2 O(3) non-linear sigma model in two-dimension

This is a theory with action (Euclidean metric $\eta = \mathrm{diag}(+1,+1)$)

$$S = \frac{1}{2g^2}\int d^2x \sum_k \partial_\mu\phi_k\partial^\mu\phi_k \tag{57}$$

$$= \frac{1}{2g^2}\int d^2x \sum_k |\partial_\mu\phi_k|^2 \tag{58}$$

where the field had three components $\phi(x,y) = (\phi_1(x,y), \phi_2(x,y), \phi_3(x,y))$ and lives on a sphere with unit radius, i.e.

$$\phi_1^2 + \phi_2^2 + \phi_3^2 = 1 \tag{59}$$

2.1 Discretization of the field

Assumption: cubic equidistant lattice ($a = a_x = a_y$)

$$S = \frac{1}{2g^2}\int dx\, dy (\partial_x\phi_1)^2 + (\partial_x\phi_2)^2 + (\partial_x\phi_3)^2 + (\partial_y\phi_1)^2 + (\partial_y\phi_2)^2 + (\partial_y\phi_3)^2 \tag{60}$$

$$\simeq \frac{a^2}{2g^2}\sum_{ij}\sum_k \frac{(\phi^{(k)}_{i+1,j} - \phi^{(k)}_{i,j})^2}{a^2} + \frac{(\phi^{(k)}_{i,j+1} - \phi^{(k)}_{i,j})^2}{a^2} \tag{61}$$

$$\simeq \frac{a^2}{2g^2}\sum_{ij}\sum_k \frac{(\phi^{(k)}_{i+1,j})^2}{a^2} + \frac{(\phi^{(k)}_{i,j})^2}{a^2} + \frac{(\phi^{(k)}_{i,j+1})^2}{a^2} + \frac{(\phi^{(k)}_{i,j})^2}{a^2} - 2\frac{\phi^{(k)}_{i+1,j}\phi^{(k)}_{i,j}}{a^2} - 2\frac{\phi^{(k)}_{i,j+1}\phi^{(k)}_{i,j}}{a^2} \tag{62}$$

$$\simeq \frac{a^2}{2g^2}\sum_{ij}\frac{4}{a^2} - \frac{a^2}{g^2}\sum_k\sum_{ij}\frac{\phi^{(k)}_{i+1,j}\phi^{(k)}_{i,j}}{a^2} + \frac{\phi^{(k)}_{i,j+1}\phi^{(k)}_{i,j}}{a^2} \tag{63}$$

$$\simeq \frac{2N_s^2}{g^2} - \frac{1}{g^2}\sum_k\sum_{ij}\phi^{(k)}_{i+1,j}\phi^{(k)}_{i,j} + \phi^{(k)}_{i,j+1}\phi^{(k)}_{i,j} \tag{64}$$

we neglect the first term (constant offset) and use

$$S_{ij} = -\frac{1}{g^2}\sum_k \phi^{(k)}_{i+1,j}\phi^{(k)}_{i,j} + \phi^{(k)}_{i,j+1}\phi^{(k)}_{i,j} \tag{65}$$

$$S_{\mathrm{tot}} = \sum_{ij} S_{ij} \tag{66}$$

$$= -\frac{1}{g^2}\sum_k\sum_{ij}\phi^{(k)}_{i+1,j}\phi^{(k)}_{i,j} + \phi^{(k)}_{i,j+1}\phi^{(k)}_{i,j} \tag{67}$$

$$Z = \int d\phi_1 d\phi_2 d\phi_3 \exp\left[-\frac{1}{2g^2}\int dx\, dy \sum_{k,\mu}|\partial_\mu\phi_k|^2\right] \tag{68}$$

$$= \sum_{\mathrm{configs}} \exp\left[-S_{\mathrm{tot}}(\phi^{(k)}_{i,j})\right] \tag{69}$$

$$\langle\phi(x_1)\phi(x_2)\rangle = \frac{\int d\phi^{(1)}d\phi^{(2)}\phi^{(3)}\ \sum_{k'}\phi^{(k')}(x_1)\phi^{(k')}(x_2)\exp\left[-\frac{1}{2g^2}\int dx\, dy\sum_{k,\mu}|\partial_\mu\phi^{(k)}|^2\right]}{\int d\phi^{(1)}d\phi^{(2)}\phi^{(3)}\ \exp\left[-\frac{1}{2g^2}\int dx\, dy\sum_{k,\mu}|\partial_\mu\phi^{(k)}|^2\right]} \tag{70}$$

$$C_{r_j} = \frac{\sum_{\mathrm{configs}}\frac{1}{N_{r_j}}\left(\sum_k\phi^{(k)}\phi^{(k)}\right)\exp\left[-S_{\mathrm{tot}}(\phi^{(k)}_{i,j})\right]}{\sum_{\mathrm{configs}}\exp\left[-S_{\mathrm{tot}}(\phi^{(k)}_{i,j})\right]} \tag{71}$$

2.2 Random field configuration at lattice point

To generate a random field at a point of the lattice - obeying the $O(3)$ constraint we use polar coordinates. As the field vectors should be uniformly distributed in S^2 we use the uniformly independent distributed angle $\varphi \in [0, 2\pi)$ and $z = \cos\vartheta \in [-1, +1)$

$$\varphi = 2\pi\,\mathcal{U}^{(1)}_{[0,1]} \tag{72}$$

$$z = \cos\vartheta = 2\,\mathcal{U}^{(2)}_{[0,1]} - 1 \qquad \rightarrow \qquad \sin\vartheta = \sqrt{1-\cos^2\theta} \tag{73}$$

$$\rightarrow \phi^{(1)} = \sin\vartheta\,\cos\varphi \tag{74}$$

$$\rightarrow \phi^{(2)} = \sin\vartheta\,\sin\varphi \tag{75}$$

$$\rightarrow \phi^{(3)} = \cos\vartheta \tag{76}$$

2.3 Code

Algorithm 1 Metropolis Monte Carlo for O(3) nonlinear sigma model in 2D

Assumption - cubic equidistant lattice
Initialize a $N_x \times N_y$ lattice with random unit vectors $\vec{\phi}_{i,j} = \phi^{(k)}_{i,j} \in \mathbb{S}^2$
Initialize a $N_x \times N_y$ lattice with local action $S_{i,j}$
Calculate total action $S_{\text{tot}} = \sum_{ij} S_{ij}$
Initialize $Z = 0$
Initialize all 2-point observables $c_{r_k} = 0$ with $r_k \in \left\{\sqrt{k_x^2 + k_y^2},\ 0 \le k_x \le {}^{N_x}\!/_2, 0 \le k_y \le {}^{N_y}\!/_2\right\}$

Calculate count of distance occurrence N_{r_j}
for step = 1 to stepMax **do**
    Choose a random site on lattice (x, y)
    Backup current field: $\vec{\phi}_{\text{old}} = \vec{\phi}_{x,y}$ and total action $S_{\text{tot(old)}} = S_{\text{tot}}$
    Propose new field: $\phi^{(k)}_{x,y}$ = random unit vector on the sphere
    Update local action S_{ij} for the discretized action of this model we actually only need to update action at three points $S_{x,y}, S_{x-1,y}, S_{x,y-1}$
    recompute total action S_{tot}
    Compute energy difference: $\Delta S = S_{\text{tot}} - S_{\text{tot(old)}}$
    if $\Delta S \le 0$ **then**
        Accept: $\vec{\phi}_{x,y}$
    else
        Draw $r \sim \mathcal{U}_{[0,1]}$
        if $r < \exp(-\Delta S)$ **then**
            Accept: $\vec{\phi}_{x,y}$
        else
            Reject: $\vec{\phi}_{x,y} = \vec{\phi}_{\text{old}}$
        end if
    end if
    if step > stepMin **then**
        $Z = Z + \exp(-S_{\text{tot}})$
        for all possible lattice distances $r_j = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$ **do**
            $c_{r_j} = c_{r_j} + \frac{1}{N_{r_j}} \sum_{|p_1-p_2|=r_j} \left(\sum_k \phi^{(k)}_{j_{x1},j_{y1}} \phi^{(k)}_{j_{x2},j_{y2}}\right) \exp(-S_{\text{tot}})$
        end for
    end if
end for
Calculate $C_{r_j} = \frac{c_{r_j}}{Z}$

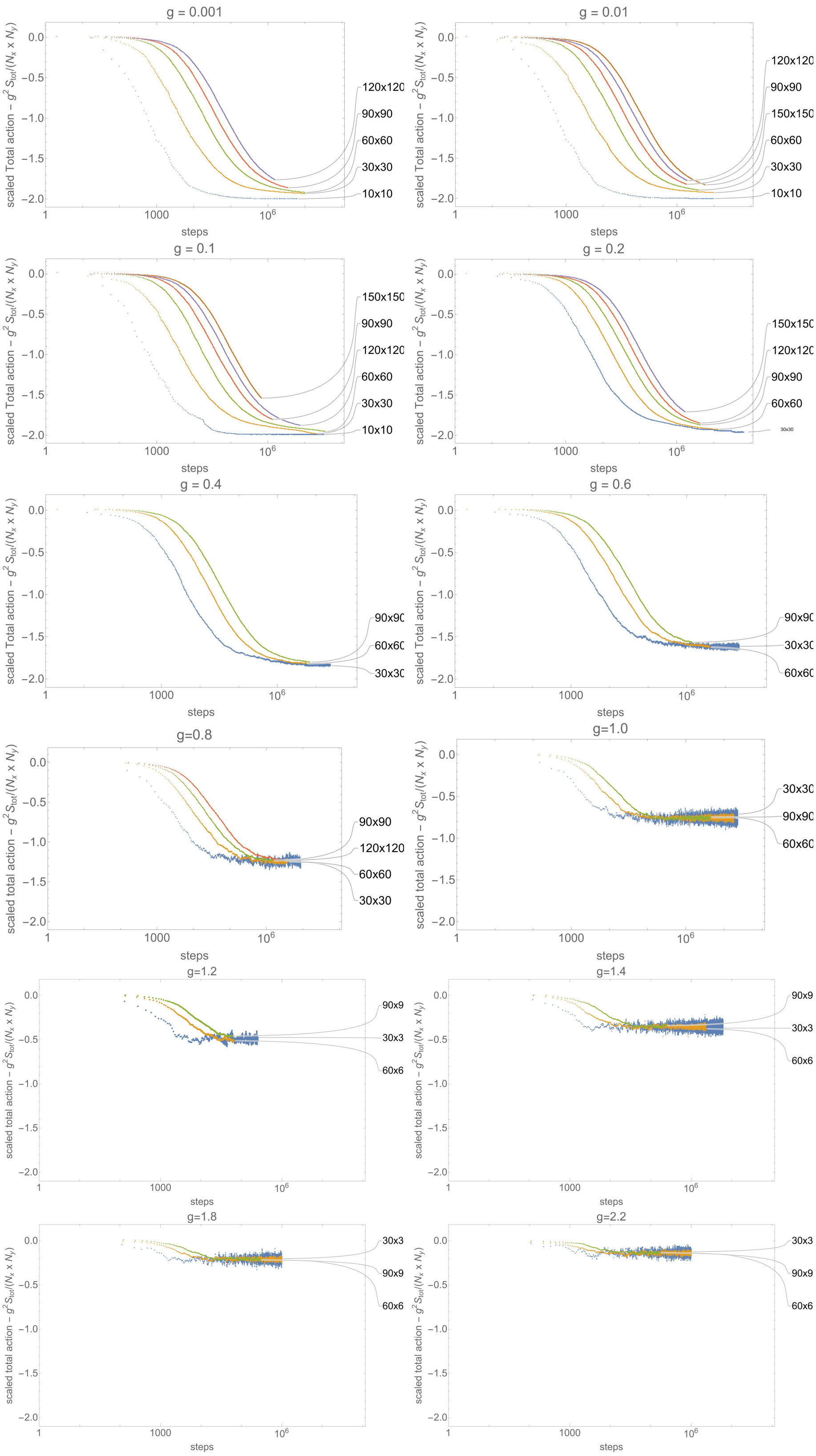

Figure 1: convergence of $g^2 S_{\mathrm{tot}}/(N_x \times N_y)$ for various g and various lattice sizes.

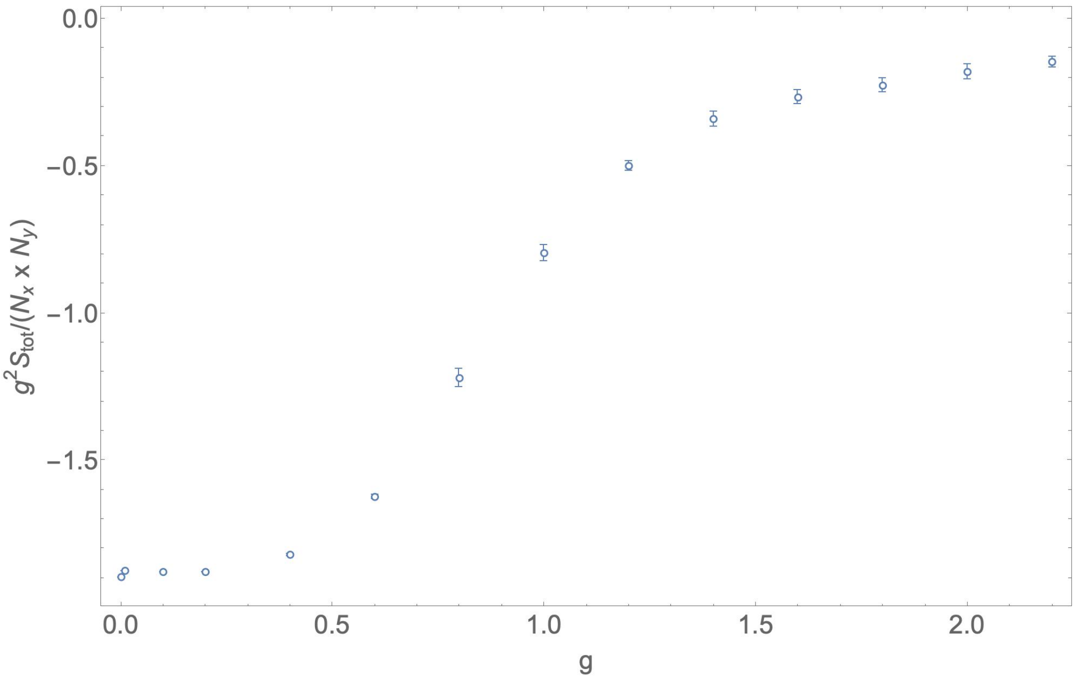

Figure 2: Mean and standard deviation of $g^2 S_{\mathrm{tot}}/(N_x \times N_y)$ at equilibrium (samplesize 1,000 after 10^7 steps) for various g for lattice size (30×30)

2.4 First results

2.5 Extract mass gap from $C(r)$

$$C(r) = f(r)e^{-mr} \tag{77}$$

$$\rightarrow \log \frac{C(r)}{C(r+1)} = \log \frac{f(r)e^{-mr}}{f(r)e^{-mr}e^{-m}} = m \tag{78}$$