0.0.1 Definitions

Group

- closure: $a, b \in G \to a \circ b \in G$
- associativity: $a \circ b) \circ c = a \circ (b \circ c)$
- identity: $\exists e \in G$ s.t. $a \circ e = e \circ a = a$
- inverse: $\forall a \in G,\ \exists a^{-1} \in G$ s.t. $a \circ a^{-1} = a^{-1} \circ a = e$

Lie Group

Continuous group

Group Representation

- $D_R : G \to \mathrm{GL}(n)$ s.t. $D_R(a)D_R(b) = a \circ b$
- $D(e) = 1_{n\times n}$
- $D(a^{-1}) = D(a)^{-1}$

Lie Algebra Representation

- $\pi : \mathfrak{g} \to \mathrm{Mat}(n)$ s.t. $\pi([A, B]) = [\pi(A), \pi(B)]$

Lie Algebra vs Lie Group

Lie group element $a(\theta)$, representation of group element $D_R(a(\theta))$, representation of Lie algebra generator $\pi_R(A^\mu)$

$$D_R(a(\theta)) = e^{i\theta_\mu \pi_R(A^\mu)} \tag{1}$$

$$\pi_R(A^\mu) = -i\, \frac{\partial D_R}{\partial \theta_\mu}\bigg|_{\theta=0} \tag{2}$$

Matrix Exponentials

$$e^X = \sum_{n=0} \frac{1}{n!} X^n \tag{3}$$

$$\det e^X = e^{\mathrm{tr} X} \tag{4}$$

$$\left(e^X\right)^{-1} = e^{-X} \tag{5}$$

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X,Y]]-\frac{1}{12}[Y,[X,Y]]+\ldots} \tag{6}$$

Irreducible representation

$D_{R_1} \oplus D_{R_2} \neq D_R$

Casimir element/operator

Object (element of the center of the universal enveloping algebra of a Lie algebra) that commutes with all generators of the Lie algebra

0.0.2 Representation facts you should know as a physicist

0.0.3 SU(2) and Quantum mechanics of spin 1/2

- Definitions I: Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{7}$$

- Observe
  - $[\sigma_k, \sigma_l] = 2i\epsilon_{klj}\sigma_j$
  - $g_i = -\frac{i}{2}\sigma_i \quad \rightarrow \quad [\sigma_k, \sigma_l] = 2i\epsilon_{klj}\sigma_j$
  - $L_i = \frac{1}{2}\sigma_i \quad \rightarrow \quad [L_k, L_l] = i\epsilon_{klj}L_j$
  - $e^{\alpha_k \sigma_k} = e^{-i\alpha_k L_k}$ form all $SU(2)$ matrices, this means g_i (or L_i) are generators of $\mathfrak{su}(2)$

- Definitions II:

$$\text{Spin up state} \quad |+{}^1\!/\!_2\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{8}$$

$$\text{Spin down state} \quad |-{}^1\!/\!_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{9}$$

$$\text{Ladder up operator} \quad L_{z+} = ig_x - g_y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{10}$$

$$\text{Ladder down operator} \quad L_{z-} = ig_x + g_y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{11}$$

$$\text{(hermitian) Spin operator} \quad L_z = ig_z = \begin{pmatrix} {}^1\!/\!_2 & 0 \\ 0 & -{}^1\!/\!_2 \end{pmatrix} \tag{12}$$

$$\text{Casimir operator} \quad L^2 = -(g_x)^2 - (g_y)^2 - (g_z)^2 = \begin{pmatrix} {}^3\!/\!_4 & 0 \\ 0 & {}^3\!/\!_4 \end{pmatrix} \tag{13}$$

$$= L_{z+}^\dagger L_{z+} + L_z^2 + L_z \tag{14}$$

$$= L_{z-}^\dagger L_{z-} + L_z^2 - L_z \tag{15}$$

$$\text{Commutators} \quad [L_{z+}, L_{z-}] = 2L_z \tag{16}$$

$$[L_z, L_{z+}] = +L_{z+} \tag{17}$$

$$[L_z, L_{z-}] = -L_{z-} \tag{18}$$

- Results

	$\lvert -{}^1\!/\!_2\rangle$	$\lvert +{}^1\!/\!_2\rangle$
L_{z+}	$\lvert +{}^1\!/\!_2\rangle$	0
L_{z-}	0	$\lvert -{}^1\!/\!_2\rangle$
L_z	$-{}^1\!/\!_2\lvert -{}^1\!/\!_2\rangle$	$+{}^1\!/\!_2\lvert +{}^1\!/\!_2\rangle$
L^2	$+{}^3\!/\!_4\lvert -{}^1\!/\!_2\rangle$	$+{}^3\!/\!_4\lvert +{}^1\!/\!_2\rangle$

- Now we can show for eigenvectors $|m\rangle$ (at least for $m = \pm{}^1\!/\!_2$)

$$L_z|m\rangle = m|m\rangle \tag{19}$$

$$\rightarrow L_{z+}|m\rangle = \sqrt{j(j+1) - m(m+1)}|m+1\rangle \tag{20}$$

$$\rightarrow L_{z-}|m\rangle = \sqrt{j(j+1) - m(m-1)}|m-1\rangle \tag{21}$$

$$\rightarrow L^2|m\rangle = j(j+1)|m\rangle \tag{22}$$

meaning $m = -j, ..., +j$.

Constructing of higher spin representation based on spin 1/2

1. Select a dimension $n = 2j+1$

2. Define n cartesian basis vectors

$$|m=+j\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad |m=j-1\rangle = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \qquad ..., \qquad |m=-j\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}, \tag{23}$$

3. Calculate action of ladder operators

L_z - eigenstate	L_{z+}	L_{z-}
$\|-j\rangle$	$\sqrt{2}\sqrt{j}\|-j+1\rangle$	0
$\|-j+1\rangle$	$\sqrt{4}\sqrt{j-1/2}\|-j+2\rangle$	$\sqrt{2}\sqrt{j}\|-j\rangle$
$\|-j+2\rangle$	$\sqrt{6}\sqrt{j-1}\|-j+3\rangle$	$\sqrt{4}\sqrt{j-1/2}\|-j+1\rangle$
$\|-j+3\rangle$	$\sqrt{8}\sqrt{j-3/2}\|-j+4\rangle$	$\sqrt{6}\sqrt{j-1}\|-j+3\rangle$
$\|-j+4\rangle$	$\sqrt{10}\sqrt{j-4}\|-j+5\rangle$	$\sqrt{8}\sqrt{j-3/2}\|-j+4\rangle$
...	...	...
$\|j-1\rangle$	$\sqrt{2}\sqrt{j}\|j\rangle$	$\sqrt{4}\sqrt{j-1/2}\|j-2\rangle$
$\|j\rangle$	0	$\sqrt{2}\sqrt{j}\|j-1\rangle$

4. Calculate ladder operator matrix elements ($|m_i\rangle$ and $|m_k\rangle$ are orthogonal)

$\langle m_i\|L_{z+}\|m_k\rangle$	$\|-j\rangle$	$\|-j+1\rangle$	...	$\|j-1\rangle$	$\|j\rangle$
$\langle -j\|$	0	0	...	0	0
$\langle -j+1\|$	$\sqrt{2}\sqrt{j}$	0	...	0	0
$\langle -j+2\|$	0	$\sqrt{4}\sqrt{j-1/2}$			
			$\ddots$		
...	...	...			
$\langle j-1\|$	0	0	...	0	0
$\langle j\|$	0	0	...	$\sqrt{2}\sqrt{j}$	0

$\langle m_i\|L_{z-}\|m_k\rangle$	$\|-j\rangle$	$\|-j+1\rangle$	...	$\|j-1\rangle$	$\|j\rangle$
$\langle -j\|$	0	$\sqrt{2}\sqrt{j}$	...	0	0
$\langle -j+1\|$	0	0	...	0	0
$\langle -j+2\|$	0	0			
			$\ddots$		
...	...	...			
$\langle j-1\|$	0	0	...	0	$\sqrt{2}\sqrt{j}$
$\langle j\|$	0	0	...	0	0

$\langle m_i\|L_z\|m_k\rangle$	$\|-j\rangle$	$\|-j+1\rangle$	...	$\|j-1\rangle$	$\|j\rangle$
$\langle -j\|$	$-j$	0	...	0	0
$\langle -j+1\|$	0	$-j+1$	...	0	0
$\langle -j+2\|$	0	0			
			$\ddots$		
...	...	...			
$\langle j-1\|$	0	0	...	$j-1$	0
$\langle j\|$	0	0	...	0	j

5. Now calculate the generators via

$$g_x = \frac{1}{2i}(L_{z-} + L_{z+}) \tag{24}$$

$$= \frac{1}{2i}\begin{pmatrix} 0 & \sqrt{2}\sqrt{j} & \dots & 0 & 0 \\ \sqrt{2}\sqrt{j} & 0 & \dots & 0 & 0 \\ \dots & & & & \\ 0 & 0 & \dots & 0 & \sqrt{2}\sqrt{j} \\ 0 & 0 & \dots & \sqrt{2}\sqrt{j} & 0 \end{pmatrix} \tag{25}$$

$$g_y = \frac{1}{2}(L_{z-} - L_{z+}) \tag{26}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & \sqrt{2}\sqrt{j} & \dots & 0 & 0 \\ -\sqrt{2}\sqrt{j} & 0 & \dots & 0 & 0 \\ \dots & & & & \\ 0 & 0 & \dots & 0 & \sqrt{2}\sqrt{j} \\ 0 & 0 & \dots & -\sqrt{2}\sqrt{j} & 0 \end{pmatrix} \tag{27}$$

$$g_z = -iL_z \tag{28}$$

$$= -i\begin{pmatrix} -j & 0 & \dots & 0 & 0 \\ 0 & -j+1 & \dots & 0 & 0 \\ \dots & & & & \\ 0 & 0 & \dots & j-1 & 0 \\ 0 & 0 & \dots & 0 & j \end{pmatrix} \tag{29}$$

6. Now calculate group elements via $M_i = e^{\theta g_i}$

$$M_z = \begin{pmatrix} e^{i\theta j} & 0 & 0 & \dots & 0 & 0 \\ 0 & e^{i\theta(j-1)} & 0 & \dots & 0 & 0 \\ \dots & & & & & \\ 0 & 0 & 0 & \dots & e^{i\theta(-j+1)} & 0 \\ 0 & 0 & 0 & \dots & 0 & e^{i\theta(-j)} \end{pmatrix} \tag{30}$$

7. Remark I:
   - massless spin 1: only $|1,-1\rangle$ and $|1,+1\rangle$ exist (left right polarized), $|1,0\rangle$ corresponds to longitudinal polarization (not possible for massless particles)
   - massless spin 2: only $|2,-2\rangle$ and $|2,+2\rangle$ exist

8. Remark II:
   - By complexifying $\mathfrak{sl}(2,\mathbb{C})_{\mathbf{C}} = \mathfrak{su}(2)_{\mathbf{C}} \oplus \mathfrak{su}(2)_{\mathbf{C}}$

$$\mathfrak{sl}(2,\mathbb{C}): J_1, J_2, J_3, K_1, K_2, K_3 \quad \rightarrow \quad A_i = \frac{1}{2}(J_i + iK_i), B_i = \frac{1}{2}(J_i - iK_i) \tag{31}$$

$$\begin{array}{ccc} \mathfrak{su}(2)_{\mathbf{C}} & \oplus & \mathfrak{su}(2)_{\mathbf{C}} \\ A_i = \frac{1}{2}(J_i + iK_i) & & B_i = \frac{1}{2}(J_i - iK_i) \\ [A_i, A_j] = \epsilon_{ijk}A_k & & [B_i, B_j] = \epsilon_{ijk}B_k \\ A_+ = iA_1 - A_2 & & B_+ = iB_1 - B_2 \\ A_- = iA_1 + A_2 & & B_- = iB_1 + B_2 \\ A = iA_3 & & B = iB_3 \end{array} \tag{32}$$

   so Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ irreps are labeled by two $\mathfrak{su}(2)$ irreps (j_L, j_R)
   - For the Lie group SL(2,ℂ) irreps are also labeled by (j_L, j_R) and can be written as $\mathrm{SL}(2,\mathbf{C})_L \otimes \mathrm{SL}(2,\mathbf{C})_R$

Tensorproducts of representations

- Lie algebra $\mathfrak{g}$ is the tangent space at identity element of Lie group G manifold

- $g_{xy} \in \mathfrak{g}$, $R_{xy}(\theta) \in G$ with $R_{xy}(\theta = 0) = 1$

$$R_{xy}(\theta) = e^{g_{xy}\theta} \quad \leftrightarrow \quad \left[\frac{dR_{xy}(\theta)}{d\theta}\right]_{\theta=0} = \left[g_{xy} e^{R_{xy}\theta}\right]_{\theta=0} = g_{xy} \tag{33}$$

- (bit odd - tensor product of two Lie group elements - whats the meaning?) $A(t) = e^{at}, B(t) = e^{bt}$

$$\left[\frac{d}{dt}(A(t) \otimes B(t))\right]_{t=0} = \left[\frac{dA(t)}{dt} \otimes B(t) + \frac{A(t) \otimes dB(t)}{dt}\right]_{t=0} = a \otimes 1 + 1 \otimes b \tag{34}$$

$$\rightarrow A(t) \otimes B(t) = e^{(a \otimes 1 + 1 \otimes b)t} \tag{35}$$

- Lie group representations $\rho_1 \rightarrow \mathrm{GL}(m), \rho_2 \rightarrow \mathrm{GL}(n)$ and $A \in G$ then the **tensor product of the group representations** is $\rho_1 \otimes \rho_2 \rightarrow \mathrm{GL}(m \cdot n)$

$$(\rho_1 \otimes \rho_2)A = \rho_1(A) \otimes 1 + 1 \otimes \rho_2(A) \tag{36}$$

- Lie group algebra $\pi_1 \rightarrow \mathrm{Mat}(m), \pi_2 \rightarrow \mathrm{Mat}(n)$ and $a \in \mathfrak{g}$ then the **tensor product of the algebra representations** is $\pi_1 \otimes \pi_2 \rightarrow \mathrm{Mat}(m \cdot n)$

$$(\pi_1 \otimes \pi_2)a = \pi_1(a) \otimes 1 + 1 \otimes \pi_2(a) \tag{37}$$

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$$L(t) = e^{\pi_1(l)t} \tag{38}$$

$$K(t) = e^{\pi_2(k)t} \tag{39}$$

$$L(t) \otimes K(t) = e^{(l \otimes 1_{m\times m} + 1_{n \times n} \otimes k)t} \tag{40}$$

- Now consider: $\mathfrak{su}(2) \otimes \mathfrak{su}(2)$ which actually means (use the same representation on both $\mathfrak{su}(2)$ and THEN build then tensor product) then

$$(\pi \otimes \pi)g_{xy} = \pi(g_{xy}) \otimes 1 + 1 \otimes \pi(g_{xy}) \tag{41}$$

so we rewrite in short for the $\mathfrak{su}(2) \otimes \mathfrak{su}(2)$ elements (multi-particle operators)

$$\overbrace{g_z}^{\mathfrak{su}(2)\otimes\mathfrak{su}(2)} \simeq \overbrace{g_z}^{\mathfrak{su}(2)} \otimes 1 + 1 \otimes \overbrace{g_z}^{\mathfrak{su}(2)} \tag{42}$$

$$g_x \simeq g_x \otimes 1 + 1 \otimes g_x \tag{43}$$

$$g_y \simeq g_y \otimes 1 + 1 \otimes g_y \tag{44}$$

$$\rightarrow g_\pm \simeq g_\pm \otimes 1 + 1 \otimes g_\pm \tag{45}$$

$$\rightarrow g_z \simeq g_z \otimes 1 + 1 \otimes g_z \tag{46}$$

BUT

$$g^2 = -(g_x)^2 - (g_y)^2 - (g_z)^2 \tag{47}$$
$$= g_+^\dagger g_+ + g_z^2 + g_z \tag{48}$$
$$= (g_+ \otimes 1 + 1 \otimes g_+)^\dagger (g_+ \otimes 1 + 1 \otimes g_+) + (g_z \otimes 1 + 1 \otimes g_z)(g_z \otimes 1 + 1 \otimes g_z) + (g_z \otimes 1 + 1 \otimes g_z) \tag{49}$$
$$= (g_+^\dagger \otimes 1 + 1 \otimes g_+^\dagger)(g_+ \otimes 1 + 1 \otimes g_+) + (g_z \otimes 1 + 1 \otimes g_z)(g_z \otimes 1 + 1 \otimes g_z) + (g_z \otimes 1 + 1 \otimes g_z) \tag{50}$$
$$= (g_- \otimes 1 + 1 \otimes g_-)(g_+ \otimes 1 + 1 \otimes g_+) + (g_z \otimes 1 + 1 \otimes g_z)(g_z \otimes 1 + 1 \otimes g_z) + (g_z \otimes 1 + 1 \otimes g_z) \tag{51}$$
$$= (g_- g_+ \otimes 1) + (g_- \otimes g_+) + (g_+ \otimes g_-) + (1 \otimes g_- g_+) \tag{52}$$
$$+ (g_z^2 \otimes 1) + (g_z \otimes g_z) + (g_z \otimes g_z) + (1 \otimes g_z^2) \tag{53}$$
$$+ (g_z \otimes 1) + (1 \otimes g_z) \tag{54}$$
$$= (g^2 \otimes 1 + 1 \otimes g^2) + 2(g_z \otimes g_z) + (g_+ \otimes g_-) + (g_- \otimes g_+) \tag{55}$$

then

$$g_z(|m_1\rangle \otimes |m_2\rangle) = (g_z \otimes 1 + 1 \otimes g_z)(|m_1\rangle \otimes |m_2\rangle) \tag{56}$$
$$= (g_z \otimes 1)(|m_1\rangle \otimes |m_2\rangle) + (1 \otimes g_z)(|m_1\rangle \otimes |m_2\rangle) \tag{57}$$
$$= (g_z|m_1\rangle \otimes 1|m_2\rangle) + (1|m_1\rangle \otimes g_z|m_2\rangle) \tag{58}$$
$$= m_1|m_1\rangle \otimes |m_2\rangle + m_2|m_1\rangle \otimes |m_2\rangle \tag{59}$$
$$= (m_1 + m_2)|m_1\rangle \otimes |m_2\rangle \tag{60}$$

- Now couple two $j = 1/2$ reps - meaning $({}^1/_2 \times {}^1/_2)$

$$g_+|-{}^1/_2\rangle = \sqrt{j(j+1) - m_j(m_j+1)}|+{}^1/_2\rangle \tag{61}$$
$$g_-|+{}^1/_2\rangle = \sqrt{j(j+1) - m_j(m_j-1)}|-{}^1/_2\rangle \tag{62}$$
$$|\uparrow\uparrow\rangle \equiv |+{}^1/_2\rangle \otimes |+{}^1/_2\rangle \tag{63}$$
$$|\uparrow\downarrow\rangle \equiv |+{}^1/_2\rangle \otimes |-{}^1/_2\rangle \tag{64}$$
$$|\downarrow\uparrow\rangle \equiv |-{}^1/_2\rangle \otimes |+{}^1/_2\rangle \tag{65}$$
$$|\downarrow\downarrow\rangle \equiv |-{}^1/_2\rangle \otimes |-{}^1/_2\rangle \tag{66}$$

	$\lvert\downarrow\downarrow\rangle$	$\lvert\downarrow\uparrow\rangle$	$\lvert\uparrow\downarrow\rangle$	$\lvert\uparrow\uparrow\rangle$
g_z	-1	0	0	1
g_+	$\lvert\uparrow\downarrow\rangle + \lvert\downarrow\uparrow\rangle$	$\lvert\uparrow\uparrow\rangle$	$\lvert\uparrow\uparrow\rangle$	0
$g_z g_+$	0	1	1	-
g_-	0	$\lvert\downarrow\downarrow\rangle$	$\lvert\downarrow\downarrow\rangle$	$\lvert\uparrow\downarrow\rangle + \lvert\downarrow\uparrow\rangle$
$g_z g_+$	-	-1	-1	0

then going up with ladder operator - we find only 3 states $|\downarrow\downarrow\rangle \equiv |j=1, m_j=-1\rangle, |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \equiv |j=1, m_j=0\rangle, |\uparrow\uparrow\rangle \equiv |j=1, m_j=+1\rangle$

$$|\downarrow\downarrow\rangle = |\downarrow\downarrow\rangle \quad \rightarrow \quad g_z(|\downarrow\downarrow\rangle) = -1|\downarrow\downarrow\rangle \tag{67}$$
$$g_+|\downarrow\downarrow\rangle = |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \quad \rightarrow \quad g_z(g_+|\downarrow\downarrow\rangle) = 0\, g_+|\downarrow\downarrow\rangle \tag{68}$$
$$g_+g_+|\downarrow\downarrow\rangle = g_+(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \tag{69}$$
$$= 2|\uparrow\uparrow\rangle \quad \rightarrow \quad g_z(g_+g_+|\downarrow\downarrow\rangle) = +1\, g_+g_+|\downarrow\downarrow\rangle \tag{70}$$
$$\rightarrow j = 1 \text{ (triplett)} \tag{71}$$

so (let's try $|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \equiv |j=0, m_j=0\rangle$)

$$g_+(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) = 0 + |\uparrow\uparrow\rangle - |\uparrow\uparrow\rangle - 0 = 0 \tag{72}$$
$$g_-(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) = |\downarrow\downarrow\rangle + 0 - 0 - |\downarrow\downarrow\rangle = 0 \tag{73}$$
$$g_z(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) = 0\,(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \tag{74}$$
$$\rightarrow j = 0 \text{ (singlet)} \tag{75}$$

Conclusion: the tensor product of the 2D representations splits into a 3D and a 1D representation

$$2 \otimes 2 = 3 \oplus 1 \qquad \text{or alternatively} \qquad {}^1\!/_2 \otimes {}^1\!/_2 = 1 \oplus 0 \tag{76}$$

Basis transformation

	J	1	1	0	1
	M_J	+1	0	0	-1
		$\|1,1\rangle$	$\|1,0\rangle$	$\|0,0\rangle$	$\|1,-1\rangle$
m_1, m_2		$\|\uparrow\uparrow\rangle$	$\frac{1}{\sqrt{2}}(\|\downarrow\uparrow\rangle + \|\uparrow\downarrow\rangle)$	$\frac{1}{\sqrt{2}}(\|\downarrow\uparrow\rangle - \|\uparrow\downarrow\rangle)$	$\|\downarrow\downarrow\rangle$
$\|+\frac{1}{2}\rangle \otimes \|+\frac{1}{2}\rangle$	$\|\uparrow\uparrow\rangle$	1	0	0	0
$\|+\frac{1}{2}\rangle \otimes \|-\frac{1}{2}\rangle$	$\|\uparrow\downarrow\rangle$	0	$1/\sqrt{2}$	$1/\sqrt{2}$	0
$\|-\frac{1}{2}\rangle \otimes \|+\frac{1}{2}\rangle$	$\|\downarrow\uparrow\rangle$	0	$1/\sqrt{2}$	$-1/\sqrt{2}$	0
$\|-\frac{1}{2}\rangle \otimes \|-\frac{1}{2}\rangle$	$\|\downarrow\downarrow\rangle$	0	0	0	1

$1/2 \times 1/2$		1 +1	1 0	0 0	1 -1
+1/2	+1/2	1			
+1/2	-1/2		$1/\sqrt{2}$	$1/\sqrt{2}$	
-1/2	+1/2		$1/\sqrt{2}$	$-1/\sqrt{2}$	
-1/2	-1/2				1

- Now couple two $j_1 = {}^3\!/_2$ with $j_2 = 1$

$$j_1 = {}^3\!/_2 \quad \rightarrow \quad \begin{cases} g_+|-{}^3\!/_2\rangle = \sqrt{3}|-{}^1\!/_2\rangle \\ g_+|-{}^1\!/_2\rangle = 2|+{}^1\!/_2\rangle \\ g_+|+{}^1\!/_2\rangle = \sqrt{3}|+{}^3\!/_2\rangle \end{cases} \tag{77}$$
$$j_2 = 1 \quad \rightarrow \quad \begin{cases} g_+|-1\rangle = \sqrt{2}|0\rangle \\ g_+|0\rangle = \sqrt{2}|+1\rangle \end{cases} \tag{78}$$
$$\tag{79}$$

Calculating the ladder up $g_+|m_1, m_2\rangle$ starting with $|-{}^3/_2, -1\rangle$

$$|-{}^3/_2,-1\rangle = |\mathbf{-{}^3/_2,-1}\rangle \tag{80}$$
$$g_z|-{}^3/_2,-1\rangle = -\frac{5}{2}|-{}^3/_2,-1\rangle \tag{81}$$
$$\to \text{Normalization} = 1, M_J = -{}^5/_2, J = {}^5/_2 \tag{82}$$
$$g_+|-{}^3/_2,-1\rangle = \sqrt{\mathbf{3}}|\mathbf{-{}^1/_2,-1}\rangle + \sqrt{\mathbf{2}}|\mathbf{-{}^3/_2,0}\rangle \tag{83}$$
$$g_z(\sqrt{3}|-{}^1/_2,-1\rangle + \sqrt{2}|-{}^3/_2,0\rangle) = -\frac{3}{2}(\sqrt{3}|-{}^1/_2,-1\rangle + \sqrt{2}|-{}^3/_2,0\rangle) \tag{84}$$
$$\to \text{Normalization} = \frac{1}{\sqrt{5}}, M_J = -{}^3/_2, J = {}^5/_2 \tag{85}$$
$$(g_+)^2|-{}^3/_2,-1\rangle = 2\sqrt{3}|+{}^1/_2,-1\rangle + \sqrt{2}\sqrt{3}|-{}^1/_2,0\rangle + \sqrt{3}\sqrt{2}|-{}^1/_2,0\rangle + \sqrt{2}\sqrt{2}|-{}^3/_2,+1\rangle \tag{86}$$
$$= \sqrt{\mathbf{12}}|\mathbf{+{}^1/_2,-1}\rangle + \sqrt{\mathbf{24}}|\mathbf{-{}^1/_2,0}\rangle + \sqrt{\mathbf{4}}|\mathbf{-{}^3/_2,+1}\rangle \tag{87}$$
$$\to \text{Normalization} = \frac{1}{\sqrt{40}}, M_J = -{}^1/_2, J = {}^5/_2 \tag{88}$$
$$(g_+)^3|-{}^3/_2,-1\rangle = \sqrt{3}\sqrt{12}|+{}^3/_2,-1\rangle + \sqrt{2}\sqrt{12}|+{}^1/_2,0\rangle + 2\sqrt{24}|+{}^1/_2,0\rangle \tag{89}$$
$$+ \sqrt{2}\sqrt{24}|-{}^1/_2,+1\rangle + \sqrt{3}\sqrt{4}|-{}^1/_2,+1\rangle + 0 \tag{90}$$
$$= \sqrt{\mathbf{36}}|\mathbf{+{}^3/_2,-1}\rangle + \sqrt{\mathbf{216}}|\mathbf{+{}^1/_2,0}\rangle + \sqrt{\mathbf{108}}|\mathbf{-{}^1/_2,+1}\rangle \tag{91}$$
$$\to \text{Normalization} = \frac{1}{\sqrt{360}}, M_J = +{}^1/_2, J = {}^5/_2 \tag{92}$$
$$(g_+)^4|-{}^3/_2,-1\rangle = 0 + \sqrt{2}\sqrt{36}|+{}^3/_2,0\rangle + \sqrt{3}\sqrt{216}|+{}^3/_2,0\rangle + \sqrt{2}\sqrt{216}|+{}^1/_2,+1\rangle + 2\sqrt{108}|+{}^1/_2,+1\rangle + 0 \tag{93}$$
$$= \sqrt{\mathbf{1152}}|\mathbf{+{}^3/_2,0}\rangle + \sqrt{\mathbf{1728}}|\mathbf{+{}^1/_2,+1}\rangle \tag{94}$$
$$\to \text{Normalization} = \frac{1}{\sqrt{2880}}, M_J = +{}^3/_2, J = {}^5/_2 \tag{95}$$
$$(g_+)^5|-{}^3/_2,-1\rangle = 0 + \sqrt{2}\sqrt{1152}|+{}^3/_2,1\rangle + \sqrt{3}\sqrt{1728}|+{}^3/_2,+1\rangle + 0 \tag{96}$$
$$= \sqrt{14400}|+{}^3/_2,1\rangle \tag{97}$$
$$\to \text{Normalization} = \frac{1}{\sqrt{14400}}, M_J = +{}^5/_2, J = {}^5/_2 \tag{98}$$
$$(g_+)^6|-{}^3/_2,-1\rangle = 0 \tag{99}$$

Constructing next ladder

$$g^2(\sqrt{c}|-{}^1/_2,-1\rangle + \sqrt{1-c}|-{}^3/_2,0\rangle) \tag{100}$$
$$= ((g^2\otimes 1 + 1\otimes g^2) + 2(g_z\otimes g_z) + (g_+\otimes g_-) + (g_-\otimes g_+))(\sqrt{c}|-{}^1/_2,-1\rangle + \sqrt{1-c}|-{}^3/_2,0\rangle) \tag{101}$$
$$= \left(\frac{3}{2}\frac{5}{2} + \frac{1}{2}\frac{3}{2}\right)(\sqrt{c}|-{}^1/_2,-1\rangle + \sqrt{1-c}|-{}^3/_2,0\rangle) + 2\left(-\frac{3}{2}\right)(\sqrt{c}|-{}^1/_2,-1\rangle + \sqrt{1-c}|-{}^3/_2,0\rangle) \tag{102}$$
$$+ (0\sqrt{c}|-{}^1/_2,-1\rangle + \sqrt{3}\sqrt{2}\sqrt{1-c}|-{}^1/_2,-1\rangle) + (\sqrt{3}\sqrt{2}\sqrt{c}|-{}^3/_2,0\rangle + 0\sqrt{1-c}|-{}^3/_2,0\rangle) \tag{103}$$
$$= \frac{18}{4}(\sqrt{c}|-{}^1/_2,-1\rangle + \sqrt{1-c}|-{}^3/_2,0\rangle) - 3(\sqrt{c}|-{}^1/_2,-1\rangle + \sqrt{1-c}|-{}^3/_2,0\rangle) \tag{104}$$
$$+ \sqrt{6}\sqrt{1-c}|-{}^1/_2,-1\rangle) + \sqrt{6}\sqrt{c}|-{}^3/_2,0\rangle \tag{105}$$

	J / M_J	5/2 / +5/2	5/2 / +3/2	3/2 / +3/2	5/2 / +1/2	3/2 / +1/2	1/2 / +1/2	5/2 / −1/2	3/2 / −1/2	1/2 / −1/2	5/2 / −3/2	3/2 / −3/2	5/2 / −5/2
m_1	m_2	1											
+3/2	+1	1											
+3/2	0		$\sqrt{2/5}$										
+1/2	+1		$\sqrt{3/5}$										
+3/2	−1				$\sqrt{1/10}$								
+1/2	0				$\sqrt{3/5}$								
−1/2	+1				$\sqrt{3/10}$								
+1/2	−1							$\sqrt{3/10}$					
−1/2	0							$\sqrt{3/5}$					
−3/2	+1							$\sqrt{1/10}$					
−1/2	−1										$\sqrt{3}/\sqrt{5}$		
−3/2	0										$\sqrt{2}/\sqrt{5}$		
−3/2	−1												1

0.0.4 SU(2) and up to SO(3) and SL(2,C)

1. 3D rotations
   (a) of vectors via 3D representation of SO(3)

$$[\mathrm{SO}(3)]\vec{r} = [\mathrm{SO}(3)]\begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{106}$$

   (b) Alternatively use a 2D representation of SU(2) and a **Pauli vector**

$$[\mathrm{SU}(2)]\begin{pmatrix} z & x-iy \\ x+iy & z \end{pmatrix}[\mathrm{SU}(2)]^\dagger \tag{107}$$

   (c) Or just via a **Pauli spinor**

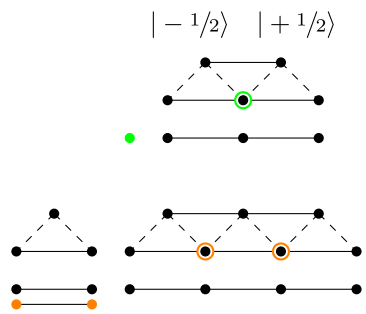
$$[\mathrm{SU}(2)]\begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} \tag{108}$$

   (d) Representations:
      i. SU(2) - one irrep per dimension (or per j) - starting from $j = 1/2$ - for given j define ladder operators, calculate all $|m_j\rangle$ and matrix elements $\langle m_1|J_{\pm,z}|m_2\rangle$, then use the matrices $J_{\pm,z}$ got get back $J_{1,2,3}$
      ii. Building Tensor products of irreps

$\boxed{\text{spin: } {}^1\!/\!_2 \otimes {}^1\!/\!_2}$

dim: $\mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{3}$

$\boxed{\text{spin: } ({}^1\!/\!_2 \otimes {}^1\!/\!_2) \otimes {}^1\!/\!_2}$



dim: $(\mathbf{2} \otimes \mathbf{2}) \otimes \mathbf{2} = (\mathbf{1} \oplus \mathbf{3}) \otimes \mathbf{2} = (\mathbf{1} \otimes \mathbf{2}) \oplus (\mathbf{3} \otimes \mathbf{2}) = (\mathbf{2}) \oplus (\mathbf{2} \oplus \mathbf{4}) = \mathbf{2} \oplus \mathbf{2} \oplus \mathbf{4}$

Figure 1: 2d ($j = {}^1\!/\!_2$) irrep represented by line with two nodes at the end - tensor product: stack another line with nodes symmetrically below each existing node

      iii. By building multiple tensor products of the spin ${}^1\!/\!_2$ representation and splitting them into irreps we can generate all other irreps (see table)
      iv. SO(3) - one irrep per odd dimension
2. Lorentz trafos of 4-vectors

Representations	**1**	**2**	**3**	**4**	**5**	**6**	**7**	...	Result
$\mathbf{2}$	-	1	-	-	-	-	-		$\mathbf{2}$
$\mathbf{2}\otimes\mathbf{2}$	1	-	1	-	-	-	-		$\mathbf{1}\oplus\mathbf{3}$
$\mathbf{2}\otimes\mathbf{2}\otimes\mathbf{2}$	-	2	-	1	-	-	-		$\mathbf{2}\oplus\mathbf{2}\oplus\mathbf{4}$
$\mathbf{2}\otimes\mathbf{2}\otimes\mathbf{2}\otimes\mathbf{2}$	2	-	3	-	1	-	-		$\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{3}\oplus\mathbf{3}\oplus\mathbf{3}\oplus\mathbf{5}$

Table 1: Splitting the tensor products of the spin ½ representation into irreps - all other irres can be generates

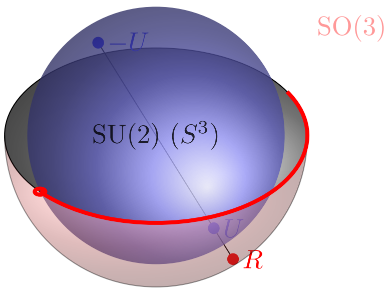


Figure 2: SU(2)$\sim S^3$ = double cover of SO(3): U and $-U$ ...

(a) via 4D representation of SO$^+$(1,3)

$$[\mathrm{SO}^+(1,3)]\begin{pmatrix}ct\\x\\y\\z\end{pmatrix} \tag{109}$$

(b) Alternatively use a 2D representation of SL(2,$\mathbb{C}$) and a **Weyl vector**

$$[\mathrm{SL}(2,\mathbf{C})]\begin{pmatrix}ct+z & x-iy\\ x+iy & ct+z\end{pmatrix}[\mathrm{SL}(2,\mathbf{C})]^\dagger \tag{110}$$

(c) Or just via a **Weyl spinor**

$$[\mathrm{SL}(2,\mathbb{C})]\begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix} \tag{111}$$

(d) Representations:
   i. SL(2,$\mathbb{C}$) - two 2D irreps (½,0) and (0,½)
   ii. SO(3) - one irrep per odd dimension

0.0.5 Overview

$\mathbb{F}$	GL($n,\mathbb{F}$)	SL($n,\mathbb{F}$)	U(n)	SU(n)	O(n)	SO(n)	SO$^+$(1,3)
$\mathbb{R}$	n^2	n^2-1	-	-	$n(n-1)/2$	$n(n-1)/2$	?
$\mathbb{C}$	$2n^2$	$2(n^2-1)$	n^2	n^2-1	$n(n-1)$	$n(n-1)$	?

Table 2: Dimensions of common Lie groups (number of independent real parameters)

Observation: dim(SO($n,\mathbb{F}$)) = dim(O($n,\mathbb{F}$)) - sign that SO(n) is not connected

Group	matrix	condition1	condition2
SU(2)≃Spin(3) double cover of SO(3)	$U \in \mathbb{C}^{2\times 2}$	$U^\dagger U = 1$	$\det U = +1$
SO(3)≃ $\mathbb{R}P^3$	$R \in \mathbb{R}^{3\times 3}$	$R^T R = 1$	$\det R = +1$
SL(2,ℂ)≃Spin(1,3) double cover of SO$^+$(1,3)	$L \in {}^{2\times 2}$	-	$\det L = +1$
SO$^+$(1,3)≃ PLS(2,ℂ)	$\Lambda \in \mathbb{R}^{4\times 4}$	$\Lambda^T \eta \Lambda = \eta$	det = +1 $\Lambda^0_0 > 0$

Group	matrix	condition1	condition2	
$\mathfrak{su}(2)$	$g \in \mathbb{C}^{2\times 2}$	$g^\dagger = -g$	$\operatorname{tr} g = 0$	g_1, g_2, g_3 $[g_i, g_j] = i\epsilon_{ijk} g_k$
$\mathfrak{so}(3)$	$g \in \mathbb{R}^{3\times 3}$	$g^T = -g$	$\operatorname{tr} g = 0$	g_1, g_2, g_3 $[g_i, g_j] = i\epsilon_{ijk} g_k$
$\mathfrak{sl}(2,\mathbb{C})$	$M \in \mathbb{C}^{2\times 2}$	-	$\mathrm{tr} M = 0$	$M^{01}, M^{02}, M^{03}, M^{12}, M^{23}, M^{31}$ with $M_{\mu\nu} = -M_{\nu\mu}$ $[M^{\mu\nu}, M^{\rho\sigma}] = i(\eta^{\nu\rho} M^{\mu\sigma} - \eta^{\mu\rho} M^{\nu\sigma} - \eta^{\nu\sigma} M^{\mu\rho} + \eta^{\mu\sigma} M^{\nu\rho})$
$\mathfrak{so}(1,3)$	$M \in \mathbb{R}^{4\times 4}$	$\eta M \eta = -M$	?	$J^1, J^2, J^3, K^1, K^2, K^3$ with $K^i = M^{i0}, J^i = \frac{1}{2}\epsilon^{ijk} M^{jk}$ $[J^i, J^j] = i\epsilon^{ijk} J^k$ $[K^i, K^j] = -i\epsilon^{ijk} J^k$ $[J^i, K^j] = i\epsilon^{ijk} K^k$

Dimension	1	2	3	4	5
Spin	0	½	1	3/2	2
$\mathfrak{su}(2)$ irreps.	1	1	1	1	1
$\mathfrak{so}(2)$ irreps.	1	0	1	0	1

0.0.6 Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{112}$$

Properties

- Determinate $\det \sigma_i = 1$
- $\sigma_1 \sigma_2 \sigma_3 = i\sigma_0$
- Traceless $\operatorname{tr} \sigma_i = 0$
- Hermitian $\sigma_i^\dagger = \sigma_i$
- Square to identity $(\sigma_i)^2 = \sigma_0 = 1_{2\times 2}$
- Commutator $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$
- Anti-commute $\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij} 1_{2\times 2}$
- Wipf-relation $\sigma_i \sigma_j = \delta_{ij}\sigma_0 + i\epsilon_{ijk}\sigma_k$

0.0.7 Grassmann (exterior) algebras

Definition

- L is an n-dimensional vector space over field $\mathbb{K}$
- For $p = 0, 1, 2, ...$ we call $\bigwedge^p L$ the p-vector space on L

- $\bigwedge^0 L = \mathbb{K}$
- $\bigwedge^1 L = L$
- $\bigwedge^2 L = \{\sum_i \alpha_i(u_i \wedge v_i)\}\ \forall u_i, v_i \in L$ with
  - $(\alpha_1 u_1 + \alpha_2 u_2) \wedge v = \alpha_1 u_1 \wedge v + \alpha_2 u_2 \wedge v$
  - $u \wedge (\beta_1 v_1 + \beta_2 v_2) = \beta_1 u \wedge v_1 + \beta_2 u \wedge v_2$
  - $u \wedge v = -v \wedge u$
- For $(2 \leq p \leq n)$ we define $\bigwedge^p L = \{\sum \alpha(u_1 \wedge u_2 \wedge ... \wedge u_p)\}\ \forall u_i \in L$ with
  - $u_1 \wedge ...u_k..u_l... \wedge u_p = -u_1 \wedge ...u_l..u_k... \wedge u_p$

– calculation example

$$\mathbf{u} \wedge \mathbf{v} = \sum_{i,j} (u^i \mathbf{e}_i) \wedge (v^j \mathbf{e}_j) \tag{113}$$

$$= \sum_{i,j} (u^i v^j)(\mathbf{e}_i \wedge \mathbf{e}_j) \tag{114}$$

$$= \sum_{i<j} (u^i v^j - u^j v^i)(\mathbf{e}_i \wedge \mathbf{e}_j) \tag{115}$$

– $\dim \bigwedge^p L = \binom{n}{p}$

– Exterior product (for obvious reasons we use again the $\wedge$): $\forall u \in \bigwedge^p L, v \in \bigwedge^q L$ then $\wedge : u, v \to u \wedge v \in \bigwedge^{p+q} L$

- $(u_1 \wedge ... \wedge u_p) \wedge (v_1 \wedge ... \wedge v_q) = u_1 \wedge ... \wedge u_p \wedge v_1 \wedge ... \wedge v_q$
- if $p + q > n$ we obtain 0

– Grassmann algebra is the pair $(\bigwedge(L), \wedge)$ of the vector space $\bigwedge(L) = \bigoplus_{k=0}^{\infty} \bigwedge^k L$ and the exterior (wedge) product

0.0.8 Tensor algebra

1. $T(V)$ algebra of contravariant tensors over vector space V (with basis $\{\mathbf{e}_1, ..., \mathbf{e}_n\}$) contains
   - $T_0 V = \mathbb{R}$: 1 scalar
   - $T_1(V) = V$: n vectors $\{\mathbf{e}_1, ..., \mathbf{e}_n\}$
   - $T_2(V) = V \otimes V$: n^2 2-tensors $\{\mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2}\}$
   - $T_k(V) = V \otimes ... \otimes V$: n^k k-tensors $\{\mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \otimes ... \otimes \mathbf{e}_{i_k}\}$

   then $T(V) = T_0(V) \oplus T_1(V) \oplus T_2(V) \oplus T_3(V) \oplus ... = \bigoplus_{k=0}^{\infty} T_k(V)$
2. Algebra of covariant tensors over vector space V
   - $T^*(V) = T^0(V) \oplus T^1(V) \oplus ... = \mathbb{R} \oplus V^* \oplus ... = \bigoplus_{k=0}^{\infty} T^k(V)$
3. If we require $\mathbf{e}_k \otimes \mathbf{e}_k = 0$ (which implies $\mathbf{e}_i \otimes \mathbf{e}_j = -\mathbf{e}_j \otimes \mathbf{e}_i$) then $T(V)$ is identical with the Grassmann algebra (the Grassmann algebra $\bigwedge(V)$ is the quotient algebra of the tensor algebra $T(V)$ by the two-sided ideal I generated by all elements $v \otimes v \in V$)

$$\bigwedge(V) = T(V)/I \tag{116}$$

0.0.9 Clifford algebras

Clifford algebras over $\mathbb{R}$

1. Definition Clifford algebra

- V vector space over $\mathbb{R}$ with symmetric bilinear form $g = g(\alpha\mathbf{u}, \beta\mathbf{v}) = \alpha\beta\, g(\mathbf{u}, \mathbf{v}) \to \mathbb{R}$
- $\mathcal{A}$ associative algebra with unity $1_{\mathcal{A}}$
  * meaning $\mathcal{A}$ is a vector space itself
    · $\forall a, b, c \in \mathcal{A} :\ (a + b) + c = a + (b + c)$
    · $\exists 0_{\mathcal{A}} :\ a + 0_{\mathcal{A}} = a$
    · $\forall a \in \mathcal{A} : \exists\, (-a) \in A$
    · $a + b = b + a \in A$
    · $\alpha(a + b) = \alpha a + \alpha b$
    · $(\alpha + \beta)a = \alpha a + \beta a$
    · $(\alpha \cdot \beta)a = \alpha(\beta a)$
    · $1_{\mathbf{R}} a = a$
  * there exists a associative bilinear map $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$
    · $a(bc) = (ac)b$
    · $1_{\mathcal{A}} a = a = a 1_{\mathcal{A}}$
- Linear mapping $\gamma : V \to \mathcal{A}$ with

$$\gamma(\mathbf{u})\gamma(\mathbf{v}) + \gamma(\mathbf{v})\gamma(\mathbf{u}) = 2g(\mathbf{u}, \mathbf{v})1_{\mathcal{A}} \tag{117}$$

$$\to \gamma(\mathbf{u})^2 = g(\mathbf{u}, \mathbf{u})1_{\mathcal{A}} \tag{118}$$

- Then $(\mathcal{A}, \gamma)$ is a Clifford algebra for (V, g) when $\mathcal{A}$ is generated by $\{\gamma(\mathbf{v}) | \mathbf{v} \in V\}$ and $\{s1_{\mathcal{A}} | s \in \mathbb{R}\}$

2. Simplification
   - V has an orthogonal basis $\{\mathbf{e}_1, ..., \mathbf{e}_n\}$ meaning
     * $g(\mathbf{e}_i, \mathbf{e}_j) = 0\ \forall i \neq j$
     * $g(\mathbf{e}_i, \mathbf{e}_i) = \pm 1\ \forall i$
   - Linear mapping can be simplified to
     * $\gamma(\sum_i \alpha^i \mathbf{e}_i) = \sum_i \alpha^i \gamma(\mathbf{e}_i)$
     * $\gamma(\mathbf{e}_i)^2 = \pm 1_{\mathcal{A}}$
   - $\mathcal{A}$ is generated by

$$s\,1_{\mathcal{A}} + \alpha^1\gamma(\mathbf{e}_1) + ... + \alpha^n\gamma(\mathbf{e}_n) + \tag{119}$$

$$+ \beta^{12}\gamma(\mathbf{e}_1)\gamma(\mathbf{e}_2) + ... + \beta^{n-1,n}\gamma(\mathbf{e}_{n-1})\gamma(\mathbf{e}_n) + \quad + \delta^{123}\gamma(\mathbf{e}_1)\gamma(\mathbf{e}_2)\gamma(\mathbf{e}_3) + ... \tag{120}$$

3. Two fundamental properties
   (a) square of any object is $\pm 1_{\mathcal{A}}$
   (b) objects anti-commute
4. Naming convention

$$\mathrm{Cl}(n, m) = \begin{cases} \text{Number of objects that square to -1} \\ \text{Number of objects that square to +1} \end{cases} \tag{121}$$

5. Examples
   - $\mathrm{Cl}(0, 1) \simeq \mathbb{C}$ (complex numbers)
   - $\mathrm{Cl}(0, 2) \simeq \mathbb{H}$ (quaternions, $k = ij$)

- $\mathrm{Cl}(0,3) \simeq$ APS (Algebra of physical space: $\{1_{APS}, \sigma_1, \sigma_2, \sigma_3, \sigma_1\sigma_2, \sigma_2\sigma_3, \sigma_3\sigma_1, \sigma_1\sigma_2\sigma_3\}$)

$$s\,1_{APS} + \alpha^1\gamma(\mathbf{e}_1) + \alpha^2\gamma(\mathbf{e}_2) + \alpha^3\gamma(\mathbf{e}_3) + \beta^{12}\gamma(\mathbf{e}_1)\gamma(\mathbf{e}_2) + \beta^{23}\gamma(\mathbf{e}_2)\gamma(\mathbf{e}_3) + \beta^{31}\gamma(\mathbf{e}_3)\gamma(\mathbf{e}_1) + p\gamma(\mathbf{e}_1)\gamma(\mathbf{e} \tag{122}$$

$$= s\,1_{APS} + \alpha^1\sigma_1 + \alpha^2\sigma_2 + \alpha^3\sigma_3 + \beta^{12}\sigma_1\sigma_2 + \beta^{23}\sigma_2\sigma_3, +\beta^{31}\sigma_3\sigma_1 + p\sigma_1\sigma_2\sigma_3 \tag{123}$$

$$= (s + ip)\,1_{APS} + (\alpha^1 + i\beta^{23})\sigma_1 + (\alpha^2 + i\beta^{31})\sigma_2 + (\alpha^3 + i\beta^{23})\sigma_3 + \in \mathrm{Cl(0,3)} \tag{124}$$

- $\mathrm{Cl}(3,1) \simeq$ STA (spacetime algebra: $(\gamma^0)^2 = 1, (\gamma^k)^2 = -1$)
- $\mathrm{Cl}(0,1)$ then $uv = u \cdot v + u \wedge v$

Clifford algebras over $\mathbb{C}$

...

Projectors of Clifford algebras

- Projectors - general definition and properties:
  * $P^2 = P$ (second projection does NOT change result)
  * If P is a projector so is $1 - P$
  * Orthogonal projectors if $P_iP_j = 0$
  * Then P and $1 - P$ are orthogonal
  * If P_i and P_j are orthogonal projectors then $P_i + P_j$ is also a projector
  * A projector is called minimal if it can not be written as a sum of two others (kind of wrong - if P_1, P_2 are minimal and orthogonal then $P_1 + P_2 = Q$ is also a projector but $P1 = Q + (-P_2)$!?!?)
- Take $U \in \mathrm{Cl}(0,3)$ with $UU = U^2 = \|U\|^2 = 1$ then

$$P_{U+} = \frac{1}{2}(1 + U), \qquad P_{U-} = \frac{1}{2}(1 - U) \tag{125}$$

  are orthogonal projectors in Cl(0,3)

- Example $U = \sigma_3$ because $\sigma_3^2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)^2 = 1$

$$P_{z+} = \frac{1}{2}(1 + \sigma_3) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{126}$$

$$P_{z-} = \frac{1}{2}(1 - \sigma_3) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{127}$$

  then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} P_{z+} = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \tag{128}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} P_{z-} = \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} \tag{129}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (P_{z+} + P_{z-}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{130}$$

Ideals of Clifford algebras

Ideal - general definition and properties

- subalgebra of Clifford algebra
  - Sum of elements of ideal remains in ideal
  - Product of element of ideal with any element of the algebra end up within ideal
- Example Cl(0,3): a left ideal is for example $\left(\begin{smallmatrix}\alpha & 0\\ \beta & 0\end{smallmatrix}\right)$ because

$$\begin{pmatrix}\alpha_1 & 0\\ \beta_1 & 0\end{pmatrix} + \begin{pmatrix}\alpha_2 & 0\\ \beta_2 & 0\end{pmatrix} = \begin{pmatrix}\alpha_1+\alpha_2 & 0\\ \beta_1+\beta_2 & 0\end{pmatrix} \tag{131}$$

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\alpha & 0\\ \beta & 0\end{pmatrix} = \begin{pmatrix}a\alpha+b\beta & 0\\ c\alpha+d\beta & 0\end{pmatrix} \tag{132}$$

- FACT: Projector of a Clifford algebra action on every element of the algebra generates and ideal - as seen above

$$\mathrm{Cl}(1,3)P_{z+} \to \begin{pmatrix}a & 0\\ c & 0\end{pmatrix} \tag{133}$$

$$\mathrm{Cl}(1,3)P_{z-} \to \begin{pmatrix}0 & b\\ 0 & d\end{pmatrix} \tag{134}$$

- Minimal ideals are generated by minimal projectors - which is equivalent to say that the ideal doe not contain any smaller subideal (except the trivial ideal $\{0\}$)

Spinors are members of minimal left ideals in Clifford algebras

Pauli spinor can be promoted to be a member of the Clifford algebra Cl(3,0) by

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} \to \begin{pmatrix}\alpha & 0\\ \beta & 0\end{pmatrix} \qquad \text{(minimal left ideal)} \tag{135}$$

0.0.10 Preliminary observations

Pauli vector

Definition: complex 2×2 matrix associated with $\mathbf{x}\in\mathbb{R}$

$$x^k\sigma_k = x\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + y\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + z\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{136}$$

$$= \begin{pmatrix}z & x-iy\\ x+iy & -z\end{pmatrix} \tag{137}$$

Properties

- Projections $(x^k\sigma_k)\sigma_j + \sigma_j(x^k\sigma_k) = 2x_j 1_{2\times 2}$
- Traceless $\mathrm{tr}\,(x^k\sigma_k) = 0$
- Hermitian $(x^k\sigma_k)^\dagger = x^k\sigma_k$
- Determinant $\det(x^k\sigma_k) = -\|\mathbf{x}\|^2$
- $(x^k\sigma_k)^2 = \|\mathbf{x}\|^2 1_{2\times 2}$

Observe 3d rotation R of 3-vector $\mathbf{x}$ can be written as unitary trafo of Pauli vector with SU(2) matrix U or matrix $-U$ (meaning SU(2) = double cover of rotation group SO(3))

$$R(\theta)x = x' \tag{138}$$

$$U(\theta)(x\cdot\sigma)U(\theta)^\dagger = x'\cdot\sigma \tag{139}$$

$$(-U(\theta))\,(x\cdot\sigma)\left(-U(\theta)^\dagger\right) = x'\cdot\sigma \tag{140}$$

where $-U(\theta) = U(\theta+2\pi)$. Explicitly

$$\underbrace{\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}}_{[\mathrm{SO(3)}]\mathbf{x}} = \begin{pmatrix}x\cos\theta - y\sin\theta\\ x\sin\theta + y\cos\theta\\ z\end{pmatrix} \tag{141}$$

$$\underbrace{\begin{pmatrix}e^{-i\theta/2} & 0\\ 0 & e^{+i\theta/2}\end{pmatrix}\begin{pmatrix}z & x-iy\\ x+iy & -z\end{pmatrix}\begin{pmatrix}e^{-i\theta/2} & 0\\ 0 & e^{+i\theta/2}\end{pmatrix}^\dagger}_{[\mathrm{SU(2)}](\text{Pauli vector})[\mathrm{SU(2)}]^\dagger} = \begin{pmatrix}z & e^{-i\theta}(x-iy)\\ e^{i\theta}(x+iy) & -z\end{pmatrix} \tag{142}$$

Pauli spinor

- Element of a complex vector space $\begin{bmatrix}\xi^1\\ \xi^2\end{bmatrix}$
- obtained from $\begin{bmatrix}z & x-iy\\ x+iy & -z\end{bmatrix} = \xi\otimes\xi^{\mathrm{dual}} = \begin{bmatrix}\xi^1\\ \xi^2\end{bmatrix}[-\xi^2\ \xi^1] = \begin{bmatrix}-\xi^1\xi^2 & (\xi_1)^2\\ -(\xi^2)^2 & \xi^1\xi^2\end{bmatrix}$
- with $\xi^1 = \sqrt{x-iy}$ and $\xi^2 = \sqrt{-x-iy}$
- This requires $x^2+y^2+z^2 = 0$ (weird) - so $x, y, z \in \mathbb{C}$!!!
- No unique - only ξ^2/ξ^1 is
- Rotation by θ

$$\underbrace{[\mathrm{SU(2)}]\begin{bmatrix}\xi^1\\ \xi^2\end{bmatrix}}_{\theta/2-\text{rotation}}\ \underbrace{[-\xi^2\ \xi^1][\mathrm{SU(2)}]^\dagger}_{\theta/2-\text{rotation}} \tag{143}$$

- (Dual) spinor basis: $\mathbf{s}_1 = \begin{bmatrix}1\\ 0\end{bmatrix}$, $\quad\mathbf{s}_2 = \begin{bmatrix}0\\ 1\end{bmatrix}$, $\quad\mathfrak{s}^1 = \begin{bmatrix}1 & 0\end{bmatrix}$, $\quad\mathfrak{s}^2 = \begin{bmatrix}0 & 1\end{bmatrix}$
- For Pauli spinor $\xi = \begin{bmatrix}\xi^1\\ \xi^2\end{bmatrix} = \xi^1\mathbf{s}_1 + \xi^2\mathbf{s}_2 = \xi^a\mathbf{s}_a$ the the associated dual spinor is

$$\xi^{\mathrm{dual}} = \xi^\dagger \tag{144}$$

$$= [\xi^{1*}\ \xi^{2*}] \tag{145}$$

$$= \xi^{1*}\mathfrak{s}^1 + \xi^{2*}\mathfrak{s}^2 \tag{146}$$

$$\equiv (\xi^{\mathrm{dual}})_1\mathfrak{s}^1 + (\xi^{\mathrm{dual}})_2\mathfrak{s}^2 \tag{147}$$

- a, b spinor indices (row and column index of Pauli matrix) and k vector index ${\sigma_k}^a{}_b$
- Then $\xi = \begin{bmatrix}\xi^1\\ \xi^2\end{bmatrix} = \xi^1\mathbf{s}_1 + \xi^2\mathbf{s}_2 = \xi^a\mathbf{s}_a$

$$\mathbf{s}_1\otimes\mathfrak{s}^2 = \begin{bmatrix}1\\ 0\end{bmatrix}\otimes[0\ 1] = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} \tag{148}$$

$$\sigma_k = {\sigma_k}^a{}_b\mathbf{s}_a\otimes\mathfrak{s}^b \tag{149}$$

$$x^k\sigma_k = x^k{\sigma_k}^a{}_b\mathbf{s}_a\otimes\mathfrak{s}^b = \begin{bmatrix}x^k{\sigma_k}^1{}_1 & x^k{\sigma_k}^1{}_2\\ x^k{\sigma_k}^2{}_1 & x^k{\sigma_k}^2{}_2\end{bmatrix} \tag{150}$$

- Inner product

$$\xi^{\text{dual}}\chi \equiv \xi^\dagger\chi \tag{151}$$
$$= \begin{bmatrix}\xi^{1*} & \xi^{2*}\end{bmatrix}\begin{bmatrix}\chi^1\\ \chi^2\end{bmatrix} \tag{152}$$
$$= \begin{bmatrix}\xi^{1*} & \xi^{2*}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\chi^1\\ \chi^2\end{bmatrix} \tag{153}$$
$$= (\xi^1)^*\chi^1 + (\xi^2)^*\chi^2 \tag{154}$$

  this way $([\text{SU}(2)]\xi)^\dagger[\text{SU}(2)]\chi) = (\xi^\dagger[\text{SU}(2)]^\dagger[\text{SU}(2)]\chi) = \xi^\dagger\chi$
- Pauli spinors have no chirality
- Pauli vector $X^a{}_b = x^k\sigma_k{}^a{}_b$ has only two spinor indices (one vector index k is replaces by two spinor indices a, b) - so a spinor is kind of a tensor of rank 1/2 ...
- as a vector is related to a Pauli vector which is a product of two spinors - we could say that a spinor is the square root of a vector...

Weyl vector

Definition: complex 2×2 matrix associated with $\mathbf{x} \in \mathbb{R}$

$$x^\mu\sigma_\mu = ct\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + x\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + y\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + z\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{155}$$
$$= \begin{pmatrix}ct+z & x-iy\\ x+iy & ct-z\end{pmatrix} \tag{156}$$

Properties

-
- Hermitian $(x^\mu\sigma_\mu)^\dagger = x^\mu\sigma_\mu$
- Determinant $\det(x^\mu\sigma_\mu) = (ct)^2 - \|\mathbf{x}\|^2$
- ...

Observe boost Λ of 4-vector x can be written as trafo of Weyl vector with SL(2,$\mathbb{C}$) matrix L or matrix $-L$ (meaning SL(2,$\mathbb{C}$) = double cover of Lorentz group SO$^+$(1,3))

$$\underbrace{\begin{pmatrix}\cosh\phi & -\sinh\phi & 0 & 0\\ \sinh\phi & \cosh\phi & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix}ct\\ x\\ y\\ z\end{pmatrix}}_{[\text{SO}^+(1,3)]\mathbf{x}} = \ldots \tag{157}$$

$$\underbrace{\begin{pmatrix}e^{-i\phi/2} & 0\\ 0 & e^{+i\phi/2}\end{pmatrix}\begin{pmatrix}ct+z & x-iy\\ x+iy & ct-z\end{pmatrix}\begin{pmatrix}e^{-i\phi/2} & 0\\ 0 & e^{+i\phi/2}\end{pmatrix}^\dagger}_{[\text{SL}(2,\mathbb{C})](\text{Weyl vector})[\text{SL}(2,\mathbb{C})]^\dagger} = \ldots \tag{158}$$

Weyl spinor

- Element of a complex vector space $\begin{bmatrix}\psi^1\\ \psi^2\end{bmatrix}$
- obtained from $\begin{bmatrix}ct+z & x-iy\\ x+iy & ct-z\end{bmatrix} = \psi_{\text{Left}}\otimes(\psi_{\text{Right}})^{\text{dual}} = \begin{bmatrix}\psi^1\\ \psi^2\end{bmatrix}\begin{bmatrix}(\psi^1)^* & (\psi^2)^*\end{bmatrix} = \begin{bmatrix}|\psi^1|^2 & \psi^1(\psi^2)^*\\ \psi^2(\psi^1)^* & |\psi^2|^2\end{bmatrix}$

– With $\psi_1 = e^{i\theta_1}\sqrt{ct+z}$ and $\psi_2 = e^{i\theta_1+\arctan(y/x)}\sqrt{ct-z}$

– For Weyl a left handed spinor $\psi = \begin{bmatrix}\phi^1\\ \phi^2\end{bmatrix}$ the associated dual spinor is defined via symplectic form $\epsilon = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$

$$\psi^{\text{dual}} \equiv \psi^T\epsilon \tag{159}$$
$$= \begin{bmatrix}\psi^1 & \psi^2\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} \tag{160}$$
$$= \begin{bmatrix}-\psi^2 & \psi^1\end{bmatrix} \tag{161}$$

– Inner product (for left handed spinor) defined via

$$\psi^{\text{dual}}\phi \equiv \psi^T\epsilon\phi \tag{162}$$
$$= \begin{bmatrix}\psi^1 & \psi^2\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}\phi^1\\ \phi^2\end{bmatrix} \tag{163}$$
$$= \begin{bmatrix}-\psi^2 & \psi^1\end{bmatrix}\begin{bmatrix}\phi^1\\ \phi^2\end{bmatrix} \tag{164}$$
$$= \psi^1\phi^2 - \psi^2\psi^1 \tag{165}$$

therefore $\psi^T\epsilon\phi = -\phi\psi^T\epsilon$ and $\phi^T\epsilon\phi = 0$

– Left handed (dual) spinor basis: $\mathbf{s}_1 = \begin{bmatrix}1\\ 0\end{bmatrix}, \quad \mathbf{s}_2 = \begin{bmatrix}0\\ 1\end{bmatrix}, \quad \mathfrak{s}^1 = \begin{bmatrix}1 & 0\end{bmatrix}, \quad \mathfrak{s}^2 = \begin{bmatrix}0 & 1\end{bmatrix}$

– Right handed (dual) spinor basis: $\mathbf{s}_{\dot{1}} = \begin{bmatrix}1 & 0\end{bmatrix}, \quad \mathbf{s}_{\dot{2}} = \begin{bmatrix}0 & 1\end{bmatrix}, \quad \mathfrak{s}^{\dot{1}} = \begin{bmatrix}1\\ 0\end{bmatrix}, \quad \mathfrak{s}^{\dot{2}} = \begin{bmatrix}0\\ 1\end{bmatrix}$

– Weyl spinors can have left or right chirality and transform using the left or right handed SU(2,ℂ) representation

$$\psi_{\text{Left}} \to [\text{SU}(2,\mathbb{C})_{\text{Left}}]\psi_{\text{Left}} \tag{166}$$
$$\psi_{\text{Right}} \to [\text{SU}(2,\mathbb{C})_{\text{Right}}]\psi_{\text{Right}} \tag{167}$$

– Van der Warden notation left spinor ψ^a, left dual spinor ψ_a, right dual spinor $\psi^{\dot{a}}$ and right spinor $\psi_{\dot{a}}$

$$\psi_{\text{Left}} = \begin{bmatrix}\psi^1\\ \psi^2\end{bmatrix} \tag{168}$$

$$= \psi^a ... \tag{169}$$

$$\to [\text{SL}(2,\mathbb{C})_{\text{Left}}]\psi^a ... \tag{170}$$

$$(\psi_{\text{Left}})^{\text{dual}} \equiv (\psi_{\text{Left}})^T \epsilon = \begin{bmatrix}\psi^1\\ \psi^2\end{bmatrix}^T \begin{bmatrix}0 & +1\\ -1 & 0\end{bmatrix} = \begin{bmatrix}-\psi^2 & \psi^1\end{bmatrix} \tag{171}$$

$$= [\psi_1 \; \psi_2] \tag{172}$$

$$= \psi_a ... \tag{173}$$

$$\to (\psi_{\text{Left}})^T \epsilon [\text{SU}(2,\mathbb{C})_{\text{Left}}]^{-1} \tag{174}$$

$$(\psi_{\text{Right}})^{\text{dual}} \equiv (\psi_{\text{Left}})^* = \begin{bmatrix}(\psi^1)^*\\ (\psi^2)^*\end{bmatrix} \tag{175}$$

$$= \begin{bmatrix}\psi^{\dot{1}}\\ \psi^{\dot{2}}\end{bmatrix} \tag{176}$$

$$= \psi^{\dot{a}} ... \tag{177}$$

$$\to [\text{SL}(2\mathbb{C})_{\text{Left}}]^* (\psi_{\text{Left}})^* \tag{178}$$

$$\psi_{\text{Right}} \equiv (\psi_{\text{Left}})^\dagger \epsilon = \begin{bmatrix}\psi^1\\ \psi^2\end{bmatrix}^\dagger \begin{bmatrix}0 & +1\\ -1 & 0\end{bmatrix} = \begin{bmatrix}-(\psi^2)^* & (\psi^1)^*\end{bmatrix} \tag{179}$$

$$= [\psi_{\dot{1}} \; \psi_{\dot{2}}] \tag{180}$$

$$= \psi_{\dot{a}} ... \tag{181}$$

$$\to (\psi_{\text{Left}})^T \epsilon ([\text{SU}(2,\mathbb{C})_{\text{Left}}]^{-1})^* \tag{182}$$

$$\begin{array}{ccc} \psi_{\text{Left}} & \xrightarrow{T\epsilon} & (\psi_{\text{Left}})^{\text{dual}} \\ \downarrow{*} & & \downarrow{*} \\ \psi_{\text{Right}} & \xrightarrow{T\epsilon^{-1}} & (\psi_{\text{Right}})^{\text{dual}} \end{array}$$

–

Dirac spinor

Left Weyl spinor | Right Weyl spinor | Dirac Spinor

$$\begin{bmatrix}\psi^1\\ \psi^2\end{bmatrix} \quad \oplus \quad \begin{bmatrix}\phi_{\dot{1}} & \phi_{\dot{2}}\end{bmatrix}^T \quad = \quad \begin{bmatrix}\psi^1\\ \psi^2\\ \phi_{\dot{1}}\\ \phi_{\dot{2}}\end{bmatrix}$$

$$\to \quad \begin{bmatrix} [\text{SU}(2,\mathbf{C})] & \begin{matrix}0 & 0\\ 0 & 0\end{matrix} \\ \begin{matrix}0 & 0\\ 0 & 0\end{matrix} & [\text{SU}(2,\mathbf{C})]^{-1\dagger} \end{bmatrix} \begin{bmatrix}\psi^1\\ \psi^2\\ \phi_{\dot{1}}\\ \phi_{\dot{2}}\end{bmatrix}$$

0.0.11 SO(2)

Group of rotations in two dimensions - therefore rotations are naturally given by a 2×2 matrix R with parameter α (and the generator X)

$$R = \begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix} \qquad -iX = \frac{\partial R}{\partial \alpha}\bigg|_{\alpha=0} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \tag{183}$$

acting on vectors (x, y). This is therefore also a 2-dimensional (real) representation of SO(2) - it is even an irrep. In a complex space the vector can be written as $z = x + iy$ and the rotation is represented by $e^{i\alpha}$ - which serves as a one dimensional complex representation.

There are actually infinitely many (non-equivalent) 1-dimensional standard irreps

$$D^k(\alpha) = e^{-ik\alpha},\ k = 0, \pm 1, \pm 2, ... \tag{184}$$

0.0.12 SO(3) - What we know from quantum mechanics

The angular momentum algebra is given by $[J_i, J_j] = i\hbar\varepsilon_{ijk}J_k$. We know that

$$J^2|jm\rangle = j(j+1)|jm\rangle \qquad j \in \left\{0, \frac{1}{2}, 1, \frac{3}{2}, ...\right\}, \quad m = -j, ..., j \tag{185}$$

$$J_z|jm\rangle = m|jm\rangle \tag{186}$$

meaning that J^2 and J_3 can be diagonalized at the same time. For each j there is a $2j+1$ dimensional irrep on the Hilbert space. The subspace spanned by the states $\{|jm\rangle\}_{m\in\{-j,...,j\}}$ is called $\mathfrak{h}_j$. The states of two added angular momenta j_1 and j_2 are in the space $\mathfrak{h}_{j_1j_2} = \mathfrak{h}_{j_1} \otimes \mathfrak{h}_{j_2}$ spanned by the tensor product of the eigenstates of $(J^2_{j_1}, J_{j_1,3})$ and $(J^2_{j_2}, J_{j_2,3})$

$$|j_1m_1j_2m_2\rangle = |j_1, m_1\rangle \otimes |j_2, m_2\rangle \tag{187}$$

The operators J^2, J_3, $J^2_{j_1}$ and $J^2_{j_2}$ commute which means they share one set of eigenfunctions $|j_1, j_2, j, m\rangle$ which also spans $\mathfrak{h}_{j_1,j_2}$. Both basis set are connected by the Clebsch-Gordon coefficients

$$|j_1, j_2, j, m\rangle = \sum_{m_1,m_2} \langle j_1, m_1, j_2, m_2|j_1, j_2, j, m\rangle|j_1m_1, j_2, m_2\rangle \tag{188}$$

The dimension of the Product space is given by

$$\dim(\mathfrak{h}_{j_1} \otimes \mathfrak{h}_{j_2}) = (2j_1 + 1)(2j_2 + 1). \tag{189}$$

The tensor product representations decomposes as (Clebsch-Gordan decomposition)

$$\mathfrak{h}_{j_1} \otimes \mathfrak{h}_{j_1} \cong \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} \mathfrak{h}_j \tag{190}$$

$$= \mathfrak{h}_{j_1+j_2} \oplus \mathfrak{h}_{j_1+j_2-1} \oplus ... \oplus \mathfrak{h}_{j_1-j_2+1} \oplus \mathfrak{h}_{|j_1-j_2|} \tag{191}$$

Examples

$$j_1 = \frac{1}{2}, j_2 = \frac{1}{2} \quad \rightarrow \quad 2 \otimes 2 = 1 \oplus 3 \tag{192}$$

$$j_1 = 1, j_2 = 1 \quad \rightarrow \quad 3 \otimes 3 = 1 \oplus 3 \oplus 5 \tag{193}$$

0.0.13 SO(3)

Definition: Group of linear transformations that does NOT change length of vectors

$$v^2 = \vec{v}^T\vec{v} \tag{194}$$

$$= (R\vec{v})^T(R\vec{v}) \tag{195}$$

$$= \vec{v}^T R^T R\vec{v} \tag{196}$$

$$\rightarrow R^T R = I \tag{197}$$

therefore rotations around the 3 coordinate axis are naturally given by three 3×3 matrices R_i (with the generators X_i).

SO(3)	$R \in \mathbb{R}^{3\times3}$	$R^{-1} = R^T$	$\det R = +1$
$\mathfrak{so}(3)$	$g \in \mathbb{R}^{3\times3}$	$g^T = -g$	$\mathrm{tr}\, g = 0$

Spin 1 (3 dimensional) representation

Called spin 1 representation because the transformation matrices acting on vectors

$$R_3 = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} = e^{-iX_3\alpha} \qquad \leftrightarrow \qquad g_3 = -iX_3 = \frac{\partial R}{\partial \alpha}\bigg|_{\alpha=0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{198}$$

$$R_2 = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix} = e^{-iX_2\alpha} \qquad \leftrightarrow \qquad g_2 = -iX_2 = \frac{\partial R}{\partial \alpha}\bigg|_{\alpha=0} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \tag{199}$$

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} = e^{-iX_1\alpha} \qquad \leftrightarrow \qquad g_1 = -iX_1 = \frac{\partial R}{\partial \alpha}\bigg|_{\alpha=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \tag{200}$$

which also a 3-dimensional representation of SO(3). The generators obey the commutation relation

$$[X_i, X_i] = i\varepsilon_{ijk}X_k \tag{201}$$

They form the $\mathfrak{so}(3)$ (Lie algebra of SO(3)).

Spin 0 (1 dimensional) representation

The X_i are now 1×1 matrices (numbers) and as the Lie algebra must be independent of representation the only choice is

$$X_3 = 0 \qquad \to \qquad R_3 = 1 \tag{202}$$
$$X_2 = 0 \qquad \to \qquad R_2 = 1 \tag{203}$$
$$X_1 = 0 \qquad \to \qquad R_1 = 1 \tag{204}$$

2 dimensional representation

There is NO 2 dimensional representation

Spin 1/2 representation

Lie algebra

$$\tilde{g}_1 = -\frac{1}{2}\sigma_2\sigma_3 = -\frac{i}{2}\sigma_1 = -\frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{205}$$

$$\tilde{g}_2 = -\frac{1}{2}\sigma_3\sigma_1 = -\frac{i}{2}\sigma_2 = -\frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{206}$$

$$\tilde{g}_3 = -\frac{1}{2}\sigma_1\sigma_2 = -\frac{i}{2}\sigma_3 = -\frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{207}$$

with

$$[\tilde{g}_i, \tilde{g}_k] = \epsilon_{ijk}\tilde{g}_k \tag{208}$$

0.0.14 SU(2)

Definition: Unitary transformation like a complex rotation - so the condition is

$$U^\dagger U = I \qquad \text{or} \qquad U^\dagger = U^{-1}$$

SU(2)	$U \in \mathbb{C}^{2\times 2}$	$U^{-1} = U^\dagger$	$\det U = +1$
$\mathfrak{su}(2)$	$M \in \mathbb{C}^{2\times 2}$	$M = -M^\dagger$	$\operatorname{tr} M = 0$

Spin 1/2 representation

Construction of a generic SU(2) matrix ($a, b, c, d \in \mathbb{C}$)

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad ad - bc = 1$$
$$\to U^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$
$$\to U^\dagger = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}$$

then with $U^\dagger = U^{-1}$ we have three conditions

$$\bar{a} = d \tag{209}$$
$$\bar{b} = -c \tag{210}$$
$$1 = ad - bc = a\bar{a} + b\bar{b} \tag{211}$$

And therefore $a, b \in \mathbb{C}$ and $a\bar{a} + b\bar{b} = a_1^2 + a_2^2 + b_1^2 + b_2^2 = 1$ (so SU(2) is shaped like the 3-sphere S_3)

$$\begin{aligned}
U &= \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} = \begin{pmatrix} a_1 + ia_2 & b_1 + ib_2 \\ -b_1 + ib_2 & a_1 - ia_2 \end{pmatrix} \\
&= a_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_2 i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b_1 i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + b_2 i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= a_1 I + a_2 i\sigma_3 + b_1 i\sigma_2 + b_2 i\sigma_1 \\
&= \sqrt{1 - a_2^2 - b_1^2 - b_2^2 I} + a_2 i\sigma_3 + b_1 i\sigma_2 + b_2 i\sigma_1
\end{aligned}$$

Finding the generators

$$M_3 = \left.\frac{\partial U}{\partial a_2}\right|_{\ldots=0} = \left.\frac{-2a_2}{2\sqrt{1 - a_2^2 - b_1^2 - b_2^2}} I + i\sigma_3\right|_{\ldots=0} = i\sigma_3 = \sigma_1\sigma_2$$
$$M_2 = \left.\frac{\partial U}{\partial b_1}\right|_{\ldots=0} = \left.\frac{-2b_1}{2\sqrt{1 - a_2^2 - b_1^2 - b_2^2}} I + i\sigma_2\right|_{\ldots=0} = i\sigma_2 = \sigma_3\sigma_1$$
$$M_1 = \left.\frac{\partial U}{\partial b_1}\right|_{\ldots=0} = \left.\frac{-2b_2}{2\sqrt{1 - a_2^2 - b_1^2 - b_2^2}} I + i\sigma_1\right|_{\ldots=0} = i\sigma_1 = \sigma_2\sigma_3$$

We observe that the 3 generators are identical with the 3 bivectors of the Clifford algebra Cl(3). The general form of the generators is $a, b, c \in \mathbb{R}$

$$M = \begin{pmatrix} ic & b + ia \\ -b + ia & -ic \end{pmatrix} \tag{212}$$

The actual (rescaled) generators are then

$$\tilde{g}_1 = -\frac{1}{2}\sigma_2\sigma_3 = -\frac{i}{2}\sigma_1 = -\frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{213}$$

$$\tilde{g}_2 = -\frac{1}{2}\sigma_3\sigma_1 = -\frac{i}{2}\sigma_2 = -\frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{214}$$

$$\tilde{g}_3 = -\frac{1}{2}\sigma_1\sigma_2 = -\frac{i}{2}\sigma_3 = -\frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{215}$$

$$\rightarrow [\tilde{g}_i, \tilde{g}_k] = i\epsilon_{ijk}\tilde{g}_k \tag{216}$$

We observe that the 3 generators are also related to the unit quaternions

Calculating the $SU(2)$ elements from the generators

$$U_3(\theta) = e^{\theta g_3} = e^{-\frac{\theta}{2}\sigma_1\sigma_2} = \ldots = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{+i\theta/2} \end{pmatrix} \tag{217}$$

$$U_2(\theta) = e^{\theta g_2} = e^{-\frac{\theta}{2}\sigma_3\sigma_1} = \ldots = \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix} \tag{218}$$

$$U_1(\theta) = e^{\theta g_1} = e^{-\frac{\theta}{2}\sigma_2\sigma_3} = \ldots = \begin{pmatrix} \cos\theta/2 & -i\sin\theta/2 \\ -i\sin\theta/2 & \cos\theta/2 \end{pmatrix} \tag{219}$$

We observe that $U_k(2\pi) = -1$ and not 1, and $U_k(4\pi) = +1$ (because of double cover)

0.0.15 SO$^+$(1,3) - Lorentz group

Definition: trafo that leaves ds^2 invariant

SO$^+$(1,3)	$\Lambda \in \mathbb{R}^{4\times4}$	$\Lambda^T\eta\Lambda = \eta$	$\det = +1, \Lambda^0_0 > 0$
$\mathfrak{so}(1,3)$	$g \in \mathbb{R}^{4\times4}$	$M = -\eta M\eta$	?

Spin 1 (4 dim representation) representation

Called spin 1 representation because the transformation matrices acting on (four) vectors

Boosts

$$\Lambda_{01}(\phi) = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \leftrightarrow \quad M_{01} = \frac{\partial\Lambda_{01}}{\partial_\phi}|_{\phi=0} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{220}$$

$$\Lambda_{02}(\phi) = \begin{pmatrix} \cosh\phi & 0 & -\sinh\phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sinh\phi & 0 & \cosh\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \leftrightarrow \quad M_{02} = \frac{\partial\Lambda_{02}}{\partial_\phi}|_{\phi=0} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{221}$$

$$\Lambda_{03}(\phi) = \begin{pmatrix} \cosh\phi & 0 & 0 & -\sinh\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\phi & 0 & 0 & \cosh\phi \end{pmatrix} \quad \leftrightarrow \quad M_{03} = \frac{\partial\Lambda_{02}}{\partial_\phi}|_{\phi=0} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{222}$$

Rotations

$$\Lambda_{12}(\theta)=\begin{pmatrix}1&0&0&0\\0&\cos\theta&-\sin\theta&0\\0&\sin\theta&\cos\theta&0\\0&0&0&1\end{pmatrix}\quad\leftrightarrow\quad M_{12}=\frac{\partial\Lambda_{01}}{\partial\theta}|_{\theta=0}=\begin{pmatrix}0&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&0\end{pmatrix}\tag{223}$$

$$\Lambda_{31}(\theta)=\begin{pmatrix}1&0&0&0\\0&\cos\theta&0&\sin\theta\\0&0&1&0\\0&-\sin\theta&0&\cos\theta\end{pmatrix}\quad\leftrightarrow\quad M_{31}=\frac{\partial\Lambda_{31}}{\partial\theta}|_{\theta=0}=\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&-1&0&0\end{pmatrix}\tag{224}$$

$$\Lambda_{23}(\theta)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\cos\theta&-\sin\theta\\0&0&\sin\theta&\cos\theta\end{pmatrix}\quad\leftrightarrow\quad M_{23}=\frac{\partial\Lambda_{23}}{\partial\theta}|_{\theta=0}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}\tag{225}$$

Generic generator - 3 symmetric boosts and 3 antisymmetric rotations

$$M=\begin{pmatrix}0&\eta_1&\eta_2&\eta_3\\\eta_1&0&-\theta_3&\theta_2\\\eta_2&\theta_3&0&-\theta_1\\\eta_3&-\theta_2&\theta_1&0\end{pmatrix}\tag{226}$$

(½,0)⊗(0,½)=(½,½) representation

From Weyl spinor transformation

$$[L]\begin{bmatrix}\psi^1\\\psi^2\end{bmatrix}\otimes[(\psi^1)^*\ (\psi^2)^*][L^\dagger]\simeq[\mathrm{SL}(2,C)_{\mathrm{Left}}]\begin{bmatrix}ct+z&x-iy\\c+iy&ct-z\end{bmatrix}[\mathrm{SL}(2,C)^\dagger_{\mathrm{Right}})^{-1}]^{-1}\tag{227}$$

to 4-vector transformation

$$\begin{pmatrix}ct'\\x'\\y'\\z'\end{pmatrix}=\underbrace{\frac{1}{2}\begin{bmatrix}1&0&0&1\\0&1&1&0\\0&i&-i&0\\1&0&0&-1\end{bmatrix}\underbrace{\begin{bmatrix}\alpha&\beta\\\gamma&\delta\end{bmatrix}}_{\mathrm{SL}(2,\mathbb{C})_{\mathrm{Left}}}\otimes\underbrace{\begin{bmatrix}\alpha^*&\gamma^*\\\beta^*&\delta^*\end{bmatrix}}_{\mathrm{SL}(2,\mathbb{C})_{\mathrm{Right}}}\begin{bmatrix}1&0&0&1\\0&1&-i&0\\0&1&i&0\\1&0&0&-1\end{bmatrix}}_{\mathrm{SO}(1,3)}\begin{pmatrix}ct\\x\\y\\z\end{pmatrix}\tag{228}$$

Spin 1/2 representations

– Left-handed (0,1/2) representation

$$J_1=-\frac{1}{2}\sigma_2\sigma_3=\frac{1}{2}\begin{pmatrix}0&-i\\-i&0\end{pmatrix}\qquad K_1=-\frac{1}{2}\sigma_1=\frac{1}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\tag{229}$$

$$J_2=-\frac{1}{2}\sigma_3\sigma_1=\frac{1}{2}\begin{pmatrix}0&-1\\1&0\end{pmatrix}\qquad K_2=-\frac{1}{2}\sigma_2=\frac{1}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\tag{230}$$

$$J_3=-\frac{1}{2}\sigma_1\sigma_2=\frac{1}{2}\begin{pmatrix}-i&0\\0&i\end{pmatrix}\qquad K_3=-\frac{1}{2}\sigma_3=\frac{1}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\tag{231}$$

– Right handed (1/2, 0) representation (start with left and do a parity transform $\sigma_i\to-\sigma_i$

then the boost change sign but the rotation remain unchanged

$$J_1 = -\frac{1}{2}\sigma_2\sigma_3 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \qquad K_1 = +\frac{1}{2}\sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{232}$$

$$J_2 = -\frac{1}{2}\sigma_3\sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad K_2 = +\frac{1}{2}\sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{233}$$

$$J_3 = -\frac{1}{2}\sigma_1\sigma_2 = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \qquad K_3 = +\frac{1}{2}\sigma_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{234}$$

– with Lie algebra

$$[J_i, J_j] = \epsilon_{ijk} J_k \tag{235}$$

$$[K_i, K_j] = -\epsilon_{ijk} J_k \tag{236}$$

$$[J_i, K_j] = \epsilon_{ijk} K_k \tag{237}$$

– Group elements

  * Rotations: $L \sim e^{\theta J_i}$ are unitary $L^\dagger = L^{-1}$

$$\Psi_L \to L\Psi_L \tag{238}$$

$$\Psi_R \to (L^\dagger)^{-1} = L\Psi_L \tag{239}$$

  * Boosts: $L \sim e^{\eta K_i}$ are hermitian $L^\dagger = L$

$$\Psi_L \to L\Psi_L \tag{240}$$

$$\Psi_R \to (L^\dagger)^{-1} = L\Psi_L^{-1} \tag{241}$$

0.0.16 SL(2,$\mathbb{C}$)

– Generic group element $a, b, c, d \in \mathbf{C}$

$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \text{with } ad - bc = 1 \tag{242}$$

SL(2,$\mathbb{C}$)	$L \in {}^{2\times 2}$	-	det$L = +1$
$\mathfrak{sl}(2, \mathbf{C})$	$M \in \mathbb{C}^{2\times 2}$	-	tr$M = 0$

– Generic **generator** $\alpha, \beta, \gamma \in \mathbf{C}$

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \tag{243}$$

$$= \begin{pmatrix} \alpha_1 + i\alpha_2 & \beta_1 + i\beta_2 \\ \gamma_1 + i\gamma_2 & -\alpha_1 - i\alpha_2 \end{pmatrix} \tag{244}$$

$$= \alpha_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - ... \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + ... \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + ... \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} + ... \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + ... \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \tag{245}$$

$$= \underbrace{...\sigma_1 + ...\sigma_2 + ...\sigma_3}_{\substack{\text{3 hermitean } M^\dagger = M \\ \text{Cl(3) vectors generating boosts}}} + \underbrace{...\sigma_1\sigma_2 + ...\sigma_3\sigma_1 + ...\sigma_2\sigma_3}_{\substack{\text{3 anti-hermitean } M^\dagger = -M \\ \text{Cl(3) bivectors generating rotations}}} \tag{246}$$

– So boost generators are $(\sigma_1, \sigma_2, \sigma_3)$ and rotation generators are $(\sigma_2\sigma_3, \sigma_3\sigma_1, \sigma_1\sigma_2) \equiv -i(\sigma_1, \sigma_2, \sigma_3)$

- therefore

$$\mathfrak{sl}(2,\mathbf{C}) = \mathfrak{su}(2) \otimes (\pm i)\mathfrak{su}(2) \tag{247}$$
$$\equiv \mathfrak{su}(2)_{\mathbb{C}} \tag{248}$$
$$\rightarrow \mathfrak{sl}(2,\mathbf{C})_{\mathbf{C}} = \mathfrak{su}(2)_{\mathbb{C}} \otimes \mathfrak{su}(2)_{\mathbb{C}} \tag{249}$$

- This implies the following possibility

$$A_j = \frac{1}{2}(J_j + iK_j) \qquad [A_i, A_j] = \epsilon_{ijk}A_k, \qquad \mathfrak{su}(2)_{\mathbb{C}} \tag{250}$$
$$B_j = \frac{1}{2}(J_j - iK_j) \qquad [B_i, B_j] = \epsilon_{ijk}B_k, \qquad \mathfrak{su}(2)_{\mathbb{C}} \tag{251}$$
$$\rightarrow [A_i, B_j] = 0 \tag{252}$$

Spin ½ representation

Lie algebra (there are two representations)

$$M_{12} = -\frac{1}{2}\sigma_1\sigma_2, M_{23} = -\frac{1}{2}\sigma_2\sigma_3, M_{31} = -\frac{1}{2}\sigma_3\sigma_1 \tag{253}$$
$$M_{01} = -\frac{1}{2}\sigma_1, M_{02} = -\frac{1}{2}\sigma_2, M_{02} = -\frac{1}{2}\sigma_3 \tag{254}$$

and

$$M_{12} = -\frac{1}{2}\sigma_1\sigma_2, M_{23} = -\frac{1}{2}\sigma_2\sigma_3, M_{31} = -\frac{1}{2}\sigma_3\sigma_1 \tag{255}$$
$$M_{01} = +\frac{1}{2}\sigma_1, M_{02} = +\frac{1}{2}\sigma_2, M_{02} = +\frac{1}{2}\sigma_3 \tag{256}$$

Group elements (there are two representation)

1. Spin ½ left-chiral (½,0) representation

   Rotations (unitary)
   $L_{ij}^\dagger = L_{ij}^{-1}$

   $$L_{12} = e^{\theta M_{12}} = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{+i\theta/2} \end{pmatrix}$$
   $$L_{23} = e^{\theta M_{23}} = \begin{pmatrix} \cos\theta/2 & -i\sin\theta/2 \\ -i\sin\theta/2 & \cos\theta/2 \end{pmatrix}$$
   $$L_{31} = e^{\theta M_{31}} = \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix}$$

   Boosts (hermitian)
   $L_{0i}^\dagger = L_{0i}$

   $$L_{03} = e^{\theta M_{03}} = \begin{pmatrix} e^{-\theta/2} & 0 \\ 0 & e^{+\theta/2} \end{pmatrix}$$
   $$L_{01} = e^{\theta M_{01}} = \begin{pmatrix} \cosh\theta/2 & -\sinh\theta/2 \\ -\sinh\theta/2 & \cosh\theta/2 \end{pmatrix}$$
   $$L_{02} = e^{\theta M_{02}} = \begin{pmatrix} \cosh\theta/2 & i\sinh\theta/2 \\ -i\sinh\theta/2 & \cosh\theta/2 \end{pmatrix}$$

2. Spin ½ right-chiral (0,½) representation
   Observation:
   - parity transformation $\sigma_i \rightarrow -\sigma_i$
   - does NOT change the Lie algebra
   - it changes the elements ($M_{0i} \rightarrow -M_{0i}, M_{ij} \rightarrow M_{ij}$) but NOT the brackets
   - BUT this creates a second non-trivial representation of the algebra and the group
   - The left handed representation $(L_L)_{\mu\nu}$ can be transformed into the right handed one $(L_R)_{\mu\nu}$ by complex conjugating and inverting $L_R = (L_L)^\dagger)^{-1}$
     * Rotations: $L \sim e^{\theta J_i}$ are unitary $L^\dagger = L^{-1}$

$$\Psi_L \rightarrow L\Psi_L \tag{257}$$
$$\Psi_R \rightarrow (L^\dagger)^{-1} = L\Psi_L \tag{258}$$

* Boosts: $L \sim e^{\eta K_i}$ are hermitian $L^\dagger = L$

$$\Psi_L \to L\Psi_L \tag{259}$$
$$\Psi_R \to (L^\dagger)^{-1} = L\Psi_L^{-1} \tag{260}$$

Rotations (unitary)
$L_{ij}^\dagger = L_{ij}^{-1}$
$$L_{12} = e^{\theta M_{12}} = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{+i\theta/2} \end{pmatrix}$$
$$L_{23} = e^{\theta M_{23}} = \begin{pmatrix} \cos\theta/2 & -i\sin\theta/2 \\ -i\sin\theta/2 & \cos\theta/2 \end{pmatrix}$$
$$L_{31} = e^{\theta M_{31}} = \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix}$$

Boosts (hermitian)
$L_{0i}^\dagger = L_{0i}$
$$L_{03} = e^{-\theta M_{03}} = \begin{pmatrix} e^{+\theta/2} & 0 \\ 0 & e^{-\theta/2} \end{pmatrix}$$
$$L_{01} = e^{-\theta M_{01}} = \begin{pmatrix} \cosh\theta/2 & \sinh\theta/2 \\ \sinh\theta/2 & \cosh\theta/2 \end{pmatrix}$$
$$L_{02} = e^{-\theta M_{02}} = \begin{pmatrix} \cosh\theta/2 & -i\sinh\theta/2 \\ i\sinh\theta/2 & \cosh\theta/2 \end{pmatrix}$$

We see that we can get from the left to the right representation by

$$(L_{\mathrm{L},ij}^\dagger)^{-1} = L_{\mathrm{R},ij} \tag{261}$$

The representations are NOT equivalent because

$$(L^\dagger)^{-1} \neq C^{-1}LC \tag{262}$$

Spin (½,0)⊕(0,½) representation

Left Weyl spinor — Right Weyl spinor — Dirac Spinor

$$L\begin{bmatrix}\psi^1\\ \psi^2\end{bmatrix} \quad \oplus \quad (L^\dagger)^{-1}\begin{bmatrix}\phi_{\dot 1}\\ \phi_{\dot 2}\end{bmatrix} \quad = \quad \begin{bmatrix} L & 0 \\ 0 & (L^\dagger)^{-1}\end{bmatrix}\begin{bmatrix}\psi^1\\ \psi^2\\ \phi_{\dot 1}\\ \phi_{\dot 2}\end{bmatrix}$$

Projectors onto Weyl spinors

$$P_L = \frac{1-\gamma^5}{2} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}, \qquad P_R = \frac{1+\gamma^5}{2} = \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} \tag{263}$$

Spin (½,0)⊗(0,½) representation

Four-vectors

0.0.17 R^{1,3}×O(1,3) - Poincare group

0.0.18 Spin(N)

Spin group - definition

– ...

0.0.19 Algebra of physical space (Pauli algebra) Cl(0,3)

– Cl(0,3)[1234] is generated by $\{1_{APS}, \gamma(\mathbf{e}_1), \gamma(\mathbf{e}_2), \gamma(\mathbf{e}_3)\}$ (and products of them) with

$$\gamma(\mathbf{e}_i)\gamma(\mathbf{e}_j) + \gamma(\mathbf{e}_j)\gamma(\mathbf{e}_i) = 2\delta_{ij}1_{APS} \tag{264}$$

$$\rightarrow \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}1_{APS} \tag{265}$$

– This algebra can be represented by the Pauli-matrices via $\mathbf{e}_i \rightarrow \gamma(\mathbf{e}_i) = \sigma_i$ with

$$\sigma_1^2 = \sigma_2^2 = \sigma_2^2 = +1 \cdot 1_{APS} \qquad \begin{Bmatrix} 0 \times & -1 \\ 3 \times & +1 \end{Bmatrix} \rightarrow \mathrm{Cl}(0,3) \tag{266}$$

– Then an arbitrary element can be written as $(s, a^i, b^{ij}, p \in \mathbb{R})$

$$s1_{APS} + a^1\gamma(\mathbf{e}_1) + a^2\gamma(\mathbf{e}_2) + a^3\gamma(\mathbf{e}_3) + b^{12}\gamma(\mathbf{e}_1)\gamma(\mathbf{e}_2) + b^{23}\gamma(\mathbf{e}_2)\gamma(\mathbf{e}_3) + b^{31}\gamma(\mathbf{e}_3)\gamma(\mathbf{e}_1) + p\gamma(\mathbf{e}_1)\gamma(\mathbf{e}_2)\gamma(\mathbf{e}_2) \tag{267}$$

$$= s1_{APS} + a^1\sigma_1 + a^2\sigma_2 + a^3\sigma_3 + b^{12}\sigma_1\sigma_2 + b^{23}\sigma_2\sigma_3 + b^{31}\sigma_3\sigma_1 + p\sigma_1\sigma_2\sigma_3 \tag{268}$$

$$= \begin{pmatrix} (s+a^3) + i(p+b^{12}) & (a^1+b^{31}) + i(b^{23}-a^2) \\ (a^1-b^{31}) + i(b^{23}+a^2) & (s-a^3) + i(p-b^{12}) \end{pmatrix} \tag{269}$$

– This algebra can be represented by the Pauli-matrices

1 trivector (pseudoscalar)		$\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} = \sigma_1\sigma_2\sigma_3$	
3 bivectors (pseudovectors)	$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \sigma_2\sigma_3 = I\sigma_1 = I$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \sigma_3\sigma_1 = I\sigma_2$	$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \sigma_1\sigma_2 = I\sigma_3$
3 vectors	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1$	$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_2$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3$
1 scalar		$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \sigma_0$	

$$(\sigma_1\sigma_2\sigma_3)^2 = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3 = -\sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_3 = \sigma_1\sigma_2\sigma_1\sigma_2\sigma_3\sigma_3 = -\sigma_1\sigma_1\sigma_2\sigma_2\sigma_3\sigma_3 = -1 \tag{270}$$

– Connection to spinors (minimal left ideal)

$$|\xi\rangle = \begin{bmatrix} \xi^1 \\ \xi^2 \end{bmatrix} \simeq \begin{pmatrix} \xi^1 & 0 \\ \xi^2 & 0 \end{pmatrix} = \xi^1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \xi^2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{271}$$

$$= \xi^1 \frac{1}{2}(1+\sigma_3) + \xi^2 \frac{1}{2}(\sigma_1 - i\sigma_2) \tag{272}$$

$$= \xi^1 \frac{1}{2}(1+\sigma_3) + \xi^2 \sigma_1 \frac{1}{2}(1+\sigma_3) \tag{273}$$

$$= \xi^1|+z\rangle + \xi^2|-z\rangle \tag{274}$$

[1] da Rocha R and Vaz J, An introduction to Clifford algebras and spinors - Oxford University Press (2016)

[2] da Rocha R and Vaz J, Revisiting Clifford algebras and spinors I: the twistor group SU(2,2) in the Dirac algebra and some other remarks (math-ph/0412074)

[3] da Rocha R and Vaz J, Revisiting Clifford algebras and spinors II: Weyl spinors in Cl3,0 and Cl0,3 and the Dirac equation (math-ph/0412075).

[4] da Rocha R and Vaz J, Revisiting Clifford algebras and spinors III: conformal structures and twistors in the paravector model of spacetime (math-ph/0412076).

– Projectors

$$P_{z+} = \frac{1}{2}(1+\sigma_3) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} P_{z+} = \begin{pmatrix} \alpha & 0 \\ \gamma & 0 \end{pmatrix} \tag{275}$$

$$P_{z-} = \frac{1}{2}(1-\sigma_3) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} P_{z-} = \begin{pmatrix} 0 & \beta \\ 0 & \gamma \end{pmatrix} \tag{276}$$

$$\rightarrow P_{z+} + P_{z-} = 1 \tag{277}$$

so we see $|+z\rangle = P_{z+}$, $|-z\rangle = \sigma_1 P_{z+}$ and

$$1_{APS} P_{z+} = \sigma_3 P_{z+} = \frac{1}{2}(1+\sigma_3) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = P_{z+} \tag{278}$$

$$\sigma_1 P_{z+} = \sigma_1\sigma_3 P_{z+} = \frac{1}{2}(\sigma_1 + i\sigma_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \sigma_1 P_{z+} \tag{279}$$

$$\sigma_2 P_{z+} = \sigma_2\sigma_3 P_{z+} = i\frac{1}{2}(\sigma_1 + i\sigma_2) = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix} = i\sigma_1 P_{z+} \tag{280}$$

$$\sigma_1\sigma_2 P_{z+} = \sigma_1\sigma_2\sigma_3 P_{z+} = i\frac{1}{2}(1+\sigma_3) = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} = iP_{z+} \tag{281}$$

Generalize (reddefinition of states falling from the sky)

$$P_{x+} = \frac{1}{2}(1+\sigma_1), \qquad P_{x-} = \frac{1}{2}(1-\sigma_1) \tag{282}$$

$$P_{y+} = \frac{1}{2}(1+\sigma_2), \qquad P_{y-} = \frac{1}{2}(1-\sigma_2) \tag{283}$$

$$|+z\rangle = P_{z+} \tag{284}$$

$$|-z\rangle = \sigma_x P_{z+} \tag{285}$$

$$|+x\rangle = \sqrt{2} P_{x+} P_{z+} \tag{286}$$

$$|-x\rangle = \sqrt{2} P_{x-} P_{z+} \tag{287}$$

$$|+y\rangle = \sqrt{2} P_{y+} P_{z+} \tag{288}$$

$$|-y\rangle = \sqrt{2} P_{y-} P_{z+} \tag{289}$$

Spinor

$$\begin{pmatrix} a+id & 0 \\ c+ib & 0 \end{pmatrix} = (a+id)P_{z+} + (c+ib)\sigma_1 P_{z+} \tag{290}$$

$$\simeq (a+id)|+z\rangle + (c+ib)|-z\rangle \tag{291}$$

Physics $\hat{S}_z = \frac{\hbar}{2}\sigma_z$

$$\sigma_z|+z\rangle = \sigma_z P_{z+} = P_{z+} = +|+z\rangle \quad \rightarrow \quad \hat{S}_z|+z\rangle = +\frac{\hbar}{2}|+z\rangle \tag{292}$$

$$\sigma_z|-z\rangle = \sigma_z\sigma_x P_{z+} = -\sigma_x P_{z+} = -|-z\rangle \quad \rightarrow \quad \hat{S}_z|-z\rangle = -\frac{\hbar}{2}|-z\rangle \tag{293}$$

$$\sigma_x|+x\rangle = +|+x\rangle \quad \rightarrow \quad \hat{S}_x|-x\rangle = -\frac{\hbar}{2}|-x\rangle \tag{294}$$

$$\sigma_x|-x\rangle = -|-x\rangle \quad \rightarrow \quad \hat{S}_x|+x\rangle = +\frac{\hbar}{2}|+x\rangle \tag{295}$$

$$\sigma_x|+y\rangle = +|+y\rangle \quad \rightarrow \quad \hat{S}_y|-y\rangle = -\frac{\hbar}{2}|-y\rangle \tag{296}$$

$$\sigma_x|-y\rangle = -|-y\rangle \quad \rightarrow \quad \hat{S}_y|+y\rangle = +\frac{\hbar}{2}|+y\rangle \tag{297}$$

$$\tag{298}$$

Bra spinor - minimal left ideal

$$|\xi\rangle \to \begin{bmatrix}\xi^1\\ \xi^2\end{bmatrix} \simeq \begin{bmatrix}\xi^1 & 0\\ \xi^2 & 0\end{bmatrix} = \xi^1|+z\rangle + \xi^2|-z\rangle \tag{299}$$

Ket spinor - minimal right ideal

$$\langle\xi| \to \begin{bmatrix}\xi^1\\ \xi^2\end{bmatrix}^\dagger = [\xi^{1*}\ \xi^{2*}] \simeq \begin{bmatrix}\xi^1 & 0\\ \xi^2 & 0\end{bmatrix}^\dagger = \begin{bmatrix}\xi^{1*} & \xi^{2*}\\ 0 & 0\end{bmatrix} \tag{300}$$

$$\begin{aligned}
\langle\xi| &= \xi^{1*}\langle+z| + \xi^{2*}\langle-z| && (301)\\
\langle\xi| = |\xi\rangle^\dagger &= (\xi^1|+z\rangle + \xi^2|-z\rangle)^\dagger && (302)\\
&= \xi^{1*}|+z\rangle^\dagger + \xi^{2*}|-z\rangle^\dagger && (303)\\
& && (304)
\end{aligned}$$

– Rotation

$$\begin{bmatrix}\mathrm{SU(2)}\\ \theta/2\end{bmatrix}\begin{bmatrix}\xi^1\\ \xi^2\end{bmatrix} \simeq \begin{bmatrix}\mathrm{Spin(3)}\\ \theta/2\end{bmatrix}\begin{pmatrix}\xi^1 & 0\\ \xi^2 & 0\end{pmatrix} \tag{305}$$

0.0.20 Spacetime algebra Cl(3,1)

– Cl(1,3) is generated by $\{1_{APS}, \gamma(\mathbf{e}_0), \gamma(\mathbf{e}_1), \gamma(\mathbf{e}_2), \gamma(\mathbf{e}_3)\}$ and via $\mathbf{e}_i \to \gamma(\mathbf{e}_i) = \gamma^i$ meaning $\{1_{APS}, \gamma^0, \gamma^1, \gamma^2, \gamma^3\}$ (and products of them) with

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta_{\mu\nu}1_{APS} \tag{306}$$

– So the algebra can be represented by the Dirac-matrices with

$$(\gamma^0)^2 = 1_{APS}, \qquad (\gamma^k)^2 = -1_{APS} \qquad \begin{Bmatrix}3\times & -1\\ 1\times & +1\end{Bmatrix} \to \mathrm{Cl}(3,1) \tag{307}$$

– Then an arbitrary element can be written as $(s, a_i, b_{ij}, c_{ijk}, d_{0123} \in \mathbb{R})$

$$\begin{aligned}
&s1_{APS} + a_0\gamma^0 + a_1\gamma^1 + a_2\gamma^2 + a_3\gamma^3 + && (308)\\
&\quad + b_{01}\gamma^0\gamma^1 + b_{02}\gamma^0\gamma^2 + b_{03}\gamma^0\gamma^3 + b_{12}\gamma^1\gamma^2 + b_{13}\gamma^1\gamma^3 + b_{23}\gamma^2\gamma^3 && (309)\\
&\quad + c_{012}\gamma^0\gamma^1\gamma^2 + c_{013}\gamma^0\gamma^1\gamma^3 + c_{123}\gamma^1\gamma^2\gamma^3 + d_{0123}\gamma^0\gamma^1\gamma^2\gamma^3 && (310)
\end{aligned}$$

with $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

1 quadvector (pseudoscalar)				$\gamma^0\gamma^1\gamma^2\gamma^3 = I$			
4 trivector (pseudoscalar)		$\gamma^1\gamma^2\gamma^3 = I\gamma^0$	$\gamma^0\gamma^2\gamma^3 = I\gamma^1$		$\gamma^0\gamma^1\gamma^3 = I\gamma^2$	$\gamma^0\gamma^1\gamma^2 = I\gamma^3$	
6 bivectors	$\gamma^0\gamma^1$	$\gamma^0\gamma^2$	$\gamma^0\gamma^3$		$\gamma^1\gamma^2$	$\gamma^1\gamma^3$	$\gamma^2\gamma^3$
4 vectors		γ^0	γ^1		γ^2	γ^3	
1 scalar				1			

with $I^2 = -1$ but

$$\gamma^0 I = -I\gamma^0, \quad \gamma^1 I = -I\gamma^1, \quad \gamma^2 I = -I\gamma^2, \quad \gamma^3 I = -I\gamma^3, \tag{311}$$

Representations:

1. Weyl/Chiral rep
2. Dirac/Mass rep

0.0.21 Dirac-Clifford algebra $\mathbb{C} \otimes \mathbf{Cl}(3,1)$

0.0.22 SU(3)

0.0.23 Lorentz group O(1,3)

The generators are a generalisation of the 3d rotations

$$\begin{aligned}
J^{\mu\nu} &= i(x^\mu \partial^\nu - x^\nu \partial^\mu) && (312)\\
&= i(x^\mu g^{\alpha\nu}\partial_\alpha - x^\nu g^{\alpha\mu}\partial_\alpha) && (313)\\
J^{\mu\nu} x^\rho &= i(x^\mu g^{\alpha\nu}\delta^\rho_\alpha - x^\nu g^{\alpha\mu}\delta^\rho_\alpha) && (314)\\
&= i(\delta^\mu_\sigma x^\sigma g^{\alpha\nu}\delta^\rho_\alpha - \delta^\nu_\sigma x^\sigma g^{\alpha\mu}\delta^\rho_\alpha) && (315)\\
&= i(\delta^\mu_\sigma g^{\rho\nu} - \delta^\nu_\sigma g^{\rho\mu}) x^\sigma && (316)\\
&= (J^{\mu\nu})^\rho{}_\sigma x^\sigma && (317)
\end{aligned}$$

meaning there is a four dimensional representation of the Lorentz Lie algebra.

$$(J^{\mu\nu})^\rho{}_\sigma = i(\delta^\mu_\sigma g^{\rho\nu} - \delta^\nu_\sigma g^{\rho\mu}) \qquad (318)$$

$$(319)$$

Finite-dimensional Representations

- $\mathbb{R}$ 1-dim - trivial representation $J^{\mu\nu} = 0$
- $\mathbb{R}^4$ 4-dim - vector representation $(J^{\mu\nu})^\rho{}_\sigma = i(\delta^\mu_\sigma g^{\rho\nu} - \delta^\nu_\sigma g^{\rho\mu})$
- $\mathbb{R}^6$ 6-dim - adjoint representation $(J^a)^b{}_c = -if^{ab}{}_c$
- $\mathbb{C}^2$ 2-dim - left handed Weyl spinor rep. $J^{\mu\nu} = S^{\mu\nu}$ with $S^{ij} = \frac{1}{2}\epsilon^{ijk}\sigma^k$ and $S^{0i} = -\frac{i}{2}\sigma^i$
- $\mathbb{C}^2$ 2-dim - right handed Weyl spinor rep. $J^{\mu\nu} = S^{\mu\nu}$ with $S^{ij} = \frac{1}{2}\epsilon^{ijk}\sigma^k$ and $S^{0i} = \frac{i}{2}\sigma^i$
- $\mathbb{C}^4$ 4-dim - Dirac spinor rep. $J^{\mu\nu} = S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$

The group elements are $\Lambda = \exp(-i\omega_{\mu\nu} J^{\mu\nu}/2)$.

There are the obvious tensor representations for tensors of first and second order

$$\begin{aligned}
[D(\Lambda)]^\alpha_\beta &= \Lambda^\alpha_\beta \quad \to \quad V^\alpha = [D(\Lambda)]^\alpha_\beta V^\beta = \Lambda^\alpha_\beta V^\beta && (320)\\
[D(\Lambda)]_{\alpha\beta}^{\gamma\delta} &= \Lambda_\alpha^\gamma \Lambda_\beta^\delta \quad \to \quad T_{\alpha\beta} = [D(\Lambda)]_{\alpha\beta}^{\gamma\delta} T_{\gamma\delta} = \Lambda_\alpha^\gamma \Lambda_\beta^\delta T_{\gamma\delta} && (321)
\end{aligned}$$

which are 4 and 16 dimensional.

Infinitesimal Lorentz transformations can be written as

$$\Lambda^\alpha_\beta = \delta^\alpha_\beta + \omega^\alpha_\beta \qquad (|\omega^\alpha_\beta| \ll 1). \qquad (322)$$

The first order approximation gives an additional restriction

$$\begin{aligned}
\eta_{\mu\nu} &= \eta_{\alpha\beta}\Lambda^\alpha_\mu \Lambda^\beta_\nu = \eta_{\alpha\beta}(\delta^\alpha_\mu + \omega^\alpha_\mu)(\delta^\beta_\nu + \omega^\beta_\nu) = \eta_{\mu\nu} + \eta_{\mu\beta}\omega^\beta_\nu + \eta_{\alpha\nu}\omega^\alpha_\mu && (323)\\
&\to \omega_{\mu\nu} = -\omega_{\nu\mu} && (324)
\end{aligned}$$

which implies six independent components. As the four dimensional representation of the infinitesimal transformation is close to unity it can then be written as

$$D(\Lambda) = D(1+\omega) = 1 + \frac{1}{2}\omega^{\alpha\beta}\sigma_{\alpha_\beta} \qquad (325)$$

where the six ω components correspond to the six matrices $\sigma_{01}, \sigma_{02}, \sigma_{03}, \sigma_{12}, \sigma_{13}, \sigma_{23}$ which are the generators of the group.

Finite dimensional irreps of the Lorentz group are labeled by two parameters (μ, ν) with

$$\mu, \nu \in \left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2, ...\right\}. \tag{326}$$

and have dimension $(2\mu + 1)(2\nu + 1)$

$$\begin{aligned} M^2 &= \mu(\mu + 1) \\ N^2 &= \nu(\nu + 1) \\ j &\in |\mu - \nu|, ..., (\mu + \nu) \end{aligned}$$

SL(2,$\mathbb{C}$) irrep	dim	j	example
$(0,0) \equiv (0,0) \otimes (0,0)$	1	0	Scalar
$(\frac{1}{2},0) \equiv (\frac{1}{2},0) \otimes (0,0)$	2	$\frac{1}{2}$	Left-handed Weyl spinor
$(0,\frac{1}{2}) \equiv (0,0) \otimes (0,\frac{1}{2})$	2	$\frac{1}{2}$	Right-handed Weyl spinor
$(\frac{1}{2},\frac{1}{2}) \equiv (\frac{1}{2},0) \otimes (0,\frac{1}{2})$	4	0,1	4-Vector A^μ
$(1,0) \equiv (1,0) \otimes (0,0)$	3	1	Self-dual 2-form
$(0,1) \equiv (0,0) \otimes (0,1)$	3	1	Anti-self-dual 2-form
$(1,1) \equiv (1,0) \otimes (0,1)$	9	0,1,2	Traceless symmetric 2^{nd} rank tensor

rep	dim	j	example
$(\frac{1}{2},0) \oplus (0,\frac{1}{2})$	-	-	Dirac bispinor $\psi^\alpha \quad \alpha \in \{1,2,3,4\}$
$(\frac{1}{2},\frac{1}{2}) \otimes \left[(\frac{1}{2},0) \oplus (0,\frac{1}{2})\right] = (1,\frac{1}{2}) \otimes (\frac{1}{2},1)$	-	-	Rarita-Schwinger field $\psi^\alpha \quad \alpha \in \{1,2,3,4\}$
$(1,0) \oplus (0,1)$	-	-	Parity invariant field of 2-forms
$(\frac{3}{2},0) \oplus (0,\frac{3}{2})$	-	-	Gravitino