^{-2r/a_0}dr \tag{16.174}$$
$$= -\frac{4\pi q}{\pi a_0^3}\frac{a_0^3}{8}\int_0^\infty z^2 e^{-z}dz \tag{16.175}$$
$$= -\frac{4\pi q}{\pi a_0^3}\frac{a_0^3}{8}\Gamma(3) \tag{16.176}$$
$$= -q \tag{16.177}$$

16.2.4 Exercise 1.6 Simple capacitors

(a) Assuming only front and back surfaces contribute

$$2E_x A = \frac{Q}{\epsilon_0} \tag{16.178}$$
$$\to \quad E_x = \frac{Q}{2\epsilon_0 A} \tag{16.179}$$
$$\to \quad \phi = -\frac{Q}{2\epsilon_0 A}x \tag{16.180}$$
$$\to \quad \phi_{\text{tot}}(x) = -\frac{Q}{2\epsilon_0 A}x - \frac{-Q}{2\epsilon_0 A}(d-x) \tag{16.181}$$
$$= -\frac{Q}{2\epsilon_0 A}(x-(d-x)) \tag{16.182}$$
$$= -\frac{Q}{2\epsilon_0 A}(2x-d) \tag{16.183}$$
$$\to \quad C = \frac{Q}{\Delta\phi} = \frac{Q}{-\frac{Q}{2\epsilon_0 A}(-d-d)} \tag{16.184}$$
$$= \epsilon_0\frac{A}{d} \tag{16.185}$$

(b) The outer sphere does not contribute to the total potential as it is field free

$$4\pi r^2 E_r = \frac{Q}{\epsilon_0} \tag{16.186}$$
$$\rightarrow \quad E_r = \frac{Q}{4\pi\epsilon_0 r^2} \tag{16.187}$$
$$\rightarrow \quad \phi = \frac{Q}{4\pi\epsilon_0 r} \tag{16.188}$$
$$\rightarrow \quad \phi_{\text{tot}} = \frac{Q}{4\pi\epsilon_0 r} \quad (a < r < b) \tag{16.189}$$
$$\rightarrow \quad C = \frac{Q}{\Delta\phi} = \frac{Q}{\frac{Q}{4\pi\epsilon_0 b} - \frac{Q}{4\pi\epsilon_0 a}} \tag{16.190}$$
$$= \epsilon_0 \frac{4\pi ab}{b-a} \tag{16.191}$$

(c)

$$2\pi r L E_r = \frac{Q}{\epsilon_0} \tag{16.192}$$
$$\rightarrow \quad E_r = \frac{Q}{2\pi r L \epsilon_0} \tag{16.193}$$
$$\rightarrow \quad \phi = -\frac{Q}{2\pi L\epsilon_0} \log r \tag{16.194}$$
$$\rightarrow \quad \phi_{\text{tot}} = -\frac{Q}{2\pi L\epsilon_0} \log r \quad (a < r < b) \tag{16.195}$$
$$\rightarrow \quad C = \frac{Q}{\Delta\phi} = \frac{Q}{-\frac{Q}{2\pi L\epsilon_0}\log b + \frac{Q}{2\pi L\epsilon_0}\log a} \tag{16.196}$$
$$= \frac{2\pi L\epsilon_0}{\log a/b} \tag{16.197}$$

(d) ...

16.2.5 Exercise 1.7 Capacity of two parallel cylinders

Gauss law for one cylinder

$$\oint_S \vec{E} \cdot \vec{n} dA = \frac{1}{\epsilon_0} \int_V \rho(x) d^3x \tag{16.198}$$
$$2\pi r L E_r = \frac{\rho_1 L}{\epsilon_0} \tag{16.199}$$
$$E_r = \frac{\rho}{2\pi\epsilon_0 r} \tag{16.200}$$
$$\phi = -\frac{\rho}{2\pi\epsilon_0} \ln r \tag{16.201}$$

For $d \gg a_{1,2}$ the potential of one cylinder on the surface of the second cylinder is constant - which means that the potential can be approximated by the sum of the potential of both cylinders (no need to make it complicated)

$$\begin{aligned}
\phi(\vec{r}) &= \phi_1 + \phi_2 && (16.202)\\
&= -\frac{\rho_1}{2\pi\epsilon_0}\ln|\vec{r}| - \frac{\rho_2}{2\pi\epsilon_0}\ln|\vec{r}-\vec{d}| && (16.203)\\
&= -\frac{\rho}{2\pi\epsilon_0}\ln|\vec{r}| + \frac{\rho}{2\pi\epsilon_0}\ln|\vec{r}-\vec{d}| && (16.204)\\
&= -\frac{\rho}{2\pi\epsilon_0}\left(\ln|\vec{r}| - \ln|\vec{r}-\vec{d}|\right) && (16.205)\\
&= -\frac{\rho}{2\pi\epsilon_0}\ln\frac{|\vec{r}|}{|\vec{r}-\vec{d}|} && (16.206)\\
&= -\frac{\rho}{\pi\epsilon_0}\ln\sqrt{\frac{|\vec{r}|}{|\vec{r}-\vec{d}|}} && (16.207)
\end{aligned}$$

Then the potential difference between to surfaces is given by (with $\vec{n} = \vec{d}/d$ and $\rho = \rho_1 = -\rho_2$)

$$\begin{aligned}
\Delta\phi &= \phi(a_1\vec{n}) - \phi((d-a_2)\vec{n}) && (16.208)\\
&= -\frac{\rho}{\pi\epsilon_0}\left(\ln\sqrt{\frac{a_1}{d-a_1}} - \ln\sqrt{\frac{d-a_2}{a_2}}\right) && (16.209)\\
&= \frac{\rho}{\pi\epsilon_0}\left(\ln\sqrt{\frac{d-a_1}{a_1}} + \ln\sqrt{\frac{d-a_2}{a_2}}\right) && (16.210)\\
&\simeq \frac{\rho}{\pi\epsilon_0}\left(\ln\sqrt{\frac{d}{a_1}} + \ln\sqrt{\frac{d}{a_2}}\right) && (16.211)\\
&\simeq \frac{\rho}{\pi\epsilon_0}\ln\frac{d}{\sqrt{a_1 a_2}} && (16.212)
\end{aligned}$$

With $C = Q/U$ we have

$$C = \frac{\rho L}{\Delta\phi} = \frac{\pi\epsilon_0 L}{\ln\frac{d}{\sqrt{a_1 a_2}}} \tag{16.213}$$

which is the desired result. The numbers are 0.49mm, 1.47mm and 4.92mm.

16.2.6 Exercise 1.8 Energy of capacitors

$$W = \frac{1}{2}\int\rho(x)\phi(x)d^3x = -\frac{\epsilon_0}{2}\int\phi\triangle\phi d^3x = \frac{\epsilon_0}{2}\int(\nabla\phi)^2 d^3x = \frac{\epsilon_0}{2}\int|\vec{E}|^2 d^3x \tag{16.214}$$

(a) With $\vec{E}_{\text{tot}} = -\nabla\phi_{\text{tot}}$ and $Q = C\cdot U$

$$\begin{aligned}
W_{\text{plate}} &= \frac{\epsilon_0}{2}\cdot\left(\frac{Q}{\epsilon_0 A}\right)^2\cdot(Ad) = \frac{Q^2 d}{2\epsilon_0 A} && (16.215)\\
&= \frac{U^2 d}{2\epsilon_0 A}\left(\frac{\epsilon_0 A}{d}\right)^2 = \frac{\epsilon_0 A U^2}{2d} && (16.216)
\end{aligned}$$

$$\begin{aligned}
W_{\text{sphere}} &= \frac{\epsilon_0}{2}4\pi\int_a^b r^2\frac{Q^2}{16\pi^2\epsilon_0^2 r^4}dr = \frac{Q^2}{8\pi\epsilon_0}\left(\frac{1}{b}-\frac{1}{a}\right) && (16.217)\\
&= \frac{U^2}{8\pi\epsilon_0}\left(\frac{a-b}{ab}\right)\cdot\left(\epsilon_0\frac{4\pi ab}{b-a}\right)^2 = 2\pi\epsilon_0 U^2\frac{ab}{b-a} && (16.218)
\end{aligned}$$

$$W_{\text{cylinder}} = \frac{\epsilon_0}{2} 2\pi L \int_a^b \left(\frac{Q}{2\pi\epsilon_0 L r}\right)^2 r\, dr = \frac{Q^2}{4\pi\epsilon_0 L} \log\frac{b}{a} \tag{16.219}$$

$$= \frac{U^2}{4\pi\epsilon_0 L} \log\frac{b}{a} \left(\frac{2\pi\epsilon_0 L}{\log b/a}\right)^2 = \frac{\pi\epsilon_0 L U^2}{\log b/a} \tag{16.220}$$

(b)

$$w_{\text{plate}} = \text{const} \tag{16.221}$$

$$w_{\text{sphere}} \sim r^{-4} \tag{16.222}$$

$$w_{\text{cylinder}} \sim r^{-2} \tag{16.223}$$

16.2.7 Exercise 12.1 Lagrangian of point charge

1. With $U^\alpha = \frac{dx_\alpha}{ds}$

$$L = -\frac{mU_\alpha U^\alpha}{2} - \frac{q}{c} U_\alpha A^\alpha \tag{16.224}$$

$$\frac{\partial L}{\partial x_\beta} = -\frac{q}{c} U_\alpha \frac{\partial A^\alpha}{\partial x_\beta} \tag{16.225}$$

$$\frac{\partial L}{\partial U_\beta} = -mU^\beta - \frac{q}{c} A^\beta \tag{16.226}$$

$$-m\frac{d}{ds}\left(\frac{dU^\beta}{ds}\right) - \frac{q}{c}\frac{dA^\beta}{ds} + \frac{q}{c} U_\alpha \frac{\partial A^\alpha}{\partial x_\beta} = 0 \tag{16.227}$$

$$m\frac{d^2 x^\beta}{ds^2} + \frac{q}{c}\frac{dA^\beta}{ds} - \frac{q}{c}\frac{dx_\alpha}{ds}\frac{\partial A^\alpha}{\partial x_\beta} = 0 \tag{16.228}$$

$$m\frac{d^2 x^\beta}{ds^2} + \frac{q}{c}\left(\frac{\partial A^\beta}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial s}\right) - \frac{q}{c}\frac{dx_\alpha}{ds}\frac{\partial A^\alpha}{\partial x_\beta} = 0 \tag{16.229}$$

$$m\frac{d^2 x^\beta}{ds^2} + \frac{q}{c}\frac{\partial x^\alpha}{\partial s}\left(\frac{\partial A^\beta}{\partial x^\alpha} - \frac{\partial A^\alpha}{\partial x_\beta}\right) = 0 \tag{16.230}$$

$$m\frac{d^2 x^\beta}{ds^2} + \frac{q}{c}\frac{\partial x^\alpha}{\partial s} F^{\alpha\beta} = 0 \tag{16.231}$$

2. Bit of a odd sign convention for the canonical momentum

$$P^\beta = -\frac{\partial L}{\partial U_\beta} = mU^\beta + \frac{q}{c} A^\beta \quad \to \quad U^\beta = \frac{1}{m}\left(P^\beta - \frac{q}{c} A^\beta\right) \tag{16.232}$$

$$H = P^\alpha U_\alpha + L \tag{16.233}$$

$$= P^\alpha \frac{1}{m}\left(P_\alpha - \frac{q}{c}A_\alpha\right) - \frac{m}{2}\frac{1}{m}\left(P_\alpha - \frac{q}{c}A_\alpha\right)\frac{1}{m}\left(P_\alpha - \frac{q}{c}A_\alpha\right) - \frac{q}{c}\frac{1}{m}\left(P_\alpha - \frac{q}{c}A_\alpha\right)A^\alpha \tag{16.234}$$

$$= \frac{1}{2m}\left(P^\alpha - \frac{q}{c}A^\alpha\right)\left(P_\alpha - \frac{q}{c}A_\alpha\right) \tag{16.235}$$

In space-time coordinates we can write

$$H = \frac{1}{2m}\left((p_0)^2 - \vec{p}^2 + \frac{q^2}{c^2}[\phi^2 - \vec{A}^2] + \frac{2q}{c}[\vec{p}\cdot\vec{A} - p^0\phi]\right) \tag{16.236}$$

$$= \frac{1}{2m}\left((\gamma mc)^2 - (\gamma m\vec{v})^2 + \frac{q^2}{c^2}[\phi^2 - \vec{A}^2] + \frac{2q}{c}[\gamma m\vec{v}\cdot\vec{A} - \gamma mc\phi]\right) \tag{16.237}$$

$$= \frac{\gamma^2 mc^2}{2}\left(1 - \frac{\vec{v}^2}{c^2}\right) + \frac{q^2}{2mc^2}[\phi^2 - \vec{A}^2] + q\gamma[\frac{1}{c}\vec{v}\cdot\vec{A} - \phi] \tag{16.238}$$

$$= \frac{mc^2}{2} + \frac{q^2}{2mc^2}[\phi^2 - \vec{A}^2] + q\gamma[\frac{1}{c}\vec{v}\cdot\vec{A} - \phi] \tag{16.239}$$

16.3 Schwinger - Classical Electrodynamics

16.3.1 Exercise 31.1 Potentials of moving point charge

$$w = z - vt \to \frac{\partial}{\partial z} = \frac{\partial w}{\partial z}\frac{\partial}{\partial w} \tag{16.240}$$

$$\to \frac{\partial^2}{\partial z^2} = \frac{\partial^2 w}{\partial z^2}\frac{\partial}{\partial w} + \left(\frac{\partial w}{\partial z}\right)^2 \frac{\partial^2}{\partial w^2} = \frac{\partial^2}{\partial w^2} \tag{16.241}$$

$$\to \frac{\partial^2}{\partial t^2} = \frac{\partial^2 w}{\partial t^2}\frac{\partial}{\partial w} + \left(\frac{\partial w}{\partial t}\right)^2 \frac{\partial^2}{\partial w^2} = v^2\frac{\partial^2}{\partial w^2} \tag{16.242}$$

then

$$\triangle - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial w^2} - \frac{v^2}{c^2}\frac{\partial^2}{\partial w^2} \tag{16.243}$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(1 - \frac{v^2}{c^2}\right)\frac{\partial^2}{\partial w^2} \tag{16.244}$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial u^2} \tag{16.245}$$

with with $u = w/\sqrt{1 - v^2/c^2}$. The wave equation can then be rewritten

$$-\Box\phi = 4\pi\rho \tag{16.246}$$

$$= 4\pi e\delta(x)\delta(y)\delta(z - vt) \tag{16.247}$$

$$-\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial u^2}\right)\phi = 4\pi e\delta(x)\delta(y)\delta\left(\sqrt{1 - \frac{v^2}{c^2}}u\right) \tag{16.248}$$

$$= \frac{4\pi}{\sqrt{1 - \frac{v^2}{c^2}}}e\delta(x)\delta(y)\delta\left(u\right) \tag{16.249}$$

Using the Green function of the Coulomb equation (13.3) we obtain

$$\phi = \frac{e}{\sqrt{1 - \frac{v^2}{c^2}}\sqrt{u^2 + x^2 + y^2}} \tag{16.250}$$

$$= \frac{e}{\sqrt{w^2 + (1 - \frac{v^2}{c^2})(x^2 + y^2)}} \tag{16.251}$$

$$= \frac{e}{\sqrt{(z - vt)^2 + (1 - \frac{v^2}{c^2})(x^2 + y^2)}} \tag{16.252}$$

For the vector potential we can calculate similarly

$$-\Box\vec{A} = 4\pi\frac{\vec{j}}{c} \tag{16.253}$$

$$-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial u^2}\right)\vec{A} = 4\pi e\frac{\vec{v}}{c}\delta(x)\delta(y)\delta\left(\sqrt{1-\frac{v^2}{c^2}}u\right) \tag{16.254}$$

$$= \frac{4\pi}{\sqrt{1-\frac{v^2}{c^2}}}e\frac{\vec{v}}{c}\delta(x)\delta(y)\delta\left(u\right) \tag{16.255}$$

which gives $\vec{A} = \vec{v}/c\phi$.

16.3.2 Exercise 31.2 Fields of moving point charge

$$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} \tag{16.256}$$

$$= \frac{e}{2}\left((z-vt)^2+(1-\frac{v^2}{c^2})(x^2+y^2)\right)^{-3/2}[(1-\frac{v^2}{c^2})2x,(1-\frac{v^2}{c^2})2y,2(z-vt)(1-\frac{v^2}{c^2})] \tag{16.257}$$

$$= e(1-\frac{v^2}{c^2})\left((z-vt)^2+(1-\frac{v^2}{c^2})(x^2+y^2)\right)^{-3/2}[x,y,(z-vt)] \tag{16.258}$$

$$\vec{B} = \nabla\times\vec{A} \tag{16.259}$$

$$= -e\frac{v}{c}(1-\frac{v^2}{c^2})\left((z-vt)^2+(1-\frac{v^2}{c^2})(x^2+y^2)\right)^{-3/2}[y,x,0] \tag{16.260}$$

16.3.3 Exercise 31.4 Wave equation for fields

With

$$\nabla\times\vec{B} = \frac{1}{c}\frac{\partial}{\partial t}\vec{E}+\frac{4\pi}{c}\vec{j}_e \tag{16.261}$$

$$\nabla\cdot\vec{E} = 4\pi\rho_e \tag{16.262}$$

$$-\nabla\times\vec{E} = \frac{1}{c}\frac{\partial}{\partial t}\vec{B}+\frac{4\pi}{c}\vec{j}_m \tag{16.263}$$

$$\nabla\cdot\vec{B} = 4\pi\rho_m \tag{16.264}$$

we obtain

$$\nabla\times\nabla\times\vec{B} = \nabla(\nabla\cdot\vec{B}) - \triangle\vec{B} \tag{16.265}$$

$$= 4\pi\nabla\rho_m - \triangle\vec{B} \tag{16.266}$$

$$= \frac{1}{c}\frac{\partial}{\partial t}\nabla\times\vec{E}+\frac{4\pi}{c}\nabla\times\vec{j}_e \tag{16.267}$$

$$= -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{B}-\frac{4\pi}{c^2}\frac{\partial}{\partial t}\vec{j}_m+\frac{4\pi}{c}\nabla\times\vec{j}_e \tag{16.268}$$

$$\to -\Box\vec{B} = -4\pi\nabla\rho_m+\frac{4\pi}{c}(\nabla\times\vec{j}_e-\frac{1}{c}\frac{\partial}{\partial t}\vec{j}_m) \tag{16.269}$$

$$
\begin{aligned}
\nabla \times \nabla \times \vec{E} &= \nabla(\nabla \cdot \vec{E}) - \triangle \vec{E} && (16.270)\\
&= 4\pi \nabla \rho_e - \triangle \vec{E} && (16.271)\\
&= -\frac{1}{c}\frac{\partial}{\partial t}\nabla \times \vec{B} - \frac{4\pi}{c}\nabla \times \vec{j}_m && (16.272)\\
&= -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} - \frac{4\pi}{c^2}\frac{\partial}{\partial t}\vec{j}_e - \frac{4\pi}{c}\nabla \times \vec{j}_m && (16.273)\\
&\to -\Box \vec{E} = -4\pi \nabla \rho_e + \frac{4\pi}{c}(\nabla \times \vec{j}_m - \frac{1}{c}\frac{\partial}{\partial t}\vec{j}_e) && (16.274)
\end{aligned}
$$

16.3.4 Exercise 31.5 Lienard-Wiechert potentials

$$
\begin{aligned}
\phi(\vec{r},t) &= \int d\vec{r}' dt' \frac{\delta(\frac{1}{c}|\vec{r}-\vec{r}'| - (t-t'))}{|\vec{r}-\vec{r}'|}\rho(\vec{r}',t') && (16.275)\\
&= \int d\vec{r}' dt' \frac{\delta(t'-t+\frac{1}{c}|\vec{r}-\vec{r}'|)}{|\vec{r}-\vec{r}'|} e\delta(\vec{r}' - \vec{r}_B(t')) && (16.276)\\
&= e\int d\vec{r}' \frac{1}{|\vec{r}-\vec{r}'|}\delta(\vec{r}' - \vec{r}_B(t - \frac{1}{c}|\vec{r}-\vec{r}'|)) && (16.277)
\end{aligned}
$$

$$
\begin{aligned}
\delta(\vec{r}' - \vec{r}_B(t - \frac{1}{c}|\vec{r}-\vec{r}'|)) &= \delta(f(\vec{r}')) && (16.278)\\
&= \sum \frac{\delta(\vec{r}')}{|f'(\vec{r}')|} && (16.279)
\end{aligned}
$$

$$
= e\int d\vec{r}' \frac{1}{|\vec{r}-\vec{r}'|}\sum \qquad (16.280)
$$

16.3.5 Exercise 38.1 Total radiated power

We observe

$$
\frac{\lambda}{(1+\lambda\beta)^4} = -\frac{1}{\beta(1+\lambda\beta)^4} + \frac{1}{\beta(1+\lambda\beta)^3}. \qquad (16.281)
$$

Then

$$
\begin{aligned}
f(\lambda) &= \frac{2}{(1+\lambda\beta)^3}\left(-\frac{\beta^2}{2} + \frac{\lambda\beta}{8}\frac{\beta^2-1}{1+\lambda\beta}\right) && (16.282)\\
&= -\beta^2\frac{1}{(1+\lambda\beta)^3} + \frac{\beta(\beta^2-1)}{4}\frac{\lambda}{(1+\lambda\beta)^4} && (16.283)\\
&= \left(-\beta^2 + \frac{\beta^2-1}{4}\right)\frac{1}{(1+\lambda\beta)^3} - \frac{(\beta^2-1)}{4}\frac{1}{(1+\lambda\beta)^4} && (16.284)\\
\int_{-1}^{1} f(\lambda)d\lambda &= -\frac{1+3\lambda^2}{4} && (16.285)
\end{aligned}
$$

16.4 Smythe - Static and Dynamic Electricity

16.4.1 Exercise 1.1 Two coaxial rings and a point charge

Total charge of an axial ringlike charge distribution

$$\begin{aligned} Q &= \int \rho_0(\varphi')\delta(z'-0)\delta(r'-a)d\varphi' dz' dr' && (16.286)\\ &= 2\pi a \rho_0 && (16.287)\end{aligned}$$

which means that the 1-dimensional charge density is $\rho_0 = Q/2\pi a$. The axial potential of a single ring is then

$$\begin{aligned} \phi(z) &= \frac{1}{4\pi\varepsilon_0}\int \frac{\rho_0\delta(z'-0)\delta(r'-a)}{\sqrt{a^2+z^2}} r d\varphi' dz' dr' && (16.288)\\ &= \frac{1}{4\pi\varepsilon_0} 2\pi a\rho_0 \frac{1}{\sqrt{a^2+z^2}} && (16.289)\\ &= \frac{Q}{4\pi\varepsilon_0}\frac{1}{\sqrt{a^2+z^2}} && (16.290)\end{aligned}$$

therefore we get for the energies

$$\begin{aligned} W_1 &= \frac{qQ_1}{4\pi\varepsilon_0}\frac{1}{a} + \frac{qQ_2}{4\pi\varepsilon_0}\frac{1}{\sqrt{a^2+b^2}} && (16.291)\\ W_2 &= \frac{qQ_1}{4\pi\varepsilon_0}\frac{1}{\sqrt{a^2+b^2}} + \frac{qQ_2}{4\pi\varepsilon_0}\frac{1}{a} && (16.292)\end{aligned}$$

solving the linear system for the charges $Q_{1,2}$ we obtain

$$\begin{aligned} Q_1 &= \frac{4\pi\varepsilon_0}{qb^2}\sqrt{a^2+b^2}\left(\sqrt{a^2+b^2}W_1 - aW_2\right) && (16.293)\\ Q_2 &= \frac{4\pi\varepsilon_0}{qb^2}\sqrt{a^2+b^2}\left(-aW_1 + \sqrt{a^2+b^2}W_2\right). && (16.294)\end{aligned}$$

16.4.2 Exercise 1.3 Flux of two point charges through circle

For the flux we have

$$\begin{aligned} N &\equiv \int \vec{E}\cdot d\vec{A} && (16.295)\\ &= \int E\cos(\vec{E},\vec{n})\, dA && (16.296)\\ &= 2\pi\int \frac{q}{4\pi\varepsilon_0(a^2+r^2)}\frac{a}{\sqrt{a^2+r^2}} rdr - 2\pi\int \frac{Q}{4\pi\varepsilon_0(a^2+r^2)}\frac{a}{\sqrt{a^2+r^2}} rdr && (16.297)\\ &= \frac{2\pi a}{4\pi\varepsilon_0}(q-Q)\int_0^a \frac{1}{(a^2+r^2)^{3/2}} rdr && (16.298)\\ &= \frac{1}{4\varepsilon_0}(q-Q)\left(2-\sqrt{2}\right) && (16.299)\end{aligned}$$

therefore

$$Q = q - \frac{4N\varepsilon_0}{2-\sqrt{2}}. \qquad (16.300)$$

16.4.3 Exercise 1.4 Concentric charged rings

The axial potential of a single ring is with radius a and charge $Q = 2\pi a\rho_0$ is

$$\begin{aligned}\phi(x) &= \frac{1}{4\pi\varepsilon_0}\int \frac{\rho_0\delta(z'-0)\delta(r'-a)}{\sqrt{a^2+x^2}} r d\varphi' dz' dr' && (16.301)\\ &= \frac{1}{4\pi\varepsilon_0} 2\pi a\rho_0 \frac{1}{\sqrt{a^2+x^2}} && (16.302)\\ &= \frac{Q}{4\pi\varepsilon_0}\frac{1}{\sqrt{a^2+x^2}} && (16.303)\end{aligned}$$

The total potential and the resulting electrical field is therefore

$$\begin{aligned}\phi(x) &= -\frac{Q}{4\pi\varepsilon_0}\frac{1}{\sqrt{a_1^2+x^2}} + \frac{\sqrt{27}Q}{4\pi\varepsilon_0}\frac{1}{\sqrt{a_2^2+x^2}} && (16.304)\\ E_x &= -\frac{\partial\phi}{\partial x} && (16.305)\\ &= \frac{Qx}{4\pi\varepsilon_0}\left(-\frac{1}{(a_2^2+x^2)^{3/2}} + \frac{\sqrt{27}}{(a_2^2+x^2)^{3/2}}\right) && (16.306)\end{aligned}$$

which only vanishes for

$$x = 0, \pm\sqrt{\frac{-3a_1^2 + a2^2}{2}}. \qquad (16.307)$$

Due to the radial symmetry the other field components at this points vanish too.

16.4.4 Exercise 12.1 Linear quadrupole

$$\begin{aligned}\beta &= \omega\sqrt{\mu\epsilon} && (16.308)\\ q_{zz}^{(2)} &= a^2 q\sin\omega t && (16.309)\\ 8\pi\epsilon\vec{Z}_{zz} &= a^2 q\sin\omega t\left(\frac{\beta}{r} - \frac{j}{r^2}\right)(\vec{r}_1\cos\theta - \vec{\theta}\sin\theta)\cos\theta e^{-j\beta r} && (16.310)\\ & && (16.311)\end{aligned}$$

16.5 WEINBERG - Foundation of Modern Physics

16.5.1 Problem 6 - Power of Carnot AC

$$\begin{aligned}P_{\text{therm}} &= 10\text{kW} && (16.312)\\ \eta &= 1 - \frac{293}{313} = 0.064 && (16.313)\\ P &= \frac{P_{\text{therm}}}{\eta} = 156.5\text{kW} && (16.314)\end{aligned}$$

16.5.2 Problem 7 - Electrolysis of water

For each O_2 molecule 4 elementary charges are needed (because O^{2-})

$$I = \frac{\Delta Q}{\Delta t} = e\frac{N_e}{\Delta t} \tag{16.315}$$

$$m_{O_2} = \frac{N_e}{2 \cdot 2} M_{O_2} \cdot u \tag{16.316}$$

$$= \frac{I \cdot \Delta t}{4e} M_{O_2} \cdot u \tag{16.317}$$

$$\Delta t = \frac{4em_{O_2}}{IM_{O_2}u} = 3,000\,\mathrm{s} \tag{16.318}$$

16.6 Thorne, Blandford - Modern Classical Physics

16.6.1 Exercise 1.1 Practice: Energy Change for Charged Particle

With $E = p^2/2m$ and (1.7c) we obtain

$$\frac{dE}{dt} = \frac{d}{dt}\frac{p^2}{2m} = \frac{2\vec{p} \cdot d\vec{p}/dt}{2m} \tag{16.319}$$

$$= \frac{q}{m}\vec{p} \cdot (\vec{E} + \vec{v} \times \vec{B}) \tag{16.320}$$

$$= q\vec{v} \cdot (\vec{E} + \vec{v} \times \vec{B}) \tag{16.321}$$

$$= q\vec{v} \cdot \vec{E}. \tag{16.322}$$

As $\vec{v} \times \vec{B}$ is orthogonal to $\vec{v}$ (and $\vec{B}$) the scalar product $\vec{v} \cdot (\vec{v} \times \vec{B})$ vanishes.

16.6.2 Exercise 1.2 Practice: Particle Moving in a Circular Orbit

(a) With

$$\frac{d\vec{n}}{ds} = \frac{\vec{n}' - \vec{n}}{R \cdot d\phi} = \frac{\vec{v}' - \vec{v}}{vR \cdot d\phi} \tag{16.323}$$

we can calculate the norm

$$\left|\frac{d\vec{n}}{ds}\right| = \frac{\sqrt{v^2 + v^2 - 2v^2\cos(d\phi)}}{vR \cdot d\phi} = \frac{v\sqrt{1 - \cos(d\phi)}}{vR \cdot d\phi} = \frac{v\sqrt{2[1 - \cos(d\phi)]}}{vR \cdot d\phi} \tag{16.324}$$

$$\approx \frac{vd\phi}{vR \cdot d\phi} = \frac{1}{R} \tag{16.325}$$

and the scalar product

$$\frac{d\vec{n}}{ds} \cdot \vec{n} = \frac{\vec{n}' \cdot \vec{n} - \vec{n} \cdot \vec{n}}{R \cdot d\phi} = \frac{n^2\cos(d\phi) - n^2}{vR \cdot d\phi} \tag{16.326}$$

$$\approx \frac{(1 - d\phi^2/2) - 1}{vR \cdot d\phi} = \frac{d\phi}{2vR} \tag{16.327}$$

which vanished for $d\phi \to 0$ and therefore implies that $d\vec{n}$ is orthogonal to $\vec{n}$ (and therefore points to the center).

(b) From (a) we know

$$\vec{R} = R^2\frac{d\vec{n}}{ds} = R^2\frac{d\vec{v}}{v \cdot ds} = R^2\frac{d\vec{v}}{v \cdot ds} = \frac{R^2}{v}\frac{d\vec{v}}{dt}\frac{dt}{ds} = \left(\frac{R}{v}\right)^2\vec{a} \tag{16.328}$$

Taking the absolute value we have

$$R = \frac{R^2}{v^2} a \quad \rightarrow \quad R = \frac{v^2}{a} \tag{16.329}$$

and therefore

$$\vec{R} = \frac{R^2}{v^2} \vec{a} = \frac{v^4}{v^2 a^2} \vec{a} = \left(\frac{v}{a}\right)^2 \vec{a}. \tag{16.330}$$

16.6.3 Exercise 1.3 Derivation: Component Manipulation Rules

1. (1.9g I) - using (1.9b), (1.9a) and (1.9c)

$$\mathbf{A} \cdot \mathbf{B} = (A_j \mathbf{e}_j) \cdot (B_k \mathbf{e}_k) = A_j B_k \mathbf{e}_j \cdot \mathbf{e}_k = A_j B_k \delta_{jk} = A_j B_j \tag{16.331}$$

2. (1.9g II) - using (1.9d) and (1.5a)

$$\mathbf{T} = T_{ijk} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \tag{16.332}$$
$$\mathbf{T}(\mathbf{A}, \mathbf{B}, \mathbf{C}) = T_{ijk} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k (\mathbf{A}, \mathbf{B}, \mathbf{C}) \tag{16.333}$$
$$= T_{ijk} (\mathbf{A} \cdot \mathbf{e}_i)(\mathbf{B} \cdot \mathbf{e}_j)(\mathbf{C} \cdot \mathbf{e}_k) \tag{16.334}$$
$$= T_{ijk} A_i B_j C_k \tag{16.335}$$

3. (1.9h) - using (1.9d), (1.6b), (1.9a) and (1.5a)

$$\mathbf{R} = R_{abcd} \mathbf{e}_a \otimes \mathbf{e}_b \otimes \mathbf{e}_c \otimes \mathbf{e}_d \tag{16.336}$$
$$\text{1\&3 contraction}(\mathbf{R}) = R_{abcd} (\mathbf{e}_a \cdot \mathbf{e}_c) \mathbf{e}_b \otimes \mathbf{e}_d \tag{16.337}$$
$$= R_{abcd} \delta_{ac} \mathbf{e}_b \otimes \mathbf{e}_d \tag{16.338}$$
$$= R_{abad} \mathbf{e}_b \otimes \mathbf{e}_d \tag{16.339}$$
$$\text{components of } [\text{1\&3 contraction}(\mathbf{R})] = R_{abad} \mathbf{e}_b \otimes \mathbf{e}_d (\mathbf{e}_j, \mathbf{e}_k) \tag{16.340}$$
$$= R_{abad} (\mathbf{e}_b \cdot \mathbf{e}_j)(\mathbf{e}_d \cdot \mathbf{e}_k) \tag{16.341}$$
$$= R_{abad} \delta_{bj} \delta_{dk} \tag{16.342}$$
$$= R_{ajak} \tag{16.343}$$

16.6.4 Exercise 1.4 Example and Practice: Numerics of Component Manipulations

$$\mathbf{C} = \mathbf{S}(\mathbf{A}, \mathbf{B}, \_) \tag{16.344}$$
$$= S_{ijk} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k (\mathbf{A}, \mathbf{B}, \_) \tag{16.345}$$
$$= S_{ijk} (\mathbf{A} \cdot \mathbf{e}_i)(\mathbf{B} \cdot \mathbf{e}_j) \mathbf{e}_k \tag{16.346}$$
$$= S_{ijk} A_i B_j \mathbf{e}_k \tag{16.347}$$

$$C_k = S_{11k} A_1 B_1 + S_{12k} A_1 B_2 \tag{16.348}$$
$$C_1 = 0, \quad C_2 = 0, \quad C_3 = S_{123} A_1 B_2 = 15 \tag{16.349}$$

$$\mathbf{D} = \mathbf{S}(\mathbf{A}, \_, \mathbf{B}) \tag{16.350}$$
$$= S_{ijk} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k (\mathbf{A}, \_, \mathbf{B}) \tag{16.351}$$
$$= S_{ijk} (\mathbf{A} \cdot \mathbf{e}_i)(\mathbf{B} \cdot \mathbf{e}_k) \mathbf{e}_j \tag{16.352}$$
$$= S_{ijk} A_i B_k \mathbf{e}_j \tag{16.353}$$

$$D_j = S_{1j1}A_1B_1 + S_{1j2}A_1B_2 = 0 \tag{16.354}$$

$$\begin{aligned}
\mathbf{W} &= \mathbf{A} \otimes \mathbf{B} && (16.355)\\
&= (A_i\mathbf{e}_i) \otimes (B_j\mathbf{e}_j) && (16.356)\\
&= A_iB_j\mathbf{e}_i \otimes \mathbf{e}_j && (16.357)
\end{aligned}$$

$$W_{11} = 12, \quad W_{12} = 15, \tag{16.358}$$

16.6.5 Exercise 1.5 Practice: Meaning of Slot-Naming Index Notation

(a) Somewhat guessing

$$\begin{aligned}
A_iB_{jk} &\to A_iB_{jk}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k && (16.359)\\
&= (A_i\mathbf{e}_i) \otimes (B_{jk}\mathbf{e}_j \otimes \mathbf{e}_k) && (16.360)\\
&= A(_) \otimes B(_,_) && (16.361)
\end{aligned}$$

$$\begin{aligned}
A_iB_{ji} &\to A_iB_{ji}\mathbf{e}_j && (16.362)\\
&= (\mathbf{A}\cdot\mathbf{e}_i)B_{ji}\mathbf{e}_j && (16.363)\\
&= B_{ji}\mathbf{e}_j \otimes \mathbf{e}_i(_,\mathbf{A}) && (16.364)\\
&= \mathbf{B}(_,\mathbf{A}) && (16.365)
\end{aligned}$$

$$S_{ijk} = S_{kji} \to ... \tag{16.366}$$

$$A_iB_i = AiB_jg_{ij} \to \mathbf{A}\cdot\mathbf{B} = \mathbf{g}(\mathbf{A},\mathbf{B}) \tag{16.367}$$

(b) Applying the standard machinery

$$\begin{aligned}
\mathbf{T}(_,_,\mathbf{A}) &= T_{ijk}\mathbf{e}_i \otimes \mathbf{e}_j(\mathbf{A}\cdot\mathbf{e}_k) && (16.368)\\
&= T_{ijk}A_k\mathbf{e}_i \otimes \mathbf{e}_j && (16.369)\\
&\to T_{ijk}A_k && (16.370)
\end{aligned}$$

$$\begin{aligned}
\mathbf{S}(\mathbf{B},_) &= S_{ab}(\mathbf{B}\cdot\mathbf{e}_a)\mathbf{e}_b && (16.371)\\
&= S_{ab}B_a\mathbf{e}_b && (16.372)\\
\mathbf{T}(_,\mathbf{S}(\mathbf{B},_),_) &= T_{ijk}\mathbf{e}_i \otimes \mathbf{e}_k(S_{ab}B_a\mathbf{e}_b\cdot\mathbf{e}_j) && (16.373)\\
&= T_{ijk}\mathbf{e}_i \otimes \mathbf{e}_k(S_{ab}B_a\delta_{bj}) && (16.374)\\
&= T_{ijk}S_{aj}B_a\mathbf{e}_i \otimes \mathbf{e}_k && (16.375)\\
&\to T_{ijk}S_{aj} && (16.376)
\end{aligned}$$

16.6.6 Exercise 1.15 Practice: Geometrized Units

(a) $t_P = \sqrt{G\hbar} \quad \to \quad \sqrt{\frac{G\hbar}{c^5}} = 5.39\cdot10^{-44}\text{s} \equiv 1.61\cdot10^{-35}\text{m}$

(b) $E = 2mc^2$

(c)

(d)

(e) $1\text{m} \equiv 3.33\cdot10^{-9}\text{s}$ and $1\text{yr} \equiv 9.45\cdot10^{15}\text{m}$

16.6.7 Exercise 3.3 Practice and Example: Regimes of Particulate and Wave - Like Behavior

(a) The Schwarzschild radius of the BH is

$$R_S = \frac{2GM}{c^2} = 44,466\text{m} \tag{16.377}$$

which gives a disk radius of $R = 7R_S = 311\text{km}$. With

$$F_{\text{Earth}} = \frac{dP}{dA} = \frac{dW}{dA\,dt} = \frac{dN \cdot E_{ph}c}{dA \cdot dl} = \left(\frac{dN}{d\mathcal{V}_x}\right)_{\text{Earth}} \cdot E_{\text{ph}}c \tag{16.378}$$

$$\left(\frac{dN}{d\mathcal{V}_x}\right)_{\text{Earth}} = \frac{F_{\text{Earth}}}{cE_{\text{ph}}} = 0.00104\text{m}^{-3} \tag{16.379}$$

$$F_{\text{CX1}} = \frac{r^2}{R^2}F_{\text{Earth}} \tag{16.380}$$

$$\left(\frac{dN}{d\mathcal{V}_x}\right)_{\text{CX1}} = \frac{F_{\text{CX1}}}{cE_{\text{ph}}} = \frac{r^2}{R^2}\frac{F_{\text{Earth}}}{cE_{\text{ph}}} = 3.72 \cdot 10^{25}\text{m}^{-3} \tag{16.381}$$

The momentum of the photons is $p = E/c$.

The mean occupation number is then

$$\eta = \frac{h^3}{g_s}\mathcal{N} = \frac{h^3}{g_s}\frac{dN}{d\mathcal{V}_x d\mathcal{V}_p} = \tag{16.382}$$

16.6.8 Exercise 4.1 Example:Canonical Transformation

(a) Simple calculation

$$p_j = \frac{\partial F}{\partial q_j} = \sum_i \frac{\partial f_i}{\partial q_j}P_i \tag{16.383}$$

$$Q_j = \frac{\partial F}{\partial P_j} = f_j \tag{16.384}$$

(b) With

$$\dot{p} = -\frac{\partial H}{\partial q} \qquad \dot{q} = \frac{\partial H}{\partial p} \tag{16.385}$$

then

$$\frac{\partial H}{\partial Q} \tag{16.386}$$

16.6.9 Exercise 5.4 Example and Derivation: Adiabatic Index for Ideal Gas

With the Gibbs fundamental form

$$dE = \delta Q - pdV + \mu dN \tag{16.387}$$

$$= TdS - pdV + \mu dN \tag{16.388}$$

$$\frac{dE}{dT}_{V,N} = T\left(\frac{\partial S}{\partial T}\right)_{V,N} \tag{16.389}$$

then

$$C_V = T\left(\frac{\partial S}{\partial T}\right)_{V,N} \tag{16.390}$$
$$= \left(\frac{\partial E}{\partial T}\right)_{V,N} \tag{16.391}$$
$$\tag{16.392}$$

16.6.10 Exercise 7.1 Practice: Group and Phase Velocities

With the definition of phase and group velocities

$$\vec{v}_{ph} = \frac{\omega}{k}\frac{\vec{k}}{k} \tag{16.393}$$
$$\vec{v}_g = \nabla_k \omega \tag{16.394}$$

$$\omega_1(\vec{k}) = C|\vec{k}| \tag{16.395}$$
$$\to \vec{v}_{ph} = \frac{C|\vec{k}|}{k}\frac{\vec{k}}{k} = C\frac{\vec{k}}{k} \tag{16.396}$$
$$\to \vec{v}_g = C\frac{2\vec{k}}{2\sqrt{k^2}} = C\frac{\vec{k}}{k} \tag{16.397}$$
$$\omega_2(\vec{k}) = \sqrt{g|\vec{k}|} \tag{16.398}$$
$$\to \vec{v}_{ph} = \frac{\sqrt{g|\vec{k}|}}{k}\frac{\vec{k}}{k} = \sqrt{\frac{g}{k}}\frac{\vec{k}}{k} \tag{16.399}$$
$$\to \vec{v}_g = \sqrt{g}\frac{1}{2\sqrt{|\vec{k}|}}\frac{\vec{k}}{k} = \frac{1}{2}\sqrt{\frac{g}{k}}\frac{\vec{k}}{k} \tag{16.400}$$
$$\omega_3(\vec{k}) = \sqrt{\frac{D}{\Lambda}}\vec{k}^2 \tag{16.401}$$
$$\to \vec{v}_{ph} = \sqrt{\frac{D}{\Lambda}}\frac{\vec{k}^2}{k}\frac{\vec{k}}{k} = \sqrt{\frac{D}{\Lambda}}k\frac{\vec{k}}{k} \tag{16.402}$$
$$\to \vec{v}_g = \sqrt{\frac{D}{\Lambda}}2\vec{k} = 2\sqrt{\frac{D}{\Lambda}}k\frac{\vec{k}}{k} \tag{16.403}$$
$$\omega_4(\vec{k}) = \vec{a}\cdot\vec{k} \tag{16.404}$$
$$\to \vec{v}_{ph} = \frac{\vec{a}\cdot\vec{k}}{k}\frac{\vec{k}}{k} = \left(\vec{a}\cdot\frac{\vec{k}}{k}\right)\frac{\vec{k}}{k} \tag{16.405}$$
$$\to \vec{v}_g = \vec{a} \tag{16.406}$$

16.6.11 Exercise 7.2 Example: Gaussian Wave Packet and Its Dispersion

(a) Taylor expansion of the dispersion relation gives

$$\omega = \Omega(k) = \omega(k_0) + \left.\frac{\partial\omega(k)}{\partial k}\right|_{k=k_0}(k-k_0) + \frac{1}{2}\left.\frac{\partial^2\omega(k)}{\partial k^2}\right|_{k=k_0}(k-k_0)^2 \tag{16.407}$$
$$= \omega(k_0) + V_g|_{k=k_0}(k-k_0) + \frac{1}{2}\left.\frac{\partial V_g(k)}{\partial k}\right|_{k=k_0}(k-k_0)^2. \tag{16.408}$$

(b) The wave packet can then be written as

$$\psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk A(k) e^{i\alpha(k)} e^{i(kx-\omega t)} \tag{16.409}$$
$$= \frac{C}{2\pi}\int_{-\infty}^{\infty} dk e^{-\frac{(k-k_0)^2}{2\Delta k^2}} e^{i[\alpha_0 - x_0(k-k0)]} e^{i(kx-[\omega_0+V_g(k-k_0)+\frac{1}{2}V_g'(k-k_0)^2]t)} \tag{16.410}$$
$$= \frac{C}{2\pi}\int_{-\infty}^{\infty} dk e^{-\frac{(k-k_0)^2}{2\Delta k^2}} e^{i(\alpha_0+k_0x-\omega_0 t-(V_g t - x + x_0)(k-k_0)-\frac{1}{2}V_g' t(k-k_0)^2)} \tag{16.411}$$
$$= \frac{C}{2\pi} e^{i(\alpha_0+k_0x-\omega_0 t)}\int_{-\infty}^{\infty} dk e^{-i(V_g t-x+x_0)(k-k_0)} e^{-\frac{1}{2}(k-k_0)^2\left(\frac{1}{\Delta k^2}+iV_g' t\right)} \tag{16.412}$$
$$= \frac{C}{2\pi} e^{i(\alpha_0+k_0x-\omega_0 t)}\int_{-\infty}^{\infty} d\kappa e^{i(x-x_0-V_g t)\kappa} e^{-\frac{1}{2}\kappa^2\left(\frac{1}{\Delta k^2}+iV_g' t\right)} \tag{16.413}$$
$$\tag{16.414}$$

(c) With

$$\int_{-\infty}^{\infty} dy e^{-(a+ic)y^2} e^{-iby} = \sqrt{\frac{\pi}{a^2+c^2}}\sqrt{a-ic}\, e^{-\frac{b^2}{4(a^2+c^2)}(a-ic)} \qquad a>0, a,b,c\in\mathbb{R} \tag{16.415}$$

and the substitutions $a = \frac{1}{2\,\Delta k^2}$, $c = \frac{V_g' t}{2}$ and

$$a^2+c^2 = \frac{1}{4\,\Delta k^2}\frac{1}{\Delta k^2}\left(1+[V_g'(\Delta k)^2 t]^2\right) \tag{16.416}$$
$$= \frac{1}{4\,\Delta k^2}L^2 \tag{16.417}$$
$$= \frac{a}{2}L^2 \tag{16.418}$$

we obtain

$$\psi(x,t) = \frac{C}{2\pi} e^{i(\alpha_0+k_0x-\omega_0 t)}\sqrt{\frac{\pi}{aL^2}}\sqrt{a-ic}\, e^{-\frac{ab^2}{4(a^2+c^2)}} e^{-\frac{(-ic)b^2}{4(a^2+c^2)}} \tag{16.419}$$
$$= \frac{C}{2\pi} e^{i(\alpha_0+k_0x-\omega_0 t)} e^{\frac{2icb^2}{4aL^2}}\sqrt{\frac{\pi}{aL^2}}\sqrt{a-ic}\, e^{-\frac{(x-x_0-V_g t)^2}{2L^2}} \tag{16.420}$$

and therefore (with $|\sqrt{a-ic}| = \sqrt{|a-ic|} = \sqrt{\sqrt{aL^2}} = a^{1/4}\sqrt{L}$)

$$|\psi(x,t)| = \frac{C}{2\pi}\sqrt{\frac{\pi}{aL^2}} a^{1/4}\sqrt{L}\, e^{-\frac{(x-x_0-V_g t)^2}{2L^2}} \tag{16.421}$$
$$= \frac{C}{2\pi}\sqrt{\frac{\pi}{\sqrt{a}L}}\, e^{-\frac{(x-x_0-V_g t)^2}{2L^2}} \tag{16.422}$$
$$= \frac{C}{2}\sqrt{\frac{1}{\pi\sqrt{a}}}\frac{1}{\sqrt{L}}\, e^{-\frac{(x-x_0-V_g t)^2}{2L^2}}. \tag{16.423}$$

(d) At $t=0$ the packets width in position space is $L = 1/\Delta k$ while the width in momentum space is Δk which means the product is $\Delta x\cdot\Delta k = 1$.

(e) With the group velocity

$$V_g = \frac{1}{2}\sqrt{\frac{g}{k_0}} \tag{16.424}$$
$$V_g' = \frac{\partial V_g}{\partial k}|_{k=k_0} = -\frac{1}{4}\sqrt{\frac{g}{k_0^3}} \tag{16.425}$$

the width of the package is proportional to

$$L = \frac{1}{\Delta k}\sqrt{1 + \left(V_g'(\Delta k)^2 t\right)^2} \tag{16.426}$$

$$\rightarrow T_D = \frac{\sqrt{3}}{V_g'(\Delta k)^2} \tag{16.427}$$

$$\rightarrow T_D = \frac{4}{\Delta k^2}\sqrt{\frac{3k_0^3}{g}}. \tag{16.428}$$

The condition for the spread limitation is

$$S_{\text{HI-CA}} \leq V_g \cdot T_D \tag{16.429}$$

$$= \frac{1}{2}\sqrt{\frac{g}{k_0}}\frac{4}{\Delta k^2}\sqrt{\frac{3k_0^3}{g}} \tag{16.430}$$

$$= 2\sqrt{3}\frac{k_0}{\Delta k^2} \tag{16.431}$$

16.6.12 Exercise 7.3 Derivation and Example: Amplitude Propagation for Dispersionless Waves Expressed as Constancy of Something along a Ray

(a)

(b)

(c)

(d)

16.6.13 Exercise 7.4 Example: Energy Density and Flux, and Adiabatic Invariant, or a Dispersionless Wave

(a) For a generic Lagrangian density $\mathcal{L}$ we find

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial x_i}\right)}\delta\left(\frac{\partial\psi}{\partial x_i}\right) \tag{16.432}$$

$$= \frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial x_i}\right)}\frac{\partial}{\partial x_i}(\delta\psi) \tag{16.433}$$

$$\rightarrow \delta\int\mathcal{L}d^4x = \int\delta\mathcal{L}d^4x \tag{16.434}$$

$$= \int\left[\frac{\partial\mathcal{L}}{\partial\psi} - \frac{\partial}{\partial x_i}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial x_i}\right)}\right)\right]\delta\psi \tag{16.435}$$

$$\rightarrow 0 = \frac{\partial\mathcal{L}}{\partial\psi} - \frac{\partial}{\partial x_i}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial x_i}\right)}\right) \tag{16.436}$$

the general Euler-Lagrange equation. For the given density we can calculate the derivatives

$$\mathcal{L} = W\left[\frac{1}{2}\left(\frac{\partial\psi}{\partial t}\right)^2 - \frac{1}{2}C^2\left(\nabla\psi\right)^2\right] \tag{16.437}$$

$$\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial t}\right)} = W\frac{\partial\psi}{\partial t} \tag{16.438}$$

$$\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial x_i}\right)} = -WC^2\frac{\partial\psi}{\partial x_i} \tag{16.439}$$

and obtain

$$\frac{\partial}{\partial t}\left(W\frac{\partial\psi}{\partial t}\right) - \frac{\partial}{\partial x_i}\left(WC^2\frac{\partial\psi}{\partial x_i}\right) = 0. \tag{16.440}$$

(b) Using the definitions we obtain

$$\frac{\partial U}{\partial t} = \frac{\partial^2\psi}{\partial t^2}\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial t)} + \frac{\partial\psi}{\partial t}\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial t)}\right) - \frac{\partial\mathcal{L}}{\partial t} \tag{16.441}$$

$$\frac{\partial F_j}{\partial x_j} = \frac{\partial^2\psi}{\partial t\partial x_j}\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_j)} + \frac{\partial\psi}{\partial t}\frac{\partial}{\partial x_j}\left(\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_j)}\right) \tag{16.442}$$

and therefore

$$\frac{\partial U}{\partial t} + \frac{\partial F_j}{\partial x_j} = \frac{\partial^2\psi}{\partial t^2}\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial t)} + \frac{\partial\psi}{\partial t}\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial t)}\right) - \frac{\partial\mathcal{L}}{\partial t} + \frac{\partial^2\psi}{\partial t\partial x_j}\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_j)} + \frac{\partial\psi}{\partial t}\frac{\partial}{\partial x_j}\left(\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_j)}\right) \tag{16.443}$$

$$= \frac{\partial^2\psi}{\partial t^2}\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial t)} + \frac{\partial\psi}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\psi}\right) - \frac{\partial\mathcal{L}}{\partial t} + \frac{\partial^2\psi}{\partial t\partial x_j}\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_j)} \tag{16.444}$$

$$= \frac{\partial\psi}{\partial t}\left(-\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial t)}\right) + \frac{\partial\psi}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\psi}\right) - \frac{\partial\mathcal{L}}{\partial t} + \frac{\partial\psi}{\partial t}\left(-\frac{\partial}{\partial x_i}\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_j)}\right) \tag{16.445}$$

$$= \frac{\partial\psi}{\partial t}\left(-\frac{\partial\mathcal{L}}{\partial\psi}\right) + \frac{\partial\psi}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\psi}\right) - \frac{\partial\mathcal{L}}{\partial t} \tag{16.446}$$

$$= -\frac{\partial\mathcal{L}}{\partial t} \tag{16.447}$$

(c) Substituting $\mathcal{L}$ into the definitions yields

$$U = \frac{\partial\psi}{\partial t}\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial t)} - \mathcal{L} \tag{16.448}$$

$$= W\left(\frac{\partial\psi}{\partial t}\right)^2 - \mathcal{L} \tag{16.449}$$

$$= W\left[\frac{1}{2}\left(\frac{\partial\psi}{\partial t}\right)^2 + \frac{1}{2}C^2\left(\nabla\psi\right)^2\right] \tag{16.450}$$

$$F_j = \frac{\partial\psi}{\partial t}\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_j)} \tag{16.451}$$

$$= -\frac{\partial\psi}{\partial t}WC^2\frac{\partial\psi}{\partial x_j}. \tag{16.452}$$

(d) The momentum density is given by

$$\pi = \frac{\partial \mathcal{L}}{\partial \frac{\partial \phi}{\partial t}} = W \frac{\partial \phi}{\partial t} \tag{16.453}$$

$$\Pi = \int \pi d^3x = \int W \frac{\partial \phi}{\partial t} d^3x \tag{16.454}$$

$$J = \int_0^{\omega/2\pi} L dt = \int_0^{\omega/2\pi} \int \mathcal{L} d^3x dt \tag{16.455}$$

$$= \tag{16.456}$$

16.6.14 Exercise 8.1 Practice: Convolutions and Fourier Transforms

(a) With $f_1(x) = e^{-\frac{x^2}{2\sigma^2}}$ and $f_2(x) = e^{-\frac{x}{h}}\theta(x)$ we obtain

$$F_1(k) = \int_{-\infty}^{\infty} f_1(x) e^{-ikx} dx \tag{16.457}$$

$$= \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} e^{-ikx} dx \tag{16.458}$$

$$= e^{-\frac{k^2\sigma^2}{2}} \int_{-\infty}^{\infty} e^{-\left(\frac{x}{\sqrt{2}\sigma} + \frac{ik\sigma}{\sqrt{2}}\right)^2} dx \tag{16.459}$$

$$= e^{-\frac{k^2\sigma^2}{2}} \sqrt{2\sigma^2} \int_{-\infty}^{\infty} e^{-y^2} dy \tag{16.460}$$

$$= \sqrt{2\pi\sigma^2} e^{-\frac{\sigma^2 k^2}{2}} \tag{16.461}$$

$$F_2(k) = \int_{-\infty}^{\infty} f_2(x) e^{-ikx} dx \tag{16.462}$$

$$= \int_0^{\infty} e^{-\frac{x}{h}} e^{-ikx} dx \tag{16.463}$$

$$= -\frac{1}{h} e^{-\frac{x}{h}} e^{-ikx} \Big|_0^{\infty} - \int_0^{\infty} \left(-\frac{1}{h}\right) e^{-\frac{x}{h}} \frac{1}{(-ik)} e^{-ikx} dx \tag{16.464}$$

$$= \frac{1}{h} - \frac{1}{ihk} \int_0^{\infty} e^{-\frac{x}{h}} e^{-ikx} dx \tag{16.465}$$

$$= \ldots \tag{16.466}$$

$$= \frac{1}{\frac{1}{h} + ik} \tag{16.467}$$

(b)

(c)

$$f_1 \otimes f_2 = \int_{-\infty}^{\infty} f_2(y-x) f_1(x) dx \tag{16.468}$$

$$= \int_{-\infty}^{\infty} e^{-\frac{y-x}{h}} \theta(y-x) e^{-\frac{x^2}{2\sigma^2}} dx \tag{16.469}$$

$$= \int_{-\infty}^{y} e^{-\frac{y-x}{h}} e^{-\frac{x^2}{2\sigma^2}} dx \tag{16.470}$$

$$= \ldots \tag{16.471}$$

Not done yet

16.6.15 Exercise 11.9 Derivation: Sag in a Cantilever

(a) For a cantilever with Young's modulus E, density ρ, width w and height h the weight per length is given by

$$W = \rho g w h \tag{16.472}$$

and

$$D \equiv E \int z^2 \, dydz = Ew \left. \frac{z^3}{3} \right|_{-h/2}^{h/2} \tag{16.473}$$

$$= Ew \frac{h^3}{3} \frac{2}{8} = \frac{1}{12} Ewh^3. \tag{16.474}$$

We now solve

$$\frac{d^4\eta}{dx^4} = \frac{W}{D} \tag{16.475}$$

$$= \frac{12\rho g}{Eh^2} \tag{16.476}$$

with $\eta(0) = 0$, $\eta'(0 = 0)$, $\eta''(l) = 0$ and $\eta'''(l) = 0$ and obtain

$$\eta'''(x) = \frac{W}{D}(x + c_3) \tag{16.477}$$

$$\eta''(x) = \frac{W}{D}\left(\frac{x^2}{2} + c_3 x + c_2\right) \tag{16.478}$$

$$\eta'(x) = \frac{W}{D}\left(\frac{x^3}{3} + c_3\frac{x^2}{2} + c_2 x + c_1\right) \tag{16.479}$$

$$\eta(x) = \frac{W}{D}\left(\frac{x^4}{24} + c_3\frac{x^3}{6} + c_2\frac{x^2}{2} + c_1 x + c_0\right) \tag{16.480}$$

using the boundary conditions we see

$$\eta(0) = 0 \quad \to \quad c_0 = 0 \tag{16.481}$$

$$\eta'(0) = 0 \quad \to \quad c_1 = 0 \tag{16.482}$$

$$\eta'''(l) = 0 \quad \to \quad c_3 = -l \tag{16.483}$$

$$\eta''(l) = 0 \quad \to \quad c_2 = \frac{l^2}{2} \tag{16.484}$$

and therefore

$$\eta(x) = \frac{W}{D}\left(\frac{1}{24}x^4 - \frac{l}{6}x^3 + \frac{l^2}{4}x^2\right) \tag{16.485}$$

$$\eta(l) = \frac{W}{D}\frac{l^4}{8} = \frac{3\rho g l^4}{2Eh^2} \tag{16.486}$$

(b) Now we need to solve

$$\frac{d^4\eta}{dx^4} = \frac{1}{D}W(x). \tag{16.487}$$

The solution for the special case in (a) was $\eta \sim Wx^4 \sim \int Wz^3 \, dz$ so we try the ansatz

$$\eta(x) = \frac{1}{6D}\int_0^x (x - z)^3 W(z) dz \tag{16.488}$$

Calculating the 4th derivative we see that our ansatz is correct.

16.6.16 Exercise 13.1 Example: Earth's Atmosphere

(a) With $PV = Nk_BT$, $\rho = \frac{\mu m_p N}{V}$ and assuming $g = \text{const}$ we obtain

$$\nabla P = \rho \mathbf{g} \tag{16.489}$$
$$\frac{dP}{dz} = -\rho g \tag{16.490}$$
$$= -\frac{\mu m_p N}{V} g \tag{16.491}$$
$$= -\mu m_p g \frac{P}{k_B T} \tag{16.492}$$
$$\tag{16.493}$$

which can be solved by

$$\frac{dP}{P} = -\frac{\mu m_p g}{k_B T} \tag{16.494}$$
$$P(z) = P_0 \exp\left(-\frac{\mu m_p g}{k_B T} z\right). \tag{16.495}$$

With $\mu = 0.2 \cdot 2 \cdot 16 + 0.8 \cdot 2 \cdot 14 +$ ($20\%O_2/80\%N_2$) and $T = 220\text{K}$ we have

$$H = 6,400\text{m} \tag{16.496}$$
$$P(16\text{km}) = 0.083\text{bar} \tag{16.497}$$
$$\frac{P(35\text{km})}{P(16\text{km})} = 0.052 \tag{16.498}$$

(b) The isentropic condition $P \sim \rho^\gamma$ acts as an additional condition on top of the equations of state. It can be rewritten as

$$P\rho^{-\gamma} = \text{const} \tag{16.499}$$
$$PV^\gamma = \text{const} \tag{16.500}$$
$$P\left(\frac{T}{P}\right)^\gamma = \text{const} \tag{16.501}$$
$$TP^{\frac{1-\gamma}{\gamma}} = \text{const.} \tag{16.502}$$

Differentiating the last equation gives

$$\frac{dT}{dz} P^{\frac{1-\gamma}{\gamma}} + \left(\frac{1-\gamma}{\gamma}\right) P^{\frac{1-2\gamma}{\gamma}} \frac{dP}{dz} T = 0 \tag{16.503}$$
$$\rightarrow \frac{dT}{dz} = -\left(\frac{1-\gamma}{\gamma}\right) \frac{T}{P} \frac{dP}{dz} \tag{16.504}$$

Inserting the

$$\frac{dP}{dz} = -\mu m_p g \frac{P}{k_B T} \tag{16.505}$$

which we calculated in (a) we obtain

$$\frac{dT}{dz} = \left(\frac{1-\gamma}{\gamma}\right) \frac{\mu m_p g}{k_B}. \tag{16.506}$$

With this we calculate a lapse rate of 9.76K km^{-1}.

16.6.17 Exercise 13.2 Practise: Weight in Vacuum

$$F_b = \rho_{\text{air}} g V_{\text{body}} \tag{16.507}$$
$$= \rho_{\text{air}} g \frac{m_{\text{body}}}{\rho_{\text{body}}} \tag{16.508}$$
$$= 1\text{N} \tag{16.509}$$

where we used a mass of 100kg and $\rho_{\text{air}}/\rho_{\text{body}} = 0.001$.

16.6.18 Exercise 13.4 Example: Polytropes — The Power of Dimensionless Variables

(a) From

$$\frac{dP}{dr} = -\rho\frac{Gm}{r^2} \quad \rightarrow \quad m = -\frac{r^2}{G\rho}\frac{dP}{dr} \tag{16.510}$$
$$\frac{dm}{dr} = 4\pi\rho r^2 \tag{16.511}$$

we obtain by differentiation

$$\frac{d^2P}{dr^2} = -G\frac{(\frac{d\rho}{dr}m + \rho\frac{dm}{dr})r^2 - 2r\rho m}{r^4} \tag{16.512}$$
$$= -\frac{G}{r^4}\left(\left[\frac{d\rho}{dr}m + \rho\frac{dm}{dr}\right]r^2 - 2r\rho m\right) \tag{16.513}$$
$$= -\frac{G}{r^4}\left(\left[-\frac{r^2}{G\rho}\frac{dP}{dr}\frac{d\rho}{dr} + \rho 4\pi\rho r^2\right]r^2 + \frac{r^2}{G\rho}\frac{dP}{dr}2r\rho\right) \tag{16.514}$$
$$= \left(\frac{1}{\rho}\frac{d\rho}{dr} - \frac{2}{r}\right)\frac{dP}{dr} - 4\pi G\rho^2 \tag{16.515}$$

(b) With the polytropic equation of state $P = K\rho^{1+1/n}$ we find for the derivatives of P

$$\frac{dP}{dr} = K\left(1 + \frac{1}{n}\right)\rho^{1/n}\frac{d\rho}{dr} \tag{16.516}$$
$$\frac{d^2P}{dr^2} = K\left(1 + \frac{1}{n}\right)\rho^{1/n}\left[\frac{1}{n}\rho^{-1}\left(\frac{d\rho}{dr}\right)^2 + \frac{d^2\rho}{dr^2}\right] \tag{16.517}$$

and therefore

$$\frac{1}{n}\rho^{-1}\left(\frac{d\rho}{dr}\right)^2 + \frac{d^2\rho}{dr^2} = \left(\frac{1}{\rho}\frac{d\rho}{dr} - \frac{2}{r}\right)\frac{d\rho}{dr} - \frac{n}{1+n}\frac{4\pi G\rho^{2-1/n}}{K} \tag{16.518}$$
$$\frac{d^2\rho}{dr^2} = \left(\frac{n-1}{n}\frac{1}{\rho}\frac{d\rho}{dr} - \frac{2}{r}\right)\frac{d\rho}{dr} - \frac{n}{1+n}\frac{4\pi G}{K}\rho^{2-1/n} \tag{16.519}$$
$$\frac{d^2\rho}{dr^2} = \frac{n-1}{n}\frac{1}{\rho}\left(\frac{d\rho}{dr}\right)^2 - \frac{2}{r}\frac{d\rho}{dr} - \frac{n}{1+n}\frac{4\pi G}{K}\rho^{2-1/n}. \tag{16.520}$$

(c) With

$$\rho(r) = \rho_c\theta^\alpha(r) \tag{16.521}$$
$$\frac{d\rho}{dr} = \rho_c\alpha\theta^{\alpha-1}\frac{d\theta}{dr} \tag{16.522}$$
$$\left(\frac{d\rho}{dr}\right)^2 = \rho_c^2\alpha^2\theta^{2(\alpha-1)}\left(\frac{d\theta}{dr}\right)^2 \tag{16.523}$$
$$\frac{d^2\rho}{dr^2} = \rho_c\alpha(\alpha-1)\theta^{\alpha-2}\left(\frac{d\theta}{dr}\right)^2 + \rho_c\alpha\theta^{\alpha-1}\frac{d^2\theta}{dr^2} \tag{16.524}$$

we can rewrite the differential equation as

$$\rho_c\alpha(\alpha-1)\theta^{\alpha-2}\left(\frac{d\theta}{dr}\right)^2+\rho_c\alpha\theta^{\alpha-1}\frac{d^2\theta}{dr^2} \tag{16.525}$$

$$=\frac{n-1}{n}\frac{1}{\rho_c\theta^\alpha}\rho_c^2\alpha^2\theta^{2(\alpha-1)}\left(\frac{d\theta}{dr}\right)^2-\frac{2}{r}\rho_c\alpha\theta^{\alpha-1}\frac{d\theta}{dr}-\frac{n}{1+n}\frac{4\pi G}{K}\rho_c^{2-1/n}\theta^{\alpha(2-1/n)} \tag{16.526}$$

and see that for $n=\alpha$ the $(d\theta/dr)^2$ terms and the left and right side cancel out.

(d) With $n=\alpha$ the simplified equation is given by

$$\frac{d^2\theta}{dr^2}+\frac{2}{r}\frac{d\theta}{dr}+\frac{4\pi G\rho_c^{1-1/n}}{(n+1)K}\theta^n=0 \tag{16.527}$$

$$\frac{d^2\theta}{dr^2}+\frac{2}{r}\frac{d\theta}{dr}+\frac{4\pi G}{(n+1)K\rho_c^{1/n-1}}\theta^n=0 \tag{16.528}$$

(e) With

$$r=a\xi \tag{16.529}$$

$$\frac{d\theta}{dr}=\frac{d\theta}{d\xi}\frac{d\xi}{dr}=\frac{1}{a}\frac{d\theta}{d\xi} \tag{16.530}$$

$$\frac{d^2\theta}{dr^2}=\frac{1}{a^2}\frac{d^2\theta}{d\xi^2} \tag{16.531}$$

we obtain

$$\frac{d^2\theta}{d\xi^2}+\frac{2}{\xi}\frac{d\theta}{d\xi}+a^2\frac{4\pi G}{(n+1)K\rho_c^{1/n-1}}\theta^n=0 \tag{16.532}$$

which for $a^{-2}=\frac{4\pi G}{(n+1)K\rho_c^{1/n-1}}$ gives the Lane-Emden equation in standard form

$$\frac{d^2\theta}{d\xi^2}+\frac{2}{\xi}\frac{d\theta}{d\xi}+\theta^n=0 \tag{16.533}$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)=-\theta^n. \tag{16.534}$$

(f)
- $\theta(\xi=0)=1$

$$\rightarrow\quad \rho(r=0)=\rho_c \tag{16.535}$$

- $\theta'(\xi=0)=0$

$$\rightarrow\quad \frac{dP}{dr}=K\left(\frac{n+1}{n}\right)\rho^{1/n}\frac{d\rho}{dr} \tag{16.536}$$

$$=K\left(\frac{n+1}{n}\right)(\rho_c^{1/n}\theta)\frac{d(\rho_c\theta^n)}{d\xi}\frac{d\xi}{dr} \tag{16.537}$$

$$=K\left(\frac{n+1}{n}\right)\rho_c^{1+1/n}\theta n\theta^{n-1}\frac{d\theta}{d\xi}\frac{d\xi}{dr} \tag{16.538}$$

$$=K(n+1)\rho_c^{1+1/n}\theta^n\frac{d\theta}{d\xi}\frac{1}{a} \tag{16.539}$$

$$\rightarrow\ \left.\frac{dP}{dr}\right|_{r=0}=0 \tag{16.540}$$

(g) The mass integral can be rewritten by using the Lane-Emden equation

$$M = 4\pi \int_0^R \rho(r) r^2 dr \tag{16.541}$$
$$= 4\pi\rho_c a^3 \int_0^{\xi_1} \theta^n \xi^2 d\xi \tag{16.542}$$
$$= -4\pi\rho_c a^3 \int_0^{\xi_1} \frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) d\xi \tag{16.543}$$
$$= -4\pi\rho_c a^3 \left[\xi^2 \frac{d\theta}{d\xi}\right]_0^{\xi_1} \tag{16.544}$$
$$= -4\pi\rho_c a^3 \xi_1^2 \theta'(\xi_1). \tag{16.545}$$

For the radius R we find

$$R = a\xi_1 \tag{16.546}$$
$$= \left[\frac{(n+1)K\rho_c^{1/n-1}}{4\pi G}\right]^{\frac{1}{2}} \xi_1 \tag{16.547}$$
$$= \left[\frac{(n+1)K}{4\pi G}\right]^{\frac{1}{2}} \rho_c^{(1-n)/2n} \xi_1 \tag{16.548}$$
$$\tag{16.549}$$

$$\xi_1 = \left[\frac{(n+1)K}{4\pi G}\right]^{-\frac{1}{2}} \rho_c^{(n-1)/2n} R. \tag{16.550}$$

Furthermore we can write (somewhat arbitrarily)

$$\xi_1^2 = \xi_1^2 1^{\frac{3-n}{1-n}} = \xi_1^2 \left(\frac{R}{a\xi_1}\right)^{\frac{3-n}{1-n}} \tag{16.551}$$
$$= \xi_1^{2-\frac{3-n}{1-n}} \left(\frac{1}{a}\right)^{\frac{3-n}{1-n}} R^{\frac{3-n}{1-n}} \tag{16.552}$$
$$= \xi_1^{\frac{n+1}{n-1}} a^{\frac{3-n}{n-1}} R^{\frac{3-n}{1-n}} \tag{16.553}$$

which results in

$$M = 4\pi\rho_c a^3 \cdot \xi_1^{\frac{n+1}{n-1}} a^{\frac{3-n}{n-1}} R^{\frac{3-n}{1-n}} \theta'(\xi_1) \tag{16.554}$$
$$= 4\pi\rho_c a^{2n/(n-1)} \cdot \xi_1^{\frac{n+1}{n-1}} R^{\frac{3-n}{1-n}} \theta'(\xi_1) \tag{16.555}$$
$$= 4\pi\rho_c \left[\frac{(n+1)K\rho_c^{1/n-1}}{4\pi G}\right]^{n/(n-1)} \cdot \xi_1^{\frac{n+1}{n-1}} R^{\frac{3-n}{1-n}} \theta'(\xi_1) \tag{16.556}$$
$$= 4\pi R^{\frac{3-n}{1-n}} \left[\frac{(n+1)K}{4\pi G}\right]^{n/(n-1)} \cdot \xi_1^{\frac{n+1}{n-1}} \theta'(\xi_1) \tag{16.557}$$

which is an expression without ρ_c.

(h) For $n = 1$ we have

$$R = \left[\frac{K}{2\pi G}\right]^{1/2} \xi_1 \tag{16.558}$$

which means R is independent of mass and central pressure and therefore constant for all objects. So we conclude $R_S = R_J$.

For $n = 1$ have have $\theta(\xi) = \sin\xi/\xi$ and find

$$\theta' = \frac{\xi\cos\xi - \sin\xi}{\xi^2} \tag{16.559}$$

$$\xi_1 = \pi \tag{16.560}$$

$$\theta'(\xi_1) = -1/\pi. \tag{16.561}$$

Therefore

$$R = \pi\left[\frac{K}{2\pi G}\right]^{1/2} \tag{16.562}$$

$$M = 4\pi^2\left[\frac{K}{2\pi G}\right]^{3/2}\rho_c = 4\pi^2\left[\frac{R}{\pi}\right]^3\rho_c \tag{16.563}$$

$$= \frac{4R^3}{\pi}\rho_c \tag{16.564}$$

$$\to \rho_c = \frac{\pi M}{4R^3} = \frac{\pi}{3}\frac{\pi M}{4\frac{\pi}{3}R^3} = \frac{\pi^2}{3}\rho_{\text{avg}} \tag{16.565}$$

which gives $\rho_{c,J} = 4.6\cdot 10^{12}\text{kg/m}^3$ and $\rho_{c,S} = 1.3\cdot 10^{12}\text{kg/m}^3$.

16.6.19 Exercise 13.5 Example: Shape of a Constant-Density, Spinning Planet

(a) The gravitational potential is given by the integral of the mass distribution $\rho(\vec{r})$

$$\Phi(\vec{r}) = -G\int\frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}d^3\vec{r}' \tag{16.566}$$

$$= -2\pi G\int\frac{\rho(\vec{r}')}{\sqrt{r^2+r'^2-2rr'\cos\theta}}r'^2\sin\theta d\theta dr' \tag{16.567}$$

$$= -\frac{2\pi G\rho}{r}\int_0^R\frac{1}{\sqrt{1+(r'/r)^2-2r'/r\cos\theta}}d(\cos\theta)r'^2dr' \tag{16.568}$$

$$= -\frac{2\pi G\rho}{r}\int_0^R r'\left(r+r'-\sqrt{(r-r')^2}\right)dr'. \tag{16.569}$$

For $\vec{r}$ inside the mass distribution the integral needs to be split

$$\Phi(\vec{r}) = -\frac{2\pi G\rho}{r}\left[\int_0^r r'\,(2r')\,dr' + \int_r^R r'\,(2r)\,dr'\right] \tag{16.570}$$

$$= -\frac{4\pi G\rho}{r}\left[\int_0^r r'^2dr' + r\int_r^R r'dr'\right] \tag{16.571}$$

$$= -\frac{4\pi G\rho}{r}\left[\frac{r^3}{3} + r\frac{1}{2}(R^2-r^2)\right] \tag{16.572}$$

$$= -4\pi G\rho\left[\frac{r^2}{3} + \frac{1}{2}(R^2-r^2)\right] \tag{16.573}$$

$$= -4\pi G\rho\left[-\frac{r^2}{6} + \frac{R^2}{2}\right] \tag{16.574}$$

$$= \frac{2\pi G\rho}{3}\left[r^2 - 3R^2\right] \tag{16.575}$$

(b) For the centrifugal force and potential we find

$$F_{\text{cen}} = m\frac{v^2}{\varpi} = m\Omega^2\varpi \tag{16.576}$$
$$\Phi_{\text{cen}} = -\frac{1}{2}(\Omega\varpi)^2 \tag{16.577}$$
$$= -\frac{1}{2}(\Omega r\cos\theta)^2 \tag{16.578}$$
$$= -\frac{1}{2}(\Omega r\sin\theta')^2 \tag{16.579}$$
$$= -\frac{1}{2}(\vec{\Omega}\times\vec{r})^2 \tag{16.580}$$

we results in

$$\Phi = \frac{2\pi G\rho r^2}{3} - 2\pi G\rho R^2 - \frac{1}{2}(\Omega r\cos\theta)^2 \tag{16.581}$$
$$= \frac{2\pi G\rho r^2}{3} - 2\pi G\rho R^2 - \frac{1}{2}\Omega^2 r^2\cos^2\theta \tag{16.582}$$
$$= \frac{2\pi G\rho r^2}{3} - 2\pi G\rho R^2 - \frac{\Omega^2}{3}r^2\frac{1}{2}3\cos^2\theta \tag{16.583}$$
$$= \frac{2\pi G\rho r^2}{3} - 2\pi G\rho R^2 - \frac{\Omega^2 r^2}{3} - \frac{\Omega^2}{3}r^2\frac{1}{2}(3\cos^2\theta - 1) \tag{16.584}$$
$$= \frac{2\pi G\rho r^2}{3} - 2\pi G\rho R^2 - \frac{\Omega^2 r^2}{3} - \frac{\Omega^2}{3}r^2 P_2(\cos\theta) \tag{16.585}$$
$$\tag{16.586}$$

(c)

(d)

(e)

16.6.20 Exercise 13.7 Problem: A Hole in My Bucket

Applying the Bernoulli equation

$$\frac{1}{2}\rho v^2 + \rho g h = \text{const} \tag{16.587}$$

to the hole and the water surface we get

$$\frac{1}{2}\rho v_{\text{hole}}^2 = \rho g h + \frac{1}{2}\rho v_{\text{surf}}^2. \tag{16.588}$$

The change in volume is given by (neglecting limitations from the Hagen-Poiseulle equation but having a hole significantly smaller than the bucket surface - with hole has the same size the assumption of a static pressure does not make sense)

$$\frac{dV}{dt} = A_{\text{bucket}}v_{\text{surf}} = A_{\text{hole}}v_{\text{hole}} = A_{\text{bucket}}\frac{dh}{dt} \tag{16.589}$$
$$\rightarrow \quad v_{\text{hole}} = \frac{A_{\text{bucket}}}{A_{\text{hole}}}\frac{dh}{dt} \tag{16.590}$$
$$\rightarrow \quad v_{\text{surf}} = \frac{dh}{dt}. \tag{16.591}$$

With this the Bernoulli equation turns into

$$\frac{1}{2}\left(\frac{A_{\text{bucket}}}{A_{\text{hole}}}\right)^2\left(\frac{dh}{dt}\right)^2 = gh + \frac{1}{2}\left(\frac{dh}{dt}\right)^2 \tag{16.592}$$

$$\left(\frac{dh}{dt}\right)^2 = \frac{2g}{\left(\frac{A_{\text{bucket}}}{A_{\text{hole}}}\right)^2 - 1}h \tag{16.593}$$

$$= \frac{2A_{\text{hole}}^2 g}{A_{\text{bucket}}^2 - A_{\text{hole}}^2}h \tag{16.594}$$

$$\tag{16.595}$$

which can be solved by

$$\frac{dh}{\sqrt{h}} = -\sqrt{\frac{2A_{\text{hole}}^2 g}{A_{\text{bucket}}^2 - A_{\text{hole}}^2}}dt \tag{16.596}$$

$$2\sqrt{h} = -\sqrt{\frac{2A_{\text{hole}}^2 g}{A_{\text{bucket}}^2 - A_{\text{hole}}^2}} \cdot t + 2\sqrt{H_0} \tag{16.597}$$

$$h(t) = \left(\sqrt{H_0} - \sqrt{\frac{A_{\text{hole}}^2 g}{2(A_{\text{bucket}}^2 - A_{\text{hole}}^2)}} \cdot t\right)^2 \tag{16.598}$$

$$\approx \left(\sqrt{H_0} - \frac{A_{\text{hole}}}{A_{\text{surface}}}\sqrt{\frac{g}{2}} \cdot t\right)^2 \tag{16.599}$$

For the time T to empty the bucket we solve for $h(T) = 0$ and obtain

$$T = \sqrt{\frac{2H_0}{g}}\frac{A_{\text{bucket}}}{A_{\text{hole}}}. \tag{16.600}$$

in the case of small holes or buckets with thick walls D the Hagen-Poiseulle should be taken into account.

16.6.21 Exercise 14.1 Practice: Constant-Angular-Momentum Flow - Relative Motion of Fluid Elements

Taylor expansion of the components of the velocity field gives

$$v_j(\mathbf{x} + \boldsymbol{\xi}) = v_j(\mathbf{x}) + \left(\left.\frac{\partial v_j(\mathbf{y})}{\partial y_i}\right|_{y=x} \xi_i\right). \tag{16.601}$$

For the vector we can then write

$$\mathbf{v}(\mathbf{x} + \boldsymbol{\xi}) = \mathbf{v}(\mathbf{x}) + \left(\left.\frac{\partial v_j(\mathbf{y})}{\partial y_i}\right|_{y=x} \xi_i\right)\mathbf{e}_j \tag{16.602}$$

$$\nabla_{\boldsymbol{\xi}}\mathbf{v} \equiv \mathbf{v}(\mathbf{x} + \boldsymbol{\xi}) - \mathbf{v}(\mathbf{x}) \tag{16.603}$$

$$= \boldsymbol{\xi} \cdot \nabla\mathbf{v} \tag{16.604}$$

For the constant-angular-momentum flow we have

$$\mathbf{v} = \frac{1}{\varpi^2}\mathbf{j} \times \mathbf{x} \tag{16.605}$$

$$= \frac{1}{\varpi^2}j\varpi\mathbf{e}_\phi = \frac{j}{\varpi}\mathbf{e}_\phi. \tag{16.606}$$

The only non-vanishing component of $\nabla\mathbf{v}$ is

$$\frac{\partial v_\phi}{\partial \varpi} = -\frac{j}{\varpi^2}. \tag{16.607}$$

- tangential: $\boldsymbol{\xi} = \varpi\, d\phi \mathbf{e}_\phi = d\epsilon\, \mathbf{e}_\phi$ Not done yet
- radial: $\boldsymbol{\xi} = d\varpi\, \mathbf{e}_\varpi = d\epsilon\, \mathbf{e}_\varpi$ Not done yet

16.6.22 Exercise 14.2 Practice: Vorticity and Incompressibility

Vorticity: $\boldsymbol{\omega} = \nabla \times \mathbf{v}$. Compressibility: $\rho = \text{const}$

$$\frac{d\rho}{dt} + \rho\nabla \cdot \mathbf{v} = 0 \quad \rightarrow \quad \nabla \cdot \mathbf{v} = 0 \tag{16.608}$$

(a)

$$\nabla \times \mathbf{v} = \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\mathbf{e}_z = 0 \cdot \mathbf{e}_z \tag{16.609}$$

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 2y \tag{16.610}$$

(b)

$$\nabla \times \mathbf{v} = -2y \cdot \mathbf{e}_z \tag{16.611}$$

$$\nabla \cdot \mathbf{v} = 0 \tag{16.612}$$

(c)

$$\nabla \times \mathbf{v} = \frac{1}{\varpi}\left(\frac{\partial(\varpi v_\phi)}{\partial \varpi} - \frac{\partial v_\varpi}{\partial \phi}\right)\mathbf{e}_z = 2 \cdot \mathbf{e}_z \tag{16.613}$$

$$\nabla \cdot \mathbf{v} = \frac{1}{\varpi}\left(\frac{\partial(\varpi v_\varpi)}{\partial \varpi} + \frac{\partial v_\phi}{\partial \phi}\right) = 0 \tag{16.614}$$

(d)

$$\nabla \times \mathbf{v} = 0 \cdot \mathbf{e}_z \tag{16.615}$$

$$\nabla \cdot \mathbf{v} = 0 \tag{16.616}$$

16.6.23 Exercise 16.9 Example: Breaking of a Dam

The PDEs

$$h_t + hv_x + vh_x = 0 \tag{16.617}$$

$$v_t + vv_x + gh_x = 0 \tag{16.618}$$

can be written as

$$Au_t + Bu_x = 0 \tag{16.619}$$

$$u = \begin{pmatrix} h \\ v \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} v & h \\ g & v \end{pmatrix} \tag{16.620}$$

Now

$$vh_t + vhv_x + v^2 h_x = 0 \tag{16.621}$$
$$hv_t + hvv_x + ghh_x = 0 \tag{16.622}$$
$$\rightarrow (hv)_t = hv_t + vh_t \tag{16.623}$$
$$= -2vhv_x - v^2 h_x - ghh_x \tag{16.624}$$
$$= -\partial_x \left(h \left[v^2 + \frac{1}{2} gh \right] \right) \tag{16.625}$$

Not done yet

16.7 Walter - Astronautics

16.7.1 Problem 1.1 - Balloon Propulsion

For the mass flow rate we have

$$\dot{m} = \rho \dot{V} \approx \rho A_t v_t \stackrel{!}{=} \frac{\rho V}{T} \quad \rightarrow \quad v_t = \frac{V}{A_t T} = 20\text{m/s} \tag{16.626}$$

and the speed of sound in a diatomic gas ($f = 5$, $\rho_0 = 1.225\text{kg/m}^3$, $P_0 = 101.3 \cdot 10^3 Pa$) is

$$c = \sqrt{\kappa \frac{p}{\rho}} = \sqrt{\frac{f+2}{f} \frac{P}{\rho}} = 340\text{m/s} \tag{16.627}$$

which justifies $v_t \ll c$. Newtons second law gives for the momentum thrust

$$F_e = \frac{dp}{dt} = \dot{m} v_t = \frac{\rho V}{T} \frac{V}{A_t T} = \frac{\rho}{A_t} \left(\frac{V}{T} \right)^2 = 0.0258\text{N} \tag{16.628}$$

From the Bernoulli equation we can obtain the pressure difference

$$P = P_0 + \frac{\rho}{2} v_t^2 \quad \rightarrow \quad P - P_0 = \frac{\rho}{2} v_t^2 \tag{16.629}$$

and can then calculate the pressure thrust

$$F_p = A_t (P - P_0) = \frac{A_t \rho}{2} v_t^2 = \frac{\rho V^2}{2 A_t T^2} = 0.0129\text{N} \tag{16.630}$$

and see $F_e = 2F_p$.

16.7.2 Problem 1.2 - Nozzle Exit Area of an SSME

For the total thrust we have in vacuum and at sea level we have

$$F_{\text{SL}} = A_t (P - P_0) + \dot{m} v_t \tag{16.631}$$
$$F_{\text{V}} = A_t (P - 0) + \dot{m} v_t \tag{16.632}$$

which implies with $P_0 = 101.3\text{Pa}$

$$A_t = \frac{F_{\text{V}} - F_{\text{SL}}}{P_0} = 4.55\text{m}^2 \tag{16.633}$$

16.7.3 Problem 1.3 - Proof of $\eta_{\mathbf{VDF}} \leq 1$

$$\langle \nu_e \rangle_\mu = \frac{\int_0^{\pi/2} \nu_e(\theta) \cdot \mu(\theta) \sin\theta \, d\theta}{\int_0^{\pi/2} \mu(\theta) \sin\theta \, d\theta} \tag{16.634}$$

$$\langle \nu_e \rangle_\mu^2 \leq \langle \nu_e^2 \rangle_\mu \tag{16.635}$$

Not done yet

16.7.4 Problem 4.1 - Gas Velocity-Pressure Relation in a Nozzle

- Using the ideal gas equation $pV = NkT$ we have for a adiabatic process

$$pV^\kappa = p\left(\frac{NkT}{p}\right)^\kappa \tag{16.636}$$
$$= p^{1-\kappa} T^\kappa \tag{16.637}$$
$$= \text{const} \tag{16.638}$$
$$\rightarrow \quad p^{\frac{1-\kappa}{\kappa}} T = p_0^{\frac{1-\kappa}{\kappa}} T_0 \tag{16.639}$$

and with $pV = nRT$ we obtain more conservation laws for adiabatic processes

$$\rho = \frac{m}{V} = \frac{nM_p}{V} = \frac{M_p p}{RT} \quad \rightarrow \quad p = \frac{R}{M_p}\rho T \tag{16.640}$$
$$(\rho T)^{\frac{1-\kappa}{\kappa}} T = \text{const} \tag{16.641}$$
$$\rho^{1-\kappa} T = \text{const} \tag{16.642}$$

as well as

$$\rho^{1-\kappa} T = \text{const} \tag{16.643}$$
$$\rho^{1-\kappa}\left(\frac{p}{\rho}\right) = \text{const} \tag{16.644}$$
$$\rho^{-\kappa} p = \text{const} \tag{16.645}$$
$$\rho p^{-\frac{1}{\kappa}} = \text{const} \tag{16.646}$$

We obtain with $\kappa = \frac{2+n}{n}$ for the energy conversion efficiency

$$\eta = 1 - \frac{T}{T_0} = 1 - \left(\frac{p}{p_0}\right)^{\frac{\kappa-1}{\kappa}} = 1 - \left(\frac{\rho}{\rho_0}\right)^{\kappa-1} \tag{16.647}$$
$$= 1 - \left(\frac{p}{p_0}\right)^{\frac{2}{n+2}} = 1 - \left(\frac{\rho}{\rho_0}\right)^{\frac{2}{n}} \tag{16.648}$$

- Energy conservation along the engine axis gives

$$\frac{1}{2} m_p v_0^2 + m_p c_p T_0 = \frac{1}{2} m_p v^2 + m_p c_p T. \tag{16.649}$$

with $v_0 = 0$ we obtain. for the gas flow velocity

$$v^2 = 2c_p(T_0 - T) = 2c_p T_0 \eta = 2\left(\frac{\kappa}{\kappa - 1}\frac{R}{M_p}\right) T_0 \eta \tag{16.650}$$

which is called the St. Venant-Wantzel equation. Differentiating yields

$$2v\frac{dv}{dp} = 2\left(\frac{\kappa}{\kappa - 1}\frac{R}{M_p}\right) T_0 \frac{d\eta}{dp} \tag{16.651}$$

- The continuity equation is given by

$$\dot{m}_p = \rho v A \quad \rightarrow \quad v = \frac{\dot{m}_p}{A\rho} \tag{16.652}$$

Now we can combine all parts

$$\begin{aligned}
\frac{dv}{dp} &= \frac{1}{v}\left(\frac{\kappa}{\kappa-1}\frac{R}{M_p}\right) T_0 \frac{d\eta}{dp} && (16.653)\\
&= -\frac{A\rho}{\dot{m}_p}\left(\frac{\kappa}{\kappa-1}\frac{R}{M_p}\right) T_0 \frac{\kappa-1}{\kappa}\left(\frac{p}{p_0}\right)^{-\frac{1}{\kappa}}\frac{1}{p_0} && (16.654)\\
&= -\frac{A\rho}{\dot{m}_p}\left(\frac{\kappa}{\kappa-1}\frac{R}{M_p}\right)\frac{\kappa-1}{\kappa}\left(\frac{p}{p_0}\right)^{-\frac{1}{\kappa}}\frac{M_p}{R\rho_0} && (16.655)\\
&= -\frac{A}{\dot{m}_p}\left(\frac{p}{p_0}\right)^{-\frac{1}{\kappa}}\frac{\rho}{\rho_0} && (16.656)\\
&= -\frac{A}{\dot{m}_p} && (16.657)
\end{aligned}$$

and obtain

$$dv = -\frac{A}{\dot{m}_p} dp. \tag{16.658}$$

16.7.5 Problem 4.2 - Approximation of the Infinite-Expansion Coefficient

$$\begin{aligned}
C_\infty &\equiv (n+2)\sqrt{\frac{n^n}{(n+1)^{n+1}}} && (16.659)\\
&= \frac{n+2}{\sqrt{n+1}}\left(\frac{n}{n+1}\right)^{n/2} && (16.660)\\
&= \frac{n+2}{\sqrt{n+1}}\left(1+\frac{1}{n}\right)^{-n/2} && (16.661)\\
&= \frac{n+2}{\sqrt{n+1}}\frac{4096}{6561}\left[1+\frac{1}{18}\left(1+\log\frac{2^{27}}{3^{18}}\right)(n-8)+O\left(n^2\right)\right] && (16.662)\\
&\approx 0.624\frac{n+2}{\sqrt{n+1}} && (16.663)
\end{aligned}$$

16.7.6 Problem 7.1 - Solutions of Poisson's Equation

$$\triangle U = 4\pi\gamma\rho \tag{16.664}$$

$$\begin{aligned}
\triangle_x G(x) &= 4\pi\gamma\delta(x) && (16.665)\\
G(x) &= \frac{1}{\sqrt{2\pi}}\int d^n y\, g(y) e^{-ixy} && (16.666)\\
\triangle_x G(x) = \frac{1}{\sqrt{2\pi}}\int d^n y\, g(y)(-iy)^2 e^{-ixy} &\equiv 4\pi\gamma\delta(x) && (16.667)\\
\rightarrow g(y) &= -\frac{e^{-i..}}{y^2} && (16.668)
\end{aligned}$$

16.8 Sutton, Biblarz - Rocket Propulsion Elements

16.8.1 Problem 2.1

a)

$$F = \frac{dp}{dt} = \frac{dm}{dt} v = \frac{50}{60} \text{kg/s} \cdot 200 \text{m/s} = 166.66 N \tag{16.669}$$

b) changing the reference frame

$$\frac{dm}{dt} = A\rho v \quad \rightarrow \quad A = \frac{1}{v\rho}\frac{dm}{dt} \tag{16.670}$$

$$F = A\rho(v-u)\cdot(v-u) = \frac{dm}{dt}\frac{v-u}{v}(v-u) = 144.322\text{N} \tag{16.671}$$

16.8.2 Problem 2.2

a) effective velocity

$$v_{\text{eff}} = \frac{F}{\dot{m}} = 2,300 \text{m/s} \tag{16.672}$$

b) kinetic jet energy per unit flow of propellant

$$\rho_{\text{kin}} = \frac{1}{2} v_{\text{eff}}^2 = 2.646 \text{kJ/kg} \tag{16.673}$$

c) internal efficiency

$$\eta_{\text{int}} = \frac{\frac{1}{2} m v_{\text{eff}}^2}{\eta_{\text{comb}} \frac{Q_{\text{chem}}}{m} m} = \frac{\frac{1}{2} v_{\text{eff}}^2}{\eta_{\text{comb}} c_{\text{chem}}} = 38.3 \tag{16.674}$$

16.9 Strogatz - Nonlinear Dynamics and Chaos

16.10 Pierrehumbert - Principles of Planetary Climate

16.10.1 Problem 1.10 - Energy of a comet

$$E = \frac{1}{2}mv^2 = 2 \cdot 10^{17}\,\mathrm{J} \tag{16.675}$$

16.10.2 Problem 1.11 - Mass of Titan

$$g = G\frac{M}{R^2} \qquad \rightarrow \quad M = \frac{gR^2}{G} = 1.35 \cdot 10^{23}\,\mathrm{kg} \tag{16.676}$$

16.10.3 Problem 1.11 - Moon falling onto earth

Momentum conservation gives $\vec{p}_M = -\vec{p}_E$ and therefore

$$-G\frac{M_E M_M}{d} = -G\frac{M_E M_M}{R_E + R_M} + \frac{p_M^2}{2M_M} + \frac{p_E^2}{2M_E} \tag{16.677}$$

$$-G\frac{M_E M_M}{d} = -G\frac{M_E M_M}{R_E + R_M} + \frac{p_M^2 M_E + p_E^2 M_M}{2M_M M_E} \tag{16.678}$$

$$-G\frac{M_E M_M}{d} = -G\frac{M_E M_M}{R_E + R_M} + \frac{p_M^2}{2\mu} \tag{16.679}$$

then

$$p_M^2 = -2\mu G\frac{M_E M_M\left[(R_E + R_M) - d\right]}{d(R_E + R_M)} \tag{16.680}$$

$$= 2G\frac{(M_E M_M)^2\left[d - (R_E + R_M)\right]}{(M_E + M_M)d(R_E + R_M)} \tag{16.681}$$

$$v_M = \sqrt{2G\frac{(M_E M_M)^2\left[d - (R_E + R_M)\right]}{M_M^2(M_E + M_M)d(R_E + R_M)}} \tag{16.682}$$

$$= \sqrt{2G\frac{M_E^2\left[d - (R_E + R_M)\right]}{(M_E + M_M)d(R_E + R_M)}} \tag{16.683}$$

$$= \sqrt{2G\frac{M_E}{(1 + \frac{M_M}{M_E})}\left[\frac{1}{R_E + R_M} - \frac{1}{d}\right]} \tag{16.684}$$

therefore

$$v_{\mathrm{impact}} = v_M + v_E = v_M + \frac{v_M M_M}{M_E} = v_M\left(1 + \frac{M_M}{M_E}\right) \tag{16.685}$$

$$= \sqrt{2GM_E\left(1 + \frac{M_M}{M_E}\right)\left[\frac{1}{R_E + R_M} - \frac{1}{d}\right]} \tag{16.686}$$

$$= 9805\,\mathrm{m/s} \tag{16.687}$$

16.10.4 Problem 2.1 - Force on pressured spaceship

$$F = p \cdot A = 4\pi r^2 p = 12.6 \cdot 10^7\,\mathrm{N} \tag{16.688}$$

16.10.5 Problem 2.2 - Hollow metal sphere

Force on one half sphere

$$\begin{aligned} F_\perp &= \Delta p r^2 \int d\phi \int_0^{\pi/2} d\theta \sin\theta \cdot \cos\theta && (16.689)\\ &= 2\pi \Delta p r^2 \int_0^{\pi/2} d\theta \sin\theta \cdot \cos\theta && (16.690)\\ &= \pi r^2 \Delta p && (16.691) \end{aligned}$$

16.10.6 Problem 2.3 - Earths atmosphere

Ideal gas equation of state

$$\begin{aligned} pV &= NkT \qquad \rightarrow \qquad \frac{N}{V} = \frac{p}{kT} && (16.692)\\ \rho_{0\mathrm{C}} &= \frac{M}{V} = \frac{m_{N_2/O_2} N}{V} && (16.693)\\ &= \frac{(0.8 \cdot 2 \cdot 14 + 0.2 \cdot 2 \cdot 16) m_u p}{kT} && (16.694)\\ &= 1.26\,\mathrm{kg/m^3} && (16.695)\\ \rho_{50\mathrm{C}} &= 1.07\,\mathrm{kg/m^3} && (16.696) \end{aligned}$$

$$\begin{aligned} F_{\mathrm{lift}} &= gV(\rho_{0\mathrm{C}} - \rho_{50\mathrm{C}}) && (16.697)\\ &= \frac{4}{3}\pi R^3 g(\rho_{0\mathrm{C}} - \rho_{50\mathrm{C}}) && (16.698)\\ &= 22.2\,\mathrm{N} && (16.699) \end{aligned}$$

16.10.7 Problem 2.5 - Bicycle tire

$$\begin{aligned} M_{\mathrm{gas}} &= \rho V && (16.700)\\ &= \frac{m_{\mathrm{gas}} m_u p V}{kT} && (16.701)\\ M_{\mathrm{tire+air}} &= 0.1143\,\mathrm{kg} && (16.702)\\ M_{\mathrm{tire+CO2}} &= 0.1219\,\mathrm{kg} && (16.703)\\ M_{\mathrm{tire+He}} &= 0.1019\,\mathrm{kg} && (16.704)\\ F &= \frac{m_{\mathrm{air}} m_u p}{kT} V g + m_{\mathrm{tire}} g - \frac{m_{\mathrm{air}} m_u p_0}{kT} g V && (16.705)\\ &= 1.09\,\mathrm{N} && (16.706) \end{aligned}$$

16.10.8 Problem 2.6 - Density of gases on planets

$$\begin{aligned} \rho_{\mathrm{CO2,Mars}} &= \frac{m_{\mathrm{CO2}} m_u p}{kT} = 0.0144\,\mathrm{kg/m^3} && (16.707)\\ \rho_{\mathrm{N2,Titan}} &= \frac{m_{\mathrm{CO2}} m_u p}{kT} = 5.315\,\mathrm{kg/m^3} && (16.708)\\ \rho_{\mathrm{CO2,Venus}} &= \frac{m_{\mathrm{CO2}} m_u p}{kT} = 66.04\,\mathrm{kg/m^3} && (16.709)\\ & && (16.710) \end{aligned}$$

Chapter 17

Simulations of Cosmic Structure Formation - MAUCA 2018

17.0.1 Exercise 2

1. Summary of Friedmann equations We start with the total energy density

$$\rho = \frac{3H_0^2}{8\pi G}\left[\Omega_\Lambda + \Omega_m\left(\frac{a_0}{a}\right)^3 + \Omega_r\left(\frac{a_0}{a}\right)^4\right] \tag{17.1}$$

Using the Friedman equation we get

$$\dot{a}^2 + K = \frac{8\pi G\rho a^2}{3} \tag{17.2}$$

$$\dot{a}^2 - \Omega_K a_0^2 H_0^2 = a^2 H_0^2\left[\Omega_\Lambda + \Omega_m\left(\frac{a_0}{a}\right)^3 + \Omega_r\left(\frac{a_0}{a}\right)^4\right] \tag{17.3}$$

$$\dot{a}^2 = a^2 H_0^2\left[\Omega_\Lambda + \Omega_m\left(\frac{a_0}{a}\right)^3 + \Omega_r\left(\frac{a_0}{a}\right)^4 + \Omega_K\left(\frac{a_0}{a}\right)^2\right] \tag{17.4}$$

where we used $\Omega_K = -K/(a_0H_0)^2$. With $a(t_0) = a_0$ and $H_0 \equiv \frac{\dot{a}(t_0)}{a(t_0)}$ we find a constraint on the Ω parameters

$$\Omega_\Lambda + \Omega_m + \Omega_r + \Omega_K = 1 \tag{17.5}$$

Then with $x = a/a_0$

$$\dot{x}^2 = x^2 H_0^2\left[\Omega_\Lambda + \Omega_m x^{-3} + \Omega_r x^{-4} + \Omega_K x^{-2}\right] \tag{17.6}$$

$$\frac{dx}{dt} = H_0\sqrt{\Omega_\Lambda x^2 + \Omega_m x^{-1} + \Omega_r x^{-2} + \Omega_K} \tag{17.7}$$

$$H_0 dt = \frac{dx}{\sqrt{\Omega_\Lambda x^2 + \Omega_m x^{-1} + \Omega_r x^{-2} + \Omega_K}} \tag{17.8}$$

2. Solutions

$$\dot{x} = H_0\sqrt{\Omega_\Lambda x^2 + \Omega_m x^{-1} + \Omega_r x^{-2} + \Omega_K} \tag{17.9}$$

- $k = 0, \Omega_\Lambda = 1, \Omega_m = \Omega_r = 0$

$$\dot{x} = H_0 x \quad \rightarrow \quad x = e^{H_0 t} \tag{17.10}$$

- $k = 0, \Omega_m = 1, \Omega_\Lambda = \Omega_r = 0$

$$\dot{x} = H_0 x^{-1/2} \quad \rightarrow \quad x = \left(1 + \frac{3}{2}H_0 t\right)^{2/3} \tag{17.11}$$

- $k = 0, \Omega_r = 1, \Omega_\Lambda = \Omega_m = 0$

$$\dot{x} = H_0 x^{-1} \quad \rightarrow \quad x = 1 + H_0 t \tag{17.12}$$

Chapter 18

Doodling I

18.1 Basic Principles

Relativistic QFT unifies quantum mechanics ($\hbar$) and special relativity (c). We will put $\hbar = 1 = c$.

- Relativistic Invariance

  Coordinates and how they transform

$$x^\mu = (ct, \vec{x}), \qquad \partial_\mu = \frac{\partial}{\partial x^\mu} \tag{18.1}$$

$$x^\mu \to x'^\mu = a^\mu + \Lambda^\mu{}_\nu x^\nu \tag{18.2}$$

  All transformations (a,Λ) - translations and Lorentz transformations - build the Poincare group $\mathcal{P}$.

$$x^\mu x_\mu = x^\mu x^\nu \eta_{\mu\nu}, \qquad \eta = \mathrm{diag}(1, -1, -1, -1) \tag{18.3}$$

  Invariance requirement

$$x'^2 \stackrel{!}{=} x^2 \quad \Rightarrow \quad \Lambda_\mu^{\ \rho} \Lambda_\nu^{\ \sigma} \eta_{\rho\sigma} \stackrel{!}{=} \eta_{\mu\nu} \tag{18.4}$$

18.2 Representation Theory of the Poincare group

- Relevant because: elementary particles = unitary, irreducible representation of the Poincare group $\mathcal{P}$.
- But there are no finite dimensional unitary representations of non-compact groups $\to \infty$-dimensional representation (need Hilbert space).
- So each transformation

$$x^\mu \to x'^\mu = a^\mu + \Lambda^\mu{}_\nu x^\nu \tag{18.5}$$

  will be represented by some unitary operator $U(a, \Lambda)$ (with $U^\dagger = U^{-1}$) acting on some infinitely dimensional Hilbert space.
- Let's look at pure translations first

$$U(a, 1) \equiv U(a) = e^{ia^\mu P_\mu} \tag{18.6}$$

  with the generators (the momentum operator) of the translations $P_\mu = (H, \vec{P})$ with needs to be hermitian $P_\mu = P_\mu^\dagger$.

- For the pure Lorentz transformations we get

$$U(0,\Lambda) \equiv U(\Lambda) = e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}} \tag{18.7}$$

with the generators of the Lorenz transformations $M^{\mu\nu}$ which need to obey $M_{\mu\nu} = M^{\dagger}_{\mu\nu}$. They can be identified by $M^{12} \equiv J^3$, $M^{23} \equiv J^1$ and $M^{31} \equiv J^2$ with the angular momentum operator $\vec{J}$ (generating rotations) and generators of the Lorentz boosts M_{0i}.

- Lets derive the commutation operator of the generators. Translations commute therefore

$$U(a_1)U(a_2) = U(a_1 + a_2) = U(a_2)U(a_1) \quad \Rightarrow \quad \boxed{[P_\mu, P_\nu] = 0} \tag{18.8}$$

In general

$$(a_1, \Lambda_1) \circ (a_2, \Lambda_2) = (a_1 + \Lambda_1 a_2, \Lambda_1\Lambda_2) \tag{18.9}$$

Then the representation need to obey

$$U(a_1, \Lambda_1)U(a_2, \Lambda_2) = U(a_1 + \Lambda_1 a_2, \Lambda_1\Lambda_2) \tag{18.10}$$

which for $\Lambda_1 = \Lambda$, $\Lambda_2 = \Lambda^{-1}$, $a_2 = a$ and $a_1 = 0$

$$U(0,\Lambda)U(a,\Lambda^{-1}) = U(\Lambda a, 1) \tag{18.11}$$

$$\rightarrow U(\Lambda)U(a)U(\Lambda^{-1}) = U(\Lambda a) \tag{18.12}$$

In lowest order this results in

$$\boxed{[M_{\rho\sigma}, P_\mu] = i\eta_{\mu\rho}P_\sigma - i\eta_{\mu\sigma}P_\rho} \tag{18.13}$$

$$\boxed{[M^{\mu\nu}, M^{\rho\sigma}] = i\eta^{\mu\rho}M^{\nu\sigma} + i\eta^{\nu\sigma}M^{\mu\rho} - i\eta^{\mu\nu}M^{\rho\sigma} - i\eta^{\rho\sigma}M^{\mu\nu}} \tag{18.14}$$

From this we can recover $[J^i, J^k] = i\varepsilon^{ijk}J^k$, $[H, \vec{P}] = 0$, $[H, \vec{J}] = 0$ (angular momentum conservation) and $[H, M_{0i}] \neq 0$ (not conserved).

How to classify irreps:

- recall QM: SO(3) use the Casimir operator $\vec{J}^2$ (operator that commutes with all generators, J_1, J_2, J_3)
  - use it to label all irreducible representations
  - use J_3 to label the states within the representation
- Casimir operators for $\mathcal{P}$ - must be relativistic invariant and commute with all generators, P^μ and $M^{\mu\nu}$
  1. $\mathcal{M}^2 = P_\mu P^\mu$ which is related to the mass (squared)
  2. $\mathcal{W}^2 = W_\mu W^\mu$ with the Pauli-Lubanski vector $W^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}M_{\nu\rho}P_\sigma$ which is related to the spin (we see $W^\mu P_\mu = 0$)
- Distinguish between mass $m^2 = 0$ and $m^2 \neq 0$
  1. Case $m^2 \neq 0$ - then in the rest frame $P^\mu = (m, 0, 0, 0)$ and because $W^\mu P_\mu = 0$ we see $W^\mu = (0, \vec{W})$ and $[W^\mu, W^\nu] = i\varepsilon^{\mu\nu\rho\sigma}W_\rho P_\sigma$. With $\vec{S} = \frac{1}{m}\vec{W}$ one can show $[S^i, S^j] = i\varepsilon^{ijk}S^k$ where the $\vec{S}$ generate the SO(3) called the little group - which is a subgroup of the Lorentz group SO(1,3).
     - The irreps are labelled by $[m^2, s]$ where $\vec{S}^2 = s(s+1)$

- The states are labelled $|[m^2, s], \vec{p}, s_3\rangle$

2. Case $m^2 = 0$ - no rest frame but $P^\mu = (k, 0, 0, k)$ and because $W^\mu P_\mu = 0$ we see $W^\mu = (W^3, W^1, W^2, W^3)$. We can show $[W^1, W^2] = 0$, $[W^3, W^1] = ikW^2$ and $[W^3, W^2] = -ikW^1$. The $\vec{W}$ generate the little group ISO(2) with only relevant cases $W^1 = W^2 = 0$ which makes $W^\mu = (W^3, 0, 0, W^3)$ and ISO(2) acting on the components (0,0). Then $W^\mu = \lambda P^\mu$ where λ is called helicity $= \vec{J} \cdot \vec{P}/|\vec{P}|$. Now using the CPT theorem
   - C: particles→antiparticles
   - P: $\vec{P} \to -\vec{P}, \vec{J} \to \vec{J}$
   - T: $\vec{P} \to -\vec{P}, \vec{J} \to -\vec{J}$
   - therefore CPT: particles→antiparticles, $\lambda \to -\lambda$

   we find
   - Gauge Theories (QED, QCD, ...): photon - charge free spin $s = 1$ particle - two helicity states ± 1 but can not have helicity $\lambda = 0$
   - Supergravity: gravitino - spin $s = 3/2$, helicity $\lambda = \pm 3/2$, but $\pm 1/2$ are missing
   - Einstein gravity: graviton - spin $s = 2$, helicity $\lambda = \pm 2$, but ± 1 are missing
   - No consistent interacting theories known for $s \geq 5/2$

   Remark: Gauge invariance is the reason for the missing helicity states. This restriction is too complicated to avoid more helicity states for higher s.

3. Summary - Representations of the Poincare groups done by Wigner

cell1	cell2	cell3
cell4	cell5	cell6
cell7	cell8	cell9

18.2.1 QFT of a free scalar field

Chapter 19

Doodling II

1. Harmonic osci

$$H = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 \qquad \text{with } [\hat{x}, \hat{p}] = i \tag{19.1}$$

$$= \omega\left(a^\dagger a + \frac{1}{2}\right) \qquad \text{with } [a, a^\dagger] = 1 \tag{19.2}$$

   and in the Heisenberg picture

$$i\frac{\partial}{\partial t}a = [a, H] = ... = \omega a \quad \rightarrow \quad a(t) = a(0)e^{-i\omega t} \tag{19.3}$$

2. Simplest Lorentz-invariant equation of motion (photon - massless, spin 0) - $\Box A_\mu = 0$ now just consider one component

$$\Box\phi = (\partial_{tt} - \triangle)\phi = 0 \tag{19.4}$$

$$\phi(\vec{x}, t) = a_p(t)e^{i\vec{p}\vec{x}} \quad \rightarrow \quad (\partial_{tt} + \vec{p}^2)a_p(t) = 0 \tag{19.5}$$

$$\phi(\vec{x}, t) = a_p e^{-i\omega t + i\vec{p}\vec{x}} \quad \rightarrow \quad \omega = \sqrt{\vec{p}^2} \tag{19.6}$$

   then

$$\phi_0(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3}\left(a_p(t)e^{i\vec{p}\vec{x}} + a_p^\dagger(t)e^{-i\vec{p}\vec{x}}\right) \tag{19.7}$$

$$= \int \frac{d^3p}{(2\pi)^3}\left(a_p e^{ipx} + a_p^* e^{-ipx}\right) \tag{19.8}$$

3. **Conclusion**: - as relativistic equation of motion is equivalent to multiple harmonic osci's then relativistic Hamiltonian should be the sum of osci's

$$H_0 = \int \frac{d^3p}{(2\pi)^3}\omega_p\left(a_p^\dagger a_p + \frac{1}{2}\right) \tag{19.9}$$

   Physical interpretation - Many quantum mechanical systems - one for each $\vec{p}$ - n-th excitation of osci $\vec{p}$ represents n (non-interacting) particles which makes sense as $\vec{p}$-excitations have equal spacings.

   For the free solution we superimpose the solutions in the Schroedinger picture

$$\phi_0(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\left(a_p e^{i\vec{p}\vec{x}} + a_p^\dagger e^{-i\vec{p}\vec{x}}\right) \tag{19.10}$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(a_p e^{i\vec{p}\vec{x}} + a_p^\dagger e^{-i\vec{p}\vec{x}}\right) \tag{19.11}$$

and in the Heisenberg picture ($px = \omega_p t - \vec{p}\vec{x}$) the free is given by

$$\phi_0(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p e^{ipx} + a_p^\dagger e^{-ipx}\right) \tag{19.12}$$

4. Generalize the harmonic osci math for the new system

$$\begin{aligned}
[a, a^\dagger] = 1 \quad &\to \quad [a_k, a_p^\dagger] = (2\pi)^3 \delta^3(\vec{p} - \vec{k}) &(19.13)\\
a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \quad &\to \quad a_p^\dagger|0\rangle = \frac{1}{\sqrt{2\omega_p}}|\vec{p}\rangle &(19.14)\\
a^\dagger + a = \sqrt{2m\omega}x \quad &\to \quad \phi_0(\vec{x})|0\rangle = |\vec{x}\rangle &(19.15)\\
&\to \quad \mathbb{I} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} |\vec{p}\rangle\langle\vec{p}| &(19.16)
\end{aligned}$$

now one can calculate things and compare to expected results

$$\begin{aligned}
\langle\vec{p}|\vec{k}\rangle &= ... &(19.17)\\
\langle\vec{p}|\phi_0(\vec{x})|0\rangle &= ... &(19.18)\\
[H_0, \phi_0(\vec{x},t)] &= ... &(19.19)
\end{aligned}$$

We can see that $\Box\phi_0(x) = 0$ and by adjusting the dispersion relation to $\omega_p = \sqrt{\vec{p}^2 + m^2}$ the field operator satisfies $(\Box + m^2)\phi_0(x) = 0$

The one-particle limit (first quantization limit) is

$$\begin{aligned}
\langle x| &= \langle 0|\phi(\vec{x},t) &(19.20)\\
\psi(x) = \langle x|\psi\rangle &= \langle 0|\phi(\vec{x},t)|\psi\rangle &(19.21)
\end{aligned}$$

then

$$\begin{aligned}
i\partial_t\psi(x) &= i\partial_t\langle 0|\phi(\vec{x},t)|\psi\rangle &(19.22)\\
&= i\langle 0|\partial_t\phi(\vec{x},t)|\psi\rangle \quad \text{with} \quad \partial_{tt}\phi_0 = (\nabla^2 - m^2)\phi_0 &(19.23)\\
&= i\langle 0|\sqrt{\nabla^2 - m^2}\phi_0(\vec{x},t)|\psi\rangle &(19.24)\\
&= \langle 0|\sqrt{m^2 - \nabla^2}\phi_0(\vec{x},t)|\psi\rangle &(19.25)\\
&= \sqrt{m^2 - \nabla^2}\psi &(19.26)\\
&\approx \left(m - \frac{\nabla^2}{2m} + ...\right)\psi &(19.27)
\end{aligned}$$

5. Adding interactions $H = H_0 + H_{\text{int}}$ keep the notation

$$\phi(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p(t) e^{ipx} + a_p^\dagger(t) e^{-ipx}\right) \tag{19.28}$$

and assume at any fixed time t the interaction theory operators $a_p(t)$ and $a_p^\dagger(t)$ have the same commutation algebra as the free ones. This means at any given time t_0 they are identical $a_p(t_0) = a_p$ and $\phi(\vec{x},t_0) = \phi_0(\vec{x},t_0)$.

$$\int d^4p = \int dp^0 \int \left(d^3p\ \delta(p^2 - m^2)\theta(p^0)\right) \tag{19.29}$$

$$= \int dp^0 \int \left(d^3p\ \delta[(p^0)^2 - (\vec{p}^2 - m^2)]\theta(p^0)\right) \tag{19.30}$$

$$= \int dp^0 \int \left(d^3p\ \sum_{\mathrm{zero}_k} \frac{\delta(p^0 + \mathrm{zero}_k)}{|2p_0|_{\mathrm{zero}_k}}\theta(p^0)\right) \tag{19.31}$$

$$= \int dp^0 \int \left(d^3p\ \left[\frac{\delta(p^0 + \sqrt{\vec{p}^2 + m^2})}{|-2\sqrt{\vec{p}^2 + m^2}|} + \frac{\delta(p^0 - \sqrt{\vec{p}^2 + m^2})}{|2\sqrt{\vec{p}^2 + m^2}|}\right]\theta(p^0)\right) \tag{19.32}$$

$$= \int dp^0 \int \left(d^3p\ \frac{\delta(p^0 - \sqrt{\vec{p}^2 + m^2})}{2\sqrt{\vec{p}^2 + m^2}}\right) \tag{19.33}$$

$$= \int \frac{d^3p}{2\omega_p} \tag{19.34}$$

Chapter 20

Doodling III

Fundamental ingredients for a quantum theory are a set of states $\{|\psi\rangle\}$ and operators $\{\mathcal{O}\}$. The time development is governed by a Hamilton operator

$$i\hbar\partial_t|\psi\rangle = H|\psi\rangle \tag{20.1}$$

Lets assume that momentum eigenstates are simultaneously eigenstates of H then a simple relativistic theory looks like

$$H|\vec{p}\rangle = E_{\vec{p}}|\vec{p}\rangle \tag{20.2}$$

$$E_{\vec{p}} = +\sqrt{\vec{p}^2c^2 + m^2c^4} \tag{20.3}$$

The time evolution of the wave function is given by

$$\psi(\vec{p},t) = e^{-iE_{\vec{p}}t}\psi(\vec{p},0) \tag{20.4}$$

$$\psi(\vec{x},t) = \int d^3\vec{p}\, e^{i\vec{p}\vec{x}}\psi(\vec{p},t) \tag{20.5}$$

$$= \int d^3\vec{p}\, e^{-i(E_{\vec{p}}t-\vec{p}\vec{x})}\psi(\vec{p},0) \tag{20.6}$$

$$= \frac{1}{(2\pi)^3}\int d^3\vec{p}\, e^{-i(E_{\vec{p}}t-\vec{p}\vec{x})}\int d^3\vec{y}e^{-i\vec{p}\vec{y}}\psi(\vec{y},0) \tag{20.7}$$

$$= \int d^3\vec{y}\left[\frac{1}{(2\pi)^3}\int d^3\vec{p}\, e^{-i(E_{\vec{p}}t-\vec{p}(\vec{x}-\vec{y}))}\right]\psi(\vec{y},0) \tag{20.8}$$

$$\psi(\vec{x},t) = \int d^3\vec{y}\, G(\vec{x}-\vec{y},t)\psi(\vec{y},0) \tag{20.9}$$

Causality of the theory is guaranteed if the commutator of two operators/observables (associated with points x and y in space time) commute if the points are space-like separated

$$|x-y| < 0 \quad\rightarrow\quad [\mathcal{O}_i,\mathcal{O}_j] = 0. \tag{20.10}$$

Localizing a particle in a small region L means

$$p \sim \frac{\hbar}{L} \tag{20.11}$$

$$E = \sqrt{m^2c^4 + p^2c^2} = pc\sqrt{1+\frac{m^2c^2}{p^2}} \tag{20.12}$$

The L at which the momentum contribution becomes comparable to the rest energy of the particle

$$mc^2 = pc = \frac{\hbar c}{L} \quad\rightarrow\quad L_c = \frac{\hbar}{mc} \tag{20.13}$$

is called Compton wavelength at which a relativistic theory is required and creation of particles and antiparticles appears.

This is therefore the method of choice to produce particles. A collision of two particles localizes a large amount of energy in a small region - creating particles

$$p\bar{p} \to X\bar{X} + ... \tag{20.14}$$

Important general principles

- CPT invariance
- Spin-statistic theorem
- Interactions of particles with higher spin rather quite constrainted
  1. for lower spins $s = 0, 1/2$ the only restrictions are locality and Lorentz invariance
  2. the constrains are so restrictive that there are no relativistic quantum particle with $s > 2$

Chapter 21

Finance stuff for Aki

Stochastic ODE for Geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{21.1}$$

Solving it via Ito's Lemma gives

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right) \tag{21.2}$$

The transition probability for the price going from S_0 at time $t = 0$ to S_t at time t (with fixed σ and μ) is given by (only stating the result)

$$f(S_t, t, S_0, t; \mu, \sigma) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma\sqrt{t}S_t} \exp\left(-\frac{\left[\log\frac{S_t}{S_0} - \left(\mu - \frac{1}{2}\sigma^2\right)t\right]^2}{2\sigma^2 t}\right) \tag{21.3}$$

You numbers are

- $S_{max} = 0.9\ S_0$ price down 10%
- $\sigma = 0.24$
- $\mu = 0.05$ maybe 5% discount rate
- $t = 1/12$ meaning one month

so market being down 10% means integrate over the tail of the probability density

$$\begin{align} p &= \int_0^{0.9S_0} f(S_t, t, S_0, t; \mu, \sigma) dS_t \tag{21.4}\\ &= 0.061 \tag{21.5}\\ &= 6.1\% \tag{21.6}\end{align}$$

Chapter 22

Companion for Dyson QFT book

1. Calculating 2.1 (9)

$$\frac{\partial \psi}{\partial t} = -\sum_k c\alpha^k \frac{\partial \psi}{\partial x_k} - i\frac{mc^2}{\hbar}\beta\psi \tag{22.1}$$

$$\frac{\partial \psi^*}{\partial t} = -\sum_k c\frac{\partial \psi^*}{\partial x_k}\alpha^{k*} + i\frac{mc^2}{\hbar}\psi^*\beta^* \tag{22.2}$$

then

$$\frac{\partial \rho}{\partial t} = \frac{\partial \psi *}{\partial t}\psi + \psi^*\frac{\partial \psi}{\partial t} \tag{22.3}$$

$$= -\sum_k c\left(\frac{\partial \psi^*}{\partial x_k}\alpha^{k*}\psi + \psi^*\alpha^k\frac{\partial \psi}{\partial x_k}\right) + i\frac{mc^2}{\hbar}(\psi^*\beta^*\psi - \psi^*\beta\psi) \tag{22.4}$$

$$\overset{\beta=\beta^*}{=} -\sum_k c\left(\frac{\partial \psi^*}{\partial x_k}\alpha^{k*}\psi + \psi^*\alpha^k\frac{\partial \psi}{\partial x_k}\right) \tag{22.5}$$

$$\overset{\alpha=\alpha^*}{=} -\sum_k c\left(\frac{\partial \psi^*}{\partial x_k}\alpha^{k}\psi + \psi^*\alpha^k\frac{\partial \psi}{\partial x_k}\right) \tag{22.6}$$

$$= -c\partial_k(\psi^*\alpha^k\psi) \tag{22.7}$$

so $j_k = \psi^*\alpha^k\psi$.

Chapter 23

Companion for Banks QFT book

1. Obtaining (1.2)

$$
\begin{aligned}
p &= (\omega_p, \vec{p}) \\
\langle \vec{p} | \vec{q} \rangle &= N_p^2 \cdot \delta^3(\vec{p} - \vec{q}) \\
\mathbb{I} &= C \int d^3\vec{p} | \vec{p} \rangle \langle \vec{p} | \quad \to | q \rangle = C \int d^3\vec{p} | p \rangle \langle p | q \rangle \\
| \vec{y} \rangle &= C \int d^3\vec{p} | \vec{p} \rangle \langle \vec{p} | \vec{y} \rangle = C \int d^3\vec{p} | \vec{p} \rangle e^{-i\vec{p}\cdot\vec{y}} \\
H | \vec{p} \rangle &= \omega_p | \vec{p} \rangle \\
A_{\mathrm{AE}} &= \int d^4x d^4y J_{\mathrm{A}}(x) J_{\mathrm{B}}(y) \cdot C^2 \int d^3\vec{p} \int d^3\vec{q} \langle \vec{p} | e^{-H(x^0 - y^0)} | \vec{q} \rangle e^{i\vec{q}\cdot\vec{x}} e^{-i\vec{p}\cdot\vec{y}} \\
&= \int d^4x d^4y J_{\mathrm{A}}(x) J_{\mathrm{B}}(y) \cdot C^2 \int d^3\vec{p} \int d^3\vec{q} \langle \vec{p} | e^{-\omega_q(x^0 - y^0)} | \vec{q} \rangle e^{i(\vec{q}\cdot\vec{x} - \vec{p}\cdot\vec{y})} \\
&= \int d^4x d^4y J_{\mathrm{A}}(x) J_{\mathrm{B}}(y) \cdot C^2 \int d^3\vec{p} \int d^3\vec{q} \langle \vec{p} | \vec{q} \rangle e^{-\omega_q(x^0 - y^0)} e^{i(\vec{q}\cdot\vec{x} - \vec{p}\cdot\vec{y})} \\
&= \int d^4x d^4y J_{\mathrm{A}}(x) J_{\mathrm{B}}(y) \cdot C^2 \int d^3\vec{p} N_p^2 e^{-\omega_p(x^0 - y^0)} e^{i\vec{p}(\vec{x} - \cdot\vec{y})} \\
&= \int d^4x d^4y J_{\mathrm{A}}(x) J_{\mathrm{B}}(y) \cdot C^2 \int d^3\vec{p} N_p^2 e^{-ip(x-y))}
\end{aligned}
$$

2.

Chapter 24

Companion for Baumann Cosmology book

Redefine radial coordinate $d\chi = dr/\sqrt{1 - kr^2/R_0^2}$

$$ds^2 = -c^2dt^2 + a^2(t)[d\chi^2 + S_k^2(\chi)d\Omega^2] \tag{24.1}$$

$$\to d\chi = \frac{c\,dt}{a(t)} \tag{24.2}$$

$$\lambda_0 = \frac{a(t_0)}{a(t_1)}\lambda_1 \tag{24.3}$$

$$z = \frac{\lambda_0 - \lambda_1}{\lambda_1} = \frac{a(t_0)}{a(t_1)} - 1 \tag{24.4}$$

$$\to 1 + z = \frac{1}{a(t_1)} \tag{24.5}$$

$$\to dz = \frac{-\dot{a}(t)}{a(t)^2}dt \tag{24.6}$$

$$\to \frac{a(t)}{\dot{a}(t)}dz = -\frac{1}{a(t)}dt \tag{24.7}$$

$$\to \frac{dz}{H(t)} = -\frac{dt}{a(t)} \tag{24.8}$$

Using $a(t_0) = 1$

$$a(t_1) = a(t_0) + \dot{a}(t_0)(t_1 - t_0) + \frac{1}{2}\ddot{a}(t_0)(t_1 - t_0)^2 + ... \tag{24.9}$$

$$= 1 + H_0(t_1 - t_0) - \frac{1}{2}q_0H_0^2(t_1 - t_0)^2 + ... \tag{24.10}$$

with $H_0 = \dot{a}(t_0)/a(t_0)$ and $q_0 = -\ddot{a}(t_0)/(a(t_0)H_0^2)$. For $z < 1$

$$1 - z \approx \frac{1}{1+z} = a(t_1) = 1 + H_0(t_1 - t_0) + ... \tag{24.11}$$

$$\to z = H_0(t_0 - t_1) + ... \tag{24.12}$$

$$\to cz \approx H_0c(t_0 - t_1) \tag{24.13}$$

$$\to v \approx H_0 d \tag{24.14}$$

$$1 + z = \frac{1}{a(t_1)} \tag{24.15}$$

$$= \frac{1}{1 + H_0(t_1 - t_0) - \frac{1}{2}q_0 H_0^2 (t_1 - t_0)^2 + ...} \tag{24.16}$$

$$= 1 - H(t_1 - t_0) + \frac{1}{2}(2 + q)H^2(t_1 - t_0)^2 + ... \tag{24.17}$$

Chapter 25

Back of the envelop physics

25.0.1 1.3 Anharmonic oscillator

A particle of mass m moves along the x-axis in a potential $U(x) = bx^4$ Compute the oscillation period T exactly. Compare the result with the estimate obtained using dimensional analysis.

Solution

$$E = \frac{mv^2}{2} + bx^4 \quad \rightarrow \quad x_m = (E/b)^{\frac{1}{4}} \tag{25.1}$$

$$= \frac{m(dx/dt)^2}{2} + bx^4 \tag{25.2}$$

$$dt = \frac{dx}{\sqrt{\frac{2}{m}(E - bx^4)}} \tag{25.3}$$

Therefore we obtain the period from integration of a quarter of the oscillation

$$T = 4\int_0^{(E/b)^{1/4}} \frac{dx}{\sqrt{\frac{2}{m}(E - bx^4)}} \tag{25.4}$$

$$= \frac{4}{\sqrt{\frac{mE}{2}}} \int_0^{(E/b)^{1/4}} \frac{dx}{\sqrt{(1 - \frac{b}{E}x^4)}} \tag{25.5}$$

$$= \frac{4}{\sqrt{\frac{2E}{m}}} \left(\frac{E}{b}\right)^{1/4} \int_0^1 \frac{dz}{\sqrt{1 - z^4}} \qquad \text{with } z = \left(\frac{b}{E}\right)^{1/4} x \tag{25.6}$$

$$= \frac{4}{\sqrt{\frac{2E}{m}}} \left(\frac{E}{b}\right)^{1/4} \int_0^1 \frac{dz}{\sqrt{1 - z^4}} \tag{25.7}$$

$$= \frac{\sqrt{m}}{(Eb)^{1/4}} 2\sqrt{2} \int_0^1 \frac{dz}{\sqrt{1 - z^4}} \tag{25.8}$$

25.0.2 1.4 Design anharmonic oscillator

Design a simple mechanical device, made of springs and straight frictionless rails, which leads to an (approximate) x^4 potential for the one-dimensional motion of a point particle.

Solution
Lets use a spring of length L orthogonal to the rail - repulsion force is then

$$
\begin{aligned}
F &= k\left(\sqrt{L^2+x^2}-L\right)\sin\phi && (25.9)\\
&= k\left(\sqrt{L^2+x^2}-L\right)\frac{x}{\sqrt{L^2+x^2}} && (25.10)\\
&= kx\left(1-\frac{1}{\sqrt{1+(x/L)^2}}\right) && (25.11)\\
&\approx kx\left(1-\left[1-\frac{x^2}{2L^2}+\frac{3x^4}{8L^4}+...\right]\right) && (25.12)\\
&\approx k\left(\frac{x^3}{2L^2}-\frac{3x^5}{8L^4}+...\right) && (25.13)
\end{aligned}
$$

then with $F=-\frac{\partial V}{\partial x}$ we see $V\sim x^4$.

25.0.3 1.5 Projectile motion

A football is kicked from the ground with initial velocity v and angle θ with respect to the horizontal. Neglect friction and the finite size of the ball. Discuss the range R of the ball, using dimensional analysis and guessing the θ dependence. Check and compare with an exact calculation.
Solution

$$
\begin{aligned}
E &= \frac{mv^2}{2} = mgH \rightarrow H = \frac{v^2}{2g} && (25.14)\\
D &\sim H\sin\theta && (25.15)
\end{aligned}
$$

25.0.4 2.1 Ground state energy of harmonic oscillator

Estimate the ground-state energy of the harmonic oscillator in quantum mechanics by using the uncertainty relation $p\cdot x\sim\hbar$ and minimizing the energy.
Solution

$$
\begin{aligned}
E &= \frac{p^2}{2m} = \frac{1}{2}kx^2 = \frac{1}{2}k\frac{\hbar^2}{p^2} && (25.16)\\
p^4 &= mk\hbar^2 && (25.17)\\
\Delta E_0 &= \frac{p^2}{2m} = \frac{1}{2}\sqrt{\frac{k}{m}}\hbar && (25.18)\\
&= \frac{1}{2}\hbar\omega && (25.19)
\end{aligned}
$$

25.0.5 2.2 Relativistic hydrogen

Consider the innermost electron in an atom with nuclear charge Z. At what values of Z do we have to worry about relativistic effects?
Solution
He know

$$
E = \frac{mc^2}{2}(\alpha Z)^2 \qquad (25.20)
$$

then $Z\alpha\sim 1$ would be critical.

25.0.6 2.4 Heron Formula implies Pythagoras

Show that Heron's formula for the area of a triangle implies Pythagoras' theorem.

Solution

The area of a right angled triangle is $A = ab/2$. With $s = (a + b + c)/2$ the Heron formula is

$$
\begin{aligned}
A &= \sqrt{s(s-a)(s-b)(s-c)} && (25.21)\\
&= \frac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)} && (25.22)\\
&= \frac{1}{4}\sqrt{-a^4+2a^2b^2-b^4+2a^2c^2+2b^2c^2-c^4} && (25.23)\\
&= \frac{1}{4}\sqrt{2a^2b^2-a^4-b^4+2a^2c^2+2b^2c^2-c^4} && (25.24)\\
&= \frac{1}{4}\sqrt{4a^2b^2-a^4-2a^2b^2-b^4+2a^2c^2+2b^2c^2-c^4} && (25.25)\\
&= \frac{1}{4}\sqrt{4a^2b^2-(a^2+b^2)^2+2(a^2+b^2)c^2-c^4} && (25.26)\\
&= \frac{1}{4}\sqrt{4a^2b^2-((a^2+b^2)-c^2)^2} && (25.27)\\
&= \frac{ab}{2}\sqrt{1-\frac{((a^2+b^2)-c^2)^2}{4a^2b^2}} && (25.28)
\end{aligned}
$$

implying $a^2 + b^2 - c^2 = 0$.

Chapter 26

Some stuff for later

1. BGGKY hierachy - Bartelmann - Theoretical Astrophysics An Introduction-Wiley (2013)
2. Random phase approximation/Tam Dankov approximation
   (a) Mahan - Many particle physics
   (b) Walecka - Theoretical Nuclear And Subnuclear Physics
   (c) Gell-Mann, Brueckner - Correlation Energy of an Electron Gas at High Density
3. For a Quantum field theory on a Riemann sphere with $g : S^2 \to G$ consider the action

$$\mathcal{S}_0 = \frac{1}{4\lambda^2} \int_{S^2} d^2 z \ \mathrm{tr}(g^{-1}\partial_\mu g g^{-1} \partial^\mu g) \tag{26.1}$$

   then $g^{-1}\partial_\mu g$ defines and element of the Lie algebra and $g^{-1}dg$ is the pullback of the Maurer-Cartan form to S^2 under the map defined by g.
4. Thirring Model, Thirring-Wess Model, CM-Sommerfeld Model
5. Volume measure under Lorentz trafo $x^\mu \to x'^\mu = \Lambda^\mu_\nu x^\nu$

$$d^4 x = dx^0 dx^1 dx^2 dx^3 \tag{26.2}$$
$$d^4 x' = \Lambda^\mu_0 dx^0 \, \Lambda^\nu_1 dx^1 \, \Lambda^\sigma_2 dx^2 \, \Lambda^\rho_1 dx^3 \tag{26.3}$$

   vs

$$d^4 x = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \tag{26.4}$$

6.
   - Baez review octonions https://arxiv.org/abs/math/0105155v4
   - Complex quaternions, octonions https://arxiv.org/abs/1611.09182
   - Conway, Smith - On quaternions and octonions

Chapter 27

Representations CheatSheet

27.0.1 Preliminaries

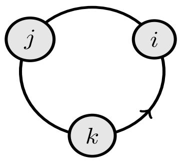
Definition 27.0.1. Number spaces $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$

- A **complex number** is an objects of the form $a + bi$ with $a, b \in \mathbb{R}$ and

$$i^2 = -1. \tag{27.1}$$

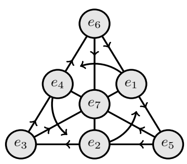
- A **quaternion** is an objects of the form $a + bi + cj + dk$ with $a, b, c, d \in \mathbb{R}$ and

$$i^2 = j^2 = k^2 = ijk = -1. \tag{27.2}$$



- An **octonion** is an objects of the form $a+bi+cj+dk+el+fm+gn+ho$ with $a, \ldots, h \in \mathbb{R}$ and $e_0 = 1, e_1 = i, \ldots, e_7 = o$

$$e_ie_j = \begin{cases} e_j, & \text{if } i = 0 \\ e_i, & \text{if } j = 0 \\ -\delta_{ij}e_0 + \varepsilon_{ijk}e_k & \text{otherwise} \end{cases} \tag{27.3}$$



Remark 27.0.1. *$\mathbb{C}$ forms a field, $\mathbb{H}$ forms a non-commutative ring*

Definition 27.0.2. The **conjugates** are defined by

$$\begin{aligned}
\bar{z} &= a - bi && (27.4)\\
\bar{q} &= a - bi - cj - dk && (27.5)\\
&= -\frac{1}{2}\left[q + iqi + jqj + kqk\right] && (27.6)\\
\bar{x} &= a - bi - cj - dk - el - fm - gn - ho && (27.7)\\
&= -\frac{1}{6}\left[x + (ix)i + (jq)j + (kq)k) + (le)l + (mf)m + (ng)n + (oh)o\right] && (27.8)
\end{aligned}$$

27.0.2 Groups theory

Definition 27.0.3. For a subgroup H of a group G **a left-coset** of the subgroup H in G is defined as the set formed by a distinct $g \in G$

$$gH = \{gh : \forall h \in H\} \tag{27.9}$$

G/H denotes the set of left cosets $\{gH : g \in G\}$ of H in G (called coset-space).

Definition 27.0.4. A subgroup N of a group G is called **normal subgroup (Normalteiler)** $N \lhd G$ if it is invariant under conjugation by members of G. Meaning

$$gng^{-1} \in N \quad \forall g \in G \tag{27.10}$$
$$gN = Ng \quad \forall g \in G \tag{27.11}$$
$$gNg^{-1} = N \quad \forall g \in G \tag{27.12}$$

Definition 27.0.5. A **simple group** is a nontrivial group whose only normal subgroups are the trivial group and the group itself.

Definition 27.0.6. Let $(G, \circ)$ and $(K, *)$ be two groups with elements $g_a \in G$ and $k_i \in K$. The **direct product** is a group $(G \otimes K, \star)$ with elements (g_a, k_i) and the multiplication rule

$$(g_a, k_i) \star (g_b, k_j) = (g_a \circ g_b, k_i * k_j). \tag{27.13}$$

Theorem 27.0.1. Every finite simple group is isomorphic to one of the following groups:

1. Z_p cyclic group of prime order
2. A_n alternating group of degree $n > 4$
3. groups of Lie type (names derived from Lie algebras with $q = p^k, m \in \mathbb{N}$
   - $A_n(q)$ Special projective linear group
   - $B_n(q), n > 1$ Commutator subgroup of $SO(2n + 1)$
   - $C_n(q), n > 2$ projective symplectic group
   - $D_n(q), n > 1$ Commutator subgroup of $SO(2n)$
   - $E_6(q), E_7(q), E_8(q), F_4(q), G_2(q)$ Chevalley group
   - ${}^2A_n(q^2), n > 1$ Special unitary group $SU(n)$
   - ${}^2B_2(2^{2m+1}))$ Suzuki Groups $Sz(2^{2m+1})$
   - ${}^2D_n(q^2),^3 D_4(q^3),^2 E_6(q^2)$ Steinberg group
   - ${}^2F_4(2^{2m+1}),^2 G_2(2^{2m+1})$ Ree group
4. 26 sporadic groups
   - Mathieu groups $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}$
   - Janko groups J_1, J_2, J_3, J_4
   - Conway groups Co_1, Co_2, Co_3
   - Fischer groups Fi_{22}, Fi_{23}, F_{3+}
   - Higman–Sims group HS
   - McLaughlin group McL
   - Held group F_7
   - Rudvalis group Ru
   - Suzuki group F_{3-}

- O'Nan group $O'N$
- Harada–Norton group F_5
- Lyons group Ly
- Thompson group F_3
- Baby Monster group F_2
- Fischer–Griess Monster group F_1

5. ${}^2F_4(2)'$ Tits group (order $2^11 \cdot 3^3 \cdot 5^2 \cdot 13 = 17,971,200$)
   - sometimes called the 27th sporadic group - but belongs for $m = 0$ to the family ${}^2F_4(2^{2m+1})'$ of commutator subgroups of ${}^2F_4(2^{2m+1})$

Figure 27.1: Periodic table of finite simple groups

Definition 27.0.7. Exceptional Lie groups

- G_2 (order 14)
- F_4 (order 52)
- E_6 (order 78)
- E_7 (order 133)
- E_8 (order 248)

Theorem 27.0.2. (Frobenius theorem, Hurwitz theorem) Any real finite-dimensional normed division algebra over the reals must be

- isomorphic to $\mathbb{R}$ or $\mathbb{C}$ if unitary and commutative (equivalently: associative and commutative)
- isomorphic to the quaternions $\mathbb{H}$ if noncommutative but associative
- isomorphic to the octonions $\mathbb{O}$ if non-associative but alternative.

Remark 27.0.2. *Projective spaces*

- $\mathfrak{so}(n+1)$ *is infinitesimal isometry of the real projective spaces* $\mathbb{RP}^n$
- $\mathfrak{su}(n+1)$ *is infinitesimal isometry of the complex projective spaces* $\mathbb{CP}^n$
- $\mathfrak{sp}(n+1)$ *is infinitesimal isometry of the quaternionic projective spaces* $\mathbb{HP}^n$
- *octonionic projective line* $\mathbb{OP}^1$ *reproduces* $\mathfrak{so}(8)$ *(already accomodated by* $\mathbb{RP}^7$*)*
- *Cayley projective plane* $\mathbb{OP}^2$ *reproduces* $\mathfrak{f}_4$*)*
- $\mathbb{OP}^n$ *for* $n > 2$ *gives nothing due to non-associativity of* $\mathbb{O}$

Remark 27.0.3. *Freudenthal-Rosenfeld-Tits magic square of Lie algebras*

$$\begin{array}{c||cccc} \mathbb{A}_1/\mathbb{A}_2 & \mathbb{R} & \mathbb{C} & \mathbb{H} & \mathbb{O} \\ \hline \mathbb{R} & \mathfrak{so}(3) & \mathfrak{su}(3) & \mathfrak{sp}(3) & \mathfrak{f}_4 \\ \mathbb{C} & \mathfrak{su}(3) & \mathfrak{su}(3)\otimes\mathfrak{su}(3) & \mathfrak{su}(6) & \mathfrak{e}_6 \\ \mathbb{H} & \mathfrak{sp}(3) & \mathfrak{su}(6) & \mathfrak{so}(12) & \mathfrak{e}_7 \\ \mathbb{O} & \mathfrak{f}_4 & \mathfrak{e}_6 & \mathfrak{e}_7 & \mathfrak{e}_8 \end{array} \tag{27.14}$$

27.0.3 Representation theory

Definition 27.0.8. A **representation** of a group $G = (\{g_i\}, \circ)$ is a mapping $g \mapsto D(g)$ of the elements $g \in G$ onto a set of linear operators with

1. $D(e) = \mathbb{I}$
2. $D(g_1)D(g_2) = D(g_1 \circ g_2)$.

This obviously implies $D(g^{-1}) = D(g)^{-1}$.

Remark 27.0.4. *A bit more formal - let G a group and V be a $\mathbb{K}$-vector space then a linear representation is a group homomorphism with $D : G \to \mathrm{GL}(V) \overset{!}{=} \mathrm{Aut}(V)$. V is then called representation space with* $\dim V$ *being the dimension of the representation and* $D(g) \in \mathrm{GL}(V)$

Definition 27.0.9. An **equivalent representation** D' of a representation D is defined by

$$D(g) \to D'(g) = S^{-1}D(g)S \qquad \forall g \in G \tag{27.15}$$

Definition 27.0.10. A representation D is called **unitary representation** if

$$D(g)^\dagger = D(g)^{-1} \qquad \forall g \in G \tag{27.16}$$

Remark 27.0.5. *For a unitary representation $D(g)^\dagger D(g) = \mathbb{I}$ an equivalent representation $D'(g) = S^{-1}D(g)S$ is only unitary*

$$\begin{aligned} D'(g)^\dagger D'(g) &= \left(S^{-1}D(g)S\right)^\dagger S^{-1}D(g)S && (27.17)\\ &= S^\dagger D(g)^\dagger (S^{-1})^\dagger S^{-1}D(g)S && (27.18)\\ &= S^\dagger D(g)^\dagger (S^\dagger)^{-1} S^{-1}D(g)S && (27.19)\\ &= S^\dagger D(g)^\dagger (SS^\dagger)^{-1} D(g)S && (27.20)\end{aligned}$$

iff S is unitary itself $SS^\dagger = \mathbb{I}$

$$D'(g)^\dagger D'(g) = S^{-1}D(g)^\dagger D(g)S = S^{-1}S = \mathbb{I}. \tag{27.21}$$

Definition 27.0.11. A representation is called a **reducible representation** if V has an invariant subspace meaning that the action of any $D(g)$ on any vector of the subspace V_P is still in the subspace. If the projection operator $P : V \to V_P$ projects to this subspace then

$$PD(g)P = D(g)P \qquad \forall g \in G \tag{27.22}$$

Remark 27.0.6. *$\forall |v\rangle \in V$ we have $P|v\rangle \in V_P$. If the subspace is invariant then any group action can not move it outside $D(g)P|v\rangle \in V_P$. But this means projecting it again would not change anything $PD(g)P|v\rangle = D(g)P|v\rangle$*

Definition 27.0.12. A representation is called an **irreducible representation** if it is not reducible.

Definition 27.0.13. A representation is called a **completely reducible representation** if it is equivalent to a representation whose matrix elements have the form

$$D(g) = \begin{pmatrix} D_1(g) & 0 & \cdots \\ 0 & D_2(g) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \tag{27.23}$$

where all $D_j(g)$ are irreducible. Representation D is is said to be the direct sum of subrepresentation D_j

$$D = D_1(g) \oplus D_2(g) \oplus \cdots \tag{27.24}$$

Definition 27.0.14. For a group of order n the n-dimensional representation D defined by

$$g_k \to |e_k\rangle \tag{27.25}$$

$$D(g_j)|e_k\rangle \overset{!}{=} |e_m\rangle \qquad \text{with } g_j \circ g_k = g_m \to |e_m\rangle \tag{27.26}$$

(where $\{|e_i\rangle\}$ is the ordinary n-dimensional cartesian basis) is called the **regular representation**. The matrices are then constructed by

$$[D(g_j)]_{ik} = \langle e_i|D(g_j)|e_k\rangle = \langle e_i|e_m\rangle. \tag{27.27}$$

Theorem 27.0.3. Every representation of a finite group is equivalent to a unitary representation.

Theorem 27.0.4. Every representation of a finite group is complete reducible.

Definition 27.0.15. Given two representations D_1 and D_2 acting on V_1 and V_2, an intertwiner between D_1 and D_2 is a linear operator $F : D_1 \to D_2$ which "commutes with G" in the sense that

$$FD_1(g) = D_2(g)F \quad \forall g \in G. \tag{27.28}$$

Chapter 28

Lie groups/algebras

Linear representation

$$g \to \tag{28.1}$$

Remark 28.0.1. *Killing classification of simple Lie groups*

- *SO(2n), SO(2n+1) - Lie algebra:* $J^T = -J$ *(skew-hermitian, trace free matrices* $GL(n, \mathbb{R})$
- *SU(n) - Lie algebra:* $J^\dagger = -J$ *(skew-hermitian, trace free matrices in* $GL(n, \mathbb{C})$
- *Sp(2n) - Lie algebra:* $J^\dagger = -J$ *(skew-hermitian matrices in* $GL(n, \mathbb{H})$

Chapter 29

Example representations

29.0.1 Cyclic group Z_2

$$\begin{array}{c||cc} Z_2 & e & p \\ \hline\hline e & e & p \\ p & p & e \end{array} \tag{29.1}$$

1d

$$D'(e) = 1, \quad D'(p) = -1 \tag{29.2}$$

29.0.2 Cyclic group Z_3

$$\begin{array}{c||ccc} Z_3 & e & a & b \\ \hline\hline e & e & a & b \\ a & a & b & e \\ b & b & e & a \end{array} \tag{29.3}$$

1d

$$D'(e) = 1, \quad D'(a) = e^{i\frac{2\pi}{3}}, \quad D'(b) = e^{i\frac{4\pi}{3}} \tag{29.4}$$

3d - regular representation

$$|e\rangle = (1,0,0)^T, \quad |a\rangle = (0,1,0)^T, \quad |b\rangle = (0,0,1)^T \tag{29.5}$$

$$D(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad D(b) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \tag{29.6}$$

29.0.3 Dihedral group $D_2 = Z_2 \otimes Z_2$

We construct the group table by utilizing the direct product rule

$$(g_a, k_i) \star (g_b, k_j) = (g_a \circ g_b, k_i * k_j) \tag{29.7}$$

which simplifies to

$$(g_a, g_i) \star (g_b, g_j) = (g_a \circ g_b, g_i \circ g_j). \tag{29.8}$$

D_2	(e,e)	(e,P)	(P,e)	(P,P)
(e,e)	(e,e)	(e,P)	(P,e)	(P,P)
(e,P)	(e,P)	(e,e)	(P,P)	(P,e)
(P,e)	(P,e)	(P,P)	(e,e)	(e,P)
(P,P)	(P,P)	(P,e)	(e,P)	(e,e)

(29.9)

29.0.4 Cyclic group Z_4

$$D'(e) = 1, \quad D'(a) = e^{i\frac{1\pi}{4}}, \quad D'(b) = e^{i\frac{2\pi}{4}}, \quad D'(c) = e^{i\frac{3\pi}{4}} \tag{29.10}$$

29.0.5 Group S_3

S_3	e	a_1	a_2	a_3	a_4	a_5
e	e	a_1	a_2	a_3	a_4	a_5
a_1	a_1	a_2	e	a_5	a_3	a_4
a_2	a_2	e	a_1	a_4	a_5	a_3
a_1	a_3	a_4	a_5	e	a_1	a_2
a_1	a_4	a_5	a_3	a_2	e	a_1
a_1	a_5	a_3	a_4	a_1	a_2	e

(29.11)

$$a_1 = (1,2,3), \quad a_2 = (3,2,1), \quad a_3 = (1,2), \quad a_4 = (2,3), \quad a_5 = (3,1) \tag{29.12}$$

2d

$$D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D(a_1) = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}, \quad D(a_2) = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}, \tag{29.13}$$

$$D(a_3) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D(a_4) = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}, \quad D(a_5) = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix} \tag{29.14}$$

Chapter 30

Richard Borcherds

1. Introduction to number theory (53)
2. Complex analysis (20)
3. Theory of numbers (20)
4. Zermelo Fraenkel axioms (8)
5. Group theory (32)
6. Rings and modules (22)
7. Galois theory (25)
8. Lie groups (11)
9. Representation theory (9)
10. Commutative Algebra (76)
11. Introduction to homological algebra (6)
12. Categories for the idle mathematician (7)
13. Modular forms (13)
14. Algebraic topology (4)
15. Algebraic geometry I: Varieties (51)
16. Algebraic geometry II: Schemes (48)
17. Algebraic geometry: Extra Topics (24)
18. Math talks (20)
19. History of science (7)

Chapter 31

How to learn ...

How to Learn Physics
big picture

- Emilio Segre, From Falling Bodies to Radio Waves: Classical Physicists and Their Discoveries, W. H. Freeman, New York, 1981.
- Emilio Segre, From X-Rays to Quarks: Modern Physicists and Their Discoveries, W. H. Freeman, San Francisco, 1980.
- Robert P. Crease and Charles C. Mann, The Second Creation: Makers of the Revolution in Twentieth-Century Physics, Rutgers University Press, New Brunswick, NJ, 1996.
- Abraham Pais, Inward Bound: of Matter and Forces in the Physical World, Clarendon Press, New York, 1986. (More technical.) Next, here are some good books to learn "the real stuff". These aren't "easy" books, but they're my favorites.

First, some very good general textbooks:

- M. S. Longair, Theoretical Concepts in Physics, Cambridge U. Press, Cambridge, 1986.
- Richard Feynman, Robert B. Leighton and Matthew Sands, The Feynman Lectures on Physics, 3 volumes, Addison-Wesley, 1989. All three volumes are now free online.
- Ian D. Lawrie, A Unified Grand Tour of Theoretical Physics, Adam Hilger, Bristol, 1990.

Classical mechanics:

- Herbert Goldstein, Charles Poole, and John Safko, Classical Mechanics, Addison Wesley, San Francisco, 2002.

Statistical mechanics:

- F. Reif, Fundamentals of Statistical and Thermal Physics, McGraw Hill, New York, 1965.

Electromagnetism:

- John David Jackson, Classical Electrodynamics, Wiley, New York, 1975.

Special relativity:

- Edwin F. Taylor, John A. Wheeler, Spacetime Physics: Introduction to Special Relativity, W. H. Freeman Press, 1992.

Quantum mechanics:

- Anthony Sudbery, Quantum Mechanics and the Particles of Nature: an Outline for Mathematicians, Cambridge University Press, Cambridge, 1986. (Not just for mathematicians!)
- Claude Cohen-Tannoudji, Bernard Diu und Franck Laloë, Quantum Mechanics (2 volumes), Wiley-Interscience, 1992.

General relativity — to get intuition for the subject before tackling the details:

- Kip S. Thorne, Black Holes and Time Warps: Einstein's Outrageous Legacy, W. W. Norton, New York, 1994.
- Robert M. Wald, Space, Time, and Gravity: the Theory of the Big Bang and Black Holes, University of Chicago Press, Chicago, 1977.
- Robert Geroch, General Relativity from A to B, University of Chicago Press, Chicago, 1978.

General relativity — for when you get serious:

- R. A. D'Inverno, Introducing Einstein's Relativity, Oxford University Press, Oxford, 1992.
- J. B. Hartle, Gravity: An Introduction to Einstein's General Relativity, Addison-Wesley, New York, 2002.
- B. F. Schutz, A First Course in General Relativity, Cambridge University Press, Cambridge, 1985.

General relativity — for when you get really serious:

- Charles W. Misner, Kip S. Thorne and John Archibald Wheeler, Gravitation, W. H. Freeman Press, San Francisco, 1973.
- Robert M. Wald, General Relativity, University of Chicago Press, Chicago, 1984.

Cosmology:

- Edward R. Harrison, Cosmology, the Science of the Universe, Cambridge University Press, Cambridge, 1981.
- M. Berry, Cosmology and Gravitation, Adam Hilger, Bristol, 1986.
- John A. Peacock, Cosmological Physics, Cambridge University Press, Cambridge, 1999. (More technical.)

Quantum field theory — to get intuition for the subject before tackling the details:

- Richard Feynman, QED: the Strange Theory of Light and Matter, Princeton University Press, Princeton, 1985.

Quantum field theory — for when you get serious:

- Michael E. Peskin and Daniel V. Schroeder, An Introduction to Quantum Field Theory, Addison-Wesley, New York, 1995. (The best modern textbook, in my opinion.)
- A. Zee, Quantum Field Theory in a Nutshell, Princeton University Press, Princeton, 2003. (Packed with wisdom told in a charmingly informal manner; not the best way to learn how to calculate stuff.)
- Warren Siegel, Fields, available for free on the arXiv.
- Mark Srednicki, Quantum Field Theory, available free on his website. (It's good to snag free textbooks while you can, if they're not on the arXiv!)

- Sidney Coleman, Physics 253: Quantum Field Theory, 1975-1976. (Not a book — it's a class! You can download free videos of this course at Harvard, taught by a brash and witty young genius.)

Quantum field theory — two classic older texts that cover a lot of material not found in Peskin and Schroeder's streamlined modern presentation:

- James D. Bjorken and Sidney D. Drell, Relativistic Quantum Mechanics, New York, McGraw-Hill, 1964.
- James D. Bjorken and Sidney D. Drell, Relativistic Quantum Fields, New York, McGraw-Hill, 1965.

Quantum field theory — for when you get really serious:

- Sidney Coleman, Aspects of Symmetry, Cambridge U. Press, 1989. (A joy to read.)
- Rudolf Haag, Local Quantum Physics: Fields, Particles, Algebras, Springer, 1992.

Quantum field theory — so even mathematicians can understand it:

- Robin Ticciati, Quantum Field Theory for Mathematicians, Cambridge University Press, Cambridge, 1999.
- Richard Borcherds and Alex Barnard, Lectures On Quantum Field Theory.

Particle physics:

- Kerson Huang, Quarks, Leptons & Gauge Fields, World Scientific, Singapore, 1982.
- L. B. Okun, Leptons and Quarks, translated from Russian by V. I. Kisin, North-Holland, 1982. (Huang's book is better on mathematical aspects of gauge theory and topology; Okun's book is better on what we actually observe particles to do.)
- T. D. Lee, Particle Physics and Introduction to Field Theory, Harwood, 1981.
- K. Grotz and H. V. Klapdor, The Weak Interaction in Nuclear, Particle, and Astrophysics, Hilger, Bristol, 1990.

While studying general relativity and quantum field theory, you should take a break now and then and dip into this book: it's a wonderful guided tour of the world of math and physics:

- Roger Penrose, The Road to Reality: A Complete Guide to the Laws of the Universe, Knopf, New York, 2005. And then, some books on more advanced topics... The interpretation of quantum mechanics:
- Roland Omnes, Interpretation of Quantum Mechanics, Princeton U. Press, Princeton, 1994.

This is a reasonable treatment of an important but incredibly controversial topic. Warning: there's no way to understand the interpretation of quantum mechanics without also being able to solve quantum mechanics problems — to understand the theory, you need to be able to use it (and vice versa). If you don't heed this advice, you'll fall prey to all sorts of nonsense that's floating around out there.

The mathematical foundations of quantum physics:

- Josef M. Jauch, Foundations of Quantum Mechanics, Addison-Wesley, 1968. (Very thoughtful and literate. Get a taste of quantum logic.)
- George Mackey, The Mathematical Foundations of Quantum Mechanics, Dover, New York, 1963. (Especially good for mathematicians who only know a little physics.)

Loop quantum gravity and spin foams:

- Carlo Rovelli, Quantum Gravity, Cambridge University Press, Cambridge, 2004. String theory:
- Barton Zwiebach, A First Course in String Theory, Cambridge U. Press, Cambridge, 2004. (The best easy introduction.)
- Katrin Becker, Melanie Becker and John H. Schwartz, String Theory and M-Theory: A Modern Introduction, Cambridge U. Press, Cambridge, 2007. (A more detailed introduction.)
- Michael B. Green, John H. Schwarz and Edward Witten, Superstring Theory (2 volumes), Cambridge U. Press, Cambridge, 1987. (The old testament.)
- Joseph Polchinski, String Theory (2 volumes), Cambridge U. Press, Cambridge, 1998. (The new testament — he's got branes.)

How to Learn Math

Math is a much more diverse subject than physics, in a way: there are lots of branches you can learn without needing to know other branches first... though you only deeply understand a subject after you see how it relates to all the others!

The study of math branches out into a dizzying variety of more advanced topics! It's hard to get the "big picture" of mathematics until you've gone fairly far into it; indeed, the more I learn, the more I laugh at my previous pathetically naive ideas of what math is "all about". But if you want a glimpse, try these books:

- F. William Lawvere and Stephen H. Schanuel, Conceptual Mathematics: a First Introduction to Categories, Cambridge University Press, 1997. (A great place to start.)
- Saunders Mac Lane, Mathematics, Form and Function, Springer, New York, 1986. (More advanced.)
- Jean Dieudonne, A Panorama of Pure Mathematics, as seen by N. Bourbaki, translated by I.G. Macdonald, Academic Press, 1982. (Very advanced — best if you know a lot of math already. Beware: many people disagree with Bourbaki's outlook.)

I haven't decided on my favorite books on all the basic math topics, but here are a few. In this list I'm trying to pick the clearest books I know, not the deepest ones — you'll want to dig deeper later:

Finite mathematics (combinatorics):

- Arthur T. Benjamin and Jennifer Quinn, Proofs that Really Count: The Art of Combinatorial Proof, The Mathematical Association of America, 2003.
- Ronald L. Graham, Donald Knuth, and Oren Patashnik, Concrete Mathematics, Addison-Wesley, Reading, Massachusetts, 1994. (Too advanced for a first course in finite mathematics, but this book is fun — quirky, full of jokes, it'll teach you tricks for counting stuff that will blow your friends minds!)

Probability theory:

- Charles M. Grinstead and J. Laurie Snell, Introduction to Probability, American Mathematical Society, Providence, Rhode Island, 1997. Also available free online at https://math.dartmouth.edu/ prob/pro

Calculus:

- Silvanus P. Thompson, Calculus Made Easy, St. Martin's Press, 1914. Also available free online at http://www.gutenberg.org/ebooks/33283. (Most college calculus texts weigh a ton; this one does not — it just gets to the point. This is how I learned calculus: my uncle gave me a copy.)

- Gilbert Strang, Calculus, Wellesley-Cambridge Press, Cambridge, 1991. Also available free online at http://ocw.mit.edu/ans7870/resources/Strang/strangtext.htm. (Another classic, with lots of applications to real-world problems.)

Multivariable calculus:

- James Nearing, Mathematical Tools for Physics, available at http://www.physics.miami.edu/ nearing/mathmethods/. See especially the sections on multvariable calculus, vector calculus 1, and vector calculus 2. (Very nice explanations!)

- George Cain and James Herod, Multivariable Calculus. Available free online at http://www.math.gatech.edu/ ca

Linear algebra:
This is a great linear algebra book if you want to understand the subject thoroughly:

- Elizabeth S. Meckes and Mark Meckes, Linear Algebra, Cambridge U. Press, 2018.

  These books are probably easier, and they're free online:

- Keith Matthews, Elementary Linear Algebra, available free online at http://www.numbertheory.org/book/.

- Jim Hefferon, Linear Algebra, available free online at http://joshua.smcvt.edu/linalg.html/.

- Robert A. Beezer, A First Course in Linear Algebra, available free online at http://linear.ups.edu/.

Ordinary differential equations — some free online books:

- Bob Terrell, Notes on Differential Equations, available free online at http://www.math.cornell.edu/ bterrell/dn.pdf. (Does both ordinary and partial differential equations.)

- James Nearing, Mathematical Tools for Physics, available at http://www.physics.miami.edu/ nearing/mathmethods/. See especially the sections on ordinary differential equations and Fourier series (which are good for solving such equations).

Partial differential equations — some free online books:

- Bob Terrell, Notes on Differential Equations, available free online at http://www.math.cornell.edu/ bterrell/dn.pdf. (Does both ordinary and partial differential equations.)

- James Nearing, Mathematical Tools for Physics, available at http://www.physics.miami.edu/ nearing/mathmethods/. See especially the section on partial differential equations.

Set theory and logic:

- Herbert B. Enderton, Elements of Set Theory, Academic Press, New York, 1977.

- Herbert B. Enderton, A Mathematical Introduction to Logic, Academic Press, New York, 2000.

- F. William Lawvere and Robert Rosebrugh, Sets for Mathematics, Cambridge U. Press, Cambridge, 2002. (An unorthodox choice, since this book takes an approach based on category theory instead of the old-fashioned Zermelo-Fraenkel axioms. But this is the wave of the future, so you might as well hop on now!)

Complex analysis:

- George Cain, Complex Analysis, available free online at http://www.math.gatech.edu/ cain/winter99/complex.h (How can you not like free online?)
- James Ward Brown and Ruel V. Churchill, Complex Variables and Applications, McGraw-Hill, New York, 2003. (A practical introduction to complex analysis.)
- Serge Lang, Complex Analysis, Springer, Berlin, 1999. (More advanced.)

Real analysis:

- Richard R. Goldberg, Methods of Real Analysis, Wiley, New York, 1976. (A gentle introduction.)
- Halsey L. Royden, Real Analysis, Prentice Hall, New York, 1988. (A bit more deep; here you get Lebesgue integration and measure spaces.)

Topology:

- James R. Munkres, Topology, James R. Munkres, Prentice Hall, New York, 1999.
- Lynn Arthur Steen and J. Arthur Seebach, Jr., Counterexamples in Topology, Dover, New York, 1995. (It's fun to see how crazy topological spaces can get: also, counterexamples help you understand definitions and theorems. But, don't get fooled into thinking this stuff is the point of topology!)

Abstract algebra:
I didn't like abstract algebra as an undergrad. Now I love it! Textbooks that seem pleasant now seemed dry as dust back then. So, I'm not confident that I could recommend an all-around textbook on algebra that my earlier self would have enjoyed. But, I would have liked these:

- Hermann Weyl, Symmetry, Princeton University Press, Princeton, New Jersey, 1983. (Before diving into group theory, find out why it's fun.)
- Ian Stewart, Galois Theory, 3rd edition, Chapman and Hall, New York, 2004. (A fun-filled introduction to a wonderful application of group theory that's often explained very badly.)

Number theory:
These are elementary textbooks; for more advanced ones read on further.

- George E. Andews, Number Theory, Dover, New York, 1994. (A good elementary introduction; don't buy the Kindle version of this edition since the equations are tiny.)
- Joseph Silverman, Friendly Introduction to Number Theory, Pearson, 2017. (Doesn't require any advanced mathematics, not even calculus.)
- Martin H. Weissman, An Illustrated Theory of Numbers, American Mathematical Society, Providence, Rhode Island, 2017. (This reveals the oft-hidden visual side of number theory.)

More Advanced Math
I'll start with some books on mathematical physics, because that's been one of my favorite subjects for a long time. Out of laziness, I'll assume you're already somewhat comfortable with the topics listed above — yes, I know that requires about 4 years of full-time work! —l and I'll pick up from there. Here's a good place to start:

- Paul Bamberg and Shlomo Sternberg, A Course of Mathematics for Students of Physics, Cambridge University, Cambridge, 1982. (A good basic introduction to modern math, actually.)

  It's also good to get ahold of these books and keep referring to them as needed:

- Robert Geroch, Mathematical Physics, University of Chicago Press, Chicago, 1985.
- Yvonne Choquet-Bruhat, Cecile DeWitt-Morette, and Margaret Dillard-Bleick, Analysis, Manifolds, and Physics (2 volumes), North-Holland, 1982 and 1989.

  Here's a free online reference book that's 787 pages long:

- Jean Claude Dutailly, Mathematics for Theoretical Physics, 2012.

Here are my favorite books on various special topics:
Group theory in physics:

- Shlomo Sternberg, Group Theory and Physics, Cambridge University Press, 1994.
- Robert Hermann, Lie Groups for Physicists, Benjamin-Cummings, 1966.
- George Mackey, Unitary Group Representations in Physics, Probability, and Number Theory, Addison-Wesley, Redwood City, California, 1989.

Lie groups, Lie algebras and their representations — in rough order of increasing sophistication:

- Brian Hall, Lie Groups, Lie Algebras, and Representations, Springer, Berlin, 2003.
- William Fulton and Joe Harris, Representation Theory — a First Course, Springer, Berlin, 1991. (A friendly introduction to finite groups, Lie groups, Lie algebras and their representations, including the classification of simple Lie algebras. One great thing is that it has many pictures of root systems, and works slowly up a ladder of examples of these before blasting the reader with abstract generalities.)
- J. Frank Adams, Lectures on Lie Groups, University of Chicago Press, Chicago, 2004. (A very elegant introduction to the theory of semisimple Lie groups and their representations, without the morass of notation that tends to plague this subject. But it's a bit terse, so you may need to look at other books to see what's really going on in here!)
- Daniel Bump, Lie Groups, Springer, Berlin, 2004. (A friendly tour of the vast and fascinating panorama of mathematics surrounding groups, starting from really basic stuff and working on up to advanced topics. The nice thing is that it explains stuff without feeling the need to prove every statement, so it can cover more territory.)

Geometry and topology for physicists — in rough order of increasing sophistication:

- Gregory L. Naber, Topology, Geometry and Gauge Fields: Foundations, Springer, Berlin, 1997.
- Chris Isham, Modern Differential Geometry for Physicists, World Scientific Press, Singapore, 1999. (Isham is an expert on general relativity so this is especially good if you want to study that.)
- Harley Flanders, Differential Forms with Applications to the Physical Sciences, Dover, New York, 1989. (Everyone has to learn differential forms eventually, and this is a pretty good place to do it.)
- Charles Nash and Siddhartha Sen, Topology and Geometry for Physicists, Academic Press, 1983. (This emphasizes the physics motivations... it's not quite as precise at points.)
- Mikio Nakahara, Geometry, Topology, and Physics, A. Hilger, New York, 1990. (More advanced.)
- Charles Nash, Differential Topology and Quantum Field Theory, Academic Press, 1991. (Still more advanced — essential if you want to understand what Witten is up to.)

Geometry and topology, straight up:

- Victor Guillemin and Alan Pollack, Differential Topology, Prentice-Hall, Englewood Cliffs, 1974.
- B. A. Dubrovin, A. T. Fomenko, and S. P. Novikov, Modern Geometry — Methods and Applications, 3 volumes, Springer, Berlin, 1990. (Lots of examples, great for building intuition, some mistakes here and there. The third volume is an excellent course on algebraic topology from a geometrical viewpoint.)

Algebraic topology:

- Allen Hatcher, Algebraic Topology, Cambridge U. Press, Cambridge, 2002. Also available free at http://www.math.cornell.edu/ hatcher/AT/ATpage.html. (An excellent modern introduction.)
- Peter May, A Concise Course in Algebraic Topology, U. of Chicago Press, Chicago, 1999. Also available free at http://www.math.uchicago.edu/ may/CONCISE/ConciseRevised.pdf. (More intense.)

Geometrical aspects of classical mechanics:

- Vladimir I. Arnol'd, Mathematical Methods of Classical Mechanics, translated by K. Vogtmann and A. Weinstein, 2nd edition, Springer, Berlin, 1989. (The appendices are somewhat more advanced and cover all sorts of nifty topics.)

Analysis and its applications to quantum physics:

- Michael Reed and Barry Simon, Methods of Modern Mathematical Physics (4 volumes), Academic Press, 1980.

And moving on to pure mathematics... Knot theory:

- Louis Kauffman, On Knots, Princeton U. Press, Princeton, 1987.
- Louis Kauffman, Knots and Physics, World Scientific, Singapore, 1991.
- Dale Rolfsen, Knots and Links, Publish or Perish, Berkeley, 1976.

Homological algebra:

- Joseph Rotman, An Introduction to Homological Algebra, Academic Press, New York, 1979. (A good introduction to an important but sometimes intimidating branch of math.)
- Charles Weibel, An Introduction to Homological Algebra, Cambridge U. Press, Cambridge, 1994. (Despite having the same title as the previous book, this goes into many more advanced topics.)

Combinatorics:

- Herbert Wilf, Generatinfunctionology, Academic Press, 1994. (Tons of fun and also available free online at https://www.math.upenn.edu/ wilf/DownldGF.html. It's good to read this after Concrete Mathematics by Graham, Knuth and Pataschnik, listed above under combinatorics.)
- Richard P. Stanley, Enumerative Combinatorics, two volumes, Cambridge U. Press, 1997. (Packed with great exercises; volume one is also available free online at http://www-math.mit.edu/ rstan/ec/ec1.

Algebraic geometry:

I found Hartshorne's famous book quite off-putting the first ten times I tried to read it. I think it's better to start by getting to know some 'classical' algebraic geometry so you see why the subject is interesting and why it's called 'geometry' before moving on to delightful modern abstractions like schemes. So, start with this introduction:

- Karen E. Smith, Lauri Kahanpää, Pekka Kekäläinen and William Traves, An Invitation to Algebraic Geometry, Springer, Berlin, 2004.

  Then try these:

- Igor R. Shafarevich, Basic Algebraic Geometry, two volumes, third edition, Springer, 2013.
- David Eisenbud and Joseph Harris, The Geometry of Schemes, Springer, 2006.
- Phillip Griffiths and Joseph Harris, Principles of Algebraic Geometry, 1994. (Especially nice if you like complex analysis, differential geometry and de Rham theory.)

Number theory:

- Kenneth Ireland and Keith Rosen, A Classical Introduction to Modern Number Theory, second edition, Springer, 1998. (A good way to catch up on some classic results in number theory while getting a taste of modern methods.)
- Yu. I. Manin and Alexei A. Panchishkin, Introduction to Modern Number Theory: Fundamental Problems, Ideas and Theories, Springer, 2007. (Much more hard-hitting, but a very useful overview of what modern number theory is like.)
- Jürgen Neukirch, Algebraic Number Theory, Springer, 2010. (A friendly introduction to class field theory.)

Category theory:

- Brendan Fong and David Spivak, Seven Sketches in Compositionality: An Invitation to Applied Category Theory. (A good first introduction to category theory through applications; available free online at http://math.mit.edu/ dspivak/teaching/sp18/7Sketches.pdf. Also see the website with videos and my online course based on this book.)
- Tom Leinster, Basic Category Theory, Cambridge Studies in Advanced Mathematics, Vol. 143, Cambridge U. Press, 2014. Also available for free on the arXiv. (A introduction for beginners that focuses on three key concepts and how they're related: adjoint functor, representable functors, and limits.)
- Emily Riehl, Category Theory in Context, Dover, New York, 2016. Also available for free on her website. (More advanced. As the title suggests, this gives many examples of how category theory is applied to other subjects in math.)

In this section, I discuss some books that

1. discuss the mathematical background for GR (differential geometry),
2. place relativity theory in the context of physics at large,
3. contain important milestones in the history of relativity theory.

I'll begin with several books in the Schaum's outline series, which, if read with discipline, can actually be a very effective way, I think, to learn some problem-solving skills. If you really are starting with linear algebra, however, you should expect to spend many months in hard labor working through these books before you are ready to being your study of GR. I am not familiar with all of the following books, but consider the one I own (the last) to be a good book.

- Seymour Lipschutz, Schaum's Outline of Linear Algebra, 2nd ed. McGraw-Hill, 1991. In print, ISBN 0-07-038007-4; list price 13.95 (paperback).
- Frank Ayres and Elliot Mendelson, Schaum's Outline of Calculus, 3rd ed. McGraw-Hill, 1990. In print, ISBN 0-07-002662-9; list price 14.95 (paperback).
- Richard Bronson, Schaum's Outline of Differential Equations, 2nd ed. McGraw-Hill, 1994. In print, ISBN 0-07-008019-4; list price 14.95 (paperback).
- Paul C. DuChateau, and D. W. Zachmann, Schaum's Outline of Partial Differential Equations McGraw-Hill, 1986. In print, ISBN 0-07-017897-6; list price 14.95 (paperback).
- Murray R. Spiegel, Advanced Mathematics for Engineers and Scientists. McGraw-Hill, 1971. In print, ISBN 0-07-060216-6; list price 14.95 (paperback).
- Martin Lipschutz, Differential Geometry. McGraw-Hill, 1969. In print, ISBN 0-07-037985-8; list price 12.95 (paperback).

A good introduction to classical differential geometry. Note well; for GR you need more advanced notions, including modern notions of manifolds, covariant, Lie, and exterior derivatives, connections, and curvature tensors.

- David C. Kay, Schaum's Outline of Tensor Calculus McGraw-Hill, 1988. In print, ISBN 0-07-033484-6; list price

  An introduction to coordinate basis tensor computations, including the metric tensor, geodesics, the Riemann tensor, with applications to classical mechanics and SR (but not GR). This won't entirely get you up to speed for GR, but like the previous book it may be useful as a supplementary text.

- John H. Hubbard, Vector Calculus, Linear Algebra and Differential Forms: A Unified Approach. Prentice Hall, 1998. In print, ISBN 0-13-657446-7;

  This book can probably serve as a substitute for all of the Schaum's books mentioned above (save the last two), with the additional bonus of introducing exterior forms early on and properly emphasizing the fact that these objects are natural, easy to understand, and easy to compute with.

- Theodore Frankel, The Geometry of Physics: An Introduction. Cambridge University Press, 1997. In print, ISBN 0-521-38334-X;

  This book is simply gorgeous. It offers a thorough and beautifully illustrated introduction to everything from riemannian geometry, Cartan geometry, symplectic geometry, differential topology and Morse Theory to vector bundles and Pontryagin and Chern classes. Applications to hamiltonian mechanics, GR, Yang-Mills theories, the Standard Model of particle physics, etc., are also sketched.

Speaking of manifolds and differential geometry, I think that one of the best all around introductions is:

- William M. Boothby, An Introduction to Differentiable Manifolds and Riemannian Geometry, 2nd ed. Academic Press, 1986. In print, ISBN 0-12-116053-X;.

  One book which is particularly well suited for background reading in GR is the outrageously expensive

- Barrett O'Neill, Semi-Riemannian Geometry with Applications to Relativity. Academic Press, 1983. In print, ISBN 0-12-526740-1;

  This book covers not only manifolds, tensors, metrics, connections, curvature, calculus of variations, homogeneous spaces, and covering spaces, but also Minkowski spacetime, the Friedmann and Schwarzschild solutions, and the singularity theorems.

  Another classic, easy to read introduction is "the great American differential geometry book":

- Michael Spivak, A Comprehensive Introduction to Differential Geometry, 5 volumes. Publish or Perish, 1979.

  1. Vol.: ISBN 0-914098-84-5;
  2. Vol.: ISBN 0-914098-85-3;
  3. Vol.: ISBN 0-914098-86-1;
  4. Vol.: ISBN 0-914098-87-X;
  5. Vol.: ISBN 0-914098-88-8; (all in hardcover only).

  This book has a somewhat fussy notation, and tends toward the verbose, but it is engaging and full of insight. Boothby is shorter but covers more, although the last volume of Spivak is a gentle introduction to Chern classes.

  A gentle introduction by popular author is:

- Frank Morgan, Riemannian Geometry: A Beginner's Guide, 2nd ed. A K Peters, 1997. In print, ISBN 1-56881-073-3;

  Another well known textbook (the author is a relativist) is:

- Barrett O'Neill, Elementary Differential Geometry, 2nd ed. Academic Press, 1997. In print, ISBN 0-12-526745-2; list price 49.95 (hardcover).

  Another well known textbook (aimed more at hamiltonian mechanics) is:

- R. Abraham, Jerrold E. Marsden, and T. Ratiu, Manifolds, Tensor Analysis, and Applications. Springer-Verlag, 1996. In print, ISBN 0-387-96790-7; list price 69.95 (hardcover).

  A cheaper alternative is:

- Richard Bishop and Samuel Goldberg, Tensor Analysis on Manifolds. Dover, 1980. In print, ISBN 0-486-64039-6; list price 8.95 (paperback).

  At a higher level, try:

- Y. Choquet-Bruhat, C. DeWitt-Morette, and M. Dillard-Bleick, Analysis, Manifolds and Physics, Pt. I: Basics. Revised ed. Elsevier Science, 1991. In print, ISBN 0-444-86017-7; list price 63.50 (paperback). Note that the first author has made important contributions to GR.

  The most influential geometry book of all time is:

- Shoshichi Kobayashi and Katsumi Nobizu, Foundations of Differential Geometry. Two volumes. John Wiley & Sons, 1996. In print, ISBN 0-471-15733-3; list price 59.95 (paperback). Not for the faint of heart.

  A textbook by the greatest geometer of all time is:

- Shiing-Shen Chern, Differential Geometry. World Scientific, 1998. In print, ISBN 981-02-2647-0; list price 26.00 (paperback). For the Russian perspective (one author is a legendary relativist), try:

- B. A. Dubrovin, A. T. Fomenko, and S. P. Novikov, Modern Geometry - Methods and Applications. 2 volumes, 2nd ed. Springer-Verlag, 1993. In print, ISBN 0-387-97663-9; list price 65.9a (hardcover).

  I am not familiar with the following book, but I like an elementary GR text by the second author:

- F. De Felice, C. J. Clarke, Relativity on Curved Manifolds. Cambridge University Press, 1992. ISBN 0-521-42908-0; list price 42.95 (paperback). Here are two pricey and extremely concise outlines of the basics of differential geometry and topology as they are used in modern physics:

- M. Nakahara, Geometry, Topology and Physics. I O P Publishing, 1990. In print, ISBN 0-85274-095-6; list price 61.00 (paperback).

- Charles Nash and Siddartha Sen. Topology and Geometry for Physicists. Academic Press, 1988 (reprint).

  In print, ISBN 0-12-514081-9; list price 58.00 (paperback). These are so dense I wouldn't recommend them for anyone without a strong background in modern physics.

  Dover has reprinted books by Levi-Civita, Schouten, and Synge on tensor calculus. These were all essential references in their day but they are now hopelessly out of date and I recommend that students spend their money on more expensive but more modern texts.

  Here are some books that may help the student place relativity theory into the grand scheme of things, physically speaking:

- I.D. Lawrie, A Unified Grand Tour of Theoretical Physics. I O P Publishing, 1990. In print, ISBN 0-85274-015-8; list price 49.00 (paperback). I like this book very much. Lawrie quite properly emphasizes the formal analogies between hamiltonian mechanics and quantum theory; the variational principle formulations of GR ties this relativity theory to both these subjects. Lawrie also emphasizes the fact that newtonian theory is not simply "wrong"; by a mere change of interpretation (and a factor of i here and a factor of h bar there) the equations of newtonian theory (as rewritten by Hamilton) go over to their quantum analogs. Needless to say these formal analogies are a great help to the working physicist.

- Richard P. Feynman, The Feynman Lectures on Physics Addison Wesley Longman, 1970. 3 Volumes. In print, available as boxed set or individual paperbacks.

  One of the great scientific expositions of all time. Full of enthusiasm and overflowing with fabulous ideas. Feynman's geometric explanation of the physical meaning of Maxwell's equation is a joy; so is his discussion of action at a distance (his revolutionary work with Wheeler). The first two volumes are particularly recommended. Note well: in volume 2, the section on SR is one of the few weak points in the book; I advise that you skip it altogether. If you must read it, not, RPF is not saying that spacetime has a Euclidean metric!

- L. D. Landau, E. M. Lifshitz and others, Course of Theoretical Physics, 8 volumes. Butterworth-Heinemann, various years.

  Vol. 1 (Mechanics) and Vol. 2 (Classical Field Theory) are particularly relevant. Well translated and quite readable for the most part. Initiated by the great Russian physicist

Lev Landau and continued after his untimely death by his disciple Lifshitz. The Russian approach to physics and math is significantly different from American ideas in many respects and it is worthwhile gaining some familiarity with Landau's vision. Unlike say Feynman's great books, this series offers many excellent exercises.

- Walter Greiner and others. A Curriculum in Theoretical Physics. Springer-Verlag, various years. Another heroic attempt to survey all of modern theoretical physics at the advanced undergraduate to second year graduate level, this time with a European perspective. Well translated from the German, very readable, with an excellent balance of theory, descriptions of "great experiments", and practical experience in computing things using the theory. Many exercises are solved in full.

  Here are some books that relate relativity theory to important subjects in mathematics:

- E. J. Flaherty, Hermitian and Kahlerian Geometry in Relativity. Springer-Verlag New York, 1976. Out of print.

  In Kahler geometry, instead of bundling tangent planes with a euclidean inner product, we bundle tangent planes with a hermitian inner product, which gives a much more "rigid" structure. But symplectic geometry may be even more important in the future; see

- M. Kauderer, Symplectic Matrices, First Order Systems and Special Relativity. World Scientific, 1994. In print, ISBN 981-02-0829-4; list price 64.00 (hardcover).

- Victor W. Guillemin and Shlomo Sternberg, Symplectic Techniques in Physics. Cambridge University Press, 1990. In print, ISBN 0-521-38990-9; list price 37.95 (paper).

- Helmut Hofer and Eduard Zehnder, Symplectic Invariants and Hamiltonian Dynamics. Birkhauser, 1994. In print, ISBN 0-8176-5066-0; list price 59.50 (hardcover).

- J. M. Souriau, Structure of Dynamical Systems: A Symplectic View of Physics. Birkhauser, 1997. Out of print.

- Dusa McDuff and Dietmar Salamon, Introduction to Symplectic Topology. Oxford University Press, 1995. In print, ISBN 0-19-851177-9; list price 90.00 (hardcover).

- A. T. Fomenko, Symplectic Geometry, 2nd ed. Gordon & Breach Publishing Group, 1995. In print, ISBN 2-88124-901-9; list price 110.00 (hardcover).

  Finally, for a glimpse of what quantum gravity may look like, try:

- J. Baez and J. Muniain, Gauge Fields, Knots and Gravity. World Scientific, 1994. In print, ISBN 981-02-2034-0; list price 43.00 (paperback).

  This book also features an excellent and concise introduction to exterior forms and a good discussion of the rather vexed terms "contravariant" and "covariant" (they way they are used in older GR books is exactly opposite to their modern meaning in mathematics!) [From the editor (DK): But I think their use in older books makes much more sense!]

  Here are some books of enduring historical interest:

- Albert Einstein and others, The Principle of Relativity Dover, 1952 reprint. In print, ISBN 0-486-60081-5; list price 7.95 (paperback).

  A collection of historic papers by Lorentz, Einstein, and others, including Einstein's 1905 paper on STR, his 1907 paper on the equivalence of mass and energy, Minkowski's 1908 paper introducing the physical interpretation of his geometry, Einstein's 1916 paper on the foundations of GTR, and early attempts to unify EM and gravitation. In particular, the paper by Weyl laid the foundation for Yang-Mills theories, and the paper by Kaluza and Klein contains the idea of "compactified dimensions" which is a key element of modern string theories.

- Hermann Weyl, Space, Time, Matter Dover, 1922. In print, ISBN 0-486-60267-2; list price 9.95 (paperback).

  Weyl was one of the great mathematicians of the early twentieth century, and one of the first to appreciate the importance of Einstein's ideas about gravitation and unified field theories. In this quirky but clearly written book, he describes the five year old theory of GR, assuming virtually no mathematical prerequisites, and attempts to go beyond it with ideas on non-riemannian connections which were several generations ahead of their time (in terms of physical application).

- Richard C. Tolman, Relativity, Thermodynamics and Cosmology Dover, 1987 reprint. In print, ISBN 0-486-65383-8; list price 13.95 (paperback).

  An important resource in the thirties and forties but by now hopelessly out of date.

- Arthur S. Eddington, Space, Time and Gravitation: An Outline of the General Theory. Cambridge University Press, 1987. In print, ISBN 0-521-33709-7; list price 24.95 (paperback).

  A classic semipopular book, by now hopelessly outdated, but written with the engaging, stylish verve that made Eddington one of the most popular science writers of his day.

- Wolfgang Pauli, Theory of Relativity. Dover, 1981 reprint. In print, ISBN 0-486-64152-X; list price 8.95 (paperback).

  This was the first book on relativity theory, written in a burst of youthful enthusiasm by the twenty year old Pauli. Needless to say, it is of purely historical interest today.

  Here is a book which is quirky but which will be valuable to some readers:

- Richard P. Feynman, Lectures on Gravitation. Addison Wesley Longman, 1995. In print, ISBN 0-201-62734-5; list price 38.43 (hardcover).

  Feynman's attempt to motivate the field equation "in the spirit of QFT"; this approach is somewhat similar to that adopted in Ohanian et al., but this book is of interest mainly for watching Feynman at play.

Histories, Biographies, and Memoirs

- Abraham Pais, Subtle Is the Lord: The Science and Life of Albert Einstein Oxford University Press, 1983. In print, ISBN 0-19-520438-7, list price 17.95 (paperback)

  This is the definitive scientific biography, written a noted physicist who personally knew Einstein, Bohr, and other key people in AE's career, and who has read every paper AE ever wrote. Features a fascinating, detailed—and fully technical—account of Einstein's heroic struggle toward his field equations.

- Roberto Torretti, Relativity and Geometry Dover, 1996 (reprint). In print, ISBN 0-486-69046-6; list price 14.95 (paperback). This is another scientific biography focusing on the work rather than the man, offering some mathematical commentary on Einstein's struggle toward the field equation, and also discussing Einstein's "philosophy".

- Don Howard and John J. Stachel, Einstein: The Formative Years, 1879-1909 Birkhauser, 1998. In print, ISBN 3-7643-4030-4, (price not available).

  Another recent biography of Einstein.

- John A. Wheeler and Kenneth Ford, Geons, Black Holes, and Quantum Foam: A Life in Physics W. W. Norton & Company, 1998. In print, ISBN 0-393-04642-7, list price 27.95 (hardcover). The autobiography of the physicist widely credited (along with Subrahmanyan Chandrasekhar) with transforming the notion of a black hole from dubious speculation into a common, and in some ways, quite well understood natural object.

The book by Kip Thorne (a former PhD student of Wheeler, who has had a distinguished career in his own right) cited above contains more information on the modern history of relativity.

- N. T. Roseveare, Mercury's Perihelion from Le Verrier to Einstein. Oxford University Press, 1982. In print, ISBN 0-19-858174-2; list price 49.95 (hardcover) Features a detailed comparison of the GR prediction of precession with astronomical observation.

  The following may also be of interest:

- B. A. Rosenfeld, The History of Non-Euclidean Geometry. Springer-Verlag, 1988. In print, ISBN 0-387-96458-4; list price 89.00 (hardcover).
- Arthur J. Miller, Albert Einstein's Special Theory of Relativity: Emergence (1905) and Early Interpretation, 1905-1911. Springer-Verlag, 1997. In print, ISBN 0-387-94870-8; list price 39.95 (hardcover).
- D. Howard and J. J. Stachel, Einstein and the History of General Relativity Birkhauser, 1989. In print, ISBN 0-8176-3392-8; list price 102.00 (hardcover).
- J. Earman, M. Janssen, and J. D. Norton, editors, The Attraction of Gravitation: New Studies in the History of General Relativity, Birkhauser, 1993. In print, ISBN 0-8176-3624-2; list price 150 (hardcover).

Philosophy and Relativity Theory
For a first book on the philosophical reaction to relativity, I'd recommend:

- Lawrence Sklar, Space, time, and spacetime. University of California Press, 1974. In print, ISBN 0-520-03174-1, list price 15.95 (paperback). Engaging and delightful. This book won a prize for the exceptionally clear quality of its exposition. As a bonus, the later chapters contain an excellent nontechnical discussion of some of the features of the spacetime geometries treated in GR.
- Hans Reichenbach, Philosophy of Space and Time, Dover, 1998 In print, ISBN 0-486-60443-8; list price 8.95 (paperback). A reprint of a (well translated) classic book. Clearly written and highly influential.

  Here are a few more recent books:

- Lawrence Sklar, Philosophy and Spacetime Physics University of California Press, 1985. In print, ISBN 0-520-06180-2; list price 13.00 (paperback).
- Michael Friedman, Foundations of Space-Time Theories: Relativistic Physics and Philosophy of Science. Princeton University Press, 1983. In print, ISBN 0-691-02039-6; list price 35.00 (paperback).
- John Earman, World Enough and Space-Time: Absolute vs. Relational Theories of Space and Time, M.I.T. Press, 1989. In print, ISBN 0-262-05040-4; list price 30.00 (hardcover).

Advanced Technical Books
I begin by listing three classic books that must be studied by every dedicated student of GR.

- Stephen W. Hawking and G. F. Ellis, The Large Scale Structure of Space-Time. Cambridge University Press, 1975. In print, ISBN 0-521-09906-4; list price 47.95 (paperback) An extremely influential classic and a standard reference; this was the first book to provide a detailed description of the revolutionary topological methods introduced by Penrose and Hawking in the early seventies.

- Roger Penrose and Wolfgang Rindler, Spinors and Space Time: Two-Spinor Calculus and Relativistic Fields, Cambridge University Press, 1984. Two Volumes. Out of print. Another standard reference, unfortunately out of print. This is the book that made Newman-Penrose tetrads and spinorial methods into a standard technique in the field.

- S. Chandrasekhar, The Mathematical Theory of Black Holes, Oxford University Press, 1998. In print, ISBN 0-19-850370-9; list price 29.95 (paperback) By common consent, one of the great scientific books of our time. This is the book on black hole physics. Not for the faint of heart.

- Robert M Wald, editor. Black holes and Relativistic Stars. University of Chicago Press, 1998. In print, ISBN 0-226-87034-0; list price 50.00 (paperback) This is the proceedings of the Chandrasekhar Memorial conference, and contains excellent survey articles by the leading experts in the field on all aspects of modern relativity theory. Particularly notable are the articles by Thorne (gravitational wave astronomy), Rees (astrophysical evidence for black holes), Penrose (censorship), Teukolsky (numerical relativity), Israel (internal structure of black holes), Wald (black hole thermodynamics), and Hawking (information paradox). Indispensable.

  Here are four recent books focusing on various specialized topics of current interest:

- Ignazio Ciufolini and John A. Wheeler, Gravitation and Inertia. Princeton University Press, 1995. In print, ISBN 0-691-03323-4; list price 49.50 (hardcover). Quirky, stylish, and inspiring. A little too concise for an introductory account, but the first half of this book features masterful summaries of the mathematical structure of GR and observational and experimental evidence. The remainder of the book focuses on Mach's principle, one of the oldest leitmotifs of GR.

- Kip S Thorne, Richard H. Price, and Douglas A. Macdonald, editors. Black holes : the Membrane Paradigm. Yale University Press, 1986. In print, ISBN 0-300-03770-8; list price 21.00 (paperback) One of the most important insights into black hole physics is that the event horizon can for many purposes be treated as a physical membrane made of a conducting material; this picture breaks down, of course, once you pass through the horizon, but it turns out to be very useful so long as you restrict yourself to physics occurring outside of the horizon. This should seem strange, because the event horizon is about as physically substantial as the International Date Line.

- John Stewart, Advanced General Relativity. Cambridge University Press, 1993. In print, ISBN 0-521-44946-4; list price 32.95 (paperback). After a swift review of the basic notions of GR, this book focuses on the theory of gravitational wave detectors, a highly topical subject because of the expected advent of gravitational wave astronomy as workable detectors such as LIGO come on line in the next few years.

- Robert M. Wald, Quantum field theory in curved spacetime and black hole thermodynamics. University of Chicago Press, 1994. In print, ISBN 0-226-87027-8; list 16.95 (paperback). John Baez considers this the premier book on semiclassical gravitation; Wald is perhaps the world's leading expert on black hole thermodynamics. Again, this is a topical subject indeed, as a glance at the Los Alamos preprint server at http://xxx.lanl.gov/ will reveal.

- N. D. Birrell, and P. C. Davies, Quantum Fields in Curved Space. Cambridge University Press, 1984. In print, ISBN 0-521-27858-9; list price 47.95 (paperback). Many students prefer this to Wald's book because of the clarity of the writing and the excellent discussion of the particle concept.

- S. A. Huggett and K. P. Tod, An Introduction to Twistor Theory, 2nd ed. Cambridge University Press, 1994. In print, ISBN 0-521-45689-4; list price 22.95 (paperback). In principle a beginning graduate level introduction to twistor theory, but I think most readers will find this book pretty tough going.

- R. S. Ward and R. O. Wells, Twistor Geometry and Field Theory. Cambridge University Press, 1991. In print, ISBN 0-521-42268-X; list print 42.95 (paper). A more mathematical book, focusing on the Penrose transform and connections with representation theory.
- Roger Penrose and Wolfgang Rindler, Spinors and Space-Time. Two volumes. Cambridge University Press, 1998 (reprint of 1986 edition). The two volume book that founded twistor theory as a branch of mathematical physics. Not for the faint of heart.

  Here are some more books on relativistic astrophysics:
- I. D. Novikov, and Ya. B. Zel'Dovich, Stars and Relativity. Dover, 1996. In print, ISBN 0-486-69424-0; list price 14.95 (paperback). A reprint of a classic book (translated from the Russian) by the two most prominent Russian relativists.
- Stuart L. Shapiro and Saul A. Teukolsky, Black Holes, White Dwarfs, and Neutron Stars: The Physics of Compact Objects. John Wiley & Sons, 1983. In print, ISBN 0-471-87316-0; list price 97.95 (paperback). Pricey, but a standard reference.
- Barrett O'Neill, The Geometry of Kerr Black Holes. A K Peters, 1995. In print, ISBN 1-56881-019-9; list price 88.00 (hardcover). Also pricey, but this will surely be the standard book on Kerr geometry for a generation to come.
- Steven Weinberg, Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity. John Wiley & Sons, 1972. In print, ISBN 0-471-92567-5; 63.50 through discount outlets. Another standard reference.

  It is appropriate to close with three volumes from the collected works of the genius who first appreciated the reality of collapsed objects and black holes:
- S. Chandrasekhar, Selected Papers: The Mathematical Theory of Black Holes and of Colliding Plane Waves. University of Chicago Press, 1991. In print, ISBN 0-226-10101-0; list price 42.00 (paperback).
- S. Chandrasekhar, Selected Papers: Relativistic Astrophysics. University of Chicago Press, 1990. In print, ISBN 0-226-10099-5; list price 36.00 (paperback).
- S. Chandrasekhar, Selected Papers: The Non-Radial Oscillations of Stars in General Relativity and Other Writings. University of Chicago Press, 1997. In print, ISBN 0-226-10104-5; list price 45.00 (paperback).

General Physics (so even mathematicians can understand it!)

- M.S. Longair: Theoretical concepts in physics, 1986. An alternative view of theoretical reasoning in physics for final-year undergrads.
- Arnold Sommerfeld: Lectures on Theoretical Physics Sommerfeld is God for mathematical physics.
- Richard Feynman: The Feynman lectures on Physics (3 vols) Highly recommended texts compiled from the undergraduate lecture course given by Feynman.
- Jearle Walker: The Flying Circus of Physics There is the entire Landau and Lifshitz series. They have volumes on classical mechanics, classical field theory, E&M, QM, QFT, statistical physics, and more. Very good series that spans the entire graduate-level curriculum.
- The New Physics edited by Paul Davies. This is one big book, and it takes time to look through topics as diverse as general relativity, astrophysics, particle theory, quantum mechanics, chaos and nonlinearity, low-temperature physics, and phase transitions. Nevertheless, this is an excellent book of recent (1989) physics articles, written by several physicists/astrophysicists.

- Richard Feynman: The Character of Physical Law In his unique no-nonsense style, Feynman lectures on what physics is all about. Down-to-earth examples keep him from straying into the kind of metaphysics of which he is often critical.
- David Mermin: Boojums all the way through: Communicating science in prosaic language
- Frank Wilczek and Betsy Devine: Longing for the Harmonies: Themes and variations from modern physics
- Greg Egan: Permutation City This is a science fiction novel which has more to say about the philosophy of physics than do most philosophers and physicists.
- Michael Crowe: A History of Vector Analysis This is a history book, covering the birth of quaternions and related concepts throughout the nineteenth century, and how these evolved into our modern vector system. Its long sentences, extremely passive style, and paucity of commas make it a laboured read—but a far easier read than the nineteenth century mathematicians and physicists whom it quotes! These early scientists probably wrote in the style of their times, but their meaning is often completely opaque to a modern reader. Although Crowe covers the history in great depth, his book has very little mathematics; for example, you won't learn anything about quaternions or vectors here. One of the few mathematical discussions—of rotating by using quaternions on page 191—gives pause for thought. It states/implies that the unit quaternions each produce a rotation of 90°. They don't; they each produce a rotation 180°. This mistake seems to be due to the players in the story, and doesn't appear to be a typographical error.

Classical Mechanics

- Herbert Goldstein: Classical Mechanics, 2nd ed, 1980. Intermediate to advanced; excellent bibliography.

  Introductory:
- The Feynman Lectures, vol 1.
- Keith Symon: Mechanics, 3rd ed., 1971 undergrad. level.
- H. Corbin and P. Stehle: Classical Mechanics, 2nd ed., 1960
- V.I. Arnold: Mathematical methods of classical mechanics, translated by K. Vogtmann and A. Weinstein, 2nd ed., 1989.

  The appendices are somewhat more advanced and cover all sorts of nifty topics. Deals with geometrical aspects of classical mechanics.
- R. Resnick and D. Halliday: Physics, vol 1, 4th Ed., 1993 Excellent introduction without much calculus. Lots of problems and review questions.
- Marion & Thornton: Classical Dynamics of Particles and Systems, 2nd ed., 1970. Undergrad level. A useful intro to classical dynamics. Not as advanced as Goldstein, but with real worked-out examples.
- A. Fetter and J. Walecka: Theoretical mechanics of particles and continua. Graduate-level text, a little less impressive than Goldstein, but sometimes a little less obtuse.
- Kiran Gupta: Classical Mechanics of Particles and Rigid Bodies (1988) At the level of Goldstein, but has many more worked problems at the end of each chapter as a good illustration of the material. Very useful for preparations for the PhD Qualifying Examination (I presume this is America only — ed.).

  Avoid: Physics for Mathematicians:

- Mechanics I, by Michael Spivak. Spivak is a well-known author of calculus and differential geometry textbooks. His book on mechanics was his attempt to show physicists how a mathematician thinks about their field. The result—at least in what I've read, the early part of the book—contains a great many comments that apparently are meant to show how much more deeply a mathematician can think about physics than any physicist could, but which only serve to show how good Spivak was at making mountains out of molehills. He sometimes misunderstood what he was doing, and laid the resulting problem at the feet of physics with an unwarranted self assurance. An example of his self confidence is his comment on page 25 that Newton need not have said that acceleration dv/dt is in the same direction as force F, because "what other direction could it have?". Well, how about F $\times$ v? Remarks such as "if you can believe that [Newton's] proof [of a particular thing] can be valid" are tiresome. The book is peppered with spelling mistakes, and on the rare occasion that its equations have been numbered, the numbering is an unprofessional mix of upper- and lower-case letters, numbers, and symbols such as asterisks, asterisks with subscripts, and even the odd set of wavy lines. Spivak's statement in the preface that the prerequisites for his book are the first two volumes of his multi-volume set on differential geometry (which is utterly unreadable for most mortals) suggests that physicists might best avoid reading this mechanics book—even though a knowledge of differential geometry is not needed for at least the first part of the book.

Classical Electromagnetism

- Jackson: Classical Electrodynamics, 2nd ed., 1975 Intermediate to advanced, the definitive graduate(US)/undergraduate(UK) text.

- Purcell: Berkeley Physics Series Vol 2. You can't beat this for the intelligent, reasonably sophisticated beginning physics student. He tells you on the very first page about the experimental proof of how charge does not vary with speed. plus...

- Chen, Min, Berkeley Physics problems with solutions.

- Reitz, Milford and Christy: Foundations of Electromagnetic Theory 4th ed., 1992.cUndergraduate level. Pretty difficult to learn from at first, but a good reference, for some calculations involving stacks of thin films and their reflectance and transmission properties, for example. It's a good, rigorous text as far as it goes, which is pretty far, but not all the way. For example, it has a great section on optical properties of a single thin film between two dielectric semi-infinite media, but no generalization to stacks of films.

- Feynman: The Feynman Lectures, Vol. 2

- Lorrain & Corson: Electromagnetism, Principles and Applications, 1979

- Resnick and Halliday: Physics, vol 2, 4th ed., 1993 Igor Irodov: Problems in Physics Excellent and extensive collection of EM problems for undergrads.

- William Smythe: Static and Dynamic Electricity, 3rd ed., 1968 For the extreme masochists. Some of the most hair-raising EM problems you'll ever see. Definitely not for the weak-of-heart.

- Landau, Lifshitz, and Pitaevskii: Electrodynamics of Continuous Media, 2nd ed., 1984. Same level as Jackson's book above, but with lots of material that is not in Jackson.

- Marion and Heald: Classical Electromagnetic Radiation, 2nd ed., 1980. Undergraduate or low-level graduate.

Quantum Mechanics

- QED: The strange theory of light and matter Richard Feynman. One need no longer be confused by this beautiful theory. Richard Feynman gives an exposition that is once again and by itself a beautiful explanation of the theory of photon-matter interactions. Taken from a popular, non-technical lecture.
- Cohen-Tannoudji: Quantum Mechanics I & II&, 1977. Introductory to intermediate.
- Liboff: Introductory Quantum Mechanics, 2nd ed., 1992 Elementary level. Makes a few mistakes.
- Sakurai: Advanced Quantum Mechanics 1967 Good as an introduction to the very basic beginnings of quantum field theory, except that it has the unfortunate feature of using "imaginary time" to make Minkowski space look euclidean.
- Sakurai: Modern Quantum Mechanics, 1985
- J. Wheeler and W. Zurek (eds.): Quantum Theory and Measurement, 1983. On the philosophical end. People who want to know about interpretations of quantum mechanics should definitely look at this collection of relevant articles.
- C. DeWitt and N. Graham: The Many Worlds Interpretation of Quantum Mechanics. Philosophical. Collection of articles.
- H. Everett: Theory of the Universal Wavefunction An exposition which has some gems on thermodynamics and probability. Worth reading for this alone.
- Bjorken and Drell: Relativistic Quantum Mechanics/ Relativistic Quantum Fields (for comments, see under Particle Physics)
- Ryder: Quantum Field Theory, 1984
- Guidry: Gauge Field Theories: an introduction with applications 1991
- Messiah: Quantum Mechanics, 1961
- Dirac: (a) Principles of QM, 4th ed., 1958 (b) Lectures in QM, 1964 (c) Lectures on Quantum Field Theory, 1966
- Itzykson and Zuber: Quantum Field Theory, 1980 Advanced level.
- Slater: Quantum theory: Address, essays, lectures. Good follow on to Schiff.

  Note: Schiff, Bjorken and Drell, Fetter and Walecka, and Slater are all volumes in "International Series in pure and Applied Physics" published by McGraw-Hill.
- Pierre Ramond: Field Theory: A Modern Primer, 2nd edition. Volume 74 in the FiP series. The so-called "revised printing" is a must, as they must've rushed the first printing of the 2nd edition because it's full of inexcusable mistakes.
- Feynman: The Feynman Lectures, Vol. 3 A non-traditional approach. A good place to get an intuitive feel for QM, if one already knows the traditional approach.
- Heitler & London: Quantum theory of molecules
- J. Bell: Speakable and Unspeakable in Quantum Mechanics, 1987 An excellent collection of essays on the philosophical aspects of QM.
- Milonni: The quantum vacuum: an introduction to quantum electrodynamics 1994.
- Holland: The Quantum Theory of Motion. A good bet for a strong foundation in QM.

- John von Neumann: Mathematical foundations of quantum mechanics, 1955. For the more mathematical side of quantum theory, especially for those who are going to be arguing about measurement theory.

- Schiff: Quantum Mechanics, 3rd ed., 1968 A little old. Not much emphasis on airy-fairy things like many worlds or excessive angst over Heisenberg UP. Straight up QM for people who want to do calculations. Introductory graduate level. Mostly Schrodinger eqn. Spin included, but only in an adjunct to Schrodinger. Not much emphasis on things like Dirac eqn, etc.

- Eisberg and Resnick: Quantum Physics of Atoms, Molecules, Solids, Nuclei, and Particles, 2nd ed., 1985. This is a basic intro. to QM, and it is excellent for undergrads. It is not thorough with the mathematics, but fills in a lot of the intuitive stuff that most textbooks do not present.

- David Saxon: Elementary Quantum Mechanics A decent undergraduate (senior level) text.

- Bethe and Jackiw: Intermediate Quantum Mechanics

- P.W.Atkins: Quanta: A Handbook of concepts Short entries, arranged alphabetically, emphasis on stuff relevant to quantum chemistry. Concentrates on the intuition and not the mathematics.

- James Peebles: Quantum Mechanics (1993) Intermediate level, based on lectures given by the author at Princeton. Very lucid exposition of the standard material with outstanding selection of mostly original problems at the end of each chapter.

Statistical Mechanics and Entropy

- David Chandler: Introduction to Modern Statistical Mechanics, 1987 Chandler's book is short, but although its discussions are dressed up as being about physics, you will gain little knowledge of statistical mechanics by reading it.

- R. Tolman: Prinicples of Statistical Mechanics. Dover

- Kittel & Kroemer: Statistical Thermodynamics Not a bad book—but, that said, it has little competition, since good books on statistical mechanics are hard to find.

- Keith Stowe: An introduction to Thermodynamics and Statistical Mechanics, 2nd ed., 2007 Stowe has written an excellent book that has plenty of physics and some very good explanations. This is worthwhile to buy as your entry into the subject. His mathematics is sometimes a little short of what you might like to see: for example, he has left a very important calculation to Appendix C, but turns it into something overly complicated there. Stowe's non-postulatory approach to the subject is far more modern and physically valid and meaningful than Callen's outdated experiment (see below) of simply postulating everything that he found convenient—or possibly didn't understand himself!

- F. Reif: Fundamentals of Statistical and Thermal Physics. Reif's book is well known. You can find much interesting and useful discussion in it, but its mathematics is generally a forest of obscure notation and unnecessary formalism, heavily cluttered by primes and overbars that add nothing. Its topics are not presented in a particularly pedagogical or clear way. Although it's a good book to refer to (once you manage to find what you are looking for), it is not for anyone wanting to learn the subject. It is a very difficult read, even for advanced physicists, on account of its cluttered notation and long discussions that don't always deliver what they promise.

- Felix Bloch: Fundamentals of Statistical Mechanics.

- Radu Balescu: Statistical Physics Graduate Level. Good description of non-equilibrium stat. mech., but difficult to read. It is all there, but often you don't realize it until after you have learned it somewhere else. Nice development in early chapters about parallels between classical and quantum statistical mechanics.

- Abrikosov, Gorkov, and Dyzaloshinski: Methods of Quantum Field Theory in Statistical Physics

- Huw Price: Time's Arrow and Archimedes' Point Semi-popular book on the direction of time, by a philosopher. It has been controversial because of its criticism of physicists such as Hawking, for their "double standards" in dealing with the old problem on the origin of the arrow of time. It is thought provoking and clearly written.

- H. Callen: Thermodynamics and an Introduction to Thermostatistics, 2nd ed., 1985.

  In the preface to this second edition, Callen described his 25-year-old postulatory approach to thermodynamics and statistical mechanics as ¨now widely accepted¨. In fact, by the time of his second edition, his approach was completely outdated, because it springs from nineteenth-century ideas of thermodynamics in which concepts such as entropy were not understood. This means that Callen simply postulated the core quantities such as entropy and temperature with essentially no context, and without providing any physical insight or analysis. It might all look streamlined, but his approach will give you no insight into the difficult and interesting questions of the subject. Callen described his approach as rendering the subject transparent and simple; but his approach comes across as obscure. For example, in the early part of the book, he insists on repeatedly writing $1/T1 = 1/T2$ for two temperatures that are ascertained to be equal, when anyone else would write $T1 = T2$. And, for what he does write, the devil is often in the details that he tends to leave out. Even at the start, when Callen introduces the concept of work, he fails to say whether he is talking about the work done on the system, or by the system, leaving the reader to work that out for himself from some irrelevant comments about the mechanical work term $-PdV$. Callen's incorrect renditions of the Taylor expansion in an appendix seem to suggest, rather oddly, that he didn't understand the difference between dx and Δx. His book includes a 20-page postscript in which he makes claims about the role of symmetry in thermodynamics; but, as far as I can tell, this section says nothing useful at all. I suspect that the reason this book is as frequently cited as it is said to be lies in its being used as the basis for a course by many lecturers who never learned the subject themselves, and hence don't realise that the book's approach is outdated. If you really want to learn the subject, use the modern statistical approach, in which entropy is defined to relate to numbers of configurations. As far as readability goes, Callen's writing tends to omit commas; but this can make his sentences tedious to read, since the reader ends up having to make two or three passes to decode what some sentences are saying. (If you use few commas yourself, study a typical sentence in Callen's book: "the intermediate states of the gas are nonequilibrium states for which the enthalpy is not defined". Callen is not singling out a special set of non-equilibrium states here; instead, enthalpy is not defined for any non-equilibrium state. He should have included a single comma, by writing ¨the intermediate states of the gas are non-equilibrium states, for which the enthalpy is not defined¨.)

- R. Pathria: Statistical Mechanics.

- D. Forster: Hydrodynamic Fluctuations, Broken Symmetry, and Correlation Functions.

- H. Stanley: Introduction to Phase Transitions and Critical Phenomena.

- S.K. Ma: Modern Theory of Critical Phenomena.

- N. Goldenfeld: Lectures on Phase Transitions and the Renormalization Group.

- J. Sethna: Statistical Mechanics: Entropy, Order Parameters, and Complexity. Apparently Sethna's book is meant to be teaching statistical mechanics; but this is not an introductory book, and it provides no real insight into statistical mechanics and entropy. (I don't know about its sections on order parameters and complexity.) It is mostly a collection of exercises for the reader, aided by the author's comments. Don't believe everything you read in it; for example, in his exercise 5.7, Sethna misinterprets the meaning of entropy to say, incorrectly, that the entropy of an isolated system remains constant in time. He incorrectly describes our universe as photon dominated on page 160, when in fact it is matter dominated. These and other instances give one the impression that Sethna is not always working in his zone of expertise. His many exercises might have some content, but they can be tedious to read, since new paragraphs are not indented in them.

Condensed Matter

- Charles Kittel: Introduction to Solid State Physics (ISSP), introductory
- Ashcroft and Mermin: Solid State Physics, intermediate to advanced
- Charles Kittel: Quantum Theory of Solids. This is from before the days of his ISSP; it is a more advanced book. At a similar level.
- Solid State Theory, by W. A. Harrison (a great bargain now that it's published by Dover)
- Theory of Solids, by Ziman.
- Fundamentals of the Theory of Metals, by Abrikosov Half of the book is on superconductivity.
- Many-Particle Physics, G. Mahan. Advanced.

Special Relativity

- Taylor and Wheeler: Spacetime Physics Still the best introduction out there.
- Relativity: Einstein's popular exposition.
- Wolfgang Rindler: Essential Relativity. Springer 1977 With a heavy bias towards astrophysics and therefore on a more moderate level formally. Quite strong on intuition.
- A.P. French: Special Relativity A thorough introductory text. Good discussion of the twin paradox, pole and the barn etc. Plenty of diagrams illustrating Lorentz-transformed coordinates, giving both an algebraic and geometrical insight to SR. (Seems to be out of print)
- Abraham Pais: Subtle is the Lord: The Science and Life of Albert Einstein
- The best technical biography of the life and work of Albert Einstein.
- Special Relativity and its Experimental Foundations Yuan Zhong Zhang Special relativity is so well established that its experimental foundation is often ignored. This book fills the gap and will be of relevance to many discussions in sci.physics.relativity

Particle Physics

- Kerson Huang: Quarks, leptons & gauge fields, World Scientific, 1982. Good on mathematical aspects of gauge theory and topology.
- L. B. Okun: Leptons and quarks, translated from Russian by V.
- I. Kisin, North-Holland, 1982.
- T. D. Lee: Particle physics and introduction to field theory.

- Itzykson: Particle Physics

- Bjorken & Drell: Relativistic Quantum Mechanics One of the more terse books. The first volume on relativistic quantum mechanics covers the subject in a blinding 300 pages. Very good if you really want to know the subject.

- Francis Halzen & Alan D. Martin: Quarks & Leptons, Beginner to intermediate, this is a standard textbook for graduate level courses. Good knowledge of quantum mechanics and special relativity is assumed. A very good introduction to the concepts of particle physics. Good examples, but not a lot of Feynman diagram calculation. For this, see Bjorken & Drell.

- Donald H. Perkins: Introduction to high energy physics Regarded by many people in the field as the best introductory text at the undergraduate level. Covers basically everything with almost no mathematics.

- Close, Marten, and Sutton: The Particle Explosion A popular exposition of the history of particle physics with terrific photography.

- Christine Sutton: Spaceship Neutrino A good, historical, largely intuitive introduction to particle physics, seen from the neutrino viewpoint.

- Mandl, Shaw: Quantum Field Theory Introductory textbook, concise and practically orientated. Used at many graduate departments as a textbook for the first course in QFT and a bare minimum for experimentalists in high energy physics. Chapters on Feynman diagrams and cross-section calculations particularly well written and useful.

- F.Gross: Relativistic Quantum Mechanics and Field Theory I am familiar with first part only (rel. QM) which I warmly recommend in conjunction with Mandl, since Klein-Gordon and Dirac Equation are explained in greater detail than in Mandl. One of my professors likes a lot the rest of the book too, but I haven't spent much time on it and can't comment. Published in 1993. S. Weinberg: The Quantum Theory of Fields, Vol I,II, 1995 The usual Weinberg stuff: refreshing, illuminating viewpoints on every page. Perhaps most suitable for graduate students who already know some basics of QFT.

- M.B. Green, J.H. Schwarz, E. Witten: Superstring Theory (2 vols) Although these two volumes do not touch the important new developments in string theories, they are still the best texts for the basics. To keep up with this fast developing subject, it is necessary to download the papers and reviews as hep-th e-prints.

- M. Kaku: Strings, Conformal Fields and Topology Just a little more up-to-date than GSW.

- Superstrings: A Theory of Everything, ed. P.C.W. Davies Through transcripts of interviews with Schwarz, Witten, Green,

- Gross, Ellis, Salam, Glashow, Feynman, and Weinberg, we learn about string theory, and how different physicists feel about its prospects as a theory of everything. This also predates the new developments that revolutionised string theory after 1993. A Pais: Inward Bound This can be regarded as a companion volume to his biography of Einstein (see special relativity section). It covers the history of particle physics through the twentieth century, but is best for the earlier half.

- R.P. Crease, C.C. Mann: The Second Creation 1996 Another history of particle physics in the twentieth century. This one is especially good on the development of the standard model. Full of personal stories taken from numerous interviews, it is difficult to put down.

- L. Lederman, D. Teresi: The God Particle: If the Universe Is the Answer, What Is the Question? 2006 This book describes the search for the Higgs Boson at Fermilab. It describes what the Higgs is and gives some background to the subject of particle physics. It also gives an account of some more general physics history.

General Relativity

- Meisner, Thorne and Wheeler: Gravitation W.H. Freeman & Co., San Francisco 1973 Usually referred to as "MTW". It has two tracks for different levels. A famous work in the subject whose main strength is probably its various asides, historical and otherwise. While it has much interesting reading, it is not a book to learn relativity from: its approach is all over the place, and it pushes gawdy notation which no one actually uses to do anything useful. Robert M. Wald: Space, Time, and Gravity: the Theory of the Big Bang and Black Holes. A good non-technical introduction, with a nice mix of mathematical rigor and comprehensible physics.
- B. Schutz: A First Course in General Relativity. A readable and useful book, to a point. The 1988 edition, at least, unfortunately has a tangled approach to its Lambda index notation that is wrong in places. Schutz goes to great lengths to convince the reader of the usefulness of one-forms, but is clearly unaware that everything he does with them can be done far more simply using vectors alone. Beware the show-stopping typos in the Riemann components for the Schwarzschild metric on page 315. The discussion about Riemann tensor signs on page 171 is also wrong, and will give you wrong results if you apply it. Indeed, that discussion is indicative of a general naïveté in the book's early mathematics as a whole.
- Weinberg: Gravitation and Cosmology A good book that takes a somewhat different approach to the subject.
- Hans Ohanian: Gravitation & Spacetime (recently back in print) For someone who actually wants to learn to work problems, ideal for self-teaching, and math is introduced as needed, rather than in a colossal blast.
- Robert Wald: General Relativity A more advanced textbook than Wald's earlier book, appropriate for an introductory graduate course in GR. It strikes just the right balance, in my opinion, between mathematical rigor and physical intuition. It has great mathematics appendices for those who care about proving theorems carefully, and a good introduction to the problems behind quantum gravity (although not to their solutions). I think it's MUCH better than either MTW or Weinberg. Clifford Will: Was Einstein Right? Putting General Relativity to the Test
- Non-technical account of the experimental support for GR, including the "classic three tests", but going well beyond them.
- Kip Thorne: Black Holes and Time Warps: Einstein's Outrageous Legacy An award-winning popular account of black holes and related objects with many historical anecdotes from the author's personal experiences. The book is famous for the final sections about time travel through wormholes. Ignore Dirac's small book on lectures in GR, unless you like reading books that have almost no discussion of their mathematical content (and almost no discussion of anything else, either). It's a sure bet that this book was only published because Dirac wrote it.

Mathematical Methods

- Morse and Feshbach: Methods of Theoretical Physics. This book used to be hard to find, but can now be bought at feshbachpublishing.com.
- Mathews and Walker: Mathematical Methods of Physics. An absolute joy for those who love math, and very informative even for those who don't. [This has been severely disputed!—ed]

- Arfken: Mathematical Methods for Physicists Academic Press Good introduction at graduate level. Not comprehensive in any area, but covers many areas widely. Arfken is to math methods what numerical recipes is to numerical methods — good intro, but not the last word.
- Zwillinger: Handbook of Differential Equations. Academic Press Kind of like CRC tables but for ODEs and PDEs. Good reference book when you've got a differential equation and want to find a solution.
- Gradshteyn and Ryzhik: Table of Integrals, Series, and Products Academic THE book of integrals. Huge, but useful when you need an integral.
- F.W. Byron and R. Fuller: Mathematics of Classical and Quantum Physics (2 vols) is a really terrific text for self-study; it is like a baby version of Morse & Feshbach.

Nuclear Physics

- Preston and Bhaduri: Structure of the Nucleus
- Blatt and Weisskopf: Theoretical Nuclear Physics
- DeShalit and Feshbach: Theoretical Nuclear Physics This is serious stuff. Also quite expensive even in paper. I think the hard cover is out of print. This is volume I (structure). Volume II (scattering) is also available.
- Satchler: Direct Nuclear Reactions
- Walecka: Theoretical Nuclear and Subnuclear Physics (1995) Covers advanced topics in theoretical nuclear physics from a modern perspective and includes results of past 20 years in a field which makes it unique. Not an easy material to read but invaluable for people seeking an updated review of the present status in the field.
- Krane: Introductory nuclear physics Introductory-to-intermediate level textbook in basic nuclear physics for senior undergraduates. Good, clear and relatively comprehensive exposition of "standard" material: nuclear models, alfa, beta, gamma radioactivity, nuclear reactions. . . Last edition issued in 1988.

Cosmology

- J. V. Narlikar: Introduction to Cosmology.1983 Jones & Bartlett Publ. For people with a solid background in physics and higher math, THE introductory text, IMHO, because it hits the balance between mathematical accuracy (tensor calculus and stuff) and intuitive clarity/geometrical models very well for grad student level. Of course, it has flaws but only noticeable by the Real Experts (TM). .
- Hawking: A Brief History of Time The ghost-written book that made Popular Science popular, but an odd mixture of easy physics and very advanced physics. Weinberg: First Three Minutes A very good book. It's pretty old, but most of the information in it is still correct.
- Timothy Ferris: Coming of Age in the Milky Way and The Whole Shebang

  More Popular Science, and very readable.
- Kolb and Turner: The Early Universe. At a more advanced level, a standard reference. As the title implies, K&T cover mostly the strange physics of very early times: it's heavy on the particle physics, and skimps on the astrophysics. There's a primer on large-scale structure, which is the most active area of cosmological research, but it's really not all that good.

- Peebles: Principles of Physical Cosmology. Comprehensive, and on the whole it's quite a good book, but it's rather poorly organized. I find myself jumping back and forth through the book whenever I want to find anything.

- Black Holes and Warped Spacetime, by William J. Kaufmann III. This is a great, fairly thorough, though non-mathematical description of black holes and spacetime as it relates to cosmology. I was impressed by how few mistakes Kaufmann makes in simplifying, while most such books tend to sacrifice accuracy for simplicity.

- M.V. Berry: Principles of Cosmology and Gravitation This is very well written, and useful as an undergrad text.

- Dennis Overbye: Lonely Hearts of the Cosmos The unfinished history of converge on Hubble's constant is presented, from the perspective of competing astrophysics rival teams and institute, along with a lot of background on cosmology (a lot on inflation, for instance). A good insight into the scientific process.

- Joseph Silk: The Big Bang I consider Silk's book an absolute must for those who want a quick run at the current state of big bang cosmology and some of the recent (1988) issues which have given so many of us lots of problems to solve. [of course that's eons out of date now—ed.] Bubbles, voids, and bumps in time: the new cosmology edited by James Cornell. This is quite a nice and relatively short read for some of the pressing issues (as of 1987-88) in astrophysical cosmology.

- T. Padmanabhan: Structure formation in the universe A no-nonsense book for those who want to calculate some problems strictly related to the formation of structure in the universe. The book even comes complete with problems at the end of each chapter. A bad thing about this book is that there isn't any coverage on clusters of galaxies and the one really big thing that annoys the hell outta me is that the bibliography for each chapter is all combined in one big bibliography towards the end of the book which makes for lots of page flipping.

- P.J.E. Peebles: The large-scale structure of the universe This is a definitive book for anyone who desires an understanding of the mathematics required to develop the theory for models of large scale structure. The essential techniques in the description of how mass is able to cluster under gravity from a smooth early universe are discussed. While I find it dry in some places, there are noteworthy sections (e.g. statistical tests, n-point correlation functions, etc.).

- Andrzej Krasinski: Inhomogeneous Cosmological Models If you are blinded by the dogma of the cosmological principle, this book is a real eye opener. A technical, historical and bibliographical survey of possible inhomogeous universes from solutions of general relativity.

- Alan Lightman and Roberta Brawer: Origins: The lives and worlds of modern cosmologists, 1990 Transcripts of interview with 27 of the most influential cosmologists from the past few decades. This book provides a unique record of how their cosmological theories have been formed.

Astronomy

- Hannu Karttunen et al. (eds.): Fundamental Astronomy. The very good book covering all of astronomy (also for absolute beginners) AND still going into a lot of detail for special work for people more involved AND presenting excellent graphics and pictures.

- Pasachoff: Contemporary Astronomy Good introductory textbook for the nontechnical reader. It gives a pretty good overview of the important topics, and it has good pictures.

- Frank Shu: The physical universe: an introduction to astronomy This is a really grand book, which covers a huge sweep of physics in its 600-odd pages. Not only does it describe the field of astronomy in great detail, but it also covers in detail the laws of classical and quantum mechanics, astrophysics and stellar evolution, cosmology, special and general relativity; and last but not least, the biochemical basis of life. In fact the last few chapters would make a great addition to a biochemist's library!

- Kenneth R. Lang: Astrophysical formulae: a compendium for the physicist and astrophysicist Here is everything you wanted to know (and more!) about astrophysical formulae on a one-line/one-paragraph/one-shot deal. Of course, the formulae come complete with references (a tad old, mind you) but it's a must for everyone who's working in astronomy and astrophysics. You learn something new every time you flip through the pages!

Plasma Physics

- (See Robert Heeter's sci.physics.fusion FAQ for details)

Numerical Methods/Simulations

- Johnson and Rees: Numerical Analysis Addison Wesley Undergraduate level broad intro.

- Numerical Recipes in X (X = C, Fortran, Pascal, etc.) Tueklosky and Press

- Young and Gregory: A survey of Numerical Mathematics Dover 2 volumes. Excellent overview at grad. level. Emphasis toward solution of elliptic PDEs, but good description of methods to get there including linear algebra, matrix techniques, ODE-solving methods, and interpolation theory. Biggest strength is it provides a coherent framework and structure to attach most commonly used numerical methods. This helps understanding about why to use one method or another. 2 volumes.

- Hockney and Eastwood: Computer Simulation Using Particles Adam Hilger Good exposition of particle-in-cell (PIC) method and extensions. Applications to plasmas, astronomy, and solid state are discussed. Emphasis is on description of algorithms. Some results shown.

- Birdsall and Langdon: Plasma Physics via Computer Simulations PIC simulation applied to plasmas. Source codes shown. First part is almost a tutorial on how to do PIC. Second part is like a series of review articles on different PIC methods.

- Tajima: Computational Plasma Physics: With Applications to Fusion and Astrophysics Addison Wesley Frontiers in physics Series. Algorithms described. Emphasis on physics that can be simulated. Applications limited to plasmas, but subject areas very broad, fusion, cosmology, solar astrophysics, magnetospheric physics, plasma turbulence, general astrophysics.

Fluid Dynamics

- D.J. Tritton: Physical Fluid Dynamics

- G.K. Batchelor: Introduction to Fluid Dynamics

- S. Chandrasekhar: Hydrodynamics and Hydromagnetic Stability

- Segel: Mathematics Applied to Continuum Mechanics Dover.

Nonlinear Dynamics, Complexity, and Chaos
There is a FAQ posted regularly to sci.nonlinear.

- Prigogine: Exploring Complexity Or any other Prigogine book. If you've read one, you read most of of them (A Poincaré recurrence maybe?).

- Guckenheimer and Holmes: Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields Springer Borderline phys./math. Advanced level. A nuts-and-bolts "how to" textbook. They let the topic provide all the razzmatazz, which is plenty if you pay attention and remember the physics that it applies to.
- Lichtenberg, A. J. and M. A. Lieberman (1982): Regular and Stochastic Motion. New York, Springer-Verlag.
- Ioos and Joseph: Elementary Stability and Bifurcation Theory. New York, Springer.
- Heinz Pagels: The Dreams Of Reason He is a very clear and interesting, captivating writer, and presents the concepts in a very intuitive way. The level is popular science, but it is still useful for physicists who know little of complexity.
- M. Mitchell Waldrop: Complexity A popular intro to the subject of spontaneous orders, complexity and so on. Covers implications for economics, biology etc and not just physics.

Optics (Classical and Quantum), Lasers

- Max Born and Emil Wolf: Principles of Optics: Electromagnetic Theory of Propagation

  Standard reference.
- Sommerfeld: For the more classically minded.
- Allen and Eberly: Optical Resonance and Two-Level Atoms. For quantum optics, the most readable but most limited.
- Goodman: Introduction to Fourier Optics. If it isn't in this book, it isn't Fourier optics. Quantum Optics and Electronics (Les Houches Summer School 1963 or 1964, but someone has claimed that Gordon and Breach, NY, are going to republish it in 1995), edited by DeWitt, Blandin, and Cohen- Tannoudji, is noteworthy primarily for Glauber's lectures, that form the basis of quantum optics as it is known today.
- Sargent, Scully, & Lamb: Laser Physics
- Yariv: Quantum Electronics
- Siegman: Lasers
- Shen: The Principles of Nonlinear Optics
- Meystre & Sargent: Elements of Quantum Optics
- Cohen-Tannoudji, Dupont-Roc, & Grynberg: Photons, Atoms and Atom-Photon Interactions.
- Hecht: Optics

  A very good introductory optics book.
- Practical Holography by Graham Saxby, Prentice Hall: New York; 1988. This is a very clear and detailed book that is an excellent introduction to holography for interested undergraduate physics people, as well as advanced readers, especially those who are interested in the practical details of making holograms and the theory behind them.

Mathematical Physics

- Lie Algebra, Topology, Knot Theory, Tensors, etc.

  These are books that are sort of talky and fun to read (but still substantial—some harder than others). These include things mathematicians can read about physics as well as vice versa. These books are different than the "bibles" one must have on hand at all times to do mathematical physics.

- Yvonne Choquet-Bruhat, Cecile DeWitt-Morette, and Margaret Dillard-Bleick: Analysis, manifolds, and physics (2 volumes) Something every mathematical physicist should have at his bedside until he knows it inside and out—but some people say it's not especially easy to read.
- Jean Dieudonne: A panorama of pure mathematics, as seen by N. Bourbaki, translated by I.G. Macdonald.

  Gives the big picture in mathematics.
- Robert Hermann: Lie groups for physicists, Benjamin-Cummings, 1966.
- George Mackey: Quantum mechanics from the point of view of the theory of group representations, Mathematical Sciences Research Institute, 1984.
- George Mackey: Unitary group representations in physics, probability, and number theory.
- Charles Nash and S. Sen: Topology and geometry for physicists.
- B. Booss and D.D. Bleecker: Topology and analysis: the Atiyah-Singer index formula and gauge-theoretic physics.
- Bamberg and S. Sternberg: A Course of Mathematics for Students of Physics
- Bishop & Goldberg: Tensor Analysis on Manifolds.
- Dodson & Poston: Tensor Geometry.
- Abraham, Marsden & Ratiu: Manifolds, Tensor Analysis and Applications.
- M. Nakahara: Topology, Geometry and Physics.
- Morandi: The Role of Topology in Classical and Quantum Physics
- Singer, Thorpe: Lecture Notes on Elementary Topology and Geometry
- L. Kauffman: Knots and Physics, World Scientific, Singapore, 1991.
- C. Yang and M. Ge: Braid group, Knot Theory & Statistical Mechanics.
- D. Kastler: C-algebras and their applications to Statistical Mechanics and Quantum Field Theory.
- Courant and Hilbert: Methods of Mathematical Physics Wiley Really a mathematics book in disguise. Emphasis on ODEs and PDEs. Proves existence, etc. Very comprehensive. 2 volumes.
- Cecille Dewitt is publishing a book on manifolds that should be out soon (maybe already is). Very high level, but supposedly of great importance for anyone needing to set the Feynman path integral in a firm foundation.
- Howard Georgi: Lie Groups for Particle Phyiscs Addison Wesley Frontiers in Physics Series.
- Synge and Schild.

Atomic Physics

- Max Born: Atomic Physics A classic, though a little old.
- Gerhard Herzberg: Atomic spectra and atomic structure, Translated with the co-operation of the author by J. W. T.Spinks. — New York, Dover publications, 1944 Old but good.
- E. U. Condon and G. H. Shortley: The theory of atomic spectra, CUP 1951

- G. K. Woodgate: Elementary atomic structure, 2d ed. Oxford:
- New York: Clarendon Press, Oxford University Press, 1983, c 1980. Introductory level.
- Alan Corney: Atomic and laser spectroscopy, Oxford, New York: Clarendon Press, 1977 Excellent,fairly advanced, large experimental bent, but good development of background. Good stuff on lasers (gas, dye)

Low Temperature Physics, Superconductivity

- The Theory of Quantum Liquids, by D. Pines and P. Nozieres Superconductivity of Metals and Alloys, P. G. DeGennes A classic introduction.
- Theory of Superconductivity, J. R. Schrieffer
- Superconductivity, M. Tinkham
- Experimental techniques in low-temperature physics, by Guy K. White. This is considered by many as a "bible" for those working in experimental low-temperature physics.

Chapter 32

Fun with names

- Gordon vs Gordan
  - Paul Gordan (1837-1912) - Clebsch-Gordan decomposition
  - Walter Gordon (1893-1939) - Klein-Gordon equation
- Lorentz vs Lorenz
  - Hendrik Lorentz (1853-1928) - Lorentz transformation, Lorentz force
  - Ludvig Lorenz (1829-1891) - Lorenz gauge
- Hertz vs Hertz
  - Heinrich Hertz (1857-1894) - Hertzian dipole antenna
  - Gustav Hertz (1887-1975) - Franck-Hertz experiment
- Bragg vs Bragg
  - William Henry Bragg (1862-1942) - Bragg equation
  - William Lawrence Bragg (1890-1971) - Bragg equation
- Klein vs Klein
  - Oskar Klein (1894-1977) - Klein-Gordon equation, Kaluza-Klein theory
  - Felix Klein (1849-1925) - Klein bottle
- Euler vs Euler
  - Hans Heinrich Euler (1909-1941) - Euler–Heisenberg Lagrangian
  - Leonhard Euler (1707-1783) - Euler's formula
- Weyl vs Weil
  - Hermann Weyl (1885-1955) - Weyl spinor, Weyl group
  - Andre Weil (1906-1998) - Weil group, Chern–Weil homomorphism
- Jordan vs Jordan vs Jordan
  - Camille Jordan (1838-1922) - Jordan normal, Jordan-Hoelder theorem
  - Wilhelm Jordan (1842-1899) - Gauss-Jordan elimination
  - Pascual Jordan (1902-1980) - Jordan algebra, Jordan Wigner transformation
- Kac vs Kac
  - Victor Kac (1943-...) - Kac–Moody algebra
  - Mark Kac (1904-1984) - Feynman–Kac formula