5 Quantum Field Theory II – Exercise sheet 3 2024-04-24

5.1 Exercise 1: BRST Quantization of Yang-Mills Theory

In the lecture, we found the following Faddeev-Popov Lagrangian for the bosonic fields $A_\mu = A_\mu^a t_a, B = B^a t_a$ and the fermionic ghost fields $c = c^a t_a, \bar{c} = \bar{c}^a t_a$

$$\mathcal{L}_{\text{FP}} = -\frac{1}{4}\langle F^{\mu\nu} F_{\mu\nu}\rangle + \frac{\xi}{2}\langle B, B\rangle + \langle B, \partial_\mu A^\mu\rangle - \langle \partial^\mu \bar{c}, D_\mu c\rangle \tag{1}$$

where $\langle , \rangle$ denotes an invariant Cartan-Killing metric. We established that this theory is invariant under the global (infinitesimal) BRST transformations

$$\begin{aligned} \delta A_\mu &= D_\mu(\theta c), \qquad \delta B = 0, \\ \delta \bar{c} &= -\theta B, \qquad \delta c = \frac{1}{2}\theta[c, c] \end{aligned}$$

with fermionic (Grassmann odd) symmetry parameter θ.

1. Compute the Euler-Lagrange equations from (1) for all fields.
2. Apply Noether's theorem to compute the current j_μ that is conserved, satisfying $\partial_\mu j^\mu = 0$, as a consequence of BRST invariance.
   Hint: The Noether trick to promote $\theta \to \epsilon(x)$, with $\epsilon(x)$ a Grassmann odd scalar on spacetime is applicable.
3. Verify that the Noether current is indeed conserved on-shell, i.e., upon using the Euler-Lagrange equations.
   Hint: Use the integrability condition obtained by taking the divergence of the field equation for A_μ, using and proving the Bianchi identity $D_\nu D_\mu F^{\mu\nu} \equiv 0$.
4. In the free theory the BRST current reduces to the expression

   $$j^\mu = \langle B, \partial^\mu c\rangle - \langle c, \partial^\mu B\rangle.$$

   Consider the conserved charge

   $$\mathcal{Q} = \int d^3x\, j^0 \tag{2}$$

   and express it in terms of A_μ and c, using the equations of motion. Then writing A_μ and c in terms of creation and annihilation operators satisfying the familiar algebra

   $$\begin{aligned} A^\mu(x) &= \sum_{\lambda = >,<,+,-} \int dk\, \left[\varepsilon_\lambda^{\mu *}(k)\, a_\lambda(k)\, e^{ikx} + \varepsilon_\lambda^\mu(k)\, a_\lambda^\dagger(k)\, e^{-ikx}\right] \\ c(x) &= \int dk\, \left[c(k)\, e^{ikx} + c^\dagger(k)\, e^{-ikx}\right] \end{aligned}$$

   show that the adjoint action of $\mathcal{Q}$ on field operators reproduces the action of the BRST operator introduced in the lecture.
5. We now view $\mathcal{Q}$ as an operator on the multi-particle Hilbert space defined by the creation and annihilation operators introduced above, satisfying the nilpotency condition $\mathcal{Q}^2 = 0$.

   There is the notion of **cohomology**, the space of $\mathcal{Q}$-*closed* vectors satisfying $\mathcal{Q}|\psi\rangle = 0$, modulo $\mathcal{Q}$-*exact* vectors of the form $\mathcal{Q}|\chi\rangle$:

   $$\mathcal{H} := \frac{\ker \mathcal{Q}}{\operatorname{im} \mathcal{Q}} = \{[|\psi\rangle] \mid \mathcal{Q}|\psi\rangle = 0\},$$

   that is, $\mathcal{H}$ consists of equivalence classes: $|\psi\rangle = [|\psi\rangle + \mathcal{Q}|\chi\rangle]$.

   Show that the cohomology $\mathcal{H}$ precisely encodes the physical states, i.e., the transverse gluon polarizations.

1. Simplifying the terms of the Lagrangian using the Lie algebra $[t_b, t_c] = f_{bc}{}^a t_a$ with $f_{bc}{}^a = -f_{ba}{}^c$ (this one is a bit of guess work) and normalization $\kappa_{ab} = \delta_{ab}$

- Yang-Mills term $-\frac{1}{4}\langle F^{\mu\nu}, F_{\mu\nu}\rangle$

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu] && (301)\\ &= (\partial_\mu A^a_\nu)t_a - (\partial_\nu A^a_\mu)t_a - A^b_\mu A^c_\nu [t_b, t_c] && (302)\\ &= (\partial_\mu A^a_\nu)t_a - (\partial_\nu A^a_\mu)t_a - A^b_\mu A^c_\nu f_{bc}{}^a t_a && (303)\\ \to F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f_{bc}{}^a A^b_\mu A^c_\nu && (304) \end{aligned}$$

then

$$\begin{aligned} -\frac{1}{4}\langle F^{\mu\nu}, F_{\mu\nu}\rangle &= -\frac{1}{4}\mathrm{tr}[F^{\mu\nu}F_{\mu\nu}] && (305)\\ &= -\frac{1}{4}\kappa_{ab}F^{\mu\nu a}F^b_{\mu\nu} && (306) \end{aligned}$$

- Nakanishi-Lautrup term $\frac{\xi}{2}\langle B, B\rangle$

$$\frac{\xi}{2}\langle B, B\rangle = \frac{\xi}{2}B^a B^a \tag{307}$$

- Gauge-fixing term $\langle B, \partial_\mu A^\mu\rangle$

$$\langle B, \partial_\mu A^\mu\rangle = B^a(\partial^\mu A^a_\mu) \tag{308}$$

- Ghost term $\langle \partial^\mu \bar{c}, D_\mu c\rangle$

$$\begin{aligned} D_\mu c &= \partial_\mu c - [A_\mu, c] && (309)\\ &= (\partial_\mu c^a)t_a - A^b_\mu c^c [t_b, t_c] && (310)\\ &= (\partial_\mu c^a)t_a - A^b_\mu c^c f_{bc}{}^a t_a && (311) \end{aligned}$$

then

$$\begin{aligned} \langle \partial^\mu \bar{c}, D_\mu c\rangle &= \langle \partial^\mu \bar{c}, D_\mu c\rangle && (312)\\ &= \langle \partial^\mu \bar{c}, \partial_\mu c - [A_\mu, c]\rangle && (313)\\ &= \langle \partial^\mu \bar{c}, \partial_\mu c\rangle - \langle \partial^\mu \bar{c}, [A_\mu, c]\rangle && (314)\\ &= \kappa_{ab}(\partial^\mu \bar{c}^a)(\partial_\mu c^b) - \kappa_{ab}(\partial^\mu \bar{c}^a)A^c_\mu c^d f_{cd}{}^b && (315)\\ &= (\partial^\mu \bar{c}^a)(\partial_\mu c^a) - f_{cd}{}^a(\partial^\mu \bar{c}^a)A^c_\mu c^d && (316) \end{aligned}$$

(a) Gauge field $A_\mu = A^a_\mu t_a$

$$\begin{aligned} \frac{\partial \mathcal{L}_{\text{FP}}}{\partial(\partial_\beta A^b_\alpha)} &= -\frac{1}{4}2(F^{\beta\alpha b} - F^{\alpha\beta b}) + B^a \delta^\alpha_\mu \delta^\mu_\beta \delta^b_a = F^{\alpha\beta b} + B^b \delta^\beta_\alpha && (317)\\ \frac{\partial \mathcal{L}_{\text{FP}}}{\partial A^b_\alpha} &= \frac{\partial}{\partial A^b_\alpha}\langle \partial^\mu \bar{c}, [A_\mu, c]\rangle + \frac{1}{4}2F^{\mu\nu a}\frac{\partial}{\partial A^b_\alpha} f_{ef}{}^a A^e_\mu A^f_\nu && (318)\\ &= \frac{\partial}{\partial A^b_\alpha}(\partial^\mu \bar{c}^d)[A_\mu, c]^d + \frac{1}{2}F^{\mu\nu a} f_{ef}{}^a(\delta^e_b \delta^\alpha_\mu A^f_\nu + A^e_\mu \delta^f_b \delta^\alpha_\nu) && (319)\\ &= \frac{\partial}{\partial A^b_\alpha}(\partial^\mu \bar{c}^d)A^e_\mu c^f f_{ef}{}^d + \frac{1}{2}(F^{\alpha\nu a} f_{bf}{}^a A^f_\nu + F^{\mu\alpha a} f_{eb}{}^a A^e_\mu) && (320)\\ &= (\partial^\mu \bar{c}^d)\delta^\alpha_\mu \delta^e_b c^f f_{ef}{}^d + F^{\alpha\nu a} f_{bf}{}^a A^f_\nu && (321)\\ &= [(\partial^\alpha \bar{c}), c] + [A_\nu, F^{\alpha\nu a}] && (322) \end{aligned}$$

then

$$\boxed{D^\mu F_{\mu\nu} - \partial_\nu B + [\partial_\nu \bar{c}, c] = 0} \tag{323}$$

(b) Nakanishi-Laudrup field $B = B^a t_a$

$$\begin{aligned} \frac{\partial \mathcal{L}_{\text{FP}}}{\partial(\partial_\mu B^b)} &= 0 && (324)\\ \frac{\partial \mathcal{L}_{\text{FP}}}{\partial B^b} &= \frac{\xi}{2}\cdot 2B^a \delta^b_a + \delta^b_a(\partial^\mu A^a_\mu) && (325)\\ &= \xi B^b + (\partial^\mu A^b_\mu) && (326) \end{aligned}$$

then

$$\boxed{B = -\frac{1}{\xi}\partial^\mu A_\mu} \tag{327}$$

(c) Ghost field $c = c^a t_a$

$$\frac{\partial \mathcal{L}_{\text{FP}}}{\partial(\partial_\nu c^b)} = -\delta^\nu_\mu \delta^b_a \partial^\mu \bar{c}^a \tag{328}$$

$$= -\partial^\nu \bar{c}^b \tag{329}$$

$$\frac{\partial \mathcal{L}_{\text{FP}}}{\partial c^b} = \frac{\partial}{\partial c^b}\langle \partial^\mu \bar{c}, [A_\mu, c]\rangle = \frac{\partial}{\partial c^b}(\partial^\mu \bar{c}^d)[A_\mu, c]^d = \frac{\partial}{\partial c^b}(\partial^\mu \bar{c}^d) A^e_\mu c^f f_{ef}^{\ \ d} \tag{330}$$

$$= (\partial^\mu \bar{c}^d) A^e_\mu f_{ef}^{\ \ d} \delta^f_b = (\partial^\mu \bar{c}^d) A^e_\mu f_{eb}^{\ \ d} \tag{331}$$

$$= -(\partial^\mu \bar{c}^d) A^e_\mu f_{ed}^{\ \ b} = -[A_\mu, (\partial^\mu \bar{c})] \tag{332}$$

then

$$\partial_\nu \partial^\nu \bar{c} + [A_\nu, (\partial^\nu \bar{c})] = 0 \tag{333}$$

$$\boxed{\to D_\nu \partial^\nu \bar{c} = 0} \tag{334}$$

(d) Anti-ghost field $\bar{c} = \bar{c}^a t_a$

$$\frac{\partial \mathcal{L}_{\text{FP}}}{\partial(\partial_\nu \bar{c}^b)} = -\delta^a_b \delta^\mu_\nu D^\mu c^b \tag{335}$$

$$= -D^\nu c^a \tag{336}$$

$$\frac{\partial \mathcal{L}_{\text{FP}}}{\partial \bar{c}^b} = 0 \tag{337}$$

then

$$\boxed{\partial_\nu D^\nu c = 0} \tag{338}$$

$$D_\nu D^\nu c = \partial_\nu D^\nu c - [A_\nu, D^\nu c] \tag{339}$$

2. Rederiving the Noether theorem (somehow I can never remember it):

- Equations of motion - $uv' = -u'v + (uv)'$

$$0 \overset{!}{=} \delta S = \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi_a}\delta\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta(\partial_\mu \phi_a) \right) \tag{340}$$

$$= \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi_a}\delta\phi_a - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\right]\delta\phi_a + \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta\phi_a\right] \right) \tag{341}$$

$$= \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\right] \right)\delta\phi_a + \int_{\partial\Omega} d^3S \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta\phi_a\right] \tag{342}$$

- Symmetry trafo of the fields (if EoM are not changing meaning $\leftrightarrow$ if δS only has changes in the boundary term $\leftrightarrow$ meaning $\mathcal{L}$ changes only by 4-divergence $\partial_\mu \mathcal{J}$)

$$\phi_a(x) \to \phi'_a(x) + \varepsilon\delta\phi_a(x) \quad \text{allowing} \quad \mathcal{L}(x) \to \mathcal{L}'(x) = \mathcal{L}(x) + \varepsilon\partial_\mu \mathcal{J}^\mu(x) \tag{343}$$

- calculating implied change $\delta\mathcal{L}$

$$\epsilon\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi_a}(\epsilon\delta\phi_a) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\partial_\mu(\epsilon\delta\phi_a) \tag{344}$$

$$= \epsilon\partial_\mu \underbrace{\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta\phi_a\right)}_{=\mathcal{J}^\mu} + \underbrace{\left(\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\right]\right)}_{=0}\delta\phi_a \tag{345}$$

With $\theta \to \epsilon(x)$

$$\delta_\theta A_\mu = \partial_\mu(\epsilon(x)c) + [A_\mu, \epsilon(x)c] \tag{346}$$

$$= \epsilon(x) D_\mu c + c(\partial_\mu \epsilon(x)) \tag{347}$$

$$\delta_\theta B = 0, \tag{348}$$

$$\delta_\theta \bar{c} = -\epsilon(x) B \tag{349}$$

$$\delta_\theta c = \frac{1}{2}\epsilon(x)[c, c] \tag{350}$$

Under this local change, the Lagrangian transforms as

$$\begin{aligned}
\delta\mathcal{L}_{\text{FP}} &= \frac{\partial\mathcal{L}_{\text{FP}}}{\partial(\partial_\nu A_\mu)}\delta A_\mu + \frac{\partial\mathcal{L}_{\text{FP}}}{\partial(\partial_\nu B)}\delta B + \frac{\partial\mathcal{L}_{\text{FP}}}{\partial(\partial_\nu c)}\delta c + \frac{\partial\mathcal{L}_{\text{FP}}}{\partial(\partial_\nu \bar{c})}\delta \bar{c} && (351)\\
&= (F^{\mu\nu}+B)[\epsilon D_\mu c + c(\partial_\nu\epsilon)] + 0\cdot 0 + (-D^\nu c)(-\epsilon B) + (-\partial_\nu\bar{c})\frac{1}{2}\epsilon[c,c] && (352)
\end{aligned}$$

then we can read off j^ν as the ϵ coefficient (up to some signs)

$$j^\nu_{\text{BRS}} = \langle F^{\mu\nu}, D_\mu c\rangle - \langle B, D^\nu c\rangle + \langle\partial^\mu\bar{c}, \frac{1}{2}[c,c]\rangle \tag{353}$$

there is also a scaling symmetry for the ghost fields $c \to e^\lambda c$, $\bar{c} \to e^{-\lambda}\bar{c}$ with current

$$j^\nu_{\text{gh}} = \langle\partial^\nu\bar{c}, c\rangle - \langle\bar{c}, D^\nu c\rangle \tag{354}$$

3. Bianchi identity

$$D_{[\lambda}F_{\mu\nu]} = 0 \quad \to \quad D_\nu D_\mu F^{\mu\nu} = 0 \tag{355}$$

and

$$\begin{aligned}
D_\mu D_\nu c - D_\nu D_\mu c &= D_\mu(\partial_\nu c - [A_\nu, c]) - D_\nu(\partial_\mu c - [A_\mu, c]) && (356)\\
&= (\cancel{\partial_\mu\partial_\nu c} - [A_\mu, \partial_\nu c] - D_\mu[A_\nu, c]) - (\cancel{\partial_\nu\partial_\mu c} - [A_\nu, \partial_\mu c] - D_\nu[A_\mu, c]) && (357)\\
&= -[A_\mu, \partial_\nu c] - \partial_\mu[A_\nu, c] + [A_\mu, [A_\nu, c]] + [A_\nu, \partial_\mu c] + \partial_\nu[A_\mu, c] - [A_\nu, [A_\mu, c]]) && (358)\\
&= [F_{\mu\nu}, c] && (359)\\
\to \langle F^{\mu\nu}, D_\nu D_\mu c\rangle &= \langle F^{\mu\nu}, D_\mu D_\nu c + [F_{\nu\mu}, c]\rangle && (360)\\
&= \langle F^{\mu\nu}, D_\mu D_\nu c\rangle + \langle F^{\mu\nu}, [F_{\nu\mu}, c]\rangle && (361)
\end{aligned}$$

Taking the 4-divergence and substituting the eom's - keeping in mind that total divergence vanish

$$\begin{aligned}
\partial_\nu j^\nu_{\text{BRS}} &= \partial_\nu\langle F^{\mu\nu}, D_\mu c\rangle - \partial_\nu\langle B, D^\nu c\rangle + \partial_\nu\langle\partial^\mu\bar{c}, \frac{1}{2}[c,c]\rangle && (362)\\
&= \langle D_\nu F^{\mu\nu}, D_\mu c\rangle + \langle F^{\mu\nu}, D_\nu D_\mu c\rangle + \underbrace{\langle\partial_\nu B, D^\nu c\rangle}_{=\langle D_\nu B, D^\nu c\rangle + \langle[A_\nu, B], D^\nu c\rangle} + \langle B, \underbrace{\partial_\nu D^\nu c}_{D_\nu D^\nu c - [A_\nu, D^\nu c]}\rangle + \ldots && (363)\\
&= \langle\underbrace{D_\nu F^{\mu\nu}}_{=\partial^\mu B - [\partial^\mu\bar{c}, c]}, D_\mu c\rangle + \langle F^{\mu\nu}, D_\nu D_\mu c\rangle + \underbrace{\langle D_\nu B, D^\nu c\rangle}_{=\frac{1}{2}\partial_\nu\langle B, D^\nu c\rangle} + \cancel{\langle[A_\nu, B], D^\nu c\rangle} + \langle B, \underbrace{D_\nu D^\nu c}_{=0}\rangle - \cancel{\langle B, [A_\nu, D^\nu c]\rangle} + \ldots && (364)\\
&= \langle\partial^\mu B, D_\mu c\rangle - \langle[\partial^\mu\bar{c}, c], D_\mu c\rangle + \ldots && (365)\\
&= \partial^\mu\langle B, D_\mu c\rangle + \langle B, \underbrace{D_\mu D^\mu c}_{=0}\rangle - \langle[\partial^\mu\bar{c}, c], D_\mu c\rangle + \ldots && (366)\\
&= -\langle\partial^\mu\bar{c}, [c, D_\mu c]\rangle && (367)\\
&= 0 && (368)
\end{aligned}$$

because last term is a 4-divergence again.

4. In the free theory

$$\begin{aligned}
j^\nu_{\text{BRS}} &= \langle F^{\mu\nu}, D_\mu c\rangle - \langle B, D^\nu c\rangle && (369)\\
&= \ldots && (370)\\
&= \langle B, \partial^\mu c\rangle - \langle c, \partial^\mu B\rangle. && (371)
\end{aligned}$$

5. I run out of time ...