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1 Intro to LFT – Exercise sheet 2 (2025-05-08)

In the following problems, $A_\mu(x)$ is a smooth SU(N) gauge field in Euclidean spacetime. In particular, this means that $A_\mu(x)$ is an element of the Lie algebra $\mathfrak{su}(N)$ associated to the Lie group SU(N); that is, it is an $N \times N$ traceless Hermitian matrix. Even though $A_\mu(x)$ can be decomposed with respect to a basis of generators, you will not need such a representation.

- The field tensor is defined as:
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]. \tag{1}$$
In particular $F_{\mu\nu}(x)$ is also an element of the Lie algebra $\mathfrak{su}(N)$.
- A gauge transformation $\Omega(x)$ is an element of the gauge group SU(N). Under a gauge transformation, the gauge field transforms as:
$$A_\mu \to A_\mu^{[\Omega]} = \Omega A_\mu \Omega^\dagger + i(\partial_\mu \Omega)\Omega^\dagger. \tag{2}$$
- Using the definition of the field tensor, one easily proves that under a gauge transformation:
$$F_{\mu\nu} \to F_{\mu\nu}^{[\Omega]} = \Omega F_{\mu\nu} \Omega^\dagger. \tag{3}$$

1.1 Exercise 1 - Path-ordered exponential

Consider a smooth parametric curve $[s_0, s_1] \ni s \mapsto \gamma(s) \in \mathbb{R}^4$ in Euclidean spacetime. The following Cauchy problem:
$$\begin{cases} \dfrac{d}{ds} P(s) = i P(s) A_\mu(\gamma(s)) \gamma'^\mu(s), \\ P(s_0) = \mathbb{I}_N, \end{cases} \tag{4}$$
where $P(s)$ is an $N \times N$ matrix, admits a unique solution for $s \in [s_0, s_1]$. The solution $P(s)$ is referred to as the *path-ordered exponential* of the gauge field A_μ along the curve γ, and it is denoted by
$$P(s) = \mathcal{P} \exp\left(i \int_{s_0}^{s} d\sigma\, \gamma'^\mu(\sigma) A_\mu(\gamma(\sigma)) \right). \tag{5}$$
In particular, $P(s_1)$ is referred to as the *parallel transport* along the curve γ, and it is often denoted by $W(\gamma)$. The following notation is also often used:
$$W(\gamma) = \mathcal{P} \exp\left(i \int_\gamma dx^\mu A_\mu(x) \right). \tag{6}$$
If γ parametrizes a segment, then $W(\gamma)$ is referred to as the *Wilson line* along γ. If γ is a closed curve, then $W(\gamma)$ is referred to as the *Wilson loop* along γ. We stress that the expressions given on the right-hand side of Eqs. (5) and (6) are just symbols, whose definition is given in terms of the solution of the Cauchy problem (4).

We now want to study some properties of the solution $P(s)$.

a.) **Special unitarity.** Using the Cauchy problem (4), prove that $P(s)$ is a special unitary matrix for every s, i.e., that $P(s)P^{\dagger}(s) = \mathbb{I}_N$ and $\det P(s) = 1$.

Hint: Write differential equations for $P(s)P^{\dagger}(s)$ and $\det P(s)$, and solve them.

b.) **Parametrization invariance.** Let $\tilde{\gamma}$ be a reparametrization of the curve γ, which preserves orientation. This is defined by providing an invertible function $[\tilde{s}_0, \tilde{s}_1] \ni \tilde{s} \mapsto h(\tilde{s}) \in [s_0, s_1]$, with the property that $h(\tilde{s}_0) = s_0$ and $h(\tilde{s}_1) = s_1$. In terms of this function, we write

$$\tilde{\gamma}(\tilde{s}) = \gamma(h(\tilde{s})). \tag{7}$$

Let $\tilde{P}(\tilde{s})$ be the unique solution of the Cauchy problem

$$\begin{cases} \dfrac{d}{d\tilde{s}}\tilde{P}(\tilde{s}) = i\tilde{P}(\tilde{s})A_{\mu}(\tilde{\gamma}(\tilde{s}))\tilde{\gamma}'^{\mu}(\tilde{s}), \\ \tilde{P}(\tilde{s}_0) = \mathbb{I}_N, \end{cases} \tag{8}$$

Prove that $\tilde{P}(\tilde{s}) = P(h(\tilde{s}))$.

Note: This equation expresses the fact that the path-ordered exponential does not depend on the particular parametrization chosen for the curve γ.

Hint: Write the differential equation for $P(h(\tilde{s}))$ and compare it with the differential equation for $\tilde{P}(\tilde{s})$.

c.) **Dependence on initial condition.** Given a generic $N \times N$ matrix M, let $R(s)$ be the unique solution of the Cauchy problem

$$\begin{cases} \dfrac{d}{ds}R(s) = iR(s)A_{\mu}(\gamma(s))\gamma'^{\mu}(s), \\ R(s_0) = M, \end{cases} \tag{9}$$

Prove that the solution of this equation is given by

$$R(s) = M\,\mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\,\gamma'^{\mu}(\sigma)A_{\mu}(\gamma(\sigma))\right). \tag{10}$$

Hint: Write the differential equation satisfied by the right-hand side of the above equation, and compare it with the differential equation for $R(s)$.

d.) **Group property.** Consider $s_0 \leq \bar{s} \leq s \leq s_1$. Prove that

$$\mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\,\gamma'^{\mu}(\sigma)A_{\mu}(\gamma(\sigma))\right) = \mathcal{P}\exp\left(i\int_{s_0}^{\bar{s}} d\sigma\,\gamma'^{\mu}(\sigma)A_{\mu}(\gamma(\sigma))\right) \cdot \mathcal{P}\exp\left(i\int_{\bar{s}}^{s} d\sigma\,\gamma'^{\mu}(\sigma)A_{\mu}(\gamma(\sigma))\right). \tag{11}$$

Provide a geometrical interpretation of this equation.

Hint: Show that both sides satisfy the same Cauchy problem.

e.) **Gauge transformation.** Prove that

$$\mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\,\gamma'^{\mu}(\sigma)A_{\mu}^{[\Omega]}(\gamma(\sigma))\right) = \Omega(\gamma(s_0)) \cdot \mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\,\gamma'^{\mu}(\sigma)A_{\mu}(\gamma(\sigma))\right) \cdot \Omega^{\dagger}(\gamma(s)). \tag{12}$$

Hint: Show that both sides of the above equation satisfy the same Cauchy problem.

f.) **Approximation as product of exponentials.** Prove that, if Δs is small,

$$\mathcal{P}\exp\left(i\int_{s}^{s+\Delta s} d\sigma\,\gamma'^{\mu}(\sigma)A_{\mu}(\gamma(\sigma))\right) = e^{i\gamma'^{\mu}(s)A_{\mu}(\gamma(s))\Delta s} + \mathcal{O}(\Delta s^2). \tag{13}$$

Use this fact to argue that the path-ordered exponential can be approximated by a product of standard exponentials of the gauge field, as in

$$\mathcal{P}\exp\left(i\int_{s_0}^{s_1} d\sigma\,\gamma'^{\mu}(\sigma)A_\mu(\gamma(\sigma))\right) = \lim_{N\to\infty}\prod_{k=0}^{N-1} e^{i\gamma'^{\mu}(\sigma_k)A_\mu(\gamma(\sigma_k))\Delta\sigma}, \tag{14}$$

where

$$\sigma_k = s_0 + k\Delta\sigma, \quad \Delta\sigma = \frac{s_1 - s_0}{N}. \tag{15}$$

Note: A full proof of this fact is beyond the scope of this exercise sheet; a plausible argument is sufficient.

a.) Using $\frac{d}{ds}P(s)$ and the fact that $A_\mu(\gamma(x))$ is hermitian (because SU(N) matrix $e^{i\mathfrak{su}(2)}$ is unitary)

$$\begin{aligned}
\frac{d}{ds}P(s)P^\dagger(s) &= \frac{dP(s)}{ds}P^\dagger(s) + P(s)\frac{dP^\dagger(s)}{ds} && (4)\\
&= [iP(s)A_\mu(\gamma(x))\gamma'^{\mu}(s)]P^\dagger(s) + P(s)[iP(s)A_\mu(\gamma(x))\gamma'^{\mu}(s)]^\dagger && (5)\\
&= iP(s)A_\mu(\gamma(x))\gamma'^{\mu}(s)P^\dagger(s) - iP(s)\gamma'^{\mu}(s)\underbrace{A_\mu^\dagger(\gamma(x))}_{=A_\mu(\gamma(x))}P^\dagger(s) && (6)\\
&= 0 && (7)
\end{aligned}$$

and the initial condition

$$\begin{aligned}
P(s_0) = \mathbb{I}_N &\to P^\dagger(s_0) = \mathbb{I} && (8)\\
&\to P(s_0)P^\dagger(s_0) = \mathbb{I} && (9)
\end{aligned}$$

we conclude

$$P(s)P^\dagger(s) = \mathbb{I}_N \qquad \forall s. \tag{10}$$

Now using the fact that A_μ is trace free

$$1 = \det \mathrm{SU}(N) = \det e^{\mathfrak{su}(N)} = e^{\mathrm{tr}(\mathfrak{su}(N))} \to \mathrm{tr}(\mathfrak{su}(N)) = 0 \tag{11}$$

and the Jacobi formula

$$\begin{aligned}
\frac{d}{ds}\det P(s) &= \det P(s)\cdot \mathrm{tr}\left(P^{-1}(s)\frac{dP(s)}{ds}\right) && (12)\\
&= \det P(s)\cdot \mathrm{tr}\left(P^{-1}(s)[iP(s)A_\mu(\gamma(x))\gamma'^{\mu}(s)]\right) && (13)\\
&= i\,\gamma'^{\mu}(s)\det P(s)\cdot \underbrace{\mathrm{tr}\left(A_\mu(\gamma(x))\right)}_{=0} && (14)\\
&= 0 && (15)
\end{aligned}$$

and the initial condition

$$P(s_0) = \mathbb{I}_N \to \det P(s_0) = 1 \tag{16}$$

to conclude that $\det P(s) = 1$

b.) Initial conditions

$$P(h(\tilde{s}_0)) = P(s_0) \tag{17}$$

and the differential equation

$$\frac{d}{d\tilde{s}}P(h(\tilde{s})) = \frac{dP(h(\tilde{s}))}{dh}\frac{dh}{d\tilde{s}} \tag{18}$$

$$= iP(h)A_\mu(\gamma(h))\underbrace{\underbrace{\gamma'^\mu(h)}_{=\frac{d\tilde{\gamma}^\mu}{dh}}\frac{dh}{d\tilde{s}}}_{=\frac{d\tilde{\gamma}^\mu}{d\tilde{s}}} \tag{19}$$

$$= iP(h(\tilde{s}))A_\mu(\gamma(h(\tilde{s})))\frac{d\tilde{\gamma}^\mu}{d\tilde{s}} \tag{20}$$

so we see that the differential equations for $\tilde{P}(\tilde{s})$ and $P(h(\tilde{s}))$ are identical.

c.) Using

$$\frac{d}{ds}\int_{g(s)}^{h(s)} f(t,s)dt = \frac{d}{ds}\left(F[h(s),s] - F[g(s),s]\right) \tag{21}$$

$$= \frac{\partial F}{\partial h}\frac{\partial h}{\partial s} + \frac{\partial F[h,s]}{\partial s} - \frac{\partial F}{\partial g}\frac{\partial g}{\partial s} - \frac{\partial F[g,s]}{\partial s} \tag{22}$$

$$= f(h(s))h'(s) - f(g(s))g'(s) + \int_{g(s)}^{h(s)}\frac{d}{ds}f(t,s)dt \tag{23}$$

and substituting the solution into the differential equations

$$\frac{dR}{ds} = \frac{d}{ds}M\mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\,\gamma'^\mu(\sigma)A_\mu(\gamma(\sigma))\right) \tag{24}$$

$$= \underbrace{M\mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\,\gamma'^\mu(s)A_\mu(\gamma(s))\right)}_{=R(s)}(i\gamma'^\mu(\sigma)A_\mu(\gamma(\sigma)))\frac{ds}{ds} \tag{25}$$

$$= iR(s)A_\mu(\gamma(s))\gamma'^\mu(s) \tag{26}$$

so we see the the solution obeys the differential equation.

d.) LHS

$$\frac{d}{ds}\mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\,\gamma'^\mu(\sigma)A_\mu(\gamma(\sigma))\right) = \mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\,\gamma'^\mu(\sigma)A_\mu(\gamma(\sigma))\right)iA_\mu(\gamma(s))\gamma'^\mu(s) \tag{27}$$

$$\rightarrow \qquad \frac{d}{ds}Q(s) = Q(s)\cdot iA_\mu(\gamma(s))\gamma'^\mu(s) \tag{28}$$

RHS

$$\frac{d}{ds}\mathcal{P}\exp\left(i\int_{s_0}^{\bar{s}} d\sigma\,\gamma'^\mu(\sigma)A_\mu(\gamma(\sigma))\right)\cdot\mathcal{P}\exp\left(i\int_{\bar{s}}^{s} d\sigma\,\gamma'^\mu(\sigma)A_\mu(\gamma(\sigma))\right) \tag{29}$$

$$= \exp\left(i\int_{s_0}^{\bar{s}} d\sigma\,\gamma'^\mu(\sigma)A_\mu(\gamma(\sigma))\right)\cdot\mathcal{P}\exp\left(i\int_{\bar{s}}^{s} d\sigma\,\gamma'^\mu(\sigma)A_\mu(\gamma(\sigma))\right)iA_\mu(\gamma(s))\gamma'^\mu(s) \tag{30}$$

$$\rightarrow \qquad \frac{d}{ds}Q(s) = Q(s)\cdot iA_\mu(\gamma(s))\gamma'^\mu(s) \tag{31}$$

e.) LHS

$$\frac{d}{ds}\left(\mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\,\gamma'^\mu(\sigma)A_\mu^{[\Omega]}(\gamma(\sigma))\right)\right) = \mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\,\gamma'^\mu(\sigma)A_\mu^{[\Omega]}(\gamma(\sigma))\right)iA_\mu^{[\Omega]}(\gamma(s))\gamma'^\mu(s) \tag{32}$$

RHS with $A_\mu^{[\Omega]} = \Omega A_\mu \Omega^\dagger + i(\partial_\mu \Omega)\Omega^\dagger$ and the adjoint $A_\mu^{[\Omega]} = \Omega A_\mu \Omega^\dagger - i(\Omega \partial_\mu \Omega^\dagger)$

$$\frac{d}{ds}\left(\Omega(\gamma(s_0)) \cdot \mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\, \gamma'^\mu(\sigma) A_\mu(\gamma(\sigma))\right) \cdot \Omega^\dagger(\gamma(s))\right) \tag{33}$$

$$= \Omega(\gamma(s_0))\left(\frac{d}{ds}\mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\, \gamma'^\mu(\sigma) A_\mu(\gamma(\sigma))\right)\right)\Omega^\dagger(\gamma(s)) \tag{34}$$

$$+ \Omega(\gamma(s_0))\mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\, \gamma'^\mu(\sigma) A_\mu(\gamma(\sigma))\right)\left(\frac{d}{ds}\Omega^\dagger(\gamma(s))\right) \tag{35}$$

$$= \Omega(\gamma(s_0))\mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\, \gamma'^\mu(\sigma) A_\mu(\gamma(\sigma))\right)(iA_\mu(\gamma(s))\gamma'^\mu(s))\,\Omega^\dagger(\gamma(s)) \tag{36}$$

$$+ \Omega(\gamma(s_0))\mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\, \gamma'^\mu(\sigma) A_\mu(\gamma(\sigma))\right)\underbrace{\left(\frac{d}{ds}\Omega^\dagger(\gamma(s))\right)}_{=\partial_\nu \Omega^\dagger \gamma'^\nu(s)} \tag{37}$$

$$= \Omega(\gamma(s_0))\mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\, \gamma'^\mu(\sigma) A_\mu(\gamma(\sigma))\right)(iA_\mu(\gamma(s))\gamma'^\mu(s))\,\Omega^\dagger(\gamma(s)) \tag{38}$$

$$+ \Omega(\gamma(s_0))\mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\, \gamma'^\mu(\sigma) A_\mu(\gamma(\sigma))\right)\gamma'^\nu(s)\partial_\nu \Omega^\dagger \tag{39}$$

$$= \Omega(\gamma(s_0))\mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\, \gamma'^\mu(\sigma) A(\gamma'^\mu(s))\right)\gamma'^\mu(\sigma)\underbrace{\left[iA_\mu(\gamma(s))\Omega^\dagger(\gamma(a)) + \partial_\mu \Omega^\dagger\right]}_{=i\Omega^\dagger(\gamma(s))A_\mu^{[\Omega]}} \tag{40}$$

$$= \Omega(\gamma(s_0))\mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\, \gamma'^\mu(\sigma) A_\mu(\gamma'^\mu(\sigma))\right)\Omega^\dagger(\gamma(s)) i\gamma'^\mu(s) A_\mu^{[\Omega]} \tag{41}$$

and therefore comparing LHS and RHS

$$\mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\, \gamma'^\mu(\sigma) A_\mu^{[\Omega]}(\gamma'^\mu(s))\right) = \Omega(\gamma(s_0))\mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\, \gamma'^\mu(\sigma) A_\mu(\gamma'^\mu(s))\right)\Omega^\dagger(\gamma(s)) \tag{42}$$

f.) With $\gamma'^\mu(\sigma)A_\mu(\gamma'^\mu(\sigma)) = \gamma'^\mu(s)A_\mu(\gamma'^\mu(s))$ and the fact that the path-ordering becomes obsolete if the integrand it constant

$$\mathcal{P}\exp\left(i\int_{s}^{s+\Delta s} d\sigma\, \gamma'^\mu(\sigma) A_\mu(\gamma'^\mu(\sigma))\right) \simeq \mathcal{P}\exp\left(i\int_{s}^{s+\Delta s} d\sigma\, \gamma'^\mu(s) A_\mu(\gamma'^\mu(s))\right) \tag{43}$$

$$\simeq \exp\left(i\,\gamma'^\mu(s) A_\mu(\gamma'^\mu(s))\int_{s}^{s+\Delta s} d\sigma\right) \tag{44}$$

$$\simeq \exp\left(i\,\gamma'^\mu(s) A_\mu(\gamma'^\mu(s))\Delta s + \mathcal{O}(\Delta s^2)\right) \tag{45}$$

For the next part we use the group property (we showed above) - splitting this into N equal sized sections $\Delta\sigma$

$$\mathcal{P}\exp\left(i\int_{s_0}^{s_1} d\sigma\, \gamma'^\mu(\sigma) A_\mu(\gamma'^\mu(\sigma))\right) \tag{46}$$

$$= \mathcal{P}\exp\left(i\int_{s_0}^{s_0+\Delta\sigma} d\sigma\, \gamma'^\mu(\sigma) A_\mu(\gamma'^\mu(\sigma))\right)\ldots\mathcal{P}\exp\left(i\int_{s_0+(N-1)\Delta\sigma}^{s_1} d\sigma\, \gamma'^\mu(\sigma) A_\mu(\gamma'^\mu(\sigma))\right) \tag{47}$$

$$\simeq \lim_{N\to\infty} \exp\left(i\,\gamma'^\mu(s_0) A_\mu(\gamma'^\mu(s_0))\Delta\sigma\right)\ldots\exp\left(i\,\gamma'^\mu(s_0+(N-1)\Delta\sigma) A_\mu(\gamma'^\mu(s_0+(N-1)\Delta\sigma))\Delta\sigma\right) \tag{48}$$

$$\simeq \lim_{N\to\infty} \exp\left(i\,\gamma'^\mu(s_0) A_\mu(\gamma'^\mu(s_0))\Delta\sigma\right)\ldots\exp\left(i\,\gamma'^\mu(s_{N-1}) A_\mu(\gamma'^\mu(s_{N-1})\Delta\sigma\right) \tag{49}$$

I think the limit is the handwavy part.

1.2 Exercise 2 - Temporal gauge

Given a gauge field $A_\mu(x)$, prove that it is always possible to find a gauge transformation $\Omega(x)$ such that $A_0^{[\Omega]}(x) = 0$ for every x (temporal gauge condition). Derive a formula for $\Omega(x)$ in terms of the path-ordered exponential.

We require (using $\Omega^{-1} = \Omega^\dagger$)

$$A_0^{[\Omega]} = \Omega A_0 \Omega^\dagger + i(\partial_0 \Omega)\Omega^\dagger \stackrel{!}{=} 0 \tag{50}$$
$$\to \Omega A_0 + i(\partial_0 \Omega) = 0 \tag{51}$$
$$\to \partial_0 \Omega = i \Omega A_0(\vec{x}, t) \tag{52}$$

so we obtained a homogeneous linear first order ODE which we can solve as usual by separation of variables

$$\Omega(\vec{x}, t) = \mathcal{P} \exp\left(i \int_{t_0}^{t} A_0(\vec{x}, s) ds \right) \tag{53}$$

2 Intro to LFT – Exercise sheet 1 (2025-04-15)

2.1 Exercise 1 - Integration of scalar function over multidimensional vector space

Since the new basis $\{b^n\}$ needs to be orthonormal - the allowed transformations are

1. permutation of the basis vectors $\{e^n\}$
2. and then a rigid rotation of the whole basis

This transformations of the basis mean ($O \in O(N)$)

$$
\begin{aligned}
e^n &= \sum_m O^n_m b^m && (54)\\
\to v &= \alpha_n e^n && (55)\\
&= \sum_m (\sum_n O^n_m \alpha_n) b^m && (56)\\
&= \sum_m \beta_m b^m && (57)\\
\to \beta_m &= \sum_n O^n_m \alpha_n && (58)
\end{aligned}
$$

Now with $d\beta_m = \sum_n O^n_m d\alpha_n$

$$
\begin{aligned}
I'(f) &= \int \left(\prod_i d\beta_i\right) f(\underbrace{\beta_k b^k}_{=\alpha_k a^k}) && (59)\\
&= \int \left(\prod_i \sum_n O^n_i d\alpha_n\right) f(\underbrace{\beta_k b^k}_{=\alpha_k a^k}) && (60)\\
&= \int |\underbrace{\det O}_{\pm 1}| \left(\prod_i d\alpha_n\right) f(\underbrace{\beta_k b^k}_{=\alpha_k a^k}) && (61)\\
&= \int \left(\prod_i d\alpha_n\right) f(\alpha_k a^k) && (62)\\
&= I(f) && (63)
\end{aligned}
$$

2.2 Exercise 2 - Multivariate Gaussian integral with a source

As usual we try to obtain a complete square

$$-\frac{1}{2}\phi^T A\phi + J^T\phi = -\frac{1}{2}\left(\phi^T A\phi - 2J^T\phi\right) \tag{64}$$
$$= -\frac{1}{2}\left((\phi+x)^T A(\phi+x)\right) + y \tag{65}$$
$$= -\frac{1}{2}\left(\phi^T A\phi + \phi^T Ax + x^T A\phi + x^T Ax\right) + y \tag{66}$$

with $x = -A^{-1}J$

$$-\frac{1}{2}\phi^T A\phi + J^T\phi = -\frac{1}{2}\left(\phi^T A\phi - \phi^T A(A^{-1}J) - (A^{-1}J)^T A\phi + (A^{-1}J)^T A(A^{-1}J)\right) + y \tag{67}$$
$$= -\frac{1}{2}\left(\phi^T A\phi - J^T\phi + \phi^T J + J^T A^{-1}J\right) + \frac{1}{2}J^T A^{-1}J \tag{68}$$
$$= -\frac{1}{2}\left(\phi^T A\phi - J^T\phi + (J^T\phi)^T + J^T A^{-1}J\right) + \frac{1}{2}J^T A^{-1}J \tag{69}$$

and therefore

$$-\frac{1}{2}\phi^T A\phi + J^T\phi = -\frac{1}{2}(\phi - A^{-1}J)^T A(\phi - A^{-1}J) + \frac{1}{2}J^T A^{-1}J \tag{70}$$

So

$$I(J) = \int d\phi\ \exp\left[-\frac{1}{2}(\phi - A^{-1}J)^T A(\phi - A^{-1}J)\right]\cdot\exp\left[\frac{1}{2}J^T A^{-1}J\right] \tag{71}$$
$$= \exp\left[\frac{1}{2}J^T A^{-1}J\right]\cdot\int d\phi\ \exp\left[-\frac{1}{2}(\phi - A^{-1}J)^T A(\phi - A^{-1}J)\right] \tag{72}$$

To calculate the remaining integral will now try to break it into a product of 1d gaussian integrals (which we know how to calculate).

- As A is real and symmetric we can write it as $A = S^T DS$ where D is diagonal (with positive eigenvalues of A on the diagonal) and S is orthogonal $S^{-1} = S^T$

$$I(0) = \int d\phi\ \exp\left[-\frac{1}{2}(\phi - A^{-1}J)^T S^T DS(\phi - A^{-1}J)\right] \tag{73}$$
$$= \int d\phi\ \exp\left[-\frac{1}{2}[S(\phi - A^{-1}J)]^T D[S(\phi - A^{-1}J)]\right] \tag{74}$$
$$= \int d\phi\ \prod_k \exp\left[-\frac{1}{2}[S(\phi - A^{-1}J)]_k^T D_{kk}[S(\phi - A^{-1}J)]_k\right] \tag{75}$$

- Now we can use the result of problem 1 - getting a new orthogonal coordinate system

$$I(0) = \int d\phi\ \prod_k \exp\left[-\frac{1}{2}[S(\phi - A^{-1}J)]_k^T D_{kk}[S(\phi - A^{-1}J)]_k\right] \tag{76}$$
$$= \prod_k \int d\psi_k\ \exp\left[-\frac{1}{2}\psi_k^T D_{kk}\psi_k\right] \tag{77}$$
$$= \prod_k \sqrt{\frac{2\pi}{D_{kk}}} \tag{78}$$
$$= \sqrt{\frac{(2\pi)^N}{detA}} \tag{79}$$

And therefore

$$I(J) = \frac{(2\pi)^{N/2}}{\sqrt{\det A}} \exp\left[\frac{1}{2} J^T A^{-1} J\right] \tag{80}$$

which is finite for every J.

2.3 Exercise 3

With

$$(A\phi)_n = \frac{2\phi_n - \phi_{n\oplus(+1)} - \phi_{n\oplus(-1)}}{a^2} \qquad (n = 1, ..., N) \tag{81}$$

we see that A is the 1d negative discretized Laplacian with periodic boundary conditions (know from finite difference numerics of the heat and Schroedinger equation)

$$A = \frac{1}{a^2}\begin{pmatrix} 2 & -1 & 0 & ... & 0 & 0 & -1 \\ -1 & 2 & -1 & ... & 0 & 0 & 0 \\ 0 & -1 & 2 & ... & 0 & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & ... & 2 & -1 & 0 \\ 0 & 0 & 0 & ... & -1 & 2 & -1 \\ -1 & 0 & 0 & ... & 0 & -1 & 2 \end{pmatrix} \tag{82}$$

This looks like the set of equations of motion for a 1D chain of atoms

$$m\ddot{x}_i = k(u_{i-1} - u_i) + k(u_{i+1} - u_i) \tag{83}$$

in matrix form.

1. We see that A is symmetric - A is positive definite if and only if all eigenvalues are strictly positive - which we might see - once we calculated the eigenvalues.

2. The periodic boundary conditions imply

$$v_0^{(p)} = v_N^{(p)} \quad \rightarrow \quad e^{iapN} = 1 \tag{84}$$

$$\rightarrow \quad apN = 2\pi k \qquad (k \in \{1, 2, 3, ..., N\}) \tag{85}$$

$$\rightarrow \quad p = \frac{2\pi k}{aN} \qquad \text{(Nyquist-Shannon-sampling theorem)} \tag{86}$$

and with $e^{iapN} = 1$ we can calculate

$$(Av^{(p)})_n = (Ae^{iapn})_n \tag{87}$$

$$= \begin{cases} \frac{1}{a^2}(2e^{iap\cdot 1} - e^{iap\cdot 2} - e^{iapN}) = \frac{1}{a^2}(2 - e^{iap} - e^{-iap})e^{iap} & (n = 1) \\ \frac{1}{a^2}(2e^{iapn} - e^{iap(n+1)} - e^{iap(n-1)}) = \frac{1}{a^2}(2 - e^{iap} - e^{-iap})e^{iapn} & \text{else} \\ \frac{1}{a^2}(2e^{iapN} - e^{iap\cdot(1)} - e^{iap\cdot(N-1)}) = \frac{1}{a^2}(2 - e^{iap} - e^{-iap})e^{iapN} & (n = N) \end{cases} \tag{88}$$

$$= \frac{1}{a^2}(2 - e^{iap} - e^{-iap})v_n^{(p)} \tag{89}$$

$$= \frac{2}{a^2}(1 - \cos ap)v_n^{(p)} \tag{90}$$

Now we can read-off the N eigenvectors and eigenvalues are ($1 \leq k \leq N$)

$$v_n^{(k)} = e^{ian\frac{2\pi k}{aN}} = e^{2\pi i \frac{n}{N} k} \tag{91}$$

$$\lambda^{(k)} = \frac{2}{a^2}\left(1 - \cos a\frac{2\pi k}{aN}\right) \tag{92}$$

$$= \frac{2}{a^2}\left(1 - \cos\frac{2\pi k}{N}\right) \tag{93}$$

$$= \frac{2}{a^2}\left(1 - \cos^2\frac{\pi k}{N} + \sin^2\frac{\pi k}{N}\right) \tag{94}$$

$$= \frac{4}{a^2}\sin^2\frac{\pi k}{N} = \left(\frac{2}{a}\sin\frac{\pi k}{N}\right)^2 \tag{95}$$

As expected - this looks like the dispersion relation $\omega(k)$ for a 1D chain of atoms. Now we see that $\lambda^{(N)} = 0$ so A is NOT positive definite but positive semi-definite.

3. Lets rewright

$$v^{(k)} = e^{2\pi i \frac{n}{N} k} = \cos\left(2\pi \frac{n}{N} k\right) + i \sin\left(2\pi \frac{n}{N} k\right) \tag{96}$$

so - using results from elementary Fourier analysis - we rewrite the completeness relation as a (finite) Fourier series

k	$\lambda^{(k)}$	(complex) $v_n^{(k)}$	(real) $u_n^{(k)}$	(real) $u_n^{(k)}$
1	$\frac{4}{a^2}\sin^2\left(\pi\frac{1}{N}\right)$	$\exp\left(2\pi\frac{in}{N}\right)$	$\sqrt{\frac{2}{N}}\cos\left(2\pi\frac{n}{N}\right)$	$\sqrt{\frac{2}{N}}\sin\left(2\pi\frac{n}{N}\right)$
2	$\frac{4}{a^2}\sin^2\left(\pi\frac{2}{N}\right)$	$\exp\left(2\pi\frac{2in}{N}\right)$	$\sqrt{\frac{2}{N}}\cos\left(2\pi\frac{2n}{N}\right)$	$\sqrt{\frac{2}{N}}\sin\left(2\pi\frac{2n}{N}\right)$
...				
$(N/2)$	$\frac{4}{a^2}$	$\exp\left(i\pi n\right)$	$\sqrt{\frac{2}{L}}(-1)^n$	0
...				
$N-2$	$\frac{4}{a^2}\sin^2\left(\pi\frac{2}{N}\right)$	$\exp\left(-2\pi\frac{2in}{N}\right)$	$\sqrt{\frac{2}{N}}\cos\left(2\pi\frac{2n}{N}\right)$	$-\sqrt{\frac{2}{N}}\sin\left(2\pi\frac{2n}{N}\right)$
$N-1$	$\frac{4}{a^2}\sin^2\left(\pi\frac{1}{N}\right)$	$\exp\left(-2\pi\frac{in}{N}\right)$	$\sqrt{\frac{2}{N}}\cos\left(2\pi\frac{n}{N}\right)$	$-\sqrt{\frac{2}{N}}\sin\left(2\pi\frac{n}{N}\right)$
N	0	1	$\sqrt{\frac{1}{N}}$	0

Table 1: Overview of eigensystems

and we see that we can drop half of the sine and cosine eigenfunctions occurring twice (up to a potential sign) - so we can drop them. So depending on N even or odd

$$u^{(N)} = \sqrt{\frac{1}{N}} \tag{97}$$

$$u^{(k)} = \sqrt{\frac{2}{N}} \cos\left(2\pi \frac{n}{N} k\right) \qquad (k = 1..[N-1/2]) \tag{98}$$

$$u^{(N/2+k)} = \sqrt{\frac{2}{N}} \sin\left(2\pi \frac{n}{N} k\right) \qquad (k = 1..[N-1/2]) \tag{99}$$

$$u^{(N/2)} = \sqrt{\frac{2}{N}}(-1)^n \qquad \text{(iff } N \text{ is even)} \tag{100}$$

It looks a bit messy in the write-up but I think its clear.

4. The eigenvalues $\tilde{\lambda}^{(k)}$ of $A + m^2 = A + 1_{N\times N} m^2$ are

$$\tilde{\lambda}^{(k)} = \lambda^{(k)} + m^2 \tag{101}$$

$$= \frac{4}{a^2}\sin^2\frac{\pi k}{N} + m^2 \tag{102}$$

while the eigenvectors are the same as for A. Using the spectral decomposition to calculate the inverse of $(A+m^2)^{-1}$ (which has the inverse eigenvalues and the same eigenvectors)

$$(A+m^2)^{-1} = \sum_k \frac{1}{\tilde{\lambda}^{(k)}} {u^{(k)}}^T u^{(k)} \tag{103}$$

$$(A+m^2)^{-1}_{ij} = \sum_k \frac{1}{\frac{4}{a^2}\sin^2\frac{\pi k}{N} + m^2} {u_i^{(k)}}^T u_j^{(k)} \tag{104}$$

I tried to find a simple expression using Mathematica - but could not find anything. Alternatively I also tried

$$\frac{1}{A+m^2} = \frac{1}{m^2}\left(1 - \frac{1}{m^2}A + \frac{1}{m^4}A^2 - \frac{1}{m^6}A^3 + ...\right) \tag{105}$$

$$\frac{1}{A+m^2}_{ij} = u^{(i)T}\frac{1}{A+m^2}u^{(j)} \tag{106}$$

$$= \frac{1}{m^2}\left(1 - \frac{1}{m^2}u^{(i)T}Au^{(j)} + \frac{1}{m^4}u^{(i)T}A^2u^{(j)} - \frac{1}{m^6}u^{(i)T}A^3u^{(j)} + ...\right) \tag{107}$$

$$= \frac{1}{m^2}\left(1 - \frac{1}{m^2}u^{(i)T}\lambda^{(j)}u^{(j)} + \frac{1}{m^4}u^{(i)T}(\lambda^{(j)})^2u^{(j)} - \frac{1}{m^6}u^{(i)T}(\lambda^{(j)})^3u^{(j)} + ...\right) \tag{108}$$

$$= \frac{1}{m^2}\left(1 - \frac{\lambda^{(j)}}{m^2}u^{(i)T}u^{(j)} + \frac{(\lambda^{(j)})^2}{m^4}u^{(i)T}u^{(j)} - \frac{(\lambda^{(j)})^3}{m^6}u^{(i)T}u^{(j)} + ...\right) \tag{109}$$

but got nowhere.

3 O(3) non-linear sigma model in two-dimension

This is a theory with action (Euclidean metric $\eta = \mathrm{diag}(+1, +1)$)

$$S = \frac{1}{2g^2} \int d^2x \sum_k \partial_\mu \phi_k \partial^\mu \phi_k \tag{110}$$

$$= \frac{1}{2g^2} \int d^2x \sum_k |\partial_\mu \phi_k|^2 \tag{111}$$

where the field had three components $\phi(x, y) = (\phi_1(x, y), \phi_2(x, y), \phi_3(x, y))$ and lives on a sphere with unit radius, i.e.

$$\phi_1^2 + \phi_2^2 + \phi_3^2 = 1 \tag{112}$$

3.1 Discretization of the field

Assumption: cubic equidistant lattice ($a = a_x = a_y$)

$$S = \frac{1}{2g^2} \int dx\, dy (\partial_x \phi_1)^2 + (\partial_x \phi_2)^2 + (\partial_x \phi_3)^2 + (\partial_y \phi_1)^2 + (\partial_y \phi_2)^2 + (\partial_y \phi_3)^2 \tag{113}$$

$$\simeq \frac{a^2}{2g^2} \sum_{ij} \sum_k \frac{(\phi^{(k)}_{i+1,j} - \phi^{(k)}_{i,j})^2}{a^2} + \frac{(\phi^{(k)}_{i,j+1} - \phi^{(k)}_{i,j})^2}{a^2} \tag{114}$$

$$\simeq \frac{a^2}{2g^2} \sum_{ij} \sum_k \frac{(\phi^{(k)}_{i+1,j})^2}{a^2} + \frac{(\phi^{(k)}_{i,j})^2}{a^2} + \frac{(\phi^{(k)}_{i,j+1})^2}{a^2} + \frac{(\phi^{(k)}_{i,j})^2}{a^2} - 2\frac{\phi^{(k)}_{i+1,j}\phi^{(k)}_{i,j}}{a^2} - 2\frac{\phi^{(k)}_{i,j+1}\phi^{(k)}_{i,j}}{a^2} \tag{115}$$

$$\simeq \frac{a^2}{2g^2} \sum_{ij} \frac{4}{a^2} - \frac{a^2}{g^2} \sum_k \sum_{ij} \frac{\phi^{(k)}_{i+1,j}\phi^{(k)}_{i,j}}{a^2} + \frac{\phi^{(k)}_{i,j+1}\phi^{(k)}_{i,j}}{a^2} \tag{116}$$

$$\simeq \frac{2N_s^2}{g^2} - \frac{1}{g^2} \sum_k \sum_{ij} \phi^{(k)}_{i+1,j}\phi^{(k)}_{i,j} + \phi^{(k)}_{i,j+1}\phi^{(k)}_{i,j} \tag{117}$$

we neglect the first term (constant offset) and use

$$S_{ij} = -\frac{1}{g^2} \sum_k \phi^{(k)}_{i+1,j}\phi^{(k)}_{i,j} + \phi^{(k)}_{i,j+1}\phi^{(k)}_{i,j} \tag{118}$$

$$S_{\mathrm{tot}} = \sum_{ij} S_{ij} \tag{119}$$

$$= -\frac{1}{g^2} \sum_k \sum_{ij} \phi^{(k)}_{i+1,j}\phi^{(k)}_{i,j} + \phi^{(k)}_{i,j+1}\phi^{(k)}_{i,j} \tag{120}$$

$$Z = \int d\phi_1 d\phi_2 d\phi_3 \exp\left[-\frac{1}{2g^2} \int dx\, dy \sum_{k,\mu} |\partial_\mu \phi_k|^2\right] \tag{121}$$

$$= \sum_{\mathrm{configs}} \exp\left[-S_{\mathrm{tot}}(\phi^{(k)}_{i,j})\right] \tag{122}$$

$$\langle \phi(x_1)\phi(x_2) \rangle = \frac{\int d\phi^{(1)} d\phi^{(2)} \phi^{(3)} \ \sum_{k'} \phi^{(k')}(x_1) \phi^{(k')}(x_2) \exp\left[-\frac{1}{2g^2} \int dx\, dy \sum_{k,\mu} |\partial_\mu \phi^{(k)}|^2\right]}{\int d\phi^{(1)} d\phi^{(2)} \phi^{(3)} \ \exp\left[-\frac{1}{2g^2} \int dx\, dy \sum_{k,\mu} |\partial_\mu \phi^{(k)}|^2\right]} \tag{123}$$

$$C_{r_j} = \frac{\sum_{\mathrm{configs}} \frac{1}{N_{r_j}} \left(\sum_k \phi^{(k)} \phi^{(k)}\right) \exp\left[-S_{\mathrm{tot}}(\phi^{(k)}_{i,j})\right]}{\sum_{\mathrm{configs}} \exp\left[-S_{\mathrm{tot}}(\phi^{(k)}_{i,j})\right]} \tag{124}$$

3.2 Random field configuration at lattice point

To generate a random field at a point of the lattice - obeying the $O(3)$ constraint we use polar coordinates. As the field vectors should be uniformly distributed in S^2 we use the uniformly independent distributed angle $\varphi \in [0, 2\pi)$ and $z = \cos\vartheta \in [-1, +1)$

$$\varphi = 2\pi\,\mathcal{U}^{(1)}_{[0,1]} \tag{125}$$

$$z = \cos\vartheta = 2\,\mathcal{U}^{(2)}_{[0,1]} - 1 \qquad \rightarrow \qquad \sin\vartheta = \sqrt{1-\cos^2\theta} \tag{126}$$

$$\rightarrow \phi^{(1)} = \sin\vartheta\,\cos\varphi \tag{127}$$

$$\rightarrow \phi^{(2)} = \sin\vartheta\,\sin\varphi \tag{128}$$

$$\rightarrow \phi^{(3)} = \cos\vartheta \tag{129}$$

3.3 Code

Algorithm 1 Metropolis Monte Carlo for O(3) nonlinear sigma model in 2D

Assumption - cubic equidistant lattice
Initialize a $N_x \times N_y$ lattice with random unit vectors $\vec{\phi}_{i,j} = \phi^{(k)}_{i,j} \in \mathbb{S}^2$
Initialize a $N_x \times N_y$ lattice with local action $S_{i,j}$
Calculate total action $S_{\text{tot}} = \sum_{ij} S_{ij}$
Initialize $Z = 0$
Initialize all 2-point observables $c_{r_k} = 0$ with $r_k \in \left\{\sqrt{k_x^2 + k_y^2},\ 0 \le k_x \le {}^{N_x}/_2, 0 \le k_y \le {}^{N_y}/_2\right\}$
Calculate count of distance occurrence N_{r_j}
for step = 1 to stepMax **do**
  Choose a random site on lattice (x, y)
  Backup current field: $\vec{\phi}_{\text{old}} = \vec{\phi}_{x,y}$ and total action $S_{\text{tot(old)}} = S_{\text{tot}}$
  Propose new field: $\phi^{(k)}_{x,y}$ = random unit vector on the sphere
  Update local action S_{ij} for the discretized action of this model we actually only need to update action at three points $S_{x,y}, S_{x-1,y}, S_{x,y-1}$
  recompute total action S_{tot}
  Compute energy difference: $\Delta S = S_{\text{tot}} - S_{\text{tot(old)}}$
  if $\Delta S \le 0$ **then**
    Accept: $\vec{\phi}_{x,y}$
  else
    Draw $r \sim \mathcal{U}_{[0,1]}$
    if $r < \exp(-\Delta S)$ **then**
      Accept: $\vec{\phi}_{x,y}$
    else
      Reject: $\vec{\phi}_{x,y} = \vec{\phi}_{\text{old}}$
    end if
  end if
  if step > stepMin **then**
    $Z = Z + \exp(-S_{\text{tot}})$
    for all possible lattice distances $r_j = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ **do**
      $c_{r_j} = c_{r_j} + \frac{1}{N_{r_j}} \sum_{|p_1 - p_2| = r_j} \left(\sum_k \phi^{(k)}_{j_{x1},j_{y1}} \phi^{(k)}_{j_{x2},j_{y2}}\right) \exp(-S_{\text{tot}})$
    end for
  end if
end for
Calculate $C_{r_j} = \frac{c_{r_j}}{Z}$

3.4 First results

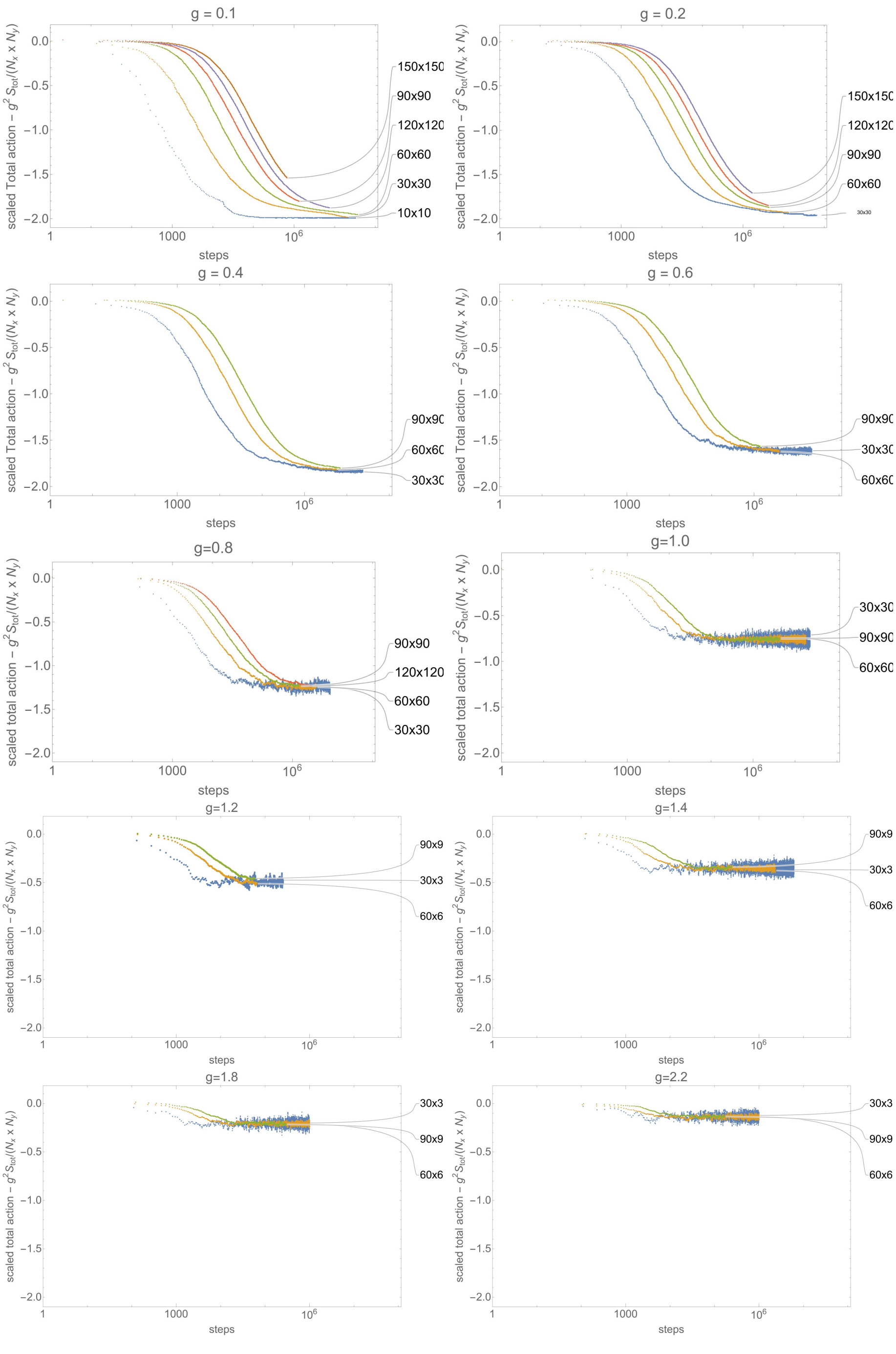


Figure 1: convergence of $g^2 S_{\mathrm{tot}}/(N_x \times N_y)$ for various g and various lattice sizes.

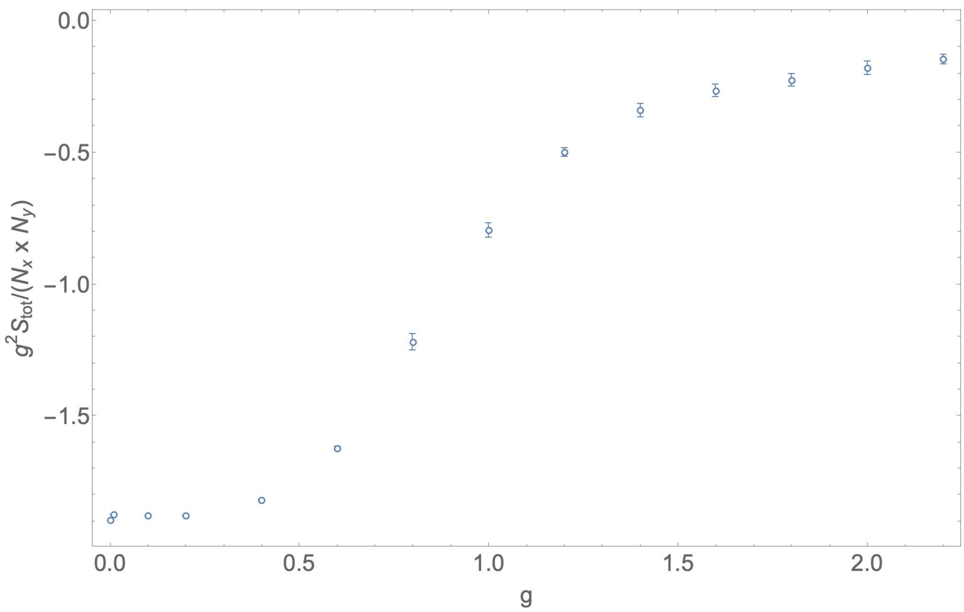


Figure 2: Mean and standard deviation of $g^2 S_{\mathrm{tot}}/(N_x \times N_y)$ at equilibrium (samplesize 1,000 after 10^7 steps) for various g for lattice size (30×30)

3.5 Extract mass gap from $C(r)$

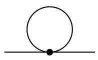
$$C(r) = f(r)e^{-mr} \tag{130}$$

$$\rightarrow \log \frac{C(r)}{C(r+1)} = \log \frac{f(r)e^{-mr}}{f(r)e^{-mr}e^{-m}} = m \tag{131}$$

4 Lectures

4.1 What is lattice field theory and why is it interesting

1. LFT is Quantum Field Theory
2. If you calculate 1-loop Feynman diagrams you can see that they are UV divergent
3. example: ϕ^4 theory



$$\simeq \int \frac{d^4p}{(2\pi)^4} \frac{1}{p_0^2 - \vec{p}^2 - m^2 + i\epsilon} \tag{132}$$

$$\xrightarrow{\text{Wick rotation}} \int \frac{d^4p}{(2\pi)^4} \frac{1}{p_0^2 + \vec{p}^2 + m^2} = \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 + m^2} \sim \int_0^\infty dp\, p^3 \frac{1}{p^2} \to +\infty \tag{133}$$

4. In QFT you need to regularize UV divergencies (i.e. introduce a UV cut-off/regulator)
5. There are multiple ways to do this:
   (a) Hard cutoff (Λ called UV regulator)

   $$\int_{|p|<\Lambda} \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 + m^2} \tag{134}$$

   allowed momentum space is a sphere

   (b) Heat kernel regularization

   $$\int \frac{d^4p}{(2\pi)^4} \frac{e^{-p^2/\Lambda^2}}{p^2 + m^2} \tag{135}$$

   (c) Dimensional regularization - D generic number of dimensions

   $$\int \frac{d^Dp}{(2\pi)^D} \frac{1}{p^2 + m^2} \simeq \int_0^\infty p^{D-1} \frac{1}{p^2 + m^2} \tag{136}$$

   $$\simeq \int_0^\infty p^{D-3} \tag{137}$$

   UV convergent for $D = 4 - 2\epsilon$, ($\epsilon > 1$), so for ϵ large enough the loop diagrams become UV finite.

   (d) The lattice is just another UV regulator

   $$\int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 + m^2} \quad \to \quad \int_{-\pi/a}^{+\pi/a} \frac{d^4p}{(2\pi)^4} \frac{1}{\sum_\mu \left(\frac{2}{a} \sin \frac{p_\mu a}{2}\right)^2 + m^2} \tag{138}$$

   $$\to \quad \lim_{a \to 0} \int_{-\pi/a}^{+\pi/a} \frac{d^4p}{(2\pi)^4} \frac{1}{\sum_\mu \left(\frac{2}{a} \frac{p_\mu a}{2}\right)^2 + m^2} \tag{139}$$

   the lattice constant a is the UV regulator $-\frac{\pi}{a} \leq p_\mu \leq +\frac{\pi}{a}$. Again hard momentum cut-off $\Lambda = \pi/a$ (but here allowed momentum space is a cube not a sphere)

6. Why is it interesting to consider the lattice regulator - isn't it just another hard cutoff?
   (a) Let's say that we are interested in QCD (theory that describes quarks and gluons)
   (b) QCD is a non-abelian gauge theory

(c) Hard spherical cut-off and heat kernel regulatization break non-abelian gauge symmetry - so they are not suitable to study QCD

(d) The only known regulators that do not break non-abelian gauge symmetry are dimensional and lattice regularization (maybe there are others?!)

(e) However dimensional regularization is defined only in conjunction with the perturbative expansion

(f) QCD is supposed to be defined beyond perturbation theory - meaning in a non-perturbative sense

- Simplified universe: only gluons and the up and down quarks (which are massless, in reality their mass is very small)
- then at any order of perturbation theory the mass of the proton is always zero
- so the proton mass is essentially a non-perturbative observable

(g) The lattice regulator is the only known regulator that allows to study QCD at the non-perturbative level

- due to running coupling constant (small at high energy, large at small energies $\sim$ 1GeV) - perturbation theory works well at high energies and non-perturbative methods need be used at low energies

4.2 What does is mean to regularize QFT on a lattice?

4.2.1 Example: ϕ^4-theory - Path integral formulation

- Action in Minkowski spacetime $\eta = \mathrm{diag}(1, \mathbf{-1}, \mathbf{-1}, \mathbf{-1})$:

$$S_{\mathrm{Minkowski}} = \int d^4x \left\{ \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{m^2}{2}\phi(x)^2 - \frac{\lambda}{4!}\phi(x)^4 \right\} \tag{140}$$

$$\mathcal{Z} = \int \mathcal{D}\phi \; e^{iS_{\mathrm{M}}(\phi)} \tag{141}$$

- Expectation values are then defined by

$$\langle P(\phi)\rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\phi \; e^{iS_{\mathrm{M}}} P(\phi) \tag{142}$$

with $\mathcal{D}\phi$ integral over all field configurations, $1/\mathcal{Z}$ normalization and e.g. $P(\phi) = \phi(x)\phi(y)$.

- One can alternatively define an action in Euclidean spacetime $\eta = \mathrm{diag}(1, \mathbf{+1}, \mathbf{+1}, \mathbf{+1})$:

$$S_{\mathrm{Euclid}} = \int d^4x \left\{ \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) + \frac{m^2}{2}\phi(x)^2 + \frac{\lambda}{4!}\phi(x)^4 \right\} \tag{143}$$

$$\mathcal{Z} = \int \mathcal{D}\phi \; e^{-S_{\mathrm{E}}(\phi)} \tag{144}$$

Why Euclidean?

(a) Minkowski n-point functions $\langle\phi(x_1)...\phi(x_2)\rangle_{\mathrm{M}}$ can be obtained from Euclidean n-point functions $\langle\phi(x_1)...\phi(x_2)\rangle_{\mathrm{E}}$ by analytically continuing in the time coordinates

$$(x_k^{(E)})_0 = i(x_k^{(M)})_0 \qquad \text{(Wick rotation)} \tag{145}$$

Euclidean & Minkowski QFT contain the same information - one can go from one to the other using the Wick rotation

(b) The lattice regulator for QFT only truly works in Euclidean space

General algorithm

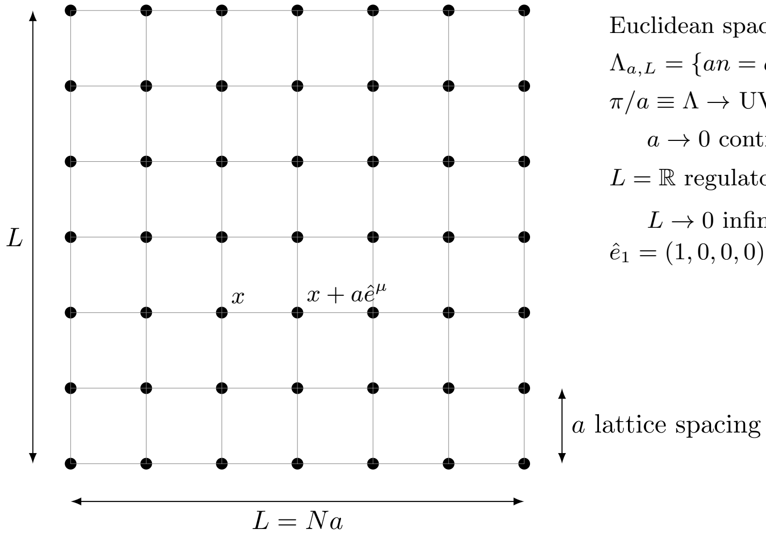
(1) Consider QFT in Euclidean

(2) Regulate using the lattice

(3) Calculate n-point functions

(4) Remove the UV regulator: $a \to 0$ (after renormalization)

(5) Wick rotate to Minkowskian



Euclidean spacetime $\mathbb{R}^4 \to$ finite lattice (regular grid)
$\Lambda_{a,L} = \{an = a(n_1, n_2, n_3, n_4)\}, n_\mu \in \mathbb{Z}, 0 \leq n_\mu < L/a \equiv N$
$\pi/a \equiv \Lambda \to$ UV regulator
 $a \to 0$ continuum limit (UV regulator is removed)
$L = \mathbb{R}$ regulator
 $L \to 0$ infinite volume limit
$\hat{e}_1 = (1,0,0,0), \hat{e}_2 = (0,1,0,0), ...$

Discretized Euclidean Action

(1) The field $\phi(x)$ is only defined for $x \in \Lambda_{a,L}$ (on point that belongs to the lattice)

(2) A (scalar) field configuration = field given at all points = one real number per lattice point = vector with N^D components

(3) Discretized forward derivative (not the only possible approximation)

(4) $\mathcal{C}$ is the space of all field configurations - vector space $\simeq \mathbb{R}^{N^4}$

$$\partial_\mu \phi(x) \to \hat{\partial}^f_\mu \phi(x) \equiv \frac{\phi(x + a\hat{e}_\mu) - \phi(x)}{a} \tag{146}$$

where $\hat{e}_\mu$ are the lattice unit vectors

(5) In order to define ∂^f_μ on a finite lattice (at boundary lattice points $z = Na\hat{e}_\mu$) we need to specify boundary conditions

- Dirichlet boundary conditions: declare $\phi(z + a\hat{e}_1) \equiv 0$
- von Neumann boundary (periodic) conditions: declare $\phi(z + a\hat{e}_1) \equiv \phi(z + a\hat{e}_1 - Na\hat{e}_1)$

(6) Discretized integral (keep in mind Riemann sum approximation $\int_a^b dx f(x) \simeq \sum_{x_i} \Delta x\, f(x_i)$)

$$\int d^4x \to \sum_{x \in \Lambda_{a,L}} a^4 \tag{147}$$

then

$$S_{\mathrm{E}} = \int d^4x \left\{ \frac{1}{2}\partial_\mu \phi(x)\partial^\mu \phi(x) + \frac{m^2}{2}\phi(x)^2 + \frac{\lambda}{4!}\phi(x)^4 \right\} \tag{148}$$

$$= \sum_{x\in\Lambda_{a,N}} a^4 \left\{ \frac{1}{2}\sum_{\mu=1}^{4} \hat{\partial}^f_\mu \phi(x)\hat{\partial}^\mu \phi(x) + \frac{m^2}{2}\phi(x)^2 + \frac{\lambda}{4!}\phi(x)^4 \right\} \tag{149}$$

(7) Using regularized path integral measure $\mathcal{D}\phi \equiv \prod_{x\in\Lambda_{a,N}} d\phi(x)$ so

$$\int \mathcal{D}\phi = \int \underbrace{\prod_{x_k\in\Lambda_{a,N}}}_{\substack{\text{all points of}\\ \text{the spactime lattice}}} \underbrace{d\phi(x_k)}_{\substack{\text{integral over } \mathbb{R}\\ \text{(all field values } \phi \text{ at } x_k)}} \tag{150}$$

this are N^4 nested **(NOT a product of)** integrals over $\mathbb{R}$ (one at each point of the lattice)

(8) To calculate a meaningful expectation value

$$\langle P(\phi)\rangle = \frac{1}{\mathcal{Z}}\int \mathcal{D}\phi\, e^{iS_{\mathrm{M}}} P(\phi) \tag{151}$$

$$\mathcal{Z} = \int \mathcal{D}\phi\, e^{-S_{\mathrm{E}}(\phi)} \tag{152}$$

we need the normalization to work so $\langle 1\rangle = 1$ which requires $\mathcal{Z}$ to be finite.

(9) On a lattice on can prove

$$\left| \int_{\mathcal{C}} \mathcal{D}\phi\, e^{-S_{\mathrm{E}}(\phi)} P(\phi) \right| < +\infty \tag{153}$$

for each $P(\phi)$ which is polynomial in the field.

Proof for $P(1)$:

$$S(\phi) = \sum_{x\in\Lambda_{a,N}} a^4 \left\{ \frac{1}{2}\sum_{\mu=1}^{4} \hat{\partial}^f_\mu \phi(x)\hat{\partial}^\mu \phi(x) + \frac{m^2}{2}\phi(x)^2 + \frac{\lambda}{4!}\phi(x)^4 \right\} \tag{154}$$

$$= \underbrace{\frac{1}{2}\sum_{x\in\Lambda} a^4 \sum_{\mu=1}^{4} \left(\hat{\partial}^f_\mu \phi(x)\right)^2}_{\geq 0} + \underbrace{\frac{m^2}{2}\sum_{x\in\Lambda} a^4 \phi(x)^2}_{\geq 0} + \underbrace{\frac{\lambda}{4!}\sum_{x\in\Lambda} a^4\phi(x)^4}_{\geq 0} \tag{155}$$

$$\geq \frac{m^2}{2}\sum_{x\in\Lambda} a^4\phi(x)^2 \tag{156}$$

$$\to\ e^{-S(\phi)} \leq e^{-\frac{m^2}{2}\sum_{x\in\Lambda} a^4\phi(x)^2} \tag{157}$$

$$\to\ \mathcal{Z} \leq \int \mathcal{D}\phi\, e^{-\frac{m^2}{2}\sum_{x\in\Lambda} a^4\phi(x)^2} \tag{158}$$

$$= \int \left\{ \prod_{x_k\in\Lambda} d\phi(x_k) \right\} \prod_{x_j\in\Lambda} e^{-\frac{m^2}{2}a^4\phi(x_j)^2} \tag{159}$$

as the integrand a product of functions of the field at lattice points $\phi(x_k)$ the nested integrals separates to a product of integrals

$$\mathcal{Z} \leq \prod_{x_k \in \Lambda} \int d\phi(x_k)\, e^{-\frac{m^2}{2} a^4 \phi(x_k)^2} \tag{160}$$

$$= \left(\int dz\; e^{-\frac{m^2 a^4}{2} z^2} \right)^{N^4} \tag{161}$$

$$= \left(\sqrt{\frac{2\pi}{m^2 a^4}} \right)^{N^4} < \infty \tag{162}$$

4.3 QFT is Quantum Mechanics for Fields

Hamiltonian for free scalar field

$$H = \int \frac{d^3p}{(2\pi)^3} E(p) a^\dagger(p) a(p) \qquad\qquad E(p) = \sqrt{m^2 + p^2} \tag{163}$$

derived from the standard classical field theoretical approach

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{164}$$

$$= \frac{1}{2}\partial_0 \phi \partial^0 \phi - \frac{1}{2}\partial_k \phi \partial^k \phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{165}$$

$$\rightarrow \pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi(x))} = \partial_0 \phi(x) \tag{166}$$

$$\rightarrow L = \int d^3x\, \mathcal{L} \tag{167}$$

$$\rightarrow H = \left\{ \frac{1}{2}\pi^2 + \frac{1}{2}\partial_k \phi \partial^k \phi + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \right\} \tag{168}$$

Question: What is the connection between the path integral formalism and the canonical formalism?

4.3.1 Quantum Mechanics Recap

First: Lets consider the case of QM of the point particle in a generic number of dimensions:

- States $\equiv$ wave functions $\psi(q) : \mathbb{R}^D \rightarrow \mathbb{C}$
- Hilbert space $\mathcal{H} = \{\psi \text{ with } ||\psi|| = \int d^Dq\, |\psi|^2 < +\infty\} \equiv L^2(\mathbb{R}^2)$
- Position operators $\hat{q}_k \psi(q) = q_k \psi(q)$ with $k = 1, ..., D$
- Momentum operators $\hat{p}_k \psi(q) = -i\partial_{q_k}\psi(q)$
- State $|\psi\rangle$
  - Eigenstate of $\hat{q}$: $|q\rangle$ with $\hat{q}|q\rangle = q|q\rangle$
  - Eigenstate of $\hat{p}$: $|p\rangle$
  - $\psi(q) \equiv \langle q|\psi\rangle \quad \rightarrow \quad \hat{q}\psi(q) = \hat{q}\langle q|\psi\rangle = q\langle q||\psi\rangle = q\psi(q)$
- $[\hat{p}_j, \hat{q}_k] = -i\delta_{ij}$ and $[\hat{q}_j, \hat{q}_k] = 0 = [\hat{p}_j, \hat{p}_k]$
- One more important operator $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q})$
  - Defines time evolution $i\partial_t|\psi_t\rangle = \hat{H}|\psi_t\rangle$

- Solution $|\psi_t\rangle = e^{i\hat{H}t}|\psi_t\rangle$ with time evolution operator $\hat{U}(t) = e^{-i\hat{H}t}$
- Initial wave function $\psi_0(q^{(i)})$, final wave function $\psi_t(q^{(f)})$

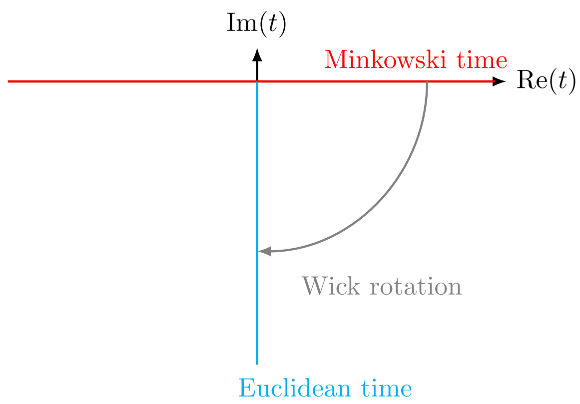
$$\psi_t(q^{(f)}) = \langle q^{(f)}|\psi_t\rangle \tag{169}$$

$$= \langle q^{(f)}|e^{-i\hat{H}t}|\psi_t\rangle \tag{170}$$

$$= \int d^D q^{(i)} \, \langle q^{(f)}|e^{-i\hat{H}t}|q^{(i)}\rangle\langle q^{(i)}|\psi_t\rangle \qquad \text{inserting identity } \; 1 = \int d^D q \, |q\rangle\langle q| \tag{171}$$

$$= \int d^D q^{(i)} \, \underbrace{\langle q^{(f)}|e^{-i\hat{H}t}|q^{(i)}\rangle}_{\text{Schroedinger kernel}} \psi_0(q^{(i)}) \tag{172}$$

- Minkowski time evolution $\langle q^{(f)}|e^{-i\hat{H}t}|q^{(i)}\rangle$
- Euclidean time evolution $t \to -it$ results in $\langle q^{(f)}|e^{-\hat{H}t}|q^{(i)}\rangle$
- **Theorem**: $\langle q^{(f)}|e^{-i\hat{H}t}|q^{(i)}\rangle$ is an analytic function as for complex t, as long as $\mathrm{Im}\, t < 0$



4.3.2 Path Integral in Quantum Mechanics

With the simple Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q}) = K(\hat{q}) + V(\hat{q}) \tag{173}$$

we obtain

$$e^{-i\hat{H}t} = e^{-i(\hat{K}+\hat{V})t} \neq e^{-i\hat{K}t}e^{-i\hat{V}t} \tag{174}$$

$$\lim_{N\to\infty} \left(e^{-i\hat{H}\frac{t}{N}}\right)^N = \lim_{N\to\infty} \left(e^{-i\frac{t}{2N}\hat{V}} e^{-i\frac{t}{N}\hat{K}} e^{-i\frac{t}{2N}\hat{V}}\right)^N \qquad \text{Trotters formula} \tag{175}$$

$$\to e^{-i\hat{H}\frac{t}{N}} = e^{-i\frac{t}{2N}\hat{V}} e^{-i\frac{t}{N}\hat{K}} e^{-i\frac{t}{2N}\hat{V}} e^{\mathcal{O}(1/N^2)} \qquad \text{for large } N \tag{176}$$

With $\tau = t/N$ and

$$\hat{T}(\tau) \equiv e^{-i\frac{\tau}{2}\hat{V}} e^{-i\tau\hat{K}} e^{-i\frac{\tau}{2}\hat{V}} = e^{-i\hat{H}\frac{t}{N}} \tag{177}$$

we can write

$$\langle q^{(f)}|e^{-i\hat{H}t}|q^{(i)}\rangle = \lim_{N\to\infty} \langle q^{(f)}| \left(\hat{T}(\tau)\right)^N |q^{(i)}\rangle \tag{178}$$

$$= \lim_{N\to\infty} \langle \underbrace{q^{(f)}}_{q^{(N\tau)}} | \underbrace{\hat{T}(\tau)\hat{T}(\tau)\cdots\hat{T}(\tau)\hat{T}(\tau)}_{N-\text{times}} | \underbrace{q^{(i)}}_{q^{(0\tau)}} \rangle \tag{179}$$

inserting $1 = \int d^D q^{(k\tau)} |q^{(k\tau)}\rangle\langle q^{(k\tau)}|$ between all $\hat{T}$

$$\langle q^{(f)}|e^{-i\hat{H}t}|q^{(i)}\rangle = \lim_{N\to\infty} \int \cdots \int d^D q^{((N-1)\tau)} \cdots d^D q^{(1\tau)} \tag{180}$$

$$\langle q^{(N\tau)}|\hat{T}(\tau)|q^{((N-1)\tau)}\rangle\langle q^{((N-1)\tau)}|\hat{T}(\tau)\cdots\hat{T}(\tau)|q^{(1\tau)}\rangle\langle q^{(1\tau)}|\hat{T}(\tau)|q^{(0\tau)}\rangle \tag{181}$$

$$= \lim_{N\to\infty} \int \underbrace{\left[\prod_{n=1}^{N-1} d^D q^{(n\tau)}\right]}_{\text{lattice path integral measure}} \left[\prod_{k=0}^{N-1} \langle q^{((k+1)\tau)}|\hat{T}(\tau)|q^{(k\tau)}\rangle\right] \tag{182}$$

Now we investigate the generic transition matrix element

$$\langle q'|\hat{T}(\tau)|q\rangle = \langle q'|e^{-i\frac{\tau}{2}\hat{V}} e^{-i\tau\hat{K}} e^{-i\frac{\tau}{2}\hat{V}}|q\rangle \tag{183}$$

$$= \langle q'|(1 - i\frac{\tau}{2}\hat{V} + ...)e^{-i\tau\hat{K}}(1 - i\frac{\tau}{2}\hat{V} + ..)|q\rangle \tag{184}$$

$$= (1 - i\frac{\tau}{2}V + ...)\langle q'|e^{-i\tau\hat{K}}|q\rangle(1 - i\frac{\tau}{2}V + ..) \tag{185}$$

$$\text{with } \hat{V}|q\rangle = V(\hat{q})|q\rangle = \sum_k (\partial_q^{(k)} V)\hat{q}^k|q\rangle = \sum_k (\partial_q^{(k)} V)q^k|q\rangle = V(q)\rangle \tag{186}$$

$$= e^{-i\frac{\tau}{2}V(q')}\langle q'|e^{-i\tau\hat{K}}|q\rangle e^{-i\frac{\tau}{2}V(q)} \tag{187}$$

Now inserting the identity and using the momentum eigenstates

$$\langle q'|e^{-i\tau\hat{K}}|q\rangle = \int \frac{d^D p}{(2\pi)^D}\langle q'|p\rangle\langle p|e^{-i\tau\hat{K}}|q\rangle = \int \frac{d^D p}{(2\pi)^D}\langle q'|p\rangle\langle p|e^{-i\tau\frac{\hat{p}^2}{2m}}|q\rangle \tag{188}$$

$$= \int \frac{d^D p}{(2\pi)^D}\langle q'|p\rangle e^{-i\tau\frac{p^2}{2m}}\langle p|q\rangle = \int \frac{d^D p}{(2\pi)^D} e^{iq'p} e^{-i\tau\frac{p^2}{2m}} e^{-ipq} \tag{189}$$

$$= \int \frac{d^D p}{(2\pi)^D} e^{-i\tau\frac{p^2}{2m} + ip(q'-q)} = \int \prod_{k=1}^{D} \frac{dp}{(2\pi)} e^{-i\tau\frac{p_k^2}{2m} + ip_k(q'_k - q_k)} \tag{190}$$

Solving 1-D integral by completing the square

$$\int \frac{dp}{(2\pi)} e^{-i\tau\frac{p^2}{2m} + ip\,\Delta q} = \int \frac{dp}{(2\pi)} e^{-\frac{i\tau}{2m}(p^2 - 2\frac{m\,\Delta q}{\tau}p)} = \int \frac{dp}{(2\pi)} e^{-\frac{i\tau}{2m}\left(p^2 - 2\frac{m\,\Delta q}{\tau}p + \frac{m^2\,\Delta q^2}{\tau^2} - \frac{m^2\,\Delta q^2}{\tau^2}\right)} \tag{191}$$

$$= \int \frac{dp}{(2\pi)} e^{-\frac{i\tau}{2m}(p^2 - \frac{m\,\Delta q}{\tau})^2 + \frac{i\tau}{2m}\frac{m^2\,\Delta q^2}{\tau^2}} = \int \frac{dp}{(2\pi)} e^{-\frac{i\tau}{2m}(p^2 - \frac{m\,\Delta q}{\tau})^2} e^{i\frac{m\,\Delta q^2}{2\tau}} \tag{192}$$

$$= e^{i\frac{m\,\Delta q^2}{2\tau}} \int \frac{d\xi}{(2\pi)} e^{-\frac{\tau}{2m}i\xi^2} \overset{i\xi^2 \to \zeta^2}{=} e^{i\frac{m\,\Delta q^2}{2\tau}} \int \sqrt{-i}\frac{d\zeta}{(2\pi)} e^{-\frac{\tau}{2m}\zeta^2} \tag{193}$$

$$= e^{i\frac{m}{2\tau}\Delta q^2} e^{-i\frac{\pi}{4}} \sqrt{\frac{2m}{\tau}}\sqrt{\pi} \qquad (\text{for } \tau > 0, t > 0) \tag{194}$$

Then (is there a power of D missing??? - or are we staying in $D = 1$)

$$\langle q'|\hat{T}(\tau)|q\rangle = e^{-i\frac{\tau}{2}V(q')}\langle q'|e^{-i\tau\hat{K}}|q\rangle e^{-i\frac{\tau}{2}V(q)} \tag{195}$$

$$= e^{-i\frac{\tau}{2}V(q')}\left(e^{i\frac{m}{2\tau}(q'-q)^2} e^{-i\frac{\pi}{4}}\sqrt{\frac{m}{2\pi\tau}}2\pi\right)e^{-i\frac{\tau}{2}V(q)} \tag{196}$$

(197)

Then collecting all together (with $\partial_s^f q(s) = \frac{q(s+\tau)-q(s)}{\tau}$

$$\langle q^{(f)}|e^{-i\hat{H}t}|q^{(i)}\rangle = \lim_{N\to\infty}\left(e^{-i\frac{\pi}{4}}\sqrt{\frac{m}{2\pi\tau}}\right)^N \int \left[\prod_{n=1}^{N-1} d^D q^{(n\tau)}\right] \prod_{k=0}^{N-1}\left[e^{-i\frac{\tau}{2}V(q(s+\tau))} e^{i\frac{m}{2}\tau\frac{(q(s+\tau)-q(s))^2}{\tau^2}} e^{-i\frac{\tau}{2}V(q(s))}\right]_{s=k\tau} \tag{198}$$

$$= \lim_{N\to\infty}\left(e^{-i\frac{\pi}{4}}\sqrt{\frac{m}{2\pi\tau}}\right)^N \int \left[\prod_{n=1}^{N-1} d^D q^{(n\tau)}\right] \exp\Bigg[i\tau\sum_{k=0}^{N-1}\frac{m}{2}\frac{(q(s+\tau)-q(s))^2}{\tau^2} - \frac{1}{2}(V(q(s+\tau)) + V(\tag{199}$$

$$= \lim_{N\to\infty}\left(e^{-i\frac{\pi}{4}}\sqrt{\frac{m}{2\pi\tau}}\right)^N \int \left[\prod_{n=1}^{N-1} d^D q^{(n\tau)}\right] \exp\left[i\int_0^t \frac{m}{2}\dot{q}(s)^2 - V(q(s))\right] \tag{200}$$

$$= \lim_{N\to\infty}\left(e^{-i\frac{\pi}{4}}\sqrt{\frac{m}{2\pi\tau}}\right)^N \int \left[\prod_{s\in\Lambda_t} d^D q(s)\right] \exp\left[iS_M(q)\right]_{q(0)=q^{(i)},q(t)=q^{(f)}} \tag{201}$$

- **Exercise 1**: Derive the path-integral formula for

$$\langle q^{(f)}|e^{itH}|q^{(i)}\rangle \qquad \text{with } t<0 \tag{202}$$

using the observation

$$\langle q^{(f)}|e^{-itH}|q^{(i)}\rangle^* = \langle q^{(f)}|e^{itH}|q^{(i)}\rangle = \langle q^{(f)}|e^{-itH}|q^{(i)}\rangle \tag{203}$$

- **Exercise 2**: Calculate

$$\langle q'|e^{-\tau\hat{K}}|q\rangle \tag{204}$$

Euclidean evolution operator $e^{-t\hat{H}}$

$$\langle q^{(f)}|e^{-t\hat{H}}|q^{(i)}\rangle = \lim_{N\to\infty}\langle q^{(f)}|(\hat{T}(\tau))^N|q^{(i)}\rangle \tag{205}$$

$$= \lim_{N\to\infty}\langle q^{(f)}|(e^{-\frac{\tau}{2}\hat{V}}e^{-\tau\hat{K}}e^{-\frac{\tau}{2}\hat{V}})^N|q^{(i)}\rangle \tag{206}$$

Define effective Hamiltonian ($t = N\tau$)

$$\lim_{N\to\infty}(\hat{T}(\tau))^N = e^{-t\hat{H}} \tag{207}$$

$$\hat{T}(\tau) = e^{-\tau\hat{H}_{\text{eff}}(\tau)} \tag{208}$$

$$\to \hat{H}_{\text{eff}}(\tau) = -\frac{1}{\tau}\log\hat{T}(\tau) \tag{209}$$

$$\langle q^{(f)}|e^{-\hat{H}t}|q^{(i)}\rangle = \lim_{N\to\infty}\left(\sqrt{\frac{m}{2\pi\tau}}\right)^{N\cdot D}\int\left[\prod_{s\in\Lambda_t} d^D q(s)\right]\exp\left[-S_E(q)\right]_{q(0)=q^{(i)},q(t)=q^{(f)}} \tag{210}$$

$$S_E(q) = \sum_{k=0}^{N-1}\tau\left[\frac{m}{2}|\partial_s^f q(s)|^2 + \frac{1}{2}V(q(s)) + \frac{1}{2}V(q(s+\tau))\right]_{s=k\tau} \tag{211}$$

4.3.3 ϕ^4-theory

Hamiltonian and canonical commutation relations are given by ($x = (x_o, \vec{x})$)

$$\hat{H} = \int d^3x\left[\frac{1}{2}\hat{\pi}(\vec{x})^2 + \frac{1}{2}\sum_{k=1}^{3}[\partial_k\hat{\phi}(\vec{x})]^2 + \frac{m^2}{2}\hat{\phi}(\vec{x})^2 + \frac{\lambda}{4!}\hat{\phi}(\vec{x})^4\right] \tag{212}$$

$$[\hat{\phi}(\vec{x}), \hat{\pi}(\vec{x})] = i\delta^{(3)}(\vec{x}-\vec{y}) \tag{213}$$

When we discretize the theory on the lattice we need a $3d$-lattice (because there is NO time in the Hamiltonian or the commutation relations

$$\Lambda_s = a(n_1, n_2, n_3) |\, 0 \leq n_k < N_s\} \tag{214}$$

$$\hat{H} = \sum_{\vec{x}\in\Lambda_s} a^3 \left[\frac{1}{2}\hat{\pi}(\vec{x})^2 + \frac{1}{2}\sum_{k=1}^{3}[\partial_k^f \hat{\phi}(\vec{x}) + \frac{m^2}{2}\hat{\phi}(\vec{x})^2 + \frac{\lambda}{4!}\hat{\phi}(\vec{x})^4\right] \tag{215}$$

$$\partial_k^f \hat{\phi}(\vec{x}) = \frac{\hat{\phi}(\vec{x}+a\hat{e}_k) - \hat{\phi}(\vec{x})}{a} \tag{216}$$

$$[\hat{\phi}(\vec{x}), \hat{\pi}(\vec{x})] = ia^{-3}\delta_{\vec{x},\vec{y}} = ia^{-3}\delta_{n_1,m_1}\delta_{n_2,m_2}\delta_{n_3,m_3} \tag{217}$$

$$\vec{x}, \vec{y} \in \Lambda_s \tag{218}$$

with $[\hat{\phi}] = \text{mass}^1$, $[\hat{\pi}] = \text{mass}^2$, $[\delta^{(3)}(\vec{x}-\vec{y})] = \text{mass}^3$. It is sufficient to index $\vec{x}$ with ONE index counter $\vec{x} \simeq x_k$ with $k = 1, ..., N_s^3$ (and not 3).

In terms of QM - ϕ^4-theory is a QM point particle in $D = N_s^3$ dimensions in an external potential $V(\hat{\phi})$

$$\hat{H} = \sum_{\vec{x}\in\Lambda_s} \frac{1}{2a^3}\hat{p}(\vec{x})^2 + \sum_{\vec{x}\in\Lambda_s} a^3 \left[\frac{1}{2}\sum_{k=1}^{3}[\partial_k^f \hat{\phi}(\vec{x}) + \frac{m^2}{2}\hat{\phi}(\vec{x})^2 + \frac{\lambda}{4!}\hat{\phi}(\vec{x})^4\right] \tag{219}$$

$$= \sum_{\vec{x}\in\Lambda_s} \frac{1}{2a^3}\hat{p}(\vec{x})^2 + V(\hat{\phi}) \tag{220}$$

$$\rightarrow \langle \phi^{(f)}| e^{-\hat{H}t} |\phi^{(i)}\rangle = \lim_{N_t\to\infty} \left(\frac{a^3}{2\pi\tau}\right)^{N_t N_s^3/2} \int \prod_{x_0\in\Lambda_t}\prod_{\vec{x}\in\Lambda_s} d\phi(x_o,\vec{x})\; e^{-S_E(\phi)} \tag{221}$$

$$S_E = \sum_{k=0}^{N-1}\sum_{\vec{x}\in\Lambda_s} \tau a^3 \left\{\frac{1}{2}(\partial_{x_o}^f \phi(\vec{x}))^2 + \sum_{\vec{x}\in\Lambda_s} a^3 \left[\frac{1}{2}\sum_{k=1}^{3}[\partial_k^f \phi(\vec{x})]^2 + \frac{m^2}{2}\phi(\vec{x})^2 + \frac{\lambda}{4!}\phi(\vec{x})^4\right]\right\}_{x_o=k\tau} \tag{222}$$

Common choice: $a = \tau$ (allowed because eventually we want to take $a, \tau \to 0$). Then for $t > 0$

$$\langle \phi^{(f)}| e^{-\hat{H}\frac{t}{\tau}} |\phi^{(i)}\rangle = \lim_{N_t\to\infty, \tau=t/N_t} = C_{a,N_s^3 N_t} \int [d\phi]_{(0,t)} e^{-S_{[0,t)}(\phi)}|_{\phi(0,\vec{x})=\phi^{(i)}(\vec{x}),\phi(t,\vec{x})=\phi^{(f)}(\vec{x})} \tag{223}$$

with

$$[d\phi]_{(0,t)} = \prod_{x_o=\tau}^{t-\tau}\prod_{\vec{x}\in\Lambda_s} d\phi(x_o,\vec{x}) \tag{224}$$

$$S_{[0,t)}(\phi) = \sum_{x_o=0}^{t-\tau} \tau a^3 \left(\frac{1}{2}(\partial_0^f\phi)^2(x) + \frac{V(x)+V(x+\tau\hat{e}_0)}{2}\right) \tag{225}$$

$$V(x) = \frac{1}{2}\sum_{k=1}^{3}(\partial_k^f\phi)^2(x) + \frac{m^2}{2}\phi(x)^2 + \frac{\lambda}{4!}\phi(x)^4 \tag{226}$$

4.3.4 Thermal QFT - QFT in thermodynamic equilibrium

- Blackbody radiation = phontons/QED in thermodynamic equilibrium
- Free energy F of a quantum system in thermal equilirium with temperature $\mathcal{T}$ (with $k_B = 1$)

$$e^{-\frac{F}{\mathcal{T}}} = \operatorname{tr} e^{-\frac{\hat{H}}{\mathcal{T}}} \tag{227}$$

$$\rightarrow F = -\mathcal{T}\log\operatorname{tr} e^{-\frac{\hat{H}}{\mathcal{T}}} \tag{228}$$

$$\rightarrow Z = \operatorname{tr} e^{-\frac{\hat{H}}{\mathcal{T}}} = \operatorname{tr} e^{-T\hat{H}} \tag{229}$$

where Euclidean time $T = 1/\mathcal{T}$ (inverse temperature)

4.3.5 Path integral formula for thermal partition function

$$\begin{aligned}
Z(T) &= \mathrm{tr}\, e^{-T\hat{H}} && (230)\\
&= \sum \langle\psi|e^{-T\hat{H}}|\psi\rangle && (231)\\
&= \sum \int [\prod_{x\in\Lambda_s} d\phi(\vec{x})]\langle\psi|\phi\rangle\langle\phi|e^{-T\hat{H}}|\psi\rangle && (232)\\
&= \int [\prod_{\vec{x}\in\Lambda_s} d\phi(\vec{x})]\langle\phi|e^{-T\hat{H}}|\phi\rangle && (233)
\end{aligned}$$

discrete version

$$\begin{aligned}
Z(T) &= \mathrm{tr}\, [\hat{T}(\tau)]^{T/\tau} && (234)\\
&= \int [\prod_{\vec{x}\in\Lambda_s} d\phi(\vec{x})]\langle\phi|[\hat{T}(\tau)]^{T/\tau}|\phi\rangle && (235)\\
&= C_{a,N_tN_s^3}\int [\prod_{\vec{x}\in\Lambda_s} d\phi(\vec{x})][\prod_{x_0=\tau}^{T-\tau}\prod_{x\in\Lambda_s} d\phi(x_0,\vec{x})]e^{-S_{[0,T)}(\phi))}|_{\phi(0,\vec{x})=\phi(T,\vec{x})=\phi(\vec{x})} && (236)\\
&= C_{a,N_tN_s^3}\int [\prod_{x_0=0}^{T-\tau}\prod_{x\in\Lambda_s} d\phi(x_0,\vec{x})]e^{-S_{[0,T)}(\phi))}|_{\phi(0,\vec{x})=\phi(T,\vec{x})} && (237)
\end{aligned}$$

4.3.6 Thermal time-ordered n-point function

- **Recall that**: at this point we have
  - states $|\phi\rangle$
  - operators $\hat{\phi}(\vec{x}), \hat{\pi}(\vec{x}), \hat{H}$

  which do NOT depend on time
- and then we have the evolution operator e^{-itH} or e^{-tH}
- **NOW:** operators in Heisenberg representation (time dependence get attached to the operator)
  - Minkowskian

$$\hat{O}(t) = e^{i\hat{H}t}\hat{O}e^{-i\hat{H}t} \tag{238}$$

$$\hat{\phi}(x_0,\vec{x}) = e^{i\hat{H}x_0}\hat{\phi}(\vec{x})e^{-i\hat{H}x_0} \tag{239}$$

  - Euclidean

$$\hat{O}(t) = e^{\hat{H}t}\hat{O}e^{-\hat{H}t} \tag{240}$$

$$\hat{\phi}(x_0,\vec{x}) = e^{\hat{H}x_0}\hat{\phi}(\vec{x})e^{i\hat{H}x_0} \tag{241}$$

1. Lowest energy levels can be extracted from from the exponential decay of the connected (vacuum) 2-point functions in Euclidean time - In practise this method is used to extract masses in lattice simulations
2. Energy levels are poles in the complex p_0-plane of connected 2-point functions (in momentum space)

4.3.7 Free scalar theory

- Discretized action (with periodic boundary conditions)

$$S = \sum_x a^4 \left\{ \frac{1}{2} \sum_\mu \partial^f_\mu \phi(x) \partial^f_\mu \phi(x) + \frac{m^2}{2} \phi(x)^2 \right\} \tag{242}$$

$$Z(0) = \int_{\mathcal{C}} d\phi \, e^{-S} \tag{243}$$

$$\langle \phi(x_1) ... \phi(x_n) \rangle = \frac{1}{Z} \int_{\mathcal{C}} d\phi \, e^{-S} \phi(x_1) ... \phi(x_n) \tag{244}$$

on configuration space $\mathcal{C} = \mathbb{R}^{N_t \cdot N_s^3}$.

- 1 dimension ($\oplus \equiv$ sum modulo Na)

$$\sum_{x/a=0}^{N-1} \partial^f_1 \phi(x) \chi(x) = \sum_{x/a=0}^{N-1} \frac{\phi(x \oplus a) - \phi(x)}{a} \chi(x) \tag{245}$$

$$= \frac{1}{a} \sum_{x/a=0}^{N-1} \phi(x \oplus a) \chi(x) - \frac{1}{a} \sum_{x/a=0}^{N-1} \phi(x) \chi(x) \tag{246}$$

$$\stackrel{x \to x \ominus a}{=} \frac{1}{a} \sum_{x/a=1}^{N} \phi(x) \chi(x \ominus a) - \frac{1}{a} \sum_{x/a=0}^{N-1} \phi(x) \chi(x) \tag{247}$$

$$= \sum_{x/a=1}^{N-1} \phi(x) \frac{\chi(x \oplus a) - \chi(x)}{a} \tag{248}$$

$$= - \sum_{x/a=0}^{N-1} \phi(x) \partial^b_1 \chi(x) \tag{249}$$

or in short with $(\phi, \chi) \equiv \sum_x \phi(x) \chi(x)$

$$(\partial^f_\mu \phi, \chi) = -(\phi, \partial^b_\mu \chi) \tag{250}$$

$$\to (\partial^f_\mu)^\dagger = -\partial^b_\mu \tag{251}$$

- Then we can rewrite the action as (using the discretized Laplacian $\hat{\Box} = \sum_\mu \partial^b_\mu \partial^f_\mu$)

$$S = \frac{a^4}{2} \sum_\mu (\partial^f_\mu \phi, \partial^f_\mu \phi) + \frac{m^2 a^4}{2} (\phi, \phi) \tag{252}$$

$$= \frac{a^4}{2} \sum_\mu (\phi, -\partial^b_\mu \partial^f_\mu \phi) + \frac{m^2 a^4}{2} (\phi, \phi) \tag{253}$$

$$= \frac{a^4}{2} \sum_\mu (\phi, -\hat{\Box} \phi) + \frac{m^2 a^4}{2} (\phi, \phi) \tag{254}$$

$$= \frac{a^4}{2} \sum_\mu (\phi, (-\hat{\Box} + m^2) \phi) \tag{255}$$

- Define a generating functional

$$Z(J) = \int_{\mathcal{C}} d\phi \, e^{-S + \sum_x a^4 J(x) \phi(x)} \tag{256}$$

and using the result of exercise 2 (sheet 1)

$$Z(J) = \int d\phi e^{-\frac{1}{2}(\phi,A\phi)+(a^4J,\phi)} = \frac{(2\pi)^{N_tN_s^3/2}}{\sqrt{\det A}} e^{\frac{1}{2}(a^4J,A^{-1}a^4J)} \tag{257}$$

$$\rightarrow A = a^4(-\hat{\Box} + m^2) \tag{258}$$

we obtain

$$Z(J) = \left(\frac{2\pi}{a^4}\right)^{N_tN_s^3/2} \frac{1}{\sqrt{\det(-\hat{\Box}+m^2)}} e^{\frac{a^4}{2}\left(J,(-\hat{\Box}+m^2)^{-1}J\right)} \tag{259}$$

- and therefore

$$\langle\phi(x_1)...\phi(x_n)\rangle = \frac{1}{Z(J)}\frac{1}{a^4}\frac{\partial}{\partial J(x_1)}...\frac{1}{a^4}\frac{\partial}{\partial J(x_n)} Z(J)\Big|_{J=0} \tag{260}$$

$$= \frac{1}{a^4}\frac{\partial}{\partial J(x_1)}...\frac{1}{a^4}\frac{\partial}{\partial J(x_n)} e^{\frac{a^4}{2}\left(J,(-\hat{\Box}+m^2)^{-1}J\right)}\Big|_{J=0} \tag{261}$$

$$= \frac{1}{a^4}\frac{\partial}{\partial J(x_1)}...\frac{1}{a^4}\frac{\partial}{\partial J(x_n)} e^{\frac{a^4}{2}\sum_{xy} J(x)(-\hat{\Box}+m^2)^{-1}(x,y)J(y)}\Big|_{J=0} \tag{262}$$

$$\rightarrow \langle\phi(x)...\phi(y)\rangle = \frac{1}{a^4}\frac{\partial}{\partial J(x)}\frac{1}{a^4}\frac{\partial}{\partial J(y)} e^{\frac{a^4}{2}\sum_{xy} J(x)(-\hat{\Box}+m^2)^{-1}(x,y)J(y)}\Big|_{J=0} \tag{263}$$

$$= ... \tag{264}$$

$$= \frac{1}{a^4}(-\hat{\Box}+m^2)^{-1}(x,y) \tag{265}$$

- Observation: We can diagonalize $(-\hat{\Box}+m^2)$ meaning we can express it in terms of
  - eigenvectors: plane waves $\psi_p(x) = e^{ixp}$ with periodic boundary conditions

$$\psi_p(x + aN_te_0) = \psi_p(x) \quad \rightarrow \quad e^{iaN_tp_0} = 1 \quad \rightarrow \quad p_0 = \frac{2\pi}{aN_t}\mathbb{Z} \tag{266}$$

$$\psi_p(x + aN_se_k) = \psi_p(x) \quad \rightarrow \quad e^{iaN_sp_k} = 1 \quad \rightarrow \quad p_k = \frac{2\pi}{aN_t}\mathbb{Z} \tag{267}$$

  - eigenvalues

- $\hat{\Box}$ is a $N_tN_s^3 \times N_tN_s^3$ matrix with $N_tN_s^3$ independent eigenvectors

$$\psi_p = e^{i(p+\frac{2\pi}{a}e_\mu)x} \tag{268}$$

$$p = \left(\frac{2\pi}{aN_t}k_0, \frac{2\pi}{aN_s}k_1, \frac{2\pi}{aN_s}k_2, \frac{2\pi}{aN_s}k_3\right) \qquad k_\mu \in \mathbb{Z} \tag{269}$$

$$\rightarrow -\frac{\pi}{a} < p_\mu \leq \frac{\pi}{a} \tag{270}$$

so there are $N_tN_s^3$ possible values of momenta

- then

$$(\psi_p, \psi_{p'}) = \sum_x (e^{ixp})^*(e^{ixp'}) = \sum_x e^{ix(p'-p)} \tag{271}$$

$$= \sum_x e^{i\sum_\mu x_\mu(p'_\mu - p_\mu)} = \prod_\mu \sum_{x_\mu/a=0}^{N_\mu-1} e^{i\frac{x_\mu}{a}a(p'_\mu-p_\mu)} \tag{272}$$

$$= ... = N_tN_s^3\delta_{pp'} \tag{273}$$

- and

$$\hat{\Box}\psi_p(x) = \sum_\mu \partial^b_\mu \partial^f_\mu \psi_p(x) \tag{274}$$

$$= \sum_\mu \frac{\psi_p(x \oplus ae_\mu) + \psi_p(x \ominus ae_\mu - 2\psi_p(x))}{a^2} = \sum_\mu \frac{e^{ip(x+ae_\mu)} + e^{ip(x-ae_\mu)} - 2e^{ipx}}{a^2} \tag{275}$$

$$= \psi_p(x) \sum_\mu \frac{e^{iap_\mu} + e^{-iap_\mu} - 2}{a^2} = \psi_p(x) \sum_\mu \frac{(e^{iap_\mu} - e^{-iap_\mu})^2}{a^2} \tag{276}$$

$$= \psi_p(x) \sum_\mu \frac{(2i\sin(ap_\mu/2))^2}{a^2} = -\psi_p(x) \sum_\mu \frac{4}{a^2} \sin^2 \frac{ap_\mu}{2} \tag{277}$$

$$\overset{a \to 0}{\to} -\psi_p \sum_\mu \frac{4}{a^2} \left(\frac{ap_\mu}{2}\right)^2 = -\psi_p \sum_\mu p_\mu^2 \tag{278}$$

- Using completeness relation regarding the orthonormal basis of eigenvectors $\{\psi_p/\sqrt{N_t N_s^3}\}$

$$(-\hat{\Box} + m^2)^{-1} = (-\hat{\Box} + m^2)^{-1} I \tag{279}$$

$$= (-\hat{\Box} + m^2)^{-1} \sum_p \frac{\psi_p}{\sqrt{N_t N_s^3}} \frac{\psi_p^\dagger}{\sqrt{N_t N_s^3}} \tag{280}$$

$$= \frac{1}{N_t N_s^3} \sum_p \frac{\psi_p \psi_p^\dagger}{\sum_\mu \frac{4}{a^2} \sin^2 \frac{ap_\mu}{2} + m^2} \tag{281}$$

$$(-\hat{\Box} + m^2)^{-1}(x,y) = \frac{1}{N_t N_s^3} \sum_p \frac{e^{ip(x-y)}}{\sum_\mu \frac{4}{a^2} \sin^2 \frac{ap_\mu}{2} + m^2} \tag{282}$$

this means the 2-point function can be written as

$$\to \langle \phi(x) ... \phi(y) \rangle = \frac{1}{a^4} (-\hat{\Box} + m^2)^{-1}(x,y) \tag{283}$$

$$= \frac{1}{TL^3} \sum_p \frac{e^{ip(x-y)}}{\sum_\mu \frac{4}{a^2} \sin^2 \frac{ap_\mu}{2} + m^2} \tag{284}$$

- **Exercise 1**: Calculate the $T \to \infty$ limit of $\langle \phi(x)...\phi(y) \rangle$ (this is related to the vacuum expectation value of the product of two fields)
- **Exercise 2**: Go to momentum space in the temporal coordinate (i.e you can set $y = 0$, $x_0 \to p_0$)
- **Exercise 3**: Find the poles in the complex p_0 place and check that they have the general structure discussed in the previous lecture (in particular, write energy levels)

4.4 Scalar QED

Complex scalar field $\phi(x) \in \mathbb{C}$ and real 4-component photon field $A_\mu(x) \in \mathbb{R}$

$$S = \int d^4x \left\{ \frac{1}{4e^2} F_{\mu\nu}^2 + \sum +\mu |D_\mu \phi|^2 + m^2 |\phi|^2 + \frac{\lambda}{4} |\phi|^4 \right\} \tag{285}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{286}$$

$$D_\mu = \partial_\mu + iA_\mu \tag{287}$$

Gauge symmetry

$$\phi(x) \to e^{i\alpha(x)}\phi(x) \tag{288}$$
$$\phi(x)^\dagger \to e^{-i\alpha(x)}\phi(x)^\dagger \tag{289}$$
$$A_\mu(x) \to A_\mu(x) - \partial_\mu\alpha(x) \tag{290}$$
$$D_\mu\phi(x) = \partial_\mu\phi + iA_\mu(x)\phi(x) \tag{291}$$
$$\to \partial_\mu\left(e^{i\alpha(x)}\phi(x)\right) + i\left(A_\mu(x) - \partial_\mu\alpha(x)\right)e^{i\alpha(x)}\phi(x) \tag{292}$$
$$\to e^{i\alpha(x)}D_\mu\phi(x) \tag{293}$$
$$F_{\mu\nu} \to \partial_\mu(A_\nu - \partial_\nu\alpha) - \partial_\nu(A_\mu - \partial_\mu\alpha) \tag{294}$$
$$\to F_{\mu\nu} \tag{295}$$

- If you just replace $\partial_\mu \to \partial_\mu^f$ and $\int d^4x \to \sum_x a^4$ - then S is NOT gauge invariant

$$D_\mu\phi(x) \simeq \partial_\mu^f\phi(x) + iA_\mu(x)\phi(x) \tag{296}$$
$$\simeq \frac{1}{a}(\phi(x+ae_\mu) - \phi(x)) + iA_\mu(x)\phi(x) \tag{297}$$
$$\to \frac{1}{a}(e^{i\alpha(x+ae_\mu)}\phi(x+ae_\mu) - e^{i\alpha(x)}\phi(x)) + iA'_\mu(x)e^{i\alpha(x)}\phi(x) \tag{298}$$
$$\stackrel{!}{=} e^{i\alpha(x)}D_\mu\phi(x) \tag{299}$$
$$= e^{i\alpha(x)}\left(\frac{1}{a}(\phi(x+ae_\mu) - \phi(x)) + iA_\mu(x)\phi(x)\right) \tag{300}$$
$$\to A'_\mu(x) = A_\mu(x) + \frac{1}{a}\left(e^{i\alpha(x)} - e^{i\alpha(x+ae_\mu)}\right)\frac{\phi(x+ae_\mu)}{ie^{i\alpha(x)}\phi(x)} \tag{301}$$

we see that requiring gauge covariance of the (naively discretized) $D_\mu\phi(x)$ implies that the gauge transformation of $A_\mu(x)$ depends on the matter field $\to$ **FAIL!**

- Guiding principle: find a discretization that preserves gauge invariance
- We want to find a formulation in which gauge fixing is not needed at all (as long as we don't care about perturbative expansions)

When (parallel) transport $\phi(x+ae_\mu)$ back to point x using the parallel transporter (which makes only sense in a continuum)

$$e^{i\int_0^a ds\, A_\mu(x+se_\mu)}\phi(x+ae_\mu) \stackrel{\text{gauge}}{\to} e^{i\int_0^a ds\,[A_\mu(x+se_\mu) - \partial_\mu\alpha(x+se_\mu)]}e^{i\alpha(x+ae_\mu)}\phi(x+ae_\mu) \tag{302}$$
$$\to e^{i\int_0^a ds\, A_\mu(x+se_\mu)}\underbrace{e^{-i\int_0^a ds\,\partial_\mu\alpha(x+se_\mu)}}_{=e^{-i(\alpha(x+ae_\mu)-\alpha(x))}}e^{i\alpha(x+ae_\mu)}\phi(x+ae_\mu) \tag{303}$$
$$\to e^{i\int_0^a ds\, A_\mu(x+se_\mu)}e^{i\alpha(x)}\phi(x+ae_\mu) \tag{304}$$
$$\to e^{i\alpha(x)}\left[e^{i\int_0^a ds\, A_\mu(x+se_\mu)}\phi(x+ae_\mu)\right] \tag{305}$$

the object transforms under a gauge transform like $\phi(x)$. Then we define

$$D_\mu^f(x) \equiv \frac{[\text{parallel transported}\,\phi(x+ae_\mu)] - \phi(x)}{a} \tag{306}$$
$$= \frac{e^{i\int_0^a ds\, A_\mu(x+se_\mu)}\phi(x+ae_\mu) - \phi(x)}{a} \tag{307}$$
$$\tag{308}$$

with properties

(1) $D^f_\mu\phi(x) \stackrel{\text{gauge}}{\to} e^{i\alpha(x)}D^f_\mu\phi(x)$ (see above)

(2) $\lim_{a\to 0} D^f_\mu\phi(x) = D_\mu\phi(x) = [\partial_\mu + iA_\mu(x)]\phi(x)$

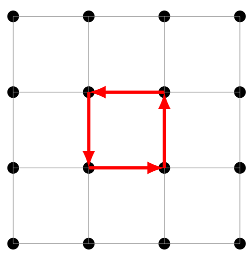
Proof of (2)

$$\begin{aligned}
\int_0^a ds\, A_\mu(x+se_\mu) &\simeq \int_0^a (A_\mu(x) + \mathcal{O}(s)) && (309)\\
&= aA_\mu(x) + \mathcal{O}(a^2) && (310)\\
\to \lim_{a\to 0} D^f_\mu\phi(x) &= \lim_{a\to 0} \frac{e^{i\int_0^a ds\, A_\mu(x+se_\mu)}\phi(x+ae_\mu) - \phi(x)}{a} && (311)\\
&= \lim_{a\to 0} \frac{e^{iaA_\mu(x)+\mathcal{O}(a^2)}[\phi(x) + a\partial_\mu\phi(x) + \mathcal{O}(a^2)] - \phi(x)}{a} && (312)\\
&= \lim_{a\to 0} \frac{(1+iaA_\mu(x)+\mathcal{O}(a^2))[\phi(x) + a\partial_\mu\phi(x) + \mathcal{O}(a^2)] - \phi(x)}{a} && (313)\\
&= \lim_{a\to 0} \frac{(\phi(x) + iaA_\mu(x)\phi(x)) + (a\partial_\mu\phi(x) + iaA_\mu(x)a\partial_\mu\phi(x)) - \phi(x)}{a} && (314)\\
&= \lim_{a\to 0} \frac{(iaA_\mu(x)\phi(x)) + a\partial_\mu\phi(x)}{a} && (315)\\
&= \partial_\mu\phi(x) + iA_\mu(x)\phi(x)) && (316)\\
&= D_\mu\phi(x) && (317)
\end{aligned}$$

On the lattice we will only need to transport matter fields form one lattice point to another lattice point. Therefore we do not need the field $A_\mu(x)$ but only the parallel transporters

$$\begin{aligned}
W(x \to x+ae_\mu) &\equiv U_\mu(x) && (318)\\
&\equiv e^{i\int_0^a ds\, A_\mu(x+se_\mu)} \in U(1) && (319)
\end{aligned}$$

- **Important:** $U_\mu(x)$ are going to be the fundamental degrees of freedom that describes the gauge field on the lattice and **NOT** $A_\mu(x)$
- These transformations ensure that the terms of the actions containing a matter field are gauge invariant.
- What about $F^2_{\mu\nu}$?
  - We need to find a discretization of $F^2_{\mu\nu}$ that depends only on $U_\mu(x)$ and not on $A_\mu(x)$
  - $F^2_{\mu\nu}$ can be written in terms of the parallel transports on the smallest squares on the lattice



$$S[U,\phi] = \sum_{x_\mu} a^4 \left\{ ? + |D_\mu\phi|^2 + m^2|\phi|^2 + \frac{\lambda_0}{4}|\phi|^4 \right\} \quad (320)$$