0.1 Schmueser - Feynman-Graphen und Eichtheorie

0.1.1 Problem 1.1

$$\sigma \cdot \mathbf{a} = \begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix} \tag{1}$$

$$(\sigma \cdot \mathbf{a})(\sigma \cdot \mathbf{b}) = \begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix} \begin{pmatrix} b_3 & b_1 - ib_2 \\ b_1 + ib_2 & -b_3 \end{pmatrix} \tag{2}$$

$$= \mathbf{a} \cdot \mathbf{b}\, 1_2 + i\sigma \cdot (\mathbf{a} \times \mathbf{b}) \tag{3}$$

0.1.2 Problem 1.2

$$\mathbf{P} \times \mathbf{P} = (-i\hbar)^2 \underbrace{(\nabla \times \nabla)}_{=0} + e^2 \underbrace{(\mathbf{A} \times \mathbf{A})}_{=0} - i\hbar e(\nabla \times \mathbf{A} + \mathbf{A} \times \nabla) \tag{4}$$

$$= -i\hbar e(\nabla \times \mathbf{A} + \mathbf{A} \times \nabla) \tag{5}$$

$$= -i\hbar e \begin{pmatrix} \partial_y A_z - \partial_z A_y + A_y \partial_z - A_z \partial_y \\ ... \\ ... \end{pmatrix} \tag{6}$$

$$= -i\hbar e \begin{pmatrix} (\partial_y A_z + A_z \partial_y) - (\partial_z A_y + A_y \partial_z) + A_y \partial_z - A_z \partial_y \\ ... \\ ... \end{pmatrix} \tag{7}$$

$$= -i\hbar e \begin{pmatrix} \partial_y A_z - \partial_z A_y \\ ... \\ ... \end{pmatrix} \tag{8}$$

$$= -i\hbar e \mathbf{B} \tag{9}$$

and therefore

$$(\sigma \cdot \mathbf{P})(\sigma \cdot \mathbf{P}) = \mathbf{P}^2\, 1_2 + e\hbar\sigma \cdot \mathbf{B} \tag{10}$$

0.2 Lancaster, Blundell - Quantum Field Theory for the gifted amateur

Exercise 1.1 - Snell's law via Fermat's principle

The light travels from point A in medium 1 to point B in medium 2. We assume a vertical medium boundary at x_0 and that the light travels within a medium in the straight line. This makes y_0 the free parameter and the the travel time is given by

$$t = \frac{s_{A0}}{c/n_1} + \frac{s_{0B}}{c/n_2} \tag{11}$$

$$= \sqrt{\frac{(x_A - x_0)^2 + (y_A - y_0)^2}{c/n_1}} + \sqrt{\frac{(x_0 - x_B)^2 + (y_0 - y_B)^2}{c/n_2}} \tag{12}$$

The local extrema of the travel time is given by

$$0 = \frac{dt}{dy_0} \tag{13}$$

$$= \frac{y_A - y_0}{s_{A0} c/n_1} + \frac{y_0 - y_B}{s_{0B} c/n_2} \tag{14}$$

$$= \frac{\sin\alpha}{c/n_1} - \frac{\sin\beta}{c/n_2} \tag{15}$$

and therefore

$$n_1 \sin\alpha = n_2 \sin\beta. \tag{16}$$

Exercise 1.2 - Functional derivatives I

- $H[f] = \int G(x,y)f(y)dy$

$$\begin{aligned}
\frac{\delta H[f]}{\delta f(z)} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\int G(x,y)(f(y)+\epsilon\delta(z-y))dy - \int G(x,y)f(y)dy\right] && (17)\\
&= \int G(x,y)\delta(z-y))dy && (18)\\
&= G(x,z) && (19)
\end{aligned}$$

- $I[f] = \int_{-1}^{1} f(x)dx$

$$\begin{aligned}
\frac{\delta^2 I[f^3]}{\delta f(x_0)\delta f(x_1)} &= \frac{\delta}{\delta f(x_0)}\frac{\delta I[f^3]}{\delta f(x_1)} && (20)\\
&= \frac{\delta}{\delta f(x_0)}\frac{\delta}{\delta f(x_1)}\int_{-1}^{1} f(x)^3 dx && (21)\\
&= \frac{\delta}{\delta f(x_0)}\frac{1}{\epsilon}\int_{-1}^{1}(f(x)+\epsilon\delta(x_1-x))^3 - f(x)^3 dx && (22)\\
&= \frac{\delta}{\delta f(x_0)}\frac{1}{\epsilon}\int_{-1}^{1}(f(x)^3+3\epsilon f(x)^2\delta(x_1-x)+\mathcal{O}(\epsilon^2)-f(x)^3 dx && (23)\\
&= \frac{\delta}{\delta f(x_0)}\begin{cases}3f(x_1)^2 & x_1\in[-1,1]\\ 0 & \text{else}\end{cases} && (24)\\
&= \begin{cases}3\frac{1}{\epsilon}[(f(x_1)-\epsilon\delta(x_0-x_1))^2 - f(x_1)^2] & x_1\in[-1,1]\\ 0 & \text{else}\end{cases} && (25)\\
&= \begin{cases}6f(x_1)\delta(x_0-x_1) & x_1\in[-1,1]\\ 0 & \text{else}\end{cases} && (26)\\
& && (27)
\end{aligned}$$

- $J[f] = \int \left(\frac{\partial f}{\partial y}\right)^2 dy$

$$\begin{aligned}
\frac{\delta J[f]}{\delta f(x)} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\int\left(\frac{\partial(f+\epsilon\delta(x-y))}{\partial y}\right)^2 dy - \int\left(\frac{\partial f}{\partial y}\right)^2 dy\right] && (28)\\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\int\left(\frac{\partial f}{\partial y}+\epsilon\frac{\partial\delta(x-y)}{\partial y}\right)^2 dy - \int\left(\frac{\partial f}{\partial y}\right)^2 dy\right] && (29)\\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\int\left(\frac{\partial f}{\partial y}\right)^2 + 2\epsilon\frac{\partial f}{\partial y}\frac{\partial\delta(x-y)}{\partial y} + \mathcal{O}(\epsilon^2) - \left(\frac{\partial f}{\partial y}\right)^2 dy\right] && (30)\\
&= 2\int\frac{\partial f}{\partial y}\frac{\partial\delta(x-y)}{\partial y}dy && (31)\\
&= \text{boundary terms} - 2\int\frac{\partial^2 f}{\partial y^2}\delta(x-y)dy && (32)\\
&= -2\int\frac{\partial^2 f}{\partial x^2} && (33)
\end{aligned}$$

Exercise 1.3 - Functional derivatives II

- $$\begin{aligned}\frac{\delta G[f]}{\delta f(x)} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int g(y,f+\epsilon\delta(x-y))-g(y,f)dy &(34)\\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int g(y,f)+\epsilon\frac{\partial g(y,f)}{\partial f}\delta(x-y))-g(y,f)dy &(35)\\ &= \frac{\partial g(x,f)}{\partial f} &(36)\end{aligned}$$

- $$\begin{aligned}\frac{\delta H[f]}{\delta f(x)} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int g(y,f+\epsilon\delta(x-y),f'+\epsilon\partial_y\delta(x-y))-g(y,f,f')dy &(37)\\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int g(y,f,f')+\epsilon\frac{\partial g(y,f,f')}{\partial f}\delta(x-y)+\epsilon\frac{\partial g(y,f,f')}{\partial f'}\partial_y\delta(x-y))-g(y,f,f')dy &(38)\\ &= \int\frac{\partial g(y,f,f')}{\partial f}\delta(x-y)+\frac{\partial g(y,f,f')}{\partial f'}\partial_y\delta(x-y))dy &(39)\\ &= \frac{\partial g(x,f,f')}{\partial f}+\int\frac{\partial g(y,f,f')}{\partial f'}\partial_y\delta(x-y))dy &(40)\\ &= \frac{\partial g(x,f,f')}{\partial f}-\int\partial_y\frac{\partial g(y,f,f')}{\partial f'}\delta(x-y))dy &(41)\\ &= \frac{\partial g(x,f,f')}{\partial f}-\partial_x\frac{\partial g(x,f,f')}{\partial f'} &(42)\end{aligned}$$

- Same as above but two times integration by parts is needed. Therefore $(-1)^2 = 1$ giving the term a final + sign.

Exercise 1.4 - Functional derivatives III

- $$\begin{aligned}\frac{\delta\phi(x)}{\delta\phi(y)} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\phi(x)+\epsilon\delta(x-y)-\phi(x)\right) &(43)\\ &= \delta(x-y) &(44)\end{aligned}$$

- $$\begin{aligned}\frac{\delta\dot\phi(t)}{\delta\phi(t_0)} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\dot\phi(t)+\epsilon\partial_t\delta(t-t_0)-\dot\phi(t)\right) &(45)\\ &= \frac{d}{dt}\delta(t-t_0) &(46)\end{aligned}$$

Exercise 1.5 - Euler-Langrange equations for elastic medium

$$\mathcal{L} = T - V \tag{47}$$

$$\frac{\partial\mathcal{L}}{\partial\psi}-\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right) = 0 \tag{48}$$

then

$$\frac{\partial \mathcal{L}}{\partial \psi} = 0 \tag{49}$$
$$\frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = \frac{\rho}{2}\int d^3x 2\frac{\partial \psi}{\partial t} \tag{50}$$
$$\frac{\partial \mathcal{L}}{\partial(\partial_k \psi)} = -\frac{\mathcal{T}}{2}\int d^3x 2\frac{\partial \psi}{\partial x^k} \tag{51}$$
$$\rightarrow -\left(\int d^3x[\rho\ddot{\psi} - \mathcal{T}\nabla^2\psi]\right) = 0 \tag{52}$$
$$\rightarrow \frac{\rho}{\mathcal{T}}\ddot{\psi} = \nabla^2\psi \tag{53}$$

Exercise 1.6 - Functional derivatives IV

$$\frac{\delta Z_0[J]}{\delta J(z_1)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\exp\left(-\frac{1}{2}\int d^4xd^4y(J(x)+\epsilon\delta(x-z_1))\Delta(x-y)(J(y)+\epsilon\delta(y-z_1))\right) \tag{54}$$
$$-\exp\left(-\frac{1}{2}\int d^4xd^4yJ(x)\Delta(x-y)J(y)\right) \tag{55}$$
$$= Z_0[J]\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\exp\left(-\frac{\epsilon}{2}\int d^4xd^4y\, J(x)\Delta(x-y)\delta(y-z_1)+\delta(x-z_1)\Delta(x-y)J(y)\right)-1\right) \tag{56}$$
$$= Z_0[J]\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(1-\frac{\epsilon}{2}\int d^4xd^4y\, J(x)\Delta(x-y)\delta(y-z_1)+\delta(x-z_1)\Delta(x-y)J(y)-1\right) \tag{57}$$
$$= -\frac{1}{2}Z_0[J]\int d^4xd^4y\, J(x)\Delta(x-y)\delta(y-z_1)+\delta(x-z_1)\Delta(x-y)J(y) \tag{58}$$
$$= -\frac{1}{2}Z_0[J]\left(\int d^4x\, J(x)\Delta(x-z_1)+\int d^4y\,\Delta(z_1-y)J(y)\right) \tag{59}$$
$$= -Z_0[J]\int d^4y\,\Delta(z_1-y)J(y) \tag{60}$$

Exercise 2.1 - Commutators of creation and annihilation operators

With $[\hat{x},\hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar$

$$[\hat{a},\hat{a}] = \frac{m\omega}{2\hbar}\left(\hat{x}\hat{x}+\frac{i}{m\omega}(\hat{x}\hat{p}+\hat{p}\hat{x})+\frac{i^2}{m^2\omega^2}\hat{p}\hat{p}\right)-\frac{m\omega}{2\hbar}\left(\hat{x}\hat{x}+\frac{i}{m\omega}(\hat{x}\hat{p}+\hat{p}\hat{x})+\frac{i^2}{m^2\omega^2}\hat{p}\hat{p}\right) \tag{61}$$
$$= 0 \tag{62}$$
$$[\hat{a}^\dagger,\hat{a}^\dagger] = ... = 0 \tag{63}$$

$$[\hat{a},\hat{a}^\dagger] = \frac{m\omega}{2\hbar}\left(\hat{x}\hat{x}+\frac{i}{m\omega}(-\hat{x}\hat{p}+\hat{p}\hat{x})-\frac{i^2}{m^2\omega^2}\hat{p}\hat{p}\right)-\frac{m\omega}{2\hbar}\left(\hat{x}\hat{x}+\frac{i}{m\omega}(\hat{x}\hat{p}-\hat{p}\hat{x})-\frac{i^2}{m^2\omega^2}\hat{p}\hat{p}\right) \tag{64}$$
$$= \frac{m\omega}{2\hbar}\frac{i}{m\omega}2(-\hat{x}\hat{p}+\hat{p}\hat{x}) \tag{65}$$
$$= \frac{i}{\hbar}(-\hat{p}\hat{x}-i\hbar+\hat{p}\hat{x}) \tag{66}$$
$$= 1 \tag{67}$$

Now the Hamiltonian

$$\hat{a}^\dagger \hat{a} = \frac{m\omega}{2\hbar}\left(\hat{x}\hat{x} + \frac{i}{m\omega}(\hat{x}\hat{p} - \hat{p}\hat{x}) - \frac{i^2}{m^2\omega^2}\hat{p}\hat{p}\right) \tag{68}$$
$$= \frac{m\omega}{2\hbar}\left(\hat{x}\hat{x} + \frac{i}{m\omega}i\hbar - \frac{i^2}{m^2\omega^2}\hat{p}\hat{p}\right) \tag{69}$$
$$= \frac{1}{2m\omega\hbar}\hat{p}^2 + \frac{m\omega}{2\hbar}\hat{x}^2 - \frac{1}{2} \tag{70}$$
$$\hat{a}^\dagger \hat{a} + \frac{1}{2} = \frac{1}{2m\omega\hbar}\hat{p}^2 + \frac{m\omega}{2\hbar}\hat{x}^2 \tag{71}$$
$$\hbar\omega\left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right) = \frac{1}{2m}\hat{p}^2 + \frac{m\omega^2}{2}\hat{x}^2 = \hat{H} \tag{72}$$

Exercise 2.2 - Perturbed harmonic oscillator

We see

$$a + a^\dagger = \sqrt{\frac{2m\omega}{\hbar}}x \tag{73}$$
$$(a + a^\dagger)^2 = \frac{2m\omega}{\hbar}x^2 \tag{74}$$
$$x^2 = \frac{\hbar}{2m\omega}(a + a^\dagger)^2 \tag{75}$$
$$x^4 = (a + a^\dagger)^2\frac{\hbar}{2m\omega}\cdot\frac{\hbar}{2m\omega}(a + a^\dagger)^2 \tag{76}$$

The first order energy perturbation is given by

$$E_n^{(1)} = \langle n|H_1|n\rangle \tag{77}$$
$$= \langle n|x^4|n\rangle \tag{78}$$
$$= \langle n|x^2 \cdot x^2|n\rangle. \tag{79}$$

By splitting H_1 the calculation gets a bit shorter. Using

$$a|n\rangle\sqrt{n}|n\rangle \qquad a^\dagger|n\rangle\sqrt{n+1}|n+1\rangle \tag{80}$$

we obtain

$$x^2|n\rangle = \frac{\hbar}{2m\omega}(a + a^\dagger)^2|n\rangle \tag{81}$$
$$= \frac{\hbar}{2m\omega}(aa^\dagger + a^\dagger a + (a^\dagger)^2 + a^2)|n\rangle \tag{82}$$
$$= \frac{\hbar}{2m\omega}\Big((n+1)|n\rangle + n|n\rangle + \sqrt{n(n-1)}|n-2\rangle + \sqrt{(n+1)(n+2)}|n+2\rangle\Big) \tag{83}$$
$$= \frac{\hbar}{2m\omega}\Big((2n+1)|n\rangle + \sqrt{n(n-1)}|n-2\rangle + \sqrt{(n+1)(n+2)}|n+2\rangle\Big) \tag{84}$$
$$\langle n|x^2 = (x^2|n\rangle)^\dagger \tag{85}$$
$$= \frac{\hbar}{2m\omega}\Big((2n+1)|n\rangle + \sqrt{n(n-1)}|n-2\rangle + \sqrt{(n+1)(n+2)}|n+2\rangle\Big) \tag{86}$$

Using the orthogonality of the unperturbed states (eigenstates of the Hamiltonian which is hermitian) we obtain

$$E_n^{(1)} = \langle n|x^2 \cdot x^2|n\rangle \tag{87}$$
$$= \frac{\hbar^2}{4m^2\omega^2}\left((2n+1)^2 + n(n-1) + (n+1)(n+2)\right) \tag{88}$$
$$= \frac{\hbar^2}{4m^2\omega^2}\left(4n^2 + 4n + 1 + n^2 - n + n^2 + 3n + 2\right) \tag{89}$$
$$= \frac{\hbar^2}{4m^2\omega^2}\left(6n^2 + 6n + 3\right) \tag{90}$$
$$= \frac{3}{4}\frac{\hbar^2}{m^2\omega^2}\left(2n^2 + 2n + 1\right) \tag{91}$$

which gives the desired result using $E_n = E_n^{(0)} + \lambda E_n^{(1)}$

Exercise 2.3 - ...

Odd notation $\tilde{x} = \hat{x}$

$$\hat{x}_j = \sqrt{\frac{\hbar}{2\omega_j m}}(\hat{a}_j + \hat{a}^\dagger_{-j}) \tag{92}$$
$$x_j = \frac{1}{\sqrt{N}}\sum_k \tilde{x}_k e^{ikja} \tag{93}$$
$$= \frac{1}{\sqrt{N}}\sqrt{\frac{\hbar}{m}}\sum_k \frac{1}{\sqrt{2\omega_k}}(\hat{a}_k + \hat{a}^\dagger_{-k})e^{ikja} \tag{94}$$
$$= \frac{1}{\sqrt{N}}\sqrt{\frac{\hbar}{m}}\sum_k \frac{1}{\sqrt{2\omega_k}}(\hat{a}_k e^{ikja} + \hat{a}^\dagger_k e^{-ikja}) \tag{95}$$

Exercise 2.4 - Wavefunction in space representation

$$\hat{a} = \sqrt{\frac{2\hbar}{m\omega}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right), \qquad \hat{a}|0\rangle = 0 \tag{96}$$
$$\to \sqrt{\frac{2\hbar}{m\omega}}\langle x|\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)|0\rangle = 0 \tag{97}$$
$$\to \sqrt{\frac{2\hbar}{m\omega}}\left(\langle x|\hat{x}|0\rangle + \frac{i}{m\omega}\langle x|\hat{p}|0\rangle\right) = 0 \tag{98}$$
$$\to \sqrt{\frac{2\hbar}{m\omega}}\left(x\langle x|0\rangle + \frac{i}{m\omega}(-i\hbar)\frac{d}{dx}\langle x|0\rangle\right) = 0 \tag{99}$$
$$\to \sqrt{\frac{2\hbar}{m\omega}}\left(x + \frac{\hbar}{m\omega}\frac{d}{dx}\right)\langle x|0\rangle = 0 \tag{100}$$

Now we can solve the ODE ($\psi_0(x) = \langle x|0\rangle$)

$$\left(x + \frac{\hbar}{m\omega}\frac{d}{dx}\right)\psi_0 = 0 \tag{101}$$

$$\int dx\, \psi_0' + \int dx\, \frac{m\omega}{\hbar} x\psi_0 = 0 \tag{102}$$

$$\frac{\psi_0'}{\psi_0} = -\frac{m\omega}{\hbar}x \tag{103}$$

$$\log\psi_0 = -\frac{m\omega}{2\hbar}x^2 + c \tag{104}$$

$$\psi_0 = Ce^{-m\omega x^2/2\hbar} \tag{105}$$

Normalization

$$\int dx\, \psi_0^*\, \psi_0 = 1 \tag{106}$$

$$C^*C\int dx\, e^{-m\omega x^2/\hbar} = 1 \tag{107}$$

$$|C|^2\sqrt{\frac{\pi\hbar}{m\omega}} = 1 \quad \rightarrow \quad C = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \tag{108}$$

Exercise 3.1 - Commutator Fourier Transformation

Bosons - commutator

$$\frac{1}{\mathcal{V}}\sum_{\mathbf{p},\mathbf{q}} e^{i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})}[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = \frac{1}{\mathcal{V}}\sum_{\mathbf{p},\mathbf{q}} e^{i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})}\delta_{\mathbf{pq}} \tag{109}$$

$$= \frac{1}{\mathcal{V}}\sum_{\mathbf{p}} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \tag{110}$$

$$= \frac{1}{L_xL_yL_z}\sum_{n_1=-N/2}^{N/2} e^{i\frac{2\pi n_1}{Na_x}(x_1-y_1)} \cdot \sum_{n_2=-N/2}^{N/2} e^{i\frac{2\pi n_2}{Na_y}(x_2-y_2)} \cdot \sum_{n_3=-N/2}^{N/2} e^{i\frac{2\pi n_3}{Na_z}(x_3-y_3)} \tag{111}$$

$$= \left(\frac{1}{L}\sum_{n=-N/2}^{N/2} e^{i\frac{2\pi n}{Na}(x-y)}\right)^3 \qquad \text{with } Na \equiv L \tag{112}$$

$$= \left(\frac{1}{L}\frac{Na}{2\pi}\sum_{p_n=-\pi/a}^{\pi/a} e^{ip_n(x-y)}\frac{2\pi}{Na}\right)^3 \qquad \text{with } \sum_{q_n} f(p_n)\Delta p = \int f(p)dp \tag{113}$$

$$= \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ip(x-y)}dp\right)^3 \qquad \text{with } N\to\infty,\, a\to 0 \tag{114}$$

$$= (\delta(x-y))^3 \tag{115}$$

$$= \delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{116}$$

with the discretization of the momentum-space $p_j = \left\{\frac{2\pi j}{Na}\right\}_{-N/2}^{N/2}$ and $\Delta p = \frac{2\pi}{Na}$.

Fermions - anticommutator

$$\{c_{\mathbf{p}}, c_{\mathbf{q}}^\dagger\} = \delta_{\mathbf{pq}} \tag{117}$$

yields same result.

Exercise 3.2 - Harmonic oscillator relations

With

$$[\hat{a}, \hat{a}^\dagger] = 1 \tag{118}$$
$$\hat{a}^\dagger \hat{a} = \hat{n} \tag{119}$$
$$\frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle = |n\rangle \tag{120}$$

Then

(a) $[\hat{a}, (\hat{a}^\dagger)^n]$

$$\hat{a}(\hat{a}^\dagger)^n = (aa^\dagger)(a^\dagger)^{n-1} \tag{121}$$
$$= (a^\dagger a + 1)(a^\dagger)^{n-1} \tag{122}$$
$$= a^\dagger a(a^\dagger)^{n-1} + (a^\dagger)^{n-1} \tag{123}$$
$$= a^\dagger a a^\dagger (a^\dagger)^{n-2} + (a^\dagger)^{n-1} \tag{124}$$
$$= a^\dagger(a^\dagger a + 1)(a^\dagger)^{n-2} + (a^\dagger)^{n-1} \tag{125}$$
$$= (a^\dagger)^2 a(a^\dagger)^{n-2} + 2(a^\dagger)^{n-1} \tag{126}$$
$$= ... \tag{127}$$
$$= (a^\dagger)^n a + n(a^\dagger)^{n-1} \tag{128}$$
$$\rightarrow [\hat{a}, (\hat{a}^\dagger)^n] = n(a^\dagger)^{n-1} \tag{129}$$

(b) $\langle 0|a^n(a^\dagger)^m|0\rangle$

If $n < m$ (similar for $n > m$) we get zero

$$\langle 0|a^n(a^\dagger)^m|0\rangle \sim \langle 1|a^{n-1}(a^\dagger)^{m-1}|1\rangle \tag{130}$$
$$\sim \langle 2|a^{n-2}(a^\dagger)^{m-2}|2\rangle \tag{131}$$
$$... \tag{132}$$
$$\sim \langle k|(a^\dagger)^{m-k}|k\rangle \tag{133}$$
$$= 0 \qquad (\langle k|a^\dagger = 0). \tag{134}$$

For $n = m$ we have with the definition

$$\frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle = |n\rangle \tag{135}$$
$$(a^\dagger)^n|0\rangle = \sqrt{n!}|n\rangle \tag{136}$$
$$\langle 0|a^n(a^\dagger)^m|0\rangle = \sqrt{n!}^2\langle n|n|\rangle \tag{137}$$
$$= n! \tag{138}$$

Therefore $\langle 0|a^n(a^\dagger)^m|0\rangle = n!\delta_{nm}$

(c) $\langle m|a^\dagger|n\rangle$

$$\frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle = |n\rangle \tag{139}$$
$$a^\dagger\frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle = a^\dagger|n\rangle \tag{140}$$
$$\frac{1}{\sqrt{n+1}}a^\dagger\frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle = \frac{1}{\sqrt{n+1}}a^\dagger|n\rangle = |n+1\rangle \tag{141}$$

then

$$\langle m|a^\dagger|n\rangle = \sqrt{n+1}\langle m|n+1\rangle \tag{142}$$
$$= \sqrt{n+1}\delta_{m,n+1} \tag{143}$$

(d) $\langle m|a|n\rangle$

$$(\langle m|a)^\dagger = a^\dagger|m\rangle \tag{144}$$
$$= \sqrt{m+1}|m+1\rangle \tag{145}$$

then

$$\langle m|a|n\rangle = \sqrt{m+1}\delta_{m+1,n} \tag{146}$$
$$= \sqrt{n}\delta_{m+1,n} \tag{147}$$

Exercise 3.2 - 3d Harmonic oscillator

Rewriting the Hamiltonian

$$H = H_1 + H_2 + H_3 \tag{148}$$
$$H_i = \frac{p_i^2}{2m} + \frac{1}{2}m\omega^2 x_i^2 \tag{149}$$

the we can reutilise the know ladder operators

$$a_i = \sqrt{\frac{m\omega}{2\hbar}}\left(x_i + \frac{i}{m\omega}p_i\right) \tag{150}$$
$$a_i^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x_i - \frac{i}{m\omega}p_i\right) \tag{151}$$

and the Hamiltonian can be obviously written as the sum

$$H = \hbar\omega\sum_k \left(a_k^\dagger a_k + \frac{1}{2}\right). \tag{152}$$

With the classic definition $\vec{L} = \vec{x} \times \vec{p}$ we see (inverting a and $a^\dagger$ to get x and p)

$$L_i = \varepsilon_{ijk}x_j p_k \tag{153}$$
$$= -i\varepsilon_{ijk}\sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{\hbar m\omega}{2}}(a_j + a_j^\dagger)(a_k - a_k^\dagger) \tag{154}$$
$$= -\frac{i\hbar}{2}\varepsilon_{ijk}(a_j a_k + a_j^\dagger a_k - a_j a_k^\dagger - a_j^\dagger a_k^\dagger) \tag{155}$$
$$= -\frac{i\hbar}{2}\varepsilon_{ijk}(a_j^\dagger a_k - \delta_{jk} - a_k^\dagger a_j) \qquad [a_j, a_k^\dagger] = \delta_{jk},\ a_j|0\rangle = 0,\ \langle 0|a_k = 0 \tag{156}$$
$$= -\frac{i\hbar}{2}(\varepsilon_{ijk}a_j^\dagger a_k - \varepsilon_{ijk}\delta_{jk} - \varepsilon_{ijk}a_k^\dagger a_j) \tag{157}$$
$$= -\frac{i\hbar}{2}(\varepsilon_{ijk}a_j^\dagger a_k - \varepsilon_{ikk} - \varepsilon_{ikj}a_j^\dagger a_k) \qquad \text{reindexing} \tag{158}$$
$$= -\frac{i\hbar}{2}(\varepsilon_{ijk}a_j^\dagger a_k + \varepsilon_{ijk}a_j^\dagger a_k) \qquad \varepsilon_{ikk} = 0 \tag{159}$$
$$= -i\hbar\varepsilon_{ijk}a_j^\dagger a_k \tag{160}$$

Now the new commutation relations

$$[b_0, b_0^\dagger] = [a_3, a_3^\dagger] = 1 = \delta_{00} \tag{161}$$
$$[b_0, b_1^\dagger] = -\frac{1}{\sqrt{2}}(a_3(a_1^\dagger + ia_2^\dagger) - (a_1^\dagger + ia_2^\dagger)a_3) \tag{162}$$
$$= -\frac{1}{\sqrt{2}}(a_3 a_1^\dagger + ia_3 a_2^\dagger - a_1^\dagger a_3 - ia_2^\dagger a_3) \tag{163}$$
$$= -\frac{1}{\sqrt{2}}(\delta_{12} + i\delta_{23}) \tag{164}$$
$$= 0 \tag{165}$$
$$[b_{-1}, b_1^\dagger] = -\frac{1}{2}((a_1 - ia_2)(a_1^\dagger - ia_2^\dagger) - (a_1^\dagger - ia_2^\dagger)(a_1 - ia_2)) \tag{166}$$
$$= -\frac{1}{2}(a_1 a_1^\dagger - ia_2 a_1^\dagger - ia_1 a_2^\dagger - a_2 a_2^\dagger - a_1^\dagger a_1 + ia_1^\dagger a_2 + ia_2^\dagger a_1 + a_2^\dagger a_2) \tag{167}$$
$$= -\frac{1}{2}(1 - i\cdot 0 - i \cdot 0 - 1) \tag{168}$$
$$= 0 \tag{169}$$
$$= \delta_{-1,1} \tag{170}$$
$$\ldots \tag{171}$$

Now the Hamiltonian with

$$b_{-1}^\dagger b_{-1} + b_1^\dagger b_1 = \frac{1}{2}(a_1^\dagger - ia_2^\dagger)(a_1 + ia_2) + \frac{1}{2}(a_1^\dagger + ia_2^\dagger)(a_1 - ia_2) \tag{172}$$
$$= \frac{1}{2}(a_1^\dagger a_1 - ia_2^\dagger a_1 + ia_1^\dagger a_2 + a_2^\dagger a_2) + \frac{1}{2}(a_1^\dagger a_1 + ia_2^\dagger a_1 - ia_1^\dagger a_2 + a_2^\dagger a_2) \tag{173}$$
$$= a_1^\dagger a_1 + a_2^\dagger a_2 \tag{174}$$

and $b_0^\dagger b_0 = a_3^\dagger a_3$ we have $H = \hbar\omega \sum(1/2 + b_m^\dagger b_m)$. While

$$-b_{-1}^\dagger b_{-1} + b_1^\dagger b_1 = -\frac{1}{2}(a_1^\dagger - ia_2^\dagger)(a_1 + ia_2) + \frac{1}{2}(a_1^\dagger + ia_2^\dagger)(a_1 - ia_2) \tag{175}$$
$$= -\frac{1}{2}(a_1^\dagger a_1 - ia_2^\dagger a_1 + ia_1^\dagger a_2 + a_2^\dagger a_2) + \frac{1}{2}(a_1^\dagger a_1 + ia_2^\dagger a_1 - ia_1^\dagger a_2 + a_2^\dagger a_2) \tag{176}$$
$$= ia_2^\dagger a_1 - ia_1^\dagger a_2 \tag{177}$$
$$= -i(-a_2^\dagger a_1 + a_1^\dagger a_2) \tag{178}$$

gives $L^3 = \hbar \sum_m m b_m^\dagger b_m$.

Exercise 5.1 - Time derivative of Lagrangian

With $\frac{\partial L}{\partial q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)$ we have

$$\frac{dL}{dt} = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial q}\dot q + \frac{\partial L}{\partial \dot q}\ddot q \tag{179}$$
$$= \frac{\partial L}{\partial t} + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)\dot q + \frac{\partial L}{\partial \dot q}\ddot q \tag{180}$$
$$= \frac{\partial L}{\partial t} + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\dot q\right) \tag{181}$$
$$= \frac{\partial L}{\partial t} + \frac{d}{dt}(p\dot q) \tag{182}$$

then

$$0 = \frac{\partial L}{\partial t} + \frac{d}{dt}\left(p\dot{q} - L\right) \tag{183}$$

and

$$\frac{\partial L}{\partial t} = -\frac{dH}{dt} \tag{184}$$

Exercise 5.3 - Commutator of Hermitian operators

In general we have

$$\begin{aligned}
[A, B]^\dagger &= (AB - BA)^\dagger & (185)\\
&= B^\dagger A^\dagger - A^\dagger B^\dagger & (186)\\
&= [B^\dagger, A^\dagger] & (187)\\
&= -[A^\dagger, B^\dagger] & (188)
\end{aligned}$$

now using $A = A^\dagger$ and $B = B^\dagger$ we obtain

$$[A, B]^\dagger = -[A, B] \tag{189}$$

Exercise 5.4 - Relativistic free particle

Taylor series expansion of the square root gives

$$\begin{aligned}
L &= -mc^2\sqrt{1 - \frac{v^2}{c^2}} & (190)\\
&\simeq -mc^2 - \frac{1}{2}mv^2 - \frac{3}{8}mv^2\frac{1}{c^2} + ... & (191)\\
&\simeq -mc^2 - \frac{1}{2}mv^2 + ... & (192)
\end{aligned}$$

Conjugated momentum

$$p = \frac{\partial L}{\partial v} = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma mv \simeq mv \tag{193}$$

Lets solve for v to get exact expression for H

$$v = \frac{cp}{m^2c^2 + p^2} \tag{194}$$

Then

$$\begin{aligned}
H &= pv - L & (195)\\
&= p\frac{cp}{m^2c^2 + p^2} + mc^2\sqrt{1 - \frac{v^2}{c^2}} & (196)\\
&= c\frac{m^2c^2 + p^2}{\sqrt{p^2 + m^2c^2}} = \sqrt{m^2c^4 + p^2c^2} & (197)\\
&\simeq mc^2 + \frac{mv^2}{2} & (198)
\end{aligned}$$

Exercise 5.6 - Relativistic free particle in EM field

Euler-Lagrange equations:

$$\frac{\partial L}{\partial x_i} = \frac{d}{dt}\frac{\partial L}{\partial v_i} \tag{199}$$

Definition of the EM potentials

$$\mathbf{E} = -\nabla V - \frac{d\mathbf{A}}{dt} \tag{200}$$
$$\mathbf{B} = \nabla \times \mathbf{A} \tag{201}$$

From Problem 5.4

$$\frac{d}{dt}\frac{\partial L}{\partial v_i} = \frac{d}{dt}(\gamma m v_i) + q\frac{d}{dt}A_i(x,t) \tag{202}$$

Lets proof the identity by calculating the single terms

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = [(\partial_k a_i)b_i + (\partial_k b_i)a_i]\mathbf{e}_k \tag{203}$$
$$(\mathbf{a}\cdot\nabla)\mathbf{b} = a_i(\partial_i b_k)\mathbf{e}_k \tag{204}$$
$$(\mathbf{b}\cdot\nabla)\mathbf{a} = b_i(\partial_i a_k)\mathbf{e}_k \tag{205}$$
$$\mathbf{b}\times(\nabla\times\mathbf{a}) = \epsilon_{kja}\epsilon_{bca}b_j(\partial_b a_c)\mathbf{e}_k \tag{206}$$
$$= (\delta_{kb}\delta_{jc} - \delta_{kc}\delta_{jb})b_j(\partial_b a_c)\mathbf{e}_k \tag{207}$$
$$= [b_c(\partial_k a_c) - b_c(\partial_c a_k)]\mathbf{e}_k \tag{208}$$
$$\mathbf{a}\times(\nabla\times\mathbf{b}) = [a_c(\partial_k b_c) - a_c(\partial_c b_k)]\mathbf{e}_k \tag{209}$$

by adding up we see

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{b}\times(\nabla\times\mathbf{a}) + \mathbf{a}\times(\nabla\times\mathbf{b}) \tag{210}$$

Now we calculate

$$\frac{\partial L}{\partial x_i} = q\frac{\partial}{\partial x_i}[\mathbf{A}\cdot\mathbf{v} - V] \tag{211}$$
$$= -q\partial_i V(x,t) + q[\nabla(\mathbf{A}(x,t)\cdot\mathbf{v})]_i \tag{212}$$
$$= -q[\nabla V]_i + q[(\mathbf{v}\cdot\nabla)\mathbf{A} + \mathbf{v}\times(\nabla\times\mathbf{A})]_i \tag{213}$$

then (combining all vector components)

$$\frac{d}{dt}(\gamma m\mathbf{v}) + q\frac{d}{dt}\mathbf{A} = q(\mathbf{v}\cdot\nabla)\mathbf{A} + q\mathbf{v}\times(\nabla\times\mathbf{A}) - q\nabla V \tag{214}$$
$$\frac{d}{dt}(\gamma m\mathbf{v}) = q\mathbf{v}\times(\nabla\times\mathbf{A}) - q\nabla V - q\left(\frac{d}{dt}\mathbf{A} - (\mathbf{v}\cdot\nabla)\mathbf{A}\right) \tag{215}$$
$$= q\mathbf{v}\times\mathbf{B} - q\nabla V - q\frac{\partial\mathbf{A}}{\partial t} \tag{216}$$
$$= q[\mathbf{v}\times\mathbf{B} + \mathbf{E}] \tag{217}$$

where we used

$$\frac{d\mathbf{A}}{dt} = \frac{\partial\mathbf{A}}{\partial t} + \frac{\partial\mathbf{A}}{\partial x_i}\frac{\partial x_i}{\partial t} \tag{218}$$
$$= \frac{\partial\mathbf{A}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{A} \tag{219}$$

Exercise 5.6 - Non-relativistic free particle in EM field

From Problem 5.4/5.5

$$p_i = \frac{\partial L}{\partial v_i} = \gamma m v_i + qA_i(x,t) \tag{220}$$
$$\mathbf{p} = \gamma m\mathbf{v} + q\mathbf{A} \tag{221}$$
$$\simeq m\mathbf{v} + q\mathbf{A} \tag{222}$$

also

$$\gamma m\mathbf{v} = \mathbf{p} - q\mathbf{A} \tag{223}$$
$$\mathbf{v} = \frac{\mathbf{p} - q\mathbf{A}}{\gamma m} \tag{224}$$
$$v^2 = \frac{(\mathbf{p} - q\mathbf{A})^2}{\gamma^2 m^2} \tag{225}$$
$$= \frac{(\mathbf{p} - q\mathbf{A})^2 c^2}{m^2c^2 + (\mathbf{p} - q\mathbf{A})^2} \tag{226}$$
$$\sqrt{1-\frac{v^2}{c^2}} = \frac{mc}{\sqrt{(\mathbf{p} - q\mathbf{A})^2 + m^2c^2}} \tag{227}$$

then

$$E = H = \mathbf{p}\cdot\dot{\mathbf{q}} - L \tag{228}$$
$$= \mathbf{p}\cdot\mathbf{v} - L \tag{229}$$
$$= (\gamma m\mathbf{v})\cdot\mathbf{v} + q\mathbf{A}\cdot\mathbf{v} - \left(-\frac{mc^2}{\gamma} + q\mathbf{A}\cdot\mathbf{v} - qV\right) \tag{230}$$
$$= (\gamma m\mathbf{v})\cdot\mathbf{v} + \frac{mc^2}{\gamma} + qV \tag{231}$$
$$= (\mathbf{p} - q\mathbf{A})\cdot\mathbf{v} + \frac{mc^2}{\gamma} + qV \tag{232}$$
$$= (\mathbf{p} - q\mathbf{A})\cdot\frac{\mathbf{p} - q\mathbf{A}}{m\gamma} + \frac{mc^2}{\gamma} + qV \tag{233}$$
$$= \left(\frac{(\mathbf{p} - q\mathbf{A})^2}{m} + mc^2\right)\sqrt{1-\frac{v^2}{c^2}} + qV \tag{234}$$
$$= \frac{1}{m}\left((\mathbf{p} - q\mathbf{A})^2 + m^2c^2\right)\frac{mc}{\sqrt{(\mathbf{p} - q\mathbf{A})^2 + m^2c^2}} + qV \tag{235}$$
$$= \sqrt{(\mathbf{p} - q\mathbf{A})^2c^2 + m^2c^4} + qV \tag{236}$$
$$= mc^2\sqrt{1 + \frac{(\mathbf{p} - q\mathbf{A})^2c^2}{m^2c^4}} + qV \tag{237}$$
$$\simeq mc^2\left(1 + \frac{(\mathbf{p} - q\mathbf{A})^2}{2m^2c^2} + ...\right) + qV \tag{238}$$
$$\simeq mc^2 + \frac{(\mathbf{p} - q\mathbf{A})^2}{2m} + qV \tag{239}$$

Exercise 6.1 - Klein-Gordon

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \phi} &= -m^2\phi && (240)\\
\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} &= \frac{1}{2}g^{\alpha\beta}(\partial_\alpha\phi)(\partial_\beta\phi) && (241)\\
&= \frac{1}{2}g^{\alpha\beta}\left[\delta^\mu_\alpha(\partial_\beta\phi) + \delta^\mu_\beta(\partial_\alpha\phi)\right] && (242)\\
&= g^{\alpha\mu}\partial_\alpha\phi && (243)\\
&= \partial^\mu\phi && (244)\\
\partial_\mu\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} &= \partial_\mu\partial^\mu\phi && (245)
\end{aligned}$$

Euler Lagrange

$$(\partial_\mu\partial^\mu + m^2)\phi = 0 \tag{246}$$

Canonical momentum

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0\phi)} = \partial^0\phi = \dot{\phi} \tag{247}$$

Hamiltonian

$$\begin{aligned}
\mathcal{H} &= \pi\dot{\phi} - \mathcal{L} && (248)\\
&= \pi^2 - \left(\frac{1}{2}\pi^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{m^2}{2}\phi^2\right) && (249)\\
&= \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2 && (250)\\
& && (251)
\end{aligned}$$

Exercise 7.1 - Klein-Gordon plus higher orders

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \phi} &= -m^2\phi - \sum_{n=1}(2n+2)\lambda_n\phi^{2n+1} && (252)\\
\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi)} &= \frac{1}{2}\frac{\partial(\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi)}{\partial(\partial_\alpha\phi)} = \frac{1}{2}(\eta^{\mu\nu}\partial_\nu\phi\delta^\alpha_\mu + \eta^{\mu\nu}\partial_\mu\phi\delta^\alpha_\nu) && (253)\\
&= \frac{1}{2}(\eta^{\alpha\nu}\partial_\nu\phi + \eta^{\mu\alpha}\partial_\mu\phi) = \partial^\alpha\phi && (254)\\
&\rightarrow \partial_\alpha\partial^\alpha\phi + m^2\phi + \sum_{n=1}(2n+2)\lambda_n\phi^{2n+1} = 0 && (255)
\end{aligned}$$

Exercise 7.2 - Klein-Gordon plus source

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \phi} &= -m^2\phi + J(x) && (256)\\
\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi)} &= \frac{1}{2}\frac{\partial(\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi)}{\partial(\partial_\alpha\phi)} = \frac{1}{2}(\eta^{\mu\nu}\partial_\nu\phi\delta^\alpha_\mu + \eta^{\mu\nu}\partial_\mu\phi\delta^\alpha_\nu) && (257)\\
&= \frac{1}{2}(\eta^{\alpha\nu}\partial_\nu\phi + \eta^{\mu\alpha}\partial_\mu\phi) = \partial^\alpha\phi && (258)\\
&\rightarrow \partial_\alpha\partial^\alpha\phi + m^2\phi - J(x) = 0 && (259)
\end{aligned}$$

Exercise 7.3 - Two interacting Klein-Gordon fields

$$\frac{\partial \mathcal{L}}{\partial \phi_i} = -m^2\phi_i - 2g(\phi_i^2 + \phi_k^2)2\phi_i \tag{260}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_i)} = \frac{1}{2}\frac{\partial(\eta^{\mu\nu}\partial_\mu\phi_i\partial_\nu\phi_i)}{\partial(\partial_\alpha\phi_i)} = \frac{1}{2}(\eta^{\mu\nu}\partial_\nu\phi_i\delta^\alpha_\mu + \eta^{\mu\nu}\partial_\mu\phi_i\delta^\alpha_\nu) \tag{261}$$

$$= \frac{1}{2}(\eta^{\alpha\nu}\partial_\nu\phi_i + \eta^{\mu\alpha}\partial_\mu\phi_i) = \partial^\alpha\phi_i \tag{262}$$

$$\to \partial_\alpha\partial^\alpha\phi_i + m^2\phi_i + 4g\phi_i(\phi_1^2+\phi_2^2) = 0 \tag{263}$$

Exercise 7.4 - Klein-Gordon again

Same calculation as in 6.1

Exercise 8.1 - Time evolution operator - NOT DONE YET

With

$$U(t_2,t_1) = e^{-iH(t_2-t_1)} \tag{264}$$

Then

(1) $U(t_1,t_1) = e^{-iH(t_1-t_1)} = e^0 = 1$

(2) $U(t_3,t_2)U(t_2,t_1) = e^{-iH(t_3-t_2)}e^{-iH(t_2-t_1)} = e^{-iH(t_3-t_2+t_2-t_1)} = e^{-iH(t_3-t_1)} = U(t_3,t_1)$

(3) $U(t_2,t_1)^{-1}$

(4)

(5)

Exercise 8.2 - Heisenberg equations of motions for ladder operators

With $[a_k, a_q^\dagger] = \delta_{kq}$ we have

$$\frac{d}{dt}a_k^\dagger = \frac{1}{i\hbar}[a_k^\dagger, H] = \frac{1}{i\hbar}\sum_n E_n[a_k^\dagger, a_n^\dagger a_n] = \frac{1}{i\hbar}\sum_n E_n(a_k^\dagger a_n^\dagger a_n - a_n^\dagger a_n a_k^\dagger) \tag{265}$$

$$= \frac{1}{i\hbar}E_k(a_k^\dagger a_k^\dagger a_k - a_k^\dagger a_k a_k^\dagger) = \frac{1}{i\hbar}E_k(a_k^\dagger a_k^\dagger a_k - a_k^\dagger(1 + a_k^\dagger a_k)) = -\frac{1}{i\hbar}E_k a_k^\dagger \tag{266}$$

then

$$a_k^\dagger = c\cdot e^{-E_k t/i\hbar} = a_k^\dagger(0)\cdot e^{-E_k t/i\hbar} \tag{267}$$

And similar

$$\frac{d}{dt}a_k = \frac{1}{i\hbar}[a_k, H] = \frac{1}{i\hbar}\sum_n E_n[a_k, a_n^\dagger a_n] = \frac{1}{i\hbar}\sum_n E_n(a_k a_n^\dagger a_n - a_n^\dagger a_n a_k) \tag{268}$$

$$= \frac{1}{i\hbar}E_k(a_k a_k^\dagger a_k - a_k^\dagger a_k a_k) = \frac{1}{i\hbar}E_k(a_k a_k^\dagger a_k - (a_k a_k^\dagger - 1)a_k) = \frac{1}{i\hbar}E_k a_k \tag{269}$$

then

$$a_k = c\cdot e^{E_k t/i\hbar} = a_k(0)\cdot e^{E_k t/i\hbar} \tag{270}$$

Exercise 10.1 - Commutator of field and energy momentum tensor

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \tag{271}$$

$$[\phi(x), P^\alpha] = \left[\phi(x), \int d^3y\, \pi(y)\partial^\alpha\phi(y) - \delta_{0\alpha}\mathcal{L}\right] \tag{272}$$

$$= \int d^3y\; [\phi(x), \pi(y)\partial^\alpha\phi(y)] - [\phi(x), \delta_{0\alpha}\mathcal{L}] \tag{273}$$

$$= \int d^3y\; [\phi(x)\pi(y)\partial^\alpha\phi(y) - \pi(y)\partial^\alpha\phi(y)\phi(x)] - [\phi(x), \delta_{0\alpha}\mathcal{L}] \tag{274}$$

$$= \int d^3y \left[\underbrace{\phi(x)\pi(y)}_{=i\delta(x-y)+\pi(y)\phi(x)} \partial^\alpha\phi(y) - \pi(y)(\partial^\alpha(\phi(y)\phi(x)) - \phi(y) \underbrace{\partial^\alpha\phi(x)}_{=\frac{\partial}{\partial y^\alpha}\phi(x)=0})\right] - [\phi(x), \delta_{0\alpha}\mathcal{L}] \tag{275}$$

$$= i\partial^\alpha\phi(x) + \int d^3y\, \pi(y)\phi(x)\partial^\alpha\phi(y) - \pi(y) \underbrace{\partial^\alpha(\phi(x)\phi(y))}_{=\phi(x)\partial^\alpha\phi(y)} - \delta_{0\alpha}[\phi(x), \mathcal{L}] \tag{276}$$

$$= i\partial^\alpha\phi(x) - \delta_{0\alpha}[\phi(x), \mathcal{L}] \tag{277}$$

Exercise 10.3 - Energy momentum tensor for scalar field

With

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 \tag{278}$$

$$\Pi^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \tag{279}$$

$$= \partial^\mu\phi \tag{280}$$

$$T^{\mu\nu} = \Pi^\mu\partial^\nu\phi - g^{\mu\nu}\mathcal{L} \tag{281}$$

$$= \partial^\mu\phi\partial^\nu\phi - \frac{1}{2}g^{\mu\nu}((\partial_\alpha\phi)^2 - m^2\phi^2) \tag{282}$$

Now

$$\partial_\mu T^{\mu\nu} = \Box\phi\partial^\nu\phi + \partial^\mu\phi\partial^\nu_\mu\phi - g^{\mu\nu}\left[(\partial_\alpha\phi)\partial_{\alpha\mu}\phi - m^2\phi\partial_\mu\phi\right] \tag{283}$$

$$= (\Box\phi + m^2\phi)\partial^\nu\phi \tag{284}$$

$$= 0 \tag{285}$$

then with $g^{00} = 1$ and $g^{0i} = 0$

$$T^{00} = (\partial^0\phi)^2 - \frac{1}{2}(\partial_\alpha\phi)^2 + \frac{1}{2}m^2\phi^2 \tag{286}$$

$$= \frac{1}{2}(\partial^0\phi)^2 + \frac{1}{2}(\partial_k\phi)^2 + \frac{1}{2}m^2\phi^2 \tag{287}$$

$$= \mathcal{H} \tag{288}$$

$$T^{0i} = \partial^0\phi\partial^i\phi \tag{289}$$

and

$$P^0 = \int d^3x T^{00} = \int d^3x \mathcal{H} \tag{290}$$

$$P^k = \int d^3x T^{0k} = \int d^3x \partial^0\phi\partial^i\phi \tag{291}$$

Exercise 11.1 - Commutator of field operators

$$[\hat{\phi}(x), \hat{\phi}(y)] = \left[\int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_p}} (\hat{a}_{\mathbf{p}} e^{-ipx} + \hat{a}^\dagger_{\mathbf{p}} e^{ipx}), \int \frac{d^3q}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_q}} (\hat{a}_{\mathbf{q}} e^{-iqy} + \hat{a}^\dagger_{\mathbf{q}} e^{iqy})\right] \quad (292)$$

$$= \iint \frac{d^3p}{(2\pi)^{3/2}} \frac{d^3q}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_q}} \frac{1}{\sqrt{2E_p}} ([\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{q}}] e^{-i(xp+yq)} + [\hat{a}^\dagger_{\mathbf{p}}, \hat{a}_{\mathbf{q}}] e^{i(px-qy)} + [\hat{a}_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{q}}] e^{i(-px+qy)} + [\hat{a}^\dagger_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{q}}] e \quad (293)$$

$$= \iint \frac{d^3p}{(2\pi)^{3/2}} \frac{d^3q}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_q}} \frac{1}{\sqrt{2E_p}} (-\delta^{(3)}(\mathbf{p}-\mathbf{q}) e^{i(px-qy)} + \delta^{(3)}(\mathbf{p}-\mathbf{q}) e^{-i(px-qy)}) \quad (294)$$

$\delta^{(3)}(\mathbf{p}-\mathbf{q}) \to \mathbf{p} = \mathbf{q}, E_p = E_q$ meaning $p = q$

$$[\hat{\phi}(x), \hat{\phi}(y)] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} (-e^{ip(x-y)} + e^{-ip(x-y)}) \quad (295)$$

$$(296)$$

0.3 van Baal - A Course in Field Theory

0.3.1 Problem 1. Violation of causality in 1+1 dimensions

(a) With $H^2 = m^2c^4 + p^2c^2$ and $p = -i\hbar\partial_x$

$$H\psi(x,t) = i\hbar\partial_t\psi(x,t) \quad (297)$$

$$H^2\psi(x,t) = -\hbar^2\partial_{tt}\psi(x,t) \quad (298)$$

$$\left(\partial_{xx} - \frac{1}{c^2}\partial_{tt} - \frac{m^2c^2}{\hbar^2}\right)\psi(x,t) = 0 \quad (299)$$

$$\left(\Box_x - \frac{m^2c^2}{\hbar^2}\right)\psi(x,t) = 0 \quad (300)$$

then we try the plane wave ansatz $\psi_k(x,t) = e^{-i(\omega_k t - kx)}$ and see

$$-k^2 + \frac{1}{c^2}\omega_k^2 - \frac{m^2c^2}{\hbar^2} = 0 \quad (301)$$

$$\to \omega_k^2 = k^2c^2 + \frac{m^2c^4}{\hbar^2} \to \omega_k = \sqrt{k^2c^2 + \frac{m^2c^4}{\hbar^2}}. \quad (302)$$

Therefore the general solution is a superposition

$$\psi(x,t) = \int dk f(k) e^{-i(\omega_k t - kx)} + g(k) e^{-i(-\omega_k t - kx)} \quad (303)$$

(b) Assume $\psi_0(x,t)$ is a solution then $\psi_0(x-y,t)$ is also a solution

$$\left(\Box_x - \frac{m^2c^2}{\hbar^2}\right)\psi_0(x,t) = 0 \quad (304)$$

$$\to \left(\Box_x - \frac{m^2c^2}{\hbar^2}\right)\psi_0(x-y,t) = 0 \quad (305)$$

then with $\psi(x,t) = \int dy\, f(y)\psi_0(x-y,t)$

$$\left(\Box_x - \frac{m^2c^2}{\hbar^2}\right)\psi(x,t) = \int dy\, f(y) \left(\Box_x - \frac{m^2c^2}{\hbar^2}\right)\psi_0(x-y,t) \quad (306)$$

$$= 0 \quad (307)$$

and

$$\psi(x,0) = \lim_{t\to 0}\int dy\, f(y)\psi_0(x-y,t) \tag{308}$$

$$= \int dy\, f(y)\delta(x-y) \tag{309}$$

$$= f(x) \tag{310}$$

Now we can use the time propagation operator

$$\psi_0(x,t) = e^{-iHt/\hbar}\psi(x,0) \tag{311}$$

$$= e^{-it\sqrt{p^2c^2+m^2c^4}/hbar}\delta(x) \tag{312}$$

$$= \frac{1}{2\pi\hbar}\int dp\, e^{-it\frac{mc^2}{\hbar}\sqrt{\frac{p^2}{m^2c^2}+1}}e^{ipx/\hbar} \tag{313}$$

and use $\cosh^2 u - \sinh^2 u = 1$ and

$$p = mc\sinh u \tag{314}$$

$$dp = mc\cosh u\; du \tag{315}$$

then

$$\psi_0(x,t) = \frac{mc}{2\pi\hbar}\int du\, e^{-it\frac{mc^2}{\hbar}\sqrt{\sinh^2 u+1}}e^{i\frac{mc}{\hbar}x\sinh u}\cosh u \tag{316}$$

$$= \frac{mc}{2\pi\hbar}\int du\, e^{-it\frac{mc^2}{\hbar}\cosh u}e^{i\frac{mc}{\hbar}x\sinh u}\cosh u \tag{317}$$

$$= \frac{mc}{2\pi\hbar}\int du\, e^{i\frac{mc}{\hbar}(x\sinh u - ct\cosh u)}\cosh u \tag{318}$$

$$= \frac{i}{2\pi c}\partial_t\int du\, e^{i\frac{mc}{\hbar}(x\sinh u - ct\cosh u)}. \tag{319}$$

Now we replace x, t by new coordinates v and z

$$x = \frac{\hbar}{mc}z\cosh v \tag{320}$$

$$ct = \frac{\hbar}{mc}z\sinh v \tag{321}$$

$$\to x^2 - c^2t^2 = \frac{\hbar^2}{m^2c^2}z^2 \tag{322}$$

then we obtain with $y = u - v$

$$\psi_0(x,t) = \frac{i}{2\pi c}\partial_t\int du\, e^{iz(\cosh v\sinh u - \sinh v\cosh u)} \tag{323}$$

$$= \frac{i}{2\pi c}\partial_t\int du\, e^{iz\sinh(u-v)} \tag{324}$$

$$= \frac{i}{2\pi c}\partial_t\int du\; [\cos(z\sinh(u-v)) + i\sin(z\sinh(u-v))] \tag{325}$$

$$= \frac{i}{2\pi c}\partial_t\int dy\; [\cos(z\sinh y) + i\sin(z\sinh y)] \tag{326}$$

$$= \frac{i}{2\pi c}\partial_t\int_{-\infty}^{\infty} dy\; \cos(z\sinh y) \tag{327}$$

$$= \frac{i}{\pi c}\partial_t\int_0^{\infty} dy\; \cos(z\sinh y) \tag{328}$$

(c)

(d)

0.4 Nastase - Introduction to Quantum Field Theory

0.4.1 Exercise 1.4 Scalar Dirac–Born–Infeld equations of motion

With

$$\begin{aligned}
\frac{\partial(\partial_\mu\phi)^2}{\partial_\nu\phi} &= \frac{\partial(\partial_\mu\phi\partial^\mu\phi)}{\partial(\partial_\nu\phi)} && (329)\\
&= \frac{\partial(\eta^{\mu\alpha}\partial_\mu\phi\partial_\alpha\phi)}{\partial(\partial_\nu\phi)} && (330)\\
&= \eta^{\mu\alpha}\frac{\partial(\partial_\mu\phi\partial_\alpha\phi)}{\partial(\partial_\nu\phi)} && (331)\\
&= \eta^{\mu\alpha}(\delta_{\mu\nu}\partial_\alpha\phi + \partial_\mu\phi\delta_{\alpha\nu}) && (332)\\
&= \eta^{\mu\alpha}\delta_{\mu\nu}\partial_\alpha\phi + \eta^{\mu\alpha}\delta_{\alpha\nu}\partial_\mu\phi && (333)\\
&= \delta^\alpha_\nu\partial_\alpha\phi + \delta^\mu_\nu\partial_\mu\phi && (334)\\
&= 2\partial_\nu\phi && (335)
\end{aligned}$$

we can calculate the parts for the Euler-Lagrange equations

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\phi} &= -\frac{1}{L^4}\frac{L^4\left[\frac{\partial g}{\partial\phi}(\partial_\mu\phi)^2 + 2m^2\phi\right]}{2\sqrt{1+L^4[g(\partial_\mu\phi)^2+m^2\phi^2]}} && (336)\\
&= -\frac{\left[\frac{\partial g}{\partial\phi}(\partial_\mu\phi)^2 + 2m^2\phi\right]}{2\sqrt{1+L^4[g(\partial_\mu\phi)^2+m^2\phi^2]}} && (337)\\
\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)} &= -\frac{1}{L^4}\frac{L^4\left[2g(\partial_\mu\phi)\delta^\mu_\nu\right]}{2\sqrt{1+L^4[g(\partial_\mu\phi)^2+m^2\phi^2]}} && (338)\\
&= -\frac{g(\partial_\nu\phi)}{\sqrt{1+L^4[g(\partial_\mu\phi)^2+m^2\phi^2]}} && (339)\\
\partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)} &= -\frac{g(\partial_\nu\partial_\nu\phi)\sqrt{1+L^4[g(\partial_\mu\phi)^2+m^2\phi^2]} - g(\partial_\nu\phi)\frac{L^4[2g(\partial_\mu\phi)(\partial_\nu\partial_\mu\phi)+2m^2\phi\partial_\nu\phi]}{2\sqrt{1+L^4[g(\partial_\mu\phi)^2+m^2\phi^2]}}}{1+L^4[g(\partial_\mu\phi)^2+m^2\phi^2]} && (340)
\end{aligned}$$

Multiplying the Euler-Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)} = 0 \tag{341}$$

by $\sqrt{1+L^4[g(\partial_\mu\phi)^2+m^2\phi^2]}$ we obtain

$$\begin{aligned}
-\frac{1}{2}\left[\frac{\partial g}{\partial\phi}(\partial_\mu\phi)^2 + 2m^2\phi\right] + g(\partial_\nu\partial_\nu\phi) + \frac{1}{2}g(\partial_\nu\phi)\frac{L^4[2g(\partial_\mu\phi)(\partial_\nu\partial_\mu\phi)+2m^2\phi\partial_\nu\phi]}{1+L^4[g(\partial_\mu\phi)^2+m^2\phi^2]} &= 0 && (342)\\
g(\Box\phi - m^2\phi) - \frac{1}{2}\frac{\partial g}{\partial\phi}(\partial_\mu\phi)^2 + gL^4\frac{g(\partial_\nu\phi)(\partial_\mu\phi)(\partial_\nu\partial_\mu\phi)+m^2\phi(\partial_\nu\phi)^2}{1+L^4[g(\partial_\mu\phi)^2+m^2\phi^2]} &= 0 && (343)
\end{aligned}$$

0.4.2 Exercise 2.1 Equations of motion for an anharmonic

With

$$p = \frac{\partial L}{\partial \dot{q}} = \dot{q} \tag{344}$$
$$H = p\dot{q} - L \tag{345}$$
$$= p^2 - \frac{p^2}{2} + \frac{\lambda}{4!}q^4 \tag{346}$$
$$= \frac{p^2}{2} + \frac{\lambda}{4!}q^4 \tag{347}$$
$$\tag{348}$$

then

$$\dot{p} = -\frac{\partial H}{\partial q} = -\frac{\lambda}{3!}q^3 \tag{349}$$
$$\dot{q} = \frac{\partial H}{\partial p} = p \tag{350}$$

Phase space path integral

$$M(q', t'; q, t) = \mathcal{D}p(t)\mathcal{D}q(t) \exp\left\{ i \int_t^{t'} dt[p(t)\dot{q}(t) - H(p(t), q(t))] \right\} \tag{351}$$
$$= \mathcal{D}p(t)\mathcal{D}q(t) \exp\left\{ i \int_t^{t'} dt[p(t)\dot{q}(t) - \frac{p(t)^2}{2} - \frac{\lambda}{4!}q(t)^4] \right\} \tag{352}$$

0.5 Mandl, Shaw - Quantum Field Theory 2e

0.5.1 Problem 1.1. Radiation field in a cube - NOT DONE YET

First checking orthogonality

$$a(a^\dagger)^n = (1 + a^\dagger a)(a^\dagger)^{n-1} \tag{353}$$
$$= (a^\dagger)^{n-1} + a^\dagger a(a^\dagger)^{n-1} \tag{354}$$
$$= (a^\dagger)^{n-1} + (a^\dagger)(1 + a^\dagger a)(a^\dagger)^{n-2} \tag{355}$$
$$= 2(a^\dagger)^{n-1} + (a^\dagger)^2 a^\dagger a(a^\dagger)^{n-2} \tag{356}$$
$$= n(a^\dagger)^{n-1} + (a^\dagger)^n a \tag{357}$$

then iteratively

$$a^2(a^\dagger)^n = n(n-1)(a^\dagger)^{n-2} + n(a^\dagger)^{n-1}a + (a^\dagger)^n a^2 \tag{358}$$
$$\ldots \tag{359}$$
$$a^n(a^\dagger)^n = n! + \ldots.a + \ldots a^2 + \ldots \tag{360}$$

so only the first term survives because of $a|0\rangle = 0$

$$\langle k|n\rangle = \langle 0|\frac{a^k}{\sqrt{k!}}\frac{(a^\dagger)^n}{\sqrt{n!}}0\rangle = \delta_{kn}. \tag{361}$$

(i)

$$\langle c|c\rangle = e^{|c|^2} \sum_{n,k} \frac{(c^*)^k c^n}{\sqrt{k!n!}} \underbrace{\langle k|n\rangle}_{\delta_{kn}} \tag{362}$$
$$= e^{-|c|^2} \sum_n \frac{|c|^{2n}}{n!} \tag{363}$$
$$= e^{-|c|^2} \sum_n \frac{(|c|^2)^n}{n!} \tag{364}$$
$$= e^{-|c|^2} e^{|c|^2} \tag{365}$$
$$= 1 \tag{366}$$

(ii) With

$$a_r(\mathbf{k})|...n_r(\mathbf{k})...\rangle = \sqrt{n_r(\mathbf{k})}|...n_r(\mathbf{k}) - 1...\rangle \tag{367}$$

then

$$a_r(\mathbf{k})|c\rangle = a_r(\mathbf{k})e^{|c|^2} \sum_{n=0}^{\infty} \frac{c^n}{\sqrt{n!}}|n\rangle \tag{368}$$
$$= e^{|c|^2} \sum_{n=0}^{\infty} \frac{c^n}{\sqrt{n!}} a_r(\mathbf{k})|n\rangle \tag{369}$$
$$= e^{|c|^2} \sum_{n=0}^{\infty} \frac{c^n}{\sqrt{n!}} \sqrt{n}|n-1\rangle \tag{370}$$
$$= c\, e^{|c|^2} \sum_{n=0}^{\infty} \frac{c^{n-1}}{\sqrt{n!}} \sqrt{n}|n-1\rangle \tag{371}$$
$$= x|c\rangle \tag{372}$$

(iii)

$$\langle c|N|c\rangle = \langle c|a^\dagger a|c\rangle \tag{373}$$
$$= \langle c|c^* c|c\rangle \tag{374}$$
$$= c^* c\langle c||c\rangle \tag{375}$$
$$= |c|^2 \tag{376}$$

(iv)

$$\langle c|N^2|c\rangle = \langle c|a^\dagger a a^\dagger a|c\rangle \tag{377}$$
$$= |c|^2\langle c|a a^\dagger|c\rangle \tag{378}$$
$$\tag{379}$$

0.5.2 Problem 1.2. Lagrangian of point particle in EM potential - NOT DONE YET

(i)

$$\frac{dL}{d\dot{\mathbf{x}}} = m\dot{\mathbf{x}} + \frac{q}{c}\mathbf{A} \tag{380}$$
$$\frac{\partial}{\partial t}\frac{dL}{d\dot{\mathbf{x}}} = m\ddot{\mathbf{x}} + \frac{q}{c}\dot{\mathbf{A}} \tag{381}$$
$$\frac{dL}{d\mathbf{x}} = \frac{q}{c}\nabla(\mathbf{A}\cdot\dot{\mathbf{x}}) - q\nabla\phi \tag{382}$$
$$= \frac{q}{c}\left[\mathbf{A}\times(\nabla\times\dot{\mathbf{x}}) + \dot{\mathbf{x}}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\dot{\mathbf{x}} + (\dot{\mathbf{x}}\cdot\nabla)\mathbf{A}\right] - q\nabla\phi \tag{383}$$
$$= \frac{q}{c}\left[0 + \dot{\mathbf{x}}\times\mathbf{B} + 0 + (\dot{\mathbf{x}}\cdot\nabla)\mathbf{A}\right] - q\nabla\phi \tag{384}$$
$$\rightarrow m\ddot{\mathbf{x}} = q\left(+\nabla\phi - \frac{1}{c}\frac{\partial}{\partial t}\dot{\mathbf{A}}\right) - \frac{q}{c}\dot{\mathbf{x}}\times\mathbf{B} - \frac{q}{c}(\dot{\mathbf{x}}\cdot\nabla)\mathbf{A} \tag{385}$$

(ii)

0.5.3 Problem 2.1 - NOT DONE YET

$$\delta S = \int d^4x\,\delta(\mathcal{L} + \partial_\alpha\Lambda^\alpha) \tag{386}$$
$$= \int d^4x\,\delta\mathcal{L} + \delta\int d^3\sigma_\alpha\Lambda^\alpha \tag{387}$$
$$= \int d^4x\,\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\phi_{,\beta}}\delta\phi_{,\beta} + \int d^3\sigma_\alpha\frac{\partial\Lambda^\alpha}{\partial\phi}\delta\phi \tag{388}$$
$$= \int d^4x\,\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi - \frac{\partial}{\partial x^\beta}\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\beta}}\right)\delta\phi + \int d^4\frac{\partial}{\partial x^\beta}\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\beta}}\delta\phi\right) + \int d^3\sigma_\alpha\frac{\partial\Lambda^\alpha}{\partial\phi}\delta\phi \tag{389}$$
$$= \int_\Omega d^4x\,\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi - \frac{\partial}{\partial x^\beta}\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\beta}}\right)\delta\phi + \int_{\partial\Omega} d^3\sigma_\beta\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\beta}}\delta\phi\right) + \int_{\partial\Omega} d^3\sigma_\alpha\frac{\partial\Lambda^\alpha}{\partial\phi}\delta\phi \tag{390}$$

as $\delta\phi$ vanishes on the boundary $\partial\Omega$ the $\Lambda^a lpha$ does not change the equation of motion.

0.6 Straumann - Relativistische Quantentheorie

0.6.1 Problem 1.11.1. Momentum and angular momentum of the radiation field

$$\mathbf{P} = \frac{1}{4\pi c}\int_V \mathbf{E}\times\mathbf{B}\,d^3x \tag{391}$$
$$\mathbf{J} = \frac{1}{4\pi c}\int_V [\mathbf{x}\times(\mathbf{E}\times\mathbf{B})]\,d^3x \tag{392}$$

In Coulomb gauge we have

$$\mathbf{E} = -\frac{1}{c}\partial_t \mathbf{A} = -\frac{1}{c}\dot{A}_l \mathbf{e}_l \tag{393}$$
$$\mathbf{B} = \nabla \times \mathbf{A} = \varepsilon_{ijk}(\partial_j A_k)\mathbf{e}_i \tag{394}$$
$$\begin{aligned}
\mathbf{E}\times\mathbf{B} &= -\frac{1}{c}\varepsilon_{nli}\mathbf{e}_n(\dot{A}_l\mathbf{e}_l)(\varepsilon_{ijk}(\partial_j A_k)\mathbf{e}_i) && (395)\\
&= -\frac{1}{c}\varepsilon_{nli}\mathbf{e}_n\dot{A}_l\varepsilon_{ijk}(\partial_j A_k)\mathbf{e}_i\mathbf{e}_l && (396)\\
&= -\frac{1}{c}\varepsilon_{nli}\mathbf{e}_n\dot{A}_l\varepsilon_{ijk}(\partial_j A_k)\delta_{il} && (397)\\
&= -\frac{1}{c}\varepsilon_{nli}\varepsilon_{ijk}(\partial_j A_k)\dot{A}_i\mathbf{e}_n && (398)\\
&= -\frac{1}{c}(\delta_{nj}\delta_{lk}-\delta_{nk}\delta_{lj})(\partial_j A_k)\dot{A}_l\mathbf{e}_n && (399)\\
&= -\frac{1}{c}((\partial_j A_k)\dot{A}_k\mathbf{e}_j-(\partial_j A_k)\dot{A}_j\mathbf{e}_k) && (400)\\
&= -\frac{1}{c}((\mathbf{e}_j\partial_j A_k)\dot{A}_k-\dot{A}_j(\partial_j A_k)\mathbf{e}_k) && (401)\\
&= -\frac{1}{c}[\nabla(\mathbf{A}\cdot\dot{\mathbf{A}})-(\dot{\mathbf{A}}\cdot\nabla)\mathbf{A}] && (402)
\end{aligned}$$

And from (1.44) and (1.33)

$$\begin{aligned}
\mathbf{A}(x,t) &= \frac{1}{\sqrt{V}}\sum_{\mathbf{k},\lambda}\sqrt{\frac{2\pi\hbar c^3}{\omega_k}}\left[a_{\mathbf{k},\lambda}\boldsymbol{\varepsilon}(k,\lambda)e^{i\mathbf{k}\cdot\mathbf{x}}+a^*_{\mathbf{k},\lambda}\boldsymbol{\varepsilon}(k,\lambda)^*e^{-i\mathbf{k}\cdot\mathbf{x}}\right] && (403)\\
&= \sum_{\mathbf{k},\lambda}\sqrt{\frac{2\pi\hbar c^3}{\omega_k}}\left[a_{\mathbf{k},\lambda}\mathbf{u}_{\mathbf{k},\lambda}(\mathbf{x})+a^*_{\mathbf{k},\lambda}\mathbf{u}^*_{\mathbf{k},\lambda}(\mathbf{x})\right] && (404)\\
& && (405)
\end{aligned}$$

0.6.2 Problem 4.5.1. Approximation for polarization potential

$$\Phi^{\mathrm{Pol}}(\mathbf{x}) = \frac{e}{(2\pi)^3}\int d^3k e^{i\mathbf{k}\cdot\mathbf{x}}\int_{4m^2}^{\infty} d\kappa^2 \frac{\Pi(x^2)}{\kappa^2(\kappa^2+\mathbf{k}^2)} \tag{406}$$

0.7 Ramond - Field Theory - A modern primer

0.7.1 Problem 1.1 A

(i) With

$$\begin{aligned}
\left(\frac{d(x+\delta x)}{dt}\right)^2 &= \left(\frac{dx}{dt}+\delta\frac{dx}{dt}\right)\left(\frac{dx}{dt}+\delta\frac{dx}{dt}\right) && (407)\\
&= \left(\frac{dx}{dt}\right)^2+2\frac{dx}{dt}\cdot\delta\frac{dx}{dt}+\left(\delta\frac{dx}{dt}\right)^2 && (408)\\
&= \left(\frac{dx}{dt}\right)^2+\frac{d}{dt}\left(\frac{dx}{dt}\delta x\right)-2\frac{d^2x}{dt^2}\delta x+\left(\delta\frac{dx}{dt}\right)^2 && (409)
\end{aligned}$$

where we integrates the second term by parts. Now we can expand the action

$$S = \int dt \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 \tag{410}$$

$$S[x+\delta x] = \int dt \frac{1}{2} m \left(\frac{d(x+\delta x)}{dt}\right)^2 \tag{411}$$

$$\delta S = -\frac{1}{2} m \int_{t_1}^{t_2} dt\, 2 \frac{dx}{dt}\frac{d\delta(x)}{dt} \tag{412}$$

$$= -\frac{1}{2} m \int_{t_1}^{t_2} dt \delta x \left(2\frac{d^2x}{dt^2}\right) + \frac{1}{2} m \frac{dx}{dt}\delta x\Big|_{t_1}^{t_2} \tag{413}$$

Assuming the equations of motion hold $\ddot{x} = 0$ an forcing the surface term to vanish (we CAN'T force $\delta x = 0$) we have

$$\frac{d}{dt}\left(\frac{dx}{dt}\right) = 0 \tag{414}$$

(ii) We could assume a velocity dependent potential is considered

$$V = \sqrt{\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2}\left(1 - \cos\frac{\sqrt{x_1^2 + x_2^2 + x_3^2}}{a}\right) \tag{415}$$

but then units would be off - so we assume v to be a constant. The

$$\delta S_V = \frac{\partial V}{\partial x_i}\delta x_i \tag{416}$$

$$= \frac{v x_i}{ar}\sin\frac{r}{a}\delta x_i \tag{417}$$

$$\to m\ddot{x}_i = \frac{v x_i}{ar}\sin\frac{r}{a}\delta x_i \tag{418}$$

Surface term

$$\left(\frac{\partial L}{\partial \dot{x}_i}\delta x_i\right)_{t_1}^{t_2} \tag{419}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = \frac{d}{dt}\frac{\partial L}{\partial p_i} = \frac{\partial L}{\partial x_i} = \frac{v x_i}{ar}\sin\frac{r}{a}\delta x_i \tag{420}$$

0.8 Muenster - Von der Quantenfeldtheorie zum Standardmodell

0.8.1 Problem 2.1 - 1

(a) The Klein-Gordon equations is given by

$$\left(\partial_\mu \partial^\mu + \frac{m^2c^2}{\hbar^2}\right)\varphi = 0 \tag{421}$$

$$\left(c^2\partial_{tt} - \triangle + \frac{m^2c^2}{\hbar^2}\right)\varphi = 0 \tag{422}$$

We make the ansatz

$$\varphi = \phi_1 + \phi_2 \tag{423}$$
$$\phi_1 = \frac{1}{2}\varphi - \alpha\partial_t\varphi \tag{424}$$
$$\phi_2 = \frac{1}{2}\varphi + \alpha\partial_t\varphi \tag{425}$$

Then we get expressions for the time derivatives

$$\phi_2 - \phi_1 = 2\alpha\partial_t\varphi \tag{426}$$
$$\to \partial_t\varphi = \frac{1}{2\alpha}(\phi_2 - \phi_1) \tag{427}$$

and

$$\partial_{tt}\varphi = c^2\left(\triangle - \frac{m^2c^2}{\hbar^2}\right)\varphi \tag{428}$$
$$= c^2\left(\triangle - \frac{m^2c^2}{\hbar^2}\right)(\phi_1 + \phi_2) \tag{429}$$

Therefore we get for $\phi_{1,2}$

$$\partial_t\phi_1 = \frac{1}{2}\partial_t\varphi - \alpha\partial_{tt}\varphi \tag{430}$$
$$= \frac{1}{2\alpha}(\phi_2 - \phi_1) - \alpha c^2\left(\triangle - \frac{m^2c^2}{\hbar^2}\right)(\phi_1 + \phi_2) \tag{431}$$
$$\partial_t\phi_2 = \frac{1}{2}\partial_t\varphi + \alpha\partial_{tt}\varphi \tag{432}$$
$$= \frac{1}{2\alpha}(\phi_2 - \phi_1) + \alpha c^2\left(\triangle - \frac{m^2c^2}{\hbar^2}\right)(\phi_1 + \phi_2) \tag{433}$$

which we can write in the form

$$i\hbar\partial_t\begin{pmatrix}\phi_1\\\phi_2\end{pmatrix} = -i\hbar\alpha c^2\left(\triangle - \frac{m^2c^2}{\hbar^2}\right)\begin{pmatrix}1 & 1\\-1 & -1\end{pmatrix}\begin{pmatrix}\phi_1\\\phi_2\end{pmatrix} + \frac{i\hbar}{2\alpha}\begin{pmatrix}-1 & 1\\-1 & 1\end{pmatrix}\begin{pmatrix}\phi_1\\\phi_2\end{pmatrix} \tag{434}$$
$$= i\hbar\begin{pmatrix}-\alpha c^2\left(\triangle - \frac{m^2c^2}{\hbar^2}\right) - \frac{1}{2\alpha} & -\alpha c^2\left(\triangle - \frac{m^2c^2}{\hbar^2}\right) + \frac{1}{2\alpha}\\ \alpha c^2\left(\triangle - \frac{m^2c^2}{\hbar^2}\right) - \frac{1}{2\alpha} & \alpha c^2\left(\triangle - \frac{m^2c^2}{\hbar^2}\right) + \frac{1}{2\alpha}\end{pmatrix}\begin{pmatrix}\phi_1\\\phi_2\end{pmatrix} \tag{435}$$

(b) Diagonalization gives

$$i\hbar\partial_t\phi = \hat{H}\phi \tag{436}$$
$$\to i\hbar\partial_t S^{-1}\phi = \underbrace{S^{-1}\hat{H}S}_{=h}S^{-1}\phi \tag{437}$$
$$\lambda_\pm = \pm\sqrt{2}c\hbar\sqrt{\Delta - \frac{m^2c^2}{\hbar^2}} \tag{438}$$
$$= \mp\sqrt{2}mc^2\sqrt{1 - \frac{\hbar^2}{m^2c^2}\triangle} \tag{439}$$

A semi-canonical choice for the parameter α is to make the $\triangle$ look like a momentum operator

$$i\hbar\alpha c^2 = -\frac{\hbar^2}{2m} \quad\to\quad \alpha = \frac{i\hbar}{2mc^2} \tag{440}$$

0.9 Peskin, Schroeder - An Introduction to Quantum Field Theory

0.9.1 Problem 2.1 - Maxwell equations

(a)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}F_{\mu\nu}F_{\alpha\beta} \tag{441}$$

$$= -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\alpha A_\beta - \partial_\beta A_\alpha) \tag{442}$$

With

$$\frac{\partial\mathcal{L}}{\partial A_\gamma} - \partial_\sigma \frac{\mathcal{L}}{\partial(\partial_\sigma A_\gamma)} = 0 \tag{443}$$

then

$$\frac{\mathcal{L}}{\partial(\partial_\sigma A_\gamma)} = -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\alpha A_\beta - \partial_\beta A_\alpha) - \frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\alpha A_\beta - \partial_\beta A_\alpha) \tag{444}$$

$$= -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}(\delta^\sigma_\mu\delta^\gamma_\nu - \delta^\sigma_\nu\delta^\gamma_\mu)(\partial_\alpha A_\beta - \partial_\beta A_\alpha) - \frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\alpha A_\beta - \partial_\beta A_\alpha) \tag{445}$$

$$= -\frac{1}{4}(\delta^{\alpha\sigma}\delta^{\beta\gamma} - \delta^{\beta\sigma}\delta^{\alpha\gamma})(\partial_\alpha A_\beta - \partial_\beta A_\alpha) - ... \tag{446}$$

$$= -\frac{1}{4}(\partial^\sigma A^\gamma - \partial^\gamma A^\sigma - \partial^\gamma A^\sigma + \partial^\sigma A^\gamma) - ... \tag{447}$$

$$= -\frac{1}{4}2F^{\sigma\gamma} - ... \tag{448}$$

$$= -F^{\sigma\gamma} \tag{449}$$

and therefore

$$\partial_\sigma F^{\sigma\gamma} = 0 \tag{450}$$

Rewriting into the common form

$$\gamma = 0 \quad \rightarrow \quad \partial_0 F^{00} + \sum_i \partial_i F^{i0} = 0 \tag{451}$$

$$\rightarrow \quad \sum_i \partial_i(-F^{0i}) = 0 \tag{452}$$

$$\rightarrow \quad \sum_i \partial_i E^i = 0 \tag{453}$$

$$\rightarrow \quad \nabla \cdot \mathbf{E} = 0 \tag{454}$$

$$\gamma = k \quad \rightarrow \quad \partial_0 F^{0k} + \sum_i \partial_i F^{ik} = 0 \tag{455}$$

$$\rightarrow \quad \partial_0(-E^k) + \sum_i \partial_i F^{ik} = 0 \tag{456}$$

$$\rightarrow \quad \partial_0(-E^k) + \sum_i \partial_i(-\epsilon_{ikm}B^m) = 0 \tag{457}$$

$$\rightarrow \quad \dot{\mathbf{E}} = \nabla \times \mathbf{B} \tag{458}$$

The other two equations come from

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{459}$$

$$\rightarrow \quad \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \tag{460}$$

(b) With the definition (2.17)

$$T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\lambda)}\partial_\nu A_\lambda - \mathcal{L}\delta^\mu_\nu \tag{461}$$

$$= -F^{\mu\lambda}\partial_\nu A_\lambda + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\delta^\mu_\nu \tag{462}$$

we rewrite

$$T^{\mu\nu} = -F^{\mu\lambda}\partial^\nu A_\lambda + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\eta^{\mu\nu} \tag{463}$$

$$\widehat{T}^{\mu\nu} = -F^{\mu\lambda}\partial^\nu A_\lambda + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\eta^{\mu\nu} + \partial_\lambda(F^{\mu\lambda}A^\nu) \tag{464}$$

$$= -F^{\mu\lambda}\partial^\nu A_\lambda + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\eta^{\mu\nu} + \underbrace{(\partial_\lambda F^{\mu\lambda})}_{=0\text{ (Maxwell)}} A^\nu + F^{\mu\lambda}(\partial_\lambda A^\nu) \tag{465}$$

$$= F^{\mu\lambda}F_\lambda^{\ \nu} + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\eta^{\mu\nu} \tag{466}$$

$$= F^{\mu\lambda}F_{\lambda\sigma}\eta^{\sigma\nu} + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\eta^{\mu\nu} \tag{467}$$

$$= F^{\uparrow\uparrow}F_{\downarrow\downarrow}\eta + \frac{1}{4}\mathrm{tr}(-F^{\uparrow\uparrow}F_{\downarrow\downarrow})\eta \tag{468}$$

and with

$$F_{\mu\nu} = F_{\downarrow\downarrow} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \qquad F^{\mu\nu} = F_{\uparrow\uparrow} = \eta F_{\downarrow\downarrow}\eta^T = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{469}$$

$$F_{\mu\nu}F^{\mu\nu} = -\mathrm{tr}(F_{\downarrow\downarrow}.F_{\uparrow\uparrow}) = 2(\mathbf{B}^2 - \mathbf{E}^2) \qquad F^{\mu\lambda}F_{\lambda\nu} = ... \tag{470}$$

we obtain

$$\widehat{T}^{\mu\nu} = \begin{pmatrix} \mathcal{E} & \mathbf{S} \\ \mathbf{S} & ... \end{pmatrix} \tag{471}$$

which looks symmetric.

0.9.2 Problem 2.2 - The complex scalar field

(a) Using $\partial_\mu\phi^*\partial^\mu\phi = \partial_\mu\phi^*\eta^{\mu\nu}\partial_\nu\phi = \partial^\mu\phi^*\partial_\mu\phi$ and $\partial^\mu = \eta^{\mu\nu}\partial_\nu = (\partial_0, -\partial_i)$ we find

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial\dot{\phi})} = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \partial^0\phi^* = \partial_0\phi^* = \dot{\phi}^* \tag{472}$$

$$\pi^* = \frac{\partial\mathcal{L}}{\partial(\partial\dot{\phi}^*)} = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi^*)} = \partial^0\phi = \partial_0\phi = \dot{\phi} \tag{473}$$

then

$$H = \int d^3x[\pi\dot{\phi} + \pi^*\dot{\phi}^* - \mathcal{L}] \tag{474}$$

$$= \int d^3x[\pi\pi^* + \pi^*\pi - \partial_\mu\phi^*\eta^{\mu\nu}\partial_\nu\phi + m^2\phi^*\phi] \tag{475}$$

$$= \int d^3x[\pi\pi^* + \pi^*\pi - (\underbrace{\dot{\phi}^*\dot{\phi}}_{=\pi\pi^*} - \nabla\phi^* \cdot \nabla\phi) + m^2\phi^*\phi] \tag{476}$$

$$= \int d^3x[\pi^*\pi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi] \tag{477}$$

Let's rewrite the Lagrangian with $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi \tag{478}$$

$$= \frac{1}{2}\partial_\mu(\phi_1 - i\phi_2)\partial^\mu(\phi_1 + i\phi_2) - \frac{1}{2}m^2(\phi_1 - i\phi_2)(\phi_1 + i\phi_2) \tag{479}$$

$$= \frac{1}{2}(\partial_\mu\phi_1\partial^\mu\phi_1 - m^2\phi_1^2) + i\frac{1}{2}(\partial_\mu\phi_2\partial^\mu\phi_2 - m^2\phi_2^2) \tag{480}$$

So we use the results for the scalar field

$$\phi_1(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_\mathbf{p}}}\left(a_\mathbf{p}e^{i\mathbf{p}\cdot\mathbf{x}} + a_\mathbf{p}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right) \tag{481}$$

$$\pi_1(\mathbf{x}) = -i\int \frac{d^3p}{(2\pi)^3\sqrt{2}}\sqrt{\omega_\mathbf{p}}\left(a_\mathbf{p}e^{i\mathbf{p}\cdot\mathbf{x}} + a_\mathbf{p}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right) \tag{482}$$

$$\phi_2(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_\mathbf{p}}}\left(b_\mathbf{p}e^{i\mathbf{p}\cdot\mathbf{x}} + b_\mathbf{p}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right) \tag{483}$$

$$\pi_2(\mathbf{x}) = -i\int \frac{d^3p}{(2\pi)^3\sqrt{2}}\sqrt{\omega_\mathbf{p}}\left(b_\mathbf{p}e^{i\mathbf{p}\cdot\mathbf{x}} + b_\mathbf{p}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right) \tag{484}$$

$$[a_\mathbf{p}, a_\mathbf{q}^\dagger] = (2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{q}) \tag{485}$$

$$[b_\mathbf{p}, b_\mathbf{q}^\dagger] = (2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{q}) \tag{486}$$

then

$$\phi(\mathbf{x}) = \frac{1}{\sqrt{2}}\int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_\mathbf{p}}}\left((a_\mathbf{p} + ib_\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}} + (a_\mathbf{p}^\dagger + ib_\mathbf{p}^\dagger)e^{-i\mathbf{p}\cdot\mathbf{x}}\right) \tag{487}$$

$$\equiv \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_\mathbf{p}}}\left(\alpha_\mathbf{p}e^{i\mathbf{p}\cdot\mathbf{x}} + \beta_\mathbf{p}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right) \tag{488}$$

$$\phi^\dagger(\mathbf{x}) = \frac{1}{\sqrt{2}\int \frac{d^3p}{(2\pi)^3}\sqrt{2\omega_\mathbf{p}}}\left((a_\mathbf{p}^\dagger - ib_\mathbf{p}^\dagger)e^{-i\mathbf{p}\cdot\mathbf{x}} + (a_\mathbf{p} - ib_\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}\right) \tag{489}$$

$$\equiv \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_\mathbf{p}}}\left(\alpha_\mathbf{p}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} + \beta_\mathbf{p}e^{i\mathbf{p}\cdot\mathbf{x}}\right) \tag{490}$$

With the new defines creation/annihilation operators

$$\alpha_\mathbf{p} = \frac{1}{\sqrt{2}}(a_\mathbf{p} + ib_\mathbf{p}) \quad \rightarrow \quad \alpha_\mathbf{p}^\dagger = \frac{1}{\sqrt{2}}(a_\mathbf{p}^\dagger - ib_\mathbf{p}^\dagger) \tag{491}$$

$$\beta_\mathbf{p} = \frac{1}{\sqrt{2}}(a_\mathbf{p} - ib_\mathbf{p}) \quad \rightarrow \quad \beta_\mathbf{p}^\dagger = \frac{1}{\sqrt{2}}(a_\mathbf{p}^\dagger + ib_\mathbf{p}^\dagger) \tag{492}$$

we can calculate their commutation relations (**assuming all the cross commutators between $a, a^\dagger$ and $b, b^\dagger$ are zero**)

$$
\begin{aligned}
[\alpha_{\mathbf{p}}, \alpha_{\mathbf{q}}] &= \frac{1}{2}[a_{\mathbf{p}} + ib_{\mathbf{p}}, a_{\mathbf{q}} + ib_{\mathbf{q}}] && (493)\\
&= \frac{1}{2}([a_{\mathbf{p}}, a_{\mathbf{q}}] + i[b_{\mathbf{p}}, a_{\mathbf{q}}] + i[a_{\mathbf{p}}, b_{\mathbf{q}}] - [b_{\mathbf{p}}, b_{\mathbf{q}}]) && (494)\\
&= \frac{1}{2}i([b_{\mathbf{p}}, a_{\mathbf{q}}] + [a_{\mathbf{p}}, b_{\mathbf{q}}]) && (495)\\
&= 0 && (496)
\end{aligned}
$$

$$
\begin{aligned}
[\alpha^\dagger_{\mathbf{p}}, \alpha^\dagger_{\mathbf{q}}] &= \frac{1}{2}([a^\dagger_{\mathbf{p}} - ib^\dagger_{\mathbf{p}}, a^\dagger_{\mathbf{q}} - ib^\dagger_{\mathbf{q}}]) && (497)\\
&= \frac{1}{2}([a^\dagger_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] - i[b^\dagger_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] - i[a^\dagger_{\mathbf{p}}, b^\dagger_{\mathbf{q}}] - [b^\dagger_{\mathbf{p}}, b^\dagger_{\mathbf{q}}]) && (498)\\
&= \frac{1}{2}(-i[b^\dagger_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] - i[a^\dagger_{\mathbf{p}}, b^\dagger_{\mathbf{q}}]) && (499)\\
&= 0 && (500)
\end{aligned}
$$

$$
\begin{aligned}
[\alpha_{\mathbf{p}}, \alpha^\dagger_{\mathbf{q}}] &= \frac{1}{2}[a_{\mathbf{p}} + ib_{\mathbf{p}}, a^\dagger_{\mathbf{q}} - ib^\dagger_{\mathbf{q}}] && (501)\\
&= \frac{1}{2}([a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] + i[b_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] - i[a_{\mathbf{p}}, b^\dagger_{\mathbf{q}}] + [b_{\mathbf{p}}, b^\dagger_{\mathbf{q}}]) && (502)\\
&= (2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{q}) + i[b_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] - i[a_{\mathbf{p}}, b^\dagger_{\mathbf{q}}] && (503)\\
&= (2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{q}) && (504)
\end{aligned}
$$

$$
\begin{aligned}
[\beta_{\mathbf{p}}, \beta^\dagger_{\mathbf{q}}] &= \frac{1}{2}[a_{\mathbf{p}} - ib_{\mathbf{p}}, a^\dagger_{\mathbf{q}} + ib^\dagger_{\mathbf{q}}] && (505)\\
&= \frac{1}{2}([a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] - i[b_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] + i[a_{\mathbf{p}}, b^\dagger_{\mathbf{q}}] + [b_{\mathbf{p}}, b^\dagger_{\mathbf{q}}]) && (506)\\
&= (2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{q}) && (507)
\end{aligned}
$$

$$
\begin{aligned}
[\alpha_{\mathbf{p}}, \beta_{\mathbf{q}}] &= \frac{1}{2}[a_{\mathbf{p}} + ib_{\mathbf{p}}, a_{\mathbf{q}} - ib_{\mathbf{q}}] && (508)\\
&= \frac{1}{2}([a_{\mathbf{p}}, a_{\mathbf{q}}] + i[a_{\mathbf{p}}, b_{\mathbf{q}}] + i[b_{\mathbf{p}}, a_{\mathbf{q}}] - [b_{\mathbf{p}}, b_{\mathbf{q}}]) && (509)\\
&= 0 && (510)
\end{aligned}
$$

$$
\begin{aligned}
[\alpha_{\mathbf{p}}, \beta^\dagger_{\mathbf{q}}] &= \frac{1}{2}[a_{\mathbf{p}} + ib_{\mathbf{p}}, a^\dagger_{\mathbf{q}} - ib^\dagger_{\mathbf{q}}] && (511)\\
&= \frac{1}{2}([a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] + i[a_{\mathbf{p}}, b^\dagger_{\mathbf{q}}] + i[b_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] - [b_{\mathbf{p}}, b^\dagger_{\mathbf{q}}]) && (512)\\
&= (2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{q}) - (2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{q}) && (513)\\
&= 0 && (514)\\
[\alpha^\dagger_{\mathbf{p}}, \beta^\dagger_{\mathbf{q}}] &= 0 && (515)
\end{aligned}
$$

As the $\phi\mathbf{x}$ is in the Schroedinger picture there is not time dependency and we can not calculate $\pi(\mathbf{x})$ - therefore we need to transform to the Heisenberg picture. To make it simple we do this first for ϕ_1 and ϕ_2 using $p \cdot x = E_p t - \mathbf{p} \cdot \mathbf{x}$ and $p^2 = E_{\mathbf{p}}^2 - \mathbf{p}^2 = m^2$ (meaning $p^0 \equiv E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$

$$\phi_1(x) = e^{iHt}\phi(\mathbf{x})e^{-iHt} \tag{516}$$
$$= ... \tag{517}$$
$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} (a_{\mathbf{p}} e^{-ipx} + a_{\mathbf{p}}^\dagger e^{ipx}) \tag{518}$$
$$\phi_2(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} (b_{\mathbf{p}} e^{-ipx} + b_{\mathbf{p}}^\dagger e^{ipx}) \tag{519}$$
$$\tag{520}$$

Here we cheated a bit - we used the result from the scalar Lagrangian - meaning using the scalar Hamiltonian. Then

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} \left(\alpha_{\mathbf{p}} e^{-ipx} + \beta_{\mathbf{p}}^\dagger e^{ipx}\right) \tag{521}$$
$$\phi^\dagger(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} \left(\alpha_{\mathbf{p}}^\dagger e^{ipx} + \beta_{\mathbf{p}} e^{-ipx}\right) \tag{522}$$

and

$$\rightarrow \quad \pi^*(x) = \dot{\phi}(x) = i \int \frac{d^3p}{(2\pi)^3\sqrt{2}} \sqrt{E_{\mathbf{p}}} \left(-\alpha_{\mathbf{p}} e^{-ipx} + \beta_{\mathbf{p}}^\dagger e^{ipx}\right) \tag{523}$$
$$\rightarrow \quad \pi(x) = \dot{\phi}^\dagger(x) = i \int \frac{d^3p}{(2\pi)^3\sqrt{2}} \sqrt{E_{\mathbf{p}}} \left(\alpha_{\mathbf{p}}^\dagger e^{ipx} - \beta_{\mathbf{p}} e^{-ipx}\right) \tag{524}$$

The only non-vanishing commutator relations for field and momentum operators are

$$[\phi(\mathbf{x},t), \pi(\mathbf{y},t)] = i \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} \int \frac{d^3q}{(2\pi)^3\sqrt{2}} \sqrt{E_{\mathbf{q}}} [\alpha_{\mathbf{p}} e^{-ipx} + \beta_{\mathbf{p}}^\dagger e^{ipx}, \alpha_{\mathbf{q}}^\dagger e^{iqy} - \beta_{\mathbf{q}} e^{-iqy}] \tag{525}$$
$$= i \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{2} \sqrt{\frac{E_{\mathbf{q}}}{E_{\mathbf{p}}}} ([\alpha_{\mathbf{p}}, \alpha_{\mathbf{q}}^\dagger] e^{-ipx+iqy} - [\beta_{\mathbf{p}}^\dagger, \beta_{\mathbf{q}}] e^{ipx-iqy}) \tag{526}$$
$$= i \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{2} \sqrt{\frac{E_{\mathbf{q}}}{E_{\mathbf{p}}}} (e^{-ipx+iqy} + e^{ipx-iqy}) (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) \tag{527}$$
$$= i \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} (e^{-ip(x-y)} + e^{ip(x-y)}) \tag{528}$$
$$= i\delta^{(3)}(\mathbf{x} - \mathbf{y}) \tag{529}$$
$$[\phi^\dagger(\mathbf{x},t), \pi^\dagger(\mathbf{y},t)] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}) \tag{530}$$

To calculate the Heisenberg equations of motion we start with

$$\nabla\phi(x) = i \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} \mathbf{p} \left(\alpha_{\mathbf{p}} e^{-ipx} - \beta_{\mathbf{p}}^\dagger e^{ipx}\right) \tag{531}$$
$$\nabla\phi^\dagger(x) = i \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} \mathbf{p} \left(-\alpha_{\mathbf{p}}^\dagger e^{ipx} + \beta_{\mathbf{p}} e^{-ipx}\right) \tag{532}$$

and then

$$i\dot{\phi}(x) = [\phi(x), H] = \left[\phi(x), \int d^3y(\pi^\dagger\pi + \nabla\phi^\dagger \cdot \nabla\phi + m^2\phi^\dagger\phi)\right] \tag{533}$$

$$= \int d^3y\pi^\dagger(y)[\phi(x), \pi(y)] \tag{534}$$

$$= i\pi^\dagger(x) \tag{535}$$

$$i\dot{\phi}^\dagger(x) = [\phi^\dagger(x), H] = \left[\phi(x), \int d^3y(\pi^\dagger\pi + \nabla\phi^\dagger \cdot \nabla\phi + m^2\phi^\dagger\phi)\right] \tag{536}$$

$$= \int d^3y[\phi^\dagger(x), \pi^\dagger(y)]\pi(y) \tag{537}$$

$$= i\pi(x) \tag{538}$$

and

$$i\dot{\pi}(x) = [\pi(x), H] = \left[\pi(x), \int d^3y(\pi^\dagger\pi + \nabla\phi^\dagger \cdot \nabla\phi + m^2\phi^\dagger\phi)\right] \tag{539}$$

$$= \left[\pi(x), \int d^3y(\pi^\dagger\pi - \triangle\phi^\dagger \cdot \phi + m^2\phi^\dagger\phi)\right] \tag{540}$$

$$= \int d^3y(-\triangle\phi^\dagger + m^2\phi^\dagger)[\pi(x), \phi(y)] \tag{541}$$

$$= i(\triangle_x - m^2)\phi^\dagger(x) \tag{542}$$

$$i\dot{\pi}^\dagger(x) = [\pi^\dagger(x), H] = \left[\pi^\dagger(x), \int d^3y(\pi^\dagger\pi + \nabla\phi^\dagger \cdot \nabla\phi + m^2\phi^\dagger\phi)\right] \tag{543}$$

$$= \left[\pi^\dagger(x), \int d^3y(\pi^\dagger\pi - \phi^\dagger \cdot \triangle\phi + m^2\phi^\dagger\phi)\right] \tag{544}$$

$$= \int d^3y[\pi^\dagger(x), \phi^\dagger(y)](-\triangle\phi + m^2\phi) \tag{545}$$

$$= i(\triangle_x - m^2)\phi(x) \tag{546}$$

resulting in

$$i\dot{\pi}(x) \quad \rightarrow \quad \ddot{\phi}^\dagger = (\triangle - m^2)\phi^\dagger \tag{547}$$

$$\rightarrow \quad (\Box + m^2)\phi^\dagger = 0 \tag{548}$$

$$i\dot{\pi}^\dagger(x) \quad \rightarrow \quad \ddot{\phi} = (\triangle - m^2)\phi \tag{549}$$

$$\rightarrow \quad (\Box + m^2)\phi = 0 \tag{550}$$

(b)

(c)

(d)

0.9.3 Problem 2.3 - Calculating $D(x-y)$

As we are calculation the vacuum expectation value we need to get the $a^\dagger$'s to the right and the a's to the left

$$
\begin{aligned}
\phi(x)\phi(y) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}(a_{\mathbf{p}}e^{-ipx}+a^\dagger_{\mathbf{p}}e^{ipx})\int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{q}}}}(a_{\mathbf{q}}e^{-iqy}+a^\dagger_{\mathbf{q}}e^{iqy}) && (551)\\
&= \iint \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{q}}}}\frac{1}{\sqrt{2E_{\mathbf{p}}}}(a_{\mathbf{p}}e^{-ipx}+a^\dagger_{\mathbf{p}}e^{ipx})(a_{\mathbf{q}}e^{-iqy}+a^\dagger_{\mathbf{q}}e^{iqy}) && (552)\\
&= \iint \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{q}}}}\frac{1}{\sqrt{2E_{\mathbf{p}}}}(a_{\mathbf{p}}a_{\mathbf{q}}e^{-ipx-iqy}+a^\dagger_{\mathbf{p}}a_{\mathbf{q}}e^{ipx-iqy}+a_{\mathbf{p}}a^\dagger_{\mathbf{q}}e^{-ipx+iqy}+a^\dagger_{\mathbf{p}}a^\dagger_{\mathbf{q}}e^{ipx+iqy}) && (553)\\
&= \iint \frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{\sqrt{4E_{\mathbf{q}}E_{\mathbf{p}}}}(a_{\mathbf{p}}a_{\mathbf{q}}e^{-ipx-iqy}+(a_{\mathbf{q}}a^\dagger_{\mathbf{p}}-(2\pi)^3\delta(\mathbf{q}-\mathbf{p}))e^{ipx-iqy}+a_{\mathbf{p}}a^\dagger_{\mathbf{q}}e^{-ipx+iqy}+a^\dagger_{\mathbf{p}}a^\dagger_{\mathbf{q}}e^{ipx+iqy}) && (554)
\end{aligned}
$$

then with $a^\dagger|0\rangle = 0$ and $\langle 0|a = 0$

$$
\begin{aligned}
\langle 0|\phi(x)\phi(y)|\rangle &= \iint \frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{\sqrt{4E_{\mathbf{q}}E_{\mathbf{p}}}}(((\langle 0|a_{\mathbf{q}}a^\dagger_{\mathbf{p}}|0\rangle - \langle 0|0\rangle(2\pi)^3\delta(\mathbf{q}-\mathbf{p}))e^{ipx-iqy}+\langle 0|a_{\mathbf{p}}a^\dagger_{\mathbf{q}}|0\rangle e^{-ipx+iqy}) && (555)\\
&= \iint \frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{\sqrt{4E_{\mathbf{q}}E_{\mathbf{p}}}}\left(\left(\frac{\langle \mathbf{q}|\mathbf{p}\rangle}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}-(2\pi)^3\delta(\mathbf{q}-\mathbf{p})\right)e^{ipx-iqy}+\frac{\langle \mathbf{p}|\mathbf{q}\rangle}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}e^{-ipx+iqy}\right) && (556)\\
&= \iint \frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{\sqrt{4E_{\mathbf{q}}E_{\mathbf{p}}}}\left(\left(\frac{2E_{\mathbf{p}}(2\pi)^3\delta^{(3)}(\mathbf{q}-\mathbf{p})}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}-(2\pi)^3\delta(\mathbf{q}-\mathbf{p})\right)e^{ipx-iqy}+\frac{2E_{\mathbf{p}}(2\pi)^3\delta^{(3)}(\mathbf{q}-\mathbf{p})}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}e^{-i} && (557)\\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{4E^2_{\mathbf{p}}}}\left(\underbrace{\left(\frac{2E_{\mathbf{p}}}{\sqrt{4E^2_{\mathbf{p}}}}-1\right)}_{=0}e^{ipx-ipy}+\frac{2E_{\mathbf{p}}}{\sqrt{4E^2_{\mathbf{p}}}}e^{-ipx+ipy}\right) && (558)\\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ip(x-y)} && (559)
\end{aligned}
$$

Now we can calculate with $x^0 - y^0 = 0$ and $\mathbf{x}-\mathbf{y}=\mathbf{r}$

$$
\begin{aligned}
D(x-y) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ip(x-y)} && (560)\\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-i(E_p(x^0-y^0)-\mathbf{p}\cdot(\mathbf{x}-\mathbf{y}))} && (561)\\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} && (562)
\end{aligned}
$$

transforming to spherical coordinates

$$D(x-y) = 2\pi \int_0^\infty \frac{p^2 dp}{(2\pi)^3} \frac{1}{2\sqrt{p^2+m^2}} \int \sin\theta\, e^{ipr\cos\theta} d\theta \tag{563}$$

$$= 2\pi \int_0^\infty \frac{p^2 dp}{(2\pi)^3} \frac{1}{2\sqrt{p^2+m^2}} \left[\frac{1}{(-ipr)} e^{ipr\cos\theta}\right]_0^\pi \tag{564}$$

$$= 2\pi \int_0^\infty \frac{p^2 dp}{(2\pi)^3} \frac{1}{2\sqrt{p^2+m^2}} \frac{1}{(-ipr)} (e^{-ipr} - e^{ipr}) \tag{565}$$

$$= \frac{i}{2(2\pi)^2 r} \int_0^\infty \frac{p\, dp}{\sqrt{p^2+m^2}} (e^{-ipr} - e^{ipr}) \tag{566}$$

$$= \frac{i}{2(2\pi)^2 r} \left(\int_0^\infty \frac{p\, dp}{\sqrt{p^2+m^2}} e^{-ipr} - \int_0^\infty \frac{p\, dp}{\sqrt{p^2+m^2}} e^{ipr} \right) \tag{567}$$

$$= \frac{i}{2(2\pi)^2 r} \left(\int_0^\infty \frac{p\, dp}{\sqrt{p^2+m^2}} e^{-ipr} - \int_0^{-\infty} \frac{(-p)\,(-dp)}{\sqrt{(-p)^2+m^2}} e^{i(-p)r} \right) \tag{568}$$

$$= \frac{i}{2(2\pi)^2 r} \int_{-\infty}^\infty \frac{p\, dp}{\sqrt{p^2+m^2}} e^{-ipr} \tag{569}$$

$$= \frac{-i}{2(2\pi)^2 r} \int_{-\infty}^\infty \frac{p\, dp}{\sqrt{p^2+m^2}} e^{ipr} \qquad (r \to -r) \tag{570}$$

Let's use contour integration (closing the contour above - $\lim_{p\to i\infty} e^{ipr} = e^{-\infty r} = 0$ so the upper half circle integral vanishes). Furthermore we see that the square root becomes zero at $\pm im$.

0.9.4 Problem 3.1 - Lorentz group

With the Lie algebra for the six generators ($J^{01}, J^{02}, J^{03}, J^{12}, J^{13}, J^{12}$ - three boosts and three rotations) are given by

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}) \tag{571}$$

and

$$L^i = \frac{1}{2}\epsilon^{ijk}J^{jk}, \qquad K^i = J^{0i} \tag{572}$$

(a) We start with calculating $[L^a, L^b]$, $[K^a, K^b]$ and $[L^a, K^b]$. Using $g^{kl} = -\delta^{kl}$ where $k = 1, 2, 3$

$$[L^a, L^b] = \frac{1}{4}[\epsilon^{ajk}J^{jk}, \epsilon^{blm}J^{lm}] \tag{573}$$

$$= \frac{1}{4}\epsilon^{ajk}\epsilon^{blm}[J^{jk}, J^{lm}] \tag{574}$$

$$= \frac{i}{4}\epsilon^{ajk}\epsilon^{blm}(g^{kl}J^{jm} - g^{jl}J^{km} - g^{km}J^{jl} + g^{jm}J^{kl}) \tag{575}$$

$$= -\frac{i}{4}(\epsilon^{ajk}\epsilon^{bkm}J^{jm} - \epsilon^{ajk}\epsilon^{bjm}J^{km} - \epsilon^{ajk}\epsilon^{blk}J^{jl} + \epsilon^{ajk}\epsilon^{blj}J^{kl}) \tag{576}$$

$$= -\frac{i}{4}(-\epsilon^{ajk}\epsilon^{bmk}J^{jm} - \epsilon^{akj}\epsilon^{bmj}J^{km} - \epsilon^{ajk}\epsilon^{blk}J^{jl} - \epsilon^{akj}\epsilon^{blj}J^{kl}) \tag{577}$$

and use $\epsilon_{abk}\epsilon^{cdk} = \delta_a^c\delta_b^d - \delta_a^d\delta_b^c$

$$[L^a, L^b] = -\frac{i}{4}[-(\delta_{ab}\delta_{jm} - \delta_{am}\delta_{jb})J^{jm} - (\delta_{ab}\delta_{km} - \delta_{am}\delta_{kb})J^{km} - (\delta_{ab}\delta_{jl} - \delta_{al}\delta_{jb})J^{jl} - (\delta_{ab}\delta_{kl} - \delta_{al}\delta_{kb})J^{kl}) \tag{578}$$

$$= -\frac{i}{4}\left[-(\delta_{ab}J^{mm} - J^{ba}) - (\delta_{ab}J^{mm} - J^{ba}) - (\delta_{ab}J^{ll} - J^{ba}) - (\delta_{ab}J^{ll} - J^{ba})\right] \tag{579}$$

as the diagonal elements of J are zero the trace J^{mm} vanishes as well and we obtain

$$
\begin{aligned}
[L^a, L^b] &= -iJ^{ba} = iJ^{ab} = i\frac{1}{2}(J^{ab} - J^{ba}) && (580)\\
&= \frac{i}{2}(\delta_{am}\delta_{bn} - \delta_{an}\delta_{bm})J^{mn} && (581)\\
&= \frac{i}{2}\epsilon_{abk}\epsilon^{mnk}J^{mn} && (582)\\
&= \frac{i}{2}\epsilon_{abk}\epsilon^{mnk}J^{mn} && (583)\\
&= i\epsilon_{abk}\frac{1}{2}\epsilon^{mnk}J^{mn} && (584)\\
&= i\epsilon_{abk}\frac{1}{2}\epsilon^{kmn}J^{mn} && (585)\\
&= i\epsilon_{abk}L^k. && (586)
\end{aligned}
$$

Now with $a, b = 1, 2, 3$

$$
\begin{aligned}
[K^a, K^b] &= [J^{0a}, J^{0b}] && (587)\\
&= i(g^{a0}J^{0b} - g^{00}J^{ab} - g^{ab}J^{00} + g^{0b}J^{a0}) && (588)\\
&= i(0 \cdot J^{0b} - 1 \cdot J^{ab} - 0 \cdot J^{00} + 0 \cdot J^{a0}) && (589)\\
&= -iJ^{ab} && (590)\\
&= ... \qquad \text{(same as last calculation above)} && (591)\\
&= -i\epsilon_{abk}L^k && (592)
\end{aligned}
$$

And

$$
\begin{aligned}
[L^a, K^b] &= \frac{1}{2}\epsilon^{ajk}[J^{jk}, J^{0b}] && (593)\\
&= \frac{i}{2}\epsilon^{ajk}(g^{k0}J^{jb} - g^{j0}J^{kb} - g^{kb}J^{j0} + g^{jb}J^{k0}) && (594)\\
&= \frac{i}{2}\epsilon^{ajk}\left(0 \cdot J^{jb} - 0 \cdot J^{kb} - g^{kb} \cdot (-K^j) + g^{jb} \cdot (-K^k)\right) && (595)\\
&= \frac{i}{2}\left(+\epsilon^{ajb}(-1)K^j - \epsilon^{abk}(-1)K^k\right) && (596)\\
&= \frac{i}{2}\left(-\epsilon^{abj}(-1)K^j - \epsilon^{abk}(-1)K^k\right) && (597)\\
&= i\epsilon^{abj}K^j && (598)
\end{aligned}
$$

Now we can finally calculate

$$
\begin{aligned}
[J_+^a, J_+^b] &= \frac{1}{4}\left([L^a, L^b] + i[L^a, K^b] + i[K^a, L^b] + i^2[K^a, K^b]\right) && (599)\\
&= \frac{1}{4}\left(i\epsilon^{abk}L^k + i \cdot i\epsilon^{abj}K^j + i \cdot i\epsilon^{abj}K^j - (-1)i\epsilon^{abk}L^k\right) && (600)\\
&= \frac{1}{4}\left(i\epsilon^{abk}L^k - \epsilon^{abj}K^j - \epsilon^{abj}K^j + i\epsilon^{abk}L^k\right) && (601)\\
&= \frac{1}{2}i\epsilon^{abk}(L^k + iK^k) && (602)\\
&= i\epsilon^{abk}J_+^k && (603)
\end{aligned}
$$

and

$$[J_-^a, J_-^b] = \frac{1}{4}\left([L^a, L^b] - i[L^a, K^b] - i[K^a, L^b] + i^2[K^a, K^b]\right) \tag{604}$$
$$= \tag{605}$$
$$[J_-^a, J_+^b] = \frac{1}{4}\left([L^a, L^b] - i[L^a, K^b] - i[K^a, L^b] - i^2[K^a, K^b]\right) \tag{606}$$
$$= \tag{607}$$

0.10 Schwartz - Quantum Field Theory and the Standard Model

0.10.1 Problem 2.2 Special relativity and colliders

1. Quick special relativity recap
$$p'^{\mu} = \Lambda^{\mu}_{\nu} p^{\nu} \quad p^{\mu}p_{\mu} = m^2c^2 \tag{608}$$
At rest
$$p^{\mu}p_{\mu} = (p^0)^2 - \vec{p}^2 = (p^0)^2 = m^2c^2 \tag{609}$$
After Lorentz trafo in x direction
$$\Lambda = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{610}$$
$$p'^{\mu} = (\gamma p^0, -\beta\gamma p^0, 0, 0) \tag{611}$$
$$\equiv \left(\frac{E}{c}, \vec{p}\right) \tag{612}$$
with $p^{\mu}p_{\mu} = m^2c^2$ we have $E^2/c^2 + \vec{p}^2 = m^2c^2$.
Now we can solve the problem
$$\frac{E_{cm}}{2} = \sqrt{m_p^2c^4 + p^2c^2} \tag{613}$$
$$\to p = \frac{1}{c}\sqrt{\frac{E_{cm}^2}{4} - m_p^2c^4} \equiv \beta\gamma m_p c \tag{614}$$
$$\to \frac{E_{cm}^2}{4} = m_p^2c^4(\beta^2\gamma^2 + 1) \tag{615}$$
$$\to \gamma = \frac{E_{cm}}{2m_pc^2} \tag{616}$$
$$\to \beta = \sqrt{1 - \left(\frac{2m_pc}{E_{cm}}\right)^2} \approx 1 - \frac{1}{2}\left(\frac{2m_pc^2}{E_{cm}}\right)^2 \tag{617}$$
$$\to c - v = 2\left(\frac{m_pc^2}{E_{cm}}\right)^2 c = 2.69\text{m/s} \tag{618}$$

2. Using the velocity addition formula
$$\Delta v = \frac{2v}{1 + \frac{v^2}{c^2}} \approx c\left(1 - 2\left[\frac{m_pc^2}{E_{cm}}\right]^4\right) \tag{619}$$

0.10.2 Problem 2.3 GZK bound

1. We are utilizing Plancks law

$$w_\nu d\nu = \frac{8\pi h\nu^3}{c^3}\frac{d\nu}{e^{h\nu/k_BT}-1} \tag{620}$$

where the spectral energy density w_ν [J m^{-3} s] gives the spacial energy density per frequency interval $d\nu$. The total radiative energy density is then given by

$$\begin{aligned}\rho_{\text{rad}} &= \frac{8\pi h}{c^3}\int_0 \infty \frac{\nu^3 d\nu}{e^{h\nu/k_BT}-1} && (621)\\ &= \frac{8\pi h}{c^3}\cdot\frac{(\pi k_BT)^4}{15h^4} && (622)\\ &= \frac{8\pi^5 k_B^4T^4}{15h^3c^3} = 0.26\text{MeV/m}^3. && (623)\end{aligned}$$

The photon density is given by

$$\begin{aligned}n_{\text{rad}} &= \int_0^\infty \frac{w_\nu}{h\nu}d\nu && (624)\\ &= \frac{8\pi}{c^3}\int\frac{\nu^2 d\nu}{e^{h\nu/k_BT}-1} && (625)\\ &= \frac{8\pi}{c^3}\cdot\frac{2\zeta(3)k_B^3T^3}{h^3} && (626)\\ &= \frac{16\pi\zeta(3)k_B^3T^3}{h^3c^3} = 416\text{cm}^{-3}. && (627)\end{aligned}$$

The average photon energy is then given by

$$\begin{aligned}E_{\text{ph}} &= \frac{\rho_{\text{rad}}}{n_{\text{rad}}} = \frac{\pi^4}{30\zeta(3)}k_BT = 0.63\text{meV} && (628)\\ \lambda_{\text{ph}} &= \frac{hc}{E_{\text{ph}}} = 1.9\text{mm} && (629)\end{aligned}$$

therefore it is called CM(icrowave)B. One obtains slightly other values if the peak of the Planck spectrum is used as definition of the average photon energy.

2. In the center-of-mass system the total momentum before and after the collision vanishes

$$\vec{p}^{cm}_{p^+} + \vec{p}^{cm}_{\gamma} = 0 = \vec{\hat{p}}^{cm}_{p^+} + \vec{\hat{p}}^{cm}_{\pi^0}. \tag{630}$$

which implies for (Lorentz-invariant) norm the systems 4-momentum $P^{cm} = p^{cm}_{p^+} + p^{cm}_{\pi^0}$

$$\begin{aligned}(P^{cm})^2 &= (E^{cm}_{p^+} + E^{cm}_{\gamma})^2 - c^2(\vec{p}^{cm}_{p^+} + \vec{p}^{cm}_{\gamma})^2 && (631)\\ &= (E^{cm}_{p^+} + E^{cm}_{\gamma})^2 && (632)\\ &= (E^{cm})^2 && (633)\\ &\overset{!}{=} (E_{p^+} + E_\gamma)^2 - c^2(\vec{p}_{p^+} + \vec{p}_\gamma)^2 && (634)\\ &\overset{!}{=} (\hat{E}_{p^+} + \hat{E}_{\pi^0})^2 - c^2(\vec{\hat{p}}_{p^+} + \vec{\hat{p}}_{\pi^0})^2 && (635)\end{aligned}$$

with $p^i = \hbar k^i = \hbar(\omega, \vec{k}) = \hbar(\omega, \frac{2\pi}{\lambda}\vec{e}_k) = h(\nu, \frac{\nu}{c}\vec{e}_k)$ and the values before

$$E_{p^+} = m_{p^+}c^2 + T_{p^+} \tag{636}$$
$$E_\gamma = h\nu \tag{637}$$
$$(\vec{p}_{p^+})^2 = \frac{1}{c^2}\left[(E_{p^+})^2 - (m_{p^+})^2 c^4\right] \tag{638}$$
$$= \frac{T_{p^+}}{c^2}\left[T_{p^+} + 2m_{p^+}c^2\right] \tag{639}$$
$$(\vec{p}_\gamma)^2 = \frac{h^2\nu^2}{c^2} \tag{640}$$

At the threshold the π^0 is created without any kinetic energy. As the total momentum is vanishing the proton also needs to be at rest

$$(E_{p^+} + E_\gamma)^2 - c^2(\vec{p}_{p^+} + \vec{p}_\gamma)^2 = \left(m_{p^+}c^2 + m_{\pi^0}c^2\right)^2 \tag{641}$$
$$E_{p^+}^2 + 2E_{p^+}E_\gamma + E_\gamma^2 - c^2\left(\vec{p}_{p^+}^2 + \vec{p}_\gamma^2 - 2\vec{p}_{p^+}\cdot\vec{p}_\gamma\right) = \left(m_{p^+}c^2 + m_{\pi^0}c^2\right)^2 \tag{642}$$
$$m_{p^+}^2c^4 + 2E_{p^+}E_\gamma + 2c^2\vec{p}_{p^+}\cdot\vec{p}_\gamma = \left(m_{p^+}c^2 + m_{\pi^0}c^2\right)^2 \tag{643}$$
$$m_{p^+}^2c^4 + 2E_{p^+}E_\gamma + 2E_\gamma\sqrt{E_{p^+}^2 - m_{p^+}^2c^2}\cos\phi = \left(m_{p^+}c^2 + m_{\pi^0}c^2\right)^2 \tag{644}$$
$$E_{p^+}E_\gamma + E_\gamma\sqrt{E_{p^+}^2 - m_{p^+}^2c^2}\cos\phi = \left(m_{p^+} + \frac{m_{\pi^0}}{2}\right)m_{\pi^0}c^4 \tag{645}$$

Now we can square the equation and solve approximately assuming $E_\gamma \ll m_{p^+}c^2$

$$E_\gamma\sqrt{E_{p^+}^2 - m_{p^+}^2c^2}\cos\phi = \left(m_{p^+} + \frac{m_{\pi^0}}{2}\right)m_{\pi^0}c^4 - E_{p^+}E_\gamma \tag{646}$$
$$E_\gamma^2\left(E_{p^+}^2 - m_{p^+}^2c^2\right)\cos^2\phi = \left(m_{p^+} + \frac{m_{\pi^0}}{2}\right)^2 m_{\pi^0}^2c^8 + (E_{p^+}E_\gamma)^2 - 2E_{p^+}E_\gamma\left(m_{p^+} + \frac{m_{\pi^0}}{2}\right)m_{\pi^0}c^4 \tag{647}$$
$$-E_\gamma^2 m_{p^+}^2c^2\cos^2\phi = \left(m_{p^+} + \frac{m_{\pi^0}}{2}\right)^2 m_{\pi^0}^2c^8 - 2E_{p^+}E_\gamma\left(m_{p^+} + \frac{m_{\pi^0}}{2}\right)m_{\pi^0}c^4 \tag{648}$$
$$E_{p^+} \approx \frac{\left(m_{p^+} + m_{\pi^0}/2\right)m_{\pi^0}c^4}{2E_\gamma} \tag{649}$$
$$= 10.8\cdot 10^{19}\text{eV} \tag{650}$$

3. By assumption the p^+ and the π^0 would rest in the CM system

$$(P^\mu)^{cm} = (p_{p^+}^\mu)^{cm} + (p_{\pi^0}^\mu)^{cm} \tag{651}$$
$$= \left([m_{p^+} + m_{\pi^0}]c^2, \vec{0}\right) \tag{652}$$
$$= \Lambda^\mu_\alpha\left[\hat{p}_{p^+}^\alpha + \hat{p}_{\pi^0}^\alpha\right] \tag{653}$$
$$= \Lambda^\mu_\alpha\left[p_{p^+}^\alpha + p_\gamma^\alpha\right] \tag{654}$$
$$\tag{655}$$

We can therefore calculate γ

$$\mu = 1: \quad 0 = \underbrace{\Lambda^1_0}_{-\gamma\beta}(E_{p^+} + E_\gamma) + \underbrace{\Lambda^1_1}_{\gamma} c(p^x_{p^+} + p^x_\gamma) \tag{656}$$

$$= -\gamma\beta(E_{p^+} + E_\gamma) + \gamma\left(\sqrt{E^2_{p^+} - m_p^2 c^4} + E_\gamma\right) \tag{657}$$

$$\to \beta = \frac{\sqrt{E^2_{p^+} - m_p^2 c^4} + E_\gamma}{E_{p^+} + E_\gamma} \approx \frac{\sqrt{E^2_{p^+} - m_p^2 c^4}}{E_{p^+}} \tag{658}$$

$$\to \gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{E_{p^+}}{m_{p^+}c^2} \tag{659}$$

which can be used to calculate the pion momentum

$$c\hat{p}_{\pi^0} = \Lambda^0_\mu (p^\mu_{\pi^0})^{cm} \tag{660}$$

$$= \Lambda^0_0 (p^0_{\pi^0})^{cm} \tag{661}$$

$$= \gamma m_{\pi^0} c^2 \tag{662}$$

$$= E_{p^+} \frac{m_{\pi^0}}{m_{p^+}}. \tag{663}$$

The $p+$ energy after the collision is then given by

$$E_{p^+} + E_\gamma = \hat{E}_{p^+} + \hat{E}_{\pi^0} \tag{664}$$

$$\to \hat{E}_{p^+} = E_{p^+} + E_\gamma - \hat{E}_{\pi^0} \tag{665}$$

$$= E_{p^+} + E_\gamma - \sqrt{m^2_{\pi^0} c^4 + \hat{\vec{p}}_{\pi^0} c^2} \tag{666}$$

$$= E_{p^+} + E_\gamma - \sqrt{m^2_{\pi^0} c^4 + E^2_{p^+} \frac{m^2_{\pi^0}}{m^2_{p^+}}} \tag{667}$$

$$= E_{p^+} + E_\gamma - m_{\pi^0} c^2 \sqrt{1 + \frac{E^2_{p^+}}{m^2_{p^+} c^4}} \tag{668}$$

$$\approx E_{p^+} - m_{\pi^0} c^2 \frac{E_{p^+}}{m_{p^+} c^2} \tag{669}$$

$$= E_{p^+} \left(1 - \frac{m_{\pi^0}}{m_{p^+}}\right) \tag{670}$$

$$\approx 0.85 \cdot E_{p^+}. \tag{671}$$

0.10.3 Problem 2.5 Compton scattering

1. the binding energy of outer(!!!) electrons is in the eV range while typical X-rays energies are in the keV range.

2. In the nonrelativistic case we have energy and momentum conservation

$$\frac{hc}{\lambda} = \frac{hc}{\lambda'} + \frac{1}{2} m_e v^2 \tag{672}$$

$$\frac{h}{\lambda} = \frac{h}{\lambda'} \cos\theta + m_e v \cos\phi \tag{673}$$

$$0 = \frac{h}{\lambda'} \sin\theta + m_e v \sin\phi \tag{674}$$

then we see

$$v = \sqrt{\frac{2hc}{m_e}\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)} = \sqrt{\frac{2hc}{m_e}\frac{\lambda' - \lambda}{\lambda\lambda'}} \tag{675}$$

and

$$\sin\phi = -\frac{h}{m_e v}\frac{1}{\lambda'}\sin\theta \tag{676}$$

$$\cos\phi = \frac{h}{m_e v}\frac{1}{\lambda'}\left(\frac{\lambda'}{\lambda} - \cos\theta\right) \tag{677}$$

$$\to 1 = \sin^2\phi + \cos^2\phi \tag{678}$$

$$= \frac{h^2}{m_e^2 v^2 \lambda'^2}\left(\sin^2\theta + \frac{\lambda'^2}{\lambda^2} - 2\frac{\lambda'}{\lambda}\cos\theta + \cos^2\theta\right) \tag{679}$$

$$= \frac{h^2}{m_e^2 v^2 \lambda'^2}\left(1 + \frac{\lambda'^2}{\lambda^2} - 2\frac{\lambda'}{\lambda}\cos\theta\right) \tag{680}$$

$$= \frac{h\lambda}{2m_e c\lambda'(\lambda' - \lambda)}\left(1 + \frac{\lambda'^2}{\lambda^2} - 2\frac{\lambda'}{\lambda}\cos\theta\right) \tag{681}$$

$$= \frac{h}{2m_e c(\lambda' - \lambda)}\left(\frac{\lambda}{\lambda'} + \frac{\lambda'}{\lambda} - 2\cos\theta\right) \tag{682}$$

$$\lambda' - \lambda \approx \frac{h}{m_e c}(1 - \cos\theta) \tag{683}$$

where we used $\lambda \approx \lambda'$.

3.

0.10.4 Problem 2.6 Lorentz invariance

1. With $\omega_k = \sqrt{\vec{k}^2 + m^2}$

$$\int_{-\infty}^{\infty} dk^0 \delta(k^2 - m^2)\theta(k^0) = \int_{-\infty}^{\infty} dk^0 \delta({k^0}^2 - [\vec{k}^2 + m^2])\theta(k^0) \tag{684}$$

$$= \frac{\theta(\omega_k)}{2\omega_k} + \frac{\theta(-\omega_k)}{2\omega_k} \tag{685}$$

$$= \frac{1}{2\omega_k} \tag{686}$$

2. Under Lorentz transformations we have $k^2 - m^2 = 0$. For orthochronous transformation we have k^0...

3. Now we can put it all together

$$\int d^4k\delta(k^2 - m^2)\theta(k^0) = \int d^3k \int dk^0 \delta(k^2 - m^2)\theta(k^0) \tag{687}$$

$$= \int \frac{d^3k}{2\omega_k} \tag{688}$$

0.10.5 Problem 2.7 Coherent states

1.

$$\partial_z\left(e^{-za^\dagger} a e^{-za^\dagger}\right) = -e^{-za^\dagger} a^\dagger a e^{-za^\dagger} + e^{-za^\dagger} a a^\dagger e^{-za^\dagger} \tag{689}$$

$$= e^{-za^\dagger}[a, a^\dagger]e^{-za^\dagger} \tag{690}$$

$$= 1 \tag{691}$$

2. Rolling the a through the $(a^\dagger)^k$ using the commutator $[a, a^\dagger] = 1$

$$a|z\rangle = ae^{za^\dagger}|0\rangle \tag{692}$$
$$= a\sum_{k=0} \frac{1}{k!} z^k (a^\dagger)^k |0\rangle \tag{693}$$
$$= a|0\rangle + \sum_{k=1} \frac{k}{k!} z^k (a^\dagger)^{k-1} |0\rangle \tag{694}$$
$$= z\sum_{n=0} \frac{1}{n!} z^n (a^\dagger)^n |0\rangle \tag{695}$$
$$= z|z\rangle \tag{696}$$

3. With $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ and using the $|z\rangle$ is an eigenstate of a we have

$$\langle n|z\rangle = \frac{1}{\sqrt{n!}}\langle 0|a^n|z\rangle = \frac{z^n}{\sqrt{n!}}\langle 0|z\rangle = \frac{z^n}{\sqrt{n!}}\langle 0|e^{za^\dagger}|0\rangle \tag{697}$$
$$= \frac{z^n}{\sqrt{n!}}\langle 0|1 + za^\dagger + \frac{1}{2}z^2(a^\dagger)^2 + ...|0\rangle \tag{698}$$
$$= \frac{z^n}{\sqrt{n!}}\langle 0|0\rangle = \frac{z^n}{\sqrt{n!}} \tag{699}$$

where we used $\langle 0|a^\dagger = 0$.

4. With

$$a + a^\dagger = \sqrt{\frac{m\omega}{2}}2q \quad \rightarrow \quad q = \frac{1}{\sqrt{2m\omega}}(a + a^\dagger) \tag{700}$$
$$a - a^\dagger = \sqrt{\frac{m\omega}{2}}2\frac{ip}{m\omega} \quad \rightarrow \quad p = -i\frac{\sqrt{m\omega}}{\sqrt{2}}(a - a^\dagger) \tag{701}$$

and $a|z\rangle = z|z\rangle$ and $\langle z|a^\dagger = \bar{z}\langle z|$

$$\langle z|q|z\rangle = \frac{1}{\sqrt{2m\omega}}\langle z|a + a^\dagger|z\rangle = \frac{1}{\sqrt{2m\omega}}\langle z|z\rangle(z + \bar{z}) \tag{702}$$
$$\langle z|p|z\rangle = -i\frac{\sqrt{m\omega}}{\sqrt{2}}\langle z|a - a^\dagger|z\rangle = -i\frac{\sqrt{m\omega}}{\sqrt{2}}\langle z|z\rangle(z - \bar{z}) \tag{703}$$
$$\langle z|q^2|z\rangle = \frac{1}{2m\omega}\langle z|aa + \underbrace{aa^\dagger}_{=1+a^\dagger a} + a^\dagger a + a^\dagger a^\dagger|z\rangle \tag{704}$$
$$= \frac{1}{2m\omega}\langle z|z\rangle\left(z^2 + 1 + 2z\bar{z} + \bar{z}^2\right) \tag{705}$$
$$\langle z|p^2|z\rangle = -\frac{m\omega}{2}\langle z|aa - \underbrace{aa^\dagger}_{=1+a^\dagger a} - a^\dagger a + a^\dagger a^\dagger|z\rangle \tag{706}$$
$$= -\frac{m\omega}{2}\langle z|z\rangle\left(z^2 - 1 - 2z\bar{z} + \bar{z}^2\right) \tag{707}$$

Therefore

$$\Delta q^2 = \langle q^2\rangle - \langle q\rangle^2 \tag{708}$$
$$= \frac{1}{2m\omega}\left(z^2 + 1 + 2z\bar{z} + \bar{z}^2\right) - \left(\frac{1}{\sqrt{2m\omega}}(z + \bar{z})\right)^2 \tag{709}$$
$$= \frac{1}{2m\omega} \tag{710}$$

and

$$\Delta p^2 = \langle p^2 \rangle - \langle p \rangle^2 \tag{711}$$
$$= -\frac{m\omega}{2}\left(z^2 - 1 - 2z\bar{z} + \bar{z}^2\right) - \left(-i\frac{\sqrt{m\omega}}{\sqrt{2}}(z-\bar{z})\right)^2 \tag{712}$$
$$= \frac{m\omega}{2} \tag{713}$$

which means

$$\Delta p \Delta q = \frac{1}{\sqrt{2m\omega}}\frac{\sqrt{m\omega}}{\sqrt{2}} = \frac{1}{2}. \tag{714}$$

5. At first let's construct the eigenstate $|w\rangle$ for a manually

$$a|w\rangle = c_w|w\rangle \tag{715}$$

Expanding the eigenstate with $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$

$$|w\rangle = \sum_n \alpha_n|n\rangle \tag{716}$$
$$a|w\rangle = \sum_n \alpha_n\sqrt{n}|n-1\rangle \stackrel{!}{=} c_w\sum_n \alpha_n|n\rangle = c_w|n\rangle \tag{717}$$
$$\rightarrow \alpha_n\sqrt{n} = c_w\alpha_{n-1} \tag{718}$$
$$\rightarrow \alpha_n = \frac{c_w}{\sqrt{n}}\alpha_{n-1} \tag{719}$$
$$|w\rangle = \sum_n \alpha_0\frac{c_w^n}{\sqrt{n!}}|n\rangle = \alpha_0\sum_n \frac{c_w^n}{n!}(a^\dagger)^n|0\rangle = \alpha_0 e^{c_w a^\dagger}|0\rangle \tag{720}$$

Now we do the same for $a^\dagger$

$$a^\dagger|v\rangle = c_v|v\rangle \tag{721}$$

Expanding the eigenstate

$$|v\rangle = \sum_n \beta_n|n\rangle \tag{722}$$
$$a^\dagger|v\rangle = \sum_n \beta_n\sqrt{n+1}|n+1\rangle \stackrel{!}{=} c_v\sum_n \beta_n|n\rangle = c_v|n\rangle \tag{723}$$
$$\rightarrow \beta_n\sqrt{n+1} = c_v\beta_{n+1} \tag{724}$$
$$\rightarrow \beta_{n+1} = \frac{\sqrt{n+1}}{c_v}\beta_n \tag{725}$$
$$|v\rangle = \sum_n \beta_0\frac{\sqrt{n!}}{c_v^n}|n\rangle = \beta_0\sum_n \frac{1}{c_v^n}(a^\dagger)^n|0\rangle \tag{726}$$

Now we calculate with $\langle 0|a^\dagger = 0$

$$\langle 0|a^\dagger|v\rangle = \beta_0\sum_n \frac{1}{c_v^n}\langle 0|(a^\dagger)^{n+1}|0\rangle \tag{727}$$
$$= \beta_0\frac{1}{c_v^0}\langle 0|a^\dagger|0\rangle \tag{728}$$
$$\tag{729}$$

0.10.6 Problem 3.1 Higher order Lagrangian

With the principle of least action

$$\delta S = \delta \int \mathcal{L} d^4x = \int \delta \mathcal{L} d^4x \tag{730}$$

we calculate

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu\phi) + \frac{\partial \mathcal{L}}{\partial(\partial_\nu\partial_\mu\phi)}\delta(\partial_\nu\partial_\mu\phi) + ... \tag{731}$$

Now we can integrate each term

$$\delta\mathcal{L}_0 = \int \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi d^4x \tag{732}$$

$$\delta\mathcal{L}_1 = \int \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu\phi) d^4x = \int \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\mu\delta\phi d^4x \tag{733}$$

$$= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi\Big|_{\partial\Omega} - \int \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi d^4x \tag{734}$$

$$\delta\mathcal{L}_2 = \int \frac{\partial \mathcal{L}}{\partial(\partial_\nu\partial_\mu\phi)}\delta(\partial_\nu\partial_\mu\phi) d^4x = \int \frac{\partial \mathcal{L}}{\partial(\partial_\nu\partial_\mu\phi)}\partial_\nu\delta\partial_\mu\phi d^4x \tag{735}$$

$$= \frac{\partial \mathcal{L}}{\partial(\partial_\nu\partial_\mu\phi)}\delta\partial_\mu\phi\Big|_{\partial\Omega} - \int \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu\partial_\mu\phi)}\delta\partial_\mu\phi d^4x \tag{736}$$

$$= \frac{\partial \mathcal{L}}{\partial(\partial_\nu\partial_\mu\phi)}\delta\partial_\mu\phi\Big|_{\partial\Omega} - \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu\partial_\mu\phi)}\delta\phi\Big|_{\partial\Omega} + \int \partial_\mu\partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu\partial_\mu\phi)}\delta\phi d^4x \tag{737}$$

Requiring that all derivatives vanish at infinity we obtain

$$\delta S = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} + \partial_\mu\partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu\partial_\mu\phi)} - ... \right)\delta\phi \tag{738}$$

and therefore

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} + \partial_\mu\partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu\partial_\mu\phi)} - ... = 0 \tag{739}$$

0.10.7 Problem 3.5 Spontaneous symmetry

$$\mathscr{L} = -\frac{1}{2}\phi\Box\phi + \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{740}$$

$$\frac{\partial \mathscr{L}}{\partial \phi} - \partial_\beta \frac{\partial \mathscr{L}}{\partial(\partial_\beta \phi)} + \partial_\mu\partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu\partial_\mu\phi)} = 0 \tag{741}$$

$$\to -\Box\phi + m^2\phi - \frac{\lambda}{3!}\phi^3 = 0 \tag{742}$$

and the Hamiltonian with $-\phi\Box\phi \sim (\partial_\mu\phi)(\partial^\mu\phi) = \eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \tag{743}$$

$$= \dot{\phi} \tag{744}$$

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} \tag{745}$$

$$= (\dot{\phi})^2 - \mathcal{L} \tag{746}$$

$$= \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{747}$$

(a)

$$m^2\phi - \frac{\lambda}{3!}\phi^3 = 0 \tag{748}$$

$$(m^2 - \frac{\lambda}{3!}\phi^2)\phi = 0 \tag{749}$$

$$\phi_0 = 0 \quad \rightarrow \quad \mathcal{H}[\phi] = 0 \tag{750}$$

$$\phi_{1,2} = \pm\sqrt{\frac{3!}{\lambda}}m \quad \rightarrow \quad \mathcal{H}[\phi] = -\frac{3m^4}{2\lambda} \tag{751}$$

(b)

(c)

0.10.8 Problem 3.6 Yukawa potential

(a) We slit the Lagranian in three parts

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}m^2A_\mu^2 - A_\mu J_\mu \tag{752}$$

$$= \mathscr{L}_F + \mathscr{L}_m + \mathscr{L}_J \tag{753}$$

with the Euler Lagrange equations

$$\frac{\partial\mathscr{L}}{\partial A_\alpha} - \partial_\beta\frac{\partial\mathscr{L}}{\partial(\partial_\beta A_\alpha)} = 0 \tag{754}$$

with

$$\frac{\partial(\partial_\mu A_\nu)}{\partial(\partial_\beta A_\alpha)} = \delta_{\mu\beta}\delta_{\nu\alpha} \tag{755}$$

we can calculate

$$\frac{\partial\mathscr{L}_m}{\partial A_\alpha} - \partial_\beta\frac{\partial\mathscr{L}_m}{\partial(\partial_\beta A_\alpha)} = m^2A_\alpha \tag{756}$$

$$\frac{\partial\mathscr{L}_J}{\partial A_\alpha} - \partial_\beta\frac{\partial\mathscr{L}_J}{\partial(\partial_\beta A_\alpha)} = -J_\alpha \tag{757}$$

$$\frac{\partial\mathscr{L}_F}{\partial A_\alpha} - \partial_\beta\frac{\partial\mathscr{L}_F}{\partial(\partial_\beta A_\alpha)} = -\frac{1}{4}\partial_\beta\left(-2F_{\mu\nu}(\delta_{\mu\beta}\delta_{\nu\alpha} - \delta_{\nu\beta}\delta_{\mu\alpha})\right) \tag{758}$$

$$= \frac{1}{4}\partial_\beta\left(2(F_{\beta\alpha} - F_{\alpha\beta})\right) \tag{759}$$

$$= \partial_\beta F_{\beta\alpha} \tag{760}$$

$$= \partial_\beta\partial_\beta A_\alpha - \partial_\beta\partial_\alpha A_\beta \tag{761}$$

to obtain (the Proca equation)

$$\Box A_\alpha - \partial_\beta\partial_\alpha A_\beta + m^2A_\alpha - J_\alpha = 0. \tag{762}$$

Now we can calculate the divergence of the equations

$$\partial_\alpha\left(\Box A_\alpha - \partial_\beta\partial_\alpha A_\beta + m^2A_\alpha - J_\alpha\right) = 0. \tag{763}$$

$$\Box\partial_\alpha A_\alpha - \partial_\alpha\partial_\alpha\partial_\beta A_\beta + m^2\partial_\alpha A_\alpha - \underbrace{\partial_\alpha J_\alpha}_{=0} = 0 \tag{764}$$

which implies $\partial_\alpha A_\alpha = 0$ and therefore

$$\Box A_\alpha + m^2A_\alpha - J_\alpha = 0. \tag{765}$$

(b) For A_0 we have for a static potential

$$(\partial_{tt} - \triangle)A_0 + m^2 A_0 - e\delta(x) = 0 \tag{766}$$
$$-\triangle A_0 + m^2 A_0 - e\delta(x) = 0. \tag{767}$$

A Fourier transformation of the equation of motion yields

$$-(ik)^2 A_0(k) + m^2 A_0(k) - e = 0 \tag{768}$$
$$\to A_0(k) = \frac{e}{k^2 + m^2} \tag{769}$$

which we can now transform back

$$A_0 = \frac{e}{(2\pi)^3} \int d^3k \frac{e^{ikx}}{k^2 + m^2} \tag{770}$$
$$= \frac{e}{4\pi r} e^{-mr} \tag{771}$$

where we used the integral evaluation from KACHELRIESS Problem 3.5.

(c)

$$\lim_{m \to 0} \frac{e}{4\pi r} e^{-mr} = \frac{e}{4\pi r} \tag{772}$$

(d) Scaling down the Coulomb potential exponentially with a characteristic length of $1/m$.

(e)

(f) We can expand and the integrate each term by parts to move over the partial derivatives

$$\mathscr{L}_F = -\frac{1}{4} F_{\mu\nu}^2 \tag{773}$$
$$= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) \tag{774}$$
$$= -\frac{1}{4}\left(\partial_\mu A_\nu \partial_\mu A_\nu - \partial_\mu A_\nu \partial_\nu A_\mu - \partial_\nu A_\mu \partial_\mu A_\nu + \partial_\nu A_\mu \partial_\nu A_\mu\right) \tag{775}$$
$$= -\frac{1}{2}\left(\partial_\mu A_\nu \partial_\mu A_\nu - \partial_\mu A_\nu \partial_\nu A_\mu\right) \tag{776}$$
$$= -\frac{1}{2}\left(-A_\nu \partial_\mu \partial_\mu A_\nu + A_\nu \partial_\nu \partial_\mu A_\mu\right) \tag{777}$$
$$= \frac{1}{2}\left(A_\mu \Box A_\mu - A_\nu \partial_\nu \underbrace{\partial_\mu A_\mu}_{=0}\right) \tag{778}$$
$$= \frac{1}{2} A_\mu \Box A_\mu \tag{779}$$

We can plug this into the full Lagrangian (renaming the summation index)

$$\mathscr{L} = \frac{1}{2} A_\mu \Box A_\mu + \frac{1}{2} m^2 A_\mu^2 - A_\mu J_\mu \tag{780}$$
$$= \frac{1}{2} A_\mu \left(\Box + m^2\right) A_\mu - A_\mu J_\mu \tag{781}$$

then we calculate the derivatives for the Euler-Lagrange equations up to second order (see problem 3.1)

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = \frac{1}{2}\Box A_\mu + m^2 A_\mu - J_\mu \tag{782}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\mu)} = 0 \tag{783}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \partial_\alpha A_\mu)} = \frac{1}{2}A_\mu \tag{784}$$

and get

$$(\Box + m^2)A_\mu = J_\mu \tag{785}$$

0.10.9 Problem 3.7 Perihelion shift of Mercury by dimensional analysis - NOT DONE YET

(a) Lets summarize the rules of dimensional analysis

varible	SI unit	equation	natural unit
c	m/s	-	1
$\hbar$	Js	-	1
Velocity	m/s	-	1
mass	kg	$E = mc^2$	E
frequency	1/s	$E = \hbar\omega$	E
time	s	$t = 2\pi/\omega$	E^{-1}
length	m	$s = ct$	E^{-1}
∂_μ	1/m	-	E
momentum	kg m/s	$E = p^2/2m$	E
action	Js	$S = Et$	1
$\mathcal{L}$	J/m^3	$S = \int d^4x\mathcal{L}$	E^4
energy density	J/m^3	$\rho = E/V$	E^4
$T^{\mu\nu}$	J/m^3	$\rho = E/V$	E^4

Now we can do a dimensions count for each term

$$\underbrace{\mathcal{L}}_{=4} = -\frac{1}{2}\underbrace{h\Box h}_{2\cdot[h]+2} + \underbrace{M_{\rm Pl}^a h^2 \Box h}_{=a+3\cdot[h]+2} - \underbrace{M_{\rm Pl}^b hT}_{b+[h]+4} \tag{786}$$

$$\rightarrow \quad [h] = 1 \tag{787}$$

$$\rightarrow \quad a = -1 \tag{788}$$

$$\rightarrow \quad b = -1 \tag{789}$$

(b) Deriving the equations of motions: keeping in mind that the Lagrangian contains second order derivatives with implies and extra term in the Euler-Lagrange equations (see problem

3.1)

$$\mathcal{L} = -\frac{1}{2}h\Box h + \frac{1}{M_{\text{Pl}}}h^2\Box h - \frac{1}{M_{\text{Pl}}}hT \tag{790}$$

$$\frac{\partial \mathcal{L}}{\partial h} = -\frac{1}{2}\cdot\Box h + 2\frac{1}{M_{\text{Pl}}}h\Box h - \frac{1}{M_{\text{Pl}}}T \tag{791}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial h)} = 0 \tag{792}$$

$$\frac{\partial \mathcal{L}}{\partial(\Box h)} = -\frac{1}{2}h + \frac{1}{M_{\text{Pl}}}h^2 \tag{793}$$

$$\rightarrow \Box h = \frac{1}{M_{\text{Pl}}}\Box(h^2) + \frac{2}{M_{\text{Pl}}}h\Box h - \frac{1}{M_{\text{Pl}}}T \tag{794}$$

which show an extra term. Alternatively we can integrate the Lagrangian by parts (neglecting the boundary terms) and get

$$\mathcal{L} = \frac{1}{2}\partial h\partial h - \frac{1}{M_{\text{Pl}}}\partial(h^2)\partial h - \frac{1}{M_{\text{Pl}}}hT \tag{795}$$

$$\frac{\partial \mathcal{L}}{\partial h} = -\frac{1}{M_{\text{Pl}}}T \tag{796}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial h)} = \Box h - \frac{1}{M_{\text{Pl}}}\Box(h^2) \tag{797}$$

$$\rightarrow \Box h = \frac{1}{M_{\text{Pl}}}\Box(h^2) - \frac{1}{M_{\text{Pl}}}T \tag{798}$$

We now assume a solution of the form

$$h = h_0 + \frac{1}{M_{\text{Pl}}}h_1 + \frac{1}{M_{\text{Pl}}^2}h_2 + ... \tag{799}$$

$$\rightarrow h^2 = h_0^2 + \frac{1}{M_{\text{Pl}}}2h_0h_1 + \frac{1}{M_{\text{Pl}}^2}(2h_0h_2 + h_1^2) + \frac{1}{M_{\text{Pl}}^3}(2h_1h_2 + 2h_0h_3) + ... \tag{800}$$

and obtain (with the Coulomb solution 3.61 and 3.61)

$$k = 0: \quad \Box h_0 = 0 \quad \rightarrow \quad h_0 = 0 \tag{801}$$

$$k = 1: \quad \Box h_1 = \Box h_0^2 - m\delta^{(3)} \tag{802}$$

$$\Box h_1 = -m\delta^{(3)} \quad \rightarrow \quad h_1 = -\frac{m}{\Box}\delta^{(3)} = \frac{m}{\triangle}\delta^{(3)} = -\frac{m}{4\pi r} \tag{803}$$

$$k = 2: \quad \Box h_2 = 2\Box h_0h_1 \quad \rightarrow \quad h_2 = 0 \tag{804}$$

$$k = 3: \quad \Box h_3 = \Box(2h_0h_2 + h_1^2) \tag{805}$$

$$\Box h_3 = \Box(h_1^2) \quad \rightarrow \quad h_3 = h_1^2 = \frac{m^2}{16\pi^2 r^2} \tag{806}$$

and therefore

$$h = -\frac{m}{4\pi r}\frac{1}{M_{\text{Pl}}} + \frac{m^2}{16\pi^2 r^2}\frac{1}{M_{\text{Pl}}^3} \tag{807}$$

$$= -\frac{m}{4\pi r}\sqrt{G_N} + \frac{m^2}{16\pi^2 r^2}\sqrt{G_N^3} \tag{808}$$

(c) The Newton potential is actually given by (and additional power of M_{Pl} is missing and we are dropping the 4π)

$$V_N = h_1\frac{1}{M_{\text{Pl}}}\cdot\frac{1}{M_{\text{Pl}}} = -\frac{Gm_{\text{Sun}}}{r} \tag{809}$$

the virial theorem implies $E_{\text{kin}} \simeq E_{\text{pot}}$ and therefore

$$\frac{1}{2}J\omega^2 \simeq \frac{G_N m_{\text{Sun}} m_{\text{Mercury}}}{R} \tag{810}$$

$$\frac{1}{2}m_{\text{Mercury}}R^2\omega^2 \simeq \frac{G_N m_{\text{Sun}} m_{\text{Mercury}}}{R} \tag{811}$$

$$\omega^2 \simeq \frac{G_N m_{\text{Sun}}}{R^3} \tag{812}$$

(d)

(e)

(f)

(g)

0.10.10 Problem 3.9 - Photon polarizations

(a) Then using the results from problem 3.6 and the corrected sign in the Lagrangian we get

$$-\frac{1}{4}(F_{\mu\nu})^2 = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) \tag{813}$$

$$= -\frac{1}{4}\left(\partial_\mu A_\nu \partial_\mu A_\nu - \partial_\mu A_\nu \partial_\nu A_\mu - \partial_\nu A_\mu \partial_\mu A_\nu + \partial_\nu A_\mu \partial_\nu A_\mu\right) \tag{814}$$

$$= -\frac{1}{2}\left(\partial_\mu A_\nu \partial_\mu A_\nu - \partial_\mu A_\nu \partial_\nu A_\mu\right) \tag{815}$$

$$= -\frac{1}{2}\left(-A_\nu \partial_\mu \partial_\mu A_\nu + A_\nu \partial_\nu \partial_\mu A_\mu\right) \tag{816}$$

$$= \frac{1}{2}\left(A_\mu \Box A_\mu - A_\nu \partial_\nu \underbrace{\partial_\mu A_\mu}_{=0}\right) \tag{817}$$

$$= \frac{1}{2}A_\mu \Box A_\mu \tag{818}$$

and therefore

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 - J_\mu A_\mu \tag{819}$$

$$= \frac{1}{2}A_\mu \Box A_\mu - J_\mu A_\mu \tag{820}$$

$$= \frac{1}{2}A_\mu \Box A_\mu - (\Box A_\mu)A_\mu \tag{821}$$

$$= -\frac{1}{2}A_\mu \Box A_\mu \tag{822}$$

The equations of motion are $\Box A_\mu = J_\mu$ which can be written in momentum space as $k^2 A_\mu(k) = J_\mu(k)$. Now let's write the Lagranian in momentum space as well

$$\mathcal{L} = \int d^4k e^{ikx} A_\mu(k) k^2 A_\mu(k) \tag{823}$$

$$= \int d^4k e^{ikx} \frac{J_\mu(k)}{k^2} k^2 \frac{J_\mu(k)}{k^2} \tag{824}$$

$$= \int d^4k e^{ikx} J_\mu(k) \frac{1}{k^2} J_\mu(k) \tag{825}$$

(b) In momentum space charge conservation is given by

$$k_\mu J_\mu = 0 \tag{826}$$
$$\omega J_0 - \kappa J_1 = 0 \tag{827}$$
$$\rightarrow \quad J_1 = \frac{\omega}{\kappa} J_0 \tag{828}$$

(c)

$$\mathcal{L} = \int d^4k e^{ikx} J_\mu(k) \frac{1}{k^2} J_\mu(k) \tag{829}$$
$$\simeq \frac{J_0^2 - J_1^2 - J_2^2 - J_3^2}{\omega^2 - \kappa^2} \tag{830}$$
$$\simeq \frac{J_0^2(1-\omega^2/\kappa^2)}{\omega^2 - \kappa^2} - \frac{J_2^2 + J_3^2}{\omega^2 - \kappa^2} \tag{831}$$
$$\simeq -\frac{J_0^2}{\kappa^2} - \frac{J_2^2 + J_3^2}{\omega^2 - \kappa^2} \tag{832}$$
$$\simeq \triangle J_0^2 - \Box(J_2^2 + J_3^2) \tag{833}$$

(d) A time derivative in the Lagrangian results in a time derivative in time derivative in the equations of motion which means a time-evolution equation. There are two causally propagating degrees of freedom J_2 and J_3.

(e) Hmmmm .... calculate the two point field correlation functions and see if they vanish outside of the light cone.

0.10.11 Problem 3.10 - Graviton polarizations - NOT DONE YET

(a) With the higher order Euler-Lagrange equations from 3.1

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} + \partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu \partial_\mu \phi)} - ... = 0 \tag{834}$$

we obtain

$$-\frac{1}{2}\Box h_{\mu\nu} + \frac{1}{M_{\mathrm{Pl}}} T_{\mu\nu} - \frac{1}{2}\Box h_{\mu\nu} = 0 \tag{835}$$
$$\rightarrow \Box h_{\mu\nu} = \frac{1}{M_{\mathrm{Pl}}} T_{\mu\nu} \tag{836}$$
$$\rightarrow h_{\mu\nu} = \frac{1}{M_{\mathrm{Pl}}} \frac{1}{\Box} T_{\mu\nu} \tag{837}$$

and

$$\mathcal{L} = -\frac{1}{2} h_{\mu\nu} \Box h_{\mu\nu} + \frac{1}{M_{\mathrm{Pl}}} h_{\mu\nu} T_{\mu\nu} \tag{838}$$
$$= -\frac{1}{2}\frac{1}{M_{\mathrm{Pl}}^2}(\frac{1}{\Box} T_{\mu\nu}) T_{\mu\nu} + \frac{1}{M_{\mathrm{Pl}}^2}(\frac{1}{\Box} T_{\mu\nu}) T_{\mu\nu} \tag{839}$$
$$= \frac{1}{2}\frac{1}{M_{\mathrm{Pl}}^2} T_{\mu\nu} \frac{1}{\Box} T_{\mu\nu} \tag{840}$$
$$\simeq \frac{1}{2}\frac{1}{M_{\mathrm{Pl}}^2} T_{\mu\nu} \frac{1}{k^2} T_{\mu\nu} \tag{841}$$
$$\tag{842}$$

(b)

(c)

(d)

0.11 Srednicki - Quantum Field Theory

0.11.1 Problem 1.2 - Schroedinger equation

$$H = \int d^3x a^\dagger(x)\left(-\frac{\hbar^2}{2m}\triangle_x + V(x)\right)a(x) + \frac{1}{2}\int d^3xd^3yV(x-y)a^\dagger(x)a^\dagger(y)a(x)a(y) \quad (843)$$

$$|\psi,t\rangle = \int d^3x_1...d^3x_n\psi(x_1,...,x_n;t)a^\dagger(x_1)...a^\dagger(x_n)|0\rangle \quad (844)$$

1. Bosons: With the commutations relation and $a|0\rangle = 0$

$$a(x)a^\dagger(x_1)...a^\dagger(x_n)|0\rangle = \left(\delta^3(x-x_1) - a^\dagger(x_1)a(x)\right)...a^\dagger(x_n)|0\rangle \quad (845)$$

$$= \sum_{k=1}^{n}(-1)^{k-1}\delta^3(x-x_k)\underbrace{a^\dagger(x_1)...a^\dagger(x_n)}_{(n-1)\times a^\dagger}|0\rangle \quad (846)$$

and similar

$$a(y)a(x)a^\dagger(x_1)...a^\dagger(x_n)|0\rangle = \sum_{j\neq k}^{n}\delta^3(x-x_k)\delta^3(y-x_j)\underbrace{a^\dagger(x_1)...a^\dagger(x_n)}_{(n-2)\times a^\dagger}|0\rangle \quad (847)$$

we obtain

$$i\hbar\frac{\partial}{\partial t}|\psi,t\rangle = \int d^3x_1...d^3x_n\frac{\partial}{\partial t}\psi(x_1,...,x_n;t)a^\dagger(x_1)...a^\dagger(x_n)|0\rangle \quad (848)$$

and

$$H|\psi,t\rangle = \sum_{k=1}^{n}a^\dagger(x_k)\left(-\frac{\hbar^2}{2m}\triangle_{x_k} + V(x_k)\right)\psi(x_1,...,x_n;t)\underbrace{a^\dagger(x_1)...a^\dagger(x_n)}_{(n-1)\times a^\dagger}|0\rangle \quad (849)$$

$$+\frac{1}{2}\sum_{j\neq k}^{n}V(x_k-x_j)\psi(x_1,...,x_n;t)a^\dagger(x_k)a^\dagger(x_j)\underbrace{a^\dagger(x_1)...a^\dagger(x_n)}_{(n-2)\times a^\dagger}|0\rangle \quad (850)$$

2. Fermions:

0.11.2 Problem 1.3 - Commutator of the number operator

Preliminary calculations (we use the boson commutation relations)

$$a^\dagger(z)a(z)a^\dagger(x) = a^\dagger(z)(\delta(x-z) + a^\dagger(x)a(z)) \quad (851)$$

$$= a^\dagger(z)\delta^3(x-z) + a^\dagger(z)a^\dagger(x)a(z) \quad (852)$$

$$= a^\dagger(z)\delta^3(x-z) + a^\dagger(x)a^\dagger(z)a(z) \quad (853)$$

and

$$a(x)a^\dagger(z)a(z) = (\delta(x-z) + a^\dagger(z)a(x))a(z) \quad (854)$$

$$= \delta^3(x-z)a(z) + a^\dagger(z)a(x)a(z) \quad (855)$$

$$= \delta^3(x-z)a(z) + a^\dagger(z)a(z)a(x) \quad (856)$$

With

$$N = \int d^3z\ a^\dagger(z)a(z) \tag{857}$$

$$H = H_1 + H_{\text{int}} \tag{858}$$

$$= \int d^3x\ a^\dagger(x)\left(-\frac{\hbar^2}{2m}\triangle_x + U(x)\right)a(x) + \frac{1}{2}\int d^3xd^3y\ V(x-y)a^\dagger(x)a^\dagger(y)a(y)a(x) \tag{859}$$

We are calculating the commutator in two parts. We start with $[N, H_1]$

$$NH_1 = \int d^3xd^3z\left(a^\dagger(z)\delta^3(x-z) + a^\dagger(x)a^\dagger(z)a(z)\right)\left(-\frac{\hbar^2}{2m}\triangle_x + U(x)\right)a(x) \tag{860}$$

$$= \int d^3xa^\dagger(x)\left(-\frac{\hbar^2}{2m}\triangle_x + U(x)\right)a(x) + \int d^3xd^3za^\dagger(x)\left(-\frac{\hbar^2}{2m}\triangle_x + U(x)\right)a^\dagger(z)a(z)a(x) \tag{861}$$

and

$$H_1N = \int d^3x\ a^\dagger(x)\left(-\frac{\hbar^2}{2m}\triangle_x + U(x)\right)(\delta^3(x-z)a(z) + a^\dagger(z)a(z)a(x)) \tag{862}$$

$$= \int d^3xa^\dagger(x)\left(-\frac{\hbar^2}{2m}\triangle_x + U(x)\right)a(x) + \int d^3xd^3za^\dagger(x)\left(-\frac{\hbar^2}{2m}\triangle_x + U(x)\right)a^\dagger(z)a(z)a(x) \tag{863}$$

therefore $[N, H_1] = 0$. For the second part $[N, H_{\text{int}}]$ we calculate

$$a_z^\dagger a_z a_x^\dagger a_y^\dagger a_y a_x = a_z^\dagger(\delta_{zx}^3 + a_x^\dagger a_z)a_y^\dagger a_y a_x \tag{864}$$

$$= \delta_{zx}^3 a_z^\dagger a_y^\dagger a_y a_x + a_z^\dagger a_x^\dagger a_z a_y^\dagger a_y a_x \tag{865}$$

$$= \delta_{zx}^3 a_y^\dagger a_z^\dagger a_y a_x + a_z^\dagger a_x^\dagger(\delta_{zy}^3 + a_y^\dagger a_z)a_y a_x \tag{866}$$

$$= \delta_{zx}^3 a_y^\dagger a_z^\dagger a_y a_x + \delta_{zy}^3 a_z^\dagger a_x^\dagger a_y a_x + a_z^\dagger a_x^\dagger a_y^\dagger a_z a_y a_x \tag{867}$$

$$= \delta_{zx}^3 a_y^\dagger a_z^\dagger a_y a_x + \delta_{zy}^3 a_x^\dagger a_z^\dagger a_y a_x + a_x^\dagger a_y^\dagger a_z^\dagger a_z a_y a_x \tag{868}$$

$$\to a_y^\dagger a_x^\dagger a_y a_x + a_x^\dagger a_y^\dagger a_y a_x + a_x^\dagger a_y^\dagger a_z^\dagger a_z a_y a_x \tag{869}$$

and

$$a_x^\dagger a_y^\dagger a_y a_x a_z^\dagger a_z = a_x^\dagger a_y^\dagger a_y(\delta_{xz}^3 + a_z^\dagger a_x)a_z \tag{870}$$

$$= \delta_{xz}^3 a_x^\dagger a_y^\dagger a_y a_z + a_x^\dagger a_y^\dagger a_y a_z^\dagger a_x a_z \tag{871}$$

$$= \delta_{xz}^3 a_x^\dagger a_y^\dagger a_z a_y + a_x^\dagger a_y^\dagger(\delta_{zy}^3 + a_z^\dagger a_y)a_x a_z \tag{872}$$

$$= \delta_{xz}^3 a_x^\dagger a_y^\dagger a_z a_y + \delta_{zy}^3 a_x^\dagger a_y^\dagger a_x a_z + a_x^\dagger a_y^\dagger a_z^\dagger a_y a_x a_z \tag{873}$$

$$= \delta_{xz}^3 a_x^\dagger a_y^\dagger a_z a_y + \delta_{zy}^3 a_x^\dagger a_y^\dagger a_z a_x + a_x^\dagger a_y^\dagger a_z^\dagger a_z a_y a_x \tag{874}$$

$$\to a_x^\dagger a_y^\dagger a_x a_y + a_x^\dagger a_y^\dagger a_y a_x + a_x^\dagger a_y^\dagger a_z^\dagger a_z a_y a_x \tag{875}$$

We therefore see that the commutator vanishes as well.

0.11.3 Problem 2.1 - Infinitesimal LT

$$g_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = g_{\rho\sigma} \tag{876}$$

$$g_{\mu\nu}\left(\delta^\mu_\rho + \delta\omega^\mu{}_\rho\right)\left(\delta^\nu{}_\sigma + \delta\omega^\nu{}_\sigma\right) = g_{\rho\sigma} \tag{877}$$

$$g_{\mu\nu}\left(\delta^\mu_\rho\delta^\nu{}_\sigma + \delta^\nu{}_\sigma \cdot \delta\omega^\mu_\rho + \delta^\mu_\rho \cdot \delta\omega^\nu{}_\sigma + \mathcal{O}(\delta\omega^2)\right) = g_{\rho\sigma} \tag{878}$$

$$g_{\rho\sigma} + g_{\mu\sigma} \cdot \delta\omega^\mu_\rho + g_{\rho\nu} \cdot \delta\omega^\nu{}_\sigma = g_{\rho\sigma} \tag{879}$$

which implies

$$\delta\omega_{\sigma\rho} + \delta\omega_{\rho\sigma} = 0 \tag{880}$$

0.11.4 Problem 2.2 - Infinitesimal LT II

Important: each $M^{\mu\nu}$ is an operator and $\delta\omega$ is just a coefficient matrix so $\delta\omega_{\mu\nu}M^{\mu\nu}$ ist a weighted sum of operators.

$$U(\Lambda^{-1}\Lambda'\Lambda) = U(\Lambda^{-1})U(\Lambda')U(\Lambda) \tag{881}$$

$$U(\Lambda^{-1}(I+\delta\omega')\Lambda) = U(\Lambda^{-1})\left(I + \frac{i}{2\hbar}\delta\omega'_{\mu\nu}M^{\mu\nu}\right)U(\Lambda) \tag{882}$$

$$U(I+\Lambda^{-1}\delta\omega'\Lambda) = I + \frac{i}{2\hbar}\delta\omega'_{\mu\nu}U(\Lambda^{-1})M^{\mu\nu}U(\Lambda) \tag{883}$$

now we calculate recalling successive LT's $(\Lambda^{-1})^{\varepsilon}{}_{\gamma}\delta\omega'^{\gamma}{}_{\beta}\Lambda^{\beta}{}_{\alpha}x^{\alpha}$

$$(\Lambda^{-1}\delta\omega'\Lambda)_{\rho\sigma} = g_{\varepsilon\rho}(\Lambda^{-1})^{\varepsilon}{}_{\mu}\delta\omega'^{\mu}{}_{\nu}\Lambda^{\nu}{}_{\sigma} \tag{884}$$

$$= g_{\varepsilon\rho}\Lambda_{\mu}{}^{\varepsilon}\delta\omega'^{\mu}{}_{\nu}\Lambda^{\nu}{}_{\sigma} \tag{885}$$

$$= \delta\omega'_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} \tag{886}$$

now we can rewrite $U(I+\Lambda^{-1}\delta\omega'\Lambda)$ and therefore

$$\delta\omega'_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}M^{\rho\sigma} = \delta\omega'_{\mu\nu}U(\Lambda^{-1})M^{\mu\nu}U(\Lambda) \tag{887}$$

As all $\delta\omega'$ components are basically independent the equation must hold for each pair μ,ν.

0.11.5 Problem 2.3 - Commutators of LT generators I

LHS:

$$U(\Lambda)^{-1}M^{\mu\nu}U(\Lambda) \simeq \left(I - \frac{i}{2\hbar}\delta\omega_{\alpha\beta}M^{\alpha\beta}\right)M^{\mu\nu}\left(I + \frac{i}{2\hbar}\delta\omega_{\rho\sigma}M^{\rho\sigma}\right) \tag{888}$$

$$\simeq M^{\mu\nu} - \frac{i}{2\hbar}\delta\omega_{\rho\sigma}(M^{\rho\sigma}M^{\mu\nu} - M^{\mu\nu}M^{\rho\sigma}) + \mathcal{O}(\delta\omega^2) \tag{889}$$

$$= M^{\mu\nu} - \frac{i}{2\hbar}\delta\omega_{\rho\sigma}[M^{\rho\sigma}, M^{\mu\nu}] \tag{890}$$

$$= M^{\mu\nu} + \frac{i}{2\hbar}\delta\omega_{\rho\sigma}[M^{\mu\nu}, M^{\rho\sigma}] \tag{891}$$

RHS:

$$\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}M^{\rho\sigma} \simeq \left(\delta^{\mu}{}_{\rho} + \delta\omega^{\mu}{}_{\rho}\right)\left(\delta^{\nu}{}_{\sigma} + \delta\omega^{\nu}{}_{\sigma}\right)M^{\rho\sigma} \tag{892}$$

$$\simeq M^{\mu\nu} + \delta^{\mu}{}_{\rho}\delta\omega^{\nu}{}_{\sigma}M^{\rho\sigma} + \delta^{\nu}{}_{\sigma}\delta\omega^{\mu}{}_{\rho}M^{\rho\sigma} \tag{893}$$

$$\simeq M^{\mu\nu} + \delta\omega^{\nu}{}_{\sigma}M^{\mu\sigma} + \delta\omega^{\mu}{}_{\rho}M^{\rho\nu} \tag{894}$$

$$\simeq M^{\mu\nu} + \delta\omega_{\alpha\sigma}g^{\alpha\nu}M^{\mu\sigma} + \delta\omega_{\alpha\rho}g^{\alpha\mu}M^{\rho\nu} \tag{895}$$

$$\simeq M^{\mu\nu} + \delta\omega_{\alpha\sigma}(g^{\alpha\nu}M^{\mu\sigma} + g^{\alpha\mu}M^{\sigma\nu}) \tag{896}$$

$$\simeq M^{\mu\nu} + \delta\omega_{\rho\sigma}(g^{\rho\nu}M^{\mu\sigma} + g^{\rho\mu}M^{\sigma\nu}) \tag{897}$$

$$\simeq M^{\mu\nu} + \frac{1}{2}\delta\omega_{\rho\sigma}\left(g^{\rho\nu}(M^{\mu\sigma} - M^{\sigma\mu}) + g^{\rho\mu}(M^{\sigma\nu} - M^{\nu\sigma})\right) \tag{898}$$

$$\simeq M^{\mu\nu} + \frac{1}{2}\delta\omega_{\rho\sigma}\left(g^{\rho\nu}M^{\mu\sigma} - g^{\nu\rho}M^{\sigma\mu} + g^{\rho\mu}M^{\sigma\nu} - g^{\mu\rho}M^{\nu\sigma}\right) \tag{899}$$

Now we use the antisymmetry of M

$$\begin{aligned}
\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}M^{\rho\sigma} &\simeq M^{\mu\nu} + \frac{1}{2}\delta\omega_{\rho\sigma}\left(g^{\nu\rho}M^{\mu\sigma} - g^{\nu\rho}M^{\sigma\mu} + g^{\rho\mu}M^{\sigma\nu} - g^{\mu\rho}M^{\nu\sigma}\right) && (900)\\
&\simeq M^{\mu\nu} - \frac{1}{2}\delta\omega_{\rho\sigma}\left(-g^{\nu\rho}M^{\mu\sigma} + g^{\nu\rho}M^{\sigma\mu} - g^{\rho\mu}M^{\sigma\nu} + g^{\mu\rho}M^{\nu\sigma}\right) && (901)\\
&\simeq M^{\mu\nu} - \frac{1}{2}\delta\omega_{\rho\sigma}\left(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma} - g^{\rho\mu}M^{\sigma\nu} + g^{\nu\rho}M^{\sigma\mu}\right) && (902)\\
&\simeq M^{\mu\nu} - \frac{1}{2}\delta\omega_{\rho\sigma}\left(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma}\right) - \frac{1}{2}\underbrace{\delta\omega_{\rho\sigma}\left(-g^{\rho\mu}M^{\sigma\nu} + g^{\nu\rho}M^{\sigma\mu}\right)}_{=\underbrace{\delta\omega_{\sigma\rho}\left(-g^{\sigma\mu}M^{\rho\nu} + g^{\nu\sigma}M^{\rho\mu}\right)}_{=-\delta\omega_{\rho\sigma}\left(-g^{\mu\sigma}\left(-M^{\nu\rho}\right)+g^{\nu\sigma}\left(-M^{\mu\rho}\right)\right)}} && (903)\\
&\simeq M^{\mu\nu} - \frac{1}{2}\delta\omega_{\rho\sigma}\left(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma}\right) - \frac{1}{2}\delta\omega_{\rho\sigma}\left(-g^{\mu\sigma}M^{\nu\rho} + g^{\nu\sigma}M^{\mu\rho}\right) && (904)\\
&\simeq M^{\mu\nu} - \frac{1}{2}\delta\omega_{\rho\sigma}\left(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma} - g^{\mu\sigma}M^{\nu\rho} + g^{\nu\sigma}M^{\mu\rho}\right) && (905)
\end{aligned}$$

As the components of $\delta\omega$ (besides the antisymmetry) are independent we get

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\hbar\left(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma} - g^{\mu\sigma}M^{\nu\rho} + g^{\nu\sigma}M^{\mu\rho}\right) \tag{906}$$

0.11.6 Problem 2.4 - Commutators of LT generators II

Preliminary calculations

$$\begin{aligned}
\epsilon_{ijk}J_k &= \varepsilon_{ijk}\frac{1}{2}\varepsilon_{kab}M^{ab} && (907)\\
&= -\frac{1}{2}\varepsilon_{kij}\varepsilon_{kab}M^{ab} && (908)\\
&= -\frac{1}{2}\left(\delta_{ia}\delta_{jb} - \delta_{ja}\delta_{ib}\right)M^{ab} && (909)\\
&= -\frac{1}{2}\left(M^{ij} - M^{ji}\right) && (910)\\
&= -M^{ij} && (911)
\end{aligned}$$

- With

$$\begin{aligned}
J_1 &= \frac{1}{2}(\varepsilon_{123}M^{23} + \varepsilon_{132}M^{32}) && (912)\\
&= \varepsilon_{123}M^{23} && (913)\\
&= M^{23} && (914)
\end{aligned}$$

  then

$$\begin{aligned}
[J_1, J_3] &= [M^{23}, M^{12}] && (915)\\
&= i\hbar\left(g^{21}M^{32} - g^{31}M^{22} - g^{22}M^{31} + g^{32}M^{21}\right) && (916)\\
&= -i\hbar g^{22}M^{31} && (917)\\
&= -i\hbar M^{31} && (918)\\
&= -i\hbar J_2 && (919)
\end{aligned}$$

- analog ...

•

$$[K^i, K^j] = [M^{i0}, M^{j0}] \tag{920}$$
$$= i\hbar\left(g^{ij}M^{00} - g^{0j}M^{i0} - g^{i0}M^{0j} + g^{00}M^{ij}\right) \tag{921}$$
$$= i\hbar\left(-\delta^{ij}M^{00} + M^{ij}\right) \tag{922}$$
$$= \begin{cases} i\hbar M^{ij} = -i\hbar\epsilon_{ijk}J_k & (i = j) \\ 0 & (i \neq j) \end{cases} \tag{923}$$

where we used the result from the preliminary calculation in the last step.

0.11.7 Problem 2.7 - Translation operator

The obvious property $T(a)T(b) = T(a+b)$. Then

$$T(\delta a + \delta b) = T(\delta a)T(\delta b) \tag{924}$$
$$= \left(1 - \frac{i}{\hbar}\delta a_\mu P^\mu\right)\left(1 - \frac{i}{\hbar}\delta b_\nu P^\nu\right) \tag{925}$$
$$\simeq 1 - \frac{i}{\hbar}(\delta a_\mu + \delta b_\mu)P^\mu + \frac{1}{\hbar^2}\delta a_\mu \delta b_\mu P^\mu P^\nu \tag{926}$$

and

$$T(\delta a + \delta b) = T(\delta b)T(\delta a) \tag{927}$$
$$= \left(1 - \frac{i}{\hbar}\delta b_\nu P^\nu\right)\left(1 - \frac{i}{\hbar}\delta a_\mu P^\mu\right) \tag{928}$$
$$\simeq 1 - \frac{i}{\hbar}(\delta a_\mu + \delta b_\mu)P^\mu + \frac{1}{\hbar^2}\delta a_\mu \delta b_\mu P^\nu P^\mu \tag{929}$$

which implies $P^\mu P^\nu = P^\nu P^\mu$.

0.11.8 Problem 2.8 - Transformation of scalar field

(a) We start with

$$U(\Lambda)^{-1}\varphi(x)U(\Lambda) = \varphi(\Lambda^{-1}x) \tag{930}$$
$$\left(1 - \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right)\varphi(x)\left(1 + \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right) = \varphi([\delta^\mu_{\ \nu} - \delta\omega^\mu_{\ \nu}]x^\nu) \tag{931}$$
$$\varphi(x) - \frac{i}{2\hbar}\delta\omega_{\mu\nu}[M^{\mu\nu}, \varphi(x)] = \varphi(x) - \delta\omega^\mu_{\ \nu}x^\nu\frac{\partial\varphi}{\partial x^\mu} \tag{932}$$
$$= \varphi(x) - \delta\omega^\mu_{\ \nu}\frac{1}{2}\left(x^\nu\frac{\partial\varphi}{\partial x^\mu} - x^\mu\frac{\partial\varphi}{\partial x^\nu}\right) \tag{933}$$
$$= \varphi(x) - \delta\omega_{\mu\nu}\frac{1}{2}\left(x^\nu\partial^\mu - x^\mu\partial^\nu\right)\varphi \tag{934}$$

and therefore

$$[\varphi, M^{\mu\nu}] = \frac{\hbar}{i}(x^\mu\partial^\nu - x^\nu\partial^\mu)\varphi \tag{935}$$

(b) (c) (d) (e) (f)

0.11.9 Problem 3.2 - Multiparticle eigenstates of the hamiltonian

With

$$|k_1...k_n\rangle = a^\dagger_{k_1}...a^\dagger_{k_n}|0\rangle \tag{936}$$

$$H = \int \widetilde{dk}\ \omega_k a^\dagger_k a_k \tag{937}$$

$$[a_k, a^\dagger_q] = \underbrace{(2\pi)^3 2\omega_k \delta^3(\vec{k}-\vec{q})}_{\delta_{kq}} \tag{938}$$

we see that the expression which needs calculating is the creation and annihilation operators. The idea is to use the commutation relations to move the a_k to the right end to use $a_k|0\rangle$

$$\begin{aligned}
a^\dagger_k a_k a^\dagger_{k_1}...a^\dagger_{k_n}|0\rangle &= a^\dagger_k(a^\dagger_{k_1}a_k + \delta_{kk_1})a^\dagger_{k_2}...a^\dagger_{k_n}|0\rangle & (939)\\
&= \delta_{kk_1}a^\dagger_k a^\dagger_{k_2}...a^\dagger_{k_n}|0\rangle + a^\dagger_k a^\dagger_{k_1}a_k a^\dagger_{k_2}...a^\dagger_{k_n}|0\rangle & (940)\\
&= ... & (941)\\
&= \sum_j \delta_{kk_j}a^\dagger_k \underbrace{a^\dagger_{k_2}...a^\dagger_{k_n}}_{(n-1)\text{ times with } a_{k_j} \text{ missing}}|0\rangle + a^\dagger_k a^\dagger_{k_1}...a^\dagger_{k_n}\underbrace{a_k|0\rangle}_{=0}. & (942)
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
H|k_1...k_n\rangle &= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\omega_k \sum_j \delta_{kk_j}a^\dagger_k a^\dagger_{k_2}...a^\dagger_{k_n}|0\rangle & (943)\\
&= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\omega_k \sum_j (2\pi)^3 2\omega_k \delta^3(\vec{k}-\vec{k}_j)a^\dagger_k a^\dagger_{k_2}...a^\dagger_{k_n}|0\rangle & (944)\\
&= \int d^3k\omega_k \sum_j \delta^3(\vec{k}-\vec{k}_j)a^\dagger_k a^\dagger_{k_2}...a^\dagger_{k_n}|0\rangle & (945)
\end{aligned}$$

which we can integrate obtaining the desired result

$$\begin{aligned}
H|k_1...k_n\rangle &= \sum_j \omega_{k_j}a^\dagger_{k_j}a^\dagger_{k_2}...a^\dagger_{k_n}|0\rangle & (946)\\
&= \left(\sum_j \omega_{k_j}\right)a^\dagger_{k_1}a^\dagger_{k_2}...a^\dagger_{k_n}|0\rangle & (947)\\
&= \left(\sum_j \omega_{k_j}\right)|k_1...k_n\rangle. & (948)
\end{aligned}$$

0.11.10 Problem 3.4 - Heisenberg equations of motion for free field

(a) For the translation operator $T(a) = e^{-iP^\mu a_\mu}$ we expand in first order

$$\begin{aligned}
T(a)^{-1}\varphi(a)T(a) &= \left(1-(-i)P^\mu a_\mu + \mathcal{O}(a^2)\right)\varphi(x)\left(1+(-i)P^\mu a_\mu + \mathcal{O}(a^2)\right) & (949)\\
&= \left(1+iP^\mu a_\mu + \mathcal{O}(a^2)\right)\varphi(x)\left(1-iP^\mu a_\mu + \mathcal{O}(a^2)\right) & (950)\\
&\simeq \varphi(x) + ia_\mu P^\mu \varphi(x) - ia_\mu \varphi(x)P^\mu & (951)\\
&\simeq \varphi(x) + ia_\mu[P^\mu, \varphi(x)] & (952)
\end{aligned}$$

for the right hand right we get

$$\varphi(x-a) \simeq \varphi(x) - \partial^\mu\varphi(x)a_\mu \tag{953}$$

and therefore

$$i[P^\mu, \varphi(x)] = -\partial^\mu \varphi(x) \tag{954}$$

(b) With $\mu = 0$ and $\partial^0 = g_{0\nu}\partial_\nu = -\partial_0$ we have

$$i[H, \varphi(x)] = -\partial^0\varphi(x) = +\partial_0\varphi(x) \tag{955}$$
$$\rightarrow \quad \dot{\varphi}(x) = i[H, \varphi(x)] \tag{956}$$

(c) We start with the hamiltonian (3.25)

$$H = \int d^3y \frac{1}{2}\Pi^2(y) + \frac{1}{2}(\nabla_y\varphi(y))^2 + \frac{1}{2}m^2\varphi(y)^2 - \Omega_0 \tag{957}$$

- Obtaining $\dot{\varphi}(x) = i[H, \varphi(x)]$

  We need to calculate (setting $x^0 = y^0$ - why can we?)

$$[\Pi^2(y), \varphi(x)] = \Pi(y)\Pi(y)\varphi(x) - \varphi(x)\Pi(y)\Pi(y) \tag{958}$$
$$= \Pi(y)\Pi(y)\varphi(x) - \Pi(y)\varphi(x)\Pi(y) + \Pi(y)\varphi(x)\Pi(y) - \varphi(x)\Pi(y)\Pi(y) \tag{959}$$
$$= \Pi(y)[\Pi(y), \varphi(x)] + [\Pi(y), \varphi(x)]\Pi(y) \tag{960}$$
$$= 2\Pi(y)(-1)i\delta^3(\vec{y} - \vec{x}) \tag{961}$$
$$[(\nabla_y\varphi(y))^2, \varphi(x)] = \nabla_y\varphi(y)\nabla_y\varphi(y)\varphi(x) - \varphi(x)\nabla_y\varphi(y)\nabla_y\varphi(y) \tag{962}$$
$$= \nabla_y\varphi(y)[\nabla_y\varphi(y), \varphi(x)] + [\nabla_y\varphi(y), \varphi(x)]\nabla_y\varphi(y) \tag{963}$$
$$= \nabla_y\varphi(y)\nabla_y[\varphi(y), \varphi(x)] + \nabla_y[\varphi(y), \varphi(x)]\nabla_y\varphi(y) \tag{964}$$
$$= 0 \tag{965}$$
$$[\varphi(y)^2, \varphi(x)] = \varphi(y)\varphi(y)\varphi(x) - \varphi(x)\varphi(y)\varphi(y) \tag{966}$$
$$= \varphi(y)\varphi(y)\varphi(x) - \varphi(y)\varphi(x)\varphi(y) + \varphi(y)\varphi(x)\varphi(y) - \varphi(x)\varphi(y)\varphi(y) \tag{967}$$
$$= \varphi(y)[\varphi(y), \varphi(x)] + [\varphi(y), \varphi(x)]\varphi(y) \tag{968}$$
$$= 0 \tag{969}$$

  then

$$\int d^3y[\Pi^2(y), \varphi(x)] = -2i\Pi(x) \tag{970}$$
$$\int d^3y[(\nabla_y\varphi(y))^2, \varphi(x)] = \int d^3y \nabla_y\varphi(y)[\nabla_y\varphi(y), \varphi(x)] + [\nabla_y\varphi(y), \varphi(x)]\nabla_y\varphi(y) \tag{971}$$
$$= 0 \tag{972}$$
$$\int d^3y[\varphi(y)^2, \varphi(x)] = 0 \tag{973}$$

  and therefore

$$\dot{\varphi}(x) = i[H, \varphi(x)] \tag{974}$$
$$= i\frac{1}{2}(-2i)\Pi(x) \tag{975}$$
$$= \Pi(x) \tag{976}$$

- Obtaining $\dot{\Pi}(x) = -i[H, \Pi(x)]$ (sign!?!)

Now we need to calculate - by using the results from above we can now shortcut a bit

$$[\Pi^2(y), \Pi(x)] = 0 \tag{977}$$
$$[(\nabla_y\varphi(y))^2, \Pi(x)] = (\nabla_y\varphi(y))(\nabla_y\varphi(y))\Pi(x) - \Pi(x)(\nabla_y\varphi(y))(\nabla_y\varphi(y)) \tag{978}$$
$$= (\nabla_y\varphi(y))[(\nabla_y\varphi(y)), \Pi(x)] - [\Pi(x), (\nabla_y\varphi(y))](\nabla_y\varphi(y)) \tag{979}$$
$$= (\nabla_y\varphi(y))\nabla_y[\varphi(y), \Pi(x)] - (\nabla_y[\Pi(x), \varphi(y)])(\nabla_y\varphi(y)) \tag{980}$$
$$= (\nabla_y\varphi(y))\nabla_y i\delta^3(\vec{x}-\vec{y}) - (\nabla_y(-i)\delta^3(\vec{x}-\vec{y}))(\nabla_y\varphi(y)) \tag{981}$$
$$= 2i(\nabla_y\delta^3(\vec{x}-\vec{y}))(\nabla_y\varphi(y)) \tag{982}$$
$$[\varphi(y)^2, \Pi(x)] = \varphi(y)\varphi(y)\Pi(x) - \Pi(x)\varphi(y)\varphi(y) \tag{983}$$
$$= \varphi(y)\varphi(y)\Pi(x) - \varphi(y)\Pi(x)\varphi(y) + \varphi(y)\Pi(x)\varphi(y) - \Pi(x)\varphi(y)\varphi(y) \tag{984}$$
$$= \varphi(y)[\varphi(y), \Pi(x)] + [\varphi(y), \Pi(x)]\varphi(y) \tag{985}$$
$$= 2i\varphi(y)\delta^3(\vec{x}-\vec{y}) \tag{986}$$

then

$$\int d^3y[\Pi^2(y), \Pi(x)] = 0 \tag{987}$$
$$\int d^3y[(\nabla_y\varphi(y))^2, \Pi(x)] = 2i\int d^3y(\nabla_y\delta^3(\vec{x}-\vec{y}))(\nabla_y\varphi(y)) \tag{988}$$
$$= -2i\int d^3y\delta^3(\vec{x}-\vec{y})(\nabla_y\nabla_y\varphi(y)) \tag{989}$$
$$= -2i\triangle_x\varphi(x) \tag{990}$$
$$\int d^3y[\varphi(y)^2, \Pi(x)] = 2i\varphi(x) \tag{991}$$

and therefore

$$\dot{\Pi}(x) = -i[H, \Pi(x)] \tag{992}$$
$$= -i\left(\frac{1}{2}(-2i)\triangle_x\varphi(x) + \frac{1}{2}m^2 2i\varphi(x)\right) \tag{993}$$
$$= -i\left(-i\triangle_x\varphi(x) + m^2 i\varphi(x)\right) \tag{994}$$
$$= -\triangle_x\varphi(x) + m^2\varphi(x) \tag{995}$$

which finally leads to (with $\Box = \partial_{tt} - \triangle$)

$$\partial^0\partial_0\varphi(x) = \partial^0\Pi(x) \tag{996}$$
$$= -\partial_0\Pi(x) \tag{997}$$
$$= -(-\triangle_x\varphi(x) + m^2\varphi(x)) \tag{998}$$
$$\to (\Box_x + m^2)\varphi(x) = 0 \tag{999}$$

(d) With

$$\vec{P} \equiv -\int d^3x\Pi(x)\nabla_x\varphi(x) \tag{1000}$$

we have to calculate

$$[\vec{P}, \varphi(y)] = -\int d^3x[\Pi(x)\nabla_x\varphi(x), \varphi(y)]. \tag{1001}$$

Let's start with

$$[\Pi(x)\nabla_x\varphi(x), \varphi(y)] = \Pi(x)\nabla_x\varphi(x)\varphi(y) - \varphi(y)\Pi(x)\nabla_x\varphi(x) \tag{1002}$$
$$= \Pi(x)\nabla_x\varphi(x)\varphi(y) - (\Pi(x)\varphi(y) + i\delta^3(\vec{x}-\vec{y}))\nabla_x\varphi(x) \tag{1003}$$
$$= \Pi(x)\nabla_x\varphi(x)\varphi(y) - \Pi(x)\varphi(y)\nabla_x\varphi(x) + i\delta^3(\vec{x}-\vec{y})\nabla_x\varphi(x) \tag{1004}$$
$$= \Pi(x)\nabla_x(\varphi(x)\varphi(y)) - \Pi(x)\nabla_x(\varphi(y)\varphi(x)) + i\delta^3(\vec{x}-\vec{y})\nabla_x\varphi(x) \tag{1005}$$
$$= \Pi(x)\nabla_x[\varphi(x), \varphi(y)] + i\delta^3(\vec{x}-\vec{y})\nabla_x\varphi(x) \tag{1006}$$
$$= i\delta^3(\vec{x}-\vec{y})\nabla_x\varphi(x) \tag{1007}$$

and then

$$[\vec{P}, \varphi(y)] = -i\int d^3x\delta^3(\vec{x}-\vec{y})\nabla_x\varphi(x) \tag{1008}$$
$$= -i\nabla_y\varphi(y) \tag{1009}$$

(e) With

$$\Pi(x) = \dot{\varphi}(x) \tag{1010}$$
$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k}(-i\omega_k)(a_k e^{ikx} - a_k^\dagger e^{-ikx}) \tag{1011}$$
$$\nabla\varphi(x) = \int \frac{d^3q}{(2\pi)^3 2\omega_k}(i\vec{q})(a_q e^{iqx} - a_q^\dagger e^{-iqx}) \tag{1012}$$
$$\tag{1013}$$

then

$$\vec{P} = -\int d^3x\Pi(x)\nabla_x\varphi(x) \tag{1014}$$
$$= -\iiint d^3x \frac{d^3k}{(2\pi)^3 2\omega_k}\frac{d^3q}{(2\pi)^3 2\omega_k}(-i\omega_k)(i\vec{q})(a_k e^{ikx} - a_k^\dagger e^{-ikx})(a_q e^{iqx} - a_q^\dagger e^{-iqx}) \tag{1015}$$
$$= -\iiint d^3x \frac{d^3k}{(2\pi)^3 2}\frac{d^3q}{(2\pi)^3 2\omega_k}\vec{q}(a_k a_q e^{i(k+q)x} - a_k^\dagger a_q e^{-i(k-q)x} - a_k a_q^\dagger e^{i(k-q)x} + a_k^\dagger a_q^\dagger e^{-i(k+q)x}) \tag{1016}$$
$$\tag{1017}$$

now we can use the commutation relations and reindex

$$= -\iiint d^3x \frac{d^3kd^3q}{4\omega_k(2\pi)^6}\vec{q}(a_k a_q e^{i(k+q)x} - a_k^\dagger a_q e^{-i(k-q)x} - (a_q^\dagger a_k + (2\pi)^3 2\omega_k\delta^3(\vec{k}-\vec{q}))e^{i(k-q)x} + a_k^\dagger a_q^\dagger e^{-i(k+q)x}) \tag{1018}$$
$$= -\iiint d^3x \frac{d^3kd^3q}{4\omega_k(2\pi)^6}\vec{q}(a_k a_q e^{i(k+q)x} + a_k^\dagger a_q^\dagger e^{-i(k+q)x}) + \iiint d^3x \frac{d^3kd^3q}{4\omega_k(2\pi)^6}\vec{q}2a_k^\dagger a_q e^{-i(k-q)x} \tag{1019}$$
$$+ \iiint d^3x \frac{d^3kd^3q}{4\omega_k(2\pi)^6}\vec{q}(2\pi)^3 2\omega_k\delta^3(\vec{k}-\vec{q})e^{i(k-q)x} \tag{1020}$$

Now we can look at the integrals individually and use the asymmetry. The first

$$-\iiint d^3x \frac{d^3kd^3q}{4\omega_k(2\pi)^6}\vec{q}(a_k a_q e^{i(k+q)x} + a_k^\dagger a_q^\dagger e^{-i(k+q)x}) = ... \tag{1021}$$
$$= 0 \tag{1022}$$

second

$$\iiint d^3x \frac{d^3kd^3q}{4\omega_k(2\pi)^6}\vec{q}(2\pi)^3 2\omega_k \delta^3(\vec{k}-\vec{q})e^{i(k-q)x} = \iiint d^3x \frac{d^3kd^3q}{2(2\pi)^3}\vec{q}\delta^3(\vec{k}-\vec{q})e^{i(k-q)x} \quad (1023)$$

$$= \iiint d^3x \frac{d^3k}{2(2\pi)^3}\vec{k} \quad (1024)$$

$$= 0 \quad (1025)$$

and third

$$\iiint d^3x \frac{d^3kd^3q}{4\omega_k(2\pi)^6}\vec{q}2a_k^\dagger a_q e^{-i(k-q)x} = \iint \frac{d^3kd^3q}{4\omega_k(2\pi)^6}\vec{q}2a_k^\dagger a_q \int d^3x\ e^{-i(k-q)x} \quad (1026)$$

$$= \iint \frac{d^3kd^3q}{4\omega_k(2\pi)^6}\vec{q}2a_k^\dagger a_q e^{-i(k-q)x} e^{-i(k^0-q^0)x^0} \int d^3x\ e^{-i(\vec{k}-\vec{q})\vec{x}} \quad (1027)$$

$$= \iint \frac{d^3kd^3q}{4\omega_k(2\pi)^6}\vec{q}2a_k^\dagger a_q e^{-i(k-q)x} e^{-i(k^0-q^0)x^0} (2\pi)^3\delta^3(\vec{k}-\vec{q}) \quad (1028)$$

$$= \int \frac{d^3k}{2\omega_k(2\pi)^3}\vec{k}a_k^\dagger a_k \quad (1029)$$

$$= \int \widetilde{d^3k}\ \vec{k}a_k^\dagger a_k \quad (1030)$$

Therefore we obtain

$$\vec{P} = \int \frac{d^3k}{2\omega_k(2\pi)^3}\vec{k}a_k^\dagger a_k \quad (1031)$$

$$= \int \widetilde{d^3k}\ \vec{k}a_k^\dagger a_k \quad (1032)$$

0.11.11 Problem 3.5 - Complex scalar field

(a) Sloppy way - Calculating the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \varphi} = -m^2\varphi^\dagger \quad (1033)$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = -\partial^\mu \varphi^\dagger \quad (1034)$$

$$\to -m^2\varphi^\dagger + \partial_\mu\partial^\mu\varphi^\dagger = 0 \quad (1035)$$

$$\to (\partial_\mu\partial^\mu - m^2)\varphi^\dagger = 0 \quad (1036)$$

Bit more rigorous with

$$\frac{\delta\phi(x_1,t_1)}{\delta\phi(x_2,t_2)} = \delta(x_1-x_2)\times\delta(t_1-t_2) \quad (1037)$$

$$\frac{\delta\partial_\mu\phi(x)}{\delta\phi(y)} = \frac{\delta}{\delta\phi(y)}\lim_{\epsilon\to 0}\frac{\phi(x_1,x_\mu+\epsilon,...,x_4)-\phi(x_1,x_2,x_3,x_4)}{\epsilon} \quad (1038)$$

$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\delta(x_\mu+\epsilon-y_\mu)-\delta(x_\mu-y_\mu)\right)\times\delta(x_1-y_1)\times ...\times\delta(x_4-y_4) \quad (1039)$$

$$= \frac{\partial}{\partial x^\mu}\delta^4(x-y) \quad (1040)$$

we get

$$S[\varphi] = \int d^4x \left(-\partial^\mu \varphi^\dagger(x)\partial_\mu\varphi(x) - m^2\varphi^\dagger(x)\varphi(x)\right) \tag{1041}$$

$$\frac{\delta S[\varphi]}{\delta\varphi(y)} = \int d^4x \left(-\partial^\mu \varphi^\dagger(x)\partial_\mu\delta^4(x-y) - m^2\varphi^\dagger(y)\delta^4(x-y)\right) \tag{1042}$$

$$= \int d^4x \left(\partial_\mu\partial^\mu \varphi^\dagger(x)\delta^4(x-y) - m^2\varphi^\dagger(x)\delta^4(x-y)\right) \tag{1043}$$

$$= (\Box_y - m^2)\varphi^\dagger(y) \tag{1044}$$

(b) With

$$\mathcal{L} = -\partial^0\varphi^\dagger\partial_0\varphi - \partial^a\varphi^\dagger\partial_a\varphi - m^2\varphi^\dagger\varphi + \Omega_0 \tag{1045}$$

$$= \partial_0\varphi^\dagger\partial_0\varphi - \partial^a\varphi^\dagger\partial_a\varphi - m^2\varphi^\dagger\varphi + \Omega_0 \tag{1046}$$

$$\Pi = \frac{\partial\mathcal{L}}{\partial\dot\varphi} = \dot\varphi^\dagger \tag{1047}$$

$$\Pi^\dagger = \frac{\partial\mathcal{L}}{\partial\dot\varphi^\dagger} = \dot\varphi \tag{1048}$$

$$\rightarrow \mathcal{H} = \Pi\dot\varphi + \Pi^\dagger\dot\varphi^\dagger - \mathcal{L} \tag{1049}$$

$$= \dot\varphi^\dagger\dot\varphi + \dot\varphi\dot\varphi^\dagger - \dot\varphi^\dagger\dot\varphi + (\nabla^a\varphi^\dagger)(\nabla_a\varphi) + m^2\varphi^\dagger\varphi - \Omega_0 \tag{1050}$$

$$= \Pi^\dagger\Pi + (\nabla^a\varphi^\dagger)(\nabla_a\varphi) + m^2\varphi^\dagger\varphi - \Omega_0 \tag{1051}$$

(c) Considering the plane wave solutions $e^{i\vec{k}\vec{x}\pm i\omega_k t}$ with

$$kx = g_{\mu\nu}k^\mu x^\nu = g_{00}k^0x^0 + g_{ik}k^ix^k = -\omega_k t + \vec{k}\vec{x} \tag{1052}$$

we have

$$\varphi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\ a_k e^{ikx} + b_k^\dagger e^{-ikx} \tag{1053}$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\ a_k e^{i\vec{k}\vec{x}-i\omega_k t} + b_k^\dagger e^{-i\vec{k}\vec{x}+i\omega_k t} \tag{1054}$$

$$e^{-iqx}\varphi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\ a_k e^{i(k-q)x} + b_k^\dagger e^{-i\vec{k}\vec{x}+i\omega_k t}e^{-iqx} \tag{1055}$$

$$\int d^3x e^{-iqx}\varphi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\ a_k \underbrace{\int d^3x e^{i(k-q)x}}_{(2\pi)^3\delta^3(\vec{k}-\vec{q})e^{-i(\omega_k-\omega_q)t}} + b_{-k}\underbrace{\int d^3x e^{i(\vec{k}-\vec{q})\vec{x}}}_{(2\pi)^3\delta^3(\vec{k}-\vec{q})} e^{i(\omega_k+\omega_q)t} \tag{1056}$$

$$= \frac{1}{2\omega_q}\left(a_q + b_{-q}^\dagger e^{2i\omega_q t}\right) \tag{1057}$$

and

$$\partial_0\varphi(\vec{x},t) = \int \frac{d^3k\ (-i\omega_k)}{(2\pi)^3 2\omega_k}\ a_k e^{i\vec{k}\vec{x}-i\omega_k t} - b_k^\dagger e^{-i\vec{k}\vec{x}+i\omega_k t} \tag{1058}$$

$$\int d^3x e^{-iqx}\partial_0\varphi(\vec{x},t) = -\frac{i}{2}\left(a_q - b_{-q}^\dagger e^{2i\omega_q t}\right) \tag{1059}$$

adding both equations gives with $\partial_0 e^{-iqx} = \partial_0 e^{-i(-\omega_k t + \vec{k}\vec{x})} = -i\omega_q e^{-iqx}$ and $f \overleftrightarrow{\partial_\mu} g = f(\partial_\mu g) - (\partial_\mu f) g$

$$a_q = \omega_q \int d^3x e^{-iqx} \varphi(\vec{x},t) + i \int d^3x e^{-iqx} \partial_0 \varphi(\vec{x},t) \tag{1060}$$
$$= i \int d^3x e^{-iqx} (-i\omega_q + \partial_0) \varphi(\vec{x},t) \tag{1061}$$
$$= i \int d^3x e^{-iqx} \overleftrightarrow{\partial_0} \varphi(\vec{x},t) \tag{1062}$$

To get b_q we solve a second set of equations for $\varphi^\dagger$

$$\varphi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \; a_k e^{ikx} + b_k^\dagger e^{-ikx} \tag{1063}$$
$$\to \varphi^\dagger(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \; a_k^\dagger e^{-ikx} + b_k e^{ikx} \tag{1064}$$
$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \; b_k e^{ikx} + a_k^\dagger e^{-ikx} \tag{1065}$$

Now b_k takes the role of a_k and we can just copy the solution

$$b_q = \omega_q \int d^3x e^{-iqx} \varphi^\dagger(\vec{x},t) + i \int d^3x e^{-iqx} \partial_0 \varphi^\dagger(\vec{x},t) \tag{1066}$$
$$= i \int d^3x e^{-iqx} (-i\omega_q + \partial_0) \varphi^\dagger(\vec{x},t) \tag{1067}$$
$$= i \int d^3x e^{-iqx} \overleftrightarrow{\partial_0} \varphi^\dagger(\vec{x},t) \tag{1068}$$

(d) Starting with the observation

$$[A,B]^\dagger = (AB)^\dagger - (BA)^\dagger \tag{1069}$$
$$= B^\dagger A^\dagger - A^\dagger B^\dagger \tag{1070}$$
$$= [B^\dagger, A^\dagger] \tag{1071}$$
$$= -[A^\dagger, B^\dagger] \tag{1072}$$

therefore the relevant commutation relations for the fields are

$$[\varphi(\vec{x},t), \varphi(\vec{y},t)] = 0 \qquad \to \qquad [\varphi^\dagger(\vec{x},t), \varphi^\dagger(\vec{y},t)] = 0 \tag{1073}$$
$$[\varphi^\dagger(\vec{x},t), \varphi(\vec{y},t)] = 0 \tag{1074}$$
$$[\Pi(\vec{x},t), \Pi(\vec{y},t)] = 0 \qquad \to \qquad [\Pi^\dagger(\vec{x},t), \Pi^\dagger(\vec{y},t)] = 0 \tag{1075}$$
$$[\Pi^\dagger(\vec{x},t), \Pi(\vec{y},t)] = 0 \tag{1076}$$
$$[\varphi(\vec{x},t), \Pi(\vec{y},t)] = i\delta^3(\vec{x}-\vec{y}) \qquad \to \qquad [\varphi^\dagger(\vec{x},t), \Pi^\dagger(\vec{y},t)] = i\delta^3(\vec{x}-\vec{y}) \tag{1077}$$
$$[\varphi^\dagger(\vec{x},t), \Pi(\vec{y},t)] = 0 \qquad \to \qquad [\varphi(\vec{x},t), \Pi^\dagger(\vec{y},t)] = 0 \tag{1078}$$

with the previous results

$$a_q = i\int d^3x e^{-iqx}(-i\omega_q + \partial_0)\varphi(\vec{x},t) \tag{1079}$$

$$= i\int d^3x e^{-iqx}(-i\omega_q\varphi(\vec{x},t) + \Pi^\dagger(\vec{x},t)) \tag{1080}$$

$$a_q^\dagger = i\int d^3x e^{iqx}(i\omega_q\varphi^\dagger(\vec{x},t) + \Pi(\vec{x},t)) \tag{1081}$$

$$b_q = i\int d^3x e^{-iqx}(-i\omega_q + \partial_0)\varphi^\dagger(\vec{x},t) \tag{1082}$$

$$= i\int d^3x e^{-iqx}(-i\omega_q\varphi(\vec{x},t) + \Pi(\vec{x},t)) \tag{1083}$$

$$b_q^\dagger = i\int d^3x e^{iqx}(i\omega_q\varphi^\dagger(\vec{x},t) + \Pi^\dagger(\vec{x},t)) \tag{1084}$$

let's calculate each of the commutators

$$[a_k, a_q^\dagger] = \iint d^3x\, d^3y\, e^{-ikx}e^{iqy}\left(\omega_k\omega_q[\varphi_x, \varphi_y^\dagger] - i\omega_q[\varphi_x, \Pi_y] + i\omega_q[\Pi_x^\dagger, \varphi_y^\dagger] + [\Pi_x^\dagger, \Pi_y]\right) \tag{1085}$$

$$= \iint d^3x\, d^3y\, e^{-i(kx-qy)}\left(-i\omega_q[\varphi_x, \Pi_y] + i\omega_q[\Pi_x^\dagger, \varphi_y^\dagger]\right) \tag{1086}$$

$$= \iint d^3x\, d^3y\, e^{-i(kx-qy)}\left(-i\omega_q i\delta^3(\vec{x}-\vec{y}) + i\omega_q(-i)\delta^3(\vec{x}-\vec{y})\right) \tag{1087}$$

$$= (\omega_q + \omega_q)\iint d^3x\, e^{-i(k-q)x} \tag{1088}$$

$$= (\omega_q + \omega_q)\,(2\pi)^3\delta^3(\vec{k}-\vec{q}) \tag{1089}$$

$$= 2\omega_q(2\pi)^3\delta^3(\vec{k}-\vec{q}) \tag{1090}$$

and so on

$$[b_k, b_q^\dagger] = ... = 2\omega_q(2\pi)^3\delta^3(\vec{k}-\vec{q}) \tag{1091}$$

(e) Now

$$H = \int d^3x\, \Pi^\dagger\Pi + (\nabla^a\varphi^\dagger)(\nabla_a\varphi) + m^2\varphi^\dagger\varphi - \Omega_0 \tag{1092}$$

$$\Pi^\dagger\Pi = \dot{\varphi}\dot{\varphi}^\dagger \tag{1093}$$

$$= \int \widetilde{d^3k}\widetilde{d^3q}(i\omega_k)(i\omega_q)\left(a_k e^{ikx} - b_k^\dagger e^{-ikx}\right)\left(a_q^\dagger e^{-iqx} - b_q e^{iqx}\right) \tag{1094}$$

$$= \int \widetilde{d^3k}\widetilde{d^3q}(-\omega_k\omega_q)\left(a_k a_q^\dagger e^{-iqx}e^{ikx} - b_k^\dagger a_q^\dagger e^{-iqx}e^{-ikx} - a_k b_q e^{iqx}e^{ikx} + b_k^\dagger b_q e^{iqx}e^{-ikx}\right) \tag{1095}$$

$$= \int \widetilde{d^3k}\widetilde{d^3q}(-\omega_k\omega_q)\left([a_q^\dagger a_k - 2\omega_k(2\pi)^3\delta^3(\vec{k}-\vec{q})]e^{-i(q-k)x} - b_k^\dagger a_q^\dagger e^{-i(q+k)x} - a_k b_q e^{i(q+k)x} + b_k^\dagger b_q e^{i(q-k)x}\right) \tag{1096}$$

$$(\nabla^a\varphi^\dagger)(\nabla_a\varphi) = \int \widetilde{d^3k}\widetilde{d^3q}(k^a q_a)\left(-a_k^\dagger e^{-ikx} + b_k e^{ikx}\right)\left(a_q e^{iqx} - b_q^\dagger e^{-iqx}\right) \tag{1097}$$

$$= \int \widetilde{d^3k}\widetilde{d^3q}(k^a q_a)\left(-a_k^\dagger a_q e^{iqx}e^{-ikx} + b_k a_q e^{iqx}e^{ikx} + a_k^\dagger b_q^\dagger e^{-iqx}e^{-ikx} - b_k b_q^\dagger e^{-iqx}e^{ikx}\right) \tag{1098}$$

$$= \int \widetilde{d^3k}\widetilde{d^3q}(k^a q_a)\left(-a_k^\dagger a_q e^{i(q-k)x} + a_q b_k e^{i(q+k)x} + a_k^\dagger b_q^\dagger e^{-i(q+k)x} - [b_q^\dagger b_k - 2\omega_k(2\pi)^3\delta^3(\vec{k}-\vec{q})]e^{-i(q-k)} \right. \tag{1099}$$

$$\varphi^\dagger\varphi = \int \widetilde{d^3k}\widetilde{d^3q}\left(a_k^\dagger e^{-ikx} + b_k e^{ikx}\right)\left(a_q e^{iqx} + b_q^\dagger e^{-iqx}\right) \tag{1100}$$

$$= \int \widetilde{d^3k}\widetilde{d^3q}\left(a_k^\dagger a_q e^{iqx}e^{-ikx} + b_k a_q e^{iqx}e^{ikx} + a_k^\dagger b_q^\dagger e^{-iqx}e^{-ikx} + b_k b_q^\dagger e^{-iqx}e^{ikx}\right) \tag{1101}$$

$$= \int \widetilde{d^3k}\widetilde{d^3q}\left(a_k^\dagger a_q e^{i(q-k)x} + a_q b_k e^{i(q+k)x} + a_k^\dagger b_q^\dagger e^{-i(q+k)x} + [b_q^\dagger b_k - 2\omega_k(2\pi)^3\delta^3(\vec{k}-\vec{q})]e^{-i(q-k)x}\right) \tag{1102}$$

then

$$H_{a^\dagger a} = \int \widetilde{d^3k}\widetilde{d^3q}\int d^3x\left[(-\omega_k\omega_q)[a_q^\dagger a_k - 2\omega_k(2\pi)^3\delta^3(\vec{k}-\vec{q})]e^{-i(q-k)x}\right] \tag{1103}$$

$$+ \int \widetilde{d^3k}\widetilde{d^3q}\int d^3x(k^a q_a)\left[-a_k^\dagger a_q e^{i(q-k)x}\right] + m^2 a_k^\dagger a_q e^{i(q-k)x} \tag{1104}$$

$$= \int \widetilde{d^3k}\widetilde{d^3q}\; a_k^\dagger a_q\left[-\omega_k\omega_q - k^a q_a + m^2\right]\int d^3x e^{i(q-k)x} \tag{1105}$$

$$- \int \widetilde{d^3k}\widetilde{d^3q}\;(-\omega_k\omega_q)2\omega_q(2\pi)^3\delta^3(\vec{q}-\vec{k})\int d^3x\; e^{i(q-k)x} \tag{1106}$$

$$= \int \widetilde{d^3k}\frac{d^3q}{(2\pi)^3 2\omega_q}\; a_k^\dagger a_q\left[-\omega_k\omega_q - k^a q_a + m^2\right](2\pi)^3\delta^3(\vec{q}-\vec{k})e^{-i(\omega_q-\omega_k)t} \tag{1107}$$

$$- \int \frac{d^3k}{(2\pi)^3 2\omega_k}\frac{1}{(2\pi)^3 2\omega_k}\;(-\omega_k^2)2\omega_k(2\pi)^3 e^{-i(\omega_k-\omega_k)t}\int d^3x \tag{1108}$$

$$= \int \widetilde{d^3k}\frac{1}{2\omega_k}\; a_k^\dagger a_k \underbrace{\left[-\omega_k^2 - \vec{k}^2 + m^2\right]}_{2\omega_k^2!?!?!} + \frac{V}{2(2\pi)^3}\int d^3k\;\omega_k \tag{1109}$$

$$= \int \widetilde{d^3k}\omega_k\; a_k^\dagger a_k + \frac{V}{2(2\pi)^3}\int d^3k\;\omega_k \tag{1110}$$

and similar for $H_{b^\dagger b}, H_{ab}, H_{a^\dagger b^\dagger}$.

$$H = \int \widetilde{d^3k}\omega_k\;(a_k^\dagger a_k + b_k^\dagger b_k) + \frac{V}{2(2\pi)^3}\int d^3k\;\omega_k \tag{1111}$$

0.11.12 Problem 4.1 - Commutator non-hermitian field

With $t = t'$ and $|\vec{x} - \vec{x}'| = r$ we have

$$
\begin{aligned}
[\varphi^+(x), \varphi^-(x')]_\pm &= \int \widetilde{dk} e^{ik(x-x')} && (1112)\\
&= \int d^3k \frac{1}{(2\pi)^3 2\omega_k} e^{ik(x-x')} && (1113)\\
&= \frac{1}{2\cdot 8\pi^3} \int d^3k \frac{1}{\sqrt{|k|^2+m^2}} e^{i[\vec{k}(\vec{x}-\vec{x}')]} && (1114)\\
&= \frac{1}{16\pi^3} \int |k|^2 dk d\phi d\theta \sin\theta \frac{1}{\sqrt{|k|^2+m^2}} e^{i|k|r\cos\theta} && (1115)\\
&= \frac{2\pi}{16\pi^3} \int |k|^2 dk \underbrace{d\theta \sin\theta}_{-d\cos\theta} \frac{1}{\sqrt{|k|^2+m^2}} e^{i|k|r\cos\theta} && (1116)\\
&= \frac{2\pi}{16\pi^3} \int |k|^2 dk \frac{1}{\sqrt{|k|^2+m^2}} \int_{-1}^{1} d\cos\theta e^{i|k|r\cos\theta} && (1117)\\
&= \frac{2\pi}{16\pi^3} \int |k|^2 dk \frac{1}{\sqrt{|k|^2+m^2}} 2\frac{\sin(|k|r)}{|k|r} && (1118)\\
&= \frac{1}{4\pi^2 r} \int_0^\infty dk \frac{|k|\sin(|k|r)}{\sqrt{|k|^2+m^2}} && (1119)
\end{aligned}
$$

With Gradshteyn, Ryzhik 7ed (8.486) - we find for the definition of the modified Bessel function K_1

$$\frac{d}{dz} K_0(z) = -K_1(z) \tag{1120}$$

and Gradshteyn, Ryzhik 7ed (3.754)

$$\int_0^\infty dx \frac{\cos(ax)}{\sqrt{\beta^2+x^2}} = K_0(a\beta) \tag{1121}$$

therefore

$$
\begin{aligned}
\frac{d}{da} K_0(a\beta) &= \int_0^\infty dx \frac{-x\sin(ax)}{\sqrt{\beta^2+x^2}} && (1122)\\
&= \beta K_0'(a\beta) && (1123)\\
&= -\beta K_1(a\beta) && (1124)\\
\to K_1(a\beta) &= \frac{1}{\beta} \int_0^\infty dx \frac{x\sin(ax)}{\sqrt{\beta^2+x^2}} && (1125)
\end{aligned}
$$

which we can use to finish the calculation

$$[\varphi^+(x), \varphi^-(x')]_\pm = \frac{1}{4\pi^2 r} m K_1(mr) \tag{1126}$$

From `https://dlmf.nist.gov/10.30` we get

$$
\begin{aligned}
\lim_{z\to 0} K_\nu(z) &\sim \frac{1}{2}\Gamma(\nu)\left(\frac{1}{2}z\right)^{-\nu} && (1127)\\
\to \lim_{z\to 0} K_1(z) &\sim \frac{1}{2}\left(\frac{1}{2}z\right)^{-1} = 1/z && (1128)
\end{aligned}
$$

and therefore

$$[\varphi^+(x), \varphi^-(x')]_\pm = \frac{1}{4\pi^2 r^2}. \tag{1129}$$

0.11.13 Problem 5.1 - LSZ reduction for complex scalar field

From Exercise 3.5 we have

$$a_q = i\int d^3x e^{-iqx} \overleftrightarrow{\partial_0}\, \varphi(\vec{x},t) \tag{1130}$$

$$a_q^\dagger = -i\int d^3x e^{iqx} \overleftrightarrow{\partial_0}\, \varphi^\dagger(\vec{x},t) \tag{1131}$$

$$b_q = i\int d^3x e^{-iqx} \overleftrightarrow{\partial_0}\, \varphi^\dagger(\vec{x},t) \tag{1132}$$

$$b_q^\dagger = -i\int d^3x e^{iqx} \overleftrightarrow{\partial_0}\, \varphi(\vec{x},t) \tag{1133}$$

then

$$a_1^\dagger(+\infty) - a_1^\dagger(-\infty) = -i\int d^3k f_1(\vec{k})\int d^4x e^{ikx}(-\Box_x + m^2)\varphi^\dagger(x) \tag{1134}$$

rearranging leads to

$$a_1^\dagger(-\infty) = a_1^\dagger(+\infty) + i\int d^3k f_1(\vec{k})\int d^4x e^{ikx}(-\Box_x + m^2)\varphi^\dagger(x) \tag{1135}$$

$$a_1(+\infty) = a_1(-\infty) + i\int d^3k f_1(\vec{k})\int d^4x e^{-ikx}(-\Box_x + m^2)\varphi(x) \tag{1136}$$

$$b_1^\dagger(-\infty) = b_1^\dagger(+\infty) + i\int d^3k f_1(\vec{k})\int d^4x e^{ikx}(-\Box_x + m^2)\varphi^\dagger(x) \tag{1137}$$

$$b_1(+\infty) = b_1(-\infty) + i\int d^3k f_1(\vec{k})\int d^4x e^{-ikx}(-\Box_x + m^2)\varphi(x) \tag{1138}$$

then we get for a, b particle scattering with the time ordering operator T (Later time to the Left)

$$\begin{aligned}
\langle f|i\rangle &= \langle 0|a_{1'}(+\infty)b_{2'}(+\infty)a_1^\dagger(-\infty)b_2^\dagger(-\infty)|0\rangle &(1139)\\
&= \langle 0|Ta_{1'}(+\infty)b_{2'}(+\infty)a_1^\dagger(-\infty)b_2^\dagger(-\infty)|0\rangle &(1140)\\
&= \langle 0|T(a_{1'}(-\infty) + i\int)(b_{2'}(-\infty) + i\int)(a_1^\dagger(+\infty) + i\int)(b_2^\dagger(+\infty) + i\int)|0\rangle &(1141)\\
&= i^4\int d^4x_1' e^{-ik_1'x_1'}(-\Box_{x_1'} + m_a^2)\int d^4x_2' e^{-ik_2'x_2'}(-\Box_{x_2'} + m_b^2)\times &(1142)\\
&\quad\times\int d^4x_1 e^{-ik_1x_1}(-\Box_{x_1} + m_a^2)\int d^4x_2 e^{-ik_2x_2}(-\Box_{x_2} + m_b^2)\langle 0|\phi_{x_1'}\phi_{x_2'}\phi_{x_1}^\dagger\phi_{x_2}^\dagger|0\rangle &(1143)
\end{aligned}$$

0.11.14 Problem 6.1 - Path integral in quantum mechanics

(a) The transition amplitude $\langle q''|e^{-iH(t''-t')}|q'\rangle$ (particle to start at q', t' and ends at position q'' at time t'') can be written in the Heisenberg picture as

$$\begin{aligned}
\langle q''|e^{-iH(t''-t')}|q'\rangle &= \langle q''|e^{-iHt''}e^{iHt''}e^{-iH(t''-t')}e^{-iHt'}e^{iHt'}|q'\rangle &(1144)\\
&= \langle q'', t''|e^{iHt''}e^{iH(t''-t')}e^{-iHt'}|q', t'\rangle &(1145)\\
&= \langle q'', t''|q', t'\rangle. &(1146)
\end{aligned}$$

Now we can do the standard path integral derivation

$$\langle q'', t''|q', t'\rangle = \int \left(\prod_{j=1}^{N} dq_j\right) \langle q''|e^{-iH\delta t}|q_N\rangle\langle q_N|e^{-iH\delta t}|q_{N-1}\rangle \ldots \langle q_1|e^{-iH\delta t}|q'\rangle \tag{1147}$$

$$= \int \left(\prod_{j=1}^{N} dq_j\right) \int \frac{dp_N}{2\pi} e^{-iH(p_N,q_N)\delta t} e^{ip_N(q'-q_N)} \ldots \int \frac{dp'}{2\pi} e^{-iH(p',q')\delta t} e^{ip'(q_1-q')} \tag{1148}$$

$$= \int \left(\prod_{j=1}^{N} dq_j\right) \left(\prod_{k=0}^{N} \frac{dp_k}{2\pi} e^{ip_k(q_{k+1}-q_k)} e^{-iH(p_k,q_k)\delta t}\right) \quad (q_0 = q', q_{N+1} = q'') \tag{1149}$$

which under Weyl ordering (see Greiner, Reinhard - field quantization) has to be replaced by

$$\langle q'', t''|q', t'\rangle = \int \left(\prod_{j=1}^{N} dq_j\right) \left(\prod_{k=0}^{N} \frac{dp_k}{2\pi} e^{ip_k(q_{k+1}-q_k)} e^{-iH(p_k,\bar{q}_k)\delta t}\right) \quad \bar{q}_k = (q_{k+1}+q_k)/2 \tag{1150}$$

$$= \int \left(\prod_{j=1}^{N} dq_j\right) \left(\prod_{k=0}^{N} \frac{dp_k}{2\pi} e^{i[p_k\dot{q}_k - H(p_k,\bar{q}_k)]\delta t}\right) \quad \dot{q}_k = (q_{k+1}-q_k)/\delta t \tag{1151}$$

$$= \int \left(\prod_{j=1}^{N} dq_j\right) \left(\prod_{k=0}^{N} \frac{dp_k}{2\pi}\right) \left(e^{i\sum_{n=0}^{N}[p_n\dot{q}_n - H(p_n,\bar{q}_n)]\delta t}\right) \tag{1152}$$

$$= \int \mathcal{D}q\mathcal{D}p \exp\left[i\int_{t'}^{t''} dt\,(p(t)\dot{q}(t) - H(p(t),q(t)))\right] \tag{1153}$$

Let's now assume $H(p,q)$ has only a quadratic term in p which is independent of q meaning

$$H(p,q) = \frac{p^2}{2m} + V(q) \tag{1154}$$

then

$$\langle q'', t''|q', t'\rangle = \int \left(\prod_{j=1}^{N} dq_j\right) \left(\prod_{k=0}^{N} \frac{dp_k}{2\pi}\right) \left(e^{i\sum_{n=0}^{N}[p_n\dot{q}_n - \frac{1}{2m}p_n^2 - V(\bar{q}_n)]\delta t}\right) \tag{1155}$$

We can evaluate a single p-integral using

$$\int_{-\infty}^{\infty} dx e^{-ax^2+bx+c} = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}+c} \tag{1156}$$

and obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dp_k \left(e^{i[p_k\dot{q}_k - \frac{1}{2m}p_k^2 - V(\bar{q}_k)]\delta t}\right) = \frac{1}{2\pi} e^{-iV(\bar{q}_k)\delta t} \int dp_k \left(e^{i[p_k\dot{q}_k - \frac{1}{2m}p_k^2]\delta t}\right) \tag{1157}$$

$$= \frac{1}{2\pi} e^{-iV(\bar{q}_k)\delta t} \sqrt{\frac{\pi}{i\frac{\delta t}{2m}}} e^{\frac{-\dot{q}_k^2\delta t^2}{4\frac{i\delta t}{2m}}} \tag{1158}$$

$$= \frac{1}{2\pi}\sqrt{\frac{2\pi m}{i\delta t}} e^{i\left(\frac{m\dot{q}_k^2}{2} - V(\bar{q}_k)\right)\delta t} \tag{1159}$$

$$= \sqrt{\frac{m}{2\pi i\delta t}} e^{iL(\bar{q}_k,\dot{q}_k)\delta t}. \tag{1160}$$

As there are $N+1$ p-integrals we have

$$\mathcal{D}q = \left(\frac{m}{2\pi i\delta t}\right)^{(N+1)/2} \prod_{j=1}^{N} dq_j \tag{1161}$$

(b) We now assume $V(q) = 0$

$$\langle q'', t''|q', t'\rangle = \int \mathcal{D}q\, e^{i\int_{t'}^{t''} dt \frac{\dot{q}^2}{2m}} \tag{1162}$$

$$= \lim_{N\to\infty} \left(\frac{m}{2\pi i\delta t}\right)^{\frac{N+1}{2}} \left(\prod_{j=1}^{N} \int_{-\infty}^{\infty} dq_j\, e^{im\frac{(q_j-q_{j+1})^2}{2\delta t^2}\delta t}\right) e^{im\frac{(q'-q_1)^2}{2\delta t}} e^{im\frac{(q_N-q'')^2}{2\delta t}} \tag{1163}$$

$$= \lim_{N\to\infty} \left(\frac{m}{2\pi i\delta t}\right)^{\frac{N+1}{2}} \left(\prod_{j=3}^{N} \int_{-\infty}^{\infty} dq_j\, e^{im\frac{(q_j-q_{j+1})^2}{2\delta t}}\right) \int dq_2 e^{im\frac{(q_2-q_3)^2}{2\delta t}} \int dq_1 e^{im\frac{(q_1-q_2)^2}{2\delta t}} e^{im\frac{(q_0-q_1)^2}{2\delta t}} \tag{1164}$$

now we can simplify the q_1-integral

$$\int_{-\infty}^{\infty} dq_1\, e^{im\frac{(q_1-q_2)^2}{2\delta t}} e^{im\frac{(q_0-q_1)^2}{2\delta t}} = \int_{-\infty}^{\infty} dq_1\, e^{\frac{im}{2\delta t}(q_0^2-2q_0q_1+q_1^2+q_1^2-2q_1q_2+q_2^2)} \tag{1165}$$

$$= e^{\frac{im}{2\delta t}(q_0^2+q_2^2)} \int_{-\infty}^{\infty} dq_1\, e^{\frac{im}{\delta t}(q_1^2-q_1(q_2+q_0))} \tag{1166}$$

$$= e^{\frac{im}{2\delta t}(q_0^2+q_2^2)} \sqrt{\frac{\pi\delta t}{m}} e^{\frac{i}{4}\left(\pi - \frac{(q_2+q_0)^2 m}{\delta t}\right)} \tag{1167}$$

$$= e^{\frac{im}{4\delta t}(q_0-q_2)^2} \sqrt{\frac{\pi\delta t}{m}}\sqrt{i} \tag{1168}$$

$$= e^{\frac{im}{4\delta t}(q_0-q_2)^2} \sqrt{\frac{i\pi\delta t}{m}} \tag{1169}$$

now simplify the q_2-integral

$$\sqrt{\frac{i\pi\delta t}{m}} \int_{-\infty}^{\infty} dq_2 e^{\frac{im}{2\delta t}(q_2-q_3)^2} e^{\frac{im}{4\delta t}(q_0-q_2)^2} = \sqrt{\frac{i\pi\delta t}{m}} \int_{-\infty}^{\infty} dq_2 e^{\frac{im}{4\delta t}(2q_2^2-4q_3q_2+2q_3^2+q_0^2-2q_0q_2+q_2^2)} \tag{1170}$$

$$= \sqrt{\frac{i\pi\delta t}{m}} \int_{-\infty}^{\infty} dq_2 e^{\frac{im}{4\delta t}(3q_2^2-(4q_3+2q_0)q_2+2q_3^2+q_0^2)} \tag{1171}$$

$$= \sqrt{\frac{i\pi\delta t}{m}} e^{\frac{im}{4\delta t}(2q_3^2+q_0^2)} \int_{-\infty}^{\infty} dq_2 e^{\frac{im}{4\delta t}(3q_2^2-(4q_3+2q_0)q_2)} \tag{1172}$$

$$= \sqrt{\frac{i\pi\delta t}{m}} e^{\frac{im}{4\delta t}(2q_3^2+q_0^2)} \sqrt{\frac{\pi 4\delta t}{3m}} e^{\frac{i}{4}\left(\pi - \frac{(4q_3+2q_0)^2 m}{12\delta t}\right)} \tag{1173}$$

$$= \sqrt{\frac{i\pi\delta t}{m}} \sqrt{\frac{4i\pi\delta t}{3m}} e^{\frac{im}{6\delta t}(q_3-q_0)^2} \tag{1174}$$

then we can extend the results (without explicitly proving)

$$\langle q'', t''|q', t'\rangle = \lim_{N\to\infty} \left(\frac{m}{2\pi i\delta t}\right)^{\frac{N+1}{2}} \prod_{j=1}^{N} \sqrt{\frac{2i\pi\delta t}{m}\frac{j}{j+1}} \cdot e^{\frac{im}{2(j+1)\delta t}(q''-q')^2} \tag{1175}$$

$$= \lim_{N\to\infty} \sqrt{\frac{m}{2\pi i\delta t}} \sqrt{\frac{1}{N+1}} \cdot e^{\frac{im}{2(N+1)\delta t}(q_{N+1}-q_0)^2} \tag{1176}$$

$$= \sqrt{\frac{m}{2\pi i(t''-t')}} \cdot e^{\frac{im(q''-q')^2}{2(t''-t')}}. \tag{1177}$$

The exponent has the dimension kg $\cdot$ m$^2/s$ which is the same as Js. So we just insert an $\hbar$

$$\langle q'', t''|q', t'\rangle = \sqrt{\frac{m}{2\pi i\hbar(t''-t')}} \cdot e^{\frac{im(q''-q')^2}{2\hbar(t''-t')}}. \tag{1178}$$

(c) Simple - with $H|k\rangle = \frac{k^2}{2m}|k\rangle$ we get

$$\begin{aligned}
\langle q'', t''|q', t'\rangle &= \langle q''|\exp(-iH(t''-t'))|q'\rangle && (1179)\\
&= \int dp \int dk \langle q''|p\rangle\langle p|\exp(-iH(t''-t'))|k\rangle\langle k|q'\rangle && (1180)\\
&= \int dp \int dk \frac{1}{\sqrt{2\pi}}e^{ipq'}\langle p|k\rangle \exp(-i\frac{k^2}{2m}(t''-t'))\frac{1}{\sqrt{2\pi}}e^{-ikq''} && (1181)\\
&= \int dp \int dk \frac{1}{\sqrt{2\pi}}e^{ipq'} \exp(-i\frac{k^2}{2m}(t''-t'))\delta(k-p)\frac{1}{\sqrt{2\pi}}e^{-ikq''} && (1182)\\
&= \frac{1}{2\pi}\int dp e^{ip(q'-q'')} \exp(-i\frac{p^2}{2m}(t''-t')) && (1183)\\
&= \frac{1}{2\pi}\sqrt{-\frac{2m\pi}{t''-t'}}e^{\frac{i}{4}\left(\pi - \frac{-2m(q''-q')^2}{t''-t'}\right)} && (1184)\\
&= \sqrt{-\frac{im}{2\pi(t''-t')}}e^{-\frac{i}{4}\frac{-2m(q''-q')^2}{t''-t'}} && (1185)\\
&= \sqrt{\frac{m}{2\pi i(t''-t')}}e^{\frac{-im(q''-q')^2}{2(t''-t')}} && (1186)
\end{aligned}$$

which is the same as in (b).

0.11.15 Problem 7.1 - Oscillator Green's function I

$$\begin{aligned}
G(t-t') &= \int_{-\infty}^{+\infty}\frac{dE}{2\pi}\frac{e^{-iE(t-t')}}{-E^2+\omega^2-i\epsilon} && (1187)\\
&= -\frac{1}{2\pi}\int_{-\infty}^{+\infty} dE \frac{e^{-iE(t-t')}}{E^2-\omega^2+i\epsilon} && (1188)
\end{aligned}$$

with

$$\begin{aligned}
E^2-\omega^2+i\epsilon &= (E+\sqrt{\omega^2-i\epsilon})(E-\sqrt{\omega^2-i\epsilon}) && (1189)\\
&= \left(E+\omega\sqrt{1-\frac{i\epsilon}{\omega^2}}\right)\left(E-\omega\sqrt{1-\frac{i\epsilon}{\omega^2}}\right) && (1190)\\
&\simeq \left(E+\omega-\frac{i\epsilon}{2\omega}\right)\left(E-\omega+\frac{i\epsilon}{2\omega^2}\right) && (1191)
\end{aligned}$$

we can simplify

$$\begin{aligned}
G(\Delta t) &= -\frac{1}{2\pi}\int_{-\infty}^{+\infty} dE e^{-iE\Delta t}\left(\frac{1}{E+\omega-\frac{i\epsilon}{2\omega}} + \frac{1}{E-\omega+\frac{i\epsilon}{2\omega}}\right) && (1192)\\
&= -\frac{1}{2\pi}\frac{1}{2\left(\omega-\frac{i\epsilon}{2\omega}\right)}\int_{-\infty}^{+\infty} dE e^{-iE\Delta t}\left(-\frac{1}{E+\omega-\frac{i\epsilon}{2\omega}} + \frac{1}{E-\omega+\frac{i\epsilon}{2\omega}}\right) && (1193)
\end{aligned}$$

Integrating along the closed contour along the lower half plane (seeing that the exponential function makes the arc part vanish - for $\Delta t > 0$) and using the residual theorem (only one pole is inside) we get (with $\epsilon \to 0$)

$$G(\Delta t) = +\frac{1}{2\pi}\frac{1}{2\left(\omega - \frac{i\epsilon}{2\omega}\right)}(2\pi i)e^{-i(\omega - \frac{i\epsilon}{2\omega})\Delta t} \tag{1194}$$

$$= \frac{i}{2\omega}e^{-i\omega\Delta t} \tag{1195}$$

For $\Delta t < 0$ we integrate along the contour of the upper plane - combining both results we get

$$G(t) = \frac{i}{2\omega}e^{-i\omega|t|} \tag{1196}$$

0.11.16 Problem 7.2 - Oscillator Green's function II

We can rewrite the Greens function using the Heaviside theta function

$$|t| = (2\theta(t) - 1)t \tag{1197}$$

$$\frac{d}{dt}|t| = 2\theta'(t)t + (2\theta(t) - 1) \tag{1198}$$

$$= 2\underbrace{\delta(t)t}_{=0} + 2\theta(t) - 1 \tag{1199}$$

$$= 2\theta(t) - 1 \tag{1200}$$

and then differentiate and use $\theta'(t) = \delta(t)$

$$G(t) = \frac{i}{2\omega}e^{-i\omega(2\theta(t)-1)t} \tag{1201}$$

$$\partial_t G(t) = \frac{i}{2\omega}e^{-i\omega(2\theta(t)-1)t}(-i\omega)(2\theta(t) - 1)) \tag{1202}$$

$$= (-i\omega)G(t)\,(2\theta(t) - 1) \tag{1203}$$

$$\partial_{tt}G(t) = (-i\omega)\partial_t G(t)\,(2\theta(t) - 1) + (-2i\omega)G(t)\delta(t) \tag{1204}$$

$$= (-i\omega)^2 G(t)\,(2\theta(t) - 1)^2 + (-2i\omega)G(t)\delta(t) \tag{1205}$$

$$= -\omega^2 G(t) + e^{-i\omega|t|}\delta(t) \tag{1206}$$

where we used $(2\theta(t) - 1)^2 \equiv 1$

$$\left(\partial_{tt} + \omega^2\right)G(t) = \left(-\omega^2 + \omega^2\right)G(t) + \delta(t) = \delta(t) \tag{1207}$$

0.11.17 Problem 7.3 - Harmonic Oscillator - Heisenberg and Schroedinger picture

(a) With $\hbar = 1$ and

$$H = \frac{1}{2}P^2 + \frac{1}{2}m\omega^2 Q^2 \tag{1208}$$

$$[Q, P] = QP - PQ = i \tag{1209}$$

$$[Q, Q] = [P, P] = 0 \tag{1210}$$

we obtain for the commutators

$$[P^2, Q] = P(PQ) - QP^2 \tag{1211}$$
$$= P(QP - i) - QP^2 \tag{1212}$$
$$= (PQ)P - Pi - QP^2 \tag{1213}$$
$$= (QP - i)P - Pi - QP^2 \tag{1214}$$
$$= -2Pi \tag{1215}$$
$$[Q^2, P] = Q(QP) - PQ^2 \tag{1216}$$
$$= Q(PQ + i) - PQ^2 \tag{1217}$$
$$= (QP)Q + iQ - PQ^2 \tag{1218}$$
$$= (PQ + i)Q + iQ - PQ^2 \tag{1219}$$
$$= 2Qi \tag{1220}$$

Then the Heisenberg equations are

$$\dot{Q}(t) = i[H, Q(t)] = i\frac{1}{2m}[P^2(t), Q(t)] = \frac{1}{m}P(t) \tag{1221}$$
$$\dot{P}(t) = i[H, P(t)] = i\frac{1}{2}m\omega^2[Q^2(t), P(t)] = -m\omega^2 Q(t) \tag{1222}$$
$$\rightarrow \ddot{Q}(t) = \frac{1}{m}\dot{P}(t) = -\omega^2 Q(t) \tag{1223}$$

with the solutions (initial conditions $Q(0) = Q, P(0) = P$)

$$Q(t) = A\cos\omega t + B\sin\omega t \qquad \rightarrow A = Q, \quad \omega B = \frac{1}{m}P \tag{1224}$$
$$= Q\cos\omega t + \frac{1}{\omega m}P\sin\omega t \tag{1225}$$
$$P(t) = m\dot{Q}(t) \tag{1226}$$
$$= -m\omega Q\sin\omega t + P\cos\omega t \tag{1227}$$

(b) Using Diracs trick from QM (rewriting H in terms of a and $a^\dagger$)

$$a = \sqrt{\frac{m\omega}{2}}(Q + \frac{i}{m\omega}P) \tag{1228}$$
$$a^\dagger = \sqrt{\frac{m\omega}{2}}(Q - \frac{i}{m\omega}P) \tag{1229}$$

we can invert the relation

$$Q = \frac{1}{\sqrt{2m\omega}}(a^\dagger + a) \tag{1230}$$
$$P = i\sqrt{\frac{m\omega}{2}}(a^\dagger - a) \tag{1231}$$

and

$$Q(t) = Q\cos\omega t + \frac{1}{\omega m}P\sin\omega t \tag{1232}$$
$$= \frac{1}{\sqrt{2m\omega}}(a^\dagger + a)\cos\omega t + \frac{1}{\omega m}i\sqrt{\frac{m\omega}{2}}(a^\dagger - a)\sin\omega t \tag{1233}$$
$$= \frac{1}{\sqrt{2m\omega}}\left((a^\dagger + a)\cos\omega t + i(a^\dagger - a)\sin\omega t\right) \tag{1234}$$
$$= \frac{1}{\sqrt{2m\omega}}\left(a^\dagger(\cos\omega t + i\sin\omega t) + a(\cos\omega t - i\sin\omega t)\right) \tag{1235}$$
$$= \frac{1}{\sqrt{2m\omega}}\left(a^\dagger e^{i\omega t} + a e^{-i\omega t}\right) \tag{1236}$$
$$P(t) = i\sqrt{\frac{m\omega}{2}}\left(a^\dagger e^{i\omega t} - a e^{-i\omega t}\right) \tag{1237}$$
$$\tag{1238}$$

(c) Now with $t_1 < t_2$ and the time ordering operator (larger time to the left)

$$\langle 0|TQ(t_1)Q(t_2)|0\rangle = \frac{1}{2m\omega}\langle 0|T\left(a^\dagger e^{i\omega t_1} + a e^{-i\omega t_1}\right)\left(a^\dagger e^{i\omega t_2} + a e^{-i\omega t_2}\right)|0\rangle \tag{1239}$$
$$= \frac{1}{2m\omega}\langle 0|\left(a^\dagger e^{i\omega t_2} + a e^{-i\omega t_2}\right)\left(a^\dagger e^{i\omega t_1} + a e^{-i\omega t_1}\right)|0\rangle \tag{1240}$$
$$= \frac{1}{2m\omega}\langle 0|a e^{-i\omega t_2} a^\dagger e^{i\omega t_1}|0\rangle \tag{1241}$$

all other terms are vanishing because of $a|0\rangle = 0$ and $\langle 0|a^\dagger = 0$. Then

$$\langle 0|TQ(t_1)Q(t_2)|0\rangle = \frac{1}{2m\omega}e^{-i\omega(t_2-t_1)}\underbrace{\langle 0|aa^\dagger|0\rangle}_{=1} \tag{1242}$$
$$= \frac{1}{2m\omega}e^{-i\omega(t_2-t_1)} \tag{1243}$$
$$\equiv \frac{1}{i}G(t_2 - t_1) \tag{1244}$$

And now the next case with $t_1 > t_2 > t_3 > t_4$

$$\langle 0|TQ(t_1)Q(t_2)Q(t_3)Q(t_4)|0\rangle = \frac{1}{(2m\omega)^2}\cdots \tag{1245}$$

0.11.18 Problem 7.4 - Harmonic Oscillator with perturbation

As $f(t)$ is a real function we have $\tilde{f}(-E) = (\tilde{f}(E))^*$ then with (7.10)

$$\langle 0|0\rangle_f = \exp\left[\frac{i}{2}\int_{-\infty}^{+\infty}\frac{dE}{2\pi}\frac{\tilde{f}(E)\tilde{f}(-E)}{-E^2+\omega^2-i\epsilon}\right] \tag{1246}$$
$$= \exp\left[\frac{i}{2}\int_{-\infty}^{+\infty}\frac{dE}{2\pi}\frac{\tilde{f}(E)\tilde{f}(E)^*}{-E^2+\omega^2-i\epsilon}\right] \tag{1247}$$

But we actually need to calculate $|\langle 0|0\rangle_f|^2$ therefore we observe with

$$e^{iz} = e^{i(x+iy)} = e^{-y}e^{ix} = e^{-y}(\cos x + i\sin x) \tag{1248}$$
$$\rightarrow \quad (e^{iz})^* = e^{-y}(\cos x - i\sin x) = e^{-y-ix}e^{-i(x-iy)} = e^{-iz^*} \tag{1249}$$
$$\langle 0|0\rangle_f = e^{iA} \quad \rightarrow \quad |\langle 0|0\rangle_f|^2 = e^{iA}(e^{iA})^* = e^{iA}e^{-iA^*} = e^{i(A-A^*)} = e^{-2\Im A} \tag{1250}$$

Now we calculate the imaginary part of the integral

$$\Im\frac{1}{4\pi}\int_{-\infty}^{+\infty}dE\frac{\tilde{f}(E)\tilde{f}(E)^*}{-E^2+\omega^2-i\epsilon} = \frac{1}{4\pi}\int_{-\infty}^{+\infty}dE\Im\frac{\tilde{f}(E)\tilde{f}(E)^*}{-E^2+\omega^2-i\epsilon} \tag{1251}$$
$$= \frac{1}{4\pi}\int_{-\infty}^{+\infty}dE\tilde{f}(E)\tilde{f}(E)^*\Im\frac{1}{-E^2+\omega^2-i\epsilon} \tag{1252}$$
$$= \frac{1}{4\pi}\int_{-\infty}^{+\infty}dE\tilde{f}(E)\tilde{f}(E)^*\Im\frac{-E^2+\omega^2+i\epsilon}{(-E^2+\omega^2)^2+\epsilon^2} \tag{1253}$$
$$= \frac{1}{4\pi}\int_{-\infty}^{+\infty}dE\tilde{f}(E)\tilde{f}(E)^*\frac{\epsilon}{(-E^2+\omega^2)^2+\epsilon^2} \tag{1254}$$
$$\simeq \frac{1}{4\pi}\int_{-\infty}^{+\infty}dE\tilde{f}(E)\tilde{f}(E)^*\pi\delta(-E^2+\omega^2) \tag{1255}$$
$$\simeq \frac{1}{4\pi}\int_{-\infty}^{+\infty}dE\tilde{f}(E)\tilde{f}(E)^*\pi\delta((\omega+E)(\omega-E)) \tag{1256}$$
$$\simeq \frac{1}{4\cdot 2\omega}(\tilde{f}(\omega)\tilde{f}(\omega)^*+\tilde{f}(-\omega)\tilde{f}(-\omega)^*) \tag{1257}$$
$$\simeq \frac{1}{8\omega}(\tilde{f}(\omega)\tilde{f}(\omega)^*+\tilde{f}(\omega)^*\tilde{f}(\omega)) \tag{1258}$$
$$\simeq \frac{1}{4\omega}\tilde{f}(\omega)\tilde{f}(\omega)^* \tag{1259}$$

then

$$|\langle 0|0\rangle_f|^2 = e^{-2\left(\frac{1}{4\omega}\right)\tilde{f}(\omega)\tilde{f}(\omega)^*} \tag{1260}$$
$$= e^{-\frac{1}{2\omega}\tilde{f}(\omega)\tilde{f}(\omega)^*} \tag{1261}$$
$$\tag{1262}$$

0.11.19 Problem 8.1 - Feynman propagator is Greens function Klein-Gordon equation

With

$$\Delta(x-x') = \frac{1}{(2\pi)^4}\int d^4k\frac{e^{ik(x-x')}}{k^2+m^2-i\epsilon} \tag{1263}$$

we have

$$(-\partial_x^2+m^2)\Delta(x-x') = \frac{1}{(2\pi)^4}\int d^4k(-i^2k^2+m^2)\frac{e^{ik(x-x')}}{k^2+m^2-i\epsilon} \tag{1264}$$
$$= \frac{1}{(2\pi)^4}\int d^4k\frac{k^2+m^2}{k^2+m^2-i\epsilon}e^{ik(x-x')} \tag{1265}$$
$$\simeq \frac{1}{(2\pi)^4}\int d^4k e^{ik(x-x')} \tag{1266}$$
$$= \delta^4(x-x') \tag{1267}$$

0.11.20 Problem 8.2 - Feynman propagator II

With $\widetilde{dk} = d^3k/((2\pi)^3 2\omega_k)$ and $\omega_k = \sqrt{\vec{k}^2 + m^2}$

$$\Delta(x - x') = \frac{1}{(2\pi)^4} \int d^4k \frac{e^{ik(x-x')}}{k^2 + m^2 - i\epsilon} \tag{1268}$$
$$= \frac{1}{(2\pi)^4} \int d^3k \int dk^0 e^{-ik^0(t-t')} \frac{e^{i\vec{k}(\vec{x}-\vec{x}')}}{-(k^0)^2 + \vec{k}^2 + m^2 - i\epsilon} \tag{1269}$$
$$= \frac{1}{(2\pi)^4} \int d^3k\, e^{i\vec{k}(\vec{x}-\vec{x}')} \int dE \frac{e^{-iE(t-t')}}{-E^2 + \vec{k}^2 + m^2 - i\epsilon} \tag{1270}$$
$$= \frac{1}{(2\pi)^4} \int d^3k\, e^{i\vec{k}(\vec{x}-\vec{x}')} 2\pi \frac{i}{2(\vec{k}^2 + m^2)} e^{-i(\vec{k}^2+m^2)|t-t'|} \tag{1271}$$

where we used exercise (7.1). Then

$$\Delta(x - x') = \frac{i}{(2\pi)^3} \int d^3k\, e^{i\vec{k}(\vec{x}-\vec{x}')} \frac{i}{2\omega_k} e^{-i\omega_k|t-t'|} \tag{1272}$$
$$= i \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{i\vec{k}(\vec{x}-\vec{x}')} e^{-i\omega_k|t-t'|} \tag{1273}$$
$$= i \int \widetilde{dk}\, e^{i\vec{k}(\vec{x}-\vec{x}') - i\omega_k|t-t'|} \tag{1274}$$
$$= i\theta(t - t') \int \widetilde{dk}\, e^{i\vec{k}(\vec{x}-\vec{x}') - i\omega_k(t-t')} + i\theta(t' - t) \int \widetilde{dk}\, e^{i\vec{k}(\vec{x}-\vec{x}') + i\omega_k(t-t')} \tag{1275}$$
$$= i\theta(t - t') \int \widetilde{dk}\, e^{ik(x-x')} + i\theta(t' - t) \int \widetilde{dk}\, e^{-i\vec{k}(\vec{x}-\vec{x}') + i\omega_k(t-t')} \tag{1276}$$
$$= i\theta(t - t') \int \widetilde{dk}\, e^{ik(x-x')} + i\theta(t' - t) \int \widetilde{dk}\, e^{-ik(x-x')} \tag{1277}$$
$$\tag{1278}$$

0.12 Coleman - Lectures of Sidney Coleman on quantum field theory

0.12.1 Problem 1.1 - Momentum space measure

Boost in z-direction

$$p_\mu = \Lambda^\nu{}_\mu p'_\nu \tag{1279}$$
$$\Lambda = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \tag{1280}$$

Combining everything using $dp_i \wedge dp_i = 0$

$$\rightarrow dp_x = dp'_x \tag{1281}$$
$$\rightarrow dp_y = dp'_y \tag{1282}$$
$$\rightarrow dp_z = -\gamma\beta\, dp'_0 + \gamma\, dp'_z \tag{1283}$$
$$= -\gamma\beta \left(\frac{\partial p'_0}{\partial p'_x} dp'_x + \frac{\partial p'_0}{\partial p'_y} dp'_y + \frac{\partial p'_0}{\partial p'_z} dp'_z \right) + \gamma\, dp'_z \tag{1284}$$
$$= -\gamma\beta \frac{1}{2\omega'_p} (2p'_x\, dp'_x + 2p'_y\, dp'_y + 2p'_z\, dp'_z) + \gamma\, dp'_z \tag{1285}$$
$$= -\gamma\beta \frac{p'_x\, dp'_x + p'_y\, dp'_y}{\omega'_p} + \gamma \left(1 - \frac{\beta}{\omega'_p} p'_z\right) dp'_z \tag{1286}$$

where we used $p'_0 = \omega'_p = \sqrt{m^2 - {p'_x}^2 - {p'_y}^2 - {p'_z}^2}$ and

$$\rightarrow \omega_p = p_0 \tag{1287}$$
$$= \gamma p'_0 - \gamma\beta\, p'_z \tag{1288}$$
$$= \gamma(\omega'_p - \beta\, p'_z) \tag{1289}$$

then

$$\frac{d^3p}{(2\pi)^3 2\omega_p} = \frac{dp_x\, dp_y\, dp_z}{(2\pi)^3 2\omega_p} \tag{1290}$$
$$= \frac{dp'_x\, dp'_y \gamma \left(1 - \frac{\beta}{\omega'_p} p'_z\right) dp'_z}{(2\pi)^3 2\gamma(\omega'_p - \beta\, p'_z)} \tag{1291}$$
$$= \frac{dp'_x\, dp'_y \gamma \left(1 - \frac{\beta}{\omega'_p} p'_z\right) dp'_z}{(2\pi)^3 2\omega'_p \gamma \left(1 - \frac{\beta}{\omega'_p}\, p'_z\right)} \tag{1292}$$
$$= \frac{dp'_x\, dp'_y\, dp'_z}{(2\pi)^3 2\omega'_p} \tag{1293}$$

0.13 Kachelriess - Quantum Fields - From the Hubble to the Planck scale

0.13.1 Problem 1.1 - Units

1. The fundamental constants are given by

$$k = 1.381 \cdot 10^{-23} \mathrm{m}^2 \mathrm{s}^{-2} \mathrm{kg}^1 \mathrm{K}^{-1} \tag{1294}$$
$$G = 6.674 \cdot 10^{-11} \mathrm{m}^3 \mathrm{s}^{-2} \mathrm{kg}^{-1} \tag{1295}$$
$$\hbar = 1.054 \cdot 10^{-34} \mathrm{m}^2 \mathrm{s}^{-1} \mathrm{kg}^1 \tag{1296}$$
$$c = 2.998 \cdot 10^{-8} \mathrm{m}^1 \mathrm{s}^{-1} \tag{1297}$$

A newly constructed Planck constant has the general form

$$X_P = c^{\alpha_c} \cdot G^{\alpha_G} \cdot \hbar^{\alpha_\hbar} \cdot k^{\alpha_k} \tag{1298}$$

and the dimension of X_P is given by $\mathrm{m}^{\beta_m} \mathrm{s}^{\beta_s} \mathrm{kg}^{\beta_{kg}} \mathrm{K}^{\beta_K}$ are determined by

$$\text{Meter} \quad \beta_m = 2\alpha_k + 3\alpha_G + 2\alpha_h + \alpha_c \tag{1299}$$
$$\text{Second} \quad \beta_s = -2\alpha_k - 2\alpha_G - \alpha_c - \alpha_h \tag{1300}$$
$$\text{Kilogram} \quad \beta_{kg} = \alpha_k - \alpha_G + \alpha_h \tag{1301}$$
$$\text{Kelvin} \quad \beta_K = -\alpha_k \tag{1302}$$

Solving the linear system gives

$$l_P = \sqrt{\frac{\hbar G}{c^3}} = 1.616 \cdot 10^{-35}\text{m} \tag{1303}$$

$$m_P = \sqrt{\frac{\hbar c}{G}} = 2.176 \cdot 10^{-8}\text{kg} \tag{1304}$$

$$t_P = \sqrt{\frac{\hbar G}{c^5}} = 5.391 \cdot 10^{-44}\text{s} \tag{1305}$$

$$T_P = \sqrt{\frac{\hbar c^5}{G k^2}} = 1.417 \cdot 10^{-32}\text{K} \tag{1306}$$

$$\tag{1307}$$

As the constants are made up from QM, SR and GR constants they indicate magnitudes at which a quantum theory of gravity is needed to make a sensible predictions.

2. We use the definition $1\text{barn} = 10^{-28}\text{m}^2$

$$1\text{cm}^2 = 10^{-4}\text{m}^2 \tag{1308}$$

$$1\text{mbarn} = 10^{-31}\text{m}^2 \tag{1309}$$

$$= 10^{-27}\text{cm}^2 \tag{1310}$$

We also have $1\text{eV} = 1.602 \cdot 10^{-19}\text{As} \cdot 1\text{V} = 1.602 \cdot 10^{-19}\text{J}$

$$E = mc^2 \quad \rightarrow \quad 1\text{kg} \cdot c^2 = 8.987 \cdot 10^{16}\text{J} = 5.609 \cdot 10^{35}\text{eV} \tag{1311}$$

$$\rightarrow \quad 1\text{GeV} = 1.782 \cdot 10^{-27}\text{kg} \tag{1312}$$

$$E = \hbar\omega \quad \rightarrow \quad \frac{1}{1\text{s}} \cdot \hbar = 1.054 \cdot 10^{-34}\text{J} = 6.582 \cdot 10^{-16}\text{eV} \tag{1313}$$

$$\rightarrow \quad 1\text{GeV}^{-1} = 6.582 \cdot 10^{-25}\text{s} \tag{1314}$$

$$E = \frac{\hbar c}{\lambda} \quad \rightarrow \quad \frac{1}{1\text{m}} \cdot \hbar c = 3.161 \cdot 10^{-26}\text{J} = 1.973 \cdot 10^{-7}\text{eV} \tag{1315}$$

$$\rightarrow \quad 1\text{GeV}^{-1} = 1.973 \cdot 10^{-16}\text{m} \tag{1316}$$

$$E \sim pc \quad \rightarrow \quad 1\text{kgms}^{-1} \cdot c = 2.998 \cdot 10^{8}\text{J} = 1.871 \cdot 10^{27}\text{eV} \tag{1317}$$

$$\rightarrow \quad 1\text{GeV} = 5.344 \cdot 10^{-19}\text{kgms}^{-1} \tag{1318}$$

therefore

$$1\text{GeV}^{-2} = (1.973 \cdot 10^{-16}\text{m})^2 \tag{1319}$$

$$= 3.893 \cdot 10^{-32}\text{m}^2 \tag{1320}$$

$$= 0.389\text{mbarn} \tag{1321}$$

0.13.2 Problem 3.2 - Maxwell Lagrangian

1. First we observe that

$$F_{\mu\nu}F^{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \tag{1322}$$

$$= (\partial_\mu A_\nu)(\partial^\mu A^\nu) - (\partial_\mu A_\nu)(\partial^\nu A^\mu) - \underbrace{(\partial_\nu A_\mu)(\partial^\mu A^\nu)}_{=(\partial_\mu A_\nu)(\partial^\nu A^\mu)} + \underbrace{(\partial_\nu A_\mu)(\partial^\nu A^\mu)}_{=(\partial_\mu A_\nu)(\partial^\mu A^\nu)} \tag{1323}$$

$$= 2\left((\partial_\mu A_\nu)(\partial^\mu A^\nu) - (\partial_\mu A_\nu)(\partial^\nu A^\mu)\right) \tag{1324}$$

$$= 2(\partial_\mu A_\nu)F^{\mu\nu}. \tag{1325}$$

The variation is then given by

$$\delta\left(F_{\mu\nu}F^{\mu\nu}\right) = 2\delta\left((\partial_\mu A_\nu)F^{\mu\nu}\right) \tag{1326}$$
$$= 2\left[\delta\left(\partial_\mu A_\nu\right)F^{\mu\nu} + (\partial_\mu A_\nu)\delta F^{\mu\nu}\right] \tag{1327}$$
$$= 2[\delta\left(\partial_\mu A_\nu\right)\underbrace{(\partial^\mu A^\nu - \partial^\nu A^\mu)}_{=F^{\mu\nu}} + (\partial_\mu A_\nu)\underbrace{(\delta(\partial^\mu A^\nu - \partial^\nu A^\mu))}_{\delta F^{\mu\nu}}] \tag{1328}$$
$$= 2\left[\delta\left(\partial_\mu A_\nu\right)\partial^\mu A^\nu - \delta\left(\partial_\mu A_\nu\right)\partial^\nu A^\mu + (\partial_\mu A_\nu)\delta(\partial^\mu A^\nu) - (\partial_\mu A_\nu)\delta(\partial^\nu A^\mu)\right] \tag{1329}$$
$$= 4\left[\delta\left(\partial_\mu A_\nu\right)\partial^\mu A^\nu - \delta\left(\partial_\mu A_\nu\right)\partial^\nu A^\mu\right] \tag{1330}$$
$$= 4(\partial^\mu A^\nu - \partial^\nu A^\mu)\ \delta(\partial_\mu A_\nu) \tag{1331}$$
$$= 4F^{\mu\nu}\ \delta(\partial_\mu A_\nu) \tag{1332}$$
$$= 4F^{\mu\nu}\ \partial_\mu(\delta A_\nu) \tag{1333}$$

We start with the source free Maxwell equations $\partial_\mu F^{\mu\nu} = 0$

$$0 = \int_\Omega d^4x\ (\delta A_\nu)\partial_\mu F^{\mu\nu} \tag{1334}$$
$$= F^{\mu\nu}(\delta A_\nu)|_{\partial\Omega} - \int_\Omega d^4x\ \underbrace{\partial_\mu(\delta A_\nu)F^{\mu\nu}}_{=\frac{1}{4}\delta(F_{\mu\nu}F^{\mu\nu})} \tag{1335}$$
$$= \int_\Omega d^4x\ \delta\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) \tag{1336}$$

and therefore $\mathscr{L}_{\text{ph}} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$.

2. So we see that the Lagrangian $\mathscr{L}_{\text{ph}} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = 2(\partial_\mu A_\nu)F^{\mu\nu}$ yields the inhomogeneous Maxwell equations

$$\frac{\partial\mathscr{L}_{\text{ph}}}{\partial A_\alpha} - \partial_\beta\frac{\partial\mathscr{L}_{\text{ph}}}{\partial(\partial_\beta A_\alpha)} = 0 \tag{1337}$$
$$-\partial_\beta\left[(2\delta_{\alpha\mu}\delta_{\beta\nu}F^{\mu\nu} + 2(\partial_\mu A_\nu)(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\nu_\alpha\delta^\mu_\beta)\right] = 0 \tag{1338}$$
$$-\partial_\beta\left[(2F^{\alpha\beta} + 2(\partial^\alpha A^\beta - \partial^\beta A^\alpha)\right] = 0 \tag{1339}$$
$$\partial_\beta(F^{\alpha\beta}) = 0 \tag{1340}$$

but not the homogeneous ones. They are fulfilled trivially - by construction of $F^{\mu\nu}$.

3. The conjugated momentum is given by

$$\pi_\mu = \frac{\partial\mathscr{L}_{\text{ph}}}{\partial\dot{A}^\mu} \tag{1341}$$
$$= F_{0\mu} \tag{1342}$$

0.13.3 Problem 3.3 - Dimension of ϕ

1. With $c = 1 = \hbar$ we see

$$E = mc^2 \rightarrow E \sim M \tag{1343}$$
$$E = \hbar\omega \rightarrow T \sim E^{-1} \sim M^{-1} \tag{1344}$$
$$s = ct \rightarrow L \sim T \sim M^{-1} \tag{1345}$$

As $\mathscr{L}$ is an action density we have

$$\mathscr{L} \sim \frac{E\cdot T}{TL^3} \sim M\cdot M^{d-1} = M^d \tag{1346}$$

From the explicit form of the scalar Lagrangian we derive

$$\mathscr{L} \sim \frac{[\phi^2]}{M^{-2}} = [\phi^2]M^{-2} \tag{1347}$$

and therefore $[\phi] = M^{(d-2)/2}$

2. Using the previous result we see

$$\lambda\phi^3 : \qquad M^d \sim [\lambda]M^{3(d-2)/2} \rightarrow d = 6 \tag{1348}$$

$$\lambda\phi^4 : \qquad M^d \sim [\lambda]M^{4(d-2)/2} \rightarrow d = 4 \tag{1349}$$

3. With

$$\mathscr{L} = \frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - \frac{1}{2}m^2\phi^2 + \lambda\phi^4 \tag{1350}$$

$$= \frac{1}{2}\eta^{\mu\nu}\left(\partial_\mu\frac{\tilde{\phi}}{\sqrt{\lambda}}\right)\left(\partial_\nu\frac{\tilde{\phi}}{\sqrt{\lambda}}\right) - \frac{1}{2}m^2\frac{\tilde{\phi}^2}{\lambda} + \lambda\frac{\tilde{\phi}^4}{\lambda^2} \tag{1351}$$

$$= \frac{1}{\lambda}\left[\frac{1}{2}\eta^{\mu\nu}(\partial_\mu\tilde{\phi})(\partial_\nu\tilde{\phi}) - \frac{1}{2}m^2\tilde{\phi}^2 + \tilde{\phi}^4\right] \tag{1352}$$

0.13.4 Problem 3.5 - Yukawa potential

Integration in spherical coordinates yields (with $x = kr$)

$$\int d^3k\frac{e^{-ik\cdot r}}{k^2+m^2} = 2\pi\int\frac{e^{-ikr\cos\theta}}{k^2+m^2}k^2\sin\theta d\theta\, dk \tag{1353}$$

$$= -2\pi\int\frac{e^{-ikr\cos\theta}}{k^2+m^2}k^2\ d(\cos\theta)\, dk \tag{1354}$$

$$= -2\pi\int\frac{k^2}{ikr}\frac{e^{-ikr\cos\theta}}{k^2+m^2}\bigg|_{-1}^{+1}\ dk \tag{1355}$$

$$= -2\pi\int\frac{k}{ir}\frac{e^{-ikr}-e^{+ikr}}{k^2+m^2}\ dk \tag{1356}$$

$$= \frac{4\pi}{r}\int_0^\infty\frac{k\sin kr}{k^2+m^2}\ dk \tag{1357}$$

$$= \frac{4\pi}{r^2}\int_0^\infty\frac{\frac{x}{r}\sin x}{\frac{x^2}{r^2}+m^2}\ dx \tag{1358}$$

$$= \frac{4\pi}{r}\int_0^\infty\frac{x\sin x}{x^2+m^2r^2}\ dx \tag{1359}$$

$$\tag{1360}$$

Now we use a small trick

$$= \frac{2\pi}{ir}\int_0^\infty \frac{x(e^{ix}-e^{-ix})}{x^2+m^2r^2}\,dx \tag{1361}$$

$$= \frac{2\pi}{ir}\left[\int_0^\infty \frac{xe^{ix}}{x^2+m^2r^2}\,dx - \int_0^\infty \frac{xe^{-ix}}{x^2+m^2r^2}\,dx\right] \tag{1362}$$

$$= \frac{2\pi}{ir}\left[\int_0^\infty \frac{xe^{ix}}{x^2+m^2r^2}\,dx - (-1)^3\int_{-\infty}^0 \frac{ye^{iy}}{y^2+m^2r^2}\,dy\right] \tag{1363}$$

$$= \frac{2\pi}{ir}\int_{-\infty}^\infty \frac{xe^{ix}}{x^2+m^2r^2}\,dx \tag{1364}$$

$$= \frac{2\pi}{ir}\int_{-\infty}^\infty \frac{xe^{ix}}{(x+imr)(x-imr)}\,dx \tag{1365}$$

$$= \frac{2\pi}{ir}\left(2\pi i\cdot \underbrace{\mathrm{Res}_{x=imr}}_{=\frac{imr\exp(i^2mr)}{2imr}} - \int_{\text{upper half circle}} ...\right) \tag{1366}$$

$$= \frac{2\pi^2}{r}e^{-mr} \tag{1367}$$

Therefore

$$\frac{1}{(2\pi)^3}\int d^3k\frac{e^{-ik\cdot r}}{k^2+m^2} = \frac{1}{4\pi r}e^{-mr} \tag{1368}$$

0.13.5 Problem 3.9 - ζ function regularization

1. Calculation the Taylor expansion (using L'Hopital's rule for the limits) we obtain

$$f(t) = \frac{t}{e^t-1} \tag{1369}$$

$$= \sum_k \left.\frac{d^kf}{dt^k}\right|_{t=0} t^k \tag{1370}$$

$$= 1-\frac{1}{2}t+\frac{1}{12}t^2-\frac{1}{12}t^4+... \tag{1371}$$

$$\stackrel{!}{=} B_0+B_1t+\frac{B_2}{2}t^2+\frac{B_3}{6}t^2+... \tag{1372}$$

$$\rightarrow B_n = \{1,-\frac{1}{2},\frac{1}{6},0,...\} \tag{1373}$$

2. Avoiding mathematical rigor we see after playing around for a while

$$\sum_{n=1}^{\infty} n e^{-an} = -\frac{d}{da}\sum_{n=1}^{\infty} e^{-an} \tag{1374}$$

$$= -\frac{d}{da}\sum_{n=1}^{\infty}\left(e^{-a}\right)^n \tag{1375}$$

$$= -\frac{d}{da}\frac{1}{1-e^{-a}} \tag{1376}$$

$$= -\frac{d}{da}\left(\frac{1}{a}\frac{a}{1-e^{-a}}\right) \tag{1377}$$

$$= -\frac{d}{da}\left(\frac{1}{a}f(t)\right) \tag{1378}$$

$$= -\frac{d}{da}\left(\frac{1}{a}\sum_{n=0}^{\infty}\frac{B_n}{n!}a^n\right) \tag{1379}$$

$$= -\frac{d}{da}\left(\frac{1}{a}\left[1-\frac{a}{2}+\frac{a^2}{12}-\frac{a^4}{720}+...\right]\right) \tag{1380}$$

$$= -\frac{d}{da}\left(\frac{1}{a}-\frac{1}{2}+\frac{a}{12}-\frac{a^3}{720}...\right) \tag{1381}$$

$$= \frac{1}{a^2}-\frac{1}{12}+\frac{a}{240}-... \tag{1382}$$

$$\overset{a\to 0}{\to} \frac{1}{a^2}-\frac{1}{12} \tag{1383}$$

3. Using the definition of the Riemann ζ function

$$\zeta(s) = \sum_{k=1}^{\infty}\frac{1}{k^s} \tag{1384}$$

0.13.6 Problem 4.1 - $Z[J]$ at order λ in ϕ^4 theory

Lets start at (4.6a) with $\mathscr{L}_I = -\lambda/4!\phi^4$

$$Z[J] = \exp\left[\mathrm{i}\int d^4x\mathscr{L}_I\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}\right)\right]\int\mathcal{D}\phi\exp\left[\mathrm{i}\int d^4x(\mathscr{L}_0+J\phi)\right] \tag{1385}$$

$$= \exp\left[\mathrm{i}\int d^4x\mathscr{L}_I\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}\right)\right]Z_0[J] \tag{1386}$$

$$= \exp\left[-\frac{\mathrm{i}\lambda}{4!}\int d^4x\left(\frac{\delta^4}{\delta J(x)^4}\right)\right]Z_0[J] \tag{1387}$$

$$= Z_0[J]-\frac{\mathrm{i}\lambda}{4!}\int d^4x\left(\frac{\delta^4 Z_0[J]}{\delta J(x)^4}\right)+\dots. \tag{1388}$$

Using (4.7)

$$Z_0[J] = Z_0[0]\exp\left[-\frac{\mathrm{i}}{2}\int d^4y d^4z J(y)\Delta_F(y-z)J(z)\right] = Z_0[0]e^{iW_0[J]} \tag{1389}$$

$$W_0[J] = -\frac{1}{2}\int d^4y d^4z J(y)\Delta_F(y-z)J(z) \tag{1390}$$

we derive (4.10) in various steps

1. Calculating $\frac{\delta W_0[J]}{\delta J(x)}$

$$\frac{\delta W_0[J]}{\delta J(x)} = -\frac{1}{2}\lim_{\epsilon\to 0}\int d^4y d^4z \frac{\left(J(y)+\epsilon\delta^{(4)}(y-x)\right)\Delta_F(y-z)\left(J(z)+\epsilon\delta^{(4)}(z-x)\right)-W_0[J]}{\epsilon} \quad (1391)$$

$$= -\frac{1}{2}\int d^4y d^4z \left[\delta^{(4)}(y-x)\Delta_F(y-z)J(z) + J(y)\Delta_F(y-z)\delta^{(4)}(z-x)\right] \quad (1392)$$

$$= -\frac{1}{2}\int d^4z \Delta_F(x-z)J(z) - \frac{1}{2}\int d^4y J(y)\Delta_F(y-x) \quad (1393)$$

$$= -\int d^4y \Delta_F(y-x)J(y) \quad (1394)$$

where we used $\Delta_F(x) = \Delta_F(-x)$.

2. Calculating $\frac{\delta^2 W_0[J]}{\delta J(x)^2}$

$$\frac{\delta^2 W_0[J]}{\delta J(x)^2} = -\int d^4y \Delta_F(y-x)\frac{\delta J(y)}{\delta J(x)} \quad (1395)$$

$$= -\int d^4y \Delta_F(y-x)\delta(y-x) \quad (1396)$$

$$= -\Delta_F(0) \quad (1397)$$

3. Calculating $\delta F[J]/\delta J(x)$ for $F[J] = f\left(W_0[J]\right)$

$$\frac{\delta F[J]}{\delta J(x)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}f(W_0[\phi(x)+\epsilon\delta(x-y)]) - f(W_0[\phi(x)]) \quad (1398)$$

$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}f(W_0[\phi(x)] + \epsilon\frac{\delta W_0}{\delta\phi}) - f(W_0[\phi(x)]) \quad (1399)$$

$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}f(W_0[\phi(x)]) + g'\epsilon\frac{\delta W_0}{\delta\phi} - f(W_0[\phi(x)]) \quad (1400)$$

$$= f'(W_0[J])\frac{\delta W_0}{\delta J} \quad (1401)$$

4. Calculating first derivative

$$\frac{\delta}{\mathrm{i}\delta J(x)}\exp\left(\mathrm{i}W_0[J]\right) = \frac{\delta W_0[J]}{\delta J(x)}\exp\left(\mathrm{i}W_0[J]\right) \quad (1402)$$

5. Calculating second derivative (using the functional derivative product rule)

$$\left(\frac{\delta}{\mathrm{i}\delta J(x)}\right)^2 \exp\left(\mathrm{i}W_0[J]\right) = \left(\left(\frac{\delta W_0[J]}{\delta J(x)}\right)^2 + \frac{1}{i}\frac{\delta^2 W_0[J]}{\delta J(x)^2}\right)\exp\left(\mathrm{i}W_0[J]\right) \quad (1403)$$

6. Calculating third derivative

$$\left(\frac{\delta}{\mathrm{i}\delta J(x)}\right)^3 \exp\left(\mathrm{i}W_0[J]\right) = \left(\left(\frac{\delta W_0[J]}{\delta J(x)}\right)^3 + \frac{3}{i}\frac{\delta^2 W_0[J]}{\delta J(x)^2}\frac{\delta W_0[J]}{\delta J(x)} + \frac{1}{i^2}\frac{\delta^3 W_0[J]}{\delta J(x)^3}\right)\exp\left(\mathrm{i}W_0[J]\right) \quad (1404)$$

7. Calculating fourth derivative

$$\left(\frac{\delta}{\mathrm{i}\delta J(x)}\right)^4 \exp\left(\mathrm{i}W_0[J]\right) = \left(\left(\frac{\delta W_0[J]}{\delta J(x)}\right)^4 + \frac{6}{i}\frac{\delta^2 W_0[J]}{\delta J(x)^2}\left(\frac{\delta W_0[J]}{\delta J(x)}\right)^2 + \frac{3}{i^2}\left(\frac{\delta^2 W_0[J]}{\delta J(x)^2}\right)^2 + \right.$$
$$\left. + \frac{4}{i^2}\frac{\delta W_0[J]}{\delta J(x)}\frac{\delta^3 W_0[J]}{\delta J(x)^3} + \frac{1}{i^3}\frac{\delta^4 W_0[J]}{\delta J(x)^4}\right) \exp\left(\mathrm{i}W_0[J]\right)$$
$$= \left(\left(\frac{\delta W_0[J]}{\delta J(x)}\right)^4 + \frac{6}{i}\frac{\delta^2 W_0[J]}{\delta J(x)^2}\left(\frac{\delta W_0[J]}{\delta J(x)}\right)^2 + \frac{3}{i^2}\left(\frac{\delta^2 W_0[J]}{\delta J(x)^2}\right)^2\right) \exp\left(\mathrm{i}W_0[J]\right)$$

8. Substituting the functional derivatives

$$\left(\frac{\delta}{\mathrm{i}\delta J(x)}\right)^4 \exp\left(\mathrm{i}W_0[J]\right) = \left[\left(\int d^4y \Delta_F(y-x)J(y)\right)^4 + 6i\Delta_F(0)\left(\int d^4y \Delta_F(y-x)J(y)\right)^2\right.$$
$$\left. +3\left(i\Delta_F(0)\right)^2\right] \exp\left(\mathrm{i}W_0[J]\right)$$

0.13.7 Problem 19.1 - Dynamical stress tensor

Preliminaries

- The Laplace expansion of the determinate by row or column is given by

$$|g| = \sum_\kappa g_{\kappa\mu} G_{\kappa\mu} \quad \text{(no sum over } \mu!) \tag{1405}$$

  with the cofactor matrix $G_{\kappa\mu}$ (matrix of determinants of minors of g).

- The inverse matrix is given by

$$g^{\alpha\beta} = \frac{1}{|g|} G_{\alpha\beta} \tag{1406}$$

- Therefore we have

$$\frac{\partial |g|}{\delta g_{\alpha\beta}} = \frac{\partial\left(\sum_\kappa g_{\kappa\beta} G_{\kappa\beta}\right)}{\delta g_{\alpha\beta}} \tag{1407}$$
$$= \delta_{\kappa\alpha} G_{\kappa\beta} \tag{1408}$$
$$= G_{\alpha\beta} \tag{1409}$$
$$= |g| g^{\alpha\beta} \tag{1410}$$

Now we can calculate

$$\delta\sqrt{|g|} = \frac{\partial\sqrt{|g|}}{\delta g_{\mu\nu}}\delta g_{\mu\nu} = \frac{1}{2\sqrt{|g|}}\frac{\partial |g|}{\delta g_{\mu\nu}}\delta g_{\mu\nu} = \frac{1}{2}\sqrt{|g|}g^{\mu\nu}\delta g_{\mu\nu} \tag{1411}$$
$$\frac{\delta\sqrt{|g(x)|}}{\delta g_{\mu\nu}(y)} = \frac{1}{2}\sqrt{|g|}\delta(x-y) \tag{1412}$$

We now use the action and definition (7.49)

$$S_\mathrm{m} = \int d^4x \sqrt{|g|}\mathscr{L}_\mathrm{m} \tag{1413}$$
$$T^{\mu\nu} = \frac{2}{\sqrt{|g|}}\frac{\delta S_\mathrm{m}}{\delta g^{\mu\nu}} \tag{1414}$$
$$= \frac{2}{\sqrt{|g|}}\int d^4x \left[\frac{1}{2}\sqrt{|g|}g^{\mu\nu}\mathscr{L}_\mathrm{m} + \sqrt{|g|}\frac{\delta\mathscr{L}_\mathrm{m}}{\delta g_{\mu\nu}}\right] \tag{1415}$$

0.13.8 Problem 19.6 - Dirac-Schwarzschild

1. (19.13) - adding the bi-spinor index might be helpful for some readers, see (B.27)
2. (19.13) vs (B.27) naming of generators $J^{\mu\nu}$ vs $\sigma_{\mu\nu}/2$

The Dirac equation in curved space is obtained (from the covariance principle) by replacing all derivatives ∂_k with covariant tetrad derivatives $\mathscr{D}_k$

$$(i\hbar\gamma^k \mathscr{D}_k + mc)\psi = 0 \tag{1416}$$

Lets start with the Schwarzschild line element

$$\begin{aligned} ds^2 &= \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2(d\vartheta^2 + \sin^2\vartheta\, d\phi^2) && (1417)\\ &= \eta_{mn} d\xi^m d\xi^n && (1418)\end{aligned}$$

with

$$d\xi^0 = \left(1 - \frac{2M}{r}\right)^{1/2} dt, \quad d\xi^1 = \left(1 - \frac{2M}{r}\right)^{-1/2} dr, \quad d\xi^2 = r d\vartheta, \quad d\xi^3 = r\sin\vartheta d\phi. \tag{1419}$$

and the tetrad fields e^m_μ can then be derived via $d\xi^m = e^m_\mu(x)dx^\mu$.

0.13.9 Problem 23.1 - Conformal transformation

For a change of coordinates we find in general

$$\begin{aligned} x^\mu &\mapsto \tilde{x}^\mu && (1420)\\ g_{\mu\nu}(x) \mapsto \tilde{g}_{\mu\nu}(\tilde{x}) &= \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(x) && (1421)\end{aligned}$$

which for $x \mapsto \tilde{x} = e^\omega x$ results in (there might be a sign error in (18.1))

$$g_{\mu\nu}(x) \mapsto \tilde{g}_{\mu\nu}(\tilde{x}) = e^{-2\omega} g_{\alpha\beta}(x) \tag{1422}$$

while for a conformal transformation we have

$$\begin{aligned} g_{\mu\nu}(x) \mapsto & \tilde{g}_{\mu\nu}(x) = \Omega^2 g_{\alpha\beta}(x) && (1423)\\ & \tilde{g}_{\mu\nu}(\tilde{x}) = \Omega^2 g_{\alpha\beta}(e^\omega x) && (1424)\end{aligned}$$

0.13.10 Problem 23.2 - Conformal transformation properties

- Christoffel symbol:

$$\begin{aligned} \tilde{g}_{\mu\nu}(x) &= \Omega^2(x) g_{\mu\nu}(x) = e^{2\omega(x)} g_{\mu\nu}(x) && (1425)\\ \tilde{g}_{\mu\nu,\alpha} &= 2\Omega\Omega_{,\alpha} g_{\mu\nu} + \Omega^2 g_{\mu\nu,\alpha} && (1426)\\ &= \Omega(2 g_{\mu\nu}\Omega_{,\alpha} + \Omega g_{\mu\nu,\alpha}) && (1427)\end{aligned}$$

  and

$$\begin{aligned} \delta^\mu_{\ \nu} &= \tilde{g}^{\mu\alpha}\tilde{g}_{\alpha\nu} = \tilde{g}^{\mu\alpha} g_{\alpha\nu}\Omega^2 && (1428)\\ \delta^\mu_{\ \nu} g^{\nu\beta} &= \tilde{g}^{\mu\alpha} g_{\alpha\nu} g^{\nu\beta}\Omega^2 && (1429)\\ g^{\mu\beta} &= \tilde{g}^{\mu\alpha}\delta^\beta_\alpha \Omega^2 && (1430)\\ &\to \tilde{g}^{\mu\beta} = \Omega^{-2} g^{\mu\beta} && (1431)\end{aligned}$$

we find by using $\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(g_{\alpha\mu,\beta} + g_{\beta\mu,\alpha} - g_{\alpha\beta,\mu} \right)$

$$\tilde{\Gamma}^{\mu}{}_{\alpha\beta} = \frac{1}{2} \tilde{g}^{\mu\nu} \left(\tilde{g}_{\alpha\nu,\beta} + \tilde{g}_{\beta\nu,\alpha} - \tilde{g}_{\alpha\beta,\nu} \right) \tag{1432}$$

$$= \frac{1}{2} \Omega^{-2} g^{\mu\nu} \left[\Omega(2g_{\alpha\nu}\Omega_{,\beta} + \Omega g_{\alpha\nu,\beta}) + \Omega(2g_{\beta\nu}\Omega_{,\alpha} + \Omega g_{\beta\nu,\alpha}) - \Omega(2g_{\alpha\beta}\Omega_{,\nu} + \Omega g_{\alpha\beta,\nu}) \right] \tag{1433}$$

$$= \Gamma^{\mu}{}_{\alpha\beta} + \Omega^{-1} g^{\mu\nu} \left[g_{\alpha\nu}\Omega_{,\beta} + g_{\beta\nu}\Omega_{,\alpha} - g_{\alpha\beta}\Omega_{,\nu} \right] \tag{1434}$$

$$= \Gamma^{\mu}{}_{\alpha\beta} + \Omega^{-1} \left[\delta^{\mu}_{\alpha}\Omega_{,\beta} + \delta^{\mu}_{\beta}\Omega_{,\alpha} - g^{\mu\nu} g_{\alpha\beta}\Omega_{,\nu} \right] \tag{1435}$$

- Ricci tensor: with

$$\Omega = e^{2\omega} \tag{1436}$$

$$\Omega^{-2}\Omega_{,\lambda} = e^{-4\omega} e^{2\omega} 2\omega_{,\lambda} \tag{1437}$$

$$= 2e^{-2\omega}\omega_{,\lambda} \tag{1438}$$

$$\Omega_{,\lambda\alpha} = \left(2e^{2\omega}\omega_{,\lambda} \right)_{,\alpha} \tag{1439}$$

$$= 4e^{2\omega}\omega_{,\lambda}\omega_{,\alpha} + 2e^{2\omega}\omega_{,\lambda\alpha} \tag{1440}$$

$$= 2e^{2\omega} \left(2\omega_{,\lambda}\omega_{,\alpha} + \omega_{,\lambda\alpha} \right) \tag{1441}$$

and

$$\partial_\lambda \tilde{\Gamma}^{\mu}{}_{\alpha\beta} = \partial_\lambda \Gamma^{\mu}{}_{\alpha\beta} - \Omega^{-2}\Omega_{,\lambda} \left[\delta^{\mu}_{\alpha}\Omega_{,\beta} + \delta^{\mu}_{\beta}\Omega_{,\alpha} - g^{\mu\nu} g_{\alpha\beta}\Omega_{,\nu} \right] + \Omega^{-1} \left[\delta^{\mu}_{\alpha}\Omega_{,\beta\lambda} + \delta^{\mu}_{\beta}\Omega_{,\alpha\lambda} - (g^{\mu\nu} g_{\alpha\beta}\Omega_{,\nu})_{,\lambda} \right] \tag{1442}$$

$$= \partial_\lambda \Gamma^{\mu}{}_{\alpha\beta} - 4\omega_{,\lambda} \left[\delta^{\mu}_{\alpha}\omega_{,\beta} + \delta^{\mu}_{\beta}\omega_{,\alpha} - g^{\mu\nu} g_{\alpha\beta}\omega_{,\nu} \right] + 2 \left[\delta^{\mu}_{\alpha} \left(2\omega_{,\beta}\omega_{,\lambda} + \omega_{,\beta\lambda} \right) + \delta^{\mu}_{\beta} \left(2\omega_{,\alpha}\omega_{,\lambda} + \omega_{,\alpha\lambda} \right) \right] \tag{1443}$$

$$- 2 \left[g^{\mu\nu}{}_{,\lambda} g_{\alpha\beta}\omega_{,\nu} + g^{\mu\nu} g_{\alpha\beta,\lambda}\omega_{,\nu} + g^{\mu\nu} g_{\alpha\beta}(2\omega_{,\nu}\omega_{,\lambda} + \omega_{,\nu\lambda}) \right] \tag{1444}$$

$$\tag{1445}$$

$$\partial_\rho \tilde{\Gamma}^{\rho}{}_{\mu\nu} = \partial_\rho \Gamma^{\rho}{}_{\mu\nu} - 4\omega_{,\rho} \left[\delta^{\rho}_{\mu}\omega_{,\nu} + \delta^{\rho}_{\nu}\omega_{,\mu} - g^{\rho\nu} g_{\mu\nu}\omega_{,\lambda} \right] + 2 \left[\delta^{\rho}_{\mu} \left(2\omega_{,\nu}\omega_{,\rho} + \omega_{,\nu\rho} \right) + \delta^{\rho}_{\nu} \left(2\omega_{,\mu}\omega_{,\rho} + \omega_{,\mu\rho} \right) \right] \tag{1446}$$

$$- 2 \left[g^{\rho\lambda}{}_{,\rho} g_{\mu\nu}\omega_{,\lambda} + g^{\rho\lambda} g_{\mu\nu,\rho}\omega_{,\lambda} + g^{\rho\lambda} g_{\mu\nu}(2\omega_{,\lambda}\omega_{,\rho} + \omega_{,\lambda\rho}) \right] \tag{1447}$$

$$= \partial_\rho \Gamma^{\rho}{}_{\mu\nu} - 4 \left[2\omega_{,\mu}\omega_{,\nu} - \omega_{,\rho} g^{\rho\nu} g_{\mu\nu}\omega_{,\lambda} \right] + 4 \left(2\omega_{,\nu}\omega_{,\mu} + \omega_{,\nu\mu} \right) \tag{1448}$$

$$- 2 \left[g^{\rho\lambda}{}_{,\rho} g_{\mu\nu}\omega_{,\lambda} + g^{\rho\lambda} g_{\mu\nu,\rho}\omega_{,\lambda} + g^{\rho\lambda} g_{\mu\nu}(2\omega_{,\lambda}\omega_{,\rho} + \omega_{,\lambda\rho}) \right] \tag{1449}$$

$$= \partial_\rho \Gamma^{\rho}{}_{\mu\nu} + 4g^{\rho\nu} g_{\mu\nu}\omega_{,\lambda}\omega_{,\rho} + 4\omega_{,\nu\mu} - 2 \left[g^{\rho\lambda}{}_{,\rho} g_{\mu\nu}\omega_{,\lambda} + g^{\rho\lambda} g_{\mu\nu,\rho}\omega_{,\lambda} + (2g^{\rho\lambda} g_{\mu\nu}\omega_{,\lambda}\omega_{,\rho} + g^{\rho\lambda} g_{\mu\nu}\omega_{,\lambda\rho}) \right] \tag{1450}$$

$$= \partial_\rho \Gamma^{\rho}{}_{\mu\nu} + 4\omega_{,\lambda}\omega_{,\mu} + 4\omega_{,\nu\mu} - 2 \left[g^{\rho\lambda}{}_{,\rho} g_{\mu\nu}\omega_{,\lambda} + g_{\mu\nu,\rho}\omega^{,\rho} + 2g_{\mu\nu}\omega^{,\rho}\omega_{,\rho} + g_{\mu\nu}\omega^{,\rho}{}_{\rho} \right] \tag{1451}$$

$$\partial_\nu \tilde{\Gamma}^{\rho}{}_{\mu\rho} = \partial_\nu \Gamma^{\rho}{}_{\mu\rho} - 4\omega_{,\nu} \left[\delta^{\rho}_{\mu}\omega_{,\rho} + \delta^{\rho}_{\rho}\omega_{,\mu} - g^{\rho\kappa} g_{\mu\rho}\omega_{,\kappa} \right] + 2 \left[\delta^{\rho}_{\mu} \left(2\omega_{,\rho}\omega_{,\nu} + \omega_{,\rho\nu} \right) + \delta^{\rho}_{\rho} \left(2\omega_{,\mu}\omega_{,\nu} + \omega_{,\mu\nu} \right) \right] \tag{1452}$$

$$- 2 \left[g^{\rho\kappa}{}_{,\nu} g_{\mu\rho}\omega_{,\kappa} + g^{\rho\kappa} g_{\mu\rho,\nu}\omega_{,\kappa} + g^{\rho\kappa} g_{\mu\rho}(2\omega_{,\kappa}\omega_{,\nu} + \omega_{,\kappa\nu}) \right] \tag{1453}$$

$$= \partial_\nu \Gamma^{\rho}{}_{\mu\rho} - 4 \left[(d+1)\omega_{,\mu}\omega_{,\nu} - \omega_{,\mu}\omega_{,\nu} \right] + 2(d+1) \left(2\omega_{,\mu}\omega_{,\nu} + \omega_{,\mu\nu} \right) \tag{1454}$$

$$- 2 \left[g^{\rho\kappa}{}_{,\nu} g_{\mu\rho}\omega_{,\kappa} + g^{\rho\kappa} g_{\mu\rho,\nu}\omega_{,\kappa} + \delta^{\kappa}_{\mu}(2\omega_{,\kappa}\omega_{,\nu} + \omega_{,\kappa\nu}) \right] \tag{1455}$$

$$= \partial_\nu \Gamma^{\rho}{}_{\mu\rho} + 4\omega_{,\mu}\omega_{,\nu} + 2(d+1)\omega_{,\mu\nu} - 2 \left[g^{\rho\kappa}{}_{,\nu} g_{\mu\rho}\omega_{,\kappa} + g^{\rho\kappa} g_{\mu\rho,\nu}\omega_{,\kappa} + (2\omega_{,\mu}\omega_{,\nu} + \omega_{,\mu\nu}) \right] \tag{1456}$$

$$= \partial_\nu \Gamma^{\rho}{}_{\mu\rho} + 2d \cdot \omega_{,\mu\nu} - 2 \left[g^{\rho\kappa}{}_{,\nu} g_{\mu\rho}\omega_{,\kappa} + g_{\mu\rho,\nu}\omega^{,\rho} \right] \tag{1457}$$

$$\tilde{\Gamma}^{\mu}{}_{\alpha\beta} = \Gamma^{\mu}{}_{\alpha\beta} + \Omega^{-1}\left[\delta^{\mu}_{\alpha}\Omega_{,\beta} + \delta^{\mu}_{\beta}\Omega_{,\alpha} - g^{\mu\nu}g_{\alpha\beta}\Omega_{,\nu}\right] \tag{1458}$$
$$\tag{1459}$$

$$\begin{aligned}
\tilde{\Gamma}^{\rho}{}_{\mu\nu}\tilde{\Gamma}^{\sigma}{}_{\rho\sigma} &= \left(\Gamma^{\rho}{}_{\mu\nu} + \Omega^{-1}\left[\delta^{\rho}_{\mu}\Omega_{,\nu} + \delta^{\rho}_{\nu}\Omega_{,\mu} - g^{\rho\lambda}g_{\mu\nu}\Omega_{,\lambda}\right]\right)\left(\Gamma^{\sigma}{}_{\rho\sigma} + d\cdot\Omega^{-1}\Omega_{,\rho}\right) && (1460)\\
&= \Gamma^{\rho}{}_{\mu\nu}\Gamma^{\sigma}{}_{\rho\sigma} + \Gamma^{\rho}{}_{\mu\nu}d\cdot\Omega^{-1}\Omega_{,\rho} + \Gamma^{\sigma}{}_{\rho\sigma}\Omega^{-1}\left[\delta^{\rho}_{\mu}\Omega_{,\nu} + \delta^{\rho}_{\nu}\Omega_{,\mu} - g^{\rho\lambda}g_{\mu\nu}\Omega_{,\lambda}\right] && (1461)\\
&\quad + d\cdot\Omega^{-2}\left[\delta^{\rho}_{\mu}\Omega_{,\nu} + \delta^{\rho}_{\nu}\Omega_{,\mu} - g^{\rho\lambda}g_{\mu\nu}\Omega_{,\lambda}\right]\Omega_{,\rho} && (1462)
\end{aligned}$$

$$\begin{aligned}
\tilde{R}_{\mu\nu} &= \tilde{R}^{\rho}{}_{\mu\rho\nu} && (1463)\\
&= \partial_{\rho}\tilde{\Gamma}^{\rho}{}_{\mu\nu} - \partial_{\nu}\tilde{\Gamma}^{\rho}{}_{\mu\rho} + \tilde{\Gamma}^{\rho}{}_{\mu\nu}\tilde{\Gamma}^{\sigma}{}_{\rho\sigma} - \tilde{\Gamma}^{\sigma}{}_{\nu\rho}\tilde{\Gamma}^{\rho}{}_{\mu\sigma} && (1464)
\end{aligned}$$

- Curvature scalar

$$\begin{aligned}
\tilde{R} &= \tilde{g}^{\mu\nu}\tilde{R}_{\mu\nu} && (1465)\\
&= \tilde{g}^{\mu\nu}\left[R_{\mu\nu} - g_{\mu\nu}\Box\omega - (d-2)\nabla_{\mu}\nabla_{\nu}\omega + (d-2)\nabla_{\mu}\omega\nabla_{\nu}\omega - (d-2)g_{\mu\nu}\nabla^{\lambda}\omega\nabla_{\lambda}\omega\right] && (1466)\\
&= \Omega^{-2}\left[R - d\Box\omega - (d-2)\Box\omega + (d-2)\nabla^{\mu}\omega\nabla_{\mu}\omega - (d-2)d\nabla^{\lambda}\omega\nabla_{\lambda}\omega\right] && (1467)\\
&= \Omega^{-2}\left[R - 2(d-1)\Box\omega - (d-2)(d-1)\nabla^{\lambda}\omega\nabla_{\lambda}\omega\right] && (1468)\\
& && (1469)
\end{aligned}$$

0.13.11 Problem 23.6 - Reflection formula

$$\Gamma(z) = \int_0^{\infty} x^{z-1}e^{-x}dx \tag{1470}$$

0.13.12 Problem 23.7 - Unruh temperature

0.13.13 Problem 24.14 - Jeans length and the speed of sound

We start with the Euler equations

$$\begin{aligned}
\frac{D\rho}{Dt} &= -\rho\nabla\cdot\vec{u} \quad\rightarrow\quad \frac{\partial\rho}{\partial t} + \vec{u}\cdot(\nabla\rho) + \rho(\nabla\cdot\vec{u}) = 0 && (1471)\\
\frac{D\vec{u}}{Dt} &= -\nabla\left(\frac{P}{\rho}\right) + \vec{g} \quad\rightarrow\quad \frac{\partial\vec{u}}{\partial t} + \vec{u}\cdot(\nabla\vec{u}) + \frac{\nabla P}{\rho} = \vec{g}. && (1472)
\end{aligned}$$

With the perturbation ansatz (small perturbation in a resting fluid)

$$\begin{aligned}
\rho &= \rho_0 + \varepsilon\rho_1(x,t) && (1473)\\
P &= P_0 + \varepsilon P_1(x,t) && (1474)\\
\vec{u} &= \varepsilon\vec{u}_1(x,t) && (1475)
\end{aligned}$$

and the Newton equation

$$\triangle\phi = 4\pi G\rho \quad\rightarrow\quad \nabla\cdot\vec{g}_1 = -4\pi G\rho_1 \tag{1476}$$

we obtain (with the EoS $P = w\rho$) in order ε

$$\frac{\partial \rho_1}{\partial t} + \rho_0(\nabla \cdot \vec{u}_1) = 0 \tag{1477}$$

$$\frac{\partial \vec{u}_1}{\partial t} + \underbrace{\frac{1}{\rho_0}\nabla P_1}_{=\frac{w}{\rho_0}\nabla \rho_1} = \vec{g}_1. \tag{1478}$$

Differentiating both (with respect to space and time) we obtain a wave equation

$$\frac{\partial^2 \rho_1}{\partial t^2} - w\triangle \rho_1 = 4\pi G \rho_0 \rho_1 \tag{1479}$$

with the speed of sound $c_s^2 = w$. Inserting the wave ansatz $\rho_1 \sim \exp[i(\vec{k} \cdot \vec{x} - \omega t)]$ yields the dispersion relation

$$\omega^2 = c_s^2 k^2 - 4\pi G \rho_0. \tag{1480}$$

For wave numbers $k_J < \sqrt{4\pi G/c_s^2}$ the ω becomes complex which gives rise to exponentially growing modes. Therefore the Jeans length is given by

$$\lambda_J = \frac{2\pi}{k_J} = c_s\sqrt{\frac{\pi}{G\rho_0}} = \sqrt{\frac{\pi w}{G\rho_0}}. \tag{1481}$$

0.13.14 Problem 25.1 - Schwarzschild metric

The simplified vacuum Einstein equations are given by

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 0 \tag{1482}$$

$$\rightarrow \quad R - \frac{1}{2}R \cdot 4 = 0 \quad \rightarrow \quad R = 0 \tag{1483}$$

$$R_{\mu\nu} = 0 \tag{1484}$$

Lets start with the metric ansatz (25.4)

$$g_{\mu\nu} = \mathrm{diag}(A(r), -B(r), -r^2, -r^2\sin^2\theta) \tag{1485}$$

$$g^{\mu\nu} = \mathrm{diag}(1/A(r), -1/B(r), -1/r^2, -1/r^2\sin^2\theta) \tag{1486}$$

The non-vanishing Chistoffel symbols are then

$$\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2}g^{\mu\kappa}(g_{\kappa\lambda,\nu} + g_{\nu\kappa,\lambda} - g_{\nu\lambda,\kappa}) \tag{1487}$$

$$\Gamma^0_{01} = \frac{A'}{2A}, \quad \Gamma^1_{00} = \frac{A'}{2B} \quad \Gamma^1_{11} = \frac{B'}{2B} \quad \Gamma^1_{22} = -\frac{r}{B} \quad \Gamma^1_{33} = \frac{r\sin^2\theta}{B} \tag{1488}$$

$$\Gamma^2_{12} = 1/r \quad \Gamma^2_{12} = -\cos\theta\sin\theta \quad \Gamma^2_{12} = 1/r \quad \Gamma^2_{12} = \cot\theta \tag{1489}$$

The non-vanishing components of the Ricci tensor are

$$R_{00} = \frac{A'}{rB} - \frac{A'^2}{4AB} - \frac{A'B'}{4B^2} + \frac{A''}{2B} \tag{1490}$$

$$R_{11} = \frac{A'^2}{4A^2} + \frac{B'}{rB} + \frac{A'B'}{4AB} - \frac{A''}{2A} \tag{1491}$$

$$R_{22} = -\frac{1}{B} + 1 - \frac{rA'}{2AB} + \frac{rB''}{2B^2} \tag{1492}$$

$$R_{33} = R_{22}\sin^2\theta \tag{1493}$$

As there are only the two unknown functions A, B we only need two vacuum equations $R_{00} = 0$ and $R_{11} = 0$. Multiplying the first by B/A and leaving the second one untouched we obtain the system

$$\frac{A'}{rA} - \frac{A'^2}{4A^2} - \frac{A'B'}{4AB} + \frac{A''}{2A} = 0 \tag{1494}$$
$$\frac{B'}{rB} + \frac{A'^2}{4A^2} + \frac{A'B'}{4AB} - \frac{A''}{2A} = 0 \tag{1495}$$

Adding bot we get $B'/B = -A'/A$ which we can substitude into the first one obtaining

$$\frac{A'}{rA} + \frac{A''}{2A} = 0 \tag{1496}$$
$$\rightarrow A'(r) = \frac{c_1}{r^2} \tag{1497}$$
$$\rightarrow A(r) = c_2 - \frac{c_1}{r} \tag{1498}$$

now we can solve for $B(r)$

$$\frac{B'}{B} = -\frac{A'}{A} \tag{1499}$$
$$\rightarrow B(r) = \frac{c_3 r}{c_1 - rc_2} = \frac{-c_3}{c_2 - \frac{c_1}{r}} \tag{1500}$$

0.13.15 Problem 26.4 - Fixed points of (26.18)

We start with

$$\text{(F1)} \qquad H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V + \rho\right) \tag{1501}$$
$$\text{(F2)} \qquad \dot{H} = -4\pi G\left[\dot{\phi}^2 + (1+w_m)\rho\right] \tag{1502}$$
$$\text{(KG)} \qquad \ddot{\phi} = -3H\dot{\phi} - V_{,\phi}. \tag{1503}$$

Using $H = \dot{a}/a$, $N = \ln(a)$ and $\lambda = -V_{,\phi}/(\sqrt{8\pi G}V)$ we obtain for the time derivatives of x and y

$$\dot{V} = \frac{dV}{d\phi}\frac{d\phi}{dt} = V_{,\phi}\dot{\phi} \tag{1504}$$
$$x = \sqrt{\frac{4}{3}\pi G}\frac{\dot{\phi}}{H} \quad \rightarrow \quad \frac{dx}{dt} = \frac{dx}{dN}\frac{d\ln(a)}{dt} = \frac{dx}{dN}H = \sqrt{\frac{4}{3}\pi G}\frac{\ddot{\phi}H - \dot{\phi}\dot{H}}{H^2} \tag{1505}$$
$$y = \sqrt{\frac{8}{3}\pi G}\frac{\sqrt{V}}{H} \quad \rightarrow \quad \frac{dy}{dt} = \frac{dy}{dN}\frac{d\ln(a)}{dt} = \frac{dy}{dN}H = \sqrt{\frac{8}{3}\pi G}\frac{\frac{V_{,\phi}\dot{\phi}}{2\sqrt{V}} - \sqrt{V}\dot{H}}{H^2}. \tag{1506}$$

With the substitutions

$$\dot{H} = -4\pi G\left[\dot{\phi}^2 + (1+w_m)\rho\right] \tag{1507}$$
$$\ddot{\phi} = -3H\dot{\phi} - V_{,\phi} \tag{1508}$$
$$V_{,\phi} = -\sqrt{8\pi G}\lambda V \tag{1509}$$
$$\rho = \frac{3H^2}{8\pi G} - \frac{1}{2}\dot{\phi}^2 - V \tag{1510}$$
$$\dot{\phi} = xH/\sqrt{\frac{4}{3}\pi G} \tag{1511}$$
$$\sqrt{V} = yH/\sqrt{\frac{8}{3}\pi G} \tag{1512}$$

we obtain

$$\frac{dx}{dN} = -3x + \frac{\sqrt{6}}{2}\lambda y^2 + \frac{3}{2}x[(1-w_m)x^2 + (1+w_m)(1-y^2)] \quad (1513)$$

$$\frac{dy}{dN} = -\frac{\sqrt{6}}{2}\lambda xy + \frac{3}{2}y[(1-w_m)x^2 + (1+w_m)(1-y^2)]. \quad (1514)$$

To find the fix points of (26.17) we need to solve

$$-3x + \frac{\sqrt{6}}{2}\lambda y^2 + \frac{3}{2}x[(1-w_m)x^2 + (1+w_m)(1-y^2)] = 0 \quad (1515)$$

$$-\frac{\sqrt{6}}{2}\lambda xy + \frac{3}{2}y[(1-w_m)x^2 + (1+w_m)(1-y^2)] = 0. \quad (1516)$$

- An obvious solution is

$$x_0 = 0, y_0 = 0. \quad (1517)$$

- Two semi-obvious solutions can be found for $y = 0$ which solves the second equation and transforms the first to the quadratic equation $x^2 - 1 = 0$ which gives

$$x_1 = +1, y_1 = 0 \quad (1518)$$

$$x_2 = -1, y_2 = 0. \quad (1519)$$

- Substituting the square bracket of the second equation into the first and simplifying the second gives

$$-3x + \frac{\sqrt{6}}{2}\lambda(x^2 + y^2) = 0 \quad (1520)$$

$$-\frac{\sqrt{6}}{2}\lambda x + \frac{3}{2}[1 + 2x^2 - (x^2 + y^2) - w_m((x^2 + x^2) - 1)] = 0. \quad (1521)$$

Now we can eliminate $x^2 + y^2$ and obtain a single quadratic equation in x

$$-\frac{\sqrt{6}}{2}\lambda x + \frac{3}{2}\left[1 + 2x^2 - \frac{\sqrt{6}}{\lambda}x - w_m\left(\frac{\sqrt{6}}{\lambda}x - 1\right)\right] = 0 \quad (1522)$$

which can be simplified to

$$x^2 - \frac{3(1+w_m) + \lambda^2}{\sqrt{6}\lambda}x + \frac{1+w_m}{2} = 0. \quad (1523)$$

This gives us two more solutions

$$x_3 = \frac{\lambda}{\sqrt{6}}, y_3 = \sqrt{1 - \frac{\lambda^2}{6}} \qquad (\lambda^2 < 6) \quad (1524)$$

$$x_4 = \sqrt{\frac{3}{2}}\frac{1+w_m}{\lambda}, y_4 = \sqrt{\frac{3}{2}}\frac{\sqrt{1-w_m^2}}{\lambda} \qquad (w_m^2 < 1). \quad (1525)$$

- Let's quickly check the stability of the fix points. The characteristic equation for the fix points of a 2d system is given by

$$\alpha^2 + a_1(x_i, y_i)\alpha + a_2(x_i, y_i) = 0 \quad (1526)$$

$$a_1(x_i, y_i) = -\left(\frac{df_x}{dx} + \frac{df_y}{dy}\right)_{x=x_i, y=y_i} \quad (1527)$$

$$a_2(x_i, y_i) = \frac{df_x}{dx}\frac{df_y}{dy} - \frac{df_x}{dy}\frac{df_y}{dx}\bigg|_{x=x_i, y=y_i} \quad (1528)$$

with the stability classification (assuming for EoS parameter $w_m^2 < 1$)

type	condition	fix point 0	fix point 1	fix point 2
saddle node	$a_2 < 0$	$-1 < w_m < 1$	$\lambda > \sqrt{6}$	$\lambda < -\sqrt{6}$
unstable node	$0 < a_2 < a_1^2/4$	-	$\lambda < \sqrt{6}$	$\lambda > -\sqrt{6}$
unstable spiral	$a_1^2/4 < a_2, a_1 < 0$	-	-	-
center	$0 < a_2, a_1 = 0$	-	-	-
stable spiral	$a_1^2/4 < a_2, a_1 > 0$	-	-	-
stable node	$0 < a_2 < a_1^2/4$	-	-	-

type	fix point 3	fix point 4
saddle node	$3(1+w_m) < \lambda^2 < 6$	-
unstable node	-	-
unstable spiral	-	-
center	-	-
stable spiral	-	$\lambda^2 > \frac{24(1+w_m)^2}{7+9w_m}$
stable node	$\lambda^2 < 3(1+w_m)$	$\lambda^2 < \frac{24(1+w_m)^2}{7+9w_m}$

0.13.16 Problem 26.5 - Tracker solution

Inserting the ansatz

$$\phi(t) = C(\alpha, n) M^{1+\nu} t^{\nu} \tag{1529}$$

into the ODE

$$\ddot{\phi} + \frac{3\alpha}{t}\dot{\phi} - \frac{M^{4+n}}{\phi^{n+1}} = 0 \tag{1530}$$

gives

$$CM^{1+\nu}\nu(\nu-1)t^{\nu-2} + CM^{1+\nu}\frac{3\alpha}{t}t^{\nu-1} - \frac{M^{4+n}}{C^{n+1}M^{(n+1)(1+\nu)}t^{\nu(n+1)}} = 0 \tag{1531}$$

$$CM^{1+\nu}\left[\nu(\nu-1) + 3\alpha\right]t^{\nu-2} - \frac{M^{3-\nu(n+1)}}{C^{n+1}}t^{-\nu(n+1)} = 0 \tag{1532}$$

From equating coefficients and powers (in t) we obtain

$$\nu = \frac{2}{2+n} \tag{1533}$$

$$C(\alpha, n) = \left(\frac{(2+n)^2}{6\alpha(2+n) - 2n}\right)^{\frac{1}{2+n}} . \tag{1534}$$

0.14 Veltman - Diagrammatica

0.14.1 Problem 1.1 - Matrix exponential

We compare

$$e^\alpha = 1 + \alpha + \frac{1}{2!}\alpha^2 + \frac{1}{3!}\alpha^3 + ... + \frac{1}{n!}\alpha^n + ... \tag{1535}$$

$$\left[1 + \frac{1}{n}\alpha\right]^n = \sum_k \binom{n}{k}\frac{1}{n^k}\alpha^k = \sum_k \frac{n!}{k!(n-k)!}\frac{1}{n^k}\alpha^k \tag{1536}$$

$$= \frac{n!}{0!(n-0)!}\frac{1}{n^0}\alpha^0 + \frac{n!}{1!(n-1)!}\frac{1}{n}\alpha^1 + \frac{n!}{2!(n-2)!}\frac{1}{n^2}\alpha^2 + \frac{n!}{3!(n-3)!}\frac{1}{n^3}\alpha^3 + ... \tag{1537}$$

$$= 1 + \alpha + \frac{1}{2!}\underbrace{\frac{n(n-1)}{n^2}}_{\to 1}\alpha^2 + \frac{1}{3!}\underbrace{\frac{n(n-1)(n-2)}{n^3}}_{\to 1}\alpha^3 + ... \tag{1538}$$

0.14.2 Problem 1.2 - Lorentz rotation

Calculating the matrix product in first order we obtain

$$RR^T = \begin{pmatrix} a^2+b^2+(g+1)^2 & a(h+1)+bc+d(g+1) & af+b(k+1)+e(g+1) & 0 \\ a(h+1)+bc+d(g+1) & c^2+d^2+(h+1)^2 & c(k+1)+de+f(h+1) & 0 \\ af+b(k+1)+e(g+1) & c(k+1)+de+f(h+1) & e^2+f^2+(k+1)^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1539}$$

$$\simeq \begin{pmatrix} 1+2g & a+d & b+e & 0 \\ a+d & 1+2h & cf & 0 \\ b+e & c+f & 1+2k & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1540}$$

This only becomes the identity for $g = h = k = 0$ as well as $a = -d$, $b = -e$ and $c = -f$.

0.15 Tong - Quantum Field Theory

0.15.1 Example Sheet 1 Oct 2007 Problem 1 - Vibrating string

Using the orthogonality of $\sin mx, \cos mx$

$$\frac{\partial y}{\partial t} = \sqrt{\frac{2}{a}} \sum_{n=1} \sin\left(\frac{n\pi x}{a}\right) \dot{q}_n \tag{1541}$$

$$\left(\frac{\partial y}{\partial t}\right)^2 = \frac{2}{a}\left(\sum_n \sin\left(\frac{n\pi x}{a}\right) \dot{q}_n\right)^2 \tag{1542}$$

$$= \frac{2}{a}\sum_n \sin^2\left(\frac{n\pi x}{a}\right) \dot{q}_n^2 + \frac{2}{a}\sum_{n,m} 2\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)\dot{q}_n\dot{q}_m \tag{1543}$$

$$\int_0^a \left(\frac{\partial y}{\partial t}\right)^2 dx = \frac{2}{a}\dot{q}_n^2 \sum_n \frac{a}{2} \tag{1544}$$

$$\frac{\partial y}{\partial x} = \sqrt{\frac{2}{a}} \sum_{n=1} \cos\left(\frac{n\pi x}{a}\right) \frac{n\pi}{a} q_n \tag{1545}$$

$$\left(\frac{\partial y}{\partial x}\right)^2 = \frac{2}{a}\left(\sum_n \cos\left(\frac{n\pi x}{a}\right) \frac{n\pi}{a} q_n\right)^2 \tag{1546}$$

$$= \frac{2}{a}\sum_n \cos^2\left(\frac{n\pi x}{a}\right) \frac{n^2\pi^2}{a^2} q_n^2 + \frac{2}{a}\sum_{n,m} 2\cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi x}{a}\right)\frac{nm\pi^2}{a^2} q_n q_m \tag{1547}$$

$$\int_0^a \left(\frac{\partial y}{\partial x}\right)^2 dx = \frac{2}{a} q_n^2 \sum_n \frac{a}{2}\left(\frac{n\pi}{a}\right)^2 \tag{1548}$$

Then we see

$$L = \sum_n \left[\frac{\sigma}{2}\dot{q}_n^2 - \frac{T}{2}\left(\frac{n\pi}{a}\right)^2 q_n^2\right] \tag{1549}$$

and therefore

$$\frac{\partial L}{\partial q_n} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_n} = 0 \tag{1550}$$

$$-\frac{T}{2}\left(\frac{n\pi}{a}\right)^2 2q_n - \frac{d}{dt}\frac{\sigma}{2} 2\dot{q}_n = 0 \tag{1551}$$

$$-T\left(\frac{n\pi}{a}\right)^2 q_n - \sigma \ddot{q}_n = 0 \tag{1552}$$

$$\ddot{q}_n + \frac{T}{\sigma}\left(\frac{n\pi}{a}\right)^2 q_n = 0 \tag{1553}$$

0.15.2 Example Sheet 1 Oct 2007 Problem 2 - Lorentz transformation of the Klein-Gordon equation

Show directly that if $\phi(x)$ satisfies the Klein-Gordon equation, then $\phi(\Lambda^{-1}x)$ also satisfies this equation for any Lorentz transformation Λ.

With $x' = \Lambda x$ or $(x = \Lambda^{-1}x')$ and

$$\phi(x) \to \phi'(x') \equiv \phi(x) = \phi(\Lambda^{-1}x') \tag{1554}$$

we need to calculate the first derivative

$$\partial'_\mu \phi'(x') = \partial'_\mu \phi(x) = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial}{\partial x^\alpha} \phi(x) \tag{1555}$$

$$= (\Lambda^{-1})^\alpha_{\ \beta} \delta^\beta_\mu \frac{\partial}{\partial x^\alpha} \phi(x) \tag{1556}$$

$$= (\Lambda^{-1})^\alpha_{\ \mu} \frac{\partial}{\partial x^\alpha} \phi(x) \tag{1557}$$

and the second derivative

$$\eta^{\mu\nu} \partial'_\nu \partial'_\mu \phi'(x') = \underbrace{\eta^{\mu\nu} (\Lambda^{-1})^\alpha_{\ \mu} (\Lambda^{-1})^\beta_{\ \nu}}_{=\eta^{\alpha\beta}} \partial_\beta \partial_\alpha \phi(x) \tag{1558}$$

$$= \eta^{\alpha\beta} \partial_\beta \partial_\alpha \phi(x) \tag{1559}$$

and therefore

$$(\partial'^\mu \partial_\mu + m^2) \phi'(x') = \partial'^\mu \partial_\mu \phi'(x') + m^2 \phi'(x') \tag{1560}$$

$$= \partial^\mu \partial_\mu \phi(x) + m^2 \phi'(x') \tag{1561}$$

$$= \partial^\mu \partial_\mu \phi(x) + m^2 \phi(x) \tag{1562}$$

$$= 0 \tag{1563}$$

0.15.3 Example Sheet 1 Oct 2007 Problem 3 - Complex Klein-Gordon field

With

$$\mathcal{L} = \eta^{\mu\nu} \partial_\mu \psi^* \partial_\nu \psi - m^2 \psi^* \psi - \frac{\lambda}{2} (\psi^* \psi)^2 \tag{1564}$$

$$\frac{\partial \mathcal{L}}{\partial \psi} = -m^2 \psi^* - \lambda (\psi^* \psi) \psi^* \tag{1565}$$

$$\frac{\partial \mathcal{L}}{\partial \psi^*} = -m^2 \psi - \lambda (\psi^* \psi) \psi \tag{1566}$$

$$\frac{\partial \mathcal{L}}{\partial (\partial_\alpha \psi)} = \eta^{\mu\nu} \partial_\mu \psi^* \delta^\alpha_\nu = \eta^{\mu\alpha} \partial_\mu \psi^* = \partial^\alpha \psi^* \tag{1567}$$

$$\frac{\partial \mathcal{L}}{\partial (\partial_\alpha \psi^*)} = \partial^\alpha \psi \tag{1568}$$

then we calculate the equation of motions

$$\partial_\alpha \partial^\alpha \psi^* + m^2 \psi^* + \lambda (\psi^* \psi) \psi^* = 0 \tag{1569}$$

$$\partial_\alpha \partial^\alpha \psi + m^2 \psi + \lambda (\psi^* \psi) \psi = 0 \tag{1570}$$

Infinitesimal variation of the Lagrangian

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \psi_a} \delta \psi_a + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi_a)} \overbrace{\delta (\partial_\mu \psi_a)}^{=\partial_\mu (\delta \psi_a)} \tag{1571}$$

$$= \left[\frac{\partial \mathcal{L}}{\partial \psi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi_a)} \right] \delta \psi_a + \partial_\mu \underbrace{\left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi_a)} \delta \psi_a \right)}_{=j^\mu} \tag{1572}$$

Lagrangian invariance - substitute infinitesimal trafo $\delta\psi, \delta\psi^*$

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\psi_a}\delta\psi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_a)}\overbrace{\delta(\partial_\mu\psi_a)}^{=\partial_\mu(\delta\psi_a)} \tag{1573}$$
$$= i\alpha\left[-m^2\underbrace{(\psi^*\psi - \psi\psi^*)}_{=0} - \lambda(\psi^*\psi)\underbrace{(\psi^*\psi - \psi\psi^*)}_{=0} + \underbrace{(\partial^\mu\psi^*)\partial_\mu\psi - (\partial^\mu\psi)\partial_\mu\psi^*}_{=0}\right] \tag{1574}$$
$$= 0 \tag{1575}$$

Noether current

$$\partial_\mu j^\mu = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_a)}\delta\psi_a\right) \tag{1576}$$
$$= \partial_\mu\left(\partial^\mu\psi^*\delta\psi + \partial^\mu\psi\delta\psi^*\right) \tag{1577}$$
$$= i\alpha\partial_\mu\left[(\partial^\mu\psi^*)\psi - (\partial^\mu\psi)\psi^*\right] \tag{1578}$$
$$= i\alpha\left[(\partial_\mu\partial^\mu\psi^*)\psi - (\partial_\mu\partial^\mu\psi)\psi^* + (\partial^\mu\psi^*)(\partial_\mu\psi) - (\partial^\mu\psi)(\partial_\mu\psi^*)\right] \tag{1579}$$
$$= i\alpha\left[(\partial_\mu\partial^\mu\psi^*)\psi - (\partial_\mu\partial^\mu\psi)\psi^*\right] \tag{1580}$$
$$= i\alpha\left[(m^2\psi^* + (\psi^*\psi)\psi^*)\psi - (m^2\psi + (\psi^*\psi)\psi)\psi^*\right] \tag{1581}$$
$$= 0 \tag{1582}$$

0.15.4 Example Sheet 1 Oct 2007 Problem 4 - Lagrangian for a triplet of real fields - NOT FINISHED

$$\frac{\partial\mathcal{L}}{\partial\phi_a} = -m^2\phi_a \tag{1583}$$
$$\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi_a)} = \eta^{\mu\nu}\partial_\mu\phi_a\delta^\alpha_\nu = \eta^{\mu\alpha}\partial_\mu\phi_a = \partial^\alpha\phi_a \tag{1584}$$
$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi_a}\delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\overbrace{\delta(\partial_\mu\phi_a)}^{=\partial_\mu(\delta\phi_a)} \tag{1585}$$
$$= -m\phi_a\theta\epsilon_{abc}n_b\phi_c + (\partial^\mu\phi_a)\theta\epsilon_{abc}n_b\partial_\mu\phi_c \tag{1586}$$
$$= \theta[\epsilon_{abc}n_b(\partial^\mu\phi_a)(\partial_\mu\phi_c) - m\epsilon_{abc}n_b\phi_a\phi_c] \tag{1587}$$
$$= \theta[-n_b\epsilon_{bac}(\partial^\mu\phi_a)(\partial_\mu\phi_c) + mn_b\epsilon_{bac}\phi_a\phi_c] \tag{1588}$$
$$= \theta[-\vec{n}\cdot(\partial_\mu\phi\times\partial_\mu\phi) + m\vec{n}\cdot(\vec{\phi}\times\vec{\phi})] \tag{1589}$$
$$= 0 \tag{1590}$$

Noether current

$$j^\mu = \theta(\partial^\mu\phi_a)\epsilon_{abc}n_b\phi_c \tag{1591}$$
$$j^0 = -\theta n_b\epsilon_{bac}\phi_c\dot{\phi}_a \tag{1592}$$

0.15.5 Example Sheet 1 Oct 2007 Problem 5 - Lorentz transformation

$$\eta_{\mu\nu}x^\mu x^\nu = \eta_{\mu\nu}x'^\mu x'^\nu \tag{1593}$$
$$= \eta_{\sigma\tau}(\Lambda^\sigma_{\ \mu}x^\mu)(\Lambda^\tau_{\ \nu}x^\nu) \tag{1594}$$
$$= \eta_{\sigma\tau}\Lambda^\sigma_{\ \mu}\Lambda^\tau_{\ \nu}x^\mu x^\nu \tag{1595}$$
$$\rightarrow \eta_{\mu\nu} = \eta_{\sigma\tau}\Lambda^\sigma_{\ \mu}\Lambda^\tau_{\ \nu} \tag{1596}$$

then

$$\eta_{\mu\nu} = \eta_{\sigma\tau}\Lambda^{\sigma}_{\ \mu}\Lambda^{\tau}_{\ \nu} \tag{1597}$$
$$= \eta_{\sigma\tau}(\delta^{\sigma}_{\ \mu} + \omega^{\sigma}_{\ \mu})(\delta^{\tau}_{\ \nu} + \omega^{\tau}_{\ \nu}) \tag{1598}$$
$$= \eta_{\sigma\tau}\delta^{\sigma}_{\ \mu}\delta^{\tau}_{\ \nu} + \eta_{\sigma\tau}\delta^{\sigma}_{\ \mu}\omega^{\tau}_{\ \nu} + \eta_{\sigma\tau}\omega^{\sigma}_{\ \mu}\delta^{\tau}_{\ \nu} + \mathcal{O}(\omega^2) \tag{1599}$$
$$\simeq \eta_{\mu\nu} + \eta_{\mu\tau}\omega^{\tau}_{\ \nu} + \eta_{\sigma\nu}\omega^{\sigma}_{\ \mu} \tag{1600}$$
$$\simeq \eta_{\mu\nu} + \omega^{\mu\nu} + \omega^{\nu\mu} \tag{1601}$$
$$\to \omega^{\mu\nu} = -\omega^{\nu\mu} \tag{1602}$$

Rotation in the $x - y$ plane (t and z are undisturbed)

$$\omega^{\mu}_{\ \nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & -\epsilon & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1603}$$

Boost in the x direction (y and z are undisturbed)

$$\omega^{\mu}_{\ \nu} = \begin{pmatrix} 0 & \epsilon & 0 & 0 \\ \epsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1604}$$

Note that ω^{μ}_{ν} for the boost is symmetric and becomes antisymmetric when $\omega_{\alpha\nu} = \eta_{\alpha\mu}\omega^{\mu}_{\ \nu}$.

0.15.6 Example Sheet 1 Oct 2007 Problem 6 - Lorentz transformation of a scalar field - NOT FINISHED

For $x' = \Lambda x$ the transformation of the scalar field is given by

$$\phi(x) \to \phi'(x') \equiv \phi(x) \tag{1605}$$
$$= \phi(\Lambda^{-1}x') \tag{1606}$$
$$\simeq \phi(x') + \partial_\mu\phi(x')[(\Lambda^{-1})^{\mu}_{\ \alpha}x'^{\alpha} - x'^{\mu}] \tag{1607}$$
$$= \phi(x') + \partial_\mu\phi(x')[(\delta^{\mu}_{\ \alpha} - \omega^{\mu}_{\ \alpha})x'^{\alpha} - x'^{\mu}] \tag{1608}$$
$$= \phi(x') - \partial_\mu\phi(x')\omega^{\mu}_{\ \alpha}x'^{\alpha} \tag{1609}$$

Checking the expression for the inverse Λ^{-1}

$$\Lambda^{-1}\Lambda = 1 \tag{1610}$$
$$(\Lambda^{-1})^{\mu}_{\ \alpha}\Lambda^{\alpha}_{\ \nu} = (\delta^{\mu}_{\ \alpha} - \omega^{\mu}_{\ \alpha})(\delta^{\alpha}_{\ \nu} + \omega^{\alpha}_{\ \nu}) \tag{1611}$$
$$= \delta^{\mu}_{\ \nu} - \omega^{\mu}_{\ \nu} + \omega^{\mu}_{\ \nu} \tag{1612}$$
$$= \delta^{\mu}_{\ \nu} \tag{1613}$$

0.15.7 Example Sheet 1 Oct 2007 Problem 7 - Energy momentum tensor field for Maxwell field - NOT DONE YET

- Checking invariance

$$\mathcal{L}' = -F'_{\mu\nu}F'^{\mu\nu} \tag{1614}$$
$$= -(\partial_\mu[A_\nu + \partial_\nu\xi] - \partial_\nu[A_\mu + \partial_\mu\xi])(\partial^\mu[A^\nu + \partial^\nu\xi] - \partial^\nu[A^\mu + \partial^\mu\xi]) \tag{1615}$$
$$= -(\partial_\mu A_\nu + \partial_\mu\partial_\nu\xi - \partial_\nu A_\mu - \partial_\nu\partial_\mu\xi)(\partial^\mu A^\nu + \partial^\mu\partial^\nu\xi - \partial^\mu\partial^\nu A^\mu - \partial^\nu\partial^\mu\xi]) \tag{1616}$$
$$= -F_{\mu\nu}F^{\mu\nu} \tag{1617}$$
$$= \mathcal{L} \tag{1618}$$

so $\mathcal{L}$ is invariant.

- Noether theorem: the action being invariant under the transform

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \epsilon G_i(A(x)) \tag{1619}$$

means that $\mathcal{L}$ can only differ by a total divergence

$$\delta\mathcal{L} = \mathcal{L}(A', \partial A') - \mathcal{L}(A, \partial A) \tag{1620}$$

$$\stackrel{!}{=} \epsilon \partial_\mu X^\mu(A(x)) \tag{1621}$$

but

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial A_\mu}\delta A_\mu + \frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\mu)}\delta(\partial_\nu A_\mu) \tag{1622}$$

$$= \frac{\partial\mathcal{L}}{\partial A_\mu}\delta A_\mu + \frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\mu)}\partial_\nu(\delta A_\mu) \tag{1623}$$

$$= \frac{\partial\mathcal{L}}{\partial A_\mu}\delta A_\mu + \partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\mu)}(\delta A_\mu)\right) - \left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\mu)}\right)(\delta A_\mu) \tag{1624}$$

$$= \partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\mu)}(\delta A_\mu)\right) \tag{1625}$$

$$= \epsilon\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\mu)}\partial_\mu X^\mu\right) \tag{1626}$$

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-

0.15.8 Example Sheet 1 Oct 2007 Problem 8 - Massive vector field

With $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 C_\mu C^\mu$

$$\frac{\partial\mathcal{L}}{\partial C_\alpha} = m^2 C^\alpha \tag{1627}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\beta C_\alpha)} = -\frac{2}{4}(\delta^\beta_\mu\delta^\alpha_\nu - \delta^\beta_\nu\delta^\alpha_\mu)F^{\mu\nu} = -\frac{1}{2}(F^{\beta\alpha} - F^{\alpha\beta}) = F^{\alpha\beta} \tag{1628}$$

resulting in the equations of motion

$$-\partial_\beta F^{\alpha\beta} + m^2 C^\alpha = 0 \tag{1629}$$

$$-\partial_\beta(\partial^\alpha C^\beta - \partial^\beta C^\alpha) + m^2 C^\alpha = 0 \tag{1630}$$

$$-\partial^\alpha\partial_\beta C^\beta + \partial_\beta\partial^\beta C^\alpha + m^2 C^\alpha = 0 \tag{1631}$$

One more differentiation ∂_α and rearranging the differential operators we see

$$-\partial_\alpha\partial^\alpha\partial_\beta C^\beta + \partial_\beta\partial^\beta\partial_\alpha C^\alpha + m^2\partial_\alpha C^\alpha = 0 \tag{1632}$$

$$\to \partial_\alpha C^\alpha = 0 \tag{1633}$$

$$\to \partial_0 C^0 = \partial_i C^i \tag{1634}$$

Therefore the equations of motions simplify

$$\partial_\beta\partial^\beta C^\alpha + m^2 C^\alpha = 0 \tag{1635}$$

$$(\partial_0\partial^0 - \partial_i\partial^i)C^\alpha + m^2 C^\alpha = 0 \tag{1636}$$

$$\partial_0\partial^0 C^\alpha - \partial_i\partial^i C^\alpha + m^2 C^\alpha = 0 \tag{1637}$$

then for $\alpha = 0$

$$\partial^0 \underbrace{\partial_0 C^0}_{=\partial_i C^i} - \partial_i \partial^i C^0 + m^2 C^0 = 0 \tag{1638}$$

$$\partial_i \partial^i C^0 - m^2 C^0 = \partial_i \dot{C}^i \qquad \text{sign missing!?!.} \tag{1639}$$

which means C^0 can be calculated from C^i by solving the PDE. Now

$$\Pi_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 C^\mu)} = F^{\mu 0} = \partial^\mu C^0 - \partial^0 C^\mu \tag{1640}$$

$$\Pi_0 = 0 \tag{1641}$$

$$\Pi_i = \partial^i C^0 - \partial^0 C^i \tag{1642}$$

then with $F^{00} = 0$

$$\mathcal{H} = \Pi_\mu \partial_0 C^\mu - \mathcal{L} \tag{1643}$$

$$= \Pi_i \partial_0 C^i + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 C_\mu C^\mu \tag{1644}$$

$$= \Pi_i (\partial_i C^0 - \Pi_i) + \frac{1}{4} F_{ij} F^{ij} + \frac{1}{4} F_{0j} F^{0j} + \frac{1}{4} F_{i0} F^{i0} - \frac{1}{2} m^2 C_\mu C^\mu \tag{1645}$$

$$= \Pi_i (\partial_i C^0 - \Pi_i) + \frac{1}{4} F_{ij} F^{ij} + \frac{1}{4} \Pi_j \Pi_j + \frac{1}{4} \Pi_i \Pi_i - \frac{1}{2} m^2 C_\mu C^\mu \tag{1646}$$

$$= -\frac{1}{2} \Pi_i \Pi_i + \Pi_i \partial_i C^0 + \frac{1}{4} F_{ij} F^{ij} - \frac{1}{2} m^2 C_\mu C^\mu \tag{1647}$$

0.15.9 Example Sheet 1 Oct 2007 Problem 9 - Scale invariance

With $x' = \lambda x$ or $(x = \lambda^{-1} x')$ and

$$\phi(x) \to \phi'(x) = \lambda^{-D} \phi(\lambda^{-1} x) \tag{1648}$$

we need to calculate the first derivative

$$\partial'_\mu \phi'(x') = \partial'_\mu \phi'(\lambda x) = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial}{\partial x^\alpha} \lambda^{-D} \phi(x) \tag{1649}$$

$$= \lambda^{-D-1} \partial_\mu \phi(x) \tag{1650}$$

then

$$S = \int d^n x \, (\partial_\mu \phi(x))(\partial^\mu \phi(x)) + \ldots \tag{1651}$$

$$\to S' = \int d^n x' \, (\partial'_\mu \phi'(x'))(\partial'^\mu \phi'(x')) + \ldots \tag{1652}$$

$$= \int \lambda^{n+1} d^n x \, \lambda^{2(-D-1)} (\partial_\mu \phi(x))(\partial^\mu \phi(x)) - \frac{1}{2} m^2 \lambda^{-2D} \phi^2 - g \lambda^{-pD} \phi^p \tag{1653}$$

$$\to \lambda^{n+1-2(D+1)} = 1 \tag{1654}$$

$$\to D = \frac{n-1}{2} \tag{1655}$$

It is a symmetry of the theory if

$$n + 1 - 2D = 0 \quad \to \quad D = \frac{n+1}{2} \quad \to \quad m = 0 \tag{1656}$$

and

$$n + 1 - pD = 0 \quad \to \quad p = \frac{n+1}{D} \quad \to \quad p = 2\frac{n+1}{n-1}. \tag{1657}$$

The scale invariant Lagrangian in 3+1 is the given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - g\phi^4 \tag{1658}$$

Now calculating the Noether current for $n = 3$, $D = 1$ and $p = 4$

$$\begin{aligned}
\delta\phi &= \lambda^{-1}\phi(\lambda^{-1}x) - \phi(x) &(1659)\\
&= \lambda^{-1}\left(\phi(x) + \partial_\alpha\phi(x)[\lambda^{-1}x^\alpha - x^\mu] + ...\right) - \phi(x) &(1660)\\
&= (\lambda^{-1} - 1)\phi(x) + \partial_\alpha x^\alpha \phi(x)(\lambda^{-1} - 1) + ... &(1661)\\
&= (\lambda^{-1} - 1)(\phi(x) + x^\alpha \partial_\alpha \phi(x)) + ... &(1662)\\
&= \frac{1-\lambda}{\lambda}(\phi(x) + x^\alpha \partial_\alpha \phi(x)) + ... &(1663)\\
&= \frac{\lambda - 1}{\lambda}(-\phi(x) - x^\alpha \partial_\alpha \phi(x)) + ... &(1664)
\end{aligned}$$

alternatively

$$\begin{aligned}
\delta\phi &= \lim_{\lambda\to 1} \frac{d\,\lambda^{-1}\phi(\lambda^{-1}x)}{d\lambda} &(1665)\\
&= -\phi(x) - x^\alpha \partial_\alpha \phi(x) &(1666)\\
\delta\mathcal{L} &= \lim_{\lambda\to 1} \frac{d\mathcal{L}(d\,\lambda^{-1}\phi(\lambda^{-1}x))}{d\lambda} &(1667)\\
&= \lim_{\lambda\to 1} \frac{d}{d\lambda}\lambda^{-4}\mathcal{L} &(1668)\\
&= \lim_{\lambda\to 1} -4\lambda^3\mathcal{L} - \partial_\mu \mathcal{L}\frac{\partial(\lambda^{-1}x^\mu)}{\partial\lambda} &(1669)\\
&= -4\mathcal{L} - x^\mu \partial_\mu \mathcal{L} &(1670)\\
&= \partial_\mu(x^\mu \mathcal{L}) &(1671)
\end{aligned}$$

then

$$\begin{aligned}
j^\mu &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi - K^\mu &(1672)\\
&= -\partial_\mu \phi(\phi(x) + x^\alpha \partial_\alpha \phi(x)) + x^\mu \mathcal{L} &(1673)
\end{aligned}$$

0.16 Liu - Relativistic Quantum Field Theory I - MIT 2023 Spring

0.16.1 Problem 1.2 - Lorentz invariance of various δ-functions

(a) Using $px = \tilde{p}\tilde{x}$ is a Lorentz scalar and

$$\eta = \Lambda\eta\Lambda^T \tag{1674}$$
$$\det\eta = \det\Lambda\cdot\det\eta\cdot\det\Lambda^T \quad\rightarrow\quad 1 = (\det\Lambda)^2 \tag{1675}$$

we see when rewriting the single δ-functions by their Fourier representation

$$\begin{aligned}
\delta^{(4)}(p) &= \delta(p^0)\delta(p^1)\delta(p^2)\delta(p^3) && (1676)\\
&= \frac{1}{2\pi}\int(-1)\cdot e^{-ip^0x^0}\,dx^0\cdot\ldots\cdot\frac{1}{2\pi}\int 1\cdot e^{ip^3x^3}\,dx^3 && (1677)\\
&= -\frac{1}{2\pi}\iiiint d^4x\, e^{ipx} && (1678)\\
&= -\frac{1}{2\pi}\iiiint \underbrace{|\det\Lambda^{-1}|^4}_{=1}\, d^4\tilde{x}\, e^{i\tilde{p}\tilde{x}} && (1679)\\
&= \frac{1}{2\pi}\int(-1)\cdot e^{-i\tilde{p}^0\tilde{x}^0}\,d\tilde{x}^0\cdot\ldots\cdot\frac{1}{2\pi}\int 1\cdot e^{i\tilde{p}^3\tilde{x}^3}\,d\tilde{x}^3 && (1680)\\
&= \delta^{(4)}(\tilde{p}) && (1681)
\end{aligned}$$

(b)

(c)

0.17 Banks - Quantum Field Theory

0.17.1 Problem 2.2 - Time evolution operator in the Dirac picture

With the definitions

$$i\partial_t U_S = (H_0 + V)U_S \tag{1682}$$
$$U_D(t,t_0) = e^{iH_0t}U_S(t,t_0)e^{-iH_0t_0} \tag{1683}$$

we can start rewriting

$$\begin{aligned}
i\partial_t U_D(t,t_0) &= i\partial_t\left(e^{iH_0t}U_S(t,t_0)e^{-iH_0t_0}\right) && (1684)\\
&= i^2H_0\underbrace{e^{iH_0t}U_S(t,t_0)e^{-iH_0t_0}}_{=U_D} + e^{iH_0t}i[\partial_t U_S(t,t_0)]e^{-iH_0t_0} && (1685)\\
&= -H_0U_D(t,t_0) + e^{iH_0t}i[\partial_t U_S(t,t_0)]e^{-iH_0t_0} && (1686)\\
&= -H_0U_D(t,t_0) + e^{iH_0t}(H_0+V)U_S(t,t_0)e^{-iH_0t_0} && (1687)\\
&= -H_0U_D(t,t_0) + H_0\underbrace{e^{iH_0t}U_S(t,t_0)e^{-iH_0t_0}}_{=U_D} + e^{iH_0t}VU_S(t,t_0)e^{-iH_0t_0} && (1688)\\
&= e^{iH_0t}VU_S(t,t_0)e^{-iH_0t_0} && (1689)\\
&= e^{iH_0t}V\underbrace{e^{-iH_0t}e^{iH_0t}}_{=1}U_S(t,t_0)e^{-iH_0t_0} && (1690)\\
&= e^{iH_0t}Ve^{-iH_0t}U_D(t,t_0) && (1691)
\end{aligned}$$

0.18 Kugo - Eichtheorie

0.18.1 Problem 1.1

With $\Lambda^{\alpha}_{\ \mu} \approx \delta^{\alpha}_{\mu} + \epsilon^{\alpha}_{\ \mu}$ we obtain

$$
\begin{aligned}
g_{\mu\nu} &= \Lambda^{\alpha}_{\ \mu}\Lambda^{\beta}_{\ \nu} g_{\alpha\beta} && (1692)\\
&\simeq \left(\delta^{\alpha}_{\mu} + \epsilon^{\alpha}_{\ \mu}\right)\left(\delta^{\beta}_{\nu} + \epsilon^{\beta}_{\ \nu}\right) g_{\alpha\beta} && (1693)\\
&\simeq g_{\mu\nu} + \epsilon^{\alpha}_{\ \mu}\delta^{\beta}_{\nu} g_{\alpha\beta} + \epsilon^{\beta}_{\ \nu}\delta^{\alpha}_{\mu} g_{\alpha\beta} + \mathcal{O}(\epsilon^2) && (1694)\\
&\simeq g_{\mu\nu} + \epsilon_{\nu\mu} + \epsilon_{\mu\nu} + \mathcal{O}(\epsilon^2) && (1695)
\end{aligned}
$$

which means that ϵ is antisymmetric $\epsilon_{\nu\mu} = -\epsilon_{\mu\nu}$ and we can write

$$
\epsilon_{\nu\mu} = \frac{1}{2}\left(\epsilon_{\nu\mu} - \epsilon_{\mu\nu}\right). \qquad (1696)
$$

The infinitesimal Poincare transformation can then be written as

$$
\begin{aligned}
x'^{\mu} &= \Lambda^{\mu}_{\ \alpha} x^{\alpha} + a^{\mu} && (1697)\\
&\simeq \left(\delta^{\mu}_{\alpha} + \epsilon^{\mu}_{\ \alpha}\right) x^{\alpha} + a^{\mu} && (1698)\\
&\simeq x^{\mu} + \epsilon^{\mu}_{\ \alpha} x^{\alpha} + a^{\mu}. && (1699)
\end{aligned}
$$

The inverted PT is then given by

$$
\begin{aligned}
x &= \Lambda^{-1}(x' - a) && (1700)\\
&= \Lambda^{-1}x' - \Lambda^{-1}a && (1701)\\
x^{\mu} &\simeq \left(\delta^{\mu}_{\alpha} - \epsilon^{\mu}_{\ \alpha}\right) x'^{\alpha} - \left(\delta^{\mu}_{\alpha} - \epsilon^{\mu}_{\ \alpha}\right) a^{\alpha} && (1702)\\
&\simeq x'^{\mu} - \epsilon^{\mu}_{\ \alpha} x'^{\alpha} - a^{\mu} + \mathcal{O}(\epsilon \cdot a) && (1703)
\end{aligned}
$$

Because of

$$
\begin{aligned}
\phi'(x') = \phi(x) \quad &\Leftrightarrow \quad \phi'(\Lambda x + a) = \phi(x) && (1704)\\
&\Leftrightarrow \quad \phi'(x) = \phi(\Lambda^{-1}(x - a)) && (1705)
\end{aligned}
$$

we can now calculate

$$
\begin{aligned}
\delta\phi(x) &\equiv \phi'(x) - \phi(x) && (1706)\\
&= \phi(\Lambda^{-1}(x-a)) - \phi(x) && (1707)\\
&\simeq \phi(x^{\mu} - \epsilon^{\mu}_{\ \alpha} x^{\alpha} - a^{\mu}) - \phi(x) && (1708)\\
&\simeq \phi(x) + \partial_{\mu}\phi(x) \cdot (-\epsilon^{\mu}_{\ \alpha} x^{\alpha} - a^{\mu}) - \phi(x) && (1709)\\
&\simeq -(a^{\mu} + \epsilon^{\mu}_{\ \alpha} x^{\alpha})\partial_{\mu}\phi(x) && (1710)\\
&\simeq -(a^{\mu} + \epsilon^{\mu\alpha} x_{\alpha})\partial_{\mu}\phi(x) && (1711)\\
&\simeq -\left(a^{\mu} + \frac{1}{2}\left(\epsilon^{\mu\alpha} - \epsilon^{\alpha\mu}\right) x_{\alpha}\right)\partial_{\mu}\phi(x) && (1712)\\
&\simeq -\left(a^{\mu}\partial_{\mu} + \frac{1}{2}\left(\epsilon^{\mu\alpha} x_{\alpha}\partial_{\mu} - \epsilon^{\alpha\mu} x_{\alpha}\partial_{\mu}\right)\right)\phi(x) && (1713)\\
&\simeq -\left(a^{\mu}\partial_{\mu} + \frac{1}{2}\left(\epsilon^{\mu\alpha} x_{\alpha}\partial_{\mu} - \epsilon^{\mu\alpha} x_{\mu}\partial_{\alpha}\right)\right)\phi(x) && (1714)\\
&\simeq i\left(a^{\mu} i\partial_{\mu} + \frac{1}{2}\epsilon^{\mu\alpha} i\left(x_{\alpha}\partial_{\mu} - x_{\mu}\partial_{\alpha}\right)\right)\phi(x) && (1715)\\
&\simeq i\left(a^{\mu} i\partial_{\mu} - \frac{1}{2}\epsilon^{\mu\alpha} i\left(x_{\mu}\partial_{\alpha} - x_{\alpha}\partial_{\mu}\right)\right)\phi(x) && (1716)\\
&\simeq i\left(a^{\mu} P_{\mu} - \frac{1}{2}\epsilon^{\mu\alpha} M_{\mu\alpha}\right)\phi(x) && (1717)
\end{aligned}
$$

Calculating the commutators

$$[P_\mu, P_\nu] = 0 \tag{1718}$$

$$\begin{aligned}
[M_{\mu\nu}, P_\rho] &= i^2(x_\mu\partial_\nu - x_\nu\partial_\mu)\partial_\rho - i^2\partial_\rho(x_\mu\partial_\nu - x_\nu\partial_\mu) && (1719)\\
&= -(x_\mu\partial_\nu - x_\nu\partial_\mu)\partial_\rho + \partial_\rho(x_\mu\partial_\nu - x_\nu\partial_\mu) && (1720)\\
&= -x_\mu\partial_\nu\partial_\rho + x_\nu\partial_\mu\partial_\rho + (\partial_\rho g_{\mu\alpha}x^\alpha)\partial_\nu + x_\mu\partial_\rho\partial_\nu - (\partial_\rho g_{\nu\alpha}x^\alpha)\partial_\mu - x_\nu\partial_\rho\partial_\mu && (1721)\\
&= (\partial_\rho g_{\mu\alpha}x^\alpha)\partial_\nu - (\partial_\rho g_{\nu\alpha}x^\alpha)\partial_\mu && (1722)\\
&= (g_{\mu\alpha}\partial_\rho x^\alpha)\partial_\nu - (g_{\nu\alpha}\partial_\rho x^\alpha)\partial_\mu && (1723)\\
&= (g_{\mu\alpha}\delta^\alpha_\rho)\partial_\nu - (g_{\nu\alpha}\delta^\alpha_\rho)\partial_\mu && (1724)\\
&= g_{\mu\rho}\partial_\nu - g_{\nu\rho}\partial_\mu && (1725)\\
&= -i(g_{\mu\rho}i\partial_\nu - g_{\nu\rho}i\partial_\mu) && (1726)\\
&= -i(g_{\mu\rho}P_\nu - g_{\nu\rho}P_\mu) && (1727)
\end{aligned}$$

$$[M_{\mu\nu}, M_{\rho,\sigma}] = \text{...painful} \tag{1728}$$

0.19 LeBellac - Quantum and Statistical Field Theory

0.19.1 Problem 1.1

Some simple geometry

$$\begin{aligned}
l &= 2a\cos\theta && (1729)\\
x &= l\sin\theta && (1730)\\
&= 2a\cos\theta\sin\theta && (1731)\\
h &= x\tan\theta && (1732)\\
&= 2a\sin^2\theta && (1733)
\end{aligned}$$

Then the potential is given by

$$\begin{aligned}
V(\phi) &= 2mga\sin^2\theta + \frac{1}{2}Ca^2(2\cos\theta - 1)^2 && (1734)\\
\frac{\partial V}{\partial\theta} &= 4mga\sin\theta\cos\theta - 2Ca^2(2\cos\theta - 1)\sin\theta && (1735)\\
&= 2a\sin\theta\,(2mg\cos\theta - Ca(2\cos\theta - 1)) && (1736)\\
&= 2a\sin\theta\,(2(mg - Ca)\cos\theta + Ca) && (1737)\\
&\to \theta_0 = 0 && (1738)\\
&\to \theta_{1,2} = \arccos\frac{Ca}{2(Ca - mg)} && (1739)
\end{aligned}$$

Stability

$$\begin{aligned}
\frac{\partial^2 V}{\partial\theta^2}(\theta_{1,2}) &= 2a(2mg - Ca) && (1740)\\
\frac{\partial^2 V}{\partial\theta^2}(\theta_0) &= 2a(2mg - Ca) && (1741)
\end{aligned}$$

0.20 de Witt - Dynamical theory of groups and fields

0.20.1 Problem 1 - Functional derivatives of actions

$$\delta F = \int dx \frac{\delta F[\phi]}{\delta \phi(x)} \cdot \delta\phi(x) \tag{1742}$$
$$= \int dx \frac{\delta F[\phi]}{\delta \phi(x)} \cdot \epsilon\delta(x-y) \tag{1743}$$
$$= \epsilon \frac{\delta F[\phi]}{\delta \phi(y)} \tag{1744}$$
$$= F[\phi + \epsilon\delta(x-y)] - F[\phi] \tag{1745}$$

which means

$$\frac{\delta F[\phi]}{\delta \phi(y)} = \lim_{\epsilon\to 0} \frac{F[\phi + \epsilon\delta(x-y)] - F[\phi]}{\epsilon} \tag{1746}$$
$$F[\phi + \epsilon\delta(x-y)] = F[\phi] + \epsilon \frac{\delta F[\phi]}{\delta \phi(y)} \tag{1747}$$
$$= F[\phi] + \epsilon \int dx \frac{\delta F[\phi]}{\delta \phi(x)} \cdot \delta(x-y) \tag{1748}$$

Now

(a) Neutral scalar meson

$$S = \int dx\ L(\varphi, \varphi_{,\mu}) \tag{1749}$$
$$= -\frac{1}{2}\int dx\ (\varphi_{,\mu}\varphi^{,\mu} + m^2\varphi^2) \tag{1750}$$
$$= -\frac{1}{2}\left(\int dx\ (\varphi_{,\mu}\varphi^{,\mu} + m^2\varphi^2)\right) \tag{1751}$$
$$= -\frac{1}{2}\left(\int dx\ \varphi_{,\mu}\varphi^{,\mu} + \int dx\ m^2\varphi^2)\right) \tag{1752}$$

Now we calculate the first part (all derivatives are with respect to x) neglecting $\mathcal{O}(\epsilon^2)$

$$\frac{\delta S_1[\varphi]}{\delta\varphi(y)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int dx\ g^{\mu\nu}(\varphi(x) + \epsilon\delta(x-y))_{,\mu}(\varphi(x) + \epsilon\delta(x-y))_{,\nu} - \int dx\ g^{\mu\nu}\varphi(x)_{,\mu}\varphi(x)_{,\nu}\right) \tag{1753}$$
$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int dx\ g^{\mu\nu}(\varphi(x)_{,\mu} + \epsilon\partial_\mu\delta(x-y))(\varphi(x)_{,\nu} + \epsilon\partial_\nu\delta(x-y)) - \int dx\ g^{\mu\nu}\varphi(x)_{,\mu}\varphi(x)_{,\nu}\right) \tag{1754}$$
$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int dx\ g^{\mu\nu}(\varphi_{,\mu}\varphi_{,\nu} + \epsilon\varphi_{,\nu}\partial_\mu\delta(x-y) + \epsilon\varphi_{,\mu}\partial_\nu\delta(x-y)) - \int dx\ g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}\right) \tag{1755}$$
$$= \int dx\, g^{\mu\nu}(\varphi_{,\nu}\partial_\mu\delta(x-y) + \varphi_{,\mu}\partial_\nu\delta(x-y)) \tag{1756}$$
$$= -\int dx\, g^{\mu\nu}(\varphi_{,\nu\mu}\delta(x-y) + \varphi_{,\mu\nu}\delta(x-y)) \tag{1757}$$
$$= -2\varphi^{,\mu}_{,\mu}(y) \tag{1758}$$

$$\frac{\delta S_2[\varphi]}{\delta\varphi(y)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}m^2\left(\int dx\ (\varphi(x)+\epsilon\delta(x-y))(\varphi(x)+\epsilon\delta(x-y)) - \int dx\ g^{\mu\nu}\varphi(x)\varphi(x)\right) \tag{1759}$$

$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}m^2\left(\int dx\ (\varphi(x)\varphi(x)+\epsilon\delta(x-y)\varphi(x)+\epsilon\varphi(x)\delta(x-y)) - \int dx\ g^{\mu\nu}\varphi(x)\varphi(x)\right) \tag{1760}$$

$$= m^2\int dx\ (\delta(x-y)\varphi(x)+\varphi(x)\delta(x-y)) \tag{1761}$$

$$= 2m^2\varphi(y) \tag{1762}$$

and therefore

$$\frac{\delta S[\varphi]}{\delta\varphi(y)} = \varphi^{;\mu}_{,\mu}(y) - m^2\varphi(y) \tag{1763}$$

(b) Neutral vector meson

$$S_1 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = (\varphi_{\nu,\mu}-\varphi_{\mu,\nu})(\varphi^{\nu,\mu}-\varphi^{\mu,\nu}) \tag{1764}$$

$$\frac{\delta S_1[\varphi]}{\delta\varphi_\alpha(y)} = -\frac{1}{4}\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int dx\,[(\varphi_\nu(x)+\epsilon\delta^\alpha_\nu\delta(x-y))_{,\mu} - (\varphi_\mu(x)+\epsilon\delta^\alpha_\mu\delta(x-y))_{,\nu}] \tag{1765}$$

$$\cdot\,[(\varphi^\nu(x)+\epsilon\delta^{\alpha\nu}\delta(x-y))^{,\mu} - (\varphi^\mu(x)+\epsilon\delta^{\alpha\mu}\delta(x-y))^{,\nu}] - [\varphi_{\nu,\mu}-\varphi_{\mu,\nu}][\varphi^{\nu,\mu}-\varphi^{\mu,\nu}] \tag{1766}$$

$$= -\frac{1}{4}\int dx\,\big((\delta^\alpha_\nu\partial_\mu\delta(x-y) - \delta^\alpha_\mu\partial_\nu\delta(x-y))[\varphi^{\nu,\mu}-\varphi^{\mu,\nu}] \tag{1767}$$

$$+[\varphi_{\nu,\mu}-\varphi_{\mu,\nu}](\delta^{\nu\alpha}\partial^\mu\delta(x-y) - \delta^{\mu\alpha}\partial^\nu\delta(x-y))\big) \tag{1768}$$

$$= -\frac{1}{4}\int dx\,(\partial_\mu\delta(x-y))[\varphi^{\alpha,\mu}-\varphi^{\mu,\alpha}] - \partial_\nu\delta(x-y))[\varphi^{\nu,\alpha}-\varphi^{\alpha,\nu}] \tag{1769}$$

$$+[\varphi^\alpha_{,\mu}-\varphi_\mu^{;\alpha}]\partial^\mu\delta(x-y) - [\varphi_\nu^{;\alpha}-\varphi^\alpha_{,\nu}]\partial^\nu\delta(x-y)\big) \tag{1770}$$

$$= \frac{1}{4}\int dx\,\delta(x-y)\,\big([\varphi^{\alpha,\mu}_{,\mu}-\varphi^{\mu,\alpha}_{,\mu}] - [\varphi^{\nu,\alpha}_{,\nu}-\varphi^{\alpha,\nu}_{,\nu}] + [\varphi^{\alpha\ ,\mu}_{,\mu}-\varphi_\mu^{;\alpha\mu}] - [\varphi_\nu^{;\alpha\nu}-\varphi^{\alpha\ ,\nu}_{,\nu}]\big) \tag{1771}$$

$$= \frac{1}{4}\int dx\,\delta(x-y)\,\big(4\varphi^{\alpha,\mu}_{,\mu} - 2\varphi^{\mu,\alpha}_{,\mu} - 2\varphi_\mu^{;\alpha\mu}\big) \tag{1772}$$

$$= \varphi(y)^{\alpha,\mu}_{,\mu} - \varphi(y)^{\mu,\alpha}_{,\mu} \tag{1773}$$

and

$$S_2 = -\frac{m^2}{2}\varphi_\mu\varphi^\mu \tag{1774}$$

$$\frac{\delta S_2[\varphi]}{\delta\varphi_\alpha(y)} = -\frac{m^2}{2}\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int dx\,\big[(\varphi_\mu+\epsilon\delta^\alpha_\mu\delta(x-y))(\varphi^\mu+\epsilon\delta^{\mu\alpha}\delta(x-y)) - \varphi_\mu\varphi^\mu\big] \tag{1775}$$

$$= -\frac{m^2}{2}\int dx\,\big[\delta^\alpha_\mu\delta(x-y)\varphi^\mu + \varphi_\mu\delta^{\mu\alpha}\delta(x-y)\big] \tag{1776}$$

$$= -\frac{m^2}{2}\int dx\,[\delta(x-y)\varphi^\alpha + \varphi^\alpha\delta(x-y)] \tag{1777}$$

$$= -m^2\varphi^\alpha(y) \tag{1778}$$

therefore

$$\frac{\delta S[\varphi]}{\delta\varphi^\alpha(y)} = \varphi(y)^{\alpha,\mu}_{,\mu} - \varphi(y)^{\mu,\alpha}_{,\mu} - m^2\varphi^\alpha \tag{1779}$$

(c) Neutral tensor meson

(d) Two-level mass spectrum
Using results from (a)

$$S_2 = \frac{1}{2}\varphi_{,\mu}\varphi^{,\mu}\frac{m_1^2+m_2^2}{m_1^2-m_2^2} \tag{1780}$$
$$\frac{\delta S_2[\varphi]}{\delta\varphi(y)} = -\frac{m_1^2+m_2^2}{m_1^2-m_2^2}\varphi^{,\mu}{}_{,\mu} \tag{1781}$$
$$S_3 = \frac{1}{2}\varphi^2\frac{m_1^2 m_2^2}{m_1^2-m_2^2} \tag{1782}$$
$$\frac{\delta S_3[\varphi]}{\delta\varphi(y)} = \frac{m_1^2 m_2^2}{m_1^2-m_2^2}\varphi \tag{1783}$$

and

$$S_1 = \varphi^{,\mu\nu}\varphi_{,\mu\nu} \tag{1784}$$
$$\frac{\delta S_1[\varphi]}{\delta\varphi(y)} = ... \tag{1785}$$
$$= \int dx\,(\partial^{\mu\nu}\delta(x-y)\varphi_{,\mu\nu} + \varphi^{,\mu\nu}\partial_{\mu\nu}\delta(x-y)) \tag{1786}$$
$$= \int dx\,\big(\delta(x-y)\varphi_{,\mu\nu}^{,\mu\nu} + \varphi^{,\mu\nu}{}_{,\mu\nu}\delta(x-y)\big) \tag{1787}$$
$$= 2\varphi^{,\mu\nu}{}_{,\mu\nu}(y) \tag{1788}$$

Resulting in

$$\frac{\delta S[\varphi]}{\delta\varphi(y)} = \frac{1}{m_1^2-m_2^2}\varphi^{,\mu\nu}{}_{,\mu\nu}(y) - \frac{m_1^2+m_2^2}{m_1^2-m_2^2}\varphi^{,\mu}{}_{,\mu} + \frac{m_1^2 m_2^2}{m_1^2-m_2^2}\varphi \tag{1789}$$
$$= \frac{1}{m_1^2-m_2^2}(\partial^\mu\partial_\mu - m_1^2)(\partial^\nu\partial_\nu - m_2^2)\varphi \tag{1790}$$

0.20.2 Problem 2 - More Lagrangians

(a) Notation is a bit odd - vector field φ^μ and scalar field φ

$$\frac{\partial L}{\partial\varphi^\beta} - \partial_\alpha\frac{\partial L}{\partial\varphi^\beta_{,\alpha}} = 0 \tag{1791}$$
$$\varphi_\beta - \frac{1}{2}\varphi_{,\beta} - \partial_\alpha\left(\frac{1}{2}\varphi\delta^{\mu\alpha}\delta_{\mu\beta}\right) = 0 \tag{1792}$$
$$\to \varphi_\beta - \varphi_{,\beta} = 0 \tag{1793}$$
$$\frac{\partial L}{\partial\varphi} - \partial_\alpha\frac{\partial L}{\partial\varphi_{,\alpha}} = 0 \tag{1794}$$
$$\frac{1}{2}\varphi^\mu{}_{,\mu} - m^2\varphi - \partial_\alpha\left(-\frac{1}{2}\varphi^\alpha\right) = 0 \tag{1795}$$
$$\to \varphi^\mu{}_{,\mu} - m^2\varphi = 0 \tag{1796}$$

now we can separate both equations of motion by

$$\varphi^\alpha - \varphi^{,\alpha} = 0 \quad \to \quad \varphi^\alpha{}_{,\alpha} - \varphi^{,\alpha}{}_{,\alpha} = 0 \tag{1797}$$
$$\varphi^\mu{}_{,\mu\alpha} - m^2\varphi_{,\alpha} = 0 \tag{1798}$$

and obtain

$$\varphi^{,\alpha}{}_{,\alpha} - m^2\varphi = 0 \tag{1799}$$

$$\varphi^{\mu}{}_{,\mu\alpha} - m^2\varphi_\alpha = 0 \qquad \text{or better} \qquad \varphi_{,\beta} = \varphi_\beta \tag{1800}$$

(b)

(c)

0.20.3 Problem 3 - Implied equations of motion

(a) Nothing to do

(b)

(c)