Book of Solutions

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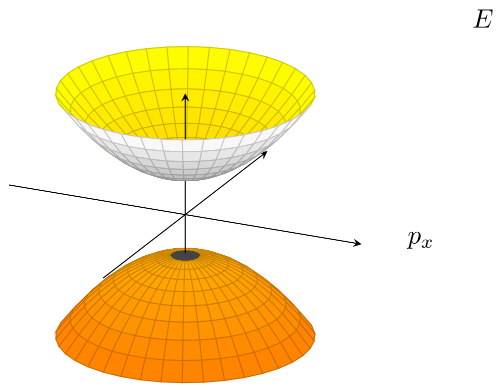
1 Introduction

There is a theory which states that if ever anyone discovers exactly what the Universe is for and why it is here, it will instantly disappear and be replaced by something even more bizarre and inexplicable. There is another theory which states that this has already happened.



Contents

2 Useful formulas

$$\left(\int_{-\infty}^{\infty} dxe^{-x^2}\right)^2 = \int_{-\infty}^{\infty} dxe^{-x^2} \cdot \int_{-\infty}^{\infty} dye^{-y^2} \tag{1}$$

$$= \int_{\mathbb{R}^2} e^{-(x^2+y^2)} dx\, dy \tag{2}$$

$$= \int_0^{2\pi} \int_0^{2\pi} e^{-r^2} rdr \tag{3}$$

$$= -2\pi \left. \frac{e^{-r^2}}{2} \right|_0^{\infty} = \pi \tag{4}$$

2.1 Common integrals

$$\int_{-\infty}^{\infty} dxe^{-ax^2} = \sqrt{\frac{\pi}{a}} \qquad a > 0, a \in \mathbb{R} \tag{5}$$

$$\int_{-\infty}^{\infty} dxe^{-ax^2+bx+c} = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}+c} \qquad a > 0, a, b, c \in \mathbb{R} \tag{6}$$

$$\int_{-\infty}^{\infty} dxe^{iax^2} = \sqrt{\frac{\pi}{a}} e^{\frac{i\pi}{4}} \qquad a > 0, a \in \mathbb{R} \tag{7}$$

2.2 Common Fourier integrals

$$\int_{-\infty}^{\infty} dye^{-ay^2} e^{-iby} = \sqrt{\frac{\pi}{a}} e^{-\frac{b^2}{4a}} \qquad a > 0, a, b \in \mathbb{R} \tag{8}$$

$$\int_{-\infty}^{\infty} dye^{iay^2} e^{-iby} = \sqrt{\frac{\pi}{a}} e^{\frac{i}{4}\left(\pi - \frac{b^2}{a}\right)} \qquad a > 0, a, b \in \mathbb{R} \tag{9}$$

$$\int_{-\infty}^{\infty} dye^{-(a+ic)y^2} e^{-iby} = \sqrt{\frac{\pi}{a+ic}} e^{-\frac{b^2}{4(a+ic)}} \qquad a > 0, a, b, c \in \mathbb{R} \tag{10}$$

$$= \sqrt{\frac{\pi}{a^2+c^2}} \sqrt{a-ic}\, e^{-\frac{b^2}{4(a^2+c^2)}(a-ic)} \tag{11}$$

2.3 Common contour integrals

$$G(t-t') = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} dE \frac{e^{-iE(t-t')}}{E^2 - \omega^2 + i\epsilon} = \frac{i}{2\omega} e^{i\omega|t|} \qquad \text{Sredniki (7.12)} \tag{12}$$

$$D(x-y) = \frac{4\pi}{(2\pi)^3} \int_0^{\infty} dp \frac{p^2 e^{i\sqrt{p^2+m^2}t}}{2\sqrt{p^2+m^2}} = \frac{1}{4\pi^2} \int_m^{\infty} dE \sqrt{E^2 - m^2} e^{iEt} \qquad \text{PS (2.51)} \tag{13}$$

$$D(x-y) = \frac{-i}{2(2\pi)^2 r} \int_{-\infty}^{+\infty} dp \frac{pe^{ipr}}{\sqrt{p^2+m^2}} = \frac{1}{4\pi^2 r} \int_m^{\infty} d\rho \frac{\rho e^{-\rho r}}{\sqrt{\rho^2 - m^2}} \qquad \text{PS (2.52)} \tag{14}$$

$$V(r) = \frac{1}{(2\pi)^2 ir} \int_{-\infty}^{\infty} dp \frac{pe^{ipr}}{p^2+m^2} = \frac{1}{4\pi r} e^{-mr} \qquad \text{PS (4.126)} \tag{15}$$

2.4 Fourier transformation

Starting from the Fourier integral theorem we have some freedom to distribute the 2π between back and forth transformation ($a, b \in \mathbb{R}$)

$$F(k) = \sqrt{\frac{|b|}{(2\pi)^{1-a}}} \int_{-\infty}^{\infty} f(x) e^{ibkx} dx \quad \leftrightarrow \quad f(x) = \sqrt{\frac{|b|}{(2\pi)^{1+a}}} \int_{-\infty}^{\infty} F(t) e^{-ibkx} dk \tag{16}$$

2.5 Delta distribution

$$x\delta(x) = 0 \tag{17}$$

$$\int \delta(x) e^{-ikx} dx = 1 \tag{18}$$

$$\int e^{ik(x-y)} dk = 2\pi\delta(x-y) \tag{19}$$

$$\begin{aligned}
\int g(x)\delta(f(x))dx &= \sum_{x_i : f(x_i)=0} \int_{x_i-\epsilon}^{x_i+\epsilon} g(x)\delta(f(x))dx && (20)\\
&= \sum_{x_i} \int_{x_i-\epsilon}^{x_i+\epsilon} g(x)\delta\left(f(x_i) + f'(x_i)(x-x_i) + \frac{1}{2}f''(x_i)(x-x_i)^2 + ...\right) dx && (21)\\
&= \sum_{x_i} \int_{x_i-\epsilon}^{x_i+\epsilon} g(x)\delta\left(f'(x_i)(x-x_i)\right) dx && (22)\\
&= \sum_{x_i} \int_{(x_i-\epsilon)f'}^{(x_i+\epsilon)f'} g\left(\frac{u}{f'(x_i)}\right)\delta\left(u - f'(x_i)x_i\right)\frac{1}{f'(x_i)} du && (23)\\
&= \sum_{x_i} \int_{(x_i-\epsilon)|f'|}^{(x_i+\epsilon)|f'|} g\left(\frac{u}{f'(x_i)}\right)\frac{1}{|f'(x_i)|}\delta\left(u - f'(x_i)x_i\right) du && (24)\\
&= \sum_{x_i} g(x_i)\frac{1}{|f'(x_i)|} && (25)
\end{aligned}$$

Important restriction: x_i are the **simple** zeros

2.6 Γ function

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dx \tag{26}$$

2.7 n-dimensional unit spheres

$$\begin{aligned}
\pi^{n/2} &= \left(\int_{-\infty}^{\infty} dt e^{-t^2}\right)^n && (27)\\
&= \int_{R^n} e^{-|x|^2} dx && (28)\\
&= \int_0^{\infty}\int_{\omega_n} e^{-r^2} r^{n-1} dr\, ds && (29)\\
&= \int_{\omega_n} ds \cdot \int_0^{\infty} e^{-r^2} r^{n-1} dr && (30)\\
&= |\omega_n| \cdot \frac{1}{2}\int_0^{\infty} e^{-\rho}\rho^{\frac{n}{2}-1} d\rho && (31)\\
&= |\omega_n| \cdot \frac{1}{2}\Gamma\left(\frac{n}{2}\right) && (32)
\end{aligned}$$

Therefore

$$\begin{aligned}
|\omega_n| &= \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} && (33)\\
V_n &= |\omega_n| \int_0^1 r^{n-1} dr && (34)\\
&= \frac{|\omega_n|}{n} && (35)
\end{aligned}$$

2.8 Laplace operator

$$\begin{aligned}
\nabla \cdot X &= \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}X^i\right) && (36)\\
(\nabla f)^i &= g^{ij}\partial_j f && (37)\\
\triangle f &= \nabla \cdot \nabla f && (38)\\
&= \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j f\right) && (39)\\
&= \sum_i \frac{\partial^2}{\partial x_j^2} && (40)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial x_i}\frac{\partial f}{\partial x_i} &= \frac{\partial}{\partial x_i}\left(\frac{\partial y_j}{\partial x_i}\frac{\partial f}{\partial y_j}\right) && (41)\\
&= \frac{\partial y_k}{\partial x_i}\frac{\partial}{\partial y_k}\left(\frac{\partial y_j}{\partial x_i}\frac{\partial f}{\partial y_j}\right) && (42)\\
&= \frac{\partial^2 y_j}{\partial x_i^2}\frac{\partial f}{\partial y_j} + \frac{\partial y_j}{\partial x_i}\frac{\partial y_k}{\partial x_i}\frac{\partial^2 f}{\partial y_j \partial y_k} && (43)\\
& && (44)
\end{aligned}$$

With $f = f(r)$ and $r = \sqrt{x_1^2 + ... + x_n^2}$ we have

$$\begin{aligned}
\triangle f(r) &= \sum_i \frac{r - x_i \frac{x_i}{r}}{r^2} \frac{\partial f}{\partial r} + \frac{x_i^2}{r^2} \frac{\partial^2 f}{\partial r^2} && (45)\\
&= \frac{nr - r}{r^2} \frac{\partial f}{\partial r} + \frac{r^2}{r^2} \frac{\partial^2 f}{\partial r^2} && (46)\\
&= \frac{(n-1)}{r} \frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial r^2} && (47)
\end{aligned}$$

2.9 Greenfunctions and PDEs

The Greensfunction $G(x, y)$ for a general PDE $D_x u(x) = f(x)$ is defined by

$$D_x G(x, y) = \delta(x - y). \qquad (48)$$

This means that general solution of the PDE can be expressed as

$$u(x) = \int G(x, y) f(y) dy \qquad (49)$$

because

$$\begin{aligned}
D_x u(x) &= D_x \int G(x, y) f(y) dy && (50)\\
&= \int D_x G(x, y) f(y) dy && (51)\\
&= \int \delta(x - y) f(y) dy && (52)\\
&= f(x) && (53)
\end{aligned}$$

Poisson equation $\triangle u(x) = f(x)$

The n-dimensional Fourier transform of $\triangle_x G(x, y) = \delta(x - y)$ and integration by parts gives

$$\begin{aligned}
\frac{1}{(2\pi)^{n/2}} \int d^n x \, \triangle_x G(x, y) e^{-ikx} &= \frac{1}{(2\pi)^{n/2}} \underbrace{\int d^n x \, \delta(x - y) e^{-ikx}}_{= e^{-iky}} && (54)\\
\frac{1}{(2\pi)^{n/2}} \int d^n x \, G(x, y) (-ik)^2 e^{-ikx} &= \frac{1}{(2\pi)^{n/2}} e^{-iky} && (55)\\
(-ik)^2 g(k) &= \frac{1}{(2\pi)^{n/2}} e^{-iky} && (56)\\
\rightarrow g(k) &= -\frac{1}{(2\pi)^{n/2}} \frac{1}{k^2} e^{-iky} && (57)
\end{aligned}$$

we can now use the Fourier transform of the Greensfunction and transform back.

- Case $n = 1$: The function has a pole at $k = 0$ and the Laurent series is given by

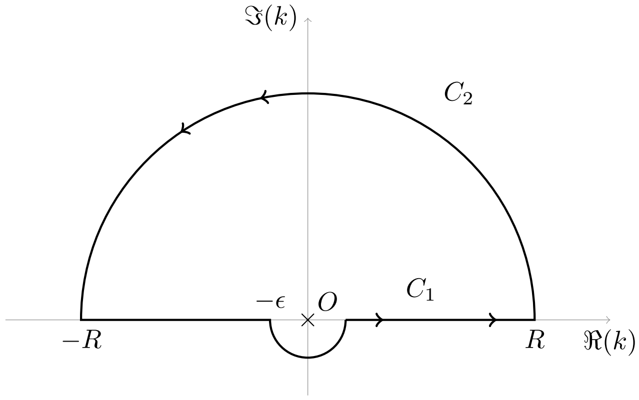
$$\frac{e^{ik(x-y)}}{k^2} = \frac{1}{k^2} + i(x - y)\frac{1}{k} - \frac{(x - y)^2}{2} - \frac{i(x - y)^3}{6} k + ... \qquad (58)$$

with Res $= i(x-y)$. We can now use the residue theorem to evaluate the integral

$$G(x,y) = -\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, \frac{e^{ik(x-y)}}{k^2} = -\frac{1}{2\pi}\int_{C_1} dk\, \frac{e^{ik(x-y)}}{k^2} \tag{59}$$

$$= -\frac{1}{2\pi}\left(\underbrace{\int_C dk\, \frac{e^{ik(x-y)}}{k^2}}_{=2\pi i \text{ Res}} - \underbrace{\int_{C_2} dk\, \frac{e^{ik(x-y)}}{k^2}}_{=0}\right) \tag{60}$$

$$= (x-y) \tag{61}$$



- Case $n = 2$:

$$G(x,y) = -\frac{1}{2\pi}\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk_1 dk_2\, \frac{e^{i(k_1(x_1-y1)+k_2(x_2-y2))}}{k_1^2+k_2^2} \tag{62}$$

$$= -\frac{1}{4\pi^2}\int_0^{\infty}\int_0^{2\pi} dk\, d\phi\, \frac{e^{ik|x-y|\cos\phi}}{k^2}k \tag{63}$$

$$= \frac{1}{2\pi}\int_0^{\infty} dk \frac{1}{k}\frac{1}{2\pi}\int_0^{2\pi} d\phi\; e^{ik|x-y|\cos\phi} \tag{64}$$

$$= \frac{1}{2\pi}\int_0^{\infty} dk\, \frac{J_0(k|x-y|)}{k} = -\frac{1}{2\pi}\int_0^{\infty|x-y|} dk'\, \frac{J_0(k')}{k'} \tag{65}$$

The last integral diverges but we try a nasty trick (?!?)

$$\frac{dG}{dx} = -\frac{1}{2\pi}\frac{d}{dx}\int_0^{\infty} dk\, \frac{J_0(k|x-y|)}{k} \tag{66}$$

$$= -\frac{1}{2\pi}\int_0^{\infty} dk\; J_1(k|x-y|) \tag{67}$$

$$= -\frac{1}{2\pi}\frac{1}{|x-y|} \tag{68}$$

Now simple integration yields

$$G(x,y) = -\frac{1}{2\pi}\log(|x-y|) \tag{69}$$

- Case $n = 3$:

$$G(x,y) = \frac{1}{(2\pi)^3}\int d^3k\,\frac{1}{k^2}e^{ik(x-y)} \tag{70}$$
$$= \frac{1}{(2\pi)^3}\int dk\underbrace{\int d\phi}_{=|\omega_2|}\int d\theta\, e^{ik|x-y|\cos\theta}\sin\theta \tag{71}$$
$$= -\frac{1}{(2\pi)^2}\int dk\int_{-1}^{+1} e^{ik|x-y|\cos\theta}d\cos\theta \tag{72}$$
$$= -\frac{1}{(2\pi)^2}\int dk\frac{e^{ik|x|}-e^{-ik|x-y|}}{ik|x-y|} \tag{73}$$
$$= -\frac{1}{2\pi^2}\int_0^\infty dk\frac{\sin k|x-y|}{k|x-y|} \tag{74}$$
$$= -\frac{1}{2\pi^2}\frac{1}{|x-y|}\int_0^\infty dk'\frac{\sin k'}{k'} \tag{75}$$
$$= -\frac{1}{4\pi}\frac{1}{|x-y|} \tag{76}$$

- Case $n > 3$: ...

Alternatively we can use the Gauss theorem with $\vec{F} = \nabla_x G(x,y)$

$$\int_V \nabla\cdot\vec{F}dx = \int_{\partial V}\vec{F}\cdot d\vec{S} \tag{77}$$
$$\int_{K_r(y)}\triangle_x G(x,y)dx = \int_{\partial_{K_r(y)}}\nabla G\cdot d\vec{S} \tag{78}$$
$$1 = \frac{\partial G(r,0)}{\partial r}|\omega_n|r^{n-1} \tag{79}$$
$$\frac{\partial G(r,0)}{\partial r} = \frac{r^{-n+1}}{|\omega_n|} \tag{80}$$
$$G(x,y) = \begin{cases} \frac{1}{|\omega_2|}\log|x-y| & n=2 \\ -\frac{1}{|\omega_n|(n-2)}\frac{1}{|x-y|^{n-2}} & n\geq 3 \end{cases} \tag{81}$$

Helmholtz equation $(\triangle + k^2)u(x) = f(x)$

The Greens function is given by $(\triangle_x + k^2)G(x,y) = \delta(x-y)$

Wave equation $\left(\frac{1}{c^2}\partial_{tt} - \triangle\right)u(x,t) = f(x,t)$

Klein-Gordon equation $\left(\frac{1}{c^2}\partial_{tt} - \triangle + \mu^2\right)u(x,t) = f(x,t)$

Feynman propagator $\left(\triangle - k^2\right)u(x) = f(x)$

Heat equation $\left(\partial_t - k\triangle\right)u(x) = f(x)$

Relativistic Heat equation $\left(\partial_{tt} + 2\gamma\partial_t - c^2\triangle\right)u(x) = f(x)$

2.10 Probability

- Hypothesis H: Steve is a librarian
- Evidence E: Steve likes reading books

Question: Whats the probability of the hypothesis is true given the evidence is true $P(H|E)$

$$P(H|E) \equiv \frac{P(E \cap H)}{P(E)} \qquad P(E|H) \equiv \frac{P(E \cap H)}{P(H)} \tag{82}$$

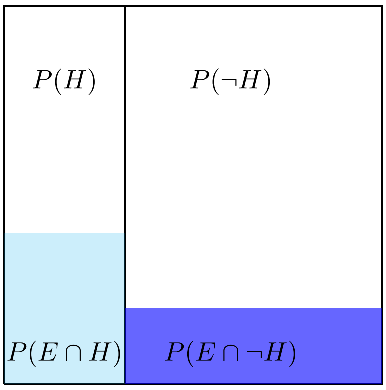
$$\rightarrow \qquad P(H|E) = \frac{P(E|H) \cdot P(H)}{P(E)} = \frac{P(H) \cdot P(E|H)}{P(H) \cdot P(E|H) + P(\neg H) \cdot P(E|\neg H)} \tag{83}$$

alternatively

$$P(H|E) = \frac{\#\text{allPeople} \cdot P(H) \cdot P(E|H)}{\#\text{allPeople} \cdot P(H) \cdot P(E|H) + \#\text{allPeople} \cdot P(\neg H) \cdot P(E|\neg H)} \tag{84}$$

$$= \frac{P(H) \cdot P(E|H)}{P(H) \cdot P(E|H) + P(\neg H) \cdot P(E|\neg H)} \tag{85}$$

$$= \frac{P(H) \cdot P(E|H)}{P(E)} \tag{86}$$



2.11 Matrices

1. inverse $A^{-1}A = \mathbb{I}$
   - therefore $\mathbb{I} = (AB)(B^{-1}A^{-1}) \quad \rightarrow \quad (AB)^{-1} = B^{-1}A^{-1}$
2. Hermitian transpose $A^\dagger = (\overline{A})^T = \overline{A^T}$
   - $(AB)^\dagger = B^\dagger A^\dagger$ therefore $\mathbb{I} = (AA^{-1})^\dagger = (A^{-1})^\dagger A^\dagger \quad \rightarrow \quad (A^\dagger)^{-1} = (A^{-1})^\dagger$

$$\langle x|Ay\rangle = \sum_k x_k^* (\vec{A}_{\text{row}k} \cdot \vec{y}) = \sum_{k,l} x_k^* A_{kl} y_l \tag{87}$$

$$\langle Bx|y\rangle = \sum_k (\vec{B}_{\text{row}k} \cdot \vec{x})^* y_k = \sum_{k,l} B_{kl}^* x_l^* y_k \tag{88}$$

3. Orthorgonal $A^T = A^{-1}$
4. Unitary $A^\dagger = A^{-1}$
5. Hermitian $A^\dagger = A$

2.12 Diagonalization

Any matrix A is called diagonalizable if there exists an invertible matrix S such that

$$D_A = S^{-1}AS \tag{89}$$

is a diagonal matrix. The diagonalizability of A is equivalent to the fact that the $\{\vec{v}_i\}$ are all linearly independent.

To find S and D_A one has to find the eigensystem $\{\lambda_i, \vec{v}_i\}$ with $A\vec{v}_i = \lambda_i \vec{v}_i$. Then $D_A S$ and S can be written as $S = (\vec{v}_1, ..., \vec{v}_n)$ and $D_A = \mathrm{diag}(\lambda_1, ..., \lambda_n)$ because $AS = (A\vec{v}_1, ..., A\vec{v}_n) = (\lambda_1\vec{v}_1, ..., \lambda_n\vec{v}_n) = SD_A$.

2.13 Functional derivatives

Let $F[\phi]$ a functional, i.e. a mapping from a Banach space $\mathcal{M}$ to the field of real or complex numbers. The functional (Frechet) derivative $\delta F[\phi]/\delta\phi$ is defined by

$$\begin{aligned}
\delta F &= \int dx \frac{\delta F[\phi]}{\delta\phi(x)} \cdot \delta\phi(x) && (90)\\
&= \int dx \frac{\delta F[\phi]}{\delta\phi(x)} \cdot \epsilon\delta(x-y) && (91)\\
&= \epsilon \frac{\delta F[\phi]}{\delta\phi(y)} && (92)\\
&= F[\phi + \epsilon\delta(x-y)] - F[\phi] && (93)
\end{aligned}$$

which means

$$\begin{aligned}
\frac{\delta F[\phi]}{\delta\phi[y]} &= \lim_{\epsilon\to 0} \frac{F[\phi+\epsilon\delta(x-y)] - F[\phi]}{\epsilon} && (94)\\
F[\phi+\epsilon\delta(x-y)] &= F[\phi] + \epsilon\frac{\delta F[\phi]}{\delta\phi(y)} && (95)\\
&= F[\phi] + \epsilon \int dx \frac{\delta F[\phi]}{\delta\phi(x)} \cdot \delta(x-y) && (96)
\end{aligned}$$

- Product rule $F[\phi] = G[\phi]H[\phi]$

$$\begin{aligned}
\frac{\delta F[\phi]}{\delta\phi(x)} &= \frac{\delta(G[\phi]H[\phi])}{\delta\phi} && (97)\\
&= \lim_{\epsilon\to 0} \frac{G[\phi+\epsilon\delta(x-y)]H[\phi+\epsilon\delta(x-y)] - G[\phi]H[\phi]}{\epsilon} && (98)\\
&= \lim_{\epsilon\to 0} \frac{\left(G[\phi]+\epsilon\frac{\delta G}{\delta\phi}\right)\left(H[\phi]+\epsilon\frac{\delta H}{\delta\phi}\right) - G[\phi]H[\phi]}{\epsilon} && (99)\\
&= \lim_{\epsilon\to 0} \frac{G[\phi]H[\phi] + \epsilon G[\phi]\frac{\delta H}{\delta\phi} + \frac{\delta G}{\delta\phi}H[\phi] + \epsilon^2\frac{\delta G}{\delta\phi}\frac{\delta H}{\delta\phi} - G[\phi]H[\phi]}{\epsilon} && (100)\\
&= G[\phi]\frac{\delta H[\phi]}{\delta\phi(x)} + \frac{\delta G[\phi]}{\delta\phi(x)}H[\phi] && (101)
\end{aligned}$$

- Chain rule $F[G[\phi]]$

$$\delta F = \int dx \frac{\delta F[G[\phi]]}{\delta \phi(x)} \delta \phi(x) \tag{102}$$

$$\delta G = \int dy \frac{\delta G[\phi]}{\delta \phi(y)} \delta \phi(y) \tag{103}$$

$$\delta F = \int dz \frac{\delta F[G]}{\delta G(z)} \delta G(z) \tag{104}$$

$$= \int dz \frac{\delta F[G]}{\delta G(z)} \int dy \frac{\delta G[\phi]}{\delta \phi(y)} \delta \phi(y) \tag{105}$$

$$= \int dy \underbrace{\int dz \frac{\delta F[G]}{\delta G(z)} \frac{\delta G[\phi]}{\delta \phi(y)}}_{= \frac{\delta F[G[\phi]]}{\delta \phi(y)}} \delta \phi(y) \tag{106}$$

$$\frac{\delta F[G[\phi]]}{\delta \phi(y)} = \int dz \frac{\delta F[G]}{\delta G(z)} \frac{\delta G[\phi]}{\delta \phi(y)} \tag{107}$$

- Chain rule (special case) $F[g[\phi]]$

$$\frac{\delta F[g[\phi]]}{\delta \phi(y)} = ... \tag{108}$$

$$= \frac{\delta F}{\delta g(\phi(y))} \frac{dg(\phi)}{d\phi(y)} \tag{109}$$

Some examples

1. $F[\phi] = \int dx \phi(x) \delta(x)$

$$\frac{\delta F[\phi]}{\delta \phi(y)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\int dx (\phi(x) + \epsilon \delta(x-y)) \delta(x)) - \int dx\, \phi(x) \delta(x) \right) \tag{110}$$

$$= \int dx\, \delta(x-y)) \delta(x) \tag{111}$$

$$= \delta(y) \tag{112}$$

2. $F[\phi] = \int dx \phi(x)$

$$\frac{\delta F[\phi]}{\delta \phi(y)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\int dx (\phi(x) + \epsilon \delta(x-y))) - \int dx\, \phi(x) \right) \tag{113}$$

$$= \int dx\, \delta(x-y) \tag{114}$$

$$= 1 \tag{115}$$

3. $F_x[\phi] = \phi(x)$

$$\frac{\delta \phi(x)}{\delta \phi(y)} = \frac{\delta F_x[\phi]}{\delta \phi(y)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left((\phi(x) + \epsilon \delta(x-y)) - \phi(x) \right) \tag{116}$$

$$= \delta(x-y) \tag{117}$$

4. $F[\phi] = \int dx \phi(x)^n$

$$\frac{\delta F[\phi]}{\delta \phi(y)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\int dx (\phi(x) + \epsilon \delta(x-y)))^n - \int dx\, \phi(x)^n \right) \tag{118}$$

$$= \int dx\, n \phi(x)^{n-1} \delta(x-y) \tag{119}$$

$$= n \phi(y)^{n-1} \tag{120}$$

5. $F[\phi] = \int dx \left(\frac{\phi(x)}{dx}\right)^n$

6. $F_y[\phi] = \int dz K(y,z)\phi(z)$

$$\frac{\delta F_y[\phi]}{\delta\phi(x)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int dz(K(y,z)(\phi(z)+\epsilon\delta(z-x)) - \int dz K(y,z)\phi(z)\right) \quad (121)$$
$$= \int dz\, K(y,z)\delta(z-x) \quad (122)$$
$$= K(y,x) \quad (123)$$

7. $F_x[\phi] = \nabla\phi(x)$

$$\frac{\delta F[\phi]}{\delta\phi(y)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\nabla_x(\phi(x)+\epsilon\delta(x-y)) - \nabla_x\phi(x)\right) \quad (124)$$
$$= \nabla_x\delta(x-y) \quad (125)$$

8. $F[\phi] = g\left(G[\phi(x)]\right)$

$$\frac{\delta F[\phi]}{\delta\phi(y)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}g(G[\phi(x)+\epsilon\delta(x-y)]) - g(G[\phi(x)]) \quad (126)$$
$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}g(G[\phi(x)] + \epsilon\frac{\delta G}{\delta\phi}) - g(G[\phi(x)]) \quad (127)$$
$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}g(G[\phi(x)]) + g'\epsilon\frac{\delta G}{\delta\phi} - g(G[\phi(x)]) \quad (128)$$
$$= \frac{\delta G}{\delta\phi}g'(G[\phi(x)]) \quad (129)$$

2.14 Space hierarchy

1. K-Vector space $(K,\oplus,\odot)$
   - set V, field K with $(K,+,\cdot)$
   - vector addition $\oplus : V\times V\to V$
   - scalar multiplication $\odot : K\times V\to V$
2. Topological vector space
   - K-vector space
   - continuous (smooth) vector addition and scalar multiplication
3. Metric (vector) space (M,d)
   - set M, metric $d : M\times M\to\mathbb{R}$
   - $d(x,y)=0 \Leftrightarrow x=y$
   - $d(x,y)=d(y,x)$
   - $d(x,y)+d(y,z)\geq d(x,z)$
   - from the requirements above follows $d(x,y)\geq 0$
4. Normed vector space $(V,\|\cdot\|)$
   - K-vector space V, norm $\|\cdot\| : V\to\mathbb{R}$
   - Typically $K\in(\mathbb{R},\mathbb{C})$ to have a definition of $|\lambda|$
   - $\|x\|\geq 0$

- $\|x\| = 0 \Leftrightarrow x = 0$
- $\|\lambda x\| = |\lambda| \|x\|$ with $\lambda \in K$
- $\|x\| + \|y\| \geq \|x + y\|$
- with $d(x, y) := \|x - y\|$ every normed vector space has also a metric
- a metric does NOT induce a always norm as the linearity/homogeneity of the norm is not guaranteed

5. Banach space (complete normed vector space)
   - normed K-vector space $(V, \|\cdot\|)$ with $K \in (\mathbb{R}, \mathbb{C})$
   - completeness: every Cauchy sequence converges (with the metric induced by the norm) to a well defined limit
   - if the space is just a metric space (without a norm) the space is called Cauchy space
6. Hilbert space (complete vector space with a scalar product)
   - K-vector space V with $K \in (\mathbb{R}, \mathbb{C})$
   - scalar product $\langle \cdot, \cdot \rangle : V \times V \to K$
   - $\langle \lambda x_1 + x_2, y \rangle = \langle \lambda x_1, y \rangle + \langle \lambda x_2, y \rangle$
   - $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$ for $\lambda \in K$
   - $\langle x, y \rangle = \overline{\langle y, x \rangle}$ which implies $\langle x, x \rangle \in \mathbb{R}$
   - $\langle x, x \rangle > 0$
   - $\langle x, x \rangle = 0 \Leftrightarrow x = 0$
   - completeness: every Cauchy sequence converges (with the metric induced by the norm which is itself induced by the scalar product) to a well defined limit
   - without completeness the space is called Pre-Hilbert space

2.15 Tensors

- For a vector $\mathbf{A}$ the expression $\mathbf{A}^2$ is the squared distance between tip and tail.
- The inner product of two vectors can then be defined by the parallelogram law

$$\mathbf{A} \cdot \mathbf{B} \equiv \frac{1}{4} \left[(\mathbf{A} + \mathbf{B})^2 - (\mathbf{A} - \mathbf{B})^2 \right] \tag{130}$$

- A rank-n tensor $\mathbf{T} = \mathbf{T}(\_, \_, \_)$ is real-valued linear function of n vectors.

$$\mathbf{T}(\alpha \mathbf{A} + \mu \mathbf{B}, \mathbf{C}, \mathbf{D}) = \alpha \mathbf{T}(\mathbf{A}, \mathbf{C}, \mathbf{D}) + \beta \mathbf{T}(\mathbf{B}, \mathbf{C}, \mathbf{D}) \tag{131}$$

- Metric tensor

$$\mathbf{g}(\mathbf{A}, \mathbf{B}) \equiv \mathbf{A} \cdot \mathbf{B} \tag{132}$$

- A vector is a tensor of rank one

$$\mathbf{A}(\mathbf{C}) \equiv \mathbf{A} \cdot \mathbf{C} \tag{133}$$

- Tensor product

$$\mathbf{A} \otimes \mathbf{B} \otimes \mathbf{C}(\mathbf{E}, \mathbf{F}, \mathbf{G}) \equiv \mathbf{A}(\mathbf{E}) \mathbf{B}(\mathbf{F}) \mathbf{C}(\mathbf{G}) = (\mathbf{A} \cdot \mathbf{E})(\mathbf{B} \cdot \mathbf{F})(\mathbf{C} \cdot \mathbf{G}) \tag{134}$$

- Contraction

$$1\&3 \text{ contraction}(\mathbf{A} \otimes \mathbf{B} \otimes \mathbf{C} \otimes \mathbf{D}) \equiv (\mathbf{A} \cdot \mathbf{C})\mathbf{B} \otimes \mathbf{D} \tag{135}$$

- Orthogonal basis

$$\mathbf{e}_j \cdot \mathbf{e}_k = \delta_{jk} \tag{136}$$

- Component expansion

$$\begin{aligned}
\mathbf{A} = A_j \mathbf{e}_j \quad &\rightarrow \quad A_j = \mathbf{A}(\mathbf{e}_j) = \mathbf{A} \cdot \mathbf{e}_j & (137)\\
\mathbf{T} = T_{abc} \mathbf{e}_a \otimes \mathbf{e}_b \otimes \mathbf{e}_c \quad &\rightarrow \quad T_{ijk} = \mathbf{T}(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k) & (138)\\
1\&3 \text{ contraction}(\mathbf{R}) \quad &\rightarrow \quad R_{ijik} & (139)\\
\mathbf{g} \quad &\rightarrow \quad g_{jk} = \mathbf{g}(\mathbf{e}_j, \mathbf{e}_k) = \mathbf{e}_j \cdot \mathbf{e}_k = \delta_{jk} & (140)
\end{aligned}$$

3 Primer special relativity

Definition of line element

$$\begin{aligned} ds^2 &= dx^\mu dx_\nu = \eta_{\mu\nu} dx^\mu dx^\nu && (141) \\ &= dx^T \eta dx && (142) \end{aligned}$$

Definition of Lorentz transformation

$$dx^\mu = \Lambda^\mu{}_\nu dx^\nu \tag{143}$$

By postulate the line element ds is invariant under Lorentz transformation

$$\begin{aligned} ds^2 &= \eta_{\mu\nu} dx^\mu dx^\nu && (144) \\ &\overset{!}{=} \eta_{\alpha\beta} \Lambda^\alpha{}_\mu dx^\mu \Lambda^\beta{}_\nu dx^\nu \quad \rightarrow \quad \eta_{\mu\nu} = \eta_{\alpha\beta} \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu && (145) \end{aligned}$$

or analog

$$\begin{aligned} ds^2 &= dx^T \eta dx && (146) \\ &\overset{!}{=} (\Lambda dx)^T \eta (\Lambda dx) && (147) \\ &= dx^T \Lambda^T \eta \Lambda dx \quad \rightarrow \quad \eta = \Lambda^T \eta \Lambda && (148) \end{aligned}$$

Observation with the eigentime $d\tau = ds/c$ and 3-velocity $dx^i = v^i dt$

$$\begin{aligned} \frac{ds^2}{d\tau^2} = c^2 &= c^2 \frac{dt^2}{d\tau^2} - \frac{dx^i}{dt} \frac{dx_i}{dt} \left(\frac{dt}{d\tau} \right)^2 && (149) \\ 1 &= \frac{dt^2}{d\tau^2} \left(1 - \frac{v^i v_i}{c^2} \right) \quad \rightarrow \quad \frac{dt}{d\tau} \equiv \gamma = \left(\sqrt{1 - \frac{v^2}{c^2}} \right)^{-1} && (150) \end{aligned}$$

Definition of 4-velocity with 3-velocity $d\vec{x} = \vec{v} dt$

$$\begin{aligned} u^\mu \equiv \frac{dx^\mu}{d\tau} &= \frac{dx^\mu}{dt} \frac{dt}{d\tau} = \quad \rightarrow \quad u^\mu u_\mu = \eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = \frac{ds^2}{d\tau^2} = c^2 && (151) \\ &= (c, \vec{v}) \gamma && (152) \end{aligned}$$

Object moving in x direction with v meaning $dx = v \cdot dt$ compared to rest frame $dx' = 0$

$$\begin{aligned} c^2 dt'^2 = ds^2 &= c^2 dt^2 - v^2 dt^2 && (153) \\ &= c^2 dt^2 \left(1 - \frac{v^2}{c^2} \right) && (154) \\ dt' = \frac{ds}{c} \equiv d\tau &= dt \sqrt{1 - \frac{v^2}{c^2}} = \frac{dt}{\gamma} && (155) \end{aligned}$$

Definition 4-momentum (using the 3-momentum $\vec{p} = \gamma m \vec{v}$)

$$\begin{aligned} p^\mu \equiv m u^\mu = (\gamma m c, \gamma m \vec{v}) = \left(\frac{E_p}{c}, \vec{p} \right) \quad &\rightarrow \quad p^\mu p_\mu = m^2 u^\mu u_\mu = m^2 c^2 && (156) \\ &\rightarrow \quad (p^0)^2 - p^i p_i = m^2 c^2 && (157) \\ &\rightarrow \quad p^0 = \sqrt{m^2 c^2 + \vec{p}^2} && (158) \\ &\rightarrow \quad E_p = \sqrt{m^2 c^4 + \vec{p}^2 c^2} && (159) \\ &\qquad\quad = \frac{mc^2}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}} && (160) \end{aligned}$$

First we observe

$$\eta_{\mu\nu} = \eta_{\alpha\beta}\Lambda^{\alpha}_{\ \mu}\Lambda^{\beta}_{\ \nu} \tag{161}$$
$$\det(\eta) = \det(\Lambda)^2 \det(\eta) \tag{162}$$
$$1 = \det(\Lambda)^2. \tag{163}$$

Now we see

$$\Lambda_{\gamma}^{\ \nu}\Lambda^{\gamma}_{\ \mu} = \eta_{\alpha\gamma}\eta^{\nu\beta}\Lambda^{\alpha}_{\ \beta}\Lambda^{\gamma}_{\ \mu} \tag{164}$$
$$= \eta^{\nu\beta}(\eta_{\alpha\gamma}\Lambda^{\alpha}_{\ \beta}\Lambda^{\gamma}_{\ \mu}) \tag{165}$$
$$= \eta^{\nu\beta}\eta_{\beta\mu} \tag{166}$$
$$= \delta^{\nu}_{\ \mu} \tag{167}$$

which means in matrix notation $\Lambda_{\gamma}^{\ \nu} = (\Lambda^{-1})^{\nu}_{\ \gamma}$. General transformation laws for tensors of first order

$$V'^{\alpha} = \Lambda^{\alpha}_{\ \beta}V^{\beta} \tag{168}$$
$$\eta_{\alpha\mu}V'^{\alpha} = \eta_{\alpha\mu}\Lambda^{\alpha}_{\ \beta}V^{\beta} = \eta_{\alpha\mu}\Lambda^{\alpha}_{\ \beta}(\eta^{\nu\beta}V_{\nu}) \tag{169}$$
$$V'_{\mu} = \Lambda_{\mu}^{\ \nu}V_{\nu} \tag{170}$$
$$\rightarrow \quad \Lambda_{\mu}^{\ \nu} = \eta_{\alpha\mu}\eta^{\nu\beta}\Lambda^{\alpha}_{\ \beta} \tag{171}$$

and second order

$$T'^{\alpha\beta} = \Lambda^{\alpha}_{\ \mu}\Lambda^{\beta}_{\ \nu}T^{\mu\nu} \tag{172}$$
$$\eta_{\alpha\delta}\eta_{\beta\gamma}T'^{\alpha\beta} = \eta_{\alpha\delta}\eta_{\beta\gamma}\Lambda^{\alpha}_{\ \mu}\Lambda^{\beta}_{\ \nu}T^{\mu\nu} = \eta_{\alpha\delta}\eta_{\beta\gamma}\Lambda^{\alpha}_{\ \mu}\Lambda^{\beta}_{\ \nu}(\eta^{\mu\rho}\eta^{\nu\sigma}T_{\rho\sigma}) \tag{173}$$
$$T'_{\delta\gamma} = \Lambda_{\delta}^{\ \rho}\Lambda_{\gamma}^{\ \sigma}T_{\rho\sigma}. \tag{174}$$

The general transformation is therefore given by

$$T'_{\mu_1\mu_2...}{}^{\nu_1\nu_2...} = \Lambda_{\mu_1}{}^{\rho_1}\Lambda_{\mu_2}{}^{\rho_2}...\Lambda^{\nu_1}{}_{\sigma_1}\Lambda^{\nu_2}{}_{\sigma_2}...T'_{\rho_1\rho_2...}{}^{\sigma_1\sigma_2...} \tag{175}$$

There exist two invariant tensors

$$\eta'_{\mu\nu} = \eta_{\alpha\beta}\Lambda^{\alpha}_{\ \mu}\Lambda^{\beta}_{\ \nu} = \Lambda_{\beta\mu}\Lambda^{\beta}_{\ \nu} = \eta_{\mu\sigma}\Lambda_{\beta}^{\ \sigma}\Lambda^{\beta}_{\ \nu} = \eta_{\mu\sigma}\delta^{\sigma}_{\ \nu} = \eta_{\mu\nu} \tag{176}$$
$$\epsilon'^{\mu\nu\rho\sigma} = \Lambda^{\mu}_{\ \alpha}\Lambda^{\nu}_{\ \beta}\Lambda^{\rho}_{\ \gamma}\Lambda^{\sigma}_{\ \delta}\epsilon'^{\alpha\beta\gamma\delta} \equiv \epsilon^{\mu\nu\rho\sigma}\det(\Lambda) = \pm\epsilon^{\mu\nu\rho\sigma} \tag{177}$$

Due to the possibility of the minus sign the Levi-Civita symbol ϵ is sometimes called pseudo-tensor.

4 Groups

4.1 Overview

$\mathbb{F}$	GL$(n,\mathbb{F})$	SL$(n,\mathbb{F})$	U(n)	SU(n)	O(n)	SO(n)
$\mathbb{R}$	n^2	n^2-1	-	-	$n(n-1)/2$	$n(n-1)/2$
$\mathbb{C}$	$2n^2$	$2(n^2-1)$	n^2	n^2-1	$n(n-1)$	$n(n-1)$

Table 1: Dimensions of common Lie groups (number of independent real parameters)

Observation: $\dim(\mathrm{SO}(n,\mathbb{F})) = \dim(\mathrm{O}(n,\mathbb{F}))$ - sign that SO(n) is not connected

4.2 SO(2)

There are infinitely many (non-equivalent) 1-dimensional standard irreps

$$D^k(\alpha) = e^{-ik\alpha},\ k = 0, \pm 1, \pm 2, ... \tag{178}$$

4.3 SO(3)

4.4 SU(2)

Finite dimensional irreps of the Lorentz group are labeled by l with

$$l \in \left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2, ...\right\}. \tag{179}$$

and have dimension $2l+1$. For two irreps with $l \geq m$ the tensor product representations decomposes as (Clebsch-Gordan decomposition)

$$\begin{aligned} V_l \otimes V_m &\cong \bigoplus_{j=l-m}^{l+m} V_j && (180)\\ &= V_{l+m} \oplus V_{l+m-1} \oplus ... \oplus V_{l-m+1} \oplus V_{l-m} && (181)\\ \dim(V_l \otimes V_m) &= (2l+1)(2m+1) && (182)\\ \dim(V_{l+m} \oplus ... \oplus V_{l-m}) &= \sum_{k=0}^{2m} 2[(l-m)+k]+1 && (183)\\ &= (2m+1)[2(l-m)+1] + 2\frac{2m(2m+1)}{2} && (184)\\ &= (2m+1)(2l+1) && (185) \end{aligned}$$

4.5 SU(3)

4.6 Lorentz group O(1,3)

There are the obvious tensor representations for tensors of first and second order

$$\begin{aligned} [D(\Lambda)]^{\alpha}_{\ \beta} = \Lambda^{\alpha}_{\ \beta} \quad &\rightarrow \quad V^{\alpha} = [D(\Lambda)]^{\alpha}_{\ \beta} V^{\beta} = \Lambda^{\alpha}_{\ \beta} V^{\beta} && (186)\\ [D(\Lambda)]_{\alpha\beta}^{\ \ \gamma\delta} = \Lambda_{\alpha}^{\ \gamma}\Lambda_{\beta}^{\ \delta} \quad &\rightarrow \quad T_{\alpha\beta} = [D(\Lambda)]_{\alpha\beta}^{\ \ \gamma\delta} T_{\gamma\delta} = \Lambda_{\alpha}^{\ \gamma}\Lambda_{\beta}^{\ \delta} T_{\gamma\delta} && (187) \end{aligned}$$

which are 4 and 16 dimensional.

Infinitesimal Lorentz transformations can be written as

$$\Lambda^{\alpha}_{\ \beta} = \delta^{\alpha}_{\ \beta} + \omega^{\alpha}_{\ \beta} \qquad (|\omega^{\alpha}_{\ \beta}| \ll 1). \tag{188}$$

The first order approximation gives an additional restriction

$$\begin{aligned} \eta_{\mu\nu} &= \eta_{\alpha\beta}\Lambda^{\alpha}_{\ \mu}\Lambda^{\beta}_{\ \nu} = \eta_{\alpha\beta}(\delta^{\alpha}_{\ \mu} + \omega^{\alpha}_{\ \mu})(\delta^{\beta}_{\ \nu} + \omega^{\beta}_{\ \nu}) = \eta_{\mu\nu} + \eta_{\mu\beta}\omega^{\beta}_{\ \nu} + \eta_{\alpha\nu}\omega^{\alpha}_{\ \mu} && (189)\\ &\rightarrow \omega_{\mu\nu} = -\omega_{\nu\mu} && (190) \end{aligned}$$

which implies six independent components. As the four dimensional representation of the infinitesimal transformation is close to unity it can then be written as

$$D(\Lambda) = D(1+\omega) = 1 + \frac{1}{2}\omega^{\alpha\beta}\sigma_{\alpha_\beta} \tag{191}$$

where the six ω components correspond to the six matrices $\sigma_{01}, \sigma_{02}, \sigma_{03}, \sigma_{12}, \sigma_{13}, \sigma_{23}$ which are the generators of the group.

Finite dimensional irreps of the Lorentz group are labeled by two parameters (μ, ν) with

$$\mu, \nu \in \left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2, ...\right\}. \tag{192}$$

and have dimension $(2\mu + 1)(2\nu + 1)$

$$\begin{aligned} M^2 &= \mu(\mu + 1) \\ N^2 &= \nu(\nu + 1) \\ j &\in |\mu - \nu|, ..., (\mu + \nu) \end{aligned}$$

irrep	dim	j	example
$(0, 0)$	1	0	Scalar
$(\frac{1}{2}, 0)$	2	$\frac{1}{2}$	Left-handed Weyl spinor
$(0, \frac{1}{2})$	2	$\frac{1}{2}$	Right-handed Weyl spinor
$(\frac{1}{2}, \frac{1}{2})$	4	0,1	4-Vector A^μ
$(1, 0)$	3	1	Self-dual 2-form
$(0, 1)$	3	1	Anti-self-dual 2-form
$(1, 1)$	9	0,1,2	Traceless symmetric 2^{nd} rank tensor

rep	dim	j	example
$(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$	-	-	Dirac bispinor $\psi^\alpha \quad \alpha \in \{1, 2, 3, 4\}$
$(\frac{1}{2}, \frac{1}{2}) \otimes \left[(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})\right]$	-	-	Rarita-Schwinger field $\psi^\alpha \quad \alpha \in \{1, 2, 3, 4\}$
$(0, 1) \oplus (0, 1)$	-	-	Parity invariant field of 2-forms

5 Mathematical

5.1 Andrews - Number theory

Problem 1.1

Lets cut the chase

$$\frac{n(n+1)(2n+1)}{6} + (n+1)^2 = (n+1)\frac{n(2n+1)+6(n+1)}{6} \tag{193}$$
$$= \frac{(n+1)}{6}(2n^2+7n+6) \tag{194}$$
$$= \frac{(n+1)}{6}(n+2)(2n+3) \tag{195}$$
$$= \frac{(n+1)}{6}(n+2)(2(n+1)+1) \tag{196}$$
$$= \frac{(n+1)(n+2)(2(n+1)+1)}{6} \tag{197}$$
$$\tag{198}$$

5.2 Bender, Orszag - Advanced Mathematical Methods for Scientists and Engineers

Problem 1.1

1. $y' = e^{x+y}$

$$\int \frac{dy}{e^y} = \int e^x dx \tag{199}$$
$$-e^{-y} = e^x + c \tag{200}$$
$$y = -\log(-e^x + c) \tag{201}$$

2. $y' = xy + x + y + 1$

$$\frac{dy}{y+1} = x + 1 \tag{202}$$
$$\log y + 1 = \frac{x^2}{2} + x + c \tag{203}$$
$$y = c' e^{x/2(x+2)} - 1 \tag{204}$$

Problem 1.2

$y'' = yy'/x$

1. Equidimensional-in-s equation

$$x = e^t \tag{205}$$
$$\frac{d}{dx} = \frac{dt}{dx}\frac{d}{dt} \tag{206}$$
$$= \frac{1}{x}\frac{d}{dt} \tag{207}$$
$$\frac{d^2}{dx^2} = \frac{dt}{dx}\frac{d}{dt}\left(\frac{1}{x}\frac{d}{dt}\right) \tag{208}$$
$$= \frac{1}{x}\left(-\frac{1}{x^2}x\frac{d}{dt} + \frac{1}{x}\frac{d^2}{dx^2}\right) \tag{209}$$
$$= \frac{1}{x^2}\left(-\frac{d}{dt} + \frac{d^2}{dt^2}\right) \tag{210}$$

now with $y = y(t)$

$$-y' + y'' = yy' \tag{211}$$

2. Autonomous equation

$$y' \equiv u(y) \tag{212}$$

$$y'' = \frac{du}{dy}\frac{dy}{dt} = \dot{u}y' \tag{213}$$

now with $u = u(y)$

$$-u + \dot{u}u = yu \tag{214}$$

$$\dot{u} = y + 1 \tag{215}$$

3. integration

$$u = \frac{y^2}{2} + y + c_0 \tag{216}$$

4. resubstitution I (with $\tan z = i\frac{e^{-iz}-e^{iz}}{e^{-iz}+e^{iz}}$)

$$y' = \frac{y^2}{2} + y + c_0 \tag{217}$$

$$t + c_3 = \int \frac{dy}{y^2/2 + y + c_0} \tag{218}$$

$$= 2\frac{1}{2\sqrt{1-2c_0}} \int dy \left(-\frac{1}{y+1+\sqrt{1-2c_0}} + \frac{1}{y+1-\sqrt{1-2c_0}} \right) \tag{219}$$

$$= \frac{1}{\sqrt{1-2c_0}} \left(-\log\left[y+1+\sqrt{1-2c_0}\right] + \log\left[y+1-\sqrt{1-2c_0}\right] \right) \tag{220}$$

$$= \frac{1}{\sqrt{1-2c_0}} \log \frac{y+1-\sqrt{1-2c_0}}{y+1+\sqrt{1-2c_0}} \tag{221}$$

$$= \frac{1}{\sqrt{1-2c_0}} \log \frac{-i\sqrt{1-2c_0}\left(-i + \frac{i(y+1)}{\sqrt{1-2c_0}}\right)}{i\sqrt{1-2c_0}\left(-i - \frac{i(y+1)}{\sqrt{1-2c_0}}\right)} \tag{222}$$

$$= \frac{1}{\sqrt{1-2c_0}} \log \frac{-\left(-i + \frac{i(y+1)}{\sqrt{1-2c_0}}\right)}{\left(-i - \frac{i(y+1)}{\sqrt{1-2c_0}}\right)} \tag{223}$$

$$= \frac{2}{\sqrt{1-2c_0}} \log \sqrt{-\frac{-i + \frac{i(y+1)}{\sqrt{1-2c_0}}}{-i - \frac{i(y+1)}{\sqrt{1-2c_0}}}} \tag{224}$$

$$= \frac{2}{i\sqrt{1-2c_0}} \arctan\left(-\frac{i(y+1)}{\sqrt{1-2c_0}} \right) \tag{225}$$

5. resubstitution II

$$\log x + c_3 = \frac{2}{i\sqrt{1-2c_0}} \arctan \frac{y+1}{i\sqrt{1-2c_0}} \tag{226}$$

$$\tan\left[\frac{\sqrt{2c_0-1}}{2}(\log x + c_3) \right] = \frac{y+1}{\sqrt{2c_0-1}} \tag{227}$$

$$y = \sqrt{2c_0-1} \tan\left[\frac{\sqrt{2c_0-1}}{2}(\log x + c_3) \right] - 1 \tag{228}$$

$$y = 2c_1 \tan\left[c_1 \log x + c_2\right] - 1 \tag{229}$$

This solution has poles at

$$\log x_P = \frac{\pi/2 + k\pi - c_2}{c_1} \tag{230}$$

while the special solution $-2/(c_4 + \log x) - 1$ has a pole at

$$\log x_P = -c_4 \tag{231}$$

???

Problem 1.10

With $y = e^{rx}$ the equation $y''' - 3y'' + 3y' - y = 0$ becomes

$$r^3 - 3r^2 + 3r - 1 = 0 \tag{232}$$

$$(r-1)^3 = 0 \tag{233}$$

then $y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x$.

Problem 1.11

We guess $y_1 = e^{-x}$ and have another guess $y_2 = e^{-x} u(x)$ we see

$$r^{-x}\left(u'' + xu'\right) = 0 \tag{234}$$

$$v' + xv = 0 \tag{235}$$

$$v = c_0 e^{-x^2/2} \tag{236}$$

$$u = c_1 \mathrm{erf}\left(\frac{x}{\sqrt{2}}\right) + c_2 \tag{237}$$

and therefore $y = c_3 e^{-x} + c_4 e^{-x}\left[\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right) + c_5\right]$

5.3 Arnol'd - A mathematical trivium

Problem 4

$$\frac{x^2+1}{x^3-x} = \frac{x^2+1}{x(x+1)(x-1)} \tag{238}$$

$$= -\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x-1} \tag{239}$$

$$\frac{d}{dx}(x+a)^{-1} = -(x+a)^{-2} \tag{240}$$

$$\frac{d^{100}}{dx^{100}}(x+a)^{-1} = 100!(x+a)^{-101} \tag{241}$$

$$\tag{242}$$

$$\frac{d^{100}}{dx^{100}}\left(\frac{x^2+1}{x^3-x}\right) = 100!\left(-\frac{1}{x^{101}} + \frac{1}{(x+1)^{101}} + \frac{1}{(x-1)^{101}}\right) \tag{243}$$

Problem 13

$$\int_1^{10} x^x dx = \int_1^{10} e^{x \log x} dx \tag{244}$$

Problem 20

$$\ddot{x} = x + A\dot{x}^2 \qquad x(0) = 1, \dot{x}(0) = 0 \tag{245}$$

Using the standard perturbation theory approach we assume $x(t) = x_0(t) + Ax_1(t) + A^2x_2(t) + ...$ Inserting into the ODE gives

$$\ddot{x}_0 + A\ddot{x}_1 + A^2\ddot{x}_2 + ... = x_0 + Ax_1 + A^2x_2 + ... + A\left(\dot{x}_0 + A\dot{x}_1 + A^2\dot{x}_2 + ...\right)^2. \tag{246}$$

Sorting by powers of A we obtain a set of ODEs

$$A^0: \qquad \ddot{x}_0 = x_0 \tag{247}$$
$$A^1: \qquad \ddot{x}_1 = x_1 + \dot{x}_0^2 \tag{248}$$
$$A^2: \qquad \ddot{x}_2 = x_2 + 2\dot{x}_0\dot{x}_1. \tag{249}$$

The first ODE can be solved directly

$$x_0 = c_1e^t + c_2e^{-t}. \tag{250}$$

The second ODE then transforms into

$$\ddot{x}_1 = x_1 + c_1^2e^{2t} + c2^2e^{-2t} - 2c_1c_2 \tag{251}$$

with the homogeneous solution

$$x_{1H} = c_3e^t + c_4e^{-t}. \tag{252}$$

For the particular solution we try the ansatz (inspired by the inhomogeneity)

$$x_{1S} = \alpha + \beta e^{2t} + \gamma e^{-2t} \tag{253}$$
$$= 2c_1c_2 + \frac{c_1^2}{3}e^{2t} + \frac{c_2^2}{3}e^{-2t} \tag{254}$$

then

$$x_1 = x_{1H} + x_{1S} \tag{255}$$
$$= c_3e^t + c_4e^{-t} + 2c_1c_2 + \frac{c_1^2}{3}e^{2t} + \frac{c_2^2}{3}e^{-2t} \tag{256}$$

Imposing initial conditions on x_0 gives

$$c_1 = c_2 = \frac{1}{2} \quad \rightarrow \quad x_0 = \cosh t \tag{257}$$
$$c_3 = c_4 = -\frac{1}{3} \quad \rightarrow \quad x_1 = -\frac{2}{3}\cosh t + \frac{1}{2} + \frac{1}{6}\cosh 2t \tag{258}$$

and therefore

$$\left.\frac{dx(t)}{dA}\right|_{A=0} = \frac{1}{2} - \frac{2}{3}\cosh t + \frac{1}{6}\cosh 2t \tag{259}$$

Problem 23

$$y' = x + \frac{x^3}{y} \tag{260}$$

Problem 85

In three dimensions we have

$$x^2 + y^2 + z^2 + xy + yz + zx = 1 \tag{261}$$

which can be written as

$$\vec{x}^T A \vec{x} = 1 \tag{262}$$

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1 \tag{263}$$

With an orthorgonal matrix S ($S^{-1} = S^T$) we can rotate the ellipsoid to line it up with the coordinate axes (choose S such that $D_A = S^{-1}AS$ is diagonal)

$$\begin{aligned} 1 &= \vec{x}^T A \vec{x} && (264) \\ &= \vec{x}^T (SS^{-1}) A (SS^{-1} \vec{x} && (265) \\ &= (\vec{x}^T S) S^{-1} AS (S^{-1} \vec{x}) && (266) \\ &= (\vec{x}^T S) S^{-1} AS (S^T \vec{x}) && (267) \\ &= (S^T \vec{x})^T S^{-1} AS (S^T \vec{x}) && (268) \\ &= (S^T \vec{x})^T D_A (S^T \vec{x}) && (269) \end{aligned}$$

For this we need to find the eigensystem $\{\vec{v}_i, \lambda_i\}$ of A. The characteristic polynomial is given by

$$\lambda^3 - 3\lambda^2 + \frac{9}{4}\lambda - \frac{1}{2} = 0. \tag{270}$$

Then

$$S = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{271}$$

$$D_A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \tag{272}$$

the length of the principal axes are therefore 4, 1 and 1.

5.4 Ahlfors - Complex Calculus

Chap 1.1

1.

$$(1+2i)^3 = 1 + 3(2i)^2 + 3 \cdot 2i + (2i)^3 \tag{273}$$
$$= 1 - 12 + 6i - 8i \tag{274}$$
$$= -11 - 2i \tag{275}$$

$$\frac{5}{-3+4i} = \frac{5(-3-4i)}{(-3+4i)(-3-4i)} \tag{276}$$
$$= \frac{-15-20i}{25} \tag{277}$$
$$= -\frac{3}{5} - \frac{4}{5}i \tag{278}$$

$$\left(\frac{2+i}{3-2i}\right)^2 = \frac{3+4i}{5-12i} \tag{279}$$
$$= \frac{(3+4i)(5+12i)}{169} \tag{280}$$
$$= \frac{15-48+20i+36i}{169} \tag{281}$$
$$= -\frac{33}{169} + \frac{56}{169}i \tag{282}$$

$$(1+i)^n + (1-i)^n = \sqrt{2}^n \left(e^{i\pi n/4} + e^{-i\pi n/4}\right) \tag{283}$$
$$= 2^{(n+1)/2} \cos \frac{n\pi}{4} \tag{284}$$

2.

$$z^4 = (x+iy)^4 \tag{285}$$
$$= x^4 + 4x^3(iy) + 6x^2(iy)^2 + 4x(iy)^3 + (iy)^4 \tag{286}$$
$$= x^4 - 6x^2y^2 + y^4 + (4x^3y - 4xy^3)i \tag{287}$$

$$1/z = \frac{x-iy}{x^2+y^2} \tag{288}$$
$$= \frac{\bar{z}}{|z|^2} \tag{289}$$

$$\frac{z-1}{z+1} = \frac{(x-1)+iy}{(x+1)+iy} \tag{290}$$
$$= \frac{x^2+y^2-1+2xyi}{(x+1)^2+y^2} \tag{291}$$

5.5 Spivak - Calculus on Manifolds

5.6 Flanders - Differential Forms with Applications to the Physical Sciences

5.7 Morse, Feshbach - Methods of mathematical physics

Problem 1.1

With

$$\cot^2\psi = \frac{\cos^2\psi}{\sin^2\psi} = \frac{\cos^2\psi}{1-\cos^2\psi} \tag{292}$$

we can obtain a quadratic equation

$$(x^2+y^2)\cos^2\psi(1-\cos^2\psi) + z^2\cos^2\psi = a^2(1-\cos^2\psi) \tag{293}$$

$$\cos^4\psi - \frac{x^2+y^2+z^2+a^2}{x^2+y^2}\cos^2\psi + \frac{a^2}{x^2+y^2} = 0 \tag{294}$$

with the solution

$$\cos^2\psi = \frac{x^2+y^2+z^2+a^2}{2(x^2+y^2)} \pm \sqrt{\frac{(x^2+y^2+z^2+a^2)^2}{4(x^2+y^2)^2} - \frac{4a^2(x^2+y^2)}{4(x^2+y^2)^2}} \tag{295}$$

$$= \frac{x^2+y^2+z^2+a^2 \pm \sqrt{(x^2+y^2+z^2+a^2)^2 - 4a^2(x^2+y^2)}}{2(x^2+y^2)} \tag{296}$$

To obtain the gradient we differentiate the surface equation implicitly with respect to x, y and z

$$2x\cos^2\psi - 2(x^2+y^2)\cos\psi\sin\psi\frac{\partial\psi}{\partial x} - 2z^2\cot\psi\csc^2\psi\frac{\partial\psi}{\partial x} = 0 \tag{297}$$

$$\to \frac{\partial\psi}{\partial x} = \psi_x = \frac{x\cos^2\psi}{z^2\cot\psi\csc^2\psi + (x^2+y^2)\sin\psi\cos\psi} \tag{298}$$

$$2y\cos^2\psi - 2(x^2+y^2)\cos\psi\sin\psi\frac{\partial\psi}{\partial x} - 2z^2\cot\psi\csc^2\psi\frac{\partial\psi}{\partial x} = 0 \tag{299}$$

$$\to \frac{\partial\psi}{\partial y} = \psi_y = \frac{y\cos^2\psi}{z^2\cot\psi\csc^2\psi + (x^2+y^2)\sin\psi\cos\psi} \tag{300}$$

$$-2(x^2+y^2)\cos\psi\sin\psi\frac{\partial\psi}{\partial z} + 2z\cot^2\psi - 2z^2\cot\psi\csc^2\psi\frac{\partial\psi}{\partial z} = 0 \tag{301}$$

$$\to \frac{\partial\psi}{\partial z} = \psi_z = \frac{z\cot^2\psi}{z^2\cot\psi\csc^2\psi + (x^2+y^2)\cos\psi\sin\psi} \tag{302}$$

The direction cosines are then given by

$$\cos\alpha = \frac{\psi_x}{\sqrt{\psi_x^2+\psi_y^2+\psi_z^2}} = \frac{2\sqrt{2}x\sin^2\psi}{\sqrt{8z^2+(x^2+y^2)(3-4\cos 2\psi+\cos 4\psi)}} \tag{303}$$

$$\cos\beta = \frac{\psi_y}{\sqrt{\psi_x^2+\psi_y^2+\psi_z^2}} = \frac{2\sqrt{2}y\sin^2\psi}{\sqrt{8z^2+(x^2+y^2)(3-4\cos 2\psi+\cos 4\psi)}} \tag{304}$$

$$\cos\gamma = \frac{\psi_z}{\sqrt{\psi_x^2+\psi_y^2+\psi_z^2}} = \frac{2\sqrt{2}z}{\sqrt{8z^2+(x^2+y^2)(3-4\cos 2\psi+\cos 4\psi)}}. \tag{305}$$

The second derivatives (for the Laplacian) can again be calculated via (lengthy) implicit differentiation and substituting the first derivatives from above. Adding them up gives zero which implies $\triangle\psi = 0$.

The surface equations $\psi = \text{const}$ can be written in form of an ellipsoid

$$\frac{x^2}{a^2 \sec^2 \psi} + \frac{y^2}{a^2 \sec^2 \psi} + \frac{z^2}{a^2 \tan^2 \psi} = 1 \tag{306}$$

which degenerates to a flat pancake for $\psi = 0, \pi$.

Problem 4.1

Standard trick

$$x = \tan \vartheta/2 \to d\theta = \frac{2dx}{1+x^2}, \sin \vartheta = \frac{2x}{1+x^2}, \cos \vartheta = \frac{1-x^2}{1+x^2} \tag{307}$$

$$\int_0^{2\pi} \frac{\sin^2 \vartheta d\vartheta}{a + b \cos \vartheta} = \int_?^? \frac{8x^3 \cdot dx}{(1+x^2)^3 (a + b\frac{1-x^2}{1+x^2})} \tag{308}$$

5.8 Woit - Quantum Theory, Groups and Representations

Problem B.1-3

Rotations of the 2D-plane

$$D^2_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \tag{309}$$

$$D^2_\phi D^2_\theta = \begin{pmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -\cos\phi\sin\theta - \cos\theta\sin\phi \\ \cos\phi\sin\theta + \cos\theta\sin\phi & \cos\theta\cos\phi - \sin\theta\sin\phi \end{pmatrix} \tag{310}$$

$$= \begin{pmatrix} \cos(\phi+\theta) & -\sin(\phi+\theta) \\ \sin(\phi+\theta) & \cos(\phi+\theta) \end{pmatrix} \tag{311}$$

$$= D^2_{\phi+\theta} \tag{312}$$

can also be represented by

$$D^1_\phi = e^{i\phi} \tag{313}$$

$$D^1_\phi D^1_\theta = e^{i\phi} e^{i\theta} = e^{i(\phi+\theta)} \tag{314}$$

$$= D^1_{\phi+\theta}. \tag{315}$$

Furthermore there is also the trivial representation

$$D^{1'}_\phi = 1 \tag{316}$$

$$D^{1'}_\phi D^1_\theta = 1 \cdot 1 = 1 \tag{317}$$

$$= D^{1'}_{\phi+\theta} \tag{318}$$

Problem B.1-4

The time evolution is given by

$$|\Psi(t)\rangle = e^{-iHt} |\Psi(0)\rangle \tag{319}$$

$$= \left(\sum_{k=0}^{\infty} \frac{(-iHt)^k}{k!} \right) |\Psi(0)\rangle \tag{320}$$

We see

$$H = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad H^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \qquad H^3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix} \tag{321}$$

and calculate

$$\sum_{k=0}^{\infty} \frac{(-it)^{2k}}{(2k)!} = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{(2k)!} = \cos(t) \tag{322}$$

$$\sum_{k=0}^{\infty} \frac{(-it)^{2k+1}}{(2k+1)!} = (-i) \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k+1}}{(2k+1)!} = -i\sin(t) \tag{323}$$

$$\sum_{k=0}^{\infty} \frac{(-i2t)^k}{k!} = \cos(2t) - i\sin(2t) = e^{-i2t} \tag{324}$$

which gives

$$e^{-iHt} = \begin{pmatrix} \cos(t) & -i\sin(t) & 0 \\ -i\sin(t) & \cos(t) & 0 \\ 0 & 0 & e^{-2it} \end{pmatrix} \tag{325}$$

and therefore

$$|\Psi(t)\rangle = \begin{pmatrix} \psi_1 \cos(t) - \psi_2 i \sin(t) \\ -\psi_1 i \sin(t) + \psi_2 \cos(t) \\ \psi_3 e^{-2it} \end{pmatrix} \tag{326}$$

. To check the result one can calculate both sides of $i\partial_t|\Psi(t)\rangle = H|\Psi(t)\rangle$.

Problem B.2-1

1. With $M = PDP^{-1}$ we have $M^2 = PDP^{-1}PDP^{-1} = PDDP^{-1}$ and see

$$e^{tM} = \sum_{k=0}^{\infty} \frac{(tM)^k}{k!} = \sum_{k=0}^{\infty} \frac{(tPDP^{-1})^k}{k!} = \sum_{k=0}^{\infty} \frac{P(tD)^k P^{-1}}{k!} \tag{327}$$

$$= P\left(\sum_{k=0}^{\infty} \frac{(tD)^k}{k!}\right) P^{-1} = Pe^{tD}P^{-1}. \tag{328}$$

The eigenvalues of M are given by

$$-\lambda^3 - (-\lambda)(-\pi^2) = 0 \quad \to \quad \lambda_1 = i\pi,\ \lambda_2 = -i\pi,\ \lambda_3 = 0 \tag{329}$$

with the eigenvectors

$$\vec{v}_1 = (-i, 1, 0) \tag{330}$$

$$\vec{v}_2 = (i, 1, 0) \tag{331}$$

$$\vec{v}_3 = (0, 0, 1) \tag{332}$$

we obtain

$$M = PDP^{-1} \tag{333}$$

$$= \begin{pmatrix} -i & i & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} i\pi & 0 & 0 \\ 0 & -i\pi & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} i/2 & 1/2 & 0 \\ -i/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{334}$$

With

$$\sum_{k=0}^{\infty} \frac{(i\pi)^k}{k!} = e^{i\pi} \tag{335}$$

$$\sum_{k=0}^{\infty} \frac{(-i\pi)^k}{k!} = e^{-i\pi} \tag{336}$$

we see

$$tD^k = \begin{pmatrix} (i\pi t)^k & 0 & 0 \\ 0 & (-i\pi t)^k & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{337}$$

$$e^{tD} = \sum_{k=0}^{\infty} \frac{(tD)^k}{k!} = \begin{pmatrix} e^{i\pi t} & 0 & 0 \\ 0 & e^{-i\pi t} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{338}$$

and therefore

$$\begin{aligned} e^{tM} &= Pe^{tD}P^{-1} && (339) \\ &= \begin{pmatrix} \frac{1}{2}(e^{-i\pi t} + e^{i\pi t}) & -\frac{1}{2}i(e^{i\pi t} - e^{-i\pi t}) & 0 \\ -\frac{1}{2}i(e^{-i\pi t} - e^{i\pi t}) & \frac{1}{2}(e^{-i\pi t} + e^{i\pi t}) & 0 \\ 0 & 0 & 1 \end{pmatrix} && (340) \\ &= \begin{pmatrix} \cos(\pi t) & \sin(\pi t) & 0 \\ -\sin(\pi t) & \cos(\pi t) & 0 \\ 0 & 0 & 1 \end{pmatrix} && (341) \end{aligned}$$

2. Brute force calculation of the matrix powers reveals

$$(tM)^2 = \begin{pmatrix} -(t\pi)^2 & 0 & 0 \\ 0 & -(t\pi)^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (tM)^3 = \begin{pmatrix} 0 & -(t\pi)^3 & 0 \\ (t\pi)^3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{342}$$

$$(tM)^4 = \begin{pmatrix} (t\pi)^4 & 0 & 0 \\ 0 & (t\pi)^4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (tM)^5 \begin{pmatrix} 0 & (t\pi)^5 & 0 \\ -(t\pi)^5 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{343}$$

With

$$\begin{aligned} 1 - \frac{1}{2!}(\pi t)^2 + \frac{1}{4!}(\pi t)^4 + ... &= \cos(\pi t) && (344) \\ \pi t - \frac{1}{3!}(\pi t)^3 + \frac{1}{5!}(\pi t)^5 + ... &= \sin(\pi t) && (345) \\ -\pi t + \frac{1}{3!}(\pi t)^3 - \frac{1}{5!}(\pi t)^5 + ... &= (-\pi t) + \frac{1}{3!}(-\pi t)^3 - \frac{1}{5!}(-\pi t)^5 + ... && (346) \\ &= \sin(-\pi t) && (347) \\ &= -\sin(\pi t) && (348) \end{aligned}$$

we obtain

$$e^{tM} = \begin{pmatrix} \cos(\pi t) & \sin(\pi t) & 0 \\ -\sin(\pi t) & \cos(\pi t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{349}$$

Problem B.2-2

For the Hamiltonian

$$H = -B_x \sigma_1 = \begin{pmatrix} 0 & -B_x \\ -B_x & 0 \end{pmatrix} \tag{350}$$

we find the eigensystem

$$E_1 = -B_x \quad |\psi_1\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{351}$$

$$E_2 = +B_x \quad |\psi_2\rangle = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{352}$$

The Hamiltonian (with full units) is given by

$$H = -g\frac{q\hbar}{2m}\frac{\sigma_1}{2}B_x \tag{353}$$

which translates into energies of

$$E_1 = -g\frac{q\hbar}{4m}B_x \tag{354}$$

$$E_2 = g\frac{q\hbar}{4m}B_x. \tag{355}$$

The time evolution is them given by

$$\begin{aligned}
|\psi(t)\rangle &= e^{-\frac{i}{\hbar}Ht}|\psi(0)\rangle &(356)\\
&= e^{-i\frac{gq}{4m}\sigma_1 t}|\psi(0)\rangle &(357)\\
&= \left[\cos\left(\frac{gq}{4m}\sigma_1 t\right) - i\sin\left(\frac{gq}{4m}\sigma_1 t\right)\right]|\psi(0)\rangle &(358)\\
&= \left[\cos\left(\frac{gq}{4m}t\right)\mathbb{I}_2 - i\sin\left(\frac{gq}{4m}t\right)\sigma_1\right]|\psi(0)\rangle &(359)\\
&= \begin{pmatrix} \cos\left(\frac{gqt}{4m}\right) & -i\sin\left(\frac{gqt}{4m}\right) \\ -i\sin\left(\frac{gqt}{4m}\right) & \cos\left(\frac{gqt}{4m}\right) \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} &(360)\\
&= \begin{pmatrix} \cos\left(\frac{gqt}{4m}\right) \\ -i\sin\left(\frac{gqt}{4m}\right) \end{pmatrix} &(361)
\end{aligned}$$

where we used $\sigma_1^{2n} = \mathbb{I}^n = \mathbb{I}$.

5.9 Baez, Muniain - Gauge Fields, Knots and Gravity

Problem I.1 - Plane waves in vacuum

With

$$\vec{\mathcal{E}} = \vec{E}e^{-i(\omega t - \vec{k}\vec{x})} \tag{362}$$

we calculate in cartesian coordinates

1. $\nabla\cdot\vec{\mathcal{E}} = 0$

$$\begin{aligned}
\nabla\cdot\vec{\mathcal{E}} &= \partial_a \mathcal{E}_a &(363)\\
&= \partial_a(e^{-i(\omega t - \vec{k}\vec{x})})E_a\vec{e}^a &(364)\\
&= \delta_{ab} i k_b E_a e^{-i(\omega t - \vec{k}\vec{x})}\vec{e}^a &(365)\\
&= i k_b E_b e^{-i(\omega t - \vec{k}\vec{x})}\vec{e}^a &(366)\\
&= 0 &(367)
\end{aligned}$$

   where we assumed $E_a = \text{const}$ and used

$$\begin{aligned}
0 &= \vec{k}\cdot\vec{E} &(368)\\
&= k_a\vec{e}^a E_a\vec{e}^a &(369)\\
&= k_a E_a &(370)
\end{aligned}$$

2. $\nabla \times \vec{\mathcal{E}} = i\frac{\partial \vec{\mathcal{E}}}{\partial t}$

$$\begin{aligned}
\nabla \times \vec{\mathcal{E}} &= \epsilon_{abc}\partial_b \mathcal{E}_c \vec{e}_a && (371)\\
&= \epsilon_{abc} E_c \vec{e}_a \partial_b (e^{-i(\omega t - \vec{k}\vec{x})}) && (372)\\
&= \epsilon_{abc} E_c \vec{e}_a \delta_{bd} i k_d e^{-i(\omega t - \vec{k}\vec{x})} && (373)\\
&= i(\epsilon_{abc} k_b E_c \vec{e}_a) e^{-i(\omega t - \vec{k}\vec{x})} && (374)\\
&= i(-i\omega E_a \vec{e}^a) e^{-i(\omega t - \vec{k}\vec{x})} && (375)\\
&= i(E_a \vec{e}^a)(-i\omega) e^{-i(\omega t - \vec{k}\vec{x})} && (376)\\
&= i\vec{E}\frac{\partial}{\partial t} e^{-i(\omega t - \vec{k}\vec{x})} && (377)\\
&= i\frac{\partial \vec{\mathcal{E}}}{\partial t} && (378)
\end{aligned}$$

where we used (typo in the book!)

$$\begin{aligned}
-i\omega \vec{E} &= \vec{k} \times \vec{E} && (379)\\
&= \epsilon_{abc} k_b E_c \vec{e}_a && (380)
\end{aligned}$$

Problem I.7 - Adding and multiplying vector fields

1. With $(v+w)f \equiv (f) + w(f)$
   (a) $(v+w)(f+g) = v(f+g) + w(f+g) = vf + vg + wf + wg = (v+w)f + (v+w)g$
   (b) $(v+w)(\alpha f) = v(\alpha f) + w(\alpha f) = \alpha v f + \alpha w f = \alpha(v+w)f$
   (c) $(v+w)(fg) = v(fg) + w(fg) = v(f)g + fv(g) + w(f)g + fw(g) = [(v+w)f]g + f[(v+w)g]$
2. With $(gv)(f) \equiv gv(f)$
   (a) $(gv)(f+h) = gv(f+h) = gv(f) + gv(h) = g(v(f) + v(h)) = gv(f) + gv(h)$
   (b) $gv(\alpha f) = gv(\alpha f) = g\alpha v(f) = \alpha g v(f)$
   (c) $(gv)(fh) = gv(fh) = g(v(f)h + fv(h)) = (gv)(f)h + f(gv)(h)$

6 Many-body physics

6.1 Coleman - Introduction to Many-Body Physics

Problem 2.1 - Specific heat capacity of a solid

Using the Boltzmann statistics and $E_n = \hbar\omega(n + \frac{1}{2})$ the energy E of a system of N_{AV} harmonic oscillators (in 3d!!) is given by

$$\begin{aligned}
E &= 3N_{AV}\frac{\sum_n E_n e^{-\frac{E_n}{k_B T}}}{\sum_n e^{-\frac{E_n}{k_B T}}} && (381)\\
&= 3N_{AV}\frac{\sum_n \left(\hbar\omega\left[n + \frac{1}{2}\right]\right) e^{-\frac{n\hbar\omega}{k_B T}} e^{-\frac{\hbar\omega}{2k_B T}}}{\sum_n e^{-\frac{n\hbar\omega}{k_B T}} e^{-\frac{\hbar\omega}{2k_B T}}} && (382)\\
&= 3N_{AV}\hbar\omega\left(\frac{1}{2} + \frac{\sum_n n e^{-\frac{n\hbar\omega}{k_B T}}}{\sum_n e^{-\frac{n\hbar\omega}{k_B T}}}\right) && (383)\\
&= 3N_{AV}\hbar\omega\left(\frac{1}{2} + \frac{1}{e^{\frac{\hbar\omega}{k_B T}} - 1}\right) && (384)
\end{aligned}$$

where we used the sum formulas

$$s_1 = \sum_{n=0} q^n = \frac{1}{1-q} \tag{385}$$

$$s_2 = \sum_{n=0} nq^n = q\frac{ds_1}{dq} = \frac{q}{(1-q)^2} \tag{386}$$

the specific heat can the be calculated as

$$C_V = \frac{dE}{dT} \tag{387}$$

$$= 3N_{AV}\hbar\omega \frac{\exp\left[\frac{\hbar\omega}{kT}\right]\frac{\hbar\omega}{kT^2}}{\left[\exp\left[\frac{\hbar\omega}{kT}\right]-1\right]^2} \tag{388}$$

$$= 3N_{AV}k\frac{x^2\exp(x)}{[\exp(x)-1]^2} \tag{389}$$

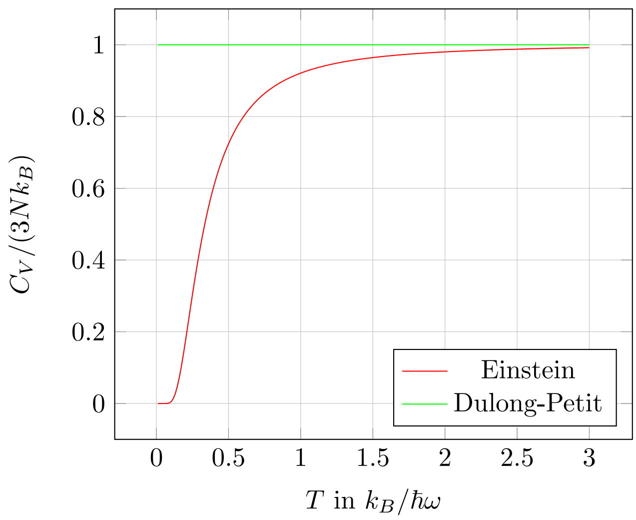
$$= 3N_{AV}k\frac{x^2}{[\exp(x/2)-\exp(-x/2)]^2} \tag{390}$$

$$= 3N_{AV}k\frac{x^2}{[\exp(x/2)-\exp(-x/2)]^2} \tag{391}$$

$$= 3N_{AV}k\left(\frac{x/2}{\sinh(x/2)}\right)^2 \tag{392}$$

$$\tag{393}$$

The Dulong-Petit rule says $k/2$ per harmonic degree of freedom which means in 3d that $C_V/N = 3k$ (for each harmonic degree there is also a kinetic one - so $f = 6$)



7 Quantum Field Theory

7.1 Lancaster, Blundell - Quantum Field Theory for the gifted amateur

Problem 1.1 - Snell's law via Fermat's principle

The light travels from point A in medium 1 to point B in medium 2. We assume a vertical medium boundary at x_0 and that the light travels within a medium in the straight line. This makes y_0 the free parameter and the the travel time is given by

$$t = \frac{s_{A0}}{c/n_1} + \frac{s_{0B}}{c/n_2} \tag{394}$$

$$= \sqrt{\frac{(x_A - x_0)^2 + (y_A - y_0)^2}{c/n_1}} + \sqrt{\frac{(x_0 - x_B)^2 + (y_0 - y_B)^2}{c/n_2}} \tag{395}$$

The local extrema of the travel time is given by

$$0 = \frac{dt}{dy_0} \tag{396}$$

$$= \frac{y_A - y_0}{s_{A0} c/n_1} + \frac{y_0 - y_B}{s_{0B} c/n_2} \tag{397}$$

$$= \frac{\sin\alpha}{c/n_1} - \frac{\sin\beta}{c/n_2} \tag{398}$$

and therefore

$$n_1 \sin\alpha = n_2 \sin\beta. \tag{399}$$

7.2 Muenster - Von der Quantenfeldtheorie zum Standardmodell

Problem 2.1 - 1

The Klein-Gordon equations is given by

$$\left(\partial_\mu \partial^\mu + \frac{m^2 c^2}{\hbar^2}\right) \varphi = 0 \tag{400}$$

$$\left(c^2 \partial_{tt} - \triangle + \frac{m^2 c^2}{\hbar^2}\right) \varphi = 0 \tag{401}$$

We make the ansatz

$$\varphi = \phi_1 + \phi_2 \tag{402}$$

$$\phi_1 = \frac{1}{2}\varphi - \alpha \partial_t \varphi \tag{403}$$

$$\phi_2 = \frac{1}{2}\varphi + \alpha \partial_t \varphi \tag{404}$$

Then we get expressions for the time derivatives

$$\phi_2 - \phi_1 = 2\alpha \partial_t \varphi \tag{405}$$

$$\to \partial_t \varphi = \frac{1}{2\alpha}(\phi_2 - \phi_1) \tag{406}$$

and

$$\partial_{tt}\varphi = c^2 \left(\triangle - \frac{m^2 c^2}{\hbar^2}\right) \varphi \tag{407}$$

$$= c^2 \left(\triangle - \frac{m^2 c^2}{\hbar^2}\right) (\phi_1 + \phi_2) \tag{408}$$

Therefore we get for $\phi_{1,2}$

$$\partial_t \phi_1 = \frac{1}{2}\partial_t \varphi - \alpha \partial_{tt}\varphi \tag{409}$$

$$= \frac{1}{2\alpha}(\phi_2 - \phi_1) - \alpha c^2 \left(\triangle - \frac{m^2 c^2}{\hbar^2}\right)(\phi_1 + \phi_2) \tag{410}$$

$$\partial_t \phi_2 = \frac{1}{2}\partial_t \varphi + \alpha \partial_{tt}\varphi \tag{411}$$

$$= \frac{1}{2\alpha}(\phi_2 - \phi_1) + \alpha c^2 \left(\triangle - \frac{m^2 c^2}{\hbar^2}\right)(\phi_1 + \phi_2) \tag{412}$$

which we can write in the form

$$i\hbar \partial_t \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = -i\hbar \alpha c^2 \left(\triangle - \frac{m^2 c^2}{\hbar^2}\right) \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} + \frac{i\hbar}{2\alpha} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{413}$$

$$= i\hbar \begin{pmatrix} -\alpha c^2 \left(\triangle - \frac{m^2c^2}{\hbar^2}\right) - \frac{1}{2\alpha} & -\alpha c^2 \left(\triangle - \frac{m^2c^2}{\hbar^2}\right) + \frac{1}{2\alpha} \\ \alpha c^2 \left(\triangle - \frac{m^2c^2}{\hbar^2}\right) - \frac{1}{2\alpha} & \alpha c^2 \left(\triangle - \frac{m^2c^2}{\hbar^2}\right) + \frac{1}{2\alpha} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{414}$$

Diagonalization gives

$$i\hbar \partial_t \phi = \hat{H}\phi \tag{415}$$

$$\to i\hbar \partial_t S^{-1}\phi = \underbrace{S^{-1}\hat{H}S}_{=h} S^{-1}\phi \tag{416}$$

$$\lambda_\pm = \sqrt{2} mc^2 \sqrt{1 - \frac{\hbar^2}{m^2c^2}\triangle} \tag{417}$$

A semi-canonical choice for the parameter α is to make the $\triangle$ look like a momentum operator

$$\imath\hbar\alpha c^2 = -\frac{\hbar^2}{2m} \quad \to \quad \alpha = \frac{i\hbar}{2mc^2} \tag{418}$$

7.3 Schwartz - Quantum Field Theory and the Standard Model

Problem 2.2 Special relativity and colliders

1. Quick special relativity recap

$$p'^\mu = \Lambda^\mu_\nu p^\nu \quad p^\mu p_\mu = m^2c^2 \tag{419}$$

At rest

$$p^\mu p_\mu = (p^0)^2 - \vec{p}^2 = (p^0)^2 = m^2c^2 \tag{420}$$

After Lorentz trafo in x direction

$$\Lambda = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{421}$$

$$p'^\mu = (\gamma p^0, -\beta\gamma p^0, 0, 0) \tag{422}$$

$$\equiv \left(\frac{E}{c}, \vec{p}\right) \tag{423}$$

with $p^\mu p_\mu = m^2c^2$ we have $E^2/c^2 + \vec{p}^2 = m^2c^2$.

Now we can solve the problem

$$\frac{E_{cm}}{2} = \sqrt{m_p^2 c^4 + p^2 c^2} \tag{424}$$

$$\rightarrow p = \frac{1}{c}\sqrt{\frac{E_{cm}^2}{4} - m_p^2 c^4} \equiv \beta\gamma m_p c \tag{425}$$

$$\rightarrow \frac{E_{cm}^2}{4} = m_p^2 c^4(\beta^2\gamma^2 + 1) \tag{426}$$

$$\rightarrow \gamma = \frac{E_{cm}}{2m_p c^2} \tag{427}$$

$$\rightarrow \beta = \sqrt{1 - \left(\frac{2m_p c}{E_{cm}}\right)^2} \approx 1 - \frac{1}{2}\left(\frac{2m_p c^2}{E_{cm}}\right)^2 \tag{428}$$

$$\rightarrow c - v = 2\left(\frac{m_p c^2}{E_{cm}}\right)^2 c = 2.69\text{m/s} \tag{429}$$

2. Using the velocity addition formula

$$\Delta v = \frac{2v}{1 + \frac{v^2}{c^2}} \approx c\left(1 - 2\left[\frac{m_p c^2}{E_{cm}}\right]^4\right) \tag{430}$$

Problem 2.3 GZK bound

1. We are utilizing Plancks law

$$w_\nu d\nu = \frac{8\pi h\nu^3}{c^3}\frac{d\nu}{e^{h\nu/k_B T} - 1} \tag{431}$$

where the spectral energy density w_ν [J m^{-3} s] gives the spacial energy density per frequency interval $d\nu$. The total radiative energy density is then given by

$$\rho_{\text{rad}} = \frac{8\pi h}{c^3}\int_0 \infty \frac{\nu^3 d\nu}{e^{h\nu/k_B T} - 1} \tag{432}$$

$$= \frac{8\pi h}{c^3}\cdot\frac{(\pi k_B T)^4}{15h^4} \tag{433}$$

$$= \frac{8\pi^5 k_B^4 T^4}{15h^3c^3} = 0.26\text{MeV/m}^3. \tag{434}$$

The photon density is given by

$$n_{\text{rad}} = \int_0^\infty \frac{w_\nu}{h\nu}d\nu \tag{435}$$

$$= \frac{8\pi}{c^3}\int\frac{\nu^2 d\nu}{e^{h\nu/k_B T} - 1} \tag{436}$$

$$= \frac{8\pi}{c^3}\cdot\frac{2\zeta(3)k_B^3 T^3}{h^3} \tag{437}$$

$$= \frac{16\pi\zeta(3)k_B^3 T^3}{h^3c^3} = 416\text{cm}^{-3}. \tag{438}$$

The average photon energy is then given by

$$E_{\text{ph}} = \frac{\rho_{\text{rad}}}{n_{\text{rad}}} = \frac{\pi^4}{30\zeta(3)}k_B T = 0.63\text{meV} \tag{439}$$

$$\lambda_{\text{ph}} = \frac{hc}{E_{\text{ph}}} = 1.9\text{mm} \tag{440}$$

therefore it is called CM(icrowave)B. One obtains slightly other values if the peak of the Planck spectrum is used as definition of the average photon energy.

2. In the center-of-mass system the total momentum before and after the collision vanishes

$$\vec{p}^{cm}_{p^+} + \vec{p}^{cm}_{\gamma} = 0 = \vec{\hat{p}}^{cm}_{p^+} + \vec{\hat{p}}^{cm}_{\pi^0}. \tag{441}$$

which implies for (Lorentz-invariant) norm the systems 4-momentum $P^{cm} = p^{cm}_{p^+} + p^{cm}_{\pi^0}$

$$\begin{aligned}
(P^{cm})^2 &= (E^{cm}_{p^+} + E^{cm}_{\gamma})^2 - c^2(\vec{p}^{cm}_{p^+} + \vec{p}^{cm}_{\gamma})^2 && (442)\\
&= (E^{cm}_{p^+} + E^{cm}_{\gamma})^2 && (443)\\
&= (E^{cm})^2 && (444)\\
&\stackrel{!}{=} (E_{p^+} + E_{\gamma})^2 - c^2(\vec{p}_{p^+} + \vec{p}_{\gamma})^2 && (445)\\
&\stackrel{!}{=} (\hat{E}_{p^+} + \hat{E}_{\pi^0})^2 - c^2(\vec{\hat{p}}_{p^+} + \vec{\hat{p}}_{\pi^0})^2 && (446)
\end{aligned}$$

with $p^i = \hbar k^i = \hbar(\omega, \vec{k}) = \hbar(\omega, \frac{2\pi}{\lambda}\vec{e}_k) = h(\nu, \frac{\nu}{c}\vec{e}_k)$ and the values before

$$\begin{aligned}
E_{p^+} &= m_{p^+}c^2 + T_{p^+} && (447)\\
E_{\gamma} &= h\nu && (448)\\
(\vec{p}_{p^+})^2 &= \frac{1}{c^2}\left[(E_{p^+})^2 - (m_{p^+})^2c^4\right] && (449)\\
&= \frac{T_{p^+}}{c^2}\left[T_{p^+} + 2m_{p^+}c^2\right] && (450)\\
(\vec{p}_{\gamma})^2 &= \frac{h^2\nu^2}{c^2} && (451)
\end{aligned}$$

At the threshold the π^0 is created without any kinetic energy. As the total momentum is vanishing the proton also needs to be at rest

$$\begin{aligned}
(E_{p^+} + E_{\gamma})^2 - c^2(\vec{p}_{p^+} + \vec{p}_{\gamma})^2 &= \left(m_{p^+}c^2 + m_{\pi^0}c^2\right)^2 && (452)\\
E^2_{p^+} + 2E_{p^+}E_{\gamma} + E^2_{\gamma} - c^2\left(\vec{p}^2_{p^+} + \vec{p}^2_{\gamma} - 2\vec{p}_{p^+}\cdot\vec{p}_{\gamma}\right) &= \left(m_{p^+}c^2 + m_{\pi^0}c^2\right)^2 && (453)\\
m^2_{p^+}c^4 + 2E_{p^+}E_{\gamma} + 2c^2\vec{p}_{p^+}\cdot\vec{p}_{\gamma} &= \left(m_{p^+}c^2 + m_{\pi^0}c^2\right)^2 && (454)\\
m^2_{p^+}c^4 + 2E_{p^+}E_{\gamma} + 2E_{\gamma}\sqrt{E^2_{p^+} - m^2_{p^+}c^2}\cos\phi &= \left(m_{p^+}c^2 + m_{\pi^0}c^2\right)^2 && (455)\\
E_{p^+}E_{\gamma} + E_{\gamma}\sqrt{E^2_{p^+} - m^2_{p^+}c^2}\cos\phi &= \left(m_{p^+} + \frac{m_{\pi^0}}{2}\right)m_{\pi^0}c^4 && (456)
\end{aligned}$$

Now we can square the equation and solve approximately assuming $E_{\gamma} \ll m_{p^+}c^2$

$$\begin{aligned}
E_{\gamma}\sqrt{E^2_{p^+} - m^2_{p^+}c^2}\cos\phi &= \left(m_{p^+} + \frac{m_{\pi^0}}{2}\right)m_{\pi^0}c^4 - E_{p^+}E_{\gamma} && (457)\\
E^2_{\gamma}\left(E^2_{p^+} - m^2_{p^+}c^2\right)\cos^2\phi &= \left(m_{p^+} + \frac{m_{\pi^0}}{2}\right)^2 m^2_{\pi^0}c^8 + (E_{p^+}E_{\gamma})^2 - 2E_{p^+}E_{\gamma}\left(m_{p^+} + \frac{m_{\pi^0}}{2}\right)m_{\pi^0}c^4 && (458)\\
-E^2_{\gamma}m^2_{p^+}c^2\cos^2\phi &= \left(m_{p^+} + \frac{m_{\pi^0}}{2}\right)^2 m^2_{\pi^0}c^8 - 2E_{p^+}E_{\gamma}\left(m_{p^+} + \frac{m_{\pi^0}}{2}\right)m_{\pi^0}c^4 && (459)\\
E_{p^+} &\approx \frac{\left(m_{p^+} + m_{\pi^0}/2\right)m_{\pi^0}c^4}{2E_{\gamma}} && (460)\\
&= 10.8\cdot 10^{19}\text{eV} && (461)
\end{aligned}$$

3. By assumption the p^+ and the π^0 would rest in the CM system

$$
\begin{aligned}
(P^\mu)^{cm} &= (p^\mu_{p^+})^{cm} + (p^\mu_{\pi^0})^{cm} && (462)\\
&= \left([m_{p^+} + m_{\pi^0}]c^2, \vec{0}\right) && (463)\\
&= \Lambda^\mu_\alpha \left[\hat{p}^\alpha_{p^+} + \hat{p}^\alpha_{\pi^0}\right] && (464)\\
&= \Lambda^\mu_\alpha \left[p^\alpha_{p^+} + p^\alpha_\gamma\right] && (465)\\
& && (466)
\end{aligned}
$$

We can therefore calculate γ

$$
\begin{aligned}
\mu = 1: \quad 0 &= \underbrace{\Lambda^1_0}_{-\gamma\beta}(E_{p^+} + E_\gamma) + \underbrace{\Lambda^1_1}_{\gamma} c(p^x_{p^+} + p^x_\gamma) && (467)\\
&= -\gamma\beta(E_{p^+} + E_\gamma) + \gamma\left(\sqrt{E^2_{p^+} - m_p^2 c^4} + E_\gamma\right) && (468)\\
\rightarrow \beta &= \frac{\sqrt{E^2_{p^+} - m_p^2 c^4} + E_\gamma}{E_{p^+} + E_\gamma} \approx \frac{\sqrt{E^2_{p^+} - m_p^2 c^4}}{E_{p^+}} && (469)\\
\rightarrow \gamma &= \frac{1}{\sqrt{1-\beta^2}} = \frac{E_{p^+}}{m_{p^+}c^2} && (470)
\end{aligned}
$$

which can be used to calculate the pion momentum

$$
\begin{aligned}
c\hat{p}_{\pi^0} &= \Lambda^0_\mu (p^\mu_{\pi^0})^{cm} && (471)\\
&= \Lambda^0_0 (p^0_{\pi^0})^{cm} && (472)\\
&= \gamma m_{\pi^0} c^2 && (473)\\
&= E_{p^+}\frac{m_{\pi^0}}{m_{p^+}}. && (474)
\end{aligned}
$$

The $p+$ energy after the collision is then given by

$$
\begin{aligned}
E_{p^+} + E_\gamma &= \hat{E}_{p^+} + \hat{E}_{\pi^0} && (475)\\
\rightarrow \hat{E}_{p^+} &= E_{p^+} + E_\gamma - \hat{E}_{\pi^0} && (476)\\
&= E_{p^+} + E_\gamma - \sqrt{m^2_{\pi^0}c^4 + \hat{\vec{p}}_{\pi^0}c^2} && (477)\\
&= E_{p^+} + E_\gamma - \sqrt{m^2_{\pi^0}c^4 + E^2_{p^+}\frac{m^2_{\pi^0}}{m^2_{p^+}}} && (478)\\
&= E_{p^+} + E_\gamma - m_{\pi^0}c^2\sqrt{1 + \frac{E^2_{p^+}}{m^2_{p^+}c^4}} && (479)\\
&\approx E_{p^+} - m_{\pi^0}c^2\frac{E_{p^+}}{m_{p^+}c^2} && (480)\\
&= E_{p^+}\left(1 - \frac{m_{\pi^0}}{m_{p^+}}\right) && (481)\\
&\approx 0.85 \cdot E_{p^+}. && (482)
\end{aligned}
$$

Problem 2.5 Compton scattering

1. the binding energy of outer(!!!) electrons is in the eV range while typical X-rays energies are in the keV range.

2. In the nonrelativistic case we have energy and momentum conservation

$$\frac{hc}{\lambda} = \frac{hc}{\lambda'} + \frac{1}{2} m_e v^2 \tag{483}$$

$$\frac{h}{\lambda} = \frac{h}{\lambda'} \cos\theta + m_e v \cos\phi \tag{484}$$

$$0 = \frac{h}{\lambda'} \sin\theta + m_e v \sin\phi \tag{485}$$

then we see

$$v = \sqrt{\frac{2hc}{m_e}\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)} = \sqrt{\frac{2hc}{m_e}\frac{\lambda' - \lambda}{\lambda\lambda'}} \tag{486}$$

and

$$\sin\phi = -\frac{h}{m_e v}\frac{1}{\lambda'}\sin\theta \tag{487}$$

$$\cos\phi = \frac{h}{m_e v}\frac{1}{\lambda'}\left(\frac{\lambda'}{\lambda} - \cos\theta\right) \tag{488}$$

$$\to 1 = \sin^2\phi + \cos^2\phi \tag{489}$$

$$= \frac{h^2}{m_e^2 v^2 \lambda'^2}\left(\sin^2\theta + \frac{\lambda'^2}{\lambda^2} - 2\frac{\lambda'}{\lambda}\cos\theta + \cos^2\theta\right) \tag{490}$$

$$= \frac{h^2}{m_e^2 v^2 \lambda'^2}\left(1 + \frac{\lambda'^2}{\lambda^2} - 2\frac{\lambda'}{\lambda}\cos\theta\right) \tag{491}$$

$$= \frac{h\lambda}{2m_e c\lambda'(\lambda' - \lambda)}\left(1 + \frac{\lambda'^2}{\lambda^2} - 2\frac{\lambda'}{\lambda}\cos\theta\right) \tag{492}$$

$$= \frac{h}{2m_e c(\lambda' - \lambda)}\left(\frac{\lambda}{\lambda'} + \frac{\lambda'}{\lambda} - 2\cos\theta\right) \tag{493}$$

$$\lambda' - \lambda \approx \frac{h}{m_e c}(1 - \cos\theta) \tag{494}$$

where we used $\lambda \approx \lambda'$.

3.

Problem 2.6 Lorentz invariance

1. With $\omega_k = \sqrt{\vec{k}^2 + m^2}$

$$\int_{-\infty}^{\infty} dk^0 \delta(k^2 - m^2)\theta(k^0) = \int_{-\infty}^{\infty} dk^0 \delta({k^0}^2 - [\vec{k}^2 + m^2])\theta(k^0) \tag{495}$$

$$= \frac{\theta(\omega_k)}{2\omega_k} + \frac{\theta(-\omega_k)}{2\omega_k} \tag{496}$$

$$= \frac{1}{2\omega_k} \tag{497}$$

2. Under Lorentz transformations we have $k^2 - m^2 = 0$. For orthochronous transformation we have k^0...

3. Now we can put it all together

$$\int d^4k \delta(k^2 - m^2)\theta(k^0) = \int d^3k \int dk^0 \delta(k^2 - m^2)\theta(k^0) \tag{498}$$

$$= \int \frac{d^3k}{2\omega_k} \tag{499}$$

Problem 2.7 Coherent states

1.

$$\partial_z\left(e^{-za^\dagger}ae^{-za^\dagger}\right) = -e^{-za^\dagger}a^\dagger ae^{-za^\dagger} + e^{-za^\dagger}aa^\dagger e^{-za^\dagger} \tag{500}$$
$$= e^{-za^\dagger}[a, a^\dagger]e^{-za^\dagger} \tag{501}$$
$$= 1 \tag{502}$$

2. Rolling the a through the $(a^\dagger)^k$ using the commutator $[a, a^\dagger] = 1$

$$a|z\rangle = ae^{za^\dagger}|0\rangle \tag{503}$$
$$= a\sum_{k=0}\frac{1}{k!}z^k(a^\dagger)^k|0\rangle \tag{504}$$
$$= a|0\rangle + \sum_{k=1}\frac{k}{k!}z^k(a^\dagger)^{k-1}|0\rangle \tag{505}$$
$$= z\sum_{n=0}\frac{1}{n!}z^n(a^\dagger)^n|0\rangle \tag{506}$$
$$= z|z\rangle \tag{507}$$

3. With $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ and using the $|z\rangle$ is an eigenstate of a we have

$$\langle n|z\rangle = \frac{1}{\sqrt{n!}}\langle 0|a^n|z\rangle = \frac{z^n}{\sqrt{n!}}\langle 0|z\rangle = \frac{z^n}{\sqrt{n!}}\langle 0|e^{za^\dagger}|0\rangle \tag{508}$$
$$= \frac{z^n}{\sqrt{n!}}\langle 0|1 + za^\dagger + \frac{1}{2}z^2(a^\dagger)^2 + ...|0\rangle \tag{509}$$
$$= \frac{z^n}{\sqrt{n!}}\langle 0|0\rangle = \frac{z^n}{\sqrt{n!}} \tag{510}$$

where we used $\langle 0|a^\dagger = 0$.

4. With

$$a + a^\dagger = \sqrt{\frac{m\omega}{2}}2q \quad \to \quad q = \frac{1}{\sqrt{2m\omega}}(a + a^\dagger) \tag{511}$$
$$a - a^\dagger = \sqrt{\frac{m\omega}{2}}2\frac{ip}{m\omega} \quad \to \quad p = -i\frac{\sqrt{m\omega}}{\sqrt{2}}(a - a^\dagger) \tag{512}$$

and $a|z\rangle = z|z\rangle$ and $\langle z|a^\dagger = \bar{z}\langle z|$

$$\langle z|q|z\rangle = \frac{1}{\sqrt{2m\omega}}\langle z|a + a^\dagger|z\rangle = \frac{1}{\sqrt{2m\omega}}\langle z|z\rangle(z + \bar{z}) \tag{513}$$
$$\langle z|p|z\rangle = -i\frac{\sqrt{m\omega}}{\sqrt{2}}\langle z|a - a^\dagger|z\rangle = -i\frac{\sqrt{m\omega}}{\sqrt{2}}\langle z|z\rangle(z - \bar{z}) \tag{514}$$
$$\langle z|q^2|z\rangle = \frac{1}{2m\omega}\langle z|aa + \underbrace{aa^\dagger}_{=1+a^\dagger a} + a^\dagger a + a^\dagger a^\dagger|z\rangle \tag{515}$$
$$= \frac{1}{2m\omega}\langle z|z\rangle\left(z^2 + 1 + 2z\bar{z} + \bar{z}^2\right) \tag{516}$$
$$\langle z|p^2|z\rangle = -\frac{m\omega}{2}\langle z|aa - \underbrace{aa^\dagger}_{=1+a^\dagger a} - a^\dagger a + a^\dagger a^\dagger|z\rangle \tag{517}$$
$$= -\frac{m\omega}{2}\langle z|z\rangle\left(z^2 - 1 - 2z\bar{z} + \bar{z}^2\right) \tag{518}$$

Therefore

$$\Delta q^2 = \langle q^2\rangle - \langle q\rangle^2 \tag{519}$$
$$= \frac{1}{2m\omega}\left(z^2+1+2z\bar{z}+\bar{z}^2\right) - \left(\frac{1}{\sqrt{2m\omega}}(z+\bar{z})\right)^2 \tag{520}$$
$$= \frac{1}{2m\omega} \tag{521}$$

and

$$\Delta p^2 = \langle p^2\rangle - \langle p\rangle^2 \tag{522}$$
$$= -\frac{m\omega}{2}\left(z^2-1-2z\bar{z}+\bar{z}^2\right) - \left(-i\frac{\sqrt{m\omega}}{\sqrt{2}}(z-\bar{z})\right)^2 \tag{523}$$
$$= \frac{m\omega}{2} \tag{524}$$

which means

$$\Delta p\Delta q = \frac{1}{\sqrt{2m\omega}}\frac{\sqrt{m\omega}}{\sqrt{2}} = \frac{1}{2}. \tag{525}$$

5. At first let's construct the eigenstate $|w\rangle$ for a manually

$$a|w\rangle = c_w|w\rangle \tag{526}$$

Expanding the eigenstate with $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$

$$|w\rangle = \sum_n \alpha_n|n\rangle \tag{527}$$
$$a|w\rangle = \sum_n \alpha_n\sqrt{n}|n-1\rangle \stackrel{!}{=} c_w\sum_n \alpha_n|n\rangle = c_w|n\rangle \tag{528}$$
$$\to \alpha_n\sqrt{n} = c_w\alpha_{n-1} \tag{529}$$
$$\to \alpha_n = \frac{c_w}{\sqrt{n}}\alpha_{n-1} \tag{530}$$
$$|w\rangle = \sum_n \alpha_0\frac{c_w^n}{\sqrt{n!}}|n\rangle = \alpha_0\sum_n \frac{c_w^n}{n!}(a^\dagger)^n|0\rangle = \alpha_0 e^{c_w a^\dagger}|0\rangle \tag{531}$$

Now we do the same for $a^\dagger$

$$a^\dagger|v\rangle = c_v|v\rangle \tag{532}$$

Expanding the eigenstate

$$|v\rangle = \sum_n \beta_n|n\rangle \tag{533}$$
$$a^\dagger|v\rangle = \sum_n \beta_n\sqrt{n+1}|n+1\rangle \stackrel{!}{=} c_v\sum_n \beta_n|n\rangle = c_v|n\rangle \tag{534}$$
$$\to \beta_n\sqrt{n+1} = c_v\beta_{n+1} \tag{535}$$
$$\to \beta_{n+1} = \frac{\sqrt{n+1}}{c_v}\beta_n \tag{536}$$
$$|v\rangle = \sum_n \beta_0\frac{\sqrt{n!}}{c_v^n}|n\rangle = \beta_0\sum_n \frac{1}{c_v^n}(a^\dagger)^n|0\rangle \tag{537}$$

Now we calculate with $\langle 0|a^\dagger = 0$

$$\langle 0|a^\dagger|v\rangle = \beta_0 \sum_n \frac{1}{c_v^n} \langle 0|(a^\dagger)^{n+1}|0\rangle \tag{538}$$
$$= \beta_0 \frac{1}{c_v^0} \langle 0|a^\dagger|0\rangle \tag{539}$$
$$\tag{540}$$

Problem 3.5 Spontaneous symmetry

1.

$$\mathscr{L} = -\frac{1}{2}\phi\Box\phi + \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{541}$$
$$\frac{\partial \mathscr{L}}{\partial \phi} - \partial_\beta \frac{\partial \mathscr{L}}{\partial(\partial_\beta \phi)} = 0 \tag{542}$$
$$\rightarrow -\frac{1}{2}\Box\phi + m^2\phi - \frac{\lambda}{3!}\phi^3 = 0 \tag{543}$$

Problem 3.1 Higher order Lagrangian

With the principle of least action

$$\delta S = \delta \int \mathcal{L} d^4x = \int \delta \mathcal{L} d^4x \tag{544}$$

we calculate

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi) + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \partial_\mu \phi)}\delta(\partial_n u \partial_\mu \phi) + ... \tag{545}$$

Now we can integrate each term

$$\delta \mathcal{L}_0 = \int \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi d^4x \tag{546}$$
$$\delta \mathcal{L}_1 = \int \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi) d^4x = \int \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\mu \delta\phi d^4x \tag{547}$$
$$= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi\Big|_{\partial\Omega} - \int \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi d^4x \tag{548}$$
$$\delta \mathcal{L}_2 = \int \frac{\partial \mathcal{L}}{\partial(\partial_\nu \partial_\mu \phi)}\delta(\partial_\nu \partial_\mu \phi) d^4x = \int \frac{\partial \mathcal{L}}{\partial(\partial_\nu \partial_\mu \phi)}\partial_\nu \delta \partial_\mu \phi d^4x \tag{549}$$
$$= \frac{\partial \mathcal{L}}{\partial(\partial_\nu \partial_\mu \phi)}\delta\partial_\mu\phi\Big|_{\partial\Omega} - \int \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu \partial_\mu \phi)}\delta\partial_\mu\phi d^4x \tag{550}$$
$$= \frac{\partial \mathcal{L}}{\partial(\partial_\nu \partial_\mu \phi)}\delta\partial_\mu\phi\Big|_{\partial\Omega} - \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu \partial_\mu \phi)}\delta\phi\Big|_{\partial\Omega} + \int \partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu \partial_\mu \phi)}\delta\phi d^4x \tag{551}$$

Requiring that all derivatives vanish at infinity we obtain

$$\delta S = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} + \partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu \partial_\mu \phi)} - ... \right) \delta\phi \tag{552}$$

and therefore

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} + \partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu \partial_\mu \phi)} - ... = 0 \tag{553}$$

Problem 3.6 Yukawa potential

(a) We slit the Lagranian in three parts

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}m^2 A_\mu^2 - A_\mu J_\mu \tag{554}$$
$$= \mathscr{L}_F + \mathscr{L}_m + \mathscr{L}_J \tag{555}$$

with the Euler Lagrange equations

$$\frac{\partial \mathscr{L}}{\partial A_\alpha} - \partial_\beta \frac{\partial \mathscr{L}}{\partial(\partial_\beta A_\alpha)} = 0 \tag{556}$$

with

$$\frac{\partial(\partial_\mu A_\nu)}{\partial(\partial_\beta A_\alpha)} = \delta_{\mu\beta}\delta_{\nu\alpha} \tag{557}$$

we can calculate

$$\frac{\partial \mathscr{L}_m}{\partial A_\alpha} - \partial_\beta \frac{\partial \mathscr{L}_m}{\partial(\partial_\beta A_\alpha)} = m^2 A_\alpha \tag{558}$$
$$\frac{\partial \mathscr{L}_J}{\partial A_\alpha} - \partial_\beta \frac{\partial \mathscr{L}_J}{\partial(\partial_\beta A_\alpha)} = -J_\alpha \tag{559}$$
$$\frac{\partial \mathscr{L}_F}{\partial A_\alpha} - \partial_\beta \frac{\partial \mathscr{L}_F}{\partial(\partial_\beta A_\alpha)} = -\frac{1}{4}\partial_\beta\left(-2F_{\mu\nu}(\delta_{\mu\beta}\delta_{\nu\alpha} - \delta_{\nu\beta}\delta_{\mu\alpha})\right) \tag{560}$$
$$= \frac{1}{4}\partial_\beta\left(2(F_{\beta\alpha} - F_{\alpha\beta})\right) \tag{561}$$
$$= \partial_\beta F_{\beta\alpha} \tag{562}$$
$$= \partial_\beta\partial_\beta A_\alpha - \partial_\beta\partial_\alpha A_\beta \tag{563}$$

to obtain (the Proca equation)

$$\Box A_\alpha - \partial_\beta\partial_\alpha A_\beta + m^2 A_\alpha - J_\alpha = 0. \tag{564}$$

Now we can calculate the divergence of the equations

$$\partial_\alpha\left(\Box A_\alpha - \partial_\beta\partial_\alpha A_\beta + m^2 A_\alpha - J_\alpha\right) = 0. \tag{565}$$
$$\Box\partial_\alpha A_\alpha - \partial_\alpha\partial_\alpha\partial_\beta A_\beta + m^2\partial_\alpha A_\alpha - \underbrace{\partial_\alpha J_\alpha}_{=0} = 0 \tag{566}$$

which implies $\partial_\alpha A_\alpha = 0$ and therefore

$$\Box A_\alpha + m^2 A_\alpha - J_\alpha = 0. \tag{567}$$

(b) For A_0 we have for a static potential

$$(\partial_{tt} - \triangle)A_0 + m^2 A_0 - e\delta(x) = 0 \tag{568}$$
$$-\triangle A_0 + m^2 A_0 - e\delta(x) = 0. \tag{569}$$

A Fourier transformation of the equation of motion yields

$$-(ik)^2 A_0(k) + m^2 A_0(k) - e = 0 \tag{570}$$
$$\to A_0(k) = \frac{e}{k^2 + m^2} \tag{571}$$

which we can now transform back

$$A_0 = \frac{e}{(2\pi)^3}\int d^3k \frac{e^{ikx}}{k^2+m^2} \tag{572}$$

$$= \frac{e}{4\pi r}e^{-mr} \tag{573}$$

where we used the integral evaluation from KACHELRIESS Problem 3.5.

(c)

$$\lim_{m\to 0}\frac{e}{4\pi r}e^{-mr} = \frac{e}{4\pi r} \tag{574}$$

(d) Scaling down the Coulomb potential exponentially with a characteristic length of $1/m$.

(e)

(f) We can expand and the integrate each term by parts to move over the partial derivatives

$$\mathscr{L}_F = -\frac{1}{4}F_{\mu\nu}^2 \tag{575}$$

$$= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) \tag{576}$$

$$= -\frac{1}{4}\left(\partial_\mu A_\nu \partial_\mu A_\nu - \partial_\mu A_\nu \partial_\nu A_\mu - \partial_\nu A_\mu \partial_\mu A_\nu + \partial_\nu A_\mu \partial_\nu A_\mu\right) \tag{577}$$

$$= -\frac{1}{2}\left(\partial_\mu A_\nu \partial_\mu A_\nu - \partial_\mu A_\nu \partial_\nu A_\mu\right) \tag{578}$$

$$= -\frac{1}{2}\left(-A_\nu \partial_\mu \partial_\mu A_\nu + A_\nu \partial_\nu \partial_\mu A_\mu\right) \tag{579}$$

$$= \frac{1}{2}\left(A_\mu \Box A_\mu - A_\nu \partial_\nu \underbrace{\partial_\mu A_\mu}_{=0}\right) \tag{580}$$

$$= \frac{1}{2}A_\mu \Box A_\mu \tag{581}$$

We can plug this into the full Lagrangian (renaming the summation index)

$$\mathscr{L} = \frac{1}{2}A_\mu \Box A_\mu + \frac{1}{2}m^2 A_\mu^2 - A_\mu J_\mu \tag{582}$$

$$= \frac{1}{2}A_\mu\left(\Box + m^2\right)A_\mu - A_\mu J_\mu \tag{583}$$

then we calculate the derivatives for the Euler-Lagrange equations up to second order (see problem 3.1)

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = \frac{1}{2}\Box A_\mu + m^2 A_\mu - J_\mu \tag{584}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\mu)} = 0 \tag{585}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \partial_\alpha A_\mu)} = \frac{1}{2}A_\mu \tag{586}$$

and get

$$(\Box + m^2)A_\mu = J_\mu \tag{587}$$

Problem 3.7 Perihelion shift of Mercury by dimensional analysis

(a) Lets summarize the rules of dimensional analysis

varible	SI unit	equation	natural unit
c	m/s	-	1
$\hbar$	Js	-	1
Velocity	m/s	-	1
mass	kg	$E = mc^2$	E
frequency	1/s	$E = \hbar\omega$	E
time	s	$t = 2\pi/\omega$	E^{-1}
length	m	$s = ct$	E^{-1}
∂_μ	1/m	-	E
momentum	kg m/s	$E = p^2/2m$	E
action	Js	$S = Et$	1
$\mathcal{L}$	J/m^3	$S = \int d^4x\mathcal{L}$	E^4
energy density	J/m^3	$\rho = E/V$	E^4
$T^{\mu\nu}$	J/m^3	$\rho = E/V$	E^4

Now we can do a dimensions count for each term

$$\underbrace{\mathcal{L}}_{=4} = -\frac{1}{2}\underbrace{h\Box h}_{2\cdot[h]+2} + \underbrace{M_{\rm Pl}^a h^2\Box h}_{=a+3\cdot[h]+2} - \underbrace{M_{\rm Pl}^b hT}_{b+[h]+4} \tag{588}$$

$$\rightarrow \quad [h] = 1 \tag{589}$$

$$\rightarrow \quad a = -1 \tag{590}$$

$$\rightarrow \quad b = -1 \tag{591}$$

(b) Deriving the equations of motions: keeping in mind that the Lagrangian contains second order derivatives with implies and extra term in the Euler-Lagrange equations (see problem 3.1)

$$\mathcal{L} = -\frac{1}{2}h\Box h + \frac{1}{M_{\rm Pl}}h^2\Box h - \frac{1}{M_{\rm Pl}}hT \tag{592}$$

$$\frac{\partial\mathcal{L}}{\partial h} = -\frac{1}{2}\cdot\Box h + 2\frac{1}{M_{\rm Pl}}h\Box h - \frac{1}{M_{\rm Pl}}T \tag{593}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial h)} = 0 \tag{594}$$

$$\frac{\partial\mathcal{L}}{\partial(\Box h)} = -\frac{1}{2}h + \frac{1}{M_{\rm Pl}}h^2 \tag{595}$$

$$\rightarrow \Box h = \frac{1}{M_{\rm Pl}}\Box(h^2) + \frac{2}{M_{\rm Pl}}h\Box h - \frac{1}{M_{\rm Pl}}T \tag{596}$$

which show an extra term. Alternatively we can integrate the Lagrangian by parts (neglecting the boundary terms) and get

$$\mathcal{L} = \frac{1}{2}\partial h\partial h - \frac{1}{M_{\rm Pl}}\partial(h^2)\partial h - \frac{1}{M_{\rm Pl}}hT \tag{597}$$

$$\frac{\partial\mathcal{L}}{\partial h} = -\frac{1}{M_{\rm Pl}}T \tag{598}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial h)} = \Box h - \frac{1}{M_{\rm Pl}}\Box(h^2) \tag{599}$$

$$\rightarrow \Box h = \frac{1}{M_{\rm Pl}}\Box(h^2) - \frac{1}{M_{\rm Pl}}T \tag{600}$$

We now assume a solution of the form

$$h = h_0 + \frac{1}{M_{\text{Pl}}} h_1 + \frac{1}{M_{\text{Pl}}^2} h_2 + \dots \tag{601}$$

$$\rightarrow h^2 = h_0^2 + \frac{1}{M_{\text{Pl}}} 2h_0 h_1 + \frac{1}{M_{\text{Pl}}^2}(2h_0 h_2 + h_1^2) + \frac{1}{M_{\text{Pl}}^3}(2h_1 h_2 + 2h_0 h_3) + \dots \tag{602}$$

and obtain (with the Coulomb solution 3.61 and 3.61)

$$k = 0: \quad \Box h_0 = 0 \quad \rightarrow \quad h_0 = 0 \tag{603}$$

$$k = 1: \quad \Box h_1 = \Box h_0^2 - m\delta^{(3)} \tag{604}$$

$$\Box h_1 = -m\delta^{(3)} \quad \rightarrow \quad h_1 = -\frac{m}{\Box}\delta^{(3)} = \frac{m}{\triangle}\delta^{(3)} = -\frac{m}{4\pi r} \tag{605}$$

$$k = 2: \quad \Box h_2 = 2\Box h_0 h_1 \quad \rightarrow \quad h_2 = 0 \tag{606}$$

$$k = 3: \quad \Box h_3 = \Box(2h_0 h_2 + h_1^2) \tag{607}$$

$$\Box h_3 = \Box(h_1^2) \quad \rightarrow \quad h_3 = h_1^2 = \frac{m^2}{16\pi^2 r^2} \tag{608}$$

and therefore

$$h = -\frac{m}{4\pi r}\frac{1}{M_{\text{Pl}}} + \frac{m^2}{16\pi^2 r^2}\frac{1}{M_{\text{Pl}}^3} \tag{609}$$

$$= -\frac{m}{4\pi r}\sqrt{G_N} + \frac{m^2}{16\pi^2 r^2}\sqrt{G_N^3} \tag{610}$$

(c) The Newton potential is actually given by (and additional power of M_{Pl} is missing and we are dropping the 4π)

$$V_N = h_1 \frac{1}{M_{\text{Pl}}} \cdot \frac{1}{M_{\text{Pl}}} = -\frac{G m_{\text{Sun}}}{r} \tag{611}$$

the virial theorem implies $E_{\text{kin}} \simeq E_{\text{pot}}$ and therefore

$$\frac{1}{2} J\omega^2 \simeq \frac{G_N m_{\text{Sun}} m_{\text{Mercury}}}{R} \tag{612}$$

$$\frac{1}{2} m_{\text{Mercury}} R^2 \omega^2 \simeq \frac{G_N m_{\text{Sun}} m_{\text{Mercury}}}{R} \tag{613}$$

$$\omega^2 \simeq \frac{G_N m_{\text{Sun}}}{R^3} \tag{614}$$

(d)

(e)

(f)

(g)

Problem 3.9 - Photon polarizations

(a) Then using the results from problem 3.6 and the corrected sign in the Lagranian we get

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 - J_\mu A_\mu \tag{615}$$

$$= \frac{1}{2} A_\mu \Box A_\mu - J_\mu A_\mu \tag{616}$$

$$= \frac{1}{2} A_\mu \Box A_\mu - (\Box A_\mu) A_\mu \tag{617}$$

$$= -\frac{1}{2} A_\mu \Box A_\mu \tag{618}$$

The equations of motion are $\Box A_\mu = J_\mu$ which can be written in momentum space as $k^2 A_\mu(k) = J_\mu(k)$. Now let's write the Lagranian in momentum space as well

$$\mathcal{L} = \int d^4k e^{ikx} A_\mu(k) k^2 A_\mu(k) \tag{619}$$

$$= \int d^4k e^{ikx} \frac{J_\mu(k)}{k^2} k^2 \frac{J_\mu(k)}{k^2} \tag{620}$$

$$= \int d^4k e^{ikx} J_\mu(k) \frac{1}{k^2} J_\mu(k) \tag{621}$$

(b) In momentum space charge conservation is given by

$$k_\mu J_\mu = 0 \tag{622}$$

$$\omega J_0 - \kappa J_1 = 0 \tag{623}$$

$$\rightarrow \quad J_1 = \frac{\omega}{\kappa} J_0 \tag{624}$$

(c)

$$\mathcal{L} = \int d^4k e^{ikx} J_\mu(k) \frac{1}{k^2} J_\mu(k) \tag{625}$$

$$\simeq \frac{J_0^2 - J_1^2 - J_3^2 - J_4^2}{\omega^2 - \kappa^2} \tag{626}$$

$$\simeq \frac{J_0^2(1 - \omega^2/\kappa^2)}{\omega^2 - \kappa^2} - \frac{J_3^2 + J_4^2}{\omega^2 - \kappa^2} \tag{627}$$

$$\simeq -\frac{J_0^2}{\kappa^2} - \frac{J_3^2 + J_4^2}{\omega^2 - \kappa^2} \tag{628}$$

$$\simeq \triangle J_0^2 - \Box(J_3^2 + J_4^2) \tag{629}$$

(d)

Problem 3.10 - Graviton polarizations

(a)

(b)

(c)

(d)

7.4 Srednicki - Quantum Field Theory

Problem 1.2 - Schroedinger equation

$$H = \int d^3x a^\dagger(x) \left(-\frac{\hbar^2}{2m} \triangle_x + V(x) \right) a(x) + \frac{1}{2} \int d^3x d^3y V(x-y) a^\dagger(x) a^\dagger(y) a(x) a(y) \tag{630}$$

$$|\psi, t\rangle = \int d^3x_1 ... d^3x_n \psi(x_1, ..., x_n; t) a^\dagger(x_1) ... a^\dagger(x_n) |0\rangle \tag{631}$$

1. Bosons: With the commutations relation and $a|0\rangle = 0$

$$a(x) a^\dagger(x_1) ... a^\dagger(x_n) |0\rangle = \left(\delta^3(x - x_1) - a^\dagger(x_1) a(x) \right) ... a^\dagger(x_n) |0\rangle \tag{632}$$

$$= \sum_{k=1}^n (-1)^{k-1} \delta^3(x - x_k) \underbrace{a^\dagger(x_1) ... a^\dagger(x_n)}_{(n-1)\times a^\dagger} |0\rangle \tag{633}$$

and similar

$$a(y)a(x)a^\dagger(x_1)...a^\dagger(x_n)|0\rangle = \sum_{j\neq k}^{n} \delta^3(x-x_k)\delta^3(y-x_j)\underbrace{a^\dagger(x_1)...a^\dagger(x_n)}_{(n-2)\times a^\dagger}|0\rangle \tag{634}$$

we obtain

$$i\hbar\frac{\partial}{\partial t}|\psi,t\rangle = \int d^3x_1...d^3x_n\frac{\partial}{\partial t}\psi(x_1,...,x_n;t)a^\dagger(x_1)...a^\dagger(x_n)|0\rangle \tag{635}$$

and

$$H|\psi,t\rangle = \sum_{k=1}^{n} a^\dagger(x_k)\left(-\frac{\hbar^2}{2m}\triangle_{x_k}+V(x_k)\right)\psi(x_1,...,x_n;t)\underbrace{a^\dagger(x_1)...a^\dagger(x_n)}_{(n-1)\times a^\dagger}|0\rangle \tag{636}$$

$$+\frac{1}{2}\sum_{j\neq k}^{n} V(x_k-x_j)\psi(x_1,...,x_n;t)a^\dagger(x_k)a^\dagger(x_j)\underbrace{a^\dagger(x_1)...a^\dagger(x_n)}_{(n-2)\times a^\dagger}|0\rangle \tag{637}$$

2. Fermions:

Problem 1.3 - Commutator of the number operator

Preliminary calculations (we use the boson commutation relations)

$$a^\dagger(z)a(z)a^\dagger(x) = a^\dagger(z)(\delta(x-z)+a^\dagger(x)a(z)) \tag{638}$$
$$= a^\dagger(z)\delta^3(x-z)+a^\dagger(z)a^\dagger(x)a(z) \tag{639}$$
$$= a^\dagger(z)\delta^3(x-z)+a^\dagger(x)a^\dagger(z)a(z) \tag{640}$$

and

$$a(x)a^\dagger(z)a(z) = (\delta(x-z)+a^\dagger(z)a(x))a(z) \tag{641}$$
$$= \delta^3(x-z)a(z)+a^\dagger(z)a(x)a(z) \tag{642}$$
$$= \delta^3(x-z)a(z)+a^\dagger(z)a(z)a(x) \tag{643}$$

With

$$N = \int d^3z\; a^\dagger(z)a(z) \tag{644}$$
$$H = H_1 + H_{\text{int}} \tag{645}$$
$$= \int d^3x\; a^\dagger(x)\left(-\frac{\hbar^2}{2m}\triangle_x+U(x)\right)a(x)+\frac{1}{2}\int d^3xd^3y\; V(x-y)a^\dagger(x)a^\dagger(y)a(y)a(x) \tag{646}$$

We are calculating the commutator in two parts. We start with $[N,H_1]$

$$NH_1 = \int d^3xd^3z\left(a^\dagger(z)\delta^3(x-z)+a^\dagger(x)a^\dagger(z)a(z)\right)\left(-\frac{\hbar^2}{2m}\triangle_x+U(x)\right)a(x) \tag{647}$$
$$= \int d^3xa^\dagger(x)\left(-\frac{\hbar^2}{2m}\triangle_x+U(x)\right)a(x)+\int d^3xd^3za^\dagger(x)\left(-\frac{\hbar^2}{2m}\triangle_x+U(x)\right)a^\dagger(z)a(z)a(x) \tag{648}$$

and

$$H_1N = \int d^3x\; a^\dagger(x)\left(-\frac{\hbar^2}{2m}\triangle_x+U(x)\right)(\delta^3(x-z)a(z)+a^\dagger(z)a(z)a(x)) \tag{649}$$
$$= \int d^3xa^\dagger(x)\left(-\frac{\hbar^2}{2m}\triangle_x+U(x)\right)a(x)+\int d^3xd^3za^\dagger(x)\left(-\frac{\hbar^2}{2m}\triangle_x+U(x)\right)a^\dagger(z)a(z)a(x) \tag{650}$$

therefore $[N, H_1] = 0$. For the second part $[N, H_{\text{int}}]$ we calculate

$$a_z^\dagger a_z a_x^\dagger a_y^\dagger a_y a_x = a_z^\dagger(\delta_{zx}^3 + a_x^\dagger a_z) a_y^\dagger a_y a_x \tag{651}$$
$$= \delta_{zx}^3 a_z^\dagger a_y^\dagger a_y a_x + a_z^\dagger a_x^\dagger a_z a_y^\dagger a_y a_x \tag{652}$$
$$= \delta_{zx}^3 a_y^\dagger a_z^\dagger a_y a_x + a_z^\dagger a_x^\dagger(\delta_{zy}^3 + a_y^\dagger a_z) a_y a_x \tag{653}$$
$$= \delta_{zx}^3 a_y^\dagger a_z^\dagger a_y a_x + \delta_{zy}^3 a_z^\dagger a_x^\dagger a_y a_x + a_z^\dagger a_x^\dagger a_y^\dagger a_z a_y a_x \tag{654}$$
$$= \delta_{zx}^3 a_y^\dagger a_z^\dagger a_y a_x + \delta_{zy}^3 a_x^\dagger a_z^\dagger a_y a_x + a_x^\dagger a_y^\dagger a_z^\dagger a_z a_y a_x \tag{655}$$
$$\to a_y^\dagger a_x^\dagger a_y a_x + a_x^\dagger a_y^\dagger a_y a_x + a_x^\dagger a_y^\dagger a_z^\dagger a_z a_y a_x \tag{656}$$

and

$$a_x^\dagger a_y^\dagger a_y a_x a_z^\dagger a_z = a_x^\dagger a_y^\dagger a_y(\delta_{xz}^3 + a_z^\dagger a_x) a_z \tag{657}$$
$$= \delta_{xz}^3 a_x^\dagger a_y^\dagger a_y a_z + a_x^\dagger a_y^\dagger a_y a_z^\dagger a_x a_z \tag{658}$$
$$= \delta_{xz}^3 a_x^\dagger a_y^\dagger a_z a_y + a_x^\dagger a_y^\dagger(\delta_{zy}^3 + a_z^\dagger a_y) a_x a_z \tag{659}$$
$$= \delta_{xz}^3 a_x^\dagger a_y^\dagger a_z a_y + \delta_{zy}^3 a_x^\dagger a_y^\dagger a_x a_z + a_x^\dagger a_y^\dagger a_z^\dagger a_y a_x a_z \tag{660}$$
$$= \delta_{xz}^3 a_x^\dagger a_y^\dagger a_z a_y + \delta_{zy}^3 a_x^\dagger a_y^\dagger a_z a_x + a_x^\dagger a_y^\dagger a_z^\dagger a_z a_y a_x \tag{661}$$
$$\to a_x^\dagger a_y^\dagger a_x a_y + a_x^\dagger a_y^\dagger a_y a_x + a_x^\dagger a_y^\dagger a_z^\dagger a_z a_y a_x \tag{662}$$

We therefore see that the commutator vanishes as well.

Problem 2.1 - Infinitesimal LT

$$g_{\mu\nu}\Lambda^\mu_{\ \rho}\Lambda^\nu_{\ \sigma} = g_{\rho\sigma} \tag{663}$$
$$g_{\mu\nu}\left(\delta^\mu_{\ \rho} + \delta\omega^\mu_{\ \rho}\right)\left(\delta^\nu_{\ \sigma} + \delta\omega^\nu_{\ \sigma}\right) = g_{\rho\sigma} \tag{664}$$
$$g_{\mu\nu}\left(\delta^\mu_{\ \rho}\delta^\nu_{\ \sigma} + \delta^\nu_{\ \sigma}\cdot\delta\omega^\mu_{\ \rho} + \delta^\mu_{\ \rho}\cdot\delta\omega^\nu_{\ \sigma} + \mathcal{O}(\delta\omega^2)\right) = g_{\rho\sigma} \tag{665}$$
$$g_{\rho\sigma} + g_{\mu\sigma}\cdot\delta\omega^\mu_{\ \rho} + g_{\rho\nu}\cdot\delta\omega^\nu_{\ \sigma} = g_{\rho\sigma} \tag{666}$$

which implies

$$\delta\omega_{\sigma\rho} + \delta\omega_{\rho\sigma} = 0 \tag{667}$$

Problem 2.2 - Infinitesimal LT II

Important: each $M^{\mu\nu}$ is an operator and $\delta\omega$ is just a coefficient matrix so $\delta\omega_{\mu\nu}M^{\mu\nu}$ ist a weighted sum of operators.

$$U(\Lambda^{-1}\Lambda'\Lambda) = U(\Lambda^{-1})U(\Lambda')U(\Lambda) \tag{668}$$
$$U(\Lambda^{-1}(I + \delta\omega')\Lambda) = U(\Lambda^{-1})\left(I + \frac{i}{2\hbar}\delta\omega'_{\mu\nu}M^{\mu\nu}\right)U(\Lambda) \tag{669}$$
$$U(I + \Lambda^{-1}\delta\omega'\Lambda) = I + \frac{i}{2\hbar}\delta\omega'_{\mu\nu}U(\Lambda^{-1})M^{\mu\nu}U(\Lambda) \tag{670}$$

now we calculate recalling successive LT's $(\Lambda^{-1})^\varepsilon_{\ \gamma}\delta\omega'^\gamma_{\ \beta}\Lambda^\beta_{\ \alpha}x^\alpha$

$$(\Lambda^{-1}\delta\omega'\Lambda)_{\rho\sigma} = g_{\varepsilon\rho}(\Lambda^{-1})^\varepsilon_{\ \mu}\delta\omega'^\mu_{\ \nu}\Lambda^\nu_{\ \sigma} \tag{671}$$
$$= g_{\varepsilon\rho}\Lambda_\mu^{\ \varepsilon}\delta\omega'^\mu_{\ \nu}\Lambda^\nu_{\ \sigma} \tag{672}$$
$$= \delta\omega'_{\mu\nu}\Lambda^\mu_{\ \rho}\Lambda^\nu_{\ \sigma} \tag{673}$$

now we can rewrite $U(I + \Lambda^{-1}\delta\omega'\Lambda)$ and therefore

$$\delta\omega'_{\mu\nu}\Lambda^\mu_{\ \rho}\Lambda^\nu_{\ \sigma}M^{\rho\sigma} = \delta\omega'_{\mu\nu}U(\Lambda^{-1})M^{\mu\nu}U(\Lambda) \tag{674}$$

As all $\delta\omega'$ components are basically independent the equation must hold for each pair μ, ν.

Problem 2.3 - Commutators of LT generators I

LHS:

$$U(\Lambda)^{-1}M^{\mu\nu}U(\Lambda) \simeq \left(I - \frac{i}{2\hbar}\delta\omega_{\alpha\beta}M^{\alpha\beta}\right)M^{\mu\nu}\left(I + \frac{i}{2\hbar}\delta\omega_{\rho\sigma}M^{\rho\sigma}\right) \tag{675}$$

$$\simeq M^{\mu\nu} - \frac{i}{2\hbar}\delta\omega_{\rho\sigma}(M^{\rho\sigma}M^{\mu\nu} - M^{\mu\nu}M^{\rho\sigma}) + \mathcal{O}(\delta\omega^2) \tag{676}$$

$$= M^{\mu\nu} - \frac{i}{2\hbar}\delta\omega_{\rho\sigma}[M^{\rho\sigma}, M^{\mu\nu}] \tag{677}$$

$$= M^{\mu\nu} + \frac{i}{2\hbar}\delta\omega_{\rho\sigma}[M^{\mu\nu}, M^{\rho\sigma}] \tag{678}$$

RHS:

$$\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}M^{\rho\sigma} \simeq \left(\delta^{\mu}{}_{\rho} + \delta\omega^{\mu}{}_{\rho}\right)\left(\delta^{\nu}{}_{\sigma} + \delta\omega^{\nu}{}_{\sigma}\right)M^{\rho\sigma} \tag{679}$$

$$\simeq M^{\mu\nu} + \delta^{\mu}{}_{\rho}\delta\omega^{\nu}{}_{\sigma}M^{\rho\sigma} + \delta^{\nu}{}_{\sigma}\delta\omega^{\mu}{}_{\rho}M^{\rho\sigma} \tag{680}$$

$$\simeq M^{\mu\nu} + \delta\omega^{\nu}{}_{\sigma}M^{\mu\sigma} + \delta\omega^{\mu}{}_{\rho}M^{\rho\nu} \tag{681}$$

$$\simeq M^{\mu\nu} + \delta\omega_{\alpha\sigma}g^{\alpha\nu}M^{\mu\sigma} + \delta\omega_{\alpha\rho}g^{\alpha\mu}M^{\rho\nu} \tag{682}$$

$$\simeq M^{\mu\nu} + \delta\omega_{\alpha\sigma}(g^{\alpha\nu}M^{\mu\sigma} + g^{\alpha\mu}M^{\sigma\nu}) \tag{683}$$

$$\simeq M^{\mu\nu} + \delta\omega_{\rho\sigma}(g^{\rho\nu}M^{\mu\sigma} + g^{\rho\mu}M^{\sigma\nu}) \tag{684}$$

$$\simeq M^{\mu\nu} + \frac{1}{2}\delta\omega_{\rho\sigma}\left(g^{\rho\nu}(M^{\mu\sigma} - M^{\sigma\mu}) + g^{\rho\mu}(M^{\sigma\nu} - M^{\nu\sigma})\right) \tag{685}$$

$$\simeq M^{\mu\nu} + \frac{1}{2}\delta\omega_{\rho\sigma}\left(g^{\rho\nu}M^{\mu\sigma} - g^{\nu\rho}M^{\sigma\mu} + g^{\rho\mu}M^{\sigma\nu} - g^{\mu\rho}M^{\nu\sigma}\right) \tag{686}$$

Now we use the antisymmetry of M

$$\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}M^{\rho\sigma} \simeq M^{\mu\nu} + \frac{1}{2}\delta\omega_{\rho\sigma}\left(g^{\nu\rho}M^{\mu\sigma} - g^{\nu\rho}M^{\sigma\mu} + g^{\rho\mu}M^{\sigma\nu} - g^{\mu\rho}M^{\nu\sigma}\right) \tag{687}$$

$$\simeq M^{\mu\nu} - \frac{1}{2}\delta\omega_{\rho\sigma}\left(-g^{\nu\rho}M^{\mu\sigma} + g^{\nu\rho}M^{\sigma\mu} - g^{\rho\mu}M^{\sigma\nu} + g^{\mu\rho}M^{\nu\sigma}\right) \tag{688}$$

$$\simeq M^{\mu\nu} - \frac{1}{2}\delta\omega_{\rho\sigma}\left(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma} - g^{\rho\mu}M^{\sigma\nu} + g^{\nu\rho}M^{\sigma\mu}\right) \tag{689}$$

$$\simeq M^{\mu\nu} - \frac{1}{2}\delta\omega_{\rho\sigma}\left(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma}\right) - \frac{1}{2}\underbrace{\delta\omega_{\rho\sigma}\left(-g^{\rho\mu}M^{\sigma\nu} + g^{\nu\rho}M^{\sigma\mu}\right)}_{=\underbrace{\delta\omega_{\sigma\rho}\left(-g^{\sigma\mu}M^{\rho\nu} + g^{\nu\sigma}M^{\rho\mu}\right)}_{=-\delta\omega_{\rho\sigma}(-g^{\mu\sigma}(-M^{\nu\rho})+g^{\nu\sigma}(-M^{\mu\rho}))}} \tag{690}$$

$$\simeq M^{\mu\nu} - \frac{1}{2}\delta\omega_{\rho\sigma}\left(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma}\right) - \frac{1}{2}\delta\omega_{\rho\sigma}\left(-g^{\mu\sigma}M^{\nu\rho} + g^{\nu\sigma}M^{\mu\rho}\right) \tag{691}$$

$$\simeq M^{\mu\nu} - \frac{1}{2}\delta\omega_{\rho\sigma}\left(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma} - g^{\mu\sigma}M^{\nu\rho} + g^{\nu\sigma}M^{\mu\rho}\right) \tag{692}$$

As the components of $\delta\omega$ (besides the antisymmetry) are independent we get

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\hbar\left(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma} - g^{\mu\sigma}M^{\nu\rho} + g^{\nu\sigma}M^{\mu\rho}\right) \tag{693}$$

Problem 2.4 - Commutators of LT generators II

Preliminary calculations

$$\begin{aligned}
\epsilon_{ijk}J_k &= \varepsilon_{ijk}\frac{1}{2}\varepsilon_{kab}M^{ab} && (694)\\
&= -\frac{1}{2}\varepsilon_{kij}\varepsilon_{kab}M^{ab} && (695)\\
&= -\frac{1}{2}\left(\delta_{ia}\delta_{jb}-\delta_{ja}\delta_{ib}\right)M^{ab} && (696)\\
&= -\frac{1}{2}\left(M^{ij}-M^{ji}\right) && (697)\\
&= -M^{ij} && (698)
\end{aligned}$$

- With

$$\begin{aligned}
J_1 &= \frac{1}{2}(\varepsilon_{123}M^{23}+\varepsilon_{132}M^{32}) && (699)\\
&= \varepsilon_{123}M^{23} && (700)\\
&= M^{23} && (701)
\end{aligned}$$

then

$$\begin{aligned}
[J_1,J_3] &= [M^{23},M^{12}] && (702)\\
&= i\hbar\left(g^{21}M^{32}-g^{31}M^{22}-g^{22}M^{31}+g^{32}M^{21}\right) && (703)\\
&= -i\hbar g^{22}M^{31} && (704)\\
&= -i\hbar M^{31} && (705)\\
&= -i\hbar J_2 && (706)
\end{aligned}$$

- analog ...
-

$$\begin{aligned}
[K^i,K^j] &= [M^{i0},M^{j0}] && (707)\\
&= i\hbar\left(g^{ij}M^{00}-g^{0j}M^{i0}-g^{i0}M^{0j}+g^{00}M^{ij}\right) && (708)\\
&= i\hbar\left(-\delta^{ij}M^{00}+M^{ij}\right) && (709)\\
&= \begin{cases} i\hbar M^{ij} = -i\hbar\epsilon_{ijk}J_k & (i=j)\\ 0 & (i\neq j)\end{cases} && (710)
\end{aligned}$$

where we used the result from the preliminary calculation in the last step.

Problem 2.7 - Translation operator

The obvious property $T(a)T(b)=T(a+b)$. Then

$$\begin{aligned}
T(\delta a+\delta b) &= T(\delta a)T(\delta b) && (711)\\
&= \left(1-\frac{i}{\hbar}\delta a_\mu P^\mu\right)\left(1-\frac{i}{\hbar}\delta b_\nu P^\nu\right) && (712)\\
&\simeq 1-\frac{i}{\hbar}(\delta a_\mu+\delta b_\mu)P^\mu+\frac{1}{\hbar^2}\delta a_\mu\delta b_\mu P^\mu P^\nu && (713)
\end{aligned}$$

and

$$\begin{aligned}
T(\delta a+\delta b) &= T(\delta b)T(\delta a) && (714)\\
&= \left(1-\frac{i}{\hbar}\delta b_\nu P^\nu\right)\left(1-\frac{i}{\hbar}\delta a_\mu P^\mu\right) && (715)\\
&\simeq 1-\frac{i}{\hbar}(\delta a_\mu+\delta b_\mu)P^\mu+\frac{1}{\hbar^2}\delta a_\mu\delta b_\mu P^\nu P^\mu && (716)
\end{aligned}$$

which implies $P^\mu P^\nu = P^\nu P^\mu$.

Problem 2.8 - Transformation of scalar field

(a) We start with

$$U(\Lambda)^{-1}\varphi(x)U(\Lambda) = \varphi(\Lambda^{-1}x) \tag{717}$$

$$\left(1 - \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right)\varphi(x)\left(1 + \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right) = \varphi([\delta^\mu_{\ \nu} - \delta\omega^\mu_{\ \nu}]x^\nu) \tag{718}$$

$$\varphi(x) - \frac{i}{2\hbar}\delta\omega_{\mu\nu}[M^{\mu\nu},\varphi(x)] = \varphi(x) - \delta\omega^\mu_{\ \nu}x^\nu\frac{\partial\varphi}{\partial x^\mu} \tag{719}$$

$$= \varphi(x) - \delta\omega^\mu_{\ \nu}\frac{1}{2}\left(x^\nu\frac{\partial\varphi}{\partial x^\mu} - x^\mu\frac{\partial\varphi}{\partial x^\nu}\right) \tag{720}$$

$$= \varphi(x) - \delta\omega_{\mu\nu}\frac{1}{2}\left(x^\nu\partial^\mu - x^\mu\partial^\nu\right)\varphi \tag{721}$$

and therefore

$$[\varphi, M^{\mu\nu}] = \frac{\hbar}{i}(x^\mu\partial^\nu - x^\nu\partial^\mu)\varphi \tag{722}$$

(b) (c) (d) (e) (f)

Problem 3.2 - Multiparticle eigenstates of the hamiltonian

With

$$|k_1...k_n\rangle = a^\dagger_{k_1}...a^\dagger_{k_n}|0\rangle \tag{723}$$

$$H = \int \widetilde{dk}\ \omega_k a^\dagger_k a_k \tag{724}$$

$$[a_k, a^\dagger_q] = \underbrace{(2\pi)^3 2\omega_k\delta^3(\vec{k} - \vec{q})}_{\delta_{kq}} \tag{725}$$

we see that the expression which needs calculating is the creation and annihilation operators. The idea is to use the commutation relations to move the a_k to the right end to use $a_k|0\rangle$

$$a^\dagger_k a_k a^\dagger_{k_1}...a^\dagger_{k_n}|0\rangle = a^\dagger_k(a^\dagger_{k_1}a_k + \delta_{kk_1})a^\dagger_{k_2}...a^\dagger_{k_n}|0\rangle \tag{726}$$

$$= \delta_{kk_1}a^\dagger_k a^\dagger_{k_2}...a^\dagger_{k_n}|0\rangle + a^\dagger_k a^\dagger_{k_1}a_k a^\dagger_{k_2}...a^\dagger_{k_n}|0\rangle \tag{727}$$

$$= ... \tag{728}$$

$$= \sum_j \delta_{kk_j}a^\dagger_k \underbrace{a^\dagger_{k_2}...a^\dagger_{k_n}}_{(n-1)\text{ times with } a_{k_j}\text{ missing}}|0\rangle + a^\dagger_k a^\dagger_{k_1}...a^\dagger_{k_n}\underbrace{a_k|0\rangle}_{=0}. \tag{729}$$

Therefore we obtain

$$H|k_1...k_n\rangle = \int\frac{d^3k}{(2\pi)^3 2\omega_k}\omega_k\sum_j\delta_{kk_j}a^\dagger_k a^\dagger_{k_2}...a^\dagger_{k_n}|0\rangle \tag{730}$$

$$= \int\frac{d^3k}{(2\pi)^3 2\omega_k}\omega_k\sum_j(2\pi)^3 2\omega_k\delta^3(\vec{k} - \vec{k}_j)a^\dagger_k a^\dagger_{k_2}...a^\dagger_{k_n}|0\rangle \tag{731}$$

$$= \int d^3k\omega_k\sum_j\delta^3(\vec{k} - \vec{k}_j)a^\dagger_k a^\dagger_{k_2}...a^\dagger_{k_n}|0\rangle \tag{732}$$

which we can integrate obtaining the desired result

$$H|k_1...k_n\rangle = \sum_j \omega_{k_j} a^\dagger_{k_j} a^\dagger_{k_2}...a^\dagger_{k_n}|0\rangle \tag{733}$$

$$= \left(\sum_j \omega_{k_j}\right) a^\dagger_{k_1} a^\dagger_{k_2}...a^\dagger_{k_n}|0\rangle \tag{734}$$

$$= \left(\sum_j \omega_{k_j}\right) |k_1...k_n\rangle. \tag{735}$$

Problem 3.4 - Heisenberg equations of motion for free field

(a) For the translation operator $T(a) = e^{-iP^\mu a_\mu}$ we expand in first order

$$\begin{aligned} T(a)^{-1}\varphi(a)T(a) &= \left(1 - (-i)P^\mu a_\mu + \mathcal{O}(a^2)\right)\varphi(x)\left(1 + (-i)P^\mu a_\mu + \mathcal{O}(a^2)\right) && (736)\\ &= \left(1 + iP^\mu a_\mu + \mathcal{O}(a^2)\right)\varphi(x)\left(1 - iP^\mu a_\mu + \mathcal{O}(a^2)\right) && (737)\\ &\simeq \varphi(x) + ia_\mu P^\mu \varphi(x) - ia_\mu \varphi(x) P^\mu && (738)\\ &\simeq \varphi(x) + ia_\mu [P^\mu, \varphi(x)] && (739) \end{aligned}$$

for the right hand right we get

$$\varphi(x-a) \simeq \varphi(x) - \partial^\mu \varphi(x) a_\mu \tag{740}$$

and therefore

$$i[P^\mu, \varphi(x)] = -\partial^\mu \varphi(x) \tag{741}$$

(b) With $\mu = 0$ and $\partial^0 = g_{0\nu}\partial_\nu = -\partial_0$ we have

$$\begin{aligned} i[H, \varphi(x)] &= -\partial^0 \varphi(x) = +\partial_0 \varphi(x) && (742)\\ \rightarrow \quad \dot\varphi(x) &= i[H, \varphi(x)] && (743) \end{aligned}$$

(c) We start with the hamiltonian (3.25)

$$H = \int d^3y \frac{1}{2}\Pi^2(y) + \frac{1}{2}(\nabla_y \varphi(y))^2 + \frac{1}{2}m^2\varphi(y)^2 - \Omega_0 \tag{744}$$

- Obtaining $\dot\varphi(x) = i[H, \varphi(x)]$

  We need to calculate (setting $x^0 = y^0$ - why can we?)

$$\begin{aligned} [\Pi^2(y), \varphi(x)] &= \Pi(y)\Pi(y)\varphi(x) - \varphi(x)\Pi(y)\Pi(y) && (745)\\ &= \Pi(y)\Pi(y)\varphi(x) - \Pi(y)\varphi(x)\Pi(y) + \Pi(y)\varphi(x)\Pi(y) - \varphi(x)\Pi(y)\Pi(y) && (746)\\ &= \Pi(y)[\Pi(y), \varphi(x)] + [\Pi(y), \varphi(x)]\Pi(y) && (747)\\ &= 2\Pi(y)(-1)i\delta^3(\vec y - \vec x) && (748)\\ [(\nabla_y\varphi(y))^2, \varphi(x)] &= \nabla_y\varphi(y)\nabla_y\varphi(y)\varphi(x) - \varphi(x)\nabla_y\varphi(y)\nabla_y\varphi(y) && (749)\\ &= \nabla_y\varphi(y)[\nabla_y\varphi(y), \varphi(x)] + [\nabla_y\varphi(y), \varphi(x)]\nabla_y\varphi(y) && (750)\\ &= \nabla_y\varphi(y)\nabla_y[\varphi(y), \varphi(x)] + \nabla_y[\varphi(y), \varphi(x)]\nabla_y\varphi(y) && (751)\\ &= 0 && (752)\\ [\varphi(y)^2, \varphi(x)] &= \varphi(y)\varphi(y)\varphi(x) - \varphi(x)\varphi(y)\varphi(y) && (753)\\ &= \varphi(y)\varphi(y)\varphi(x) - \varphi(y)\varphi(x)\varphi(y) + \varphi(y)\varphi(x)\varphi(y) - \varphi(x)\varphi(y)\varphi(y) && (754)\\ &= \varphi(y)[\varphi(y), \varphi(x)] + [\varphi(y), \varphi(x)]\varphi(y) && (755)\\ &= 0 && (756) \end{aligned}$$

then

$$\int d^3y[\Pi^2(y), \varphi(x)] = -2i\Pi(x) \tag{757}$$

$$\int d^3y[(\nabla_y\varphi(y))^2, \varphi(x)] = \int d^3y\nabla_y\varphi(y)[\nabla_y\varphi(y), \varphi(x)] + [\nabla_y\varphi(y), \varphi(x)]\nabla_y\varphi(y) \tag{758}$$

$$= 0 \tag{759}$$

$$\int d^3y[\varphi(y)^2, \varphi(x)] = 0 \tag{760}$$

and therefore

$$\dot{\varphi}(x) = i[H, \varphi(x)] \tag{761}$$

$$= i\frac{1}{2}(-2i)\Pi(x) \tag{762}$$

$$= \Pi(x) \tag{763}$$

- Obtaining $\dot{\Pi}(x) = -i[H, \Pi(x)]$ (sign!?!)

  Now we need to calculate - by using the results from above we can now shortcut a bit

$$[\Pi^2(y), \Pi(x)] = 0 \tag{764}$$

$$[(\nabla_y\varphi(y))^2, \Pi(x)] = (\nabla_y\varphi(y))(\nabla_y\varphi(y))\Pi(x) - \Pi(x)(\nabla_y\varphi(y))(\nabla_y\varphi(y)) \tag{765}$$

$$= (\nabla_y\varphi(y))[(\nabla_y\varphi(y)), \Pi(x)] - [\Pi(x), (\nabla_y\varphi(y))](\nabla_y\varphi(y)) \tag{766}$$

$$= (\nabla_y\varphi(y))\nabla_y[\varphi(y), \Pi(x)] - (\nabla_y[\Pi(x), \varphi(y)])(\nabla_y\varphi(y)) \tag{767}$$

$$= (\nabla_y\varphi(y))\nabla_y i\delta^3(\vec{x}-\vec{y}) - (\nabla_y(-i)\delta^3(\vec{x}-\vec{y}))(\nabla_y\varphi(y)) \tag{768}$$

$$= 2i(\nabla_y\delta^3(\vec{x}-\vec{y}))(\nabla_y\varphi(y)) \tag{769}$$

$$[\varphi(y)^2, \Pi(x)] = \varphi(y)\varphi(y)\Pi(x) - \Pi(x)\varphi(y)\varphi(y) \tag{770}$$

$$= \varphi(y)\varphi(y)\Pi(x) - \varphi(y)\Pi(x)\varphi(y) + \varphi(y)\Pi(x)\varphi(y) - \Pi(x)\varphi(y)\varphi(y) \tag{771}$$

$$= \varphi(y)[\varphi(y), \Pi(x)] + [\varphi(y), \Pi(x)]\varphi(y) \tag{772}$$

$$= 2i\varphi(y)\delta^3(\vec{x}-\vec{y}) \tag{773}$$

then

$$\int d^3y[\Pi^2(y), \Pi(x)] = 0 \tag{774}$$

$$\int d^3y[(\nabla_y\varphi(y))^2, \Pi(x)] = 2i\int d^3y(\nabla_y\delta^3(\vec{x}-\vec{y}))(\nabla_y\varphi(y)) \tag{775}$$

$$= -2i\int d^3y\delta^3(\vec{x}-\vec{y})(\nabla_y\nabla_y\varphi(y)) \tag{776}$$

$$= -2i\triangle_x\varphi(x) \tag{777}$$

$$\int d^3y[\varphi(y)^2, \Pi(x)] = 2i\varphi(x) \tag{778}$$

and therefore

$$\dot{\Pi}(x) = -i[H, \Pi(x)] \tag{779}$$

$$= -i\left(\frac{1}{2}(-2i)\triangle_x\varphi(x) + \frac{1}{2}m^2 2i\varphi(x)\right) \tag{780}$$

$$= -i\left(-i\triangle_x\varphi(x) + m^2 i\varphi(x)\right) \tag{781}$$

$$= -\triangle_x\varphi(x) + m^2\varphi(x) \tag{782}$$

which finally leads to (with $\Box = \partial_{tt} - \triangle$)

$$\partial^0\partial_0\varphi(x) = \partial^0\Pi(x) \tag{783}$$
$$= -\partial_0\Pi(x) \tag{784}$$
$$= -(-\triangle_x\varphi(x) + m^2\varphi(x)) \tag{785}$$
$$\to (\Box_x + m^2)\varphi(x) = 0 \tag{786}$$

(d) With

$$\vec{P} \equiv -\int d^3x\Pi(x)\nabla_x\varphi(x) \tag{787}$$

we have to calculate

$$[\vec{P}, \varphi(y)] = -\int d^3x[\Pi(x)\nabla_x\varphi(x), \varphi(y)]. \tag{788}$$

Let's start with

$$[\Pi(x)\nabla_x\varphi(x), \varphi(y)] = \Pi(x)\nabla_x\varphi(x)\varphi(y) - \varphi(y)\Pi(x)\nabla_x\varphi(x) \tag{789}$$
$$= \Pi(x)\nabla_x\varphi(x)\varphi(y) - (\Pi(x)\varphi(y) + i\delta^3(\vec{x}-\vec{y}))\nabla_x\varphi(x) \tag{790}$$
$$= \Pi(x)\nabla_x\varphi(x)\varphi(y) - \Pi(x)\varphi(y)\nabla_x\varphi(x) + i\delta^3(\vec{x}-\vec{y})\nabla_x\varphi(x) \tag{791}$$
$$= \Pi(x)\nabla_x(\varphi(x)\varphi(y)) - \Pi(x)\nabla_x(\varphi(y)\varphi(x)) + i\delta^3(\vec{x}-\vec{y})\nabla_x\varphi(x) \tag{792}$$
$$= \Pi(x)\nabla_x[\varphi(x), \varphi(y)] + i\delta^3(\vec{x}-\vec{y})\nabla_x\varphi(x) \tag{793}$$
$$= i\delta^3(\vec{x}-\vec{y})\nabla_x\varphi(x) \tag{794}$$

and then

$$[\vec{P}, \varphi(y)] = -i\int d^3x\delta^3(\vec{x}-\vec{y})\nabla_x\varphi(x) \tag{795}$$
$$= -i\nabla_y\varphi(y) \tag{796}$$

(e) With

$$\Pi(x) = \dot{\varphi}(x) \tag{797}$$
$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k}(-i\omega_k)(a_k e^{ikx} - a_k^\dagger e^{-ikx}) \tag{798}$$
$$\nabla\varphi(x) = \int \frac{d^3q}{(2\pi)^3 2\omega_k}(i\vec{q})(a_q e^{iqx} - a_q^\dagger e^{-iqx}) \tag{799}$$
$$\tag{800}$$

then

$$\vec{P} = -\int d^3x\Pi(x)\nabla_x\varphi(x) \tag{801}$$
$$= -\iiint d^3x \frac{d^3k}{(2\pi)^3 2\omega_k}\frac{d^3q}{(2\pi)^3 2\omega_k}(-i\omega_k)(i\vec{q})(a_k e^{ikx} - a_k^\dagger e^{-ikx})(a_q e^{iqx} - a_q^\dagger e^{-iqx}) \tag{802}$$
$$= -\iiint d^3x \frac{d^3k}{(2\pi)^3 2}\frac{d^3q}{(2\pi)^3 2\omega_k}\vec{q}(a_k a_q e^{i(k+q)x} - a_k^\dagger a_q e^{-i(k-q)x} - a_k a_q^\dagger e^{i(k-q)x} + a_k^\dagger a_q^\dagger e^{-i(k+q)x}) \tag{803}$$
$$\tag{804}$$

now we can use the commutation relations and reindex

$$= -\iiint d^3x \frac{d^3kd^3q}{4\omega_k(2\pi)^6}\vec{q}(a_k a_q e^{i(k+q)x} - a_k^\dagger a_q e^{-i(k-q)x} - (a_q^\dagger a_k + (2\pi)^3 2\omega_k \delta^3(\vec{k}-\vec{q}))e^{i(k-q)x} + a_k^\dagger a_q^\dagger e^{-i(k+q)x}) \tag{805}$$

$$= -\iiint d^3x \frac{d^3kd^3q}{4\omega_k(2\pi)^6}\vec{q}(a_k a_q e^{i(k+q)x} + a_k^\dagger a_q^\dagger e^{-i(k+q)x}) + \iiint d^3x \frac{d^3kd^3q}{4\omega_k(2\pi)^6}\vec{q}2a_k^\dagger a_q e^{-i(k-q)x} \tag{806}$$

$$+ \iiint d^3x \frac{d^3kd^3q}{4\omega_k(2\pi)^6}\vec{q}(2\pi)^3 2\omega_k \delta^3(\vec{k}-\vec{q})e^{i(k-q)x} \tag{807}$$

Now we can look at the integrals individually and use the asymmetry. The first

$$-\iiint d^3x \frac{d^3kd^3q}{4\omega_k(2\pi)^6}\vec{q}(a_k a_q e^{i(k+q)x} + a_k^\dagger a_q^\dagger e^{-i(k+q)x}) = ... \tag{808}$$

$$= 0 \tag{809}$$

second

$$\iiint d^3x \frac{d^3kd^3q}{4\omega_k(2\pi)^6}\vec{q}(2\pi)^3 2\omega_k \delta^3(\vec{k}-\vec{q})e^{i(k-q)x} = \iiint d^3x \frac{d^3kd^3q}{2(2\pi)^3}\vec{q}\delta^3(\vec{k}-\vec{q})e^{i(k-q)x} \tag{810}$$

$$= \iiint d^3x \frac{d^3k}{2(2\pi)^3}\vec{k} \tag{811}$$

$$= 0 \tag{812}$$

and third

$$\iiint d^3x \frac{d^3kd^3q}{4\omega_k(2\pi)^6}\vec{q}2a_k^\dagger a_q e^{-i(k-q)x} = \iint \frac{d^3kd^3q}{4\omega_k(2\pi)^6}\vec{q}2a_k^\dagger a_q \int d^3x\ e^{-i(k-q)x} \tag{813}$$

$$= \iint \frac{d^3kd^3q}{4\omega_k(2\pi)^6}\vec{q}2a_k^\dagger a_q e^{-i(k-q)x}e^{-i(k^0-q^0)x^0} \int d^3x\ e^{-i(\vec{k}-\vec{q})\vec{x}} \tag{814}$$

$$= \iint \frac{d^3kd^3q}{4\omega_k(2\pi)^6}\vec{q}2a_k^\dagger a_q e^{-i(k-q)x}e^{-i(k^0-q^0)x^0}(2\pi)^3\delta^3(\vec{k}-\vec{q}) \tag{815}$$

$$= \int \frac{d^3k}{2\omega_k(2\pi)^3}\vec{k}a_k^\dagger a_k \tag{816}$$

$$= \int \widetilde{d^3k}\ \vec{k}a_k^\dagger a_k \tag{817}$$

Therefore we obtain

$$\vec{P} = \int \frac{d^3k}{2\omega_k(2\pi)^3}\vec{k}a_k^\dagger a_k \tag{818}$$

$$= \int \widetilde{d^3k}\ \vec{k}a_k^\dagger a_k \tag{819}$$

Problem 3.5 - Complex scalar field

(a) Sloppy way - Calculating the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \varphi} = -m^2\varphi^\dagger \tag{820}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = -\partial^\mu \varphi^\dagger \tag{821}$$

$$\rightarrow -m^2\varphi^\dagger + \partial_\mu\partial^\mu\varphi^\dagger = 0 \tag{822}$$

$$\rightarrow (\partial_\mu\partial^\mu - m^2)\varphi^\dagger = 0 \tag{823}$$

Bit more rigorous with

$$\frac{\delta\phi(x_1,t_1)}{\delta\phi(x_2,t_2)} = \delta(x_1-x_2)\times\delta(t_1-t_2) \tag{824}$$

$$\frac{\delta\partial_\mu\phi(x)}{\delta\phi(y)} = \frac{\delta}{\delta\phi(y)}\lim_{\epsilon\to 0}\frac{\phi(x_1,x_\mu+\epsilon,...,x_4)-\phi(x_1,x_2,x_3,x_4)}{\epsilon} \tag{825}$$

$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\delta(x_\mu+\epsilon-y_\mu)-\delta(x_\mu-y_\mu)\right)\times\delta(x_1-y_1)\times ...\times\delta(x_4-y_4) \tag{826}$$

$$= \frac{\partial}{\partial x^\mu}\delta^4(x-y) \tag{827}$$

we get

$$S[\varphi] = \int d^4x\left(-\partial^\mu\varphi^\dagger(x)\partial_\mu\varphi(x) - m^2\varphi^\dagger(x)\varphi(x)\right) \tag{828}$$

$$\frac{\delta S[\varphi]}{\delta\varphi(y)} = \int d^4x\left(-\partial^\mu\varphi^\dagger(x)\partial_\mu\delta^4(x-y) - m^2\varphi^\dagger(y)\delta^4(x-y)\right) \tag{829}$$

$$= \int d^4x\left(\partial_\mu\partial^\mu\varphi^\dagger(x)\delta^4(x-y) - m^2\varphi^\dagger(x)\delta^4(x-y)\right) \tag{830}$$

$$= (\Box_y - m^2)\varphi^\dagger(y) \tag{831}$$

(b) With

$$\mathcal{L} = -\partial^0\varphi^\dagger\partial_0\varphi - \partial^a\varphi^\dagger\partial_a\varphi - m^2\varphi^\dagger\varphi + \Omega_0 \tag{832}$$

$$= \partial_0\varphi^\dagger\partial_0\varphi - \partial^a\varphi^\dagger\partial_a\varphi - m^2\varphi^\dagger\varphi + \Omega_0 \tag{833}$$

$$\Pi = \frac{\partial\mathcal{L}}{\partial\dot\varphi} = \dot\varphi^\dagger \tag{834}$$

$$\Pi^\dagger = \frac{\partial\mathcal{L}}{\partial\dot\varphi^\dagger} = \dot\varphi \tag{835}$$

$$\rightarrow \mathcal{H} = \Pi\dot\varphi + \Pi^\dagger\dot\varphi^\dagger - \mathcal{L} \tag{836}$$

$$= \dot\varphi^\dagger\dot\varphi + \dot\varphi\dot\varphi^\dagger - \dot\varphi^\dagger\dot\varphi + (\nabla^a\varphi^\dagger)(\nabla_a\varphi) + m^2\varphi^\dagger\varphi - \Omega_0 \tag{837}$$

$$= \Pi^\dagger\Pi + (\nabla^a\varphi^\dagger)(\nabla_a\varphi) + m^2\varphi^\dagger\varphi - \Omega_0 \tag{838}$$

(c) Considering the plane wave solutions $e^{i\vec{k}\vec{x}\pm i\omega_k t}$ with

$$kx = g_{\mu\nu}k^\mu x^\nu = g_{00}k^0x^0 + g_{ik}k^ix^k = -\omega_k t + \vec{k}\vec{x} \tag{839}$$

we have

$$\varphi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\; a_k e^{ikx} + b_k^\dagger e^{-ikx} \tag{840}$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\; a_k e^{i\vec{k}\vec{x}-i\omega_k t} + b_k^\dagger e^{-i\vec{k}\vec{x}+i\omega_k t} \tag{841}$$

$$e^{-iqx}\varphi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\; a_k e^{i(k-q)x} + b_k^\dagger e^{-i\vec{k}\vec{x}+i\omega_k t} e^{-iqx} \tag{842}$$

$$\int d^3x e^{-iqx}\varphi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\; a_k \underbrace{\int d^3x e^{i(k-q)x}}_{(2\pi)^3\delta^3(\vec{k}-\vec{q})e^{-i(\omega_k-\omega_q)t}} + b_{-k} \underbrace{\int d^3x e^{i(\vec{k}-\vec{q})\vec{x}}}_{(2\pi)^3\delta^3(\vec{k}-\vec{q})} e^{i(\omega_k+\omega_q)t} \tag{843}$$

$$= \frac{1}{2\omega_q}\left(a_q + b_{-q}^\dagger e^{2i\omega_q t}\right) \tag{844}$$

and

$$\partial_0\varphi(\vec{x},t) = \int \frac{d^3k\;(-i\omega_k)}{(2\pi)^3 2\omega_k}\; a_k e^{i\vec{k}\vec{x}-i\omega_k t} - b_k^\dagger e^{-i\vec{k}\vec{x}+i\omega_k t} \tag{845}$$

$$\int d^3x e^{-iqx}\partial_0\varphi(\vec{x},t) = -\frac{i}{2}\left(a_q - b_{-q}^\dagger e^{2i\omega_q t}\right) \tag{846}$$

adding both equations gives with $\partial_0 e^{-iqx} = \partial_0 e^{-i(-\omega_k t+\vec{k}\vec{x})} = -i\omega_q e^{-iqx}$ and $f \overleftrightarrow{\partial_\mu} g = f(\partial_\mu g) - (\partial_\mu f)g$

$$a_q = \omega_q \int d^3x e^{-iqx}\varphi(\vec{x},t) + i\int d^3x e^{-iqx}\partial_0\varphi(\vec{x},t) \tag{847}$$

$$= i\int d^3x e^{-iqx}(-i\omega_q + \partial_0)\varphi(\vec{x},t) \tag{848}$$

$$= i\int d^3x e^{-iqx} \overleftrightarrow{\partial_0} \varphi(\vec{x},t) \tag{849}$$

To get b_q we solve a second set of equations for $\varphi^\dagger$

$$\varphi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\; a_k e^{ikx} + b_k^\dagger e^{-ikx} \tag{850}$$

$$\to \varphi^\dagger(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\; a_k^\dagger e^{-ikx} + b_k e^{ikx} \tag{851}$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\; b_k e^{ikx} + a_k^\dagger e^{-ikx} \tag{852}$$

Now b_k takes the role of a_k and we can just copy the solution

$$b_q = \omega_q \int d^3x e^{-iqx}\varphi^\dagger(\vec{x},t) + i\int d^3x e^{-iqx}\partial_0\varphi^\dagger(\vec{x},t) \tag{853}$$

$$= i\int d^3x e^{-iqx}(-i\omega_q + \partial_0)\varphi^\dagger(\vec{x},t) \tag{854}$$

$$= i\int d^3x e^{-iqx} \overleftrightarrow{\partial_0} \varphi^\dagger(\vec{x},t) \tag{855}$$

(d) Starting with the observation

$$[A,B]^\dagger = (AB)^\dagger - (BA)^\dagger \tag{856}$$

$$= B^\dagger A^\dagger - A^\dagger B^\dagger \tag{857}$$

$$= [B^\dagger, A^\dagger] \tag{858}$$

$$= -[A^\dagger, B^\dagger] \tag{859}$$

therefore the relevant commutation relations for the fields are

$$[\varphi(\vec{x},t),\varphi(\vec{y},t)]=0 \quad\rightarrow\quad [\varphi^\dagger(\vec{x},t),\varphi^\dagger(\vec{y},t)]=0 \tag{860}$$
$$[\varphi^\dagger(\vec{x},t),\varphi(\vec{y},t)]=0 \tag{861}$$
$$[\Pi(\vec{x},t),\Pi(\vec{y},t)]=0 \quad\rightarrow\quad [\Pi^\dagger(\vec{x},t),\Pi^\dagger(\vec{y},t)]=0 \tag{862}$$
$$[\Pi^\dagger(\vec{x},t),\Pi(\vec{y},t)]=0 \tag{863}$$
$$[\varphi(\vec{x},t),\Pi(\vec{y},t)]=i\delta^3(\vec{x}-\vec{y}) \quad\rightarrow\quad [\varphi^\dagger(\vec{x},t),\Pi^\dagger(\vec{y},t)]=i\delta^3(\vec{x}-\vec{y}) \tag{864}$$
$$[\varphi^\dagger(\vec{x},t),\Pi(\vec{y},t)]=0 \quad\rightarrow\quad [\varphi(\vec{x},t),\Pi^\dagger(\vec{y},t)]=0 \tag{865}$$

with the previous results

$$a_q = i\int d^3x e^{-iqx}(-i\omega_q+\partial_0)\varphi(\vec{x},t) \tag{866}$$
$$= i\int d^3x e^{-iqx}(-i\omega_q\varphi(\vec{x},t)+\Pi^\dagger(\vec{x},t)) \tag{867}$$
$$a_q^\dagger = i\int d^3x e^{iqx}(i\omega_q\varphi^\dagger(\vec{x},t)+\Pi(\vec{x},t)) \tag{868}$$
$$b_q = i\int d^3x e^{-iqx}(-i\omega_q+\partial_0)\varphi^\dagger(\vec{x},t) \tag{869}$$
$$= i\int d^3x e^{-iqx}(-i\omega_q\varphi(\vec{x},t)+\Pi(\vec{x},t)) \tag{870}$$
$$b_q^\dagger = i\int d^3x e^{iqx}(i\omega_q\varphi^\dagger(\vec{x},t)+\Pi^\dagger(\vec{x},t)) \tag{871}$$

let's calculate each of the commutators

$$[a_k,a_q^\dagger] = \iint d^3x\, d^3y\, e^{-ikx}e^{iqy}\left(\omega_k\omega_q[\varphi_x,\varphi_y^\dagger]-i\omega_q[\varphi_x,\Pi_y]+i\omega_q[\Pi_x^\dagger,\varphi_y^\dagger]+[\Pi_x^\dagger,\Pi_y]\right) \tag{872}$$
$$= \iint d^3x\, d^3y\, e^{-i(kx-qy)}\left(-i\omega_q[\varphi_x,\Pi_y]+i\omega_q[\Pi_x^\dagger,\varphi_y^\dagger]\right) \tag{873}$$
$$= \iint d^3x\, d^3y\, e^{-i(kx-qy)}\left(-i\omega_q i\delta^3(\vec{x}-\vec{y})+i\omega_q(-i)\delta^3(\vec{x}-\vec{y})\right) \tag{874}$$
$$= (\omega_q+\omega_q)\iint d^3x\, e^{-i(k-q)x} \tag{875}$$
$$= (\omega_q+\omega_q)\,(2\pi)^3\delta^3(\vec{k}-\vec{q}) \tag{876}$$
$$= 2\omega_q(2\pi)^3\delta^3(\vec{k}-\vec{q}) \tag{877}$$

and so on

$$[b_k,b_q^\dagger] = ... = 2\omega_q(2\pi)^3\delta^3(\vec{k}-\vec{q}) \tag{878}$$

(e) Now

$$H = \int d^3x\, \Pi^\dagger\Pi + (\nabla^a\varphi^\dagger)(\nabla_a\varphi) + m^2\varphi^\dagger\varphi - \Omega_0 \tag{879}$$
$$\Pi^\dagger\Pi = \dot{\varphi}\dot{\varphi}^\dagger \tag{880}$$
$$= \int \widetilde{d^3k}\widetilde{d^3q}(i\omega_k)(i\omega_q)\left(a_ke^{ikx}-b_k^\dagger e^{-ikx}\right)\left(a_q^\dagger e^{-iqx}-b_qe^{iqx}\right) \tag{881}$$
$$= \int \widetilde{d^3k}\widetilde{d^3q}(-\omega_k\omega_q)\left(a_ka_q^\dagger e^{-iqx}e^{ikx}-b_k^\dagger a_q^\dagger e^{-iqx}e^{-ikx}-a_kb_qe^{iqx}e^{ikx}+b_k^\dagger b_qe^{iqx}e^{-ikx}\right) \tag{882}$$
$$= \int \widetilde{d^3k}\widetilde{d^3q}(-\omega_k\omega_q)\left([a_q^\dagger a_k-2\omega_k(2\pi)^3\delta^3(\vec{k}-\vec{q})]e^{-i(q-k)x}-b_k^\dagger a_q^\dagger e^{-i(q+k)x}-a_kb_qe^{i(q+k)x}+b_k^\dagger b_qe^{i(q-k)x}\right) \tag{883}$$

$$(\nabla^a \varphi^\dagger)(\nabla_a \varphi) = \int \widetilde{d^3k}\widetilde{d^3q}(k^a q_a) \left(-a_k^\dagger e^{-ikx} + b_k e^{ikx}\right) \left(a_q e^{iqx} - b_q^\dagger e^{-iqx}\right) \tag{884}$$

$$= \int \widetilde{d^3k}\widetilde{d^3q}(k^a q_a) \left(-a_k^\dagger a_q e^{iqx} e^{-ikx} + b_k a_q e^{iqx} e^{ikx} + a_k^\dagger b_q^\dagger e^{-iqx} e^{-ikx} - b_k b_q^\dagger e^{-iqx} e^{ikx}\right) \tag{885}$$

$$= \int \widetilde{d^3k}\widetilde{d^3q}(k^a q_a) \left(-a_k^\dagger a_q e^{i(q-k)x} + a_q b_k e^{i(q+k)x} + a_k^\dagger b_q^\dagger e^{-i(q+k)x} - [b_q^\dagger b_k - 2\omega_k (2\pi)^3 \delta^3(\vec{k} - \vec{q})] e^{-i(q-k)} \right. \tag{886}$$

$$\varphi^\dagger \varphi = \int \widetilde{d^3k}\widetilde{d^3q} \left(a_k^\dagger e^{-ikx} + b_k e^{ikx}\right) \left(a_q e^{iqx} + b_q^\dagger e^{-iqx}\right) \tag{887}$$

$$= \int \widetilde{d^3k}\widetilde{d^3q} \left(a_k^\dagger a_q e^{iqx} e^{-ikx} + b_k a_q e^{iqx} e^{ikx} + a_k^\dagger b_q^\dagger e^{-iqx} e^{-ikx} + b_k b_q^\dagger e^{-iqx} e^{ikx}\right) \tag{888}$$

$$= \int \widetilde{d^3k}\widetilde{d^3q} \left(a_k^\dagger a_q e^{i(q-k)x} + a_q b_k e^{i(q+k)x} + a_k^\dagger b_q^\dagger e^{-i(q+k)x} + [b_q^\dagger b_k - 2\omega_k (2\pi)^3 \delta^3(\vec{k} - \vec{q})] e^{-i(q-k)x}\right) \tag{889}$$

then

$$H_{a^\dagger a} = \int \widetilde{d^3k}\widetilde{d^3q} \int d^3x \left[(-\omega_k \omega_q)[a_q^\dagger a_k - 2\omega_k (2\pi)^3 \delta^3(\vec{k} - \vec{q})] e^{-i(q-k)x}\right] \tag{890}$$

$$+ \int \widetilde{d^3k}\widetilde{d^3q} \int d^3x (k^a q_a) \left[-a_k^\dagger a_q e^{i(q-k)x}\right] + m^2 a_k^\dagger a_q e^{i(q-k)x} \tag{891}$$

$$= \int \widetilde{d^3k}\widetilde{d^3q}\ a_k^\dagger a_q \left[-\omega_k \omega_q - k^a q_a + m^2\right] \int d^3x e^{i(q-k)x} \tag{892}$$

$$- \int \widetilde{d^3k}\widetilde{d^3q}\ (-\omega_k \omega_q) 2\omega_q (2\pi)^3 \delta^3(\vec{q} - \vec{k}) \int d^3x\ e^{i(q-k)x} \tag{893}$$

$$= \int \widetilde{d^3k} \frac{d^3q}{(2\pi)^3 2\omega_q}\ a_k^\dagger a_q \left[-\omega_k \omega_q - k^a q_a + m^2\right] (2\pi)^3 \delta^3(\vec{q} - \vec{k}) e^{-i(\omega_q - \omega_k)t} \tag{894}$$

$$- \int \frac{d^3k}{(2\pi)^3 2\omega_k} \frac{1}{(2\pi)^3 2\omega_k}\ (-\omega_k^2) 2\omega_k (2\pi)^3 e^{-i(\omega_k - \omega_k)t} \int d^3x \tag{895}$$

$$= \int \widetilde{d^3k} \frac{1}{2\omega_k}\ a_k^\dagger a_k \underbrace{\left[-\omega_k^2 - \vec{k}^2 + m^2\right]}_{2\omega_k^2 !?!?!} + \frac{V}{2(2\pi)^3} \int d^3k\ \omega_k \tag{896}$$

$$= \int \widetilde{d^3k} \omega_k\ a_k^\dagger a_k + \frac{V}{2(2\pi)^3} \int d^3k\ \omega_k \tag{897}$$

and similar for $H_{b^\dagger b}, H_{ab}, H_{a^\dagger b^\dagger}$.

$$H = \int \widetilde{d^3k} \omega_k\ (a_k^\dagger a_k + b_k^\dagger b_k) + \frac{V}{2(2\pi)^3} \int d^3k\ \omega_k \tag{898}$$

Problem 4.1 - Commutator non-hermitian field

With $t = t'$ and $|\vec{x} - \vec{x}'| = r$ we have

$$[\varphi^+(x), \varphi^-(x')]_\pm = \int \widetilde{dk} e^{ik(x-x')} \tag{899}$$

$$= \int d^3k \frac{1}{(2\pi)^3 2\omega_k} e^{ik(x-x')} \tag{900}$$

$$= \frac{1}{2 \cdot 8\pi^3} \int d^3k \frac{1}{\sqrt{|k|^2 + m^2}} e^{i[\vec{k}(\vec{x}-\vec{x}')]} \tag{901}$$

$$= \frac{1}{16\pi^3} \int |k|^2 dk d\phi d\theta \sin\theta \frac{1}{\sqrt{|k|^2 + m^2}} e^{i|k|r\cos\theta} \tag{902}$$

$$= \frac{2\pi}{16\pi^3} \int |k|^2 dk \underbrace{d\theta \sin\theta}_{-d\cos\theta} \frac{1}{\sqrt{|k|^2 + m^2}} e^{i|k|r\cos\theta} \tag{903}$$

$$= \frac{2\pi}{16\pi^3} \int |k|^2 dk \frac{1}{\sqrt{|k|^2 + m^2}} \int_{-1}^{1} d\cos\theta e^{i|k|r\cos\theta} \tag{904}$$

$$= \frac{2\pi}{16\pi^3} \int |k|^2 dk \frac{1}{\sqrt{|k|^2 + m^2}} 2 \frac{\sin(|k|r)}{|k|r} \tag{905}$$

$$= \frac{1}{4\pi^2 r} \int_0^\infty dk \frac{|k| \sin(|k|r)}{\sqrt{|k|^2 + m^2}} \tag{906}$$

With Gradshteyn, Ryzhik 7ed (8.486) - we find for the definition of the modified Bessel function K_1

$$\frac{d}{dz} K_0(z) = -K_1(z) \tag{907}$$

and Gradshteyn, Ryzhik 7ed (3.754)

$$\int_0^\infty dx \frac{\cos(ax)}{\sqrt{\beta^2 + x^2}} = K_0(a\beta) \tag{908}$$

therefore

$$\frac{d}{da} K_0(a\beta) = \int_0^\infty dx \frac{-x\sin(ax)}{\sqrt{\beta^2 + x^2}} \tag{909}$$

$$= \beta K_0'(a\beta) \tag{910}$$

$$= -\beta K_1(a\beta) \tag{911}$$

$$\to K_1(a\beta) = \frac{1}{\beta} \int_0^\infty dx \frac{x\sin(ax)}{\sqrt{\beta^2 + x^2}} \tag{912}$$

which we can use to finish the calculation

$$[\varphi^+(x), \varphi^-(x')]_\pm = \frac{1}{4\pi^2 r} m K_1(mr) \tag{913}$$

From `https://dlmf.nist.gov/10.30` we get

$$\lim_{z\to 0} K_\nu(z) \sim \frac{1}{2}\Gamma(\nu) \left(\frac{1}{2}z\right)^{-\nu} \tag{914}$$

$$\to \lim_{z\to 0} K_1(z) \sim \frac{1}{2} \left(\frac{1}{2}z\right)^{-1} = 1/z \tag{915}$$

and therefore

$$[\varphi^+(x), \varphi^-(x')]_\pm = \frac{1}{4\pi^2 r^2}. \tag{916}$$

Problem 5.1 - LSZ reduction for complex scalar field

From Exercise 3.5 we have

$$a_q = i\int d^3x e^{-iqx} \overleftrightarrow{\partial_0}\, \varphi(\vec{x},t) \tag{917}$$

$$a_q^\dagger = -i\int d^3x e^{iqx} \overleftrightarrow{\partial_0}\, \varphi^\dagger(\vec{x},t) \tag{918}$$

$$b_q = i\int d^3x e^{-iqx} \overleftrightarrow{\partial_0}\, \varphi^\dagger(\vec{x},t) \tag{919}$$

$$b_q^\dagger = -i\int d^3x e^{iqx} \overleftrightarrow{\partial_0}\, \varphi(\vec{x},t) \tag{920}$$

then

$$a_1^\dagger(+\infty) - a_1^\dagger(-\infty) = -i\int d^3k f_1(\vec{k}) \int d^4x e^{ikx}(-\Box_x + m^2)\varphi^\dagger(x) \tag{921}$$

rearranging leads to

$$a_1^\dagger(-\infty) = a_1^\dagger(+\infty) + i\int d^3k f_1(\vec{k}) \int d^4x e^{ikx}(-\Box_x + m^2)\varphi^\dagger(x) \tag{922}$$

$$a_1(+\infty) = a_1(-\infty) + i\int d^3k f_1(\vec{k}) \int d^4x e^{-ikx}(-\Box_x + m^2)\varphi(x) \tag{923}$$

$$b_1^\dagger(-\infty) = b_1^\dagger(+\infty) + i\int d^3k f_1(\vec{k}) \int d^4x e^{ikx}(-\Box_x + m^2)\varphi^\dagger(x) \tag{924}$$

$$b_1(+\infty) = b_1(-\infty) + i\int d^3k f_1(\vec{k}) \int d^4x e^{-ikx}(-\Box_x + m^2)\varphi(x) \tag{925}$$

then we get for a, b particle scattering with the time ordering operator T (Later time to the Left)

$$\langle f|i\rangle = \langle 0|a_{1'}(+\infty)b_{2'}(+\infty)a_1^\dagger(-\infty)b_2^\dagger(-\infty)|0\rangle \tag{926}$$

$$= \langle 0|T a_{1'}(+\infty)b_{2'}(+\infty)a_1^\dagger(-\infty)b_2^\dagger(-\infty)|0\rangle \tag{927}$$

$$= \langle 0|T(a_{1'}(-\infty) + i\int)(b_{2'}(-\infty) + i\int)(a_1^\dagger(+\infty) + i\int)(b_2^\dagger(+\infty) + i\int)|0\rangle \tag{928}$$

$$= i^4 \int d^4x_1' e^{-ik_1'x_1'}(-\Box_{x_1'} + m_a^2) \int d^4x_2' e^{-ik_2'x_2'}(-\Box_{x_2'} + m_b^2)\times \tag{929}$$

$$\times \int d^4x_1 e^{-ik_1x_1}(-\Box_{x_1} + m_a^2) \int d^4x_2 e^{-ik_2x_2}(-\Box_{x_2} + m_b^2)\langle 0|\phi_{x_1'}\phi_{x_2'}\phi_{x_1}^\dagger\phi_{x_2}^\dagger|0\rangle \tag{930}$$

Problem 6.1 - Path integral in quantum mechanics

(a) The transition amplitude $\langle q''|e^{-iH(t''-t')}|q'\rangle$ (particle to start at q', t' and ends at position q'' at time t'') can be written in the Heisenberg picture as

$$\langle q''|e^{-iH(t''-t')}|q'\rangle = \langle q''|e^{-iHt''}e^{iHt''}e^{-iH(t''-t')}e^{-iHt'}e^{iHt'}|q'\rangle \tag{931}$$

$$= \langle q'', t''|e^{iHt''}e^{iH(t''-t')}e^{-iHt'}|q', t'\rangle \tag{932}$$

$$= \langle q'', t''|q', t'\rangle. \tag{933}$$

Now we can do the standard path integral derivation

$$\langle q'', t''|q', t'\rangle = \int \left(\prod_{j=1}^{N} dq_j\right) \langle q''|e^{-iH\delta t}|q_N\rangle\langle q_N|e^{-iH\delta t}|q_{N-1}\rangle \dots \langle q_1|e^{-iH\delta t}|q'\rangle \tag{934}$$

$$= \int \left(\prod_{j=1}^{N} dq_j\right) \int \frac{dp_N}{2\pi} e^{-iH(p_N,q_N)\delta t} e^{ip_N(q'-q_N)} \dots \int \frac{dp'}{2\pi} e^{-iH(p',q')\delta t} e^{ip'(q_1-q')} \tag{935}$$

$$= \int \left(\prod_{j=1}^{N} dq_j\right) \left(\prod_{k=0}^{N} \frac{dp_k}{2\pi} e^{ip_k(q_{k+1}-q_k)} e^{-iH(p_k,q_k)\delta t}\right) \quad (q_0 = q', q_{N+1} = q'') \tag{936}$$

which under Weyl ordering (see Greiner, Reinhard - field quantization) has to be replaced by

$$\langle q'', t''|q', t'\rangle = \int \left(\prod_{j=1}^{N} dq_j\right) \left(\prod_{k=0}^{N} \frac{dp_k}{2\pi} e^{ip_k(q_{k+1}-q_k)} e^{-iH(p_k,\bar{q}_k)\delta t}\right) \quad \bar{q}_k = (q_{k+1} + q_k)/2 \tag{937}$$

$$= \int \left(\prod_{j=1}^{N} dq_j\right) \left(\prod_{k=0}^{N} \frac{dp_k}{2\pi} e^{i[p_k\dot{q}_k - H(p_k,\bar{q}_k)]\delta t}\right) \quad \dot{q}_k = (q_{k+1} - q_k)/\delta t \tag{938}$$

$$= \int \left(\prod_{j=1}^{N} dq_j\right) \left(\prod_{k=0}^{N} \frac{dp_k}{2\pi}\right) \left(e^{i\sum_{n=0}^{N}[p_n\dot{q}_n - H(p_n,\bar{q}_n)]\delta t}\right) \tag{939}$$

$$= \int \mathcal{D}q\mathcal{D}p \exp\left[i\int_{t'}^{t''} dt\,(p(t)\dot{q}(t) - H(p(t), q(t)))\right] \tag{940}$$

Let's now assume $H(p, q)$ has only a quadratic term in p which is independent of q meaning

$$H(p, q) = \frac{p^2}{2m} + V(q) \tag{941}$$

then

$$\langle q'', t''|q', t'\rangle = \int \left(\prod_{j=1}^{N} dq_j\right) \left(\prod_{k=0}^{N} \frac{dp_k}{2\pi}\right) \left(e^{i\sum_{n=0}^{N}[p_n\dot{q}_n - \frac{1}{2m}p_n^2 - V(\bar{q}_n)]\delta t}\right) \tag{942}$$

We can evaluate a single p-integral using

$$\int_{-\infty}^{\infty} dx e^{-ax^2+bx+c} = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}+c} \tag{943}$$

and obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dp_k \left(e^{i[p_k\dot{q}_k - \frac{1}{2m}p_k^2 - V(\bar{q}_k)]\delta t}\right) = \frac{1}{2\pi} e^{-iV(\bar{q}_k)\delta t} \int dp_k \left(e^{i[p_k\dot{q}_k - \frac{1}{2m}p_k^2]\delta t}\right) \tag{944}$$

$$= \frac{1}{2\pi} e^{-iV(\bar{q}_k)\delta t} \sqrt{\frac{\pi}{i\frac{\delta t}{2m}}} e^{\frac{-\dot{q}_k^2\delta t^2}{4\frac{i\delta t}{2m}}} \tag{945}$$

$$= \frac{1}{2\pi}\sqrt{\frac{2\pi m}{i\delta t}} e^{i\left(\frac{m\dot{q}_k^2}{2} - V(\bar{q}_k)\right)\delta t} \tag{946}$$

$$= \sqrt{\frac{m}{2\pi i\delta t}} e^{iL(\bar{q}_k,\dot{q}_k)\delta t}. \tag{947}$$

As there are $N+1$ p-integrals we have

$$\mathcal{D}q = \Big(\frac{m}{2\pi i\delta t}\Big)^{(N+1)/2}\prod_{j=1}^{N} dq_j \tag{948}$$

(b) We now assume $V(q)=0$

$$\langle q'', t''|q', t'\rangle = \int \mathcal{D}q\, e^{i\int_{t'}^{t''} dt \frac{\dot{q}^2}{2m}} \tag{949}$$

$$= \lim_{N\to\infty}\Big(\frac{m}{2\pi i\delta t}\Big)^{\frac{N+1}{2}}\left(\prod_{j=1}^{N}\int_{-\infty}^{\infty} dq_j\, e^{im\frac{(q_j-q_{j+1})^2}{2\delta t^2}\delta t}\right) e^{im\frac{(q'-q_1)^2}{2\delta t}} e^{im\frac{(q_N-q'')^2}{2\delta t}} \tag{950}$$

$$= \lim_{N\to\infty}\Big(\frac{m}{2\pi i\delta t}\Big)^{\frac{N+1}{2}}\left(\prod_{j=3}^{N}\int_{-\infty}^{\infty} dq_j\, e^{im\frac{(q_j-q_{j+1})^2}{2\delta t}}\right)\int dq_2 e^{im\frac{(q_2-q_3)^2}{2\delta t}}\int dq_1 e^{im\frac{(q_1-q_2)^2}{2\delta t}} e^{im\frac{(q_0-q_1)^2}{2\delta t}} \tag{951}$$

now we can simplify the q_1-integral

$$\int_{-\infty}^{\infty} dq_1\, e^{im\frac{(q_1-q_2)^2}{2\delta t}} e^{im\frac{(q_0-q_1)^2}{2\delta t}} = \int_{-\infty}^{\infty} dq_1\, e^{\frac{im}{2\delta t}(q_0^2-2q_0q_1+q_1^2+q_1^2-2q_1q_2+q_2^2)} \tag{952}$$

$$= e^{\frac{im}{2\delta t}(q_0^2+q_2^2)}\int_{-\infty}^{\infty} dq_1\, e^{\frac{im}{\delta t}(q_1^2-q_1(q_2+q_0))} \tag{953}$$

$$= e^{\frac{im}{2\delta t}(q_0^2+q_2^2)}\sqrt{\frac{\pi\delta t}{m}}e^{\frac{i}{4}\left(\pi-\frac{(q_2+q_0)^2m}{\delta t}\right)} \tag{954}$$

$$= e^{\frac{im}{4\delta t}(q_0-q_2)^2}\sqrt{\frac{\pi\delta t}{m}}\sqrt{i} \tag{955}$$

$$= e^{\frac{im}{4\delta t}(q_0-q_2)^2}\sqrt{\frac{i\pi\delta t}{m}} \tag{956}$$

now simplify the q_2-integral

$$\sqrt{\frac{i\pi\delta t}{m}}\int_{-\infty}^{\infty} dq_2 e^{\frac{im}{2\delta t}(q_2-q_3)^2} e^{\frac{im}{4\delta t}(q_0-q_2)^2} = \sqrt{\frac{i\pi\delta t}{m}}\int_{-\infty}^{\infty} dq_2 e^{\frac{im}{4\delta t}(2q_2^2-4q_3q_2+2q_3^2+q_0^2-2q_0q_2+q_2^2)} \tag{957}$$

$$= \sqrt{\frac{i\pi\delta t}{m}}\int_{-\infty}^{\infty} dq_2 e^{\frac{im}{4\delta t}(3q_2^2-(4q_3+2q_0)q_2+2q_3^2+q_0^2)} \tag{958}$$

$$= \sqrt{\frac{i\pi\delta t}{m}}e^{\frac{im}{4\delta t}(2q_3^2+q_0^2)}\int_{-\infty}^{\infty} dq_2 e^{\frac{im}{4\delta t}(3q_2^2-(4q_3+2q_0)q_2)} \tag{959}$$

$$= \sqrt{\frac{i\pi\delta t}{m}}e^{\frac{im}{4\delta t}(2q_3^2+q_0^2)}\sqrt{\frac{\pi 4\delta t}{3m}}e^{\frac{i}{4}\left(\pi-\frac{(4q_3+2q_0)^2m}{12\delta t}\right)} \tag{960}$$

$$= \sqrt{\frac{i\pi\delta t}{m}}\sqrt{\frac{4i\pi\delta t}{3m}}e^{\frac{im}{6\delta t}(q_3-q_0)^2} \tag{961}$$

then we can extend the results (without explicitly proving)

$$\langle q'', t''|q', t'\rangle = \lim_{N\to\infty}\Big(\frac{m}{2\pi i\delta t}\Big)^{\frac{N+1}{2}}\prod_{j=1}^{N}\sqrt{\frac{2i\pi\delta t}{m}\frac{j}{j+1}}\cdot e^{\frac{im}{2(j+1)\delta t}(q''-q')^2} \tag{962}$$

$$= \lim_{N\to\infty}\sqrt{\frac{m}{2\pi i\delta t}}\sqrt{\frac{1}{N+1}}\cdot e^{\frac{im}{2(N+1)\delta t}(q_{N+1}-q_0)^2} \tag{963}$$

$$= \sqrt{\frac{m}{2\pi i(t''-t')}}\cdot e^{\frac{im(q''-q')^2}{2(t''-t')}}. \tag{964}$$

The exponent has the dimension kg $\cdot$ m$^2/s$ which is the same as Js. So we just insert an $\hbar$

$$\langle q'', t''|q', t'\rangle = \sqrt{\frac{m}{2\pi i\hbar(t''-t')}} \cdot e^{\frac{im(q''-q')^2}{2\hbar(t''-t')}}. \tag{965}$$

(c) Simple - with $H|k\rangle = \frac{k^2}{2m}|k\rangle$ we get

$$\begin{aligned}
\langle q'', t''|q', t'\rangle &= \langle q''|\exp(-iH(t''-t'))|q'\rangle && (966)\\
&= \int dp \int dk \langle q''|p\rangle\langle p|\exp(-iH(t''-t'))|k\rangle\langle k|q'\rangle && (967)\\
&= \int dp \int dk \frac{1}{\sqrt{2\pi}} e^{ipq'} \langle p|k\rangle \exp(-i\frac{k^2}{2m}(t''-t'))\frac{1}{\sqrt{2\pi}}e^{-ikq''} && (968)\\
&= \int dp \int dk \frac{1}{\sqrt{2\pi}} e^{ipq'} \exp(-i\frac{k^2}{2m}(t''-t'))\delta(k-p)\frac{1}{\sqrt{2\pi}}e^{-ikq''} && (969)\\
&= \frac{1}{2\pi}\int dp e^{ip(q'-q'')} \exp(-i\frac{p^2}{2m}(t''-t')) && (970)\\
&= \frac{1}{2\pi}\sqrt{-\frac{2m\pi}{t''-t'}} e^{\frac{i}{4}\left(\pi - \frac{-2m(q''-q')^2}{t''-t'}\right)} && (971)\\
&= \sqrt{-\frac{im}{2\pi(t''-t')}} e^{-\frac{i}{4}\frac{-2m(q''-q')^2}{t''-t'}} && (972)\\
&= \sqrt{\frac{m}{2\pi i(t''-t')}} e^{\frac{-im(q''-q')^2}{2(t''-t')}} && (973)\\
& && (974)
\end{aligned}$$

which is the same as in (b).

Problem 7.1 - Oscillator Green's function I

$$\begin{aligned}
G(t-t') &= \int_{-\infty}^{+\infty} \frac{dE}{2\pi} \frac{e^{-iE(t-t')}}{-E^2+\omega^2-i\epsilon} && (975)\\
&= -\frac{1}{2\pi}\int_{-\infty}^{+\infty} dE \frac{e^{-iE(t-t')}}{E^2-\omega^2+i\epsilon} && (976)
\end{aligned}$$

with

$$\begin{aligned}
E^2-\omega^2+i\epsilon &= (E+\sqrt{\omega^2-i\epsilon})(E-\sqrt{\omega^2-i\epsilon}) && (977)\\
&= \left(E+\omega\sqrt{1-\frac{i\epsilon}{\omega^2}}\right)\left(E-\omega\sqrt{1-\frac{i\epsilon}{\omega^2}}\right) && (978)\\
&\simeq \left(E+\omega-\frac{i\epsilon}{2\omega}\right)\left(E-\omega+\frac{i\epsilon}{2\omega^2}\right) && (979)
\end{aligned}$$

we can simplify

$$\begin{aligned}
G(\Delta t) &= -\frac{1}{2\pi}\int_{-\infty}^{+\infty} dE e^{-iE\Delta t}\left(\frac{1}{E+\omega-\frac{i\epsilon}{2\omega}} + \frac{1}{E-\omega+\frac{i\epsilon}{2\omega}}\right) && (980)\\
&= -\frac{1}{2\pi}\frac{1}{2\left(\omega-\frac{i\epsilon}{2\omega}\right)}\int_{-\infty}^{+\infty} dE e^{-iE\Delta t}\left(-\frac{1}{E+\omega-\frac{i\epsilon}{2\omega}} + \frac{1}{E-\omega+\frac{i\epsilon}{2\omega}}\right) && (981)
\end{aligned}$$

Integrating along the closed contour along the lower half plane (seeing that the exponential function makes the arc part vanish - for $\Delta t > 0$) and using the residual theorem (only one pole is inside) we get (with $\epsilon \to 0$)

$$G(\Delta t) = +\frac{1}{2\pi} \frac{1}{2\left(\omega - \frac{i\epsilon}{2\omega}\right)} (2\pi i) e^{-i(\omega - \frac{i\epsilon}{2\omega})\Delta t} \tag{982}$$

$$= \frac{i}{2\omega} e^{-i\omega\Delta t} \tag{983}$$

For $\Delta t < 0$ we integrate along the contour of the upper plane - combining both results we get

$$G(t) = \frac{i}{2\omega} e^{-i\omega|t|} \tag{984}$$

Problem 7.2 - Oscillator Green's function II

We can rewrite the Greens function using the Heaviside theta function

$$|t| = (2\theta(t) - 1)t \tag{985}$$

$$\frac{d}{dt}|t| = 2\theta'(t)t + (2\theta(t) - 1) \tag{986}$$

$$= 2\underbrace{\delta(t)t}_{=0} + 2\theta(t) - 1 \tag{987}$$

$$= 2\theta(t) - 1 \tag{988}$$

and then differentiate and use $\theta'(t) = \delta(t)$

$$G(t) = \frac{i}{2\omega} e^{-i\omega(2\theta(t)-1)t} \tag{989}$$

$$\partial_t G(t) = \frac{i}{2\omega} e^{-i\omega(2\theta(t)-1)t} (-i\omega)(2\theta(t) - 1)) \tag{990}$$

$$= (-i\omega) G(t) \left(2\theta(t) - 1\right) \tag{991}$$

$$\partial_{tt} G(t) = (-i\omega)\partial_t G(t) \left(2\theta(t) - 1\right) + (-2i\omega) G(t)\delta(t) \tag{992}$$

$$= (-i\omega)^2 G(t) \left(2\theta(t) - 1\right)^2 + (-2i\omega) G(t)\delta(t) \tag{993}$$

$$= -\omega^2 G(t) + e^{-i\omega|t|}\delta(t) \tag{994}$$

where we used $(2\theta(t) - 1)^2 \equiv 1$

$$\left(\partial_{tt} + \omega^2\right) G(t) = \left(-\omega^2 + \omega^2\right) G(t) + \delta(t) = \delta(t) \tag{995}$$

Problem 7.3 - Harmonic Oscillator - Heisenberg and Schroedinger picture

(a) With $\hbar = 1$ and

$$H = \frac{1}{2}P^2 + \frac{1}{2}m\omega^2 Q^2 \tag{996}$$

$$[Q, P] = QP - PQ = i \tag{997}$$

$$[Q, Q] = [P, P] = 0 \tag{998}$$

we obtain for the commutators

$$[P^2, Q] = P(PQ) - QP^2 \tag{999}$$
$$= P(QP - i) - QP^2 \tag{1000}$$
$$= (PQ)P - Pi - QP^2 \tag{1001}$$
$$= (QP - i)P - Pi - QP^2 \tag{1002}$$
$$= -2Pi \tag{1003}$$
$$[Q^2, P] = Q(QP) - PQ^2 \tag{1004}$$
$$= Q(PQ + i) - PQ^2 \tag{1005}$$
$$= (QP)Q + iQ - PQ^2 \tag{1006}$$
$$= (PQ + i)Q + iQ - PQ^2 \tag{1007}$$
$$= 2Qi \tag{1008}$$

Then the Heisenberg equations are

$$\dot{Q}(t) = i[H, Q(t)] = i\frac{1}{2m}[P^2(t), Q(t)] = \frac{1}{m}P(t) \tag{1009}$$
$$\dot{P}(t) = i[H, P(t)] = i\frac{1}{2}m\omega^2[Q^2(t), P(t)] = -m\omega^2 Q(t) \tag{1010}$$
$$\rightarrow \ddot{Q}(t) = \frac{1}{m}\dot{P}(t) = -\omega^2 Q(t) \tag{1011}$$

with the solutions (initial conditions $Q(0) = Q, P(0) = P$)

$$Q(t) = A\cos\omega t + B\sin\omega t \qquad \rightarrow A = Q, \quad \omega B = \frac{1}{m}P \tag{1012}$$
$$= Q\cos\omega t + \frac{1}{\omega m}P\sin\omega t \tag{1013}$$
$$P(t) = m\dot{Q}(t) \tag{1014}$$
$$= -m\omega Q\sin\omega t + P\cos\omega t \tag{1015}$$

(b) Using Diracs trick from QM (rewriting H in terms of a and $a^\dagger$)

$$a = \sqrt{\frac{m\omega}{2}}(Q + \frac{i}{m\omega}P) \tag{1016}$$
$$a^\dagger = \sqrt{\frac{m\omega}{2}}(Q - \frac{i}{m\omega}P) \tag{1017}$$

we can invert the relation

$$Q = \frac{1}{\sqrt{2m\omega}}(a^\dagger + a) \tag{1018}$$
$$P = i\sqrt{\frac{m\omega}{2}}(a^\dagger - a) \tag{1019}$$

and

$$Q(t) = Q\cos\omega t + \frac{1}{\omega m}P\sin\omega t \tag{1020}$$
$$= \frac{1}{\sqrt{2m\omega}}(a^\dagger + a)\cos\omega t + \frac{1}{\omega m}i\sqrt{\frac{m\omega}{2}}(a^\dagger - a)\sin\omega t \tag{1021}$$
$$= \frac{1}{\sqrt{2m\omega}}\left((a^\dagger + a)\cos\omega t + i(a^\dagger - a)\sin\omega t\right) \tag{1022}$$
$$= \frac{1}{\sqrt{2m\omega}}\left(a^\dagger(\cos\omega t + i\sin\omega t) + a(\cos\omega t - i\sin\omega t)\right) \tag{1023}$$
$$= \frac{1}{\sqrt{2m\omega}}\left(a^\dagger e^{i\omega t} + ae^{-i\omega t}\right) \tag{1024}$$
$$P(t) = i\sqrt{\frac{m\omega}{2}}\left(a^\dagger e^{i\omega t} - ae^{-i\omega t}\right) \tag{1025}$$
$$\tag{1026}$$

(c) Now with $t_1 < t_2$ and the time ordering operator (larger time to the left)

$$\langle 0|TQ(t_1)Q(t_2)|0\rangle = \frac{1}{2m\omega}\langle 0|T\left(a^\dagger e^{i\omega t_1} + ae^{-i\omega t_1}\right)\left(a^\dagger e^{i\omega t_2} + ae^{-i\omega t_2}\right)|0\rangle \tag{1027}$$
$$= \frac{1}{2m\omega}\langle 0|\left(a^\dagger e^{i\omega t_2} + ae^{-i\omega t_2}\right)\left(a^\dagger e^{i\omega t_1} + ae^{-i\omega t_1}\right)|0\rangle \tag{1028}$$
$$= \frac{1}{2m\omega}\langle 0|ae^{-i\omega t_2}a^\dagger e^{i\omega t_1}|0\rangle \tag{1029}$$

all other terms are vanishing because of $a|0\rangle = 0$ and $\langle 0|a^\dagger = 0$. Then

$$\langle 0|TQ(t_1)Q(t_2)|0\rangle = \frac{1}{2m\omega}e^{-i\omega(t_2-t_1)}\underbrace{\langle 0|aa^\dagger|0\rangle}_{=1} \tag{1030}$$
$$= \frac{1}{2m\omega}e^{-i\omega(t_2-t_1)} \tag{1031}$$
$$\equiv \frac{1}{i}G(t_2 - t_1) \tag{1032}$$

And now the next case with $t_1 > t_2 > t_3 > t_4$

$$\langle 0|TQ(t_1)Q(t_2)Q(t_3)Q(t_4)|0\rangle = \frac{1}{(2m\omega)^2}\cdots \tag{1033}$$

Problem 7.4 - Harmonic Oscillator with perturbation

As $f(t)$ is a real function we have $\tilde{f}(-E) = (\tilde{f}(E))^*$ then with (7.10)

$$\langle 0|0\rangle_f = \exp\left[\frac{i}{2}\int_{-\infty}^{+\infty}\frac{dE}{2\pi}\frac{\tilde{f}(E)\tilde{f}(-E)}{-E^2 + \omega^2 - i\epsilon}\right] \tag{1034}$$
$$= \exp\left[\frac{i}{2}\int_{-\infty}^{+\infty}\frac{dE}{2\pi}\frac{\tilde{f}(E)\tilde{f}(E)^*}{-E^2 + \omega^2 - i\epsilon}\right] \tag{1035}$$

But we actually need to calculate $|\langle 0|0\rangle_f|^2$ therefore we observe with

$$e^{iz} = e^{i(x+iy)} = e^{-y}e^{ix} = e^{-y}(\cos x + i\sin x) \tag{1036}$$
$$\rightarrow \quad (e^{iz})^* = e^{-y}(\cos x - i\sin x) = e^{-y-ix}e^{-i(x-iy)} = e^{-iz^*} \tag{1037}$$
$$\langle 0|0\rangle_f = e^{iA} \quad \rightarrow \quad |\langle 0|0\rangle_f|^2 = e^{iA}(e^{iA})^* = e^{iA}e^{-iA^*} = e^{i(A-A^*)} = e^{-2\Im A} \tag{1038}$$

Now we calculate the imaginary part of the integral

$$\Im\frac{1}{4\pi}\int_{-\infty}^{+\infty}dE\frac{\tilde{f}(E)\tilde{f}(E)^*}{-E^2+\omega^2-i\epsilon} = \frac{1}{4\pi}\int_{-\infty}^{+\infty}dE\Im\frac{\tilde{f}(E)\tilde{f}(E)^*}{-E^2+\omega^2-i\epsilon} \tag{1039}$$

$$= \frac{1}{4\pi}\int_{-\infty}^{+\infty}dE\tilde{f}(E)\tilde{f}(E)^*\Im\frac{1}{-E^2+\omega^2-i\epsilon} \tag{1040}$$

$$= \frac{1}{4\pi}\int_{-\infty}^{+\infty}dE\tilde{f}(E)\tilde{f}(E)^*\Im\frac{-E^2+\omega^2+i\epsilon}{(-E^2+\omega^2)^2+\epsilon^2} \tag{1041}$$

$$= \frac{1}{4\pi}\int_{-\infty}^{+\infty}dE\tilde{f}(E)\tilde{f}(E)^*\frac{\epsilon}{(-E^2+\omega^2)^2+\epsilon^2} \tag{1042}$$

$$\simeq \frac{1}{4\pi}\int_{-\infty}^{+\infty}dE\tilde{f}(E)\tilde{f}(E)^*\pi\delta(-E^2+\omega^2) \tag{1043}$$

$$\simeq \frac{1}{4\pi}\int_{-\infty}^{+\infty}dE\tilde{f}(E)\tilde{f}(E)^*\pi\delta((\omega+E)(\omega-E)) \tag{1044}$$

$$\simeq \frac{1}{4\cdot 2\omega}(\tilde{f}(\omega)\tilde{f}(\omega)^*+\tilde{f}(-\omega)\tilde{f}(-\omega)^*) \tag{1045}$$

$$\simeq \frac{1}{8\omega}(\tilde{f}(\omega)\tilde{f}(\omega)^*+\tilde{f}(\omega)^*\tilde{f}(\omega)) \tag{1046}$$

$$\simeq \frac{1}{4\omega}\tilde{f}(\omega)\tilde{f}(\omega)^* \tag{1047}$$

then

$$|\langle 0|0\rangle_f|^2 = e^{-2\left(\frac{1}{4\omega}\right)\tilde{f}(\omega)\tilde{f}(\omega)^*} \tag{1048}$$

$$= e^{-\frac{1}{2\omega}\tilde{f}(\omega)\tilde{f}(\omega)^*} \tag{1049}$$

$$\tag{1050}$$

Problem 8.1 - Feynman propagator is Greens function Klein-Gordon equation

With

$$\Delta(x-x') = \frac{1}{(2\pi)^4}\int d^4k\frac{e^{ik(x-x')}}{k^2+m^2-i\epsilon} \tag{1051}$$

we have

$$(-\partial_x^2+m^2)\Delta(x-x') = \frac{1}{(2\pi)^4}\int d^4k(-i^2k^2+m^2)\frac{e^{ik(x-x')}}{k^2+m^2-i\epsilon} \tag{1052}$$

$$= \frac{1}{(2\pi)^4}\int d^4k\frac{k^2+m^2}{k^2+m^2-i\epsilon}e^{ik(x-x')} \tag{1053}$$

$$\simeq \frac{1}{(2\pi)^4}\int d^4ke^{ik(x-x')} \tag{1054}$$

$$= \delta^4(x-x') \tag{1055}$$

Problem 8.2 - Feynman propagator II

With $\widetilde{dk} = d^3k/((2\pi)^3 2\omega_k)$ and $\omega_k = \sqrt{\vec{k}^2 + m^2}$

$$\Delta(x-x') = \frac{1}{(2\pi)^4}\int d^4k \frac{e^{ik(x-x')}}{k^2+m^2-i\epsilon} \tag{1056}$$
$$= \frac{1}{(2\pi)^4}\int d^3k \int dk^0 e^{-ik^0(t-t')}\frac{e^{i\vec{k}(\vec{x}-\vec{x}')}}{-(k^0)^2+\vec{k}^2+m^2-i\epsilon} \tag{1057}$$
$$= \frac{1}{(2\pi)^4}\int d^3k\, e^{i\vec{k}(\vec{x}-\vec{x}')}\int dE \frac{e^{-iE(t-t')}}{-E^2+\vec{k}^2+m^2-i\epsilon} \tag{1058}$$
$$= \frac{1}{(2\pi)^4}\int d^3k\, e^{i\vec{k}(\vec{x}-\vec{x}')} 2\pi \frac{i}{2(\vec{k}^2+m^2)} e^{-i(\vec{k}^2+m^2)|t-t'|} \tag{1059}$$

where we used exercise (7.1). Then

$$\Delta(x-x') = \frac{i}{(2\pi)^3}\int d^3k\, e^{i\vec{k}(\vec{x}-\vec{x}')}\frac{i}{2\omega_k}e^{-i\omega_k|t-t'|} \tag{1060}$$
$$= i\int \frac{d^3k}{(2\pi)^3 2\omega_k}\, e^{i\vec{k}(\vec{x}-\vec{x}')}e^{-i\omega_k|t-t'|} \tag{1061}$$
$$= i\int \widetilde{dk}\, e^{i\vec{k}(\vec{x}-\vec{x}')-i\omega_k|t-t'|} \tag{1062}$$
$$= i\theta(t-t')\int \widetilde{dk}\, e^{i\vec{k}(\vec{x}-\vec{x}')-i\omega_k(t-t')} + i\theta(t'-t)\int \widetilde{dk}\, e^{i\vec{k}(\vec{x}-\vec{x}')+i\omega_k(t-t')} \tag{1063}$$
$$= i\theta(t-t')\int \widetilde{dk}\, e^{ik(x-x')} + i\theta(t'-t)\int \widetilde{dk}\, e^{-i\vec{k}(\vec{x}-\vec{x}')+i\omega_k(t-t')} \tag{1064}$$
$$= i\theta(t-t')\int \widetilde{dk}\, e^{ik(x-x')} + i\theta(t'-t)\int \widetilde{dk}\, e^{-ik(x-x')} \tag{1065}$$
$$\tag{1066}$$

7.5 Kachelriess - Quantum Fields - From the Hubble to the Planck scale

Problem 1.1 - Units

1. The fundamental constants are given by

$$k = 1.381 \cdot 10^{-23}\mathrm{m}^2\mathrm{s}^{-2}\mathrm{kg}^1\mathrm{K}^{-1} \tag{1067}$$
$$G = 6.674 \cdot 10^{-11}\mathrm{m}^3\mathrm{s}^{-2}\mathrm{kg}^{-1} \tag{1068}$$
$$\hbar = 1.054 \cdot 10^{-34}\mathrm{m}^2\mathrm{s}^{-1}\mathrm{kg}^1 \tag{1069}$$
$$c = 2.998 \cdot 10^{-8}\mathrm{m}^1\mathrm{s}^{-1} \tag{1070}$$

A newly constructed Planck constant has the general form

$$X_P = c^{\alpha_c} \cdot G^{\alpha_G} \cdot \hbar^{\alpha_\hbar} \cdot k^{\alpha_k} \tag{1071}$$

and the dimension of X_P is given by $\mathrm{m}^{\beta_m}\mathrm{s}^{\beta_s}\mathrm{kg}^{\beta_{kg}}\mathrm{K}^{\beta_K}$ are determined by

$$\text{Meter} \quad \beta_m = 2\alpha_k + 3\alpha_G + 2\alpha_h + \alpha_c \tag{1072}$$
$$\text{Second} \quad \beta_s = -2\alpha_k - 2\alpha_G - \alpha_c - \alpha_h \tag{1073}$$
$$\text{Kilogram} \quad \beta_{kg} = \alpha_k - \alpha_G + \alpha_h \tag{1074}$$
$$\text{Kelvin} \quad \beta_K = -\alpha_k \tag{1075}$$

Solving the linear system gives

$$l_P = \sqrt{\frac{\hbar G}{c^3}} = 1.616 \cdot 10^{-35}\text{m} \tag{1076}$$
$$m_P = \sqrt{\frac{\hbar c}{G}} = 2.176 \cdot 10^{-8}\text{kg} \tag{1077}$$
$$t_P = \sqrt{\frac{\hbar G}{c^5}} = 5.391 \cdot 10^{-44}\text{s} \tag{1078}$$
$$T_P = \sqrt{\frac{\hbar c^5}{Gk^2}} = 1.417 \cdot 10^{-32}\text{K} \tag{1079}$$
$$\tag{1080}$$

As the constants are made up from QM, SR and GR constants they indicate magnitudes at which a quantum theory of gravity is needed to make a sensible predictions.

2. We use the definition $1\text{barn} = 10^{-28}\text{m}^2$

$$1\text{cm}^2 = 10^{-4}\text{m}^2 \tag{1081}$$
$$1\text{mbarn} = 10^{-31}\text{m}^2 \tag{1082}$$
$$= 10^{-27}\text{cm}^2 \tag{1083}$$

We also have $1\text{eV} = 1.602 \cdot 10^{-19}\text{As} \cdot 1\text{V} = 1.602 \cdot 10^{-19}\text{J}$

$$E = mc^2 \quad \rightarrow \quad 1\text{kg} \cdot c^2 = 8.987 \cdot 10^{16}\text{J} = 5.609 \cdot 10^{35}\text{eV} \tag{1084}$$
$$\rightarrow \quad 1\text{GeV} = 1.782 \cdot 10^{-27}\text{kg} \tag{1085}$$
$$E = \hbar\omega \quad \rightarrow \quad \frac{1}{1\text{s}} \cdot \hbar = 1.054 \cdot 10^{-34}\text{J} = 6.582 \cdot 10^{-16}\text{eV} \tag{1086}$$
$$\rightarrow \quad 1\text{GeV}^{-1} = 6.582 \cdot 10^{-25}\text{s} \tag{1087}$$
$$E = \frac{\hbar c}{\lambda} \quad \rightarrow \quad \frac{1}{1\text{m}} \cdot \hbar c = 3.161 \cdot 10^{-26}\text{J} = 1.973 \cdot 10^{-7}\text{eV} \tag{1088}$$
$$\rightarrow \quad 1\text{GeV}^{-1} = 1.973 \cdot 10^{-16}\text{m} \tag{1089}$$
$$E \sim pc \quad \rightarrow \quad 1\text{kgms}^{-1} \cdot c = 2.998 \cdot 10^{8}\text{J} = 1.871 \cdot 10^{27}\text{eV} \tag{1090}$$
$$\rightarrow \quad 1\text{GeV} = 5.344 \cdot 10^{-19}\text{kgms}^{-1} \tag{1091}$$

therefore

$$1\text{GeV}^{-2} = (1.973 \cdot 10^{-16}\text{m})^2 \tag{1092}$$
$$= 3.893 \cdot 10^{-32}\text{m}^2 \tag{1093}$$
$$= 0.389\text{mbarn} \tag{1094}$$

Problem 3.2 - Maxwell Lagrangian

1. First we observe that

$$F_{\mu\nu}F^{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \tag{1095}$$
$$= (\partial_\mu A_\nu)(\partial^\mu A^\nu) - (\partial_\mu A_\nu)(\partial^\nu A^\mu) - \underbrace{(\partial_\nu A_\mu)(\partial^\mu A^\nu)}_{=(\partial_\mu A_\nu)(\partial^\nu A^\mu)} + \underbrace{(\partial_\nu A_\mu)(\partial^\nu A^\mu)}_{=(\partial_\mu A_\nu)(\partial^\mu A^\nu)} \tag{1096}$$
$$= 2\left((\partial_\mu A_\nu)(\partial^\mu A^\nu) - (\partial_\mu A_\nu)(\partial^\nu A^\mu)\right) \tag{1097}$$
$$= 2(\partial_\mu A_\nu)F^{\mu\nu}. \tag{1098}$$

The variation is then given by

$$\begin{aligned}
\delta\left(F_{\mu\nu}F^{\mu\nu}\right) &= 2\delta\left((\partial_\mu A_\nu)F^{\mu\nu}\right) && (1099)\\
&= 2\left[\delta\left(\partial_\mu A_\nu\right)F^{\mu\nu} + (\partial_\mu A_\nu)\delta F^{\mu\nu}\right] && (1100)\\
&= 2[\delta\left(\partial_\mu A_\nu\right)\underbrace{(\partial^\mu A^\nu - \partial^\nu A^\mu)}_{=F^{\mu\nu}} + (\partial_\mu A_\nu)\underbrace{(\delta(\partial^\mu A^\nu - \partial^\nu A^\mu))}_{\delta F^{\mu\nu}}] && (1101)\\
&= 2\left[\delta\left(\partial_\mu A_\nu\right)\partial^\mu A^\nu - \delta\left(\partial_\mu A_\nu\right)\partial^\nu A^\mu + (\partial_\mu A_\nu)\delta(\partial^\mu A^\nu) - (\partial_\mu A_\nu)\delta(\partial^\nu A^\mu)\right] && (1102)\\
&= 4\left[\delta\left(\partial_\mu A_\nu\right)\partial^\mu A^\nu - \delta\left(\partial_\mu A_\nu\right)\partial^\nu A^\mu\right] && (1103)\\
&= 4(\partial^\mu A^\nu - \partial^\nu A^\mu)\ \delta(\partial_\mu A_\nu) && (1104)\\
&= 4F^{\mu\nu}\ \delta(\partial_\mu A_\nu) && (1105)\\
&= 4F^{\mu\nu}\ \partial_\mu(\delta A_\nu) && (1106)
\end{aligned}$$

We start with the source free Maxwell equations $\partial_\mu F^{\mu\nu} = 0$

$$\begin{aligned}
0 &= \int_\Omega d^4x\ (\delta A_\nu)\partial_\mu F^{\mu\nu} && (1107)\\
&= F^{\mu\nu}(\delta A_\nu)|_{\partial\Omega} - \int_\Omega d^4x\ \underbrace{\partial_\mu(\delta A_\nu)F^{\mu\nu}}_{=\frac{1}{4}\delta(F_{\mu\nu}F^{\mu\nu})} && (1108)\\
&= \int_\Omega d^4x\ \delta\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) && (1109)
\end{aligned}$$

and therefore $\mathscr{L}_{\text{ph}} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$.

2. So we see that the Lagrangian $\mathscr{L}_{\text{ph}} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = 2(\partial_\mu A_\nu)F^{\mu\nu}$ yields the inhomogeneous Maxwell equations

$$\begin{aligned}
\frac{\partial\mathscr{L}_{\text{ph}}}{\partial A_\alpha} - \partial_\beta\frac{\partial\mathscr{L}_{\text{ph}}}{\partial(\partial_\beta A_\alpha)} &= 0 && (1110)\\
-\partial_\beta\left[(2\delta_{\alpha\mu}\delta_{\beta\nu}F^{\mu\nu} + 2(\partial_\mu A_\nu)(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\nu_\alpha\delta^\mu_\beta)\right] &= 0 && (1111)\\
-\partial_\beta\left[(2F^{\alpha\beta} + 2(\partial^\alpha A^\beta - \partial^\beta A^\alpha)\right] &= 0 && (1112)\\
\partial_\beta(F^{\alpha\beta}) &= 0 && (1113)
\end{aligned}$$

but not the homogeneous ones. They are fulfilled trivially - by construction of $F^{\mu\nu}$.

3. The conjugated momentum is given by

$$\begin{aligned}
\pi_\mu &= \frac{\partial\mathscr{L}_{\text{ph}}}{\partial\dot{A}^\mu} && (1114)\\
&= F_{0\mu} && (1115)
\end{aligned}$$

Problem 3.3 - Dimension of ϕ

1. With $c = 1 = \hbar$ we see

$$\begin{aligned}
E = mc^2 &\rightarrow E \sim M && (1116)\\
E = \hbar\omega &\rightarrow T \sim E^{-1} \sim M^{-1} && (1117)\\
s = ct &\rightarrow L \sim T \sim M^{-1} && (1118)
\end{aligned}$$

As $\mathscr{L}$ is an action density we have

$$\mathscr{L} \sim \frac{E\cdot T}{TL^3} \sim M\cdot M^{d-1} = M^d \qquad (1119)$$

From the explicit form of the scalar Lagrangian we derive

$$\mathscr{L} \sim \frac{[\phi^2]}{M^{-2}} = [\phi^2]M^{-2} \tag{1120}$$

and therefore $[\phi] = M^{(d-2)/2}$

2. Using the previous result we see

$$\lambda\phi^3: \qquad M^d \sim [\lambda]M^{3(d-2)/2} \to d = 6 \tag{1121}$$

$$\lambda\phi^4: \qquad M^d \sim [\lambda]M^{4(d-2)/2} \to d = 4 \tag{1122}$$

3. With

$$\mathscr{L} = \frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - \frac{1}{2}m^2\phi^2 + \lambda\phi^4 \tag{1123}$$

$$= \frac{1}{2}\eta^{\mu\nu}\left(\partial_\mu\frac{\tilde{\phi}}{\sqrt{\lambda}}\right)\left(\partial_\nu\frac{\tilde{\phi}}{\sqrt{\lambda}}\right) - \frac{1}{2}m^2\frac{\tilde{\phi}^2}{\lambda} + \lambda\frac{\tilde{\phi}^4}{\lambda^2} \tag{1124}$$

$$= \frac{1}{\lambda}\left[\frac{1}{2}\eta^{\mu\nu}(\partial_\mu\tilde{\phi})(\partial_\nu\tilde{\phi}) - \frac{1}{2}m^2\tilde{\phi}^2 + \tilde{\phi}^4\right] \tag{1125}$$

Problem 3.5 - Yukawa potential

Integration in spherical coordinates yields (with $x = kr$)

$$\int d^3k\frac{e^{-ik\cdot r}}{k^2+m^2} = 2\pi\int\frac{e^{-ikr\cos\theta}}{k^2+m^2}k^2\sin\theta d\theta\, dk \tag{1126}$$

$$= -2\pi\int\frac{e^{-ikr\cos\theta}}{k^2+m^2}k^2\, d(\cos\theta)\, dk \tag{1127}$$

$$= -2\pi\int\frac{k^2}{ikr}\frac{e^{-ikr\cos\theta}}{k^2+m^2}\bigg|_{-1}^{+1}\, dk \tag{1128}$$

$$= -2\pi\int\frac{k}{ir}\frac{e^{-ikr}-e^{+ikr}}{k^2+m^2}\, dk \tag{1129}$$

$$= \frac{4\pi}{r}\int_0^\infty\frac{k\sin kr}{k^2+m^2}\, dk \tag{1130}$$

$$= \frac{4\pi}{r^2}\int_0^\infty\frac{\frac{x}{r}\sin x}{\frac{x^2}{r^2}+m^2}\, dx \tag{1131}$$

$$= \frac{4\pi}{r}\int_0^\infty\frac{x\sin x}{x^2+m^2r^2}\, dx \tag{1132}$$

$$\tag{1133}$$

Now we use a small trick

$$= \frac{2\pi}{ir} \int_0^\infty \frac{x(e^{ix} - e^{-ix})}{x^2 + m^2r^2}\, dx \tag{1134}$$

$$= \frac{2\pi}{ir} \left[\int_0^\infty \frac{xe^{ix}}{x^2 + m^2r^2}\, dx - \int_0^\infty \frac{xe^{-ix}}{x^2 + m^2r^2}\, dx \right] \tag{1135}$$

$$= \frac{2\pi}{ir} \left[\int_0^\infty \frac{xe^{ix}}{x^2 + m^2r^2}\, dx - (-1)^3 \int_{-\infty}^0 \frac{ye^{iy}}{y^2 + m^2r^2}\, dy \right] \tag{1136}$$

$$= \frac{2\pi}{ir} \int_{-\infty}^\infty \frac{xe^{ix}}{x^2 + m^2r^2}\, dx \tag{1137}$$

$$= \frac{2\pi}{ir} \int_{-\infty}^\infty \frac{xe^{ix}}{(x + imr)(x - imr)}\, dx \tag{1138}$$

$$= \frac{2\pi}{ir} \left(2\pi i \cdot \underbrace{\mathrm{Res}_{x=imr}}_{= \frac{imr \exp(i^2 mr)}{2imr}} - \int_{\text{upper half circle}} \ldots \right) \tag{1139}$$

$$= \frac{2\pi^2}{r} e^{-mr} \tag{1140}$$

Therefore

$$\frac{1}{(2\pi)^3} \int d^3k \frac{e^{-ik\cdot r}}{k^2 + m^2} = \frac{1}{4\pi r} e^{-mr} \tag{1141}$$

Problem 3.9 - ζ function regularization

1. Calculation the Taylor expansion (using L'Hopital's rule for the limits) we obtain

$$f(t) = \frac{t}{e^t - 1} \tag{1142}$$

$$= \sum_k \left. \frac{d^k f}{dt^k} \right|_{t=0} t^k \tag{1143}$$

$$= 1 - \frac{1}{2}t + \frac{1}{12}t^2 - \frac{1}{12}t^4 + \ldots \tag{1144}$$

$$\stackrel{!}{=} B_0 + B_1 t + \frac{B_2}{2}t^2 + \frac{B_3}{6}t^2 + \ldots \tag{1145}$$

$$\to B_n = \{1, -\frac{1}{2}, \frac{1}{6}, 0, \ldots\} \tag{1146}$$

2. Avoiding mathematical rigor we see after playing around for a while

$$\sum_{n=1}^{\infty} ne^{-an} = -\frac{d}{da}\sum_{n=1}^{\infty} e^{-an} \tag{1147}$$
$$= -\frac{d}{da}\sum_{n=1}^{\infty} \left(e^{-a}\right)^n \tag{1148}$$
$$= -\frac{d}{da}\frac{1}{1-e^{-a}} \tag{1149}$$
$$= -\frac{d}{da}\left(\frac{1}{a}\frac{a}{1-e^{-a}}\right) \tag{1150}$$
$$= -\frac{d}{da}\left(\frac{1}{a}f(t)\right) \tag{1151}$$
$$= -\frac{d}{da}\left(\frac{1}{a}\sum_{n=0}^{\infty}\frac{B_n}{n!}a^n\right) \tag{1152}$$
$$= -\frac{d}{da}\left(\frac{1}{a}\left[1-\frac{a}{2}+\frac{a^2}{12}-\frac{a^4}{720}+...\right]\right) \tag{1153}$$
$$= -\frac{d}{da}\left(\frac{1}{a}-\frac{1}{2}+\frac{a}{12}-\frac{a^3}{720}...\right) \tag{1154}$$
$$= \frac{1}{a^2}-\frac{1}{12}+\frac{a}{240}-... \tag{1155}$$
$$\stackrel{a\to 0}{\to} \frac{1}{a^2}-\frac{1}{12} \tag{1156}$$

3. Using the definition of the Riemann ζ function

$$\zeta(s) = \sum_{k=1}^{\infty}\frac{1}{k^s} \tag{1157}$$

Problem 4.1 - $Z[J]$ at order λ in ϕ^4 theory

Lets start at (4.6a) with $\mathscr{L}_I = -\lambda/4!\phi^4$

$$Z[J] = \exp\left[\mathrm{i}\int d^4x\mathscr{L}_I\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}\right)\right]\int\mathcal{D}\phi\exp\left[\mathrm{i}\int d^4x(\mathscr{L}_0+J\phi)\right] \tag{1158}$$
$$= \exp\left[\mathrm{i}\int d^4x\mathscr{L}_I\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}\right)\right]Z_0[J] \tag{1159}$$
$$= \exp\left[-\frac{\mathrm{i}\lambda}{4!}\int d^4x\left(\frac{\delta^4}{\delta J(x)^4}\right)\right]Z_0[J] \tag{1160}$$
$$= Z_0[J]-\frac{\mathrm{i}\lambda}{4!}\int d^4x\left(\frac{\delta^4 Z_0[J]}{\delta J(x)^4}\right)+\dots. \tag{1161}$$

Using (4.7)

$$Z_0[J] = Z_0[0]\exp\left[-\frac{\mathrm{i}}{2}\int d^4yd^4zJ(y)\Delta_F(y-z)J(z)\right] = Z_0[0]e^{iW_0[J]} \tag{1162}$$
$$W_0[J] = -\frac{1}{2}\int d^4yd^4zJ(y)\Delta_F(y-z)J(z) \tag{1163}$$

we derive (4.10) in various steps

1. Calculating $\frac{\delta W_0[J]}{\delta J(x)}$

$$\frac{\delta W_0[J]}{\delta J(x)} = -\frac{1}{2}\lim_{\epsilon\to 0}\int d^4y d^4z \frac{\left(J(y)+\epsilon\delta^{(4)}(y-x)\right)\Delta_F(y-z)\left(J(z)+\epsilon\delta^{(4)}(z-x)\right)-W_0[J]}{\epsilon} \tag{1164}$$

$$= -\frac{1}{2}\int d^4y d^4z \left[\delta^{(4)}(y-x)\Delta_F(y-z)J(z)+J(y)\Delta_F(y-z)\delta^{(4)}(z-x)\right] \tag{1165}$$

$$= -\frac{1}{2}\int d^4z\Delta_F(x-z)J(z)-\frac{1}{2}\int d^4yJ(y)\Delta_F(y-x) \tag{1166}$$

$$= -\int d^4y\Delta_F(y-x)J(y) \tag{1167}$$

where we used $\Delta_F(x)=\Delta_F(-x)$.

2. Calculating $\frac{\delta^2 W_0[J]}{\delta J(x)^2}$

$$\frac{\delta^2 W_0[J]}{\delta J(x)^2} = -\int d^4y\Delta_F(y-x)\frac{\delta J(y)}{\delta J(x)} \tag{1168}$$

$$= -\int d^4y\Delta_F(y-x)\delta(y-x) \tag{1169}$$

$$= -\Delta_F(0) \tag{1170}$$

3. Calculating $\delta F[J]/\delta J(x)$ for $F[J]=f\left(W_0[J]\right)$

$$\frac{\delta F[J]}{\delta J(x)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}f(W_0[\phi(x)+\epsilon\delta(x-y)])-f(W_0[\phi(x)]) \tag{1171}$$

$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}f(W_0[\phi(x)]+\epsilon\frac{\delta W_0}{\delta\phi})-f(W_0[\phi(x)]) \tag{1172}$$

$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}f(W_0[\phi(x)])+g'\epsilon\frac{\delta W_0}{\delta\phi}-f(W_0[\phi(x)]) \tag{1173}$$

$$= f'(W_0[J])\frac{\delta W_0}{\delta J} \tag{1174}$$

4. Calculating first derivative

$$\frac{\delta}{\mathrm{i}\delta J(x)}\exp\left(\mathrm{i}W_0[J]\right) = \frac{\delta W_0[J]}{\delta J(x)}\exp\left(\mathrm{i}W_0[J]\right) \tag{1175}$$

5. Calculating second derivative (using the functional derivative product rule)

$$\left(\frac{\delta}{\mathrm{i}\delta J(x)}\right)^2\exp\left(\mathrm{i}W_0[J]\right) = \left(\left(\frac{\delta W_0[J]}{\delta J(x)}\right)^2+\frac{1}{i}\frac{\delta^2 W_0[J]}{\delta J(x)^2}\right)\exp\left(\mathrm{i}W_0[J]\right) \tag{1176}$$

6. Calculating third derivative

$$\left(\frac{\delta}{\mathrm{i}\delta J(x)}\right)^3\exp\left(\mathrm{i}W_0[J]\right) = \left(\left(\frac{\delta W_0[J]}{\delta J(x)}\right)^3+\frac{3}{i}\frac{\delta^2 W_0[J]}{\delta J(x)^2}\frac{\delta W_0[J]}{\delta J(x)}+\frac{1}{i^2}\frac{\delta^3 W_0[J]}{\delta J(x)^3}\right)\exp\left(\mathrm{i}W_0[J]\right) \tag{1177}$$

7. Calculating fourth derivative

$$\left(\frac{\delta}{\mathrm{i}\delta J(x)}\right)^4 \exp\left(\mathrm{i}W_0[J]\right) = \left(\left(\frac{\delta W_0[J]}{\delta J(x)}\right)^4 + \frac{6}{i}\frac{\delta^2 W_0[J]}{\delta J(x)^2}\left(\frac{\delta W_0[J]}{\delta J(x)}\right)^2 + \frac{3}{i^2}\left(\frac{\delta^2 W_0[J]}{\delta J(x)^2}\right)^2 + \right.$$
$$\left. + \frac{4}{i^2}\frac{\delta W_0[J]}{\delta J(x)}\frac{\delta^3 W_0[J]}{\delta J(x)^3} + \frac{1}{i^3}\frac{\delta^4 W_0[J]}{\delta J(x)^4}\right)\exp\left(\mathrm{i}W_0[J]\right)$$
$$= \left(\left(\frac{\delta W_0[J]}{\delta J(x)}\right)^4 + \frac{6}{i}\frac{\delta^2 W_0[J]}{\delta J(x)^2}\left(\frac{\delta W_0[J]}{\delta J(x)}\right)^2 + \frac{3}{i^2}\left(\frac{\delta^2 W_0[J]}{\delta J(x)^2}\right)^2\right)\exp\left(\mathrm{i}W_0[J]\right)$$

8. Substituting the functional derivatives

$$\left(\frac{\delta}{\mathrm{i}\delta J(x)}\right)^4 \exp\left(\mathrm{i}W_0[J]\right) = \left[\left(\int d^4y\Delta_F(y-x)J(y)\right)^4 + 6i\Delta_F(0)\left(\int d^4y\Delta_F(y-x)J(y)\right)^2\right.$$
$$\left. +3\left(i\Delta_F(0)\right)^2\right]\exp\left(\mathrm{i}W_0[J]\right)$$

Problem 19.1 - Dynamical stress tensor

Preliminaries

- The Laplace expansion of the determinate by row or column is given by

$$|g| = \sum_\kappa g_{\kappa\mu}G_{\kappa\mu} \quad \text{(no sum over } \mu!) \tag{1178}$$

  with the cofactor matrix $G_{\kappa\mu}$ (matrix of determinants of minors of g).

- The inverse matrix is given by

$$g^{\alpha\beta} = \frac{1}{|g|}G_{\alpha\beta} \tag{1179}$$

- Therefore we have

$$\frac{\partial|g|}{\delta g_{\alpha\beta}} = \frac{\partial\left(\sum_\kappa g_{\kappa\beta}G_{\kappa\beta}\right)}{\delta g_{\alpha\beta}} \tag{1180}$$
$$= \delta_{\kappa\alpha}G_{\kappa\beta} \tag{1181}$$
$$= G_{\alpha\beta} \tag{1182}$$
$$= |g|g^{\alpha\beta} \tag{1183}$$

Now we can calculate

$$\delta\sqrt{|g|} = \frac{\partial\sqrt{|g|}}{\delta g_{\mu\nu}}\delta g_{\mu\nu} = \frac{1}{2\sqrt{|g|}}\frac{\partial|g|}{\delta g_{\mu\nu}}\delta g_{\mu\nu} = \frac{1}{2}\sqrt{|g|}g^{\mu\nu}\delta g_{\mu\nu} \tag{1184}$$
$$\frac{\delta\sqrt{|g(x)|}}{\delta g_{\mu\nu}(y)} = \frac{1}{2}\sqrt{|g|}\delta(x-y) \tag{1185}$$

We now use the action and definition (7.49)

$$S_\mathrm{m} = \int d^4x\sqrt{|g|}\mathscr{L}_\mathrm{m} \tag{1186}$$
$$T^{\mu\nu} = \frac{2}{\sqrt{|g|}}\frac{\delta S_\mathrm{m}}{\delta g^{\mu\nu}} \tag{1187}$$
$$= \frac{2}{\sqrt{|g|}}\int d^4x\left[\frac{1}{2}\sqrt{|g|}g^{\mu\nu}\mathscr{L}_\mathrm{m} + \sqrt{|g|}\frac{\delta\mathscr{L}_\mathrm{m}}{\delta g_{\mu\nu}}\right] \tag{1188}$$

Problem 19.6 - Dirac-Schwarzschild

1. (19.13) - adding the bi-spinor index might be helpful for some readers, see (B.27)
2. (19.13) vs (B.27) naming of generators $J^{\mu\nu}$ vs $\sigma_{\mu\nu}/2$

The Dirac equation in curved space is obtained (from the covariance principle) by replacing all derivatives ∂_k with covariant tetrad derivatives $\mathscr{D}_k$

$$(i\hbar\gamma^k \mathscr{D}_k + mc)\psi = 0 \tag{1189}$$

Lets start with the Schwarzschild line element

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2(d\vartheta^2 + \sin^2\vartheta\, d\phi^2) \tag{1190}$$

$$= \eta_{mn} d\xi^m d\xi^n \tag{1191}$$

with

$$d\xi^0 = \left(1 - \frac{2M}{r}\right)^{1/2} dt, \quad d\xi^1 = \left(1 - \frac{2M}{r}\right)^{-1/2} dr, \quad d\xi^2 = r d\vartheta, \quad d\xi^3 = r\sin\vartheta d\phi. \tag{1192}$$

and the tetrad fields e^m_μ can then be derived via $d\xi^m = e^m_\mu(x)dx^\mu$.

Problem 23.1 - Conformal transformation

For a change of coordinates we find in general

$$x^\mu \mapsto \tilde{x}^\mu \tag{1193}$$

$$g_{\mu\nu}(x) \mapsto \tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(x) \tag{1194}$$

which for $x \mapsto \tilde{x} = e^\omega x$ results in (there might be a sign error in (18.1))

$$g_{\mu\nu}(x) \mapsto \tilde{g}_{\mu\nu}(\tilde{x}) = e^{-2\omega} g_{\alpha\beta}(x) \tag{1195}$$

while for a conformal transformation we have

$$g_{\mu\nu}(x) \mapsto \tilde{g}_{\mu\nu}(x) = \Omega^2 g_{\alpha\beta}(x) \tag{1196}$$

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \Omega^2 g_{\alpha\beta}(e^\omega x) \tag{1197}$$

Problem 23.2 - Conformal transformation properties

- Christoffel symbol:

$$\tilde{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x) = e^{2\omega(x)} g_{\mu\nu}(x) \tag{1198}$$

$$\tilde{g}_{\mu\nu,\alpha} = 2\Omega\Omega_{,\alpha} g_{\mu\nu} + \Omega^2 g_{\mu\nu,\alpha} \tag{1199}$$

$$= \Omega(2g_{\mu\nu}\Omega_{,\alpha} + \Omega g_{\mu\nu,\alpha}) \tag{1200}$$

and

$$\delta^\mu_\nu = \tilde{g}^{\mu\alpha}\tilde{g}_{\alpha\nu} = \tilde{g}^{\mu\alpha} g_{\alpha\nu}\Omega^2 \tag{1201}$$

$$\delta^\mu{}_\nu g^{\nu\beta} = \tilde{g}^{\mu\alpha} g_{\alpha\nu} g^{\nu\beta}\Omega^2 \tag{1202}$$

$$g^{\mu\beta} = \tilde{g}^{\mu\alpha}\delta^\beta_\alpha \Omega^2 \tag{1203}$$

$$\to \tilde{g}^{\mu\beta} = \Omega^{-2} g^{\mu\beta} \tag{1204}$$

we find by using ${\Gamma^\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}\left(g_{\alpha\mu,\beta} + g_{\beta\mu,\alpha} - g_{\alpha\beta,\mu}\right)$

$$
\begin{aligned}
{\tilde{\Gamma}^\mu}_{\alpha\beta} &= \frac{1}{2}\tilde{g}^{\mu\nu}\left(\tilde{g}_{\alpha\nu,\beta} + \tilde{g}_{\beta\nu,\alpha} - \tilde{g}_{\alpha\beta,\nu}\right) && (1205)\\
&= \frac{1}{2}\Omega^{-2} g^{\mu\nu}\left[\Omega(2g_{\alpha\nu}\Omega_{,\beta} + \Omega g_{\alpha\nu,\beta}) + \Omega(2g_{\beta\nu}\Omega_{,\alpha} + \Omega g_{\beta\nu,\alpha}) - \Omega(2g_{\alpha\beta}\Omega_{,\nu} + \Omega g_{\alpha\beta,\nu})\right] && (1206)\\
&= {\Gamma^\mu}_{\alpha\beta} + \Omega^{-1} g^{\mu\nu}\left[g_{\alpha\nu}\Omega_{,\beta} + g_{\beta\nu}\Omega_{,\alpha} - g_{\alpha\beta}\Omega_{,\nu}\right] && (1207)\\
&= {\Gamma^\mu}_{\alpha\beta} + \Omega^{-1}\left[\delta^\mu_\alpha\Omega_{,\beta} + \delta^\mu_\beta\Omega_{,\alpha} - g^{\mu\nu} g_{\alpha\beta}\Omega_{,\nu}\right] && (1208)
\end{aligned}
$$

- Ricci tensor: with

$$
\begin{aligned}
\Omega &= e^{2\omega} && (1209)\\
\Omega^{-2}\Omega_{,\lambda} &= e^{-4\omega} e^{2\omega} 2\omega_{,\lambda} && (1210)\\
&= 2e^{-2\omega}\omega_{,\lambda} && (1211)\\
\Omega_{,\lambda\alpha} &= \left(2e^{2\omega}\omega_{,\lambda}\right)_{,\alpha} && (1212)\\
&= 4e^{2\omega}\omega_{,\lambda}\omega_{,\alpha} + 2e^{2\omega}\omega_{,\lambda\alpha} && (1213)\\
&= 2e^{2\omega}\left(2\omega_{,\lambda}\omega_{,\alpha} + \omega_{,\lambda\alpha}\right) && (1214)
\end{aligned}
$$

and

$$
\begin{aligned}
\partial_\lambda {\tilde{\Gamma}^\mu}_{\alpha\beta} &= \partial_\lambda {\Gamma^\mu}_{\alpha\beta} - \Omega^{-2}\Omega_{,\lambda}\left[\delta^\mu_\alpha\Omega_{,\beta} + \delta^\mu_\beta\Omega_{,\alpha} - g^{\mu\nu} g_{\alpha\beta}\Omega_{,\nu}\right] + \Omega^{-1}\left[\delta^\mu_\alpha\Omega_{,\beta\lambda} + \delta^\mu_\beta\Omega_{,\alpha\lambda} - (g^{\mu\nu} g_{\alpha\beta}\Omega_{,\nu})_{,\lambda}\right] && (1215)\\
&= \partial_\lambda {\Gamma^\mu}_{\alpha\beta} - 4\omega_{,\lambda}\left[\delta^\mu_\alpha\omega_{,\beta} + \delta^\mu_\beta\omega_{,\alpha} - g^{\mu\nu} g_{\alpha\beta}\omega_{,\nu}\right] + 2\left[\delta^\mu_\alpha\left(2\omega_{,\beta}\omega_{,\lambda} + \omega_{,\beta\lambda}\right) + \delta^\mu_\beta\left(2\omega_{,\alpha}\omega_{,\lambda} + \omega_{,\alpha\lambda}\right)\right] && (1216)\\
&\quad - 2\left[{g^{\mu\nu}}_{,\lambda} g_{\alpha\beta}\omega_{,\nu} + g^{\mu\nu} g_{\alpha\beta,\lambda}\omega_{,\nu} + g^{\mu\nu} g_{\alpha\beta}(2\omega_{,\nu}\omega_{,\lambda} + \omega_{,\nu\lambda})\right] && (1217)\\
& && (1218)
\end{aligned}
$$

$$
\begin{aligned}
\partial_\rho {\tilde{\Gamma}^\rho}_{\mu\nu} &= \partial_\rho {\Gamma^\rho}_{\mu\nu} - 4\omega_{,\rho}\left[\delta^\rho_\mu\omega_{,\nu} + \delta^\rho_\nu\omega_{,\mu} - g^{\rho\nu} g_{\mu\nu}\omega_{,\lambda}\right] + 2\left[\delta^\rho_\mu\left(2\omega_{,\nu}\omega_{,\rho} + \omega_{,\nu\rho}\right) + \delta^\rho_\nu\left(2\omega_{,\mu}\omega_{,\rho} + \omega_{,\mu\rho}\right)\right] && (1219)\\
&\quad - 2\left[{g^{\rho\lambda}}_{,\rho} g_{\mu\nu}\omega_{,\lambda} + g^{\rho\lambda} g_{\mu\nu,\rho}\omega_{,\lambda} + g^{\rho\lambda} g_{\mu\nu}(2\omega_{,\lambda}\omega_{,\rho} + \omega_{,\lambda\rho})\right] && (1220)\\
&= \partial_\rho {\Gamma^\rho}_{\mu\nu} - 4\left[2\omega_{,\mu}\omega_{,\nu} - \omega_{,\rho} g^{\rho\nu} g_{\mu\nu}\omega_{,\lambda}\right] + 4\left(2\omega_{,\nu}\omega_{,\mu} + \omega_{,\nu\mu}\right) && (1221)\\
&\quad - 2\left[{g^{\rho\lambda}}_{,\rho} g_{\mu\nu}\omega_{,\lambda} + g^{\rho\lambda} g_{\mu\nu,\rho}\omega_{,\lambda} + g^{\rho\lambda} g_{\mu\nu}(2\omega_{,\lambda}\omega_{,\rho} + \omega_{,\lambda\rho})\right] && (1222)\\
&= \partial_\rho {\Gamma^\rho}_{\mu\nu} + 4g^{\rho\nu} g_{\mu\nu}\omega_{,\lambda}\omega_{,\rho} + 4\omega_{,\nu\mu} - 2\left[{g^{\rho\lambda}}_{,\rho} g_{\mu\nu}\omega_{,\lambda} + g^{\rho\lambda} g_{\mu\nu,\rho}\omega_{,\lambda} + (2g^{\rho\lambda} g_{\mu\nu}\omega_{,\lambda}\omega_{,\rho} + g^{\rho\lambda} g_{\mu\nu}\omega_{,\lambda\rho})\right] && (1223)\\
&= \partial_\rho {\Gamma^\rho}_{\mu\nu} + 4\omega_{,\lambda}\omega_{,\mu} + 4\omega_{,\nu\mu} - 2\left[{g^{\rho\lambda}}_{,\rho} g_{\mu\nu}\omega_{,\lambda} + g_{\mu\nu,\rho}\omega^{,\rho} + 2g_{\mu\nu}\omega^{,\rho}\omega_{,\rho} + g_{\mu\nu}{\omega^{,\rho}}_{\rho}\right] && (1224)
\end{aligned}
$$

$$
\begin{aligned}
\partial_\nu {\tilde{\Gamma}^\rho}_{\mu\rho} &= \partial_\nu {\Gamma^\rho}_{\mu\rho} - 4\omega_{,\nu}\left[\delta^\rho_\mu\omega_{,\rho} + \delta^\rho_\rho\omega_{,\mu} - g^{\rho\kappa} g_{\mu\rho}\omega_{,\kappa}\right] + 2\left[\delta^\rho_\mu\left(2\omega_{,\rho}\omega_{,\nu} + \omega_{,\rho\nu}\right) + \delta^\rho_\rho\left(2\omega_{,\mu}\omega_{,\nu} + \omega_{,\mu\nu}\right)\right] && (1225)\\
&\quad - 2\left[{g^{\rho\kappa}}_{,\nu} g_{\mu\rho}\omega_{,\kappa} + g^{\rho\kappa} g_{\mu\rho,\nu}\omega_{,\kappa} + g^{\rho\kappa} g_{\mu\rho}(2\omega_{,\kappa}\omega_{,\nu} + \omega_{,\kappa\nu})\right] && (1226)\\
&= \partial_\nu {\Gamma^\rho}_{\mu\rho} - 4\left[(d+1)\omega_{,\mu}\omega_{,\nu} - \omega_{,\mu}\omega_{,\nu}\right] + 2(d+1)\left(2\omega_{,\mu}\omega_{,\nu} + \omega_{,\mu\nu}\right) && (1227)\\
&\quad - 2\left[{g^{\rho\kappa}}_{,\nu} g_{\mu\rho}\omega_{,\kappa} + g^{\rho\kappa} g_{\mu\rho,\nu}\omega_{,\kappa} + \delta^\kappa_\mu(2\omega_{,\kappa}\omega_{,\nu} + \omega_{,\kappa\nu})\right] && (1228)\\
&= \partial_\nu {\Gamma^\rho}_{\mu\rho} + 4\omega_{,\mu}\omega_{,\nu} + 2(d+1)\omega_{,\mu\nu} - 2\left[{g^{\rho\kappa}}_{,\nu} g_{\mu\rho}\omega_{,\kappa} + g^{\rho\kappa} g_{\mu\rho,\nu}\omega_{,\kappa} + (2\omega_{,\mu}\omega_{,\nu} + \omega_{,\mu\nu})\right] && (1229)\\
&= \partial_\nu {\Gamma^\rho}_{\mu\rho} + 2d\cdot\omega_{,\mu\nu} - 2\left[{g^{\rho\kappa}}_{,\nu} g_{\mu\rho}\omega_{,\kappa} + g_{\mu\rho,\nu}\omega^{,\rho}\right] && (1230)
\end{aligned}
$$

$$\tilde{\Gamma}^{\mu}{}_{\alpha\beta} = \Gamma^{\mu}{}_{\alpha\beta} + \Omega^{-1}\left[\delta^{\mu}_{\alpha}\Omega_{,\beta} + \delta^{\mu}_{\beta}\Omega_{,\alpha} - g^{\mu\nu}g_{\alpha\beta}\Omega_{,\nu}\right] \tag{1231}$$
$$\tag{1232}$$

$$\tilde{\Gamma}^{\rho}{}_{\mu\nu}\tilde{\Gamma}^{\sigma}{}_{\rho\sigma} = \left(\Gamma^{\rho}{}_{\mu\nu} + \Omega^{-1}\left[\delta^{\rho}_{\mu}\Omega_{,\nu} + \delta^{\rho}_{\nu}\Omega_{,\mu} - g^{\rho\lambda}g_{\mu\nu}\Omega_{,\lambda}\right]\right)\left(\Gamma^{\sigma}{}_{\rho\sigma} + d\cdot\Omega^{-1}\Omega_{,\rho}\right) \tag{1233}$$
$$= \Gamma^{\rho}{}_{\mu\nu}\Gamma^{\sigma}{}_{\rho\sigma} + \Gamma^{\rho}{}_{\mu\nu}d\cdot\Omega^{-1}\Omega_{,\rho} + \Gamma^{\sigma}{}_{\rho\sigma}\Omega^{-1}\left[\delta^{\rho}_{\mu}\Omega_{,\nu} + \delta^{\rho}_{\nu}\Omega_{,\mu} - g^{\rho\lambda}g_{\mu\nu}\Omega_{,\lambda}\right] \tag{1234}$$
$$+ d\cdot\Omega^{-2}\left[\delta^{\rho}_{\mu}\Omega_{,\nu} + \delta^{\rho}_{\nu}\Omega_{,\mu} - g^{\rho\lambda}g_{\mu\nu}\Omega_{,\lambda}\right]\Omega_{,\rho} \tag{1235}$$

$$\tilde{R}_{\mu\nu} = \tilde{R}^{\rho}{}_{\mu\rho\nu} \tag{1236}$$
$$= \partial_{\rho}\tilde{\Gamma}^{\rho}{}_{\mu\nu} - \partial_{\nu}\tilde{\Gamma}^{\rho}{}_{\mu\rho} + \tilde{\Gamma}^{\rho}{}_{\mu\nu}\tilde{\Gamma}^{\sigma}{}_{\rho\sigma} - \tilde{\Gamma}^{\sigma}{}_{\nu\rho}\tilde{\Gamma}^{\rho}{}_{\mu\sigma} \tag{1237}$$

- Curvature scalar

$$\tilde{R} = \tilde{g}^{\mu\nu}\tilde{R}_{\mu\nu} \tag{1238}$$
$$= \tilde{g}^{\mu\nu}\left[R_{\mu\nu} - g_{\mu\nu}\Box\omega - (d-2)\nabla_{\mu}\nabla_{\nu}\omega + (d-2)\nabla_{\mu}\omega\nabla_{\nu}\omega - (d-2)g_{\mu\nu}\nabla^{\lambda}\omega\nabla_{\lambda}\omega\right] \tag{1239}$$
$$= \Omega^{-2}\left[R - d\Box\omega - (d-2)\Box\omega + (d-2)\nabla^{\mu}\omega\nabla_{\mu}\omega - (d-2)d\nabla^{\lambda}\omega\nabla_{\lambda}\omega\right] \tag{1240}$$
$$= \Omega^{-2}\left[R - 2(d-1)\Box\omega - (d-2)(d-1)\nabla^{\lambda}\omega\nabla_{\lambda}\omega\right] \tag{1241}$$
$$\tag{1242}$$

Problem 23.6 - Reflection formula

$$\Gamma(z) = \int_0^{\infty} x^{z-1}e^{-x}dx \tag{1243}$$

Problem 23.7 - Unruh temperature

Problem 24.14 - Jeans length and the speed of sound

We start with the Euler equations

$$\frac{D\rho}{Dt} = -\rho\nabla\cdot\vec{u} \quad \rightarrow \quad \frac{\partial\rho}{\partial t} + \vec{u}\cdot(\nabla\rho) + \rho(\nabla\cdot\vec{u}) = 0 \tag{1244}$$
$$\frac{D\vec{u}}{Dt} = -\nabla\left(\frac{P}{\rho}\right) + \vec{g} \quad \rightarrow \quad \frac{\partial\vec{u}}{\partial t} + \vec{u}\cdot(\nabla\vec{u}) + \frac{\nabla P}{\rho} = \vec{g}. \tag{1245}$$

With the perturbation ansatz (small perturbation in a resting fluid)

$$\rho = \rho_0 + \varepsilon\rho_1(x,t) \tag{1246}$$
$$P = P_0 + \varepsilon P_1(x,t) \tag{1247}$$
$$\vec{u} = \varepsilon\vec{u}_1(x,t) \tag{1248}$$

and the Newton equation

$$\triangle\phi = 4\pi G\rho \quad \rightarrow \quad \nabla\cdot\vec{g}_1 = -4\pi G\rho_1 \tag{1249}$$

we obtain (with the EoS $P = w\rho$) in order ε

$$\frac{\partial \rho_1}{\partial t} + \rho_0(\nabla \cdot \vec{u}_1) = 0 \tag{1250}$$

$$\frac{\partial \vec{u}_1}{\partial t} + \underbrace{\frac{1}{\rho_0}\nabla P_1}_{=\frac{w}{\rho_0}\nabla\rho_1} = \vec{g}_1. \tag{1251}$$

Differentiating both (with respect to space and time) we obtain a wave equation

$$\frac{\partial^2 \rho_1}{\partial t^2} - w\triangle\rho_1 = 4\pi G\rho_0\rho_1 \tag{1252}$$

with the speed of sound $c_s^2 = w$. Inserting the wave ansatz $\rho_1 \sim \exp[i(\vec{k}\cdot\vec{x} - \omega t)]$ yields the dispersion relation

$$\omega^2 = c_s^2 k^2 - 4\pi G\rho_0. \tag{1253}$$

For wave numbers $k_J < \sqrt{4\pi G/c_s^2}$ the ω becomes complex which gives rise to exponentially growing modes. Therefore the Jeans length is given by

$$\lambda_J = \frac{2\pi}{k_J} = c_s\sqrt{\frac{\pi}{G\rho_0}} = \sqrt{\frac{\pi w}{G\rho_0}}. \tag{1254}$$

Problem 26.4 - Fixed points of (26.18)

We start with

$$\text{(F1)} \qquad H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V + \rho\right) \tag{1255}$$

$$\text{(F2)} \qquad \dot{H} = -4\pi G\left[\dot{\phi}^2 + (1+w_m)\rho\right] \tag{1256}$$

$$\text{(KG)} \qquad \ddot{\phi} = -3H\dot{\phi} - V_{,\phi}. \tag{1257}$$

Using $H = \dot{a}/a$, $N = \ln(a)$ and $\lambda = -V_{,\phi}/(\sqrt{8\pi G}V)$ we obtain for the time derivatives of x and y

$$\dot{V} = \frac{dV}{d\phi}\frac{d\phi}{dt} = V_{,\phi}\dot{\phi} \tag{1258}$$

$$x = \sqrt{\frac{4}{3}\pi G}\frac{\dot{\phi}}{H} \quad \rightarrow \quad \frac{dx}{dt} = \frac{dx}{dN}\frac{d\ln(a)}{dt} = \frac{dx}{dN}H = \sqrt{\frac{4}{3}\pi G}\frac{\ddot{\phi}H - \dot{\phi}\dot{H}}{H^2} \tag{1259}$$

$$y = \sqrt{\frac{8}{3}\pi G}\frac{\sqrt{V}}{H} \quad \rightarrow \quad \frac{dy}{dt} = \frac{dy}{dN}\frac{d\ln(a)}{dt} = \frac{dy}{dN}H = \sqrt{\frac{8}{3}\pi G}\frac{\frac{V_{,\phi}\dot{\phi}}{2\sqrt{V}} - \sqrt{V}\dot{H}}{H^2}. \tag{1260}$$

With the substitutions

$$\dot{H} = -4\pi G\left[\dot{\phi}^2 + (1+w_m)\rho\right] \tag{1261}$$

$$\ddot{\phi} = -3H\dot{\phi} - V_{,\phi} \tag{1262}$$

$$V_{,\phi} = -\sqrt{8\pi G}\lambda V \tag{1263}$$

$$\rho = \frac{3H^2}{8\pi G} - \frac{1}{2}\dot{\phi}^2 - V \tag{1264}$$

$$\dot{\phi} = xH/\sqrt{\frac{4}{3}\pi G} \tag{1265}$$

$$\sqrt{V} = yH/\sqrt{\frac{8}{3}\pi G} \tag{1266}$$

we obtain

$$\frac{dx}{dN} = -3x + \frac{\sqrt{6}}{2}\lambda y^2 + \frac{3}{2}x[(1-w_m)x^2 + (1+w_m)(1-y^2)] \tag{1267}$$

$$\frac{dy}{dN} = -\frac{\sqrt{6}}{2}\lambda xy + \frac{3}{2}y[(1-w_m)x^2 + (1+w_m)(1-y^2)]. \tag{1268}$$

To find the fix points of (26.17) we need to solve

$$-3x + \frac{\sqrt{6}}{2}\lambda y^2 + \frac{3}{2}x[(1-w_m)x^2 + (1+w_m)(1-y^2)] = 0 \tag{1269}$$

$$-\frac{\sqrt{6}}{2}\lambda xy + \frac{3}{2}y[(1-w_m)x^2 + (1+w_m)(1-y^2)] = 0. \tag{1270}$$

- An obvious solution is

$$x_0 = 0, y_0 = 0. \tag{1271}$$

- Two semi-obvious solutions can be found for $y = 0$ which solves the second equation and transforms the first to the quadratic equation $x^2 - 1 = 0$ which gives

$$x_1 = +1, y_1 = 0 \tag{1272}$$

$$x_2 = -1, y_2 = 0. \tag{1273}$$

- Substituting the square bracket of the second equation into the first and simplifying the second gives

$$-3x + \frac{\sqrt{6}}{2}\lambda(x^2 + y^2) = 0 \tag{1274}$$

$$-\frac{\sqrt{6}}{2}\lambda x + \frac{3}{2}[1 + 2x^2 - (x^2 + y^2) - w_m((x^2 + x^2) - 1)] = 0. \tag{1275}$$

Now we can eliminate $x^2 + y^2$ and obtain a single quadratic equation in x

$$-\frac{\sqrt{6}}{2}\lambda x + \frac{3}{2}\left[1 + 2x^2 - \frac{\sqrt{6}}{\lambda}x - w_m\left(\frac{\sqrt{6}}{\lambda}x - 1\right)\right] = 0 \tag{1276}$$

which can be simplified to

$$x^2 - \frac{3(1+w_m) + \lambda^2}{\sqrt{6}\lambda}x + \frac{1+w_m}{2} = 0. \tag{1277}$$

This gives us two more solutions

$$x_3 = \frac{\lambda}{\sqrt{6}}, y_3 = \sqrt{1 - \frac{\lambda^2}{6}} \qquad (\lambda^2 < 6) \tag{1278}$$

$$x_4 = \sqrt{\frac{3}{2}}\frac{1+w_m}{\lambda}, y_4 = \sqrt{\frac{3}{2}}\frac{\sqrt{1-w_m^2}}{\lambda} \qquad (w_m^2 < 1). \tag{1279}$$

- Let's quickly check the stability of the fix points. The characteristic equation for the fix points of a 2d system is given by

$$\alpha^2 + a_1(x_i, y_i)\alpha + a_2(x_i, y_i) = 0 \tag{1280}$$

$$a_1(x_i, y_i) = -\left(\frac{df_x}{dx} + \frac{df_y}{dy}\right)_{x=x_i, y=y_i} \tag{1281}$$

$$a_2(x_i, y_i) = \left.\frac{df_x}{dx}\frac{df_y}{dy} - \frac{df_x}{dy}\frac{df_y}{dx}\right|_{x=x_i, y=y_i} \tag{1282}$$

with the stability classification (assuming for EoS parameter $w_m^2 < 1$)

type	condition	fix point 0	fix point 1	fix point 2
saddle node	$a_2 < 0$	$-1 < w_m < 1$	$\lambda > \sqrt{6}$	$\lambda < -\sqrt{6}$
unstable node	$0 < a_2 < a_1^2/4$	-	$\lambda < \sqrt{6}$	$\lambda > -\sqrt{6}$
unstable spiral	$a_1^2/4 < a_2, a_1 < 0$	-	-	-
center	$0 < a_2, a_1 = 0$	-	-	-
stable spiral	$a_1^2/4 < a_2, a_1 > 0$	-	-	-
stable node	$0 < a_2 < a_1^2/4$	-	-	-

type	fix point 3	fix point 4
saddle node	$3(1+w_m) < \lambda^2 < 6$	-
unstable node	-	-
unstable spiral	-	-
center	-	-
stable spiral	-	$\lambda^2 > \frac{24(1+w_m)^2}{7+9w_m}$
stable node	$\lambda^2 < 3(1+w_m)$	$\lambda^2 < \frac{24(1+w_m)^2}{7+9w_m}$

Problem 26.5 - Tracker solution

Inserting the ansatz

$$\phi(t) = C(\alpha, n) M^{1+\nu} t^{\nu} \tag{1283}$$

into the ODE

$$\ddot{\phi} + \frac{3\alpha}{t}\dot{\phi} - \frac{M^{4+n}}{\phi^{n+1}} = 0 \tag{1284}$$

gives

$$CM^{1+\nu}\nu(\nu-1)t^{\nu-2} + CM^{1+\nu}\frac{3\alpha}{t}t^{\nu-1} - \frac{M^{4+n}}{C^{n+1}M^{(n+1)(1+\nu)}t^{\nu(n+1)}} = 0 \tag{1285}$$

$$CM^{1+\nu}\left[\nu(\nu-1)+3\alpha\right]t^{\nu-2} - \frac{M^{3-\nu(n+1)}}{C^{n+1}}t^{-\nu(n+1)} = 0 \tag{1286}$$

From equating coefficients and powers (in t) we obtain

$$\nu = \frac{2}{2+n} \tag{1287}$$

$$C(\alpha, n) = \left(\frac{(2+n)^2}{6\alpha(2+n)-2n}\right)^{\frac{1}{2+n}}. \tag{1288}$$

7.6 Kugo - Eichtheorie

Problem 1.1

With $\Lambda^\alpha_{\ \mu} \approx \delta^\alpha_\mu + \epsilon^\alpha_{\ \mu}$ we obtain

$$\begin{aligned}
g_{\mu\nu} &= \Lambda^\alpha_{\ \mu}\Lambda^\beta_{\ \nu} g_{\alpha\beta} && (1289)\\
&\simeq \left(\delta^\alpha_\mu + \epsilon^\alpha_{\ \mu}\right)\left(\delta^\beta_\nu + \epsilon^\beta_{\ \nu}\right) g_{\alpha\beta} && (1290)\\
&\simeq g_{\mu\nu} + \epsilon^\alpha_{\ \mu}\delta^\beta_\nu g_{\alpha\beta} + \epsilon^\beta_{\ \nu}\delta^\alpha_\mu g_{\alpha\beta} + \mathcal{O}(\epsilon^2) && (1291)\\
&\simeq g_{\mu\nu} + \epsilon_{\nu\mu} + \epsilon_{\mu\nu} + \mathcal{O}(\epsilon^2) && (1292)
\end{aligned}$$

which means that ϵ is antisymmetric $\epsilon_{\nu\mu} = -\epsilon_{\mu\nu}$ and we can write

$$\epsilon_{\nu\mu} = \frac{1}{2}\left(\epsilon_{\nu\mu} - \epsilon_{\mu\nu}\right). \qquad (1293)$$

The infinitesimal Poincare transformation can then be written as

$$\begin{aligned}
x'^\mu &= \Lambda^\mu_{\ \alpha} x^\alpha + a^\mu && (1294)\\
&\simeq \left(\delta^\mu_\alpha + \epsilon^\mu_{\ \alpha}\right) x^\alpha + a^\mu && (1295)\\
&\simeq x^\mu + \epsilon^\mu_{\ \alpha} x^\alpha + a^\mu. && (1296)
\end{aligned}$$

The inverted PT is then given by

$$\begin{aligned}
x &= \Lambda^{-1}(x' - a) && (1297)\\
&= \Lambda^{-1}x' - \Lambda^{-1}a && (1298)\\
x^\mu &\simeq \left(\delta^\mu_\alpha - \epsilon^\mu_{\ \alpha}\right) x'^\alpha - \left(\delta^\mu_\alpha - \epsilon^\mu_{\ \alpha}\right) a^\alpha && (1299)\\
&\simeq x'^\mu - \epsilon^\mu_{\ \alpha} x'^\alpha - a^\mu + \mathcal{O}(\epsilon \cdot a) && (1300)
\end{aligned}$$

Because of

$$\begin{aligned}
\phi'(x') = \phi(x) \quad &\Leftrightarrow \quad \phi'(\Lambda x + a) = \phi(x) && (1301)\\
&\Leftrightarrow \quad \phi'(x) = \phi(\Lambda^{-1}(x - a)) && (1302)
\end{aligned}$$

we can now calculate

$$\begin{aligned}
\delta\phi(x) &\equiv \phi'(x) - \phi(x) && (1303)\\
&= \phi(\Lambda^{-1}(x-a)) - \phi(x) && (1304)\\
&\simeq \phi(x^\mu - \epsilon^\mu_{\ \alpha}x^\alpha - a^\mu) - \phi(x) && (1305)\\
&\simeq \phi(x) + \partial_\mu\phi(x)\cdot(-\epsilon^\mu_{\ \alpha}x^\alpha - a^\mu) - \phi(x) && (1306)\\
&\simeq -(a^\mu + \epsilon^\mu_{\ \alpha}x^\alpha)\partial_\mu\phi(x) && (1307)\\
&\simeq -(a^\mu + \epsilon^{\mu\alpha}x_\alpha)\partial_\mu\phi(x) && (1308)\\
&\simeq -\left(a^\mu + \frac{1}{2}\left(\epsilon^{\mu\alpha} - \epsilon^{\alpha\mu}\right)x_\alpha\right)\partial_\mu\phi(x) && (1309)\\
&\simeq -\left(a^\mu\partial_\mu + \frac{1}{2}\left(\epsilon^{\mu\alpha}x_\alpha\partial_\mu - \epsilon^{\alpha\mu}x_\alpha\partial_\mu\right)\right)\phi(x) && (1310)\\
&\simeq -\left(a^\mu\partial_\mu + \frac{1}{2}\left(\epsilon^{\mu\alpha}x_\alpha\partial_\mu - \epsilon^{\mu\alpha}x_\mu\partial_\alpha\right)\right)\phi(x) && (1311)\\
&\simeq i\left(a^\mu i\partial_\mu + \frac{1}{2}\epsilon^{\mu\alpha} i\left(x_\alpha\partial_\mu - x_\mu\partial_\alpha\right)\right)\phi(x) && (1312)\\
&\simeq i\left(a^\mu i\partial_\mu - \frac{1}{2}\epsilon^{\mu\alpha} i\left(x_\mu\partial_\alpha - x_\alpha\partial_\mu\right)\right)\phi(x) && (1313)\\
&\simeq i\left(a^\mu P_\mu - \frac{1}{2}\epsilon^{\mu\alpha}M_{\mu\alpha}\right)\phi(x) && (1314)
\end{aligned}$$

Calculating the commutators

$$[P_\mu, P_\nu] = 0 \tag{1315}$$

$$\begin{aligned}
[M_{\mu\nu}, P_\rho] &= i^2(x_\mu\partial_\nu - x_\nu\partial_\mu)\partial_\rho - i^2\partial_\rho(x_\mu\partial_\nu - x_\nu\partial_\mu) && (1316)\\
&= -(x_\mu\partial_\nu - x_\nu\partial_\mu)\partial_\rho + \partial_\rho(x_\mu\partial_\nu - x_\nu\partial_\mu) && (1317)\\
&= -x_\mu\partial_\nu\partial_\rho + x_\nu\partial_\mu\partial_\rho + (\partial_\rho g_{\mu\alpha}x^\alpha)\partial_\nu + x_\mu\partial_\rho\partial_\nu - (\partial_\rho g_{\nu\alpha}x^\alpha)\partial_\mu - x_\nu\partial_\rho\partial_\mu && (1318)\\
&= (\partial_\rho g_{\mu\alpha}x^\alpha)\partial_\nu - (\partial_\rho g_{\nu\alpha}x^\alpha)\partial_\mu && (1319)\\
&= (g_{\mu\alpha}\partial_\rho x^\alpha)\partial_\nu - (g_{\nu\alpha}\partial_\rho x^\alpha)\partial_\mu && (1320)\\
&= (g_{\mu\alpha}\delta^\alpha_\rho)\partial_\nu - (g_{\nu\alpha}\delta^\alpha_\rho)\partial_\mu && (1321)\\
&= g_{\mu\rho}\partial_\nu - g_{\nu\rho}\partial_\mu && (1322)\\
&= -i(g_{\mu\rho}i\partial_\nu - g_{\nu\rho}i\partial_\mu) && (1323)\\
&= -i(g_{\mu\rho}P_\nu - g_{\nu\rho}P_\mu) && (1324)
\end{aligned}$$

$$[M_{\mu\nu}, M_{\rho,\sigma}] = \text{...painful} \tag{1325}$$

7.7 LeBellac - Quantum and Statistical Field Theory

Problem 1.1

Some simple geometry

$$\begin{aligned}
l &= 2a\cos\theta && (1326)\\
x &= l\sin\theta && (1327)\\
&= 2a\cos\theta\sin\theta && (1328)\\
h &= x\tan\theta && (1329)\\
&= 2a\sin^2\theta && (1330)
\end{aligned}$$

Then the potential is given by

$$\begin{aligned}
V(\phi) &= 2mga\sin^2\theta + \frac{1}{2}Ca^2(2\cos\theta - 1)^2 && (1331)\\
\frac{\partial V}{\partial\theta} &= 4mga\sin\theta\cos\theta - 2Ca^2(2\cos\theta - 1)\sin\theta && (1332)\\
&= 2a\sin\theta\,(2mg\cos\theta - Ca(2\cos\theta - 1)) && (1333)\\
&= 2a\sin\theta\,(2(mg - Ca)\cos\theta + Ca) && (1334)\\
&\to \theta_0 = 0 && (1335)\\
&\to \theta_{1,2} = \arccos\frac{Ca}{2(Ca - mg)} && (1336)
\end{aligned}$$

Stability

$$\begin{aligned}
\frac{\partial^2 V}{\partial\theta^2}(\theta_{1,2}) &= 2a(2mg - Ca) && (1337)\\
\frac{\partial^2 V}{\partial\theta^2}(\theta_0) &= 2a(2mg - Ca) && (1338)
\end{aligned}$$

8 Particle Physics

8.1 Nagashima - Elementary Particle Physics Volume 1: Quantum Field Theory and Particles

Problem 2.1

1. Simple calculation

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} = \frac{1.6^2 \cdot 10^{-38}}{12 \cdot 9 \cdot 10^{-12} \cdot 10^{-34} \cdot 3 \cdot 10^8} \tag{1339}$$

$$= \frac{1}{108} 10^{-38+34} \frac{1}{10^{-12+8}} \tag{1340}$$

$$\approx \frac{1}{137} \tag{1341}$$

$$\alpha_G = G_N \frac{m_e m_p}{\hbar c} = 7 \cdot 10^{-11} \frac{10^{-30} \cdot 2 \cdot 10^{-27}}{10^{-34} \cdot 3 \cdot 10^8} \tag{1342}$$

$$= 4 \cdot 10^{-11-30-27} \frac{1}{10^{-34+8}} \tag{1343}$$

$$= 4 \cdot 10^{-42} \tag{1344}$$

2. Another simple one

$$1 = G\frac{m_P^2}{\hbar c} \quad \rightarrow \quad m_P = \sqrt{\frac{\hbar c}{G}} = 2 \cdot 10^{-8}\text{kg} \tag{1345}$$

$$E_P = m_P c^2 = \sqrt{\frac{\hbar c^5}{G}} = 2 \cdot 10^9\text{J} = 1.2 \cdot 10^{19}\text{eV} \tag{1346}$$

Problem 2.3

Basic approximation with $\Delta m = m$

$$\Delta E \cdot \Delta t \approx \frac{\hbar}{2} \tag{1347}$$

$$\Delta t \approx \frac{\hbar}{2\Delta m \cdot c^2} \tag{1348}$$

$$\Delta x = c \cdot \Delta t \approx \frac{\hbar}{2\Delta m \cdot c} \tag{1349}$$

$$\tag{1350}$$

Forgetting the factor 2 and knowing $1\text{GeV}^{-1} = 0.197 \cdot 10^{-15}\text{s}$

$$\Delta x_W = \frac{1}{80}\frac{1}{\text{GeV}} = 2.4 \cdot 10^{-18} \tag{1351}$$

$$\Delta x_Z = \frac{1}{91}\frac{1}{\text{GeV}} = 2.16 \cdot 10^{-18} \tag{1352}$$

Problem 2.4

$$\Delta m = \frac{\hbar}{\Delta x c} \tag{1353}$$

$$\Delta E = \Delta m c^2 = \frac{\hbar c}{\Delta x} \tag{1354}$$

$$\Delta E_{\text{crab}} = 2 \cdot 10^{-25}\text{eV} \tag{1355}$$

$$\Delta E_{\text{galactic}} = 2 \cdot 10^{-29}\text{eV} \tag{1356}$$

9 General Relativity

9.1 Coleman - Sidney Coleman's Lectures On Relativity

Problem 1.1

Lets simplify

$$\tau(b) - \tau(a) = \int_a^b \sqrt{c^2dt^2 - dx^2 - dy^2 - dz^2} \tag{1357}$$

$$= \int_a^b cdt\sqrt{1 - \frac{v^2}{c^2}} \tag{1358}$$

If we LT into the inertial system of Alice her proper time is simply (because $v = 0$)

$$\Delta\tau_A = \int_a^b cdt = ct \tag{1359}$$

For Bob we obtain

$$\Delta\tau_B = \int_a^b cdt\sqrt{1 - \frac{v(t)^2}{c^2}} \tag{1360}$$

where the square root is smaller than one as soon the the observer is moving. Therefore it clear that $\Delta\tau_A < \Delta\tau_B$.

9.2 Carroll - Spacetime an Geometry

Problem 1.7

1. Because the metric is symmetric

$$X^{\mu}{}_{\nu} = \eta_{\nu\alpha}X^{\mu\alpha} = X^{\mu\alpha}\eta_{\alpha\nu} \equiv X\eta \tag{1361}$$

$$= \begin{pmatrix} -2 & 0 & 1 & -1 \\ 1 & 0 & 3 & 2 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & -2 \end{pmatrix} \tag{1362}$$

2.

$$X_{\mu}{}^{\nu} = \eta_{\nu\alpha}X^{\alpha\nu} \equiv \eta X \tag{1363}$$

$$= \begin{pmatrix} -2 & 0 & -1 & 1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix} \tag{1364}$$

3.

$$X^{(\mu\nu)} = \frac{1}{2}(X^{\mu\nu} + X^{\nu\mu}) \tag{1365}$$

$$= \begin{pmatrix} 2 & -1/2 & 0. & -3/2 \\ -1/2 & 0 & 2 & 3/2 \\ 0 & 2 & 0 & 1/2 \\ -3/2 & 3/2 & 1/2 & -2 \end{pmatrix} \tag{1366}$$

4.

$$X_{\mu\nu} = \eta_{\mu\alpha}\eta_{\nu\beta}X^{\alpha\beta} \equiv \eta X \eta \tag{1367}$$

$$= \begin{pmatrix} 2 & 0 & -1 & 1 \\ 1 & 0 & 3 & 2 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & -2 \end{pmatrix} \tag{1368}$$

$$X_{[\mu\nu]} = \frac{1}{2}(X_{\mu\nu} - X_{\nu\mu}) \tag{1369}$$

$$= \begin{pmatrix} 0 & -1/2 & -1 & -1/2 \\ 1/2 & 0 & 1 & 1/2 \\ 1 & -1 & 0 & -1/2 \\ 1/2 & -1/2 & 1/2 & 0 \end{pmatrix} \tag{1370}$$

5.

$$X^{\lambda}{}_{\lambda} = \eta_{\lambda\alpha}X^{\lambda\alpha} = X^{\lambda_\alpha}\eta_{\alpha\lambda} \equiv \mathrm{Tr}(X\eta) = -4 \tag{1371}$$

$$\tag{1372}$$

6.

$$V^{\mu}V_{\mu} = V^{\mu}\eta_{\mu\nu}V^{\nu} = 7 \tag{1373}$$

7.

$$V_{\mu}X^{\mu\nu} = V^{\alpha}\eta_{\alpha\mu}X^{\mu\nu} \equiv V\eta X = (4, -2, 5, 7) \tag{1374}$$

Problem 3.3 - Christoffel symbols for diagonal metric

With $g_{\mu\nu} = \mathrm{diag}(g_{11}, g_{22}, g_{33}, g_{44})$ the inverse is given by $g^{\mu\nu} = \mathrm{diag}(1/g_{11}, 1/g_{22}, 1/g_{33}, 1/g_{44})$. Now for $\mu \neq \mu \neq \lambda$ we obtain

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}(\partial_{\mu}g_{\nu\sigma} + \partial_{\nu}g_{\mu\sigma} - \partial_{\sigma}g_{\mu\nu}) \tag{1375}$$

$$= \frac{1}{2}g^{\lambda\lambda}(\partial_{\mu}\underbrace{g_{\nu\lambda}}_{=0} + \partial_{\nu}\underbrace{g_{\mu\lambda}}_{=0} - \partial_{\lambda}\underbrace{g_{\mu\nu}}_{=0}) \tag{1376}$$

$$= 0 \tag{1377}$$

$$\Gamma^{\lambda}_{\mu\mu} = \frac{1}{2}g^{\lambda\sigma}(\partial_{\mu}g_{\mu\sigma} + \partial_{\mu}g_{\mu\sigma} - \partial_{\sigma}g_{\mu\mu}) \tag{1378}$$

$$= \frac{1}{2}g^{\lambda\lambda}(\partial_{\mu}g_{\mu\lambda} + \partial_{\mu}g_{\mu\lambda} - \partial_{\lambda}g_{\mu\mu}) \tag{1379}$$

$$= -\frac{1}{2}\frac{1}{g_{\lambda\lambda}}\partial_{\lambda}g_{\mu\mu} \tag{1380}$$

$$\begin{aligned}
\Gamma^\lambda_{\mu\lambda} &= \frac{1}{2}g^{\lambda\sigma}(\partial_\lambda g_{\mu\sigma} + \partial_\mu g_{\lambda\sigma} - \partial_\sigma g_{\lambda\mu}) && (1381)\\
&= \frac{1}{2}g^{\lambda\lambda}(\partial_\lambda g_{\mu\lambda} + \partial_\mu g_{\lambda\lambda} - \partial_\lambda g_{\lambda\mu}) && (1382)\\
&= \frac{1}{2}\frac{1}{g^{\lambda\lambda}}\partial_\mu g_{\lambda\lambda} && (1383)\\
&= \frac{1}{2}\frac{1}{\text{sgn}\cdot|g^{\lambda\lambda}|}\partial_\mu(\text{sgn}\cdot|g_{\lambda\lambda}|) && (1384)\\
&= \frac{1}{2}\frac{1}{|g^{\lambda\lambda}|}\partial_\mu(|g_{\lambda\lambda}|) && (1385)\\
&= \frac{1}{2}\partial_\mu \log|g_{\lambda\lambda}| && (1386)\\
&= \partial_\mu \log\sqrt{|g_{\lambda\lambda}|} && (1387)
\end{aligned}$$

$$\begin{aligned}
\Gamma^\lambda_{\lambda\lambda} &= \frac{1}{2}g^{\lambda\sigma}(\partial_\lambda g_{\lambda\sigma} + \partial_\lambda g_{\lambda\sigma} - \partial_\sigma g_{\lambda\lambda}) && (1388)\\
&= \frac{1}{2}g^{\lambda\lambda}(\partial_\lambda g_{\lambda\lambda} + \partial_\lambda g_{\lambda\lambda} - \partial_\lambda g_{\lambda\lambda}) && (1389)\\
&= \frac{1}{2}\frac{\partial_\lambda g_{\lambda\lambda}}{g_{\lambda\lambda}} && (1390)\\
&= \partial_\lambda \log\sqrt{|g_{\lambda\lambda}|} && (1391)
\end{aligned}$$

9.3 Poisson - A relativists toolkit

Problem 1.1 - Parallel transport on cone

1. We find the metric by using elementary geometry

$$ds^2 = dr^2 + (r\sin\alpha)^2 d\phi^2 \tag{1392}$$

2. Trying around a bit - we find

$$\begin{aligned}
x &= r\cos(\phi\sin\alpha) && (1393)\\
y &= r\sin(\phi\sin\alpha) && (1394)
\end{aligned}$$

$$\begin{aligned}
x = f(r,\phi) \quad &\rightarrow \quad dx = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \phi}d\phi && (1395)\\
& \quad dx = \cos(\phi\sin\alpha)dr - r\sin(\phi\sin\alpha)\sin\alpha\, d\phi && (1396)\\
y = g(r,\phi) \quad &\rightarrow \quad dy = \frac{\partial g}{\partial r}dr + \frac{\partial g}{\partial \phi}d\phi && (1397)\\
& \quad dy = \sin(\phi\sin\alpha)dr + r\cos(\phi\sin\alpha)\sin\alpha\, d\phi && (1398)
\end{aligned}$$

We can then simply check $ds^2 = dx^2 + dy^2$

3. The parallel transport equation for a vector A^λ along curve $x^\mu(s)$ is given by

$$\begin{aligned}
\dot{x}^\mu \nabla_\mu A^\lambda &= 0 && (1399)\\
\frac{dx^\mu}{ds}\partial_\mu A^\lambda + \Gamma^\lambda_{\mu\nu}\dot{x}^\mu A^\nu &= 0 && (1400)\\
\frac{\partial A^\lambda}{\partial s} + \Gamma^\lambda_{\mu\nu}\dot{x}^\mu A^\nu &= 0 && (1401)
\end{aligned}$$

We are moving the vector $\vec{A} = (A_r, A_\phi)$ along $\vec{x}(s) = (r, s)$ with $s \in [0, 2\pi]$ and $\dot{\vec{x}}(s) = (0, 1)$. Calculating the Christoffel symbols gives

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \tag{1402}$$
$$\Gamma^r_{\phi\phi} = -r \sin^2 \alpha \tag{1403}$$
$$\Gamma^\phi_{r\phi} = 1/r \tag{1404}$$
$$\Gamma^\phi_{\phi r} = 1/r \tag{1405}$$

and the parallel transport equations simplify to

$$\dot{A}_r + \Gamma^r_{\phi\phi} \dot{x}_\phi A_\phi = 0 \quad \rightarrow \quad \dot{A}_r - r \sin^2 \alpha A_\phi = 0 \tag{1406}$$
$$\dot{A}_\phi + \Gamma^\phi_{\phi r} \dot{x}_\phi A_r = 0 \quad \rightarrow \quad \dot{A}_\phi + \frac{1}{r} A_r = 0. \tag{1407}$$

which can be solved by Mathematica. To obtain the angle β we calculate first the norm of the vector at a given t

$$|\vec{A}(t)| = g_{\mu\nu} A^\mu(t) A^\nu(t) \tag{1408}$$
$$= A_r(0)^2 + A_\phi(0)^2 r^2 \sin^2 \alpha. \tag{1409}$$

Now we can calculate the inner product

$$\vec{A}(t = 2\pi) \cdot \vec{A}(0) = g_{\mu\nu} A^\mu(t) A^\nu(0) \tag{1410}$$
$$= (A_r(0)^2 + A_\phi(0)^2 r^2 \sin^2 \alpha) \cos(2\pi \sin \alpha) \tag{1411}$$

so $\cos \beta = \cos(2\pi \sin \alpha)$.

10 Quantum Gravity

10.1 Hartman - Lectures on Quantum Gravity and Black Holes

10.2 Ammon, Erdmenger - Gauge/Gravity Duality - Foundations and Applications

The authors use $d-1$ spacial dimension and the sign convention

$$\eta_{\mu\nu} = diag(-1, 1, ..., 1) \tag{1412}$$

which implies

$$\Box = \partial^\mu \partial_\mu = -\partial_t^2 + \triangle \tag{1413}$$

$$kx = -k^0 x^0 + \vec{k}\vec{x} \tag{1414}$$

and results in a minus sign in the KG equation.

Problem 1.1.1 - Fourier representation of free scalar field

Ansatz (because KG equation looks quite similar to wave equation) $\phi(x) = a \cdot e^{ikx}$ with $x^\mu = (t, \vec{x})$, $k^\mu = (\omega, \vec{k})$ and $a \in \mathbb{C}$ meaning

$$e^{ikx} \equiv e^{ik^\mu x_\mu} = e^{i\eta_{\mu\nu} k^\mu x^\nu} = e^{i(-k^0 x^0 + \vec{k}\vec{x})} \tag{1415}$$

Inserting into the equation of motion

$$(\Box - m^2)\phi(x) = (\partial^t \partial_t + \triangle - m^2)\phi(x) \tag{1416}$$

$$= a(-\partial_t^2 + \triangle - m^2) e^{i(-\omega t + \vec{k}\vec{x})} \tag{1417}$$

$$= a\left(\omega^2 + i^2 \vec{k}^2 - m^2\right) e^{i(-\omega t + \vec{k}\vec{x})} = 0 \tag{1418}$$

This implies $\omega^2 - \vec{k}^2 - m^2 = 0$ and therefore $\omega_k \equiv \omega = \sqrt{\vec{k}^2 + m^2}$. One particular solution is therefore $\phi(x) = a \cdot e^{ikx}|_{k^0 = \omega_k}$. The general solution is then given by a superposition

$$\phi(x) = \int d^{d-1}\vec{k} \left[a(\vec{k}) e^{ikx}\right] \tag{1419}$$

to ensure a real valued ϕx we add the conjugate complex solution

$$\phi(x) = \int d^{d-1}\vec{k} \left[a(\vec{k}) e^{ikx} + a^*(\vec{k}) e^{-ikx}\right]. \tag{1420}$$

The factor $(2\pi)^{1-d}/2\omega_k$ can be absorbed into $a(k)$.

Problem 1.1.2 - Lagrangian of self-interacting scalar field

The Lagrangian is then

$$\mathcal{L} = \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}} \tag{1421}$$

$$= -\frac{1}{2}\eta^{\mu\nu}\partial_\mu \phi(x) \partial_\nu \phi(x) - \frac{1}{2} m^2 \phi(x)^2 - \frac{g}{4!}\phi(x)^4. \tag{1422}$$

with the Euler-Lagrange equations

$$\partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi)}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0. \tag{1423}$$

Therefore

$$\partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi)} \right) = \partial_\alpha \left(-\frac{1}{2}\eta^{\mu\nu}[\delta_{\mu\alpha}\partial_\nu\phi + \partial_\mu\phi\delta_{\nu\alpha}] \right) \tag{1424}$$

$$= \partial_\alpha \left(-\frac{1}{2}\eta^{\alpha\nu}\partial_\nu\phi - \frac{1}{2}\eta^{\mu\alpha}\partial_\mu\phi \right) \tag{1425}$$

$$= -\partial_\alpha \left(\eta^{\alpha\beta}\partial_\beta\phi \right) \tag{1426}$$

$$= -\partial^\beta\partial_\beta\phi \tag{1427}$$

$$= -\Box\phi \tag{1428}$$

and

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi - \frac{g}{3!}\phi^3. \tag{1429}$$

The relevant term in the Euler-Lagrange equations is $\partial\mathcal{L}_{\text{int}}/\partial\phi = -g\phi^3/3!$. The modified equation of motion is therefore

$$(\Box - m^2)\phi(x) - \frac{g}{3!}\phi(x)^3 = 0 \tag{1430}$$

Problem 1.1.3 - Complex scalar field

$$\mathcal{L}_{\text{free}} = -\partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi \tag{1431}$$

$$= -\eta^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi - m^2\phi^*\phi \tag{1432}$$

$$= -\frac{1}{2}\eta^{\mu\nu}\partial_\mu(\phi_1 - i\phi_2)\partial_\nu(\phi_1 + i\phi_2) - \frac{1}{2}m^2(\phi_1^2 + \phi_2^2) \tag{1433}$$

$$= -\frac{1}{2}\eta^{\mu\nu}\left(\partial_\mu\phi_1\partial_\nu\phi_1 + i\partial_\mu\phi_1\partial_\nu\phi_2 - i\partial_\mu\phi_2\partial_\nu\phi_1 + \partial_\mu\phi_2\partial_\nu\phi_2\right) - \frac{1}{2}m^2(\phi_1^2 + \phi_2^2) \tag{1434}$$

$$= -\frac{1}{2}\eta^{\mu\nu}\left(\partial_\mu\phi_1\partial_\nu\phi_1 + \partial_\mu\phi_2\partial_\nu\phi_2\right) - \frac{1}{2}m^2(\phi_1^2 + \phi_2^2) \tag{1435}$$

$$= -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi_1\partial_\nu\phi_1 - \frac{1}{2}m^2\phi_1^2 - \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi_2\partial_\nu\phi_2 - \frac{1}{2}m^2\phi_2^2 \tag{1436}$$

$$= \mathcal{L}_{\text{free1}} + \mathcal{L}_{\text{free2}} \tag{1437}$$

Equations of motion for ϕ and ϕ^* are given by

$$\partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^*)} \right) - \frac{\partial \mathcal{L}}{\partial \phi^*} = 0 \tag{1438}$$

$$-\partial_\mu\partial^\mu\phi + m^2\phi = 0 \tag{1439}$$

$$(\Box - m^2)\phi = 0 \tag{1440}$$

and

$$\partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^*)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \tag{1441}$$

$$-\partial_\mu\partial^\mu\phi + m^2\phi^* = 0 \tag{1442}$$

$$(\Box - m^2)\phi^* = 0 \tag{1443}$$

Problem 1.2.1 - Time-independence of Noether charge

The conserved current is

$$\partial_\mu\mathcal{J}^\mu \equiv -\partial_0\mathcal{J}^0 + \partial_i\mathcal{J}^i = 0. \tag{1444}$$

Spacial integration using Gauss law on the right hand side gives

$$\int_{\mathbb{R}^{d-1}} d^{d-1}\vec{x}\, \partial_0 \mathcal{J}^0 = \int_{\mathbb{R}^{d-1}} d^{d-1}\vec{x}\, \partial_i \mathcal{J}^i \tag{1445}$$

$$\partial_0 \int_{\mathbb{R}^{d-1}} d^{d-1}\vec{x}\, \mathcal{J}^0 = \int_{\partial\mathbb{R}^{d-1}} dS\, \mathcal{J}^i \tag{1446}$$

$$\partial_0 \mathcal{Q} = 0 \tag{1447}$$

where we used that $\mathcal{J}^i$ is vanishing at infinity.

Problem 1.2.2 - Hamiltonian of scalar field

The Lagrangian of the real free scalar field is given by

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi(x)\partial_\nu\phi(x) - \frac{1}{2}m^2\phi(x)^2. \tag{1448}$$

The canonical momentum is therefore

$$\Pi = \frac{\partial\mathcal{L}}{\partial(\partial_t\phi)} \tag{1449}$$

$$= -\frac{1}{2}2\eta^{ti}\partial_i\phi - \frac{1}{2}2\eta^{tt}\partial_t\phi \tag{1450}$$

$$= \partial_t\phi. \tag{1451}$$

Using $\eta_{\mu\nu} = diag(-1,1,...,1)$ the Hamiltonian $\mathcal{H} = \Theta^{tt} = \eta^{t\nu}\Theta^t{}_\nu = -\Theta^t{}_t$ is

$$\Theta^t{}_t = -\frac{\partial\mathcal{L}}{\partial(\partial_t\phi)}\partial_t\phi + \mathcal{L} \tag{1452}$$

$$= -\Pi\cdot\partial_t\phi + \mathcal{L} \tag{1453}$$

and therefore

$$\mathcal{H} = \Pi\partial_t\phi - \mathcal{L} \tag{1454}$$

$$= \Pi^2 - \left(-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi(x)\partial_\nu\phi(x) - \frac{1}{2}m^2\phi(x)^2\right) \tag{1455}$$

$$= \Pi^2 - \left(\frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi(x)^2\right) \tag{1456}$$

$$= \frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi(x)^2 \tag{1457}$$

Problem 1.2.3 - Symmetric energy-momentum tensor

The Lorentz transformation

$$\Lambda^\mu_{\ \nu} = \delta^\mu_{\ \nu} + \omega^\mu_{\ \nu} \tag{1458}$$

implies the field transformation

$$\phi(x^\mu) \to \tilde{\phi}(x^\mu) = \phi(x^\mu - \omega^\mu{}_\rho x^\rho) \tag{1459}$$

$$= \phi(x^\mu) - \omega^\mu{}_\rho x^\rho\partial_\mu\phi \tag{1460}$$

under which the Lagrangian transforms as

$$\begin{aligned}
\mathcal{L} \to \tilde{\mathcal{L}} &= \mathcal{L} + \frac{\partial \mathcal{L}}{\partial x^\mu} dx^\mu && (1461)\\
&= \mathcal{L} - \omega^\nu{}_\rho x^\rho \partial_\mu (\delta^\mu_\nu \mathcal{L}) && (1462)\\
&= \mathcal{L} + \partial_\mu(\omega^\nu{}_\rho x^\rho) \cdot (\delta^\mu_\nu \mathcal{L}) - \partial_\mu(\omega^\nu{}_\rho x^\rho \delta^\mu_\nu \mathcal{L}) && (1463)\\
&= \mathcal{L} + \omega^\nu{}_\rho \delta^\rho_\mu \cdot (\delta^\mu_\nu \mathcal{L}) - \partial_\mu(\omega^\nu{}_\rho x^\rho \delta^\mu_\nu \mathcal{L}) && (1464)\\
&= \mathcal{L} + \omega^\rho{}_\rho \mathcal{L} - \partial_\mu(\omega^\nu{}_\rho x^\rho \delta^\mu_\nu \mathcal{L}) && (1465)\\
&= \mathcal{L} - \partial_\mu(\omega^\nu{}_\rho x^\rho \delta^\mu_\nu \mathcal{L}) && (1466)
\end{aligned}$$

where we used $\omega_{\mu\nu} = -\omega_{\nu\mu}$ meaning

$$\begin{aligned}
\omega^\rho{}_\rho &= \eta^{\alpha\rho}\omega_{\alpha\rho} && (1467)\\
&= \sum_\rho \eta^{0\rho}\omega_{0\rho} + \eta^{1\rho}\omega_{1\rho} + \eta^{2\rho}\omega_{2\rho} + \eta^{3\rho}\omega_{3\rho} && (1468)\\
&= 0 && (1469)
\end{aligned}$$

in the last step (as η has only diagonal elements and the diagonal elements of ω are zero). With $\delta\phi = -\omega^\mu{}_\rho x^\rho \partial_\mu \phi$ and $X^\mu = -\omega^\nu{}_\rho x^\rho \delta^\mu_\nu \mathcal{L}$ we obtain for the conserved current

$$\begin{aligned}
\mathcal{J}^\mu &= -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi + X^\mu && (1470)\\
&= -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (-\omega^\nu{}_\rho x^\rho \partial_\nu \phi) + (-\omega^\nu{}_\rho x^\rho \delta^\mu_\nu \mathcal{L}) && (1471)\\
&= (-\omega^\nu{}_\rho x^\rho)\left(-\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi + (\delta^\mu_\nu \mathcal{L})\right) && (1472)\\
&= (-\omega^\nu{}_\rho x^\rho)\Theta^\mu_\nu && (1473)\\
&= (-\eta^{\nu\alpha}\omega_{\alpha\rho} x^\rho)\Theta^\mu_\nu && (1474)\\
&= -\omega_{\alpha\rho} x^\rho \Theta^{\mu\alpha} && (1475)\\
&= -\frac{1}{2}\omega_{\alpha\rho}(x^\rho \Theta^{\mu\alpha} - x^\alpha \Theta^{\mu\rho}) && (1476)\\
&= -\frac{1}{2}\omega_{\alpha\rho} N^{\mu\rho\alpha} && (1477)
\end{aligned}$$

With $\partial_\mu \Theta^\mu_\nu = 0$ and $\partial_\mu N^{\mu\nu\rho} = 0$ we see

$$\begin{aligned}
0 &= \partial_\mu N^{\mu\nu\rho} && (1478)\\
&= \partial_\mu \left(x^\nu \Theta^{\mu\rho} - x^\rho \Theta^{\mu\nu}\right) && (1479)\\
&= (\partial_\mu x^\nu)\Theta^{\mu\rho} + x^\nu(\partial_\mu \Theta^{\mu\rho}) - (\partial_\mu x^\rho)\Theta^{\mu\nu} - x^\rho(\partial_\mu \Theta^{\mu\nu}) && (1480)\\
&= \delta^\nu_\mu \Theta^{\mu\rho} + x^\nu(\partial_\mu \Theta^{\mu\rho}) - \delta^\rho_\mu \Theta^{\mu\nu} - x^\rho(\partial_\mu \Theta^{\mu\nu}) && (1481)\\
&= \Theta^{\nu\rho} - \Theta^{\rho\nu}. && (1482)
\end{aligned}$$

which means that the (canonical) energy-momentum tensor for Poincare invariant field theories is symmetric $\Theta^{\nu\rho} = \Theta^{\rho\nu}$.

Problem 1.2.4 - Callan-Coleman-Jackiw energy-momentum tensor

For the scalar field we have with $\mathcal{L} = -\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}m^2\phi^2$

$$\begin{aligned}
\Theta^\mu_{\ \nu} &= -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi + (\delta^\mu_{\ \nu}\mathcal{L}) && (1483)\\
&= -\left(-\frac{1}{2}\eta^{\alpha\beta}\delta^\mu_\alpha\partial_\beta\phi - \frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\phi\delta^\mu_\beta\right)\partial_\nu\phi + \delta^\mu_{\ \nu}\left(-\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}m^2\phi^2\right) && (1484)\\
&= \partial^\mu\phi\partial_\nu\phi - \frac{1}{2}\delta^\mu_{\ \nu}(\partial^\beta\phi\partial_\beta\phi + m^2\phi^2) && (1485)
\end{aligned}$$

which gives in the massless case

$$\begin{aligned}
\Theta^\mu_{\ \nu,\ \text{massless}} &= \partial^\mu\phi\partial_\nu\phi - \frac{1}{2}\delta^\mu_{\ \nu}\partial^\beta\phi\partial_\beta\phi && (1486)\\
\Theta_{\mu\nu,\ \text{massless}} &= \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\partial^\beta\phi\partial_\beta\phi && (1487)
\end{aligned}$$

The new improved or Callan–Coleman–Jackiw energy-momentum tensor for a single, real, massless scalar field in d-dimensional Minkowski space is obtained by adding a term proportional to $(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box)\phi^2$ where the proportionality constant is chosen to make the tensor traceless

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\partial_\rho\phi\partial^\rho\phi - \frac{d-2}{4(d-1)}\left(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box\right)\phi^2 \qquad (1488)$$

Let us now check the properties

1. symmetric: obvious
2. conserved: we use the equation of motion $\partial^\mu\partial_\mu\phi = \Box\phi = 0$

$$\begin{aligned}
\partial_\mu T^{\mu\nu} &= (\partial_\mu\partial^\mu\phi)\partial^\nu\phi + \partial^\mu\phi(\partial_\mu\partial^\nu\phi) && (1489)\\
&\quad - \frac{1}{2}\eta^{\mu\nu}\left[(\partial_\mu\partial_\rho\phi)\partial^\rho\phi + \partial_\rho\phi(\partial_\mu\partial^\rho\phi)\right] && (1490)\\
&\quad - \frac{d-2}{4(d-1)}\Box\partial^\nu\phi^2 + \frac{d-2}{4(d-1)}\eta^{\mu\nu}\partial_\mu\Box\phi^2 && (1491)\\
&= \partial^\mu\phi(\partial_\mu\partial^\nu\phi) - \frac{1}{2}\left[(\partial^\nu\partial_\rho\phi)\partial^\rho\phi + \partial_\rho\phi(\partial^\nu\partial^\rho\phi)\right] && (1492)\\
&= 0 && (1493)
\end{aligned}$$

3. traceless:

$$\begin{aligned}
T^\mu_{\ \mu} &= \partial^\mu\phi\partial_\mu\phi - \frac{1}{2}\eta^\mu_{\ \mu}\partial_\rho\phi\partial^\rho\phi - \frac{d-2}{4(d-1)}\left(\partial^\mu\partial_\mu - \eta^\mu_{\ \mu}\Box\right)\phi^2 && (1494)\\
&= \partial^\mu\phi\partial_\mu\phi - \frac{d}{2}\partial_\rho\phi\partial^\rho\phi - \frac{d-2}{4(d-1)}\left(\partial^\mu\partial_\mu - d\cdot\partial^\mu\partial_\mu\right)\phi^2 && (1495)\\
&= \frac{2-d}{2}\partial_\rho\phi\partial^\rho\phi - \frac{d-2}{4(d-1)}(1-d)\partial^\mu\partial_\mu\phi^2 && (1496)\\
&= \frac{2-d}{2}\partial_\rho\phi\partial^\rho\phi + \frac{d-2}{4}\partial^\mu\partial_\mu\phi^2 && (1497)\\
&= \frac{2-d}{2}\partial_\rho\phi\partial^\rho\phi + \frac{d-2}{4}\partial^\mu(2\phi\partial_\mu\phi) && (1498)\\
&= \frac{2-d}{2}\left[\partial_\rho\phi\partial^\rho\phi - \partial^\mu\phi\partial_\mu\phi\right] + \frac{d-2}{2}\phi\cdot\Box\phi && (1499)\\
&= 0. && (1500)
\end{aligned}$$

Problem 1.2.5 - Noether currents of complex scalar field

$$\mathcal{L}_{\text{free}} = -\partial^\mu \phi^* \partial_\mu \phi - m^2 \phi^* \phi \tag{1501}$$
$$= -\eta^{\mu\nu} \partial_\nu \phi^* \partial_\nu \phi - m^2 \phi^* \phi \tag{1502}$$

with the field transformations

$$\phi \to \phi' = e^{i\alpha}\phi = \phi + i\alpha\phi \tag{1503}$$
$$\phi^* \to \phi^{*'} = e^{-i\alpha}\phi^* = \phi^* - i\alpha\phi^* \tag{1504}$$
$$\mathcal{L} \to \mathcal{L}' = \mathcal{L} \tag{1505}$$

we have $\delta\phi = i\alpha\phi$ and $\delta\phi^* = -i\alpha\phi^*$ and $X^\mu = 0$. With

$$\mathcal{J}^\sigma = -\frac{\partial \mathcal{L}}{\partial(\partial_\sigma \phi)}\delta\phi + X^\sigma \tag{1506}$$

we obtain the the two fields

$$\mathcal{J}^\sigma = -\frac{\partial \mathcal{L}}{\partial(\partial_\sigma \phi)}\delta\phi - \frac{\partial \mathcal{L}}{\partial(\partial_\sigma \phi^*)}\delta\phi^* \tag{1507}$$
$$= -(\eta^{\sigma\nu}\partial_\nu\phi^*)i\alpha\phi + (\eta^{\sigma\nu}\partial_\nu\phi)i\alpha\phi^* \tag{1508}$$
$$= i\alpha\left[\phi^*(\partial^\sigma\phi) - \phi(\partial^\sigma\phi^*)\right] \tag{1509}$$

Problem 1.2.6 - $O(n)$ invariance of action of n free scalar fields

For the n real scalar fields with equal mass m we have

$$\mathcal{L} = -\frac{1}{2}\sum_{j=1}^{n}\left[\eta^{\alpha\beta}(\partial_\alpha\phi_j)(\partial_\beta\phi^j) + m^2(\phi^j)^2\right] \tag{1510}$$

the action functional is then

$$S = \int d^dx \mathcal{L} \tag{1511}$$
$$= -\frac{1}{2}\sum_{j=1}^{n}\int d^dx\left[\eta^{\alpha\beta}(\partial_\alpha\phi_j)(\partial_\beta\phi^j) + m^2(\phi_j\phi^j)\right] \tag{1512}$$

With $\phi'^j = R^j_{\ k}\phi^k$ and the definition of an orthogonal matrix R (inner product is invariant under rotation)

$$x^i x_i = x^i \delta_{ij} x^j \tag{1513}$$
$$\stackrel{!}{=} R^i_{\ a} x^a \delta_{ij} R^j_{\ b} x^b \tag{1514}$$
$$= \delta_{ij} R^j_{\ b} R^i_{\ a} x^a x^b \tag{1515}$$
$$= R_{ib} R^i_{\ a} x^a x^b \tag{1516}$$

we require $R_{ib}R^i{}_a = \delta_{ba}$. Then we can recalculate the action

$$S' = -\frac{1}{2}\sum_{j=1}^{n}\int d^dx \left[\eta^{\alpha\beta}(\partial_\alpha R_{ja}\phi^a)(\partial_\beta R^j{}_b\phi^b) + m^2(R_{ja}\phi^a \cdot R^j{}_b\phi^b)\right] \tag{1517}$$

$$= -\frac{1}{2}\sum_{j=1}^{n}\int d^dx \left[\eta^{\alpha\beta}R_{ja}R^j{}_b(\partial_\alpha\phi^a)(\partial_\beta\phi^b) + m^2 R_{ja}R^j{}_b(\phi^a\cdot\phi^b)\right] \tag{1518}$$

$$= -\frac{1}{2}\sum_{b=1}^{n}\int d^dx \left[\eta^{\alpha\beta}\delta_{ab}(\partial_\alpha\phi^a)(\partial_\beta\phi^b) + m^2\delta_{ab}(\phi^a\cdot\phi^b)\right] \tag{1519}$$

$$= -\frac{1}{2}\sum_{b=1}^{n}\int d^dx \left[\eta^{\alpha\beta}(\partial_\alpha\phi_b)(\partial_\beta\phi^b) + m^2(\phi_b\cdot\phi^b)\right] \tag{1520}$$

Analog for the complex case.

Problem 1.3.1 - Field commutators of scalar field

From the field

$$\hat\phi(x) = \frac{1}{(2\pi)^{d-1}}\int\frac{d^{d-1}\vec k}{2\omega_k}\left[\hat a(\vec k)e^{ikx} + \hat a^\dagger(\vec k)e^{-ikx}\right]_{k^0=\omega_k} \tag{1521}$$

we can derive the conjugated momentum

$$\hat\Pi(x) = \partial_t\hat\phi \tag{1522}$$

$$= \frac{1}{(2\pi)^{d-1}}\int\frac{d^{d-1}\vec k}{2\omega_k}\partial_t\left[\hat a(\vec k)e^{-i\omega_k t}e^{i\vec k\vec x} + \hat a^\dagger(\vec k)e^{i\omega_k t}e^{-i\vec k\vec x}\right] \tag{1523}$$

$$= \frac{1}{(2\pi)^{d-1}}\int\frac{d^{d-1}\vec k}{2\omega_k}\left[\hat a(\vec k)(-i\omega_k)e^{ikx} + \hat a^\dagger(\vec k)(i\omega_k)e^{-ikx}\right]_{k^0=\omega_k} \tag{1524}$$

$$= \frac{i}{2(2\pi)^{d-1}}\int d^{d-1}\vec k\left[-\hat a(\vec k)e^{ikx} + \hat a^\dagger(\vec k)e^{-ikx}\right]_{k^0=\omega_k}. \tag{1525}$$

Now calculating the three commutation relations

- $[\hat\phi(t,\vec x), \hat\phi(t,\vec y)]$

$$= \frac{1}{(2\pi)^{2(d-1)}}\int\frac{d^{d-1}\vec k d^{d-1}\vec q}{4\omega_k\omega_q}\Big((\hat a(\vec k)e^{ikx} + \hat a^\dagger(\vec k)e^{-ikx})(\hat a(\vec q)e^{iqy} + \hat a^\dagger(\vec q)e^{-iqy}) - \tag{1526}$$

$$(\hat a(\vec q)e^{iqy} + \hat a^\dagger(\vec q)e^{-iqy})(\hat a(\vec k)e^{ikx} + \hat a^\dagger(\vec k)e^{-ikx})\Big) \tag{1527}$$

the bracket can then be simplified

$$(\hat a(\vec k)e^{ikx} + \hat a^\dagger(\vec k)e^{-ikx})(\hat a(\vec q)e^{iqy} + \hat a^\dagger(\vec q)e^{-iqy}) - (\hat a(\vec q)e^{iqy} + \hat a^\dagger(\vec q)e^{-iqy})(\hat a(\vec k)e^{ikx} + \hat a^\dagger(\vec k)e^{-ikx}) \tag{1528}$$

$$= [\hat a(\vec k), \hat a(\vec q)]e^{i(kx+qy)} + [\hat a(\vec k), \hat a^\dagger(\vec q)]e^{i(kx-qy)} + [\hat a^\dagger(\vec k), \hat a(\vec q)]e^{i(-kx+qy)} + [\hat a^\dagger(\vec k), \hat a^\dagger(\vec q)]e^{i(-kx-qy)} \tag{1529}$$

$$= [\hat a(\vec k), \hat a^\dagger(\vec q)]e^{i(kx-qy)} - [\hat a(\vec q), \hat a^\dagger(\vec k)]e^{i(-kx+qy)} \tag{1530}$$

$$= 2\omega_k(2\pi)^{d-1}\left(\delta^{d-1}(\vec k-\vec q)e^{i(kx-qy)} - \delta^{d-1}(\vec q-\vec k)e^{i(-kx+qy)}\right) \tag{1531}$$

where we used the given commutation relations for $\hat{a}(\vec{k})$.

$$[\hat{\phi}(t,\vec{x}), \hat{\phi}(t,\vec{y})] = \frac{1}{(2\pi)^{2(d-1)}} \int \frac{d^{d-1}\vec{k}d^{d-1}\vec{q}}{4\omega_k\omega_q} 2\omega_k(2\pi)^{d-1} \left(\delta^{d-1}(\vec{k}-\vec{q})e^{i(kx-qy)} - \delta^{d-1}(\vec{q}-\vec{k})e^{i(-kx+qy)}\right) \tag{1532}$$

$$= \frac{1}{(2\pi)^{d-1}} \int \frac{d^{d-1}\vec{k}d^{d-1}\vec{q}}{2\omega_q} \left(\delta^{d-1}(\vec{k}-\vec{q})e^{i(kx-qy)} - \delta^{d-1}(\vec{q}-\vec{k})e^{i(-kx+qy)}\right) \tag{1533}$$

$$= \frac{1}{(2\pi)^{d-1}} \int \frac{d^{d-1}\vec{k}d^{d-1}\vec{q}}{2\omega_q} \Big(\delta^{d-1}(\vec{k}-\vec{q})e^{i(-\omega_k t+\vec{k}\vec{x}-[-\omega_q t+\vec{q}\vec{y}]))} \tag{1534}$$

$$-\delta^{d-1}(\vec{q}-\vec{k})e^{-i(-\omega_k t+\vec{k}\vec{x}-[-\omega_q t+\vec{q}\vec{y}]))}\Big) \tag{1535}$$

$$= \frac{1}{(2\pi)^{d-1}} \int \frac{d^{d-1}\vec{k}d^{d-1}\vec{q}}{2\omega_q} \Big(\delta^{d-1}(\vec{k}-\vec{q})e^{i(-[\omega_k-\omega_q]t+\vec{k}\vec{x}-\vec{q}\vec{y})} \tag{1536}$$

$$-\delta^{d-1}(\vec{q}-\vec{k})e^{-i(-[\omega_k-\omega_q]t+\vec{k}\vec{x}-\vec{q}\vec{y})}\Big) \tag{1537}$$

$$= \frac{1}{(2\pi)^{d-1}} \int \frac{d^{d-1}\vec{k}}{2\omega_k} \left(e^{i\vec{k}(\vec{x}-\vec{y})} - e^{-i\vec{k}(\vec{x}-\vec{y})}\right) \tag{1538}$$

$$= \frac{1}{2\omega_k} \left(\delta^{d-1}(\vec{y}-\vec{x}) - \delta^{d-1}(\vec{x}-\vec{y})\right) \tag{1539}$$

$$= 0 \tag{1540}$$

where we used $\delta(x) = \int dk e^{-2\pi ikx}$ or $\delta^d(x) = \int \frac{d^dk}{(2\pi)^d} e^{-ikx}$.

- $[\hat{\Pi}(t,\vec{x}), \hat{\Pi}(t,\vec{y})]$ Not done yet
- $[\hat{\phi}(t,\vec{x}), \hat{\Pi}(t,\vec{y})]$ Not done yet

Problem 1.3.2 - Lorentz invariant integration measure

We use the property of the δ-function $\delta(f(x)) = \sum_i \frac{\delta(x-a_i)}{]f'(a_i)|}$ where a_i are the zeros of $f(x)$ and $\omega_k = \sqrt{\vec{k}^2+m^2}$. With $\int d^dk$ being manifestly Lorentz invariant

$$dk'^\mu = \Lambda^\mu_\nu dk^\nu \quad \rightarrow \quad \frac{dk'^\mu}{dk^\nu} = \Lambda^\mu_\nu \quad \rightarrow \quad \int d^dk' = |\det(\Lambda^\mu_\nu)| \int d^dk = \int d^dk \tag{1541}$$

$\delta^d[k^2+m^2]$ being invariant and with $k^0 = \sqrt{\vec{k}^2+m^2}$ we see that k is inside the forward light cone and remains there under orthochrone transformation ($\Theta(k^0)$ is invariant for relevant k) we are convinced that the starting expression is Lorentz invariant (integration over the upper mass shell)

$$\int d^d\vec{k}\delta^d[k^2+m^2]\Theta(k^0) = \int d^{d-1}\vec{k}\int dk^0\delta^d[k^2+m^2]\Theta(k^0) \tag{1542}$$

$$= \int d^{d-1}\vec{k}\int dk^0\delta^d[-(k^0)^2+\vec{k}^2+m^2]\Theta(k^0) \tag{1543}$$

$$= \int d^{d-1}\vec{k}\int dk^0\delta^d[\omega_k^2-(k^0)^2]\Theta(k^0) \tag{1544}$$

$$= \int d^{d-1}\vec{k}\int dk^0\left(\frac{\delta(k^0-\omega_k)}{2\omega_k}+\frac{\delta(k^0+\omega_k)}{2\omega_k}\right)\Theta(k^0) \tag{1545}$$

$$= \int \frac{d^{d-1}\vec{k}}{2\omega_k}\int dk^0\delta(k^0-\omega_k) \tag{1546}$$

$$= \int \frac{d^{d-1}\vec{k}}{2\omega_k}. \tag{1547}$$

As we started with a Lorentz invariant expression the derived measure is also invariant.

Problem 1.3.3 - Retarded Green function

$$\Delta_{\mathrm{F}} = \int \frac{d^dk}{(2\pi)^d}\frac{e^{ik(x-y)}}{k^2+m^2-i\epsilon} \tag{1548}$$

$$G_{\mathrm{R}} = \int \frac{d^dk}{(2\pi)^d}\frac{e^{ik(x-y)}}{-(k^0+i\epsilon)^2+\vec{k}^2+m^2} \tag{1549}$$

For the poles of G_{R} we have

$$-(k^0+i\epsilon)^2+\vec{k}^2+m^2 = 0 \tag{1550}$$

$$k^0 = -i\epsilon \pm \sqrt{\vec{k}^2+m^2} \tag{1551}$$

$$= -i\epsilon \pm \omega_k \tag{1552}$$

while we the poles of Δ_{F} are given by

$$-(k^0)^2+\vec{k}^2+m^2-i\epsilon = 0 \tag{1553}$$

$$k^0 = \pm\sqrt{\vec{k}^2+m^2-i\epsilon} \tag{1554}$$

$$= \pm\sqrt{\omega_k^2-i\epsilon} \tag{1555}$$

Figure 1: Poles of G_{R} (circle) and Δ_{F} (triangle)

With $|\vec{k}\rangle = a^\dagger(\vec{k})|0\rangle$ and

$$\hat{\phi}(x) = \frac{1}{(2\pi)^{d-1}}\int\frac{d^{d-1}\vec{k}}{2\omega_k}\left[\hat{a}(\vec{k})e^{ikx}+\hat{a}^\dagger(\vec{k})e^{-ikx}\right]_{k^0=\omega_k} \tag{1556}$$

we obtain

$$\hat{\phi}(x)\hat{\phi}(y) \sim \left(\hat{a}(\vec{k})e^{ikx} + \hat{a}^\dagger(\vec{k})e^{-ikx}\right)\left(\hat{a}(\vec{q})e^{iqy} + \hat{a}^\dagger(\vec{q})e^{-iqy}\right) \tag{1557}$$

$$= \hat{a}(\vec{k})\hat{a}(\vec{q})e^{i(kx+qy)} + \hat{a}(\vec{k})\hat{a}^\dagger(\vec{q})e^{-i(-kx+qy)} + \hat{a}^\dagger(\vec{k})\hat{a}(\vec{q})e^{i(-kx+qy)} + \hat{a}^\dagger(\vec{k})\hat{a}^\dagger(\vec{q})e^{-i(kx+qy)} \tag{1558}$$

$$= \hat{a}(\vec{k})\hat{a}(\vec{q})e^{i(kx+qy)} + \hat{a}(\vec{k})\hat{a}^\dagger(\vec{q})e^{-i(-kx+qy)} + \hat{a}^\dagger(\vec{k})\hat{a}^\dagger(\vec{q})e^{-i(kx+qy)} \tag{1559}$$

$$+ \left(\hat{a}(\vec{q})\hat{a}^\dagger(\vec{k}) - 2\omega_k(2\pi)^{d-1}\delta^{d-1}(\vec{q}-\vec{k})\right)e^{i(-kx+qy)} \tag{1560}$$

and therefore

$$\langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle = \frac{1}{(2\pi)^{2(d-1)}}\int \frac{d^{d-1}\vec{k}}{2\omega_k}\frac{d^{d-1}\vec{q}}{2\omega_q}\langle 0|\hat{a}(\vec{k})\hat{a}(\vec{q})|0\rangle e^{i(kx+qy)} + \langle 0|\hat{a}(\vec{k})\hat{a}^\dagger(\vec{q})|0\rangle e^{-i(-kx+qy)} \tag{1561}$$

$$+ \langle 0|\hat{a}^\dagger(\vec{k})\hat{a}^\dagger(\vec{q})|0\rangle e^{-i(kx+qy)} + \left(\langle 0|\hat{a}(\vec{q})\hat{a}^\dagger(\vec{k})|0\rangle - 2\omega_k(2\pi)^{d-1}\delta^{d-1}(\vec{q}-\vec{k})\right)e^{i(-kx+qy)} \tag{1562}$$

$$= \frac{1}{(2\pi)^{2(d-1)}}\int \frac{d^{d-1}\vec{k}}{2\omega_k}\frac{d^{d-1}\vec{q}}{2\omega_q}\langle\vec{k}|\vec{q}\rangle e^{-i(-kx+qy)} + \left(\langle\vec{q}|\vec{k}\rangle - 2\omega_k(2\pi)^{d-1}\delta^{d-1}(\vec{q}-\vec{k})\right)e^{i(-kx+qy)} \tag{1563}$$

$$\tag{1564}$$

Not done yet

Problem 1.3.4 - Feynman rules of ϕ^4 theory

Not done yet

Problem 1.3.5 - Convergence of perturbative expansion

Not done yet

Problem 1.3.6

Not done yet

Problem 1.3.7

Not done yet

Problem 1.3.8

Not done yet

11 String Theory

11.1 Zwiebach - A First Course in String Theory

11.2 Becker, Becker, Schwarz - String Theory and M-Theroy

11.3 Polchinski - String Theory Volumes 1 and 2

Problem 1.1 - Non-relativistic action limits

(a) We start with (1.2.2) and use $dt = \gamma d\tau$ and $u^\mu = \gamma(c, \vec{v})$ as well as $v \ll c$

$$\begin{aligned} S_{\text{pp}} &= -mc \int d\tau \sqrt{-\dot{X}^\mu \dot{X}_\mu} && (1565)\\ &= -mc \int d\tau \sqrt{(c^2 - v^2)\gamma^2} && (1566)\\ &= -\int mc^2 \cdot dt \sqrt{1 - \frac{v^2}{c^2}} && (1567)\\ &\approx -\int dt \cdot mc^2 \left(1 - \frac{1}{2}\frac{v^2}{c^2}\right) && (1568)\\ &= -\int dt \left(mc^2 - \frac{1}{2}mv^2\right) && (1569) \end{aligned}$$

(b) We start with (1.2.9) and $X^\mu = X^\mu(\tau, \sigma)$

$$\begin{aligned} S_{\text{NG}} &= \int_M d\tau d\sigma \mathcal{L}_{\text{NG}} && (1570)\\ &= -\frac{1}{2\pi\alpha'} \int_M d\tau d\sigma \sqrt{-\det h_{ab}} && (1571)\\ &= -\frac{1}{2\pi\alpha'} \int_M d\tau d\sigma \sqrt{-\det \partial_a X^\mu \partial_b X_\mu} && (1572)\\ & && (1573) \end{aligned}$$

Not done yet

12 Astrophysics

12.1 Carroll, Ostlie - An Introduction to Modern Astrophysics

12.2 Weinberg - Lecture on Astrophysics

Problem 1 - Hydrostatics of spherical star

Gravitational force on a mass element must be balanced by the top and bottom pressure (buoyancy)

$$\begin{aligned} F_p^{\text{top}} - F_p^{\text{bottom}} &= F_g && (1574)\\ dA \cdot p\left(r + \frac{dr}{2}\right) - dA \cdot p\left(r - \frac{dr}{2}\right) &= -g(r)\rho(r) \cdot dA \cdot dr && (1575)\\ \frac{dp}{dr} &= -g(r)\rho(r) && (1576)\\ &= -G\frac{\mathcal{M}(r)}{r^2}\rho(r) && (1577) \end{aligned}$$

and therefore

$$\rho(r)\mathcal{M}(r) = -\frac{dp}{dr}\frac{r^2}{G} \tag{1578}$$

where

$$g(r) = G\frac{\mathcal{M}(r)}{r^2} = \frac{G}{r^2}\int_0^r 4\pi\rho(r')r'^2 dr'. \tag{1579}$$

The gravitational binding energy Ω is given by

$$d\Omega = -G\frac{m_{\text{shell}}\mathcal{M}}{r} \tag{1580}$$

$$\Omega = -G\int_0^R \frac{4\pi\rho(r)\mathcal{M}(r)}{r}r^2 dr \tag{1581}$$

$$= -4\pi G\int_0^R r\rho(r)\mathcal{M}(r)dr \tag{1582}$$

$$= 4\pi\int_0^R \frac{dp}{dr}r^3 dr \tag{1583}$$

$$= 4\pi p r^3|_0^R - 3\cdot 4\pi\int_0^R p(r)r^2 dr \tag{1584}$$

$$= 4\pi p_0 R^3 - 3\left(4\pi\int_0^R p(r)r^2 dr\right) \tag{1585}$$

$$= 4\pi p_0 R^3 - 3\int_{K_R} p(\vec{r})d^3r. \tag{1586}$$

Problem 2 - CNO cycle

$$\Gamma(ii) = \Gamma(iii) = \Gamma(iv) = \Gamma(v) = \Gamma(i) \tag{1587}$$

$$\Gamma(vi) = P\cdot\Gamma(i) \tag{1588}$$

$$\Gamma(vii) = \Gamma(viii) = \Gamma(ix) = \Gamma(x) = (1-P)\cdot\Gamma(i) \tag{1589}$$

Check result!

Problem 3

Not done yet

Problem 4

Not done yet

Problem 5 - Radial density expansion for a polytrope

For the polytrope equation

$$p = K\rho^\Gamma \tag{1590}$$

we obtain

$$\frac{dp}{d\rho} = K\Gamma\rho^{\Gamma-1} \tag{1591}$$

$$= \Gamma\frac{p}{\rho} \tag{1592}$$

With equations (1.1.4/5)

$$\frac{dp}{dr} = -\frac{G\mathcal{M}(r)\rho(r)}{r^2} \quad \rightarrow \quad \mathcal{M}(r) = -\frac{p' r^2}{G\rho} \tag{1593}$$

$$\frac{d\mathcal{M}(r)}{dr} = 4\pi r^2 \rho(r) \tag{1594}$$

we can obtain a second order ODE by differentiating the first one and substituting $\mathcal{M}'$

$$\mathcal{M}' = -\frac{1}{G}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{d}{dr}p\right) \tag{1595}$$

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{d}{dr}p\right) + G\mathcal{M}' = 0 \tag{1596}$$

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{d}{dr}p\right) + 4\pi G r^2 \rho = 0 \tag{1597}$$

now we can substitute the $p = K\rho^\Gamma$ and obtain

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{d}{dr}\rho^\Gamma\right) + \frac{4\pi G}{K} r^2 \rho = 0. \tag{1598}$$

The Taylor expansion

$$\rho(r) = \rho(0)\left[1 + ar^2 + br^4 + ...\right] \tag{1599}$$

$$\rho(r)^\Gamma = \rho(0)^\Gamma \left[1 + ar^2 + br^4 + ...\right]^\Gamma \tag{1600}$$

$$= \rho(0)^\Gamma \left[1 + a\Gamma r^2 + \left(b\Gamma + \frac{1}{2}a^2\Gamma(\Gamma-1)\right) r^4 + ...\right] \tag{1601}$$

$$\frac{1}{\rho} = \frac{1}{\rho(0)}\left[1 - ar^2 + (a^2 - b)r^4 + ...\right] \tag{1602}$$

can be substituted into the ODE

$$\rho(0)^{\Gamma-1}\frac{d}{dr}\left(r^2\left[1 - ar^2 + (a^2-b)r^4 + ...\right]\left[a\Gamma 2r + \left(b\Gamma + \frac{1}{2}a^2\Gamma(\Gamma-1)\right)4r^3 + ...\right]\right) \tag{1603}$$

$$+\frac{4\pi G}{K}\rho(0)\left[r^2 + ar^4 + br^6 + ...\right] = 0. \tag{1604}$$

and sort by powers of r

$$\rho(0)^{\Gamma-1}\frac{d}{dr}\left(2\Gamma a r^3 + \left[-2\Gamma a^2 + 4\left(b\Gamma + \frac{1}{2}a^2\Gamma(\Gamma-1)\right)\right]r^5 + ...\right) + \frac{4\pi G}{K}\rho(0)\left[r^2 + ar^4 + br^6 + ...\right] = 0. \tag{1605}$$

In second order of r we obtain

$$\rho(0)^{\Gamma-1}2\Gamma a3 + \frac{4\pi G}{K}\rho(0) = 0 \tag{1606}$$

which results in

$$a = -\frac{2\pi G}{3\Gamma K \rho(0)^{\Gamma-2}} \tag{1607}$$

Problem 6

Not done yet

Problem 7

Not done yet

Problem 8

Not done yet

Problem 9

Not done yet

Problem 10

Not done yet

Problem 11 - Modified Newtonian gravity

The modified Poisson equation is given by

$$\left(\triangle + \mathcal{R}^{-2}\right)\phi = 4\pi G\rho \tag{1608}$$

with the Greens function

$$\left(\triangle + \mathcal{R}^{-2}\right)G(\vec{r}) = -\delta^3(\vec{r}). \tag{1609}$$

The Fourier transform of the Greens function

$$G(\vec{k}) = \int d^3\vec{r}\, G(\vec{r})e^{-i\vec{k}\vec{r}} \tag{1610}$$

and the field equations are given by

$$\left[k^2 + \mathcal{R}^{-2}\right]G(\vec{k}) = -1 \tag{1611}$$

$$G(\vec{k}) = \frac{1}{k^2 + \mathcal{R}^{-2}} \tag{1612}$$

$$G(\vec{x}) = \frac{1}{(2\pi)^3}\int d^3\vec{k}\frac{e^{i\vec{k}\vec{r}}}{k^2 + \mathcal{R}^{-2}} \tag{1613}$$

$$= \frac{1}{(2\pi)^3}2\pi\int_0^\infty\int_0^\pi \frac{e^{ik_r\cdot r\cos\theta}}{k_r^2 + \mathcal{R}^{-2}}k_r^2\sin\theta\, d\theta dk_r \tag{1614}$$

$$= \frac{1}{(2\pi)^3}2\pi\int_0^\infty\left[-\frac{e^{ik_r r\cos\theta}}{ik_r r}\right]_0^\pi \frac{1}{k_r^2 + \mathcal{R}^{-2}}k_r^2 dk_r \tag{1615}$$

$$= \frac{1}{2\pi^2 r}\int_0^\infty \frac{k_r\sin(k_r r)}{k_r^2 + \mathcal{R}^{-2}}dk_r \tag{1616}$$

$$\tag{1617}$$

The integral can be can be calculated using the residual theorem

$$\int_0^\infty \frac{k_r\sin(k_r r)}{k_r^2 + \mathcal{R}^{-2}}dk_r = \frac{1}{2}\int_{-\infty}^\infty \frac{k_r\sin(k_r r)}{k_r^2 + \mathcal{R}^{-2}}dk_r \tag{1618}$$

$$= \frac{1}{2}\int_{-\infty}^\infty \frac{k_r\sin(k_r r)}{(k_r + i\mathcal{R}^{-1})(k_r - i\mathcal{R}^{-1})}dk_r \tag{1619}$$

$$= \frac{1}{2}\int_{-\infty}^\infty \frac{k_r\sin(k_r r)}{2k_r}\left(\frac{1}{k_r + i\mathcal{R}^{-1}} + \frac{1}{k_r - i\mathcal{R}^{-1}}\right)dk_r \tag{1620}$$

$$= \frac{1}{4}\int_{-\infty}^\infty \frac{\sin(k_r r)}{k_r + i\mathcal{R}^{-1}}dk_r + \frac{1}{4}\int_{-\infty}^\infty \frac{\sin(k_r r)}{k_r - i\mathcal{R}^{-1}}dk_r \tag{1621}$$

Not done yet

Problem 12

Not done yet

13 Cosmology

13.1 Boerner - The Early Universe - Facts and Fiction (4th edition)

1.1 Friedman equations

1. The Friedman equations in book contain a small typo ($\rho = \varrho$)

$$(A) \qquad \ddot{R} = -\frac{4\pi}{3}(\varrho + 3p)GR + \frac{1}{3}\Lambda R \tag{1622}$$

$$(B) \qquad \dot{R}^2 = \frac{8\pi}{3}G\varrho R^2 + \frac{1}{3}\Lambda R^2 - K \tag{1623}$$

$$(C) \qquad 0 = (\varrho R^3)^{\cdot} + p(R^3)^{\cdot} \tag{1624}$$

Calculating the time derivative of (B)

$$2\dot{R}\ddot{R} = \frac{8\pi}{3}G(\dot{\varrho}R^2 + 2\varrho R\dot{R}) + \frac{2}{3}\Lambda R\dot{R} \tag{1625}$$

$$\ddot{R} = \frac{R}{3}\left(4\pi G\dot{\varrho}\frac{R}{\dot{R}} + 8\pi G\varrho + \Lambda\right) \tag{1626}$$

and simplifying (A)

$$\ddot{R} = \frac{R}{3}\left(-4\pi G(\varrho + 3p) + \Lambda\right) \tag{1627}$$

Combining both yields

$$\dot{\varrho}\frac{R}{\dot{R}} + 2\varrho = -(\varrho + 3p) \tag{1628}$$

$$\dot{\varrho}R = -3(\varrho + p)\dot{R} \tag{1629}$$

which is (C). Rearranging the order of the steps gives the other two cases.

2. From (C) we have

$$\dot{\varrho} = -3(\varrho + p)\frac{\dot{R}}{R} \tag{1630}$$

$$= -3\varrho\left(1 + k\varrho^{\gamma-1}\right)\frac{\dot{R}}{R} \tag{1631}$$

which can be rearranged and integrated

$$\frac{\dot{R}}{R} = \frac{\dot{\varrho}}{-3\varrho\left(1 + k\varrho^{\gamma-1}\right)} \tag{1632}$$

$$\to -\frac{1}{3(1-\gamma)}\log(k + \varrho^{1-\gamma}) = \log R + c \tag{1633}$$

$$\to \log(k + \varrho^{1-\gamma}) = -3(1-\gamma)\log R + c' \tag{1634}$$

$$\to k + \varrho^{1-\gamma} = e^{-3(1-\gamma)\log R + c'} \tag{1635}$$

$$\to k + \varrho^{1-\gamma} = c'' R^{-3(1-\gamma)} \tag{1636}$$

$$\to \varrho = \left(c'' R^{3(\gamma-1)} - k\right)^{1/(1-\gamma)} \tag{1637}$$

with

$$c'' = \frac{k + \varrho_0^{1-\gamma}}{R_0^{3(\gamma-1)}} \tag{1638}$$

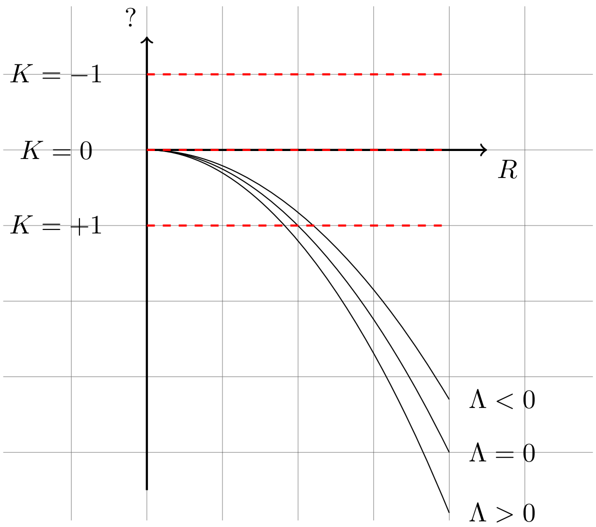
$$\rightarrow \quad \varrho = \left([k + \varrho_0^{1-\gamma}] \frac{R}{R_0}^{3(\gamma-1)} - k \right)^{1/(1-\gamma)} \tag{1639}$$

$$\rightarrow \quad \varrho = \left(k \left[\frac{R}{R_0}^{3(\gamma-1)} - 1 \right] + \left[\frac{R^3}{\varrho_0 R_0^3} \right]^{\gamma-1} \right)^{1/(1-\gamma)} \tag{1640}$$

We obtain from (B)

$$\dot{R}^2 - \left(\frac{8\pi}{3} G\varrho + \frac{1}{3}\Lambda \right) R^2 = -K \tag{1641}$$

which we can interpret as motion of a particle in a changing $-R^2$ potential.



13.2 Dodelson, Schmidt - Cosmology (2nd edition)

1.2

We start with

$$\rho_{\text{cr}} = \frac{3H_0^2}{8\pi G} \tag{1642}$$

$$H(t) = \frac{1}{a}\frac{da}{dt} \tag{1643}$$

$$H(t)^2 = \frac{8\pi G}{3} \left[\varrho(t) + \frac{\Lambda}{3} - \frac{k}{a^2} \right] \tag{1644}$$

$$= \frac{8\pi G}{3} \left[\rho(t) + \frac{\rho_{\text{cr}} - \rho(t_0)}{a^2} \right] \tag{1645}$$

$$= \frac{8\pi G}{3} \left[\Omega_m \left(\frac{a_0}{a} \right)^3 \rho_{\text{cr}} + \Omega_\Lambda \rho_{\text{cr}} + \frac{\rho_{\text{cr}} - \rho(t_0)}{a^2} \right] \tag{1646}$$

$$= H_0^2 \left[\Omega_m \left(\frac{a_0}{a} \right)^3 + \Omega_\Lambda + \frac{\rho_{\text{cr}} - \rho(t_0)}{\rho_{\text{cr}} a^2} \right] \tag{1647}$$

$$\tag{1648}$$

and assume $\rho_{\rm cr} = \rho(t_0)$ (same as Euclidean $k = 0$?!?) and $\Omega_\Lambda + \Omega_m = 1$ and $a_0 = 1$

$$dt = \frac{da}{a}\frac{1}{H(t)} \tag{1649}$$

$$= \frac{da}{a}\frac{1}{H_0\sqrt{\Omega_m\left(\frac{a_0}{a}\right)^3 + \Omega_\Lambda}} \tag{1650}$$

$$= \frac{1}{H_0}\frac{da}{a}\left[\frac{1-\Omega_\Lambda}{a^3} + \Omega_\Lambda\right]^{-1/2} \tag{1651}$$

(a) Now with $\Omega_\Lambda = 0$

$$dt = \frac{1}{H_0}\frac{da}{a}a^{3/2} = \frac{1}{H_0}da\ a^{1/2} \tag{1652}$$

$$\to t - t_i = \frac{2}{3H_0}(a^{3/2} - a_i^{3/2}) \tag{1653}$$

$$\to a(t) = \left(\frac{3H_0}{2}(t-t_i) + a_i^{3/2}\right)^{2/3} \tag{1654}$$

with $a(t=0) = 0$

$$a(t) = \left(\frac{3H_0}{2}t\right)^{2/3} \tag{1655}$$

$$\to T = \frac{2}{3H_0} \tag{1656}$$

(b) ...

1.3 Lyman-α splitting in hydrogen isotopes

The energy eigenvalues are

$$E_n = -\frac{1}{2}\mu c^2\frac{\alpha^2}{n^2} \tag{1657}$$

$$= -\frac{1}{2}\frac{m_e M_{\rm nuc}}{m_e + M_{\rm nuc}}c^2\frac{\alpha^2}{n^2} \tag{1658}$$

then

$$\Delta E_{2\to1} = -\frac{1}{2}\frac{m_e M_{\rm nuc}}{m_e + M_{\rm nuc}}c^2\alpha^2\left(\frac{1}{2^2} - \frac{1}{1^2}\right) \tag{1659}$$

$$= \frac{3}{8}\frac{m_e M_{\rm nuc}}{m_e + M_{\rm nuc}}c^2\alpha^2 \tag{1660}$$

$$= \frac{3}{8}\frac{m_e M_{\rm nuc}}{M_{\rm nuc}(1 + m_e/M_{\rm nuc})}c^2\alpha^2 \tag{1661}$$

$$= \frac{3}{8}\frac{m_e}{1 + m_e/M_{\rm nuc}}c^2\alpha^2 \tag{1662}$$

and

$$\Delta E^{\rm D}_{2\to1} = \frac{3}{8}\frac{m_e}{1 + m_e/2m_p}c^2\alpha^2 \tag{1663}$$

$$\Delta E^{\rm H}_{2\to1} = \frac{3}{8}\frac{m_e}{1 + m_e/m_p}c^2\alpha^2 \tag{1664}$$

$$\to \Delta E^{\rm D}_{2\to1} = \Delta E^{\rm H}_{2\to1}\frac{1 + m_e/m_p}{1 + m_e/2m_p} \tag{1665}$$

and with $E = hc/\lambda$

$$\lambda^{\mathrm{D}}_{2\to1} = \frac{hc}{\Delta E^{\mathrm{D}}_{2\to1}} \tag{1666}$$

$$= \frac{hc}{\Delta E^{\mathrm{H}}_{2\to1}} \frac{1 + m_e/2m_p}{1 + m_e/m_p} \tag{1667}$$

$$= \lambda^{\mathrm{H}}_{2\to1} \frac{1 + m_e/2m_p}{1 + m_e/m_p} \tag{1668}$$

$$= \lambda^{\mathrm{H}}_{2\to1} \left(1 + \frac{m_e}{2m_p}\right)\left(1 - \frac{m_e}{m_p}\right) \tag{1669}$$

$$\simeq \lambda^{\mathrm{H}}_{2\to1} \left(1 - \frac{1}{2}\frac{m_e}{m_p}\right) \tag{1670}$$

$$= 1215.67\text{Å} \tag{1671}$$

furthermore

$$c\frac{\Delta\lambda}{\lambda} = c\frac{\lambda^{\mathrm{D}}_{2\to1} - \lambda^{\mathrm{H}}_{2\to1}}{\lambda^{\mathrm{H}}_{2\to1}} \tag{1672}$$

$$= \left(1 - \frac{1}{2}\frac{m_e}{m_p}\right) \tag{1673}$$

$$= 0.999727c \tag{1674}$$

1.4 Planck law for CMB

Insider hint $1\text{MJy} = 10^6\text{Jansky} = 10^6 \cdot 10^{-26}\text{J} \cdot \text{s}^{-1} \cdot \text{Hz}^{-1} \cdot \text{m}^{-2}$. We start with $c = \lambda\nu = 2\pi\nu/k$

$$I_\nu = \frac{4\pi\hbar\nu^3}{c^2}\frac{1}{e^{2\pi\hbar\nu/k_{\mathrm{B}}T} - 1} \tag{1675}$$

which has the unit energy per area (per frequency per time are cancelling)

$$\frac{\text{Js} \cdot \text{s}^{-3}}{\text{m}^2/\text{s}^2} = \text{J} \cdot \text{m}^{-2} \tag{1676}$$

then

$$\frac{I_\nu d\nu}{d\Omega} \tag{1677}$$

14 Quantum Mechanics

14.1 Feynman, Hibbs - Quantum mechanics and path integrals 2ed

2.1

With $\dot{x} = 0$ and $\dot{x}$ =const we see

$$S = \int_{t_a}^{t_b} L dt \tag{1678}$$

$$= \frac{m}{2}\int_{t_a}^{t_b} \dot{x}^2 dt \tag{1679}$$

$$= \frac{m}{2}\left[\dot{x}x\big|_{t_a}^{t_b} - \int_{t_a}^{t_b} x\ddot{x} dt\right] \tag{1680}$$

$$= \frac{m}{2}\frac{x_b - x_a}{t_b - t_b}(x_b - x_a) \tag{1681}$$

$$= \frac{m}{2}\frac{(x_b - x_a)^2}{t_b - t_b} \tag{1682}$$

2.1

With the solution of the equation of motion

$$\ddot{x} + \omega^2 x = 0 \quad \rightarrow \quad x = x_0 \sin(\omega t + \varphi_0) = (x_0 \cos\varphi_0) \sin\omega t + (x_0 \sin\varphi_0) \cos\omega t \tag{1683}$$

$$\rightarrow \quad \dot{x} = (x_0 \omega \cos\varphi_0) \cos\omega t - (x_0 \omega \sin\varphi_0) \sin\omega t \tag{1684}$$

then with (x_a, x_b, t_a, t_b) we can solve for x_0 and φ_0

$$x_0 \cos\varphi_0 = \frac{x_a \cos\omega t_b - x_b \cos\omega t_a}{\cos\omega t_b \sin\omega t_a - \cos\omega t_a \sin\omega t_b} \tag{1685}$$

$$= \frac{x_a \cos\omega t_b - x_b \cos\omega t_a}{\sin\omega(t_a - t_b)} \tag{1686}$$

$$x_0 \sin\varphi_0 = -\frac{x_a \frac{\sin\omega t_b}{\sin\omega t_a} - x_b \tan\omega t_a}{-\sin\omega t_b + \cos\omega t_b \tan\omega t_a} \tag{1687}$$

$$= \frac{x_b \sin\omega t_a - x_a \sin\omega t_b}{\sin\omega(t_a - t_b)} \tag{1688}$$

and therefore

$$v_a = \frac{x_a \cos\omega t_b - x_b \cos\omega t_a}{\sin\omega(t_a - t_b)} \sin\omega t_a + \frac{x_b \sin\omega t_a - x_a \sin\omega t_b}{\sin\omega(t_a - t_b)} \sin\omega t_a \tag{1689}$$

$$= -\frac{1}{\sin\omega T} \left[(x_a \cos\omega t_b - x_b \cos\omega t_a) \sin\omega t_a + (x_b \sin\omega t_a - x_a \sin\omega t_b) \sin\omega t_a\right] \tag{1690}$$

$$= -\frac{1}{\sin\omega T} \left[x_a(\cos\omega t_b \sin\omega t_a - \sin\omega t_a \sin\omega t_b) + x_b(\sin^2\omega t_a - \cos\omega t_a \sin\omega t_a)\right] \tag{1691}$$

$$v_b = \frac{x_a \cos\omega t_b - x_b \cos\omega t_a}{\sin\omega(t_a - t_b)} \sin\omega t_b + \frac{x_b \sin\omega t_a - x_a \sin\omega t_b}{\sin\omega(t_a - t_b)} \sin\omega t_b \tag{1692}$$

$$= -\frac{1}{\sin\omega T} \left[x_a(\cos\omega t_b \sin\omega t_b - \sin^2\omega t_b) + x_b(\sin\omega t_a \sin\omega t_b - \cos\omega t_a \sin\omega t_b)\right] \tag{1693}$$

Now we can write

$$S = \int_{t_a}^{t_b} L dt \tag{1694}$$

$$= \frac{m}{2} \int_{t_a}^{t_b} (\dot{x}^2 - \omega^2 x^2) dt \tag{1695}$$

$$= \frac{m}{2} x_0^2 \omega^2 \int_{t_a}^{t_b} dt \left(\cos^2(\omega t + \varphi) - \sin^2(\omega t + \varphi)\right) \tag{1696}$$

$$= \frac{m}{2} x_0^2 \omega^2 \int_{t_a}^{t_b} dt \cos(2[\omega t + \varphi]) \tag{1697}$$

$$= \frac{m}{4} x_0^2 \omega \sin(2[\omega t + \varphi])|_{t_a}^{t_b} \tag{1698}$$

$$= \frac{m}{2} x_0^2 \omega \sin(\omega t + \varphi) \cos(\omega t + \varphi)|_{t_a}^{t_b} \tag{1699}$$

$$= \frac{m}{2} x \dot{x}|_{t_a}^{t_b} \tag{1700}$$

$$= \frac{m}{2} (x_b v_b - x_a v_a) \tag{1701}$$

$$= \frac{m\omega}{2 \sin\omega T} \left[(x_a^2 + x_b^2) \cos\omega T - 2 x_a x_b\right] \tag{1702}$$

14.2 Straumann - Quantenmechanik 2ed

2.1 - Spectral oscillator density

The vanishing electrical field in the surface requires for each standing wave

$$k_i = \frac{\pi}{L} n_i. \tag{1703}$$

and

$$k^2 = k_x^2 + k_y^2 + k_z^2 \tag{1704}$$

$$\Delta V = \frac{\pi^3}{L^3}. \tag{1705}$$

With $k = 2\pi/\lambda = \omega/c$ we have $dk = \frac{d\omega}{c}$ and the volume of a sphere in k-space is given by

$$V(k) = \frac{4}{3}\pi k^3 \tag{1706}$$

$$dV = 4\pi k^2 dk = 4\pi \frac{\omega^2}{c^2}\frac{d\omega}{c} = 4\pi (2\pi)^3 \frac{\nu^2}{c^3} d\nu \tag{1707}$$

The number of oscillator are then given by the number of points in the positive quadrant (all k_i positive) time two (polarization)

$$dN(\nu) = 2\frac{V(\nu)/8}{\Delta V} = L^3 \frac{8\pi}{c^3}\nu^2 d\nu \tag{1708}$$

2.2 - Energy variance of the harmonic oscillator

First we obtain an expression for T

$$E = \frac{h\nu}{e^{h\nu/kT} - 1} \quad \rightarrow \quad \frac{h\nu}{kT} = \ln\left(\frac{h\nu}{E} + 1\right) \tag{1709}$$

which we can use in

$$\frac{dS}{dE} = \frac{1}{T} = \frac{k}{h\nu}\ln\left(\frac{h\nu}{E} + 1\right) \tag{1710}$$

and take one more derivative

$$\frac{d^2 S}{dE^2} = -\frac{k}{h\nu}\frac{\frac{h\nu}{E^2}}{\frac{h\nu}{E} + 1} \tag{1711}$$

$$= -k\frac{1}{h\nu E + E^2}. \tag{1712}$$

Now we see

$$\langle(\Delta E)^2\rangle = E^2 + Eh\nu. \tag{1713}$$

3.6 - 1D molecular potential

With the given coordinate transformation we get for the single terms

$$e^{-\alpha x} = \frac{\alpha\hbar\xi}{2\sqrt{2mA}} \tag{1714}$$

$$e^{-2\alpha x} = \frac{(\alpha\hbar\xi)^2}{8mA} \tag{1715}$$

$$\frac{\partial}{\partial x} = \frac{\partial\xi}{\partial x}\frac{\partial}{\partial\xi} \tag{1716}$$

$$= -\alpha\xi\frac{\partial}{\partial\xi} \tag{1717}$$

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2\xi}{\partial x^2}\frac{\partial}{\partial\xi} + \left(\frac{\partial\xi}{\partial x}\right)^2\frac{\partial^2}{\partial\xi^2} \tag{1718}$$

$$= \alpha^2\xi\frac{\partial}{\partial\xi} + (\alpha\xi)^2\frac{\partial^2}{\partial\xi^2} \tag{1719}$$

and combined

$$-\frac{\hbar^2}{2m}\partial_{xx}\psi + A(e^{-2\alpha x} - 2e^{-\alpha x})\psi = E\psi \tag{1720}$$

$$-\frac{\hbar^2}{2m}\left(\alpha^2\xi\frac{\partial}{\partial\xi} + (\alpha\xi)^2\frac{\partial^2}{\partial\xi^2}\right)\psi + A\left(\frac{(\alpha\hbar\xi)^2}{8mA} - 2\frac{\alpha\hbar\xi}{2\sqrt{2mA}}\right)\psi = E\psi \tag{1721}$$

$$\left(\alpha^2\xi\frac{\partial}{\partial\xi} + (\alpha\xi)^2\frac{\partial^2}{\partial\xi^2}\right)\psi - \frac{2mA}{\hbar^2}\left(\frac{(\alpha\hbar\xi)^2}{8mA} - 2\frac{\alpha\hbar\xi}{2\sqrt{2mA}}\right)\psi = -\frac{2mE}{\hbar^2}\psi \tag{1722}$$

$$\left(\frac{1}{\xi}\frac{\partial}{\partial\xi} + \frac{\partial^2}{\partial\xi^2}\right)\psi - \frac{2mA}{\alpha^2\xi^2\hbar^2}\left(\frac{(\alpha\hbar\xi)^2}{8mA} - 2\frac{\alpha\hbar\xi}{2\sqrt{2mA}}\right)\psi = -\frac{2mE}{\hbar^2\alpha^2\xi^2}\psi \tag{1723}$$

$$\left(\frac{1}{\xi}\frac{\partial}{\partial\xi} + \frac{\partial^2}{\partial\xi^2}\right)\psi + \left(-\frac{1}{4} + \frac{\sqrt{2mA}}{\alpha\hbar\xi}\right)\psi = -\frac{2mE}{\hbar^2\alpha^2\xi^2}\psi \tag{1724}$$

$$\left(\frac{1}{\xi}\frac{\partial}{\partial\xi} + \frac{\partial^2}{\partial\xi^2}\right)\psi + \left(-\frac{1}{4} + \frac{n+s+\frac{1}{2}}{\xi}\right)\psi = \frac{s^2}{\xi^2}\psi \tag{1725}$$

$$\left(\frac{\partial^2}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial}{\partial\xi}\right)\psi + \left(-\frac{1}{4} + \frac{n+s+\frac{1}{2}}{\xi} - \frac{s^2}{\xi^2}\right)\psi = 0. \tag{1726}$$

The units of ξ is $\sqrt{\text{kg}\cdot\text{J}}/\text{m}^{-1}\text{Js} = 1$ so ξ in dimensionless.

1. Case $\xi \gg 1$ ($x \to -\infty$) Dropping all $1/\xi$ terms

$$\psi'' - \frac{1}{4}\psi = 0 \quad \to \quad \psi = c_1 e^{\xi/2} + c_2 e^{-\xi/2} \tag{1727}$$

2. Case $0 < \xi \ll 1$ ($x \to +\infty$) Ansatz $\psi \sim \xi^m$

$$m(m-1)\xi^{m-2} + m\xi^{m-2} - \frac{1}{4}\xi^m + \left(n+s+\frac{1}{2}\right)\xi^{m-1} - s^2\xi^{m-2} = 0 \tag{1728}$$

$$\left[(m^2-s^2) - \frac{1}{4}\xi^2 + \left(n+s+\frac{1}{2}\right)\xi\right]\xi^{m-2} = 0 \tag{1729}$$

which for small ξ becomes

$$(m^2-s^2)\xi^{m-2} = 0 \quad \to \quad \psi = \xi^{\pm s} \tag{1730}$$

With the two asymptotics we can make a physically sensible ansatz for a full solutions $\psi = \xi^s e^{-\xi/2}u(\xi)$ which leads to

$$\xi u'' + (2s+1-\xi)u' + nu = 0 \tag{1731}$$

To solve this equation we use the Sommerfeld polynomial method

$$u = \sum_k a_k \xi^k \quad \rightarrow \quad \sum_k k(k-1)a_k\xi^{k-1} + (2s+1)ka_k\xi^{k-1} - ka_k\xi^k + na_k\xi^k = 0 \tag{1732}$$

$$\sum_k (k+1)ka_{k+1}\xi^k + (2s+1)(k+1)a_{k+1}\xi^k - ka_k\xi^k + na_k\xi^k = 0 \tag{1733}$$

$$a_{k+1} = \frac{k-n}{(k+1)(2s+1+k)}a_k. \tag{1734}$$

The requirement for the series to cut off (making u a finite order polynomial) is $n_k = k$. The energies of the bound states are therefore

$$E_k = -\frac{\alpha^2\hbar^2}{2m}s_k^2 \tag{1735}$$

$$= -\frac{\alpha^2\hbar^2}{2m}\left[\frac{\sqrt{2mA}}{\alpha\hbar} - (k+1/2)\right]^2 \tag{1736}$$

$$= -A\left[1 - \frac{\alpha\hbar}{\sqrt{2mA}}(k+1/2)\right]^2 \tag{1737}$$

where the only valid k are the ones where E_k is in $[-A, 0]$.

14.3 Schwinger - Quantum Mechanics Symbolism of Atomic Measurements

2.1

Observe

$$\int_{-\infty}^{\infty} (\theta(x+a) + \theta(a-x))\, e^{ikx}dx = \int_{-a}^{a} e^{ikx}dx \tag{1738}$$

$$= \frac{1}{ik}\left(e^{ika} - e^{-ika}\right) \tag{1739}$$

$$= 2a\frac{\sin ka}{ka} \tag{1740}$$

$$\lim_{P\to\infty}\int_{-\infty}^{\infty}\frac{d\chi}{\pi}\frac{\sin\chi}{\chi}e^{ik\left(q'+\frac{\chi}{P}\right)} = \frac{1}{\pi}e^{ikq'}\lim_{P\to\infty}\int_{-\infty}^{\infty}d\chi\frac{\sin\chi}{\chi}e^{i\frac{k}{P}\chi} \tag{1741}$$

14.4 Weinberg - Quantum Mechanics 2nd edition

1.1

- The solution of for a free particle in the interval $-a < x < a$ is given by

$$\left[-\frac{\hbar^2}{2M}\frac{d^2}{dx^2} - E\right]\phi = 0 \tag{1742}$$

$$\left[\frac{d^2}{dx^2} + \frac{2ME}{\hbar^2}\right]\phi = 0 \tag{1743}$$

$$\rightarrow \phi = A\sin\left(\frac{\sqrt{2ME}}{\hbar}x\right) + B\cos\left(\frac{\sqrt{2ME}}{\hbar}x\right) \tag{1744}$$

with the two boundary conditions

$$A\sin\left(\frac{\sqrt{2ME}}{\hbar}(-a)\right)+B\cos\left(\frac{\sqrt{2ME}}{\hbar}(-a)\right)=0 \tag{1745}$$

$$A\sin\left(\frac{\sqrt{2ME}}{\hbar}a\right)+B\cos\left(\frac{\sqrt{2ME}}{\hbar}a\right)=0. \tag{1746}$$

The possible energy eigenvalues are therefore

$$A=0,\quad \frac{\sqrt{2ME_{2n+1}}}{\hbar}a=(2n+1)\frac{\pi}{2}\quad\rightarrow\quad E_{2n+1}=\frac{\pi^2\hbar^2}{8Ma^2}(2n+1)^2 \tag{1747}$$

$$\rightarrow\quad \phi=\frac{1}{\sqrt{a}}\cos\left(x\frac{\pi}{2a}(2n+1)\right) \tag{1748}$$

$$B=0,\quad \frac{\sqrt{2ME_{2n}}}{\hbar}a=2n\frac{\pi}{2}\quad\rightarrow\quad E_{2n}=\frac{\pi^2\hbar^2}{8Ma^2}(2n)^2 \tag{1749}$$

$$\rightarrow\quad \phi=\frac{1}{\sqrt{a}}\sin\left(x\frac{\pi}{2a}(2n)\right) \tag{1750}$$

where we calculated the normalization via

$$\int_{-a}^{a}\sin^2(kx)dx=\int_{-a}^{a}(1-\cos^2(kx))dx \tag{1751}$$

$$=2a-\int_{-a}^{a}\cos^2(kx)dx\quad\rightarrow\int_{-a}^{a}\sin^2(kx)dx=a. \tag{1752}$$

- Lets first calculate the normalization

$$\int_{-a}^{a}(a^2-x^2)^2dx = a^4x-2a^2\frac{x^3}{3}+\frac{x^5}{5}\Big|_{-a}^{a} \tag{1753}$$

$$=a^4(2a)-\frac{2}{3}a^2(16a^3)+\frac{1}{5}(64a^5) \tag{1754}$$

$$=\left(2-\frac{4}{3}+\frac{2}{5}\right)a^5=\frac{16}{15}a^5 \tag{1755}$$

and then obtain

$$\int_{-a}^{a}\frac{1}{\sqrt{\frac{16a^5}{15}}}\left(a^2-x^2\right)\frac{1}{\sqrt{a}}\cos\left(\frac{\pi x}{2a}\right)dx=\frac{8\sqrt{15}}{\pi^3} \tag{1756}$$

1.2

- We can write the Hamiltonian as

$$H=\frac{\vec{P}^2}{2M}+\frac{M\omega_0^2}{2}\vec{X}^2 \tag{1757}$$

$$=\sum_{k=1}^{3}\frac{p_k^2}{2M}+\frac{M\omega_0^2}{2}x_k^2 \tag{1758}$$

the energy is therefore given by

$$E_{n_1,n_2,n_3} = \hbar\omega_0 \left(n_1 + n_2 + n_3 + \frac{3}{2} \right) \quad (1759)$$

$$N_{n=n_1+n_2+n_3} = \sum_{k=0}^{n} (k+1) \quad (1760)$$

$$= \frac{n(n+1)}{2} + n + 1 \quad (1761)$$

$$= \frac{(n+1)(n+2)}{2} \quad (1762)$$

- With (1.4.5), (1.4.15) and $\omega_{01} = \omega_0$ we have

$$\vec{x}]_{01} = e^{i\omega_0 t} \sqrt{\frac{\hbar}{2M\omega_0}} \quad (1763)$$

$$A_{n=1}^{n=0} = \frac{4e^2\omega_0^3}{3c^3\hbar} |[\vec{x}]_{01}|^2 \quad (1764)$$

$$= \frac{2e^2\omega_0^2}{3c^3 M} \quad (1765)$$

where with (1.4.15).

14.5 Sakurai, Napolitano - Modern Quantum Mechanics 3rd ed

5.1 Harmonic oscillator with linear perturbation

The Hamiltonians are given by

$$\hat{H}_0 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2} m\omega_o^2 x^2 \quad (1766)$$

$$\hat{H}_1 = bx \quad (1767)$$

We remember

$$\phi_0(x) = \left(\frac{m\omega_0}{\pi\hbar} \right)^{1/4} e^{-m\omega_0 x^2/2\hbar} \quad (1768)$$

$$E_0 = \frac{1}{2}\hbar\omega_0 \quad (1769)$$

$$\phi_n(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega_0}{\pi\hbar} \right)^{1/4} e^{-m\omega_0 x^2/2\hbar} H_n \left(\sqrt{\frac{m\omega_0}{\hbar}} x \right) \quad (1770)$$

$$E_n = \hbar\omega_0 \left(n + \frac{1}{2} \right) \quad (1771)$$

1. Time independent perturbation theory gives

$$\Delta E_n^{(1)} = \langle n^{(0)} | \hat{H}_1 | n^{(0)} \rangle \quad (1772)$$

$$\Delta E_0^{(1)} = \langle 0^{(0)} | \hat{H}_1 | 0^{(0)} \rangle = 0 \quad (1773)$$

The first order energy shift vanishes because of the wave function is even and H_1 is odd. For the first order perturbation of the wave function we observe

$$H_1(x) = 2xH_0(x) \quad \rightarrow \quad \hat{H}_1|0^{(0)}\rangle = \frac{b}{2}\sqrt{2}\sqrt{\frac{\hbar}{m\omega_0}}|1^{(0)}\rangle \quad (1774)$$

$$\langle m^{(0)} | n^{(0)} \rangle = \delta_{nm} \quad (1775)$$

Now we can calculate

$$|n^{(1)}\rangle = \sum_{k\neq n} \frac{\langle k^{(0)}|\hat{H}_1|n^{(0)}\rangle}{E_n^{(0)} - E_k^{(0)}}|k^{(0)}\rangle \tag{1776}$$

$$|0^{(1)}\rangle = \frac{\langle 0^{(0)}|\hat{H}_1|1^{(0)}\rangle}{E_0^{(0)} - E_1^{(0)}}|1^{(0)}\rangle \tag{1777}$$

$$= -\frac{1}{\hbar\omega_0} b\sqrt{\frac{\hbar}{2m\omega_0}}|1^{(0)}\rangle \tag{1778}$$

$$= -b\sqrt{\frac{1}{2m\hbar\omega_0^3}}|1^{(0)}\rangle \tag{1779}$$

Second order enegy perturbation

$$\Delta E_n^{(2)} = \langle n^{(0)}|\hat{H}_1|n^{(1)}\rangle = \sum_{k\neq n} \frac{|\langle k^{(0)}|\hat{H}_1|n^{(0)}\rangle|^2}{E_n^{(0)} - E_k^{(0)}} \tag{1780}$$

$$\Delta E_0^{(2)} = \langle 0^{(0)}|\hat{H}_1|0^{(1)}\rangle \tag{1781}$$

$$= b\sqrt{\frac{\hbar}{2m\omega_0}}\langle 1^{(0)}|0^{(1)}\rangle \tag{1782}$$

$$= b\sqrt{\frac{\hbar}{2m\omega_0}}\langle 1^{(0)}| \left(-b\sqrt{\frac{1}{2m\hbar\omega_0^3}}\right)|1^{(0)}\rangle \tag{1783}$$

$$= -b^2\frac{1}{2m\omega_0^2} \tag{1784}$$

2. The linear perturbation does not change the shape of the potential - only shifts the minimum

$$V(x) = \frac{m\omega_0^2}{2}x^2 + bx = \frac{m\omega_0^2}{2}\left(x + \frac{b}{m\omega_0^2}\right)^2 - \frac{b^2}{2m\omega_0^2} \tag{1785}$$

$$\Delta E^{(\infty)} = -\frac{b^2}{2m\omega_0^2} \tag{1786}$$

So the second order gives the exact result - interesting to see if higher orders would all vanish or give oscillating contributions.

5.2 Potential well with linear slope

We will treat the slope as a perturbation with

$$\hat{H}_1 = \frac{V}{L}x \tag{1787}$$

Therefore the unperturbed wave functions are given by

$$\phi_n = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L} \qquad E_n = \frac{\pi^2\hbar^2}{2mL^2}n^2 \tag{1788}$$

Then

$$
\begin{aligned}
\Delta E_n^{(1)} &= \langle n^{(0)}|\hat{H}_1|n^{(0)}\rangle && (1789)\\
&= \frac{V}{L}\frac{2}{L}\int_0^L x\sin^2\frac{n\pi x}{L}dx && (1790)\\
&= \frac{2V}{L^2}\int_0^L x\sin^2\frac{n\pi x}{L}dx && (1791)\\
&= \frac{2V}{L^2}\int_0^L x\left(1-\cos^2\frac{n\pi x}{L}\right)dx && (1792)\\
&= \frac{2V}{L^2}\frac{L^2}{2}-\Delta E_n^{(1)} && (1793)
\end{aligned}
$$

meaning $\Delta E_n^{(1)} = V/2$.

5.3 Relativistic perturbation

We can approximate the kinetic energy by

$$
\begin{aligned}
E &= \sqrt{m^2c^4+p^2c^2} && (1794)\\
&\approx mc^2+\frac{p^2}{2m}-\frac{p^4}{8m^3c^2}+\frac{p^6}{16m^5c^4}+\cdots && (1795)\\
&= mc^2+\frac{mc^2}{2}\frac{p^2}{m^2c^2}-\frac{mc^2}{8}\frac{p^4}{m^4c^4}+\cdots && (1796)\\
&= mc^2\left(1+\frac{1}{2}\beta^2-\frac{1}{8}\beta^4+\cdots\right) && (1797)
\end{aligned}
$$

so

$$
\begin{aligned}
\hat{H}_0 &= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\frac{1}{2}m\omega^2x^2 && (1798)\\
\hat{H}_1 &= -\frac{1}{8m^3c^2}p^4 = -\frac{\hbar^4}{8m^3c^2}\frac{d^4}{dx^4} && (1799)
\end{aligned}
$$

and we remember

$$
\begin{aligned}
\phi_0(x) &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-m\omega x^2/2\hbar} && (1800)\\
E_0 &= \frac{1}{2}\hbar\omega_0 && (1801)
\end{aligned}
$$

then

$$
\begin{aligned}
\Delta E_0^{(1)} &= \langle 0^{(0)}|\hat{H}_1|0^{(0)}\rangle && (1802)\\
&= -\frac{\hbar^4}{8m^3c^2}\int_{-\infty}^{\infty}\phi_0(x)^*\frac{d^4}{d^4x}\phi_0(x)dx && (1803)\\
&= -\frac{3\hbar^2\omega^2}{32mc^2} && (1804)
\end{aligned}
$$

5.4 Diatomic atomic rotor

Hamiltonian of the problem is given by

$$
H = \frac{L^2}{2I} \quad\rightarrow\quad \hat{H} = -\frac{\hbar^2}{2I}\frac{d^2}{d\varphi^2} \tag{1805}
$$

with the unperturbed solutions

$$\phi_n^{(0)} = Ce^{in\phi} \qquad E_n^{(0)} = \frac{\hbar^2 n^2}{2I} \tag{1806}$$

where only E_0 is non-degenerate (all other are double degenerated). For the perturbation we use the Hamiltonian

$$\hat{H}_1 = Ed\cos\varphi \tag{1807}$$

Hmmm....

5.6 Two dimensional potential well

As the problem separates

$$\left(\hat{H}_x + \hat{H}_y\right)\phi_x\phi_y = (E_x + E_y)\phi_x\phi_y \tag{1808}$$

$$\phi_y\hat{H}_x\phi_x + \phi_x\hat{H}_y\phi_y = (E_x + E_y)\phi_x\phi_y \tag{1809}$$

$$\frac{\hat{H}_x\phi_x}{\phi_x} + \frac{\hat{H}_y\phi_y}{\phi_y} = (E_x + E_y) \tag{1810}$$

the wave function can be written as a product of the 1-dimensional wave functions

$$\phi_{n_x,n_y} = \sqrt{\frac{2}{L}}\sqrt{\frac{2}{L}}\sin\left(\frac{n_x\pi}{L}x\right)\sin\left(\frac{n_y\pi}{L}y\right) \tag{1811}$$

$$E_{n_x,n_y} = \frac{\pi^2\hbar^2}{2mL^2}(n_x^2 + n_y^2) \tag{1812}$$

So

$$\phi_{1,1} \quad\rightarrow\quad E_{1,1} = 2\frac{\pi^2\hbar^2}{2mL^2} \tag{1813}$$

$$\phi_{2,1},\phi_{1,2} \quad\rightarrow\quad E_{2,1} = 5\frac{\pi^2\hbar^2}{2mL^2} \tag{1814}$$

$$\phi_{2,2} \quad\rightarrow\quad E_{1,1} = 8\frac{\pi^2\hbar^2}{2mL^2} \tag{1815}$$

for the non-degenerated levels $E_{1,1}$ and $E_{2,2}$ we get

$$\Delta E_{1,1}^{(1)} = \langle 1,1^{(0)}|\hat{H}_1|1,1^{(0)}\rangle \tag{1816}$$

$$= \frac{1}{4}\lambda L^2 \tag{1817}$$

$$\Delta E_{2,2}^{(1)} = \langle 2,2^{(0)}|\hat{H}_1|2,2^{(0)}\rangle \tag{1818}$$

$$= \frac{1}{4}\lambda L^2 \tag{1819}$$

and for the degenerated levels $E_{1,2}/E_{2,1}$ we get

$$H = \begin{pmatrix} \langle 1,2^{(0)}|\hat{H}_1|1,2^{(0)}\rangle & \langle 1,2^{(0)}|\hat{H}_1|2,1^{(0)}\rangle \\ \langle 2,1^{(0)}|\hat{H}_1|1,2^{(0)}\rangle & \langle 2,1^{(0)}|\hat{H}_1|2,1^{(0)}\rangle \end{pmatrix} \tag{1820}$$

with

$$H_{aa} = \langle 1,2^{(0)}|\hat{H}_1|1,2^{(0)}\rangle = \frac{\lambda L^2}{4} \tag{1821}$$

$$H_{ab} = \langle 1,2^{(0)}|\hat{H}_1|2,1^{(0)}\rangle = \frac{256\lambda L^2}{81\pi^4} \tag{1822}$$

$$H_{bb} = \langle 2,1^{(0)}|\hat{H}_1|2,1^{(0)}\rangle = \frac{\lambda L^2}{4} \tag{1823}$$

and $\hat{H}_1 = \lambda xy$ Diagonalising the matrix H gives the perturbation

$$\Delta E^{(1)}_{12,21} = \frac{\lambda L^2}{4} - \frac{256\lambda L^2}{81\pi^4} \tag{1824}$$

$$\Delta E^{(1)}_{12,21} = \frac{\lambda L^2}{4} + \frac{256\lambda L^2}{81\pi^4} \tag{1825}$$

$$\tag{1826}$$

5.8 Quadratically perturbed harmonic oscillator

$$\hat{H}_1 = \epsilon \frac{1}{2} m\omega^2 x^2 \tag{1827}$$

$$H_0(x) = 1 \tag{1828}$$

$$H_2(x) = 4x^2 - 2 \quad \rightarrow \quad x^2 = \frac{H_2}{4} + \frac{1}{2} \tag{1829}$$

8.1 Natural units

1. Proton Mass

$$E_p = m_p c^2/e = 0.937\text{GeV} \tag{1830}$$

2. With $\Delta p \cdot \Delta x \geq \hbar/2$ and $E = \sqrt{m^2c^4 + p^2c^2} \approx pc$

$$E = \Delta pc/e = 98.6\text{MeV} \tag{1831}$$

Alternatively we have $E = \frac{\hbar c}{e \cdot dx}$ meaning $1\text{fm} = \frac{1}{197.3\text{MeV}}$ and therefore

$$E = \frac{\hbar}{2 \cdot \Delta x} c = 197.3/2\text{MeV} \tag{1832}$$

3. Solving for α, β, γ

$$M_P = G^\alpha c^\beta \hbar^\gamma \tag{1833}$$

$$= \left(\frac{\text{Nm}^2}{\text{kg}^2}\right)^\alpha \left(\frac{\text{m}}{\text{s}}\right)^\beta (\text{Js})^\gamma \tag{1834}$$

$$= \sqrt{\frac{\hbar c}{G}} \tag{1835}$$

$$E_P = \sqrt{\frac{\hbar c}{G}} c^2 \frac{1}{e} = 1.22 \cdot 10^{19}\text{GeV} \tag{1836}$$

8.2 Minkowski Metric

The definition implies that $\eta_{\lambda\nu}$ is the inverse of $\eta^{\lambda\nu}$ - simple calculation shows that they are identical. Now we can calculate

$$\eta^{\mu\lambda}\eta^{\nu\sigma}\eta_{\lambda\sigma} = \eta^{\nu\sigma}\delta^\mu_\sigma \tag{1837}$$

$$= \eta^{\nu\mu} \tag{1838}$$

and

$$a^\mu b_\mu = a_\alpha \eta^{\alpha\mu} b^\beta \eta_{\beta\mu} = a_\alpha b^\beta \delta^\alpha_\beta = a_\alpha b^\alpha \tag{1839}$$

14.6 Bethe, Jackiw - Intermediate Quantum Mechanics

1.1 Atomic units

Set $\hbar = e = m_e = 1$ and $a_B = \frac{4\pi\varepsilon_0\hbar^2}{m_e e^2} = 1$ then $4\pi\varepsilon_0 = 1$ and therefore $\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} = 1/c$

1. energy: $E_{1s} = \frac{1}{2}m_e c^2\alpha^2$ therefore 1 a.u. $= 2 \times 13.6$eV
2. momentum: $p = m_e c$ therefore 1 a.u. $= 2 \cdot 10^{-31}\text{kg} \times 3 \cdot 10^{8}\text{m/s}^2 = 2.73 \cdot 10^{-22}\text{J}$
3. angular momentum: $L = \hbar$ therefore 1 a.u. $= 1.04 \cdot 10^{-34}$Js

1.9 Exponential potential

Stepwise calculation for the verification of the solution

$$
\begin{aligned}
r^2\partial_r\psi &= u'r - u && (1840)\\
&= \frac{1}{2}\left[J_{n-1}(.) - J_{n+1}(.)\right] ar_0 e^{-\frac{r}{2r_0}}\frac{-r}{2r_0} - J_n(.) && (1841)\\
&= -\frac{1}{4}\left[J_{n-1}(.) - J_{n+1}(.)\right] are^{-\frac{r}{2r_0}} - J_n(.) && (1842)\\
& && (1843)
\end{aligned}
$$

15 General Physics

15.1 Feynman - Feynman Lectures on Physics

Section G1-1 - 1961 Sep 28 (1.16)

Section G1-2 - 1961 Sep 28 (1.15)

(a) We use the Penman equation to estimate the specific evaporation rate

$$
\begin{aligned}
\frac{dm}{dAdt} &= \frac{mR_n + \rho_{\text{air}}c_p(\delta e)g_a}{\lambda_v(m+\gamma)} && (1844)\\
&= \frac{mR_n + \rho_{\text{air}}c_p(\delta e)g_a}{\lambda_v(m + \frac{c_p p}{\lambda_v MW_{\text{ratio}}})} && (1845)\\
&\approx \frac{mR_n}{\lambda_v(m + \frac{c_p p}{\lambda_v MW_{\text{ratio}}})}. && (1846)
\end{aligned}
$$

The total time is then given by

$$
\begin{aligned}
t &= \frac{M}{\frac{dm}{dAdt}A} && (1847)\\
&= \frac{M}{\frac{dm}{dAdt}\pi r^2} && (1848)\\
&= \frac{M\lambda_v(m + \frac{c_p p}{\lambda_v MW_{\text{ratio}}})}{\pi r^2 mR_n} && (1849)
\end{aligned}
$$

with vapor the water vapor pressure

$$p_{\text{vap}} = \frac{101325\text{Pa}}{760}\exp\left[20.386 - \frac{5132K}{T}\right] \quad (1850)$$

the slope of the saturation vapor pressure

$$m = \frac{\partial p_{\text{vap}}}{\partial T} = ... \quad (1851)$$

the air heat capacity $c_p = 1.012\mathrm{Jkg}^{-1}\mathrm{K}^{-1}$, the latent heat of vaporization $\lambda_v = 2.26 \cdot 10^6\mathrm{Jkg}^{-1}$, the net irradiance $R_n = 150\mathrm{Wm}^{-2}$ (average day/night partly shade), the ratio molecular weight of water vapor/dry air $MW_{\mathrm{ratio}} = 0.622$, the pressure $p = 10^5\mathrm{Pa}$, the temperature $T = 298\mathrm{K}$, the water weight $M = 0.5\mathrm{kg}$ and the radius of the glass $r = 0.04\mathrm{m}$. This results in $t = 26$ days.

(b) With the molar mass of water $m_{H2O} = 18\,\mathrm{g}\cdot\mathrm{mol}^{-1}$

$$N = \frac{dm}{dAdt}\frac{N_A}{m_{H20}} \tag{1852}$$
$$= \frac{mR_n}{\lambda_v(m + \frac{c_p p}{\lambda_v MW_{\mathrm{ratio}}})}\frac{N_A}{m_{H20}} \tag{1853}$$
$$= 1.47 \cdot 10^{17}\mathrm{cm}^{-1}\mathrm{s}^{-1} \tag{1854}$$

(c) The total mass of water vaporizing on earth in one year is

$$M_{\mathrm{1y\ prec}} = \varepsilon_{\mathrm{ocean}} 4\pi R_E^2 \frac{dm}{dAdt} t_{1y}. \tag{1855}$$

with $\varepsilon_{\mathrm{ocean}} = 0.7$. In equilibrium this must be equal to the total amount of precipitation. So the average rainfall height is

$$h = \frac{M_{\mathrm{1y\ prec}}}{4\pi R_E^2 \rho_{\mathrm{H2O}}} \tag{1856}$$
$$= \frac{\varepsilon_{\mathrm{ocean}} t_{1y}}{\rho_{\mathrm{H2O}}}\frac{dm}{dAdt} \tag{1857}$$
$$= 947\mathrm{mm}. \tag{1858}$$

which seems reasonable (given that the solar constant is $1,361\mathrm{Wm}^{-2}$ the estimate of $R_n = 150\mathrm{Wm}^{-2}$ seems ok).

Section G-1 - 1961 Oct 5 (?.??)

(a) $\sqrt{s/g}$

(b) mL/T^2

(c) $\rho g h$

(d) $\sqrt{p/\rho}$

(e) gT (need to use the period T as c is not a material constant due to strong dispersion)

(f) $\rho g H^2$

(g) $\sqrt{R/g}$ here we assume the hemisphere rests on the table upside down - so it acts like a pendulum

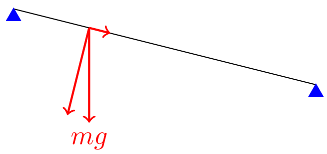
(h) $\sqrt{FL/m}$

Section G-2 - 1961 Oct 5 (?.??)

1. Equilibrium is given by condition

$$
\begin{aligned}
m_1 g &= m_2 g \sin\alpha && (1859)\\
&= m_2 g \frac{x}{\sqrt{x^2+a^2}} && (1860)\\
\rightarrow \quad m_1^2(x^2+a^2) &= m_2^2 x^2 && (1861)\\
\rightarrow \quad x &= \frac{m_1 a}{\sqrt{m_2^2-m_1^2}} && (1862)\\
& && (1863)
\end{aligned}
$$

2. General consideration



3.

4.

5.

Problem Set 3-1 - 1961 Nov 03 (3.16)

Direct measurement can be done for the

- radius of the earth $R_e = 6371\text{km}$
- orbital period of the moon $T_M = 28\text{d}$
- angular diameter of the moon $\delta = 30' = 0.5°$
- earths gravitational acceleration $g = 9.81\text{ms}^2$
- also Sputnik I orbital data can be looked up $a_{\text{satellite}} = R_E + 584\text{km}$ and $T_{\text{satellite}} = 96.2\text{min}$
- height difference between low and high tide $\Delta h = 1\text{m}$

1. We use Keplers 3rd law

$$
\begin{aligned}
\frac{a_M^3}{T_M^2} &= \frac{a_{\text{satellite}}^3}{T_{\text{satellite}}^2} && (1864)\\
a_M &= a_{\text{sat}}\left(\frac{T_M}{T_{\text{sat}}}\right)^{2/3} && (1865)
\end{aligned}
$$

then the radius of the moon is given by

$$
R_M = \frac{a_M}{2}\tan\delta = \frac{a_{\text{sat}}}{2}\left(\frac{T_M}{T_{\text{sat}}}\right)^{2/3}\tan\delta \qquad (1866)
$$

and the mass by

$$m_M = \rho_M V_M = \frac{4}{3}\pi\rho_M R_M^3 \tag{1867}$$

$$= \frac{4}{3}\pi\rho_M \left(\frac{a_{\text{sat}}}{2}\left(\frac{T_M}{T_{\text{sat}}}\right)^{2/3}\tan\delta\right)^3 \tag{1868}$$

$$= \frac{1}{6}\pi\rho_M a_{\text{sat}}^3 \left(\frac{T_M}{T_{\text{sat}}}\right)^2 \tan^3\delta \tag{1869}$$

$$\approx \frac{1}{6}\pi\rho_E a_{\text{sat}}^3 \left(\frac{T_M}{T_{\text{sat}}}\right)^2 \tan^3\delta \tag{1870}$$

where we approximated the moon by the earth mass density. From the gravitational law we can obtain the earth density by

$$g = \frac{F_g}{m} = \frac{Gm_E}{R_E^2} \quad \rightarrow \quad m_E = \frac{gR_E^2}{G} \tag{1871}$$

$$\rho_E = \frac{m_E}{V_E} = \frac{m_E}{\frac{4}{3}\pi R_E^3} = \frac{3g}{4\pi G R_E}. \tag{1872}$$

Therefore the mass of the moon is given by

$$m_M \approx \frac{g}{8GR_E} a_{\text{sat}}^3 \left(\frac{T_M}{T_{\text{sat}}}\right)^2 \tan^3\delta \tag{1873}$$

$$= 1.16 \cdot 10^{23}\text{kg}. \tag{1874}$$

2. We use Keplers 3rd law (for the earth-moon system) and the gravitational law for the earth

$$\frac{a_M^3}{T_M^2} = \frac{G(m_E + m_M)}{4\pi^2} \approx \frac{Gm_E}{4\pi^2} = \frac{a_{\text{satellite}}^3}{T_{\text{satellite}}^2} \tag{1875}$$

$$g = \frac{F_g}{m} = \frac{Gm_E}{R_E^2} \tag{1876}$$

and obtain

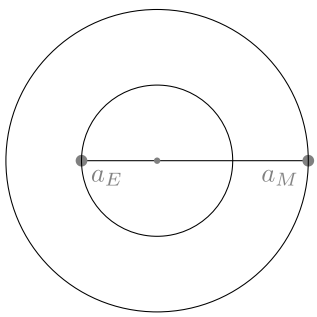
$$\frac{a_{\text{satellite}}^3}{T_{\text{satellite}}^2} = \frac{gR_E^2 + Gm_M}{4\pi^2} \tag{1877}$$

$$m_M = \frac{4\pi^2}{G}\left(\frac{a_{\text{satellite}}^3}{T_{\text{satellite}}^2} - \frac{gR_E^2}{4\pi^2}\right) \tag{1878}$$

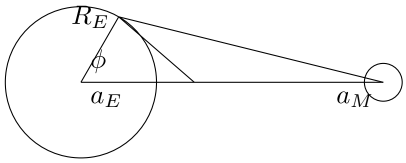
$$= 7.07 \cdot 10^{21}\text{kg}. \tag{1879}$$

This result is quite sensitive to the satellite orbital data.

3. We will use the earth tidal data. Lets assume circular orbits with $a_E + a_M = D$ which we can justify by observation (as the moon appears to have constant angular diameter). As reference system we use the center of mass of the system



$$m_E\omega^2 a_E = \frac{Gm_E m_M}{D^2} = m_M\omega^2 a_M \tag{1880}$$
$$\to a_E = \frac{m_M D}{m_E + m_M} \tag{1881}$$
$$\to \omega^2 = \frac{G(m_E + m_M)}{D^3} \tag{1882}$$
$$\to \omega^2 a_E^2 = \frac{Gm_M^2}{D(m_E + m_M)} \tag{1883}$$
$$\to \frac{R_E}{a_E} = \frac{m_E + m_M}{m_M}\frac{R_E}{D} \tag{1884}$$



The potential is then given by (gravity of moon and earth as well as the centripetal potential around the center of gravity)

$$V = V_{\text{G,moon}} + V_{\text{G,earth}} + V_{\text{cent}} \tag{1885}$$
$$= -\frac{Gm_M}{\sqrt{R_E^2 + D^2 - 2DR_E\cos\phi}} - \frac{Gm_E}{R_E} - \frac{1}{2}\omega^2(R_E^2 + a_E^2 - 2a_E R_E\cos\phi) \tag{1886}$$
$$= -\frac{Gm_M}{D\sqrt{\left(\frac{R_E}{D}\right)^2 + 1 - 2\frac{R_E}{D}\cos\phi}} - \frac{Gm_E}{R_E} - \frac{1}{2}\omega^2 a_E^2\left[\left(\frac{R_E}{a_E}\right)^2 + 1 - 2\frac{R_E}{a_E}\cos\phi\right] \tag{1887}$$
$$\approx -\frac{Gm_M}{D}\left(1 + \frac{R_E}{D}\cos\phi + \frac{3}{2}\left(\frac{R_E}{D}\right)^2\cos^2\phi\right) - \frac{Gm_E}{R_E} \tag{1888}$$
$$-\frac{1}{2}\omega^2 a_E^2\left[\left(\frac{R_E}{a_E}\right)^2 + 1 - 2\frac{R_E}{a_E}\cos\phi\right] \tag{1889}$$
$$\approx -\frac{Gm_M}{D}\left(1 + \frac{R_E}{D}\cos\phi + \frac{3}{2}\left(\frac{R_E}{D}\right)^2\cos^2\phi\right) - \frac{Gm_E}{R_E} \tag{1890}$$
$$-\frac{1}{2}\frac{Gm_M^2}{D(m_E + m_M)}\left[\left(\frac{m_E + m_M}{m_M}\frac{R_E}{D}\right)^2 + 1 - 2\frac{m_E + m_M}{m_M}\frac{R_E}{D}\cos\phi\right] \tag{1891}$$
$$\approx -\frac{Gm_M}{D} - \frac{3Gm_M}{2D}\frac{R_E^2}{D^2}\cos^2\phi - \frac{Gm_E}{R_E} \tag{1892}$$
$$-\frac{1}{2}\frac{Gm_M^2}{D(m_E + m_M)}\left[\left(\frac{m_E + m_M}{m_M}\frac{R_E}{D}\right)^2 + \frac{m_M^2 D^2}{m_M^2 D^2}\right] \tag{1893}$$
$$\approx -\frac{Gm_M}{D} - \frac{3Gm_M}{2D}\frac{R_E^2}{D^2}\cos^2\phi - \frac{Gm_E}{R_E} - \frac{G}{2}\left[(m_E + m_M)\frac{R_E^2}{D^3} + \frac{m_M^2}{m_E + m_M}\frac{1}{D}\right]. \tag{1894}$$

with the angular dependent tidal part

$$V_{\text{tidal}} = -\frac{3GR_E^2 m_M}{2D^3}\cos^2\phi. \tag{1895}$$

The tidal water surface would be formed by the the surface $r_{\text{surf}}(\phi) = R_E + h$ of constant potential. The height difference between low and high tide can then be estimated by

$$-\frac{3GR_E^2 m_M}{2D^3} = Gm_E\left(\frac{1}{R_E+h} - \frac{1}{R_E}\right) \tag{1896}$$

$$\approx Gm_E\left(\frac{1}{R_E\left(1+\frac{h}{R_E}\right)} - \frac{1}{R_E}\right) \tag{1897}$$

$$\approx \frac{Gm_E}{R_E}\left(\left(1-\frac{h}{R_E}\right)-1\right) \tag{1898}$$

which gives

$$h = \frac{3R_E^4}{2D^3}\frac{m_M}{m_E}. \tag{1899}$$

Using the results from above

$$m_E = \frac{gR_E^2}{G} \tag{1900}$$

$$\omega^2 = \frac{G(m_E+m_M)}{D^3} \tag{1901}$$

$$\rightarrow \quad D^3 = \frac{G(m_E+m_M)}{\omega^2} = G(m_E+m_M)\frac{T_M^2}{4\pi^2} \tag{1902}$$

we obtain

$$h = \frac{6\pi^2 R_E^4 T_M^2}{G(m_E+m_M)T_M^2}\frac{m_M}{m_E}. \tag{1903}$$

and can subsequently solve for m_M

$$m_M = \frac{Ghm_E^2T^2}{6\pi^2R_E^4 - Ghm_ET^2} \tag{1904}$$

$$= \frac{m_E}{\frac{6\pi^2R_E^4}{Ghm_ET^2}-1} \tag{1905}$$

$$= \frac{g^2hT_M^2R_E^2}{G(6\pi^2R_E^2 - ghT_M^2)} \tag{1906}$$

$$= \frac{gR_E^2}{G\left(\frac{6\pi^2R_E^2}{ghT_M^2}-1\right)} \tag{1907}$$

$$= 1.38\cdot 10^{23}\text{kg} \tag{1908}$$

Problem Set 3-3 - 1961 Nov 03 (3.10)

(a) We use Keplers 3rd law for the earth

$$\frac{a_E^3}{T_E^2} = \frac{G(m_S+m_E)}{4\pi^2} \approx \frac{Gm_S}{4\pi^2} \tag{1909}$$

$$\tag{1910}$$

and the stars a and b

$$\frac{a^3}{T^2} = \frac{G(m_A+m_B)}{4\pi^2} \tag{1911}$$

$$\frac{(Ra_E)^3}{(TT_E)^2} = \frac{R^3}{T^2}\frac{a_E^3}{T_E^2} = \frac{R^3}{T^2}\frac{Gm_S}{4\pi^2} = \frac{G(m_A+m_B)}{4\pi^2} \tag{1912}$$

$$\rightarrow m_A+m_B = \frac{R^3}{T^2}m_S = \frac{729}{25}m_S \tag{1913}$$

(b) For a the circular orbits we have the stability condition

$$
\begin{aligned}
m_A\omega^2 r_A = F_{AB} &= m_B\omega^2 r_B && (1914)\\
&\to m_A\omega v_A = m_B\omega v_B && (1915)\\
&\to \frac{m_A}{m_B} = \frac{v_B}{v_A} = \frac{1}{5} && (1916)
\end{aligned}
$$

with $m_B = 5m_A$ we have

$$
\begin{aligned}
m_A &= \frac{243}{50}m_S && (1917)\\
m_B &= \frac{243}{10}m_S. && (1918)
\end{aligned}
$$

Book (2.22)

The center of the spheres build a tetrahedron where each connection has to carry a third of the

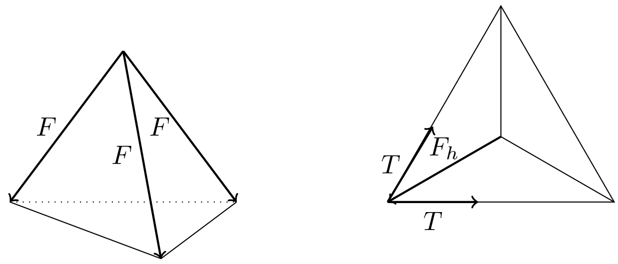


Figure 2: Problem (2.22)

weight mg

$$F = \frac{mg}{3\cos\alpha} \qquad (1919)$$

where α is the angle of the edge

$$\cos\alpha = \frac{H}{a} = \frac{\sqrt{a^2 - \left(\frac{2}{3}h\right)^2}}{a} = \frac{\sqrt{a^2 - \left(\frac{2}{3}\frac{\sqrt{3}}{2}a\right)^2}}{a} = \sqrt{2/3}. \qquad (1920)$$

The horizontal projection is then

$$F_h = F\sin\alpha = \frac{mg}{3}\tan\alpha. \qquad (1921)$$

Projecting them in the plane gives

$$
\begin{aligned}
F_h &= \sqrt{T^2 + T^2 - 2T^2\cos\frac{2\pi}{3}} && (1922)\\
\to T &= \frac{mg}{3\sqrt{6}} && (1923)
\end{aligned}
$$

including the safety margin we obtain

$$\widetilde{T} = 3T = \frac{mg}{\sqrt{6}} = 2\text{ton-wt} \qquad (1924)$$

Problem Set 3-4 - 1961 Nov 03 (?.??)

$$g_M = \frac{GM_M}{R_M^2} = \frac{4}{3}G\rho_M R_M = \frac{4}{3}G(0.537\rho_E)(0.716R_E) = 0.384\cdot g_E \qquad (1925)$$

Problem Set 1a 4-1 - 1961 Nov 10 (11.16)

For the masses we obtain

$$\widetilde{m}_i = \rho_i V_i = \frac{4}{3}\pi(kR_i)^3\rho_i \tag{1926}$$
$$= k^3 m_i \tag{1927}$$

The third Kepler law is given by

$$T^2 = a^3 \frac{4\pi}{G(M+m)} \tag{1928}$$

applying the scaling to a we have $\widetilde{a}^3 = (ka)^3$ and therefore

$$\widetilde{T}^2 = \widetilde{a}^3 \frac{4\pi}{G(\widetilde{M}+\widetilde{m})} \tag{1929}$$
$$= k^3 a^3 \frac{4\pi}{G(k^3 M + k^3 m)} \tag{1930}$$
$$= a^3 \frac{4\pi}{G(M+m)} \tag{1931}$$
$$= T^2. \tag{1932}$$

So we conclude that there is no change in T.

Problem Set 1a 4-3 - 1961 Nov 10 (??.??)

With $\vec{a} = (1, 0, 2)$ and $\vec{b} = (1, 4, 0)$

$$\cos\alpha = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|} = 1/\sqrt{85} \tag{1933}$$

Problem Set 1a 4-6 - 1961 Nov 10 (??.??)

The 3, 4, 5 triangle is rectangular with and incline angle α

$$\sin\alpha = 3/5 \tag{1934}$$
$$\cos\alpha = 4/5 \tag{1935}$$

and therefore

$$F_{A,\parallel} = M_A g(\sin\alpha - \mu_A\cos\alpha) \tag{1936}$$
$$F_{B,\parallel} = M_B g(\sin\alpha - \mu_B\cos\alpha) \tag{1937}$$

(a) Then

$$a = \frac{F_{A,\parallel} + F_{B,\parallel}}{M_A + M_B} \tag{1938}$$
$$= \frac{M_A(\sin\alpha - \mu_A\cos\alpha) + M_B(\sin\alpha - \mu_B\cos\alpha)}{M_A + M_B} g \tag{1939}$$
$$= \frac{(M_A + M_B)\sin\alpha - (M_A\mu_A + M_B\mu_B)\cos\alpha}{M_A + M_B} g \tag{1940}$$
$$= \left(\sin\alpha - \frac{M_A\mu_A + M_B\mu_B}{M_A + M_B}\cos\alpha\right) g \tag{1941}$$
$$= 4.84\text{m/s}^2 \tag{1942}$$

(b) Newton 3

$$F_{A,\parallel} - M_A a = T \tag{1943}$$
$$F_{B,\parallel} - M_B a = -T \tag{1944}$$

then

$$2T = (F_{A,\parallel} - F_{B,\parallel}) - (M_A - M_B)a \tag{1945}$$
$$T = \frac{M_A M_B}{M_A + M_B}(\mu_B - \mu_A) g \cos\alpha \tag{1946}$$
$$= 2.09\text{N} \tag{1947}$$

Problem Set 1b 3-3 - 1962 Jan 16 (??.??)

1.

2. For the angular momentum (without precision) we get

$$L = J\omega = \frac{1}{2} M R^2 \omega \tag{1948}$$

3. With a little geometry we see

$$\frac{d\vec{L}}{dt} = \vec{a} \times \vec{M} = M\vec{a} \times \vec{g} \tag{1949}$$
$$\frac{dL}{L} = \sin d\phi \approx d\phi \tag{1950}$$
$$\to \Omega_1 = \frac{d\phi}{dt} = \frac{Mag}{L} = \frac{2ag}{R^2\omega} \tag{1951}$$
$$\to \omega = \frac{2ag}{R^2\Omega_1} \tag{1952}$$

4.

Problem Set 1b 11-1 - 1962 Feb 16 (20.11)

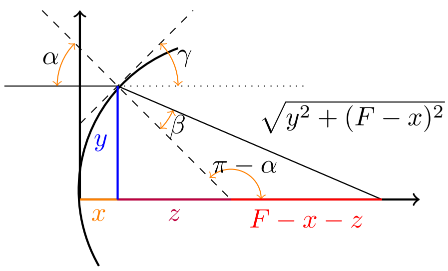


Figure 3: Problem (20.11)

We start with Snell's law

$$\sin\alpha = n \sin\beta \tag{1953}$$

and the sine law

$$\frac{\sin(\pi - \alpha)}{\sqrt{y^2 + (F-x)^2}} = \frac{\sin\beta}{F - x - z} \tag{1954}$$

we also see

$$\frac{y}{z} = \tan\alpha \tag{1955}$$

most importantly we have for the slope of the surface

$$\frac{dy}{dx} = \tan\gamma \tag{1956}$$

$$= \tan(\pi/2 - \alpha) = \cot\alpha = \frac{1}{\tan\alpha} \tag{1957}$$

Now we can put it all together

$$\frac{\sin(\alpha)}{\sqrt{y^2 + (F-x)^2}} = \frac{\sin\alpha}{n(F - x - \frac{y}{\tan\alpha})} \tag{1958}$$

$$\frac{1}{\sqrt{y^2 + (F-x)^2}} = \frac{1}{n[(F-x) - y\frac{dy}{dx}]} \tag{1959}$$

$$(F-x) - y\frac{dy}{dx} = \frac{1}{n}\sqrt{y^2 + (F-x)^2} \tag{1960}$$

The ODE can be solved by Mathematica which gives two solutions

$$y_1 = \pm\sqrt{2Fx\left(1 - \frac{1}{n}\right) - \left(1 - \frac{1}{n^2}\right)x^2} \tag{1961}$$

$$y_2 = \pm\sqrt{2Fx\left(1 + \frac{1}{n}\right) - \left(1 - \frac{1}{n^2}\right)x^2} \tag{1962}$$

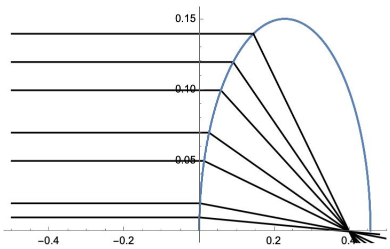


Figure 4: Light rays for solution y_1 of Feynman problem (20.11)

Book (20.14)

Using the sine law for the appropriate triangle inside the sphere (cosine law to calculate the length of one side) we obtain

$$\frac{\sin\beta}{R} = \frac{\sin\alpha}{\sqrt{R^2 + R^2 - 2R\cdot R\cos\alpha}} \tag{1963}$$

$$\frac{\sin\alpha}{nR} = \frac{\sin\alpha}{\sqrt{2}R\sqrt{1 - \cdot R\cos\alpha}} \tag{1964}$$

where we used Snells law. Simplifying further

$$\cos\alpha = 1 - \frac{n^2}{2} \tag{1965}$$

$$\sqrt{1-\sin^2\alpha} = 1 - \frac{n^2}{2} \tag{1966}$$

$$\sin^2\alpha = 1 - \left(1 - \frac{n^2}{2}\right)^2 \tag{1967}$$

$$\frac{y^2}{4R^2} = 1 - \left(1 - \frac{n^2}{2}\right)^2 \tag{1968}$$

$$\rightarrow y = 2nR\sqrt{1 - \frac{n^2}{4}} = 1.92R \tag{1969}$$

Book (20.16)

First lens

$$\frac{1}{f} = \frac{1}{g} + \frac{1}{b} \quad \rightarrow \quad b = \frac{gF}{g-F} \tag{1970}$$

$$\frac{B}{G} = \frac{b}{g} \quad \rightarrow \quad B = G\frac{b}{g} = \frac{G}{g}\frac{gF}{g-F} \tag{1971}$$

Distant object means

$$B \simeq \frac{GF}{g} \tag{1972}$$

The second lens works as a magnifying glass - focussing at infinity means the virtual picture is at infinity and therefore the object (real picture of the first lens) needs to be at the focus of the second lens. The angle is then given by

$$\tan\alpha' = \frac{B}{f} = \frac{GF}{gf} \tag{1973}$$

without lenses the angle would have been

$$\tan\alpha = \frac{G}{g}. \tag{1974}$$

Then

$$M = \tan\alpha'/\tan\alpha = F/f. \tag{1975}$$

Problem Set 1c 13-1 - 1962 May 25 (?.??)

We cheat a little and use the Lagrange formalism

$$L = T - V \tag{1976}$$

$$= \frac{m_1}{2}\dot{x}^2 + \frac{m_2}{2}\dot{y}^2 - \frac{k_1}{2}x^2 - \frac{k_2}{2}y^2 - \frac{k}{2}(x-y)^2 \tag{1977}$$

then

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0 \tag{1978}$$

gives

$$m_1\ddot{x} + k_1 x + k(x-y) = 0 \quad \rightarrow \quad \ddot{x} + \omega_0^2 x + \frac{k}{m_1}(x-y) = 0 \tag{1979}$$

$$m_2\ddot{y} + k_2 y - k(x-y) = 0 \quad \rightarrow \quad \ddot{y} + \omega_0^2 y + \frac{k}{m_2}(y-x) = 0 \tag{1980}$$

Problem Set 1c 13-2 - 1962 May 25 (?.??)

We obtain

$$-A\omega^2 + A\omega_0^2 + \frac{k}{m_1}(A - B) = 0 \tag{1981}$$

$$-B\omega^2 + B\omega_0^2 + \frac{k}{m_2}(B - A) = 0 \tag{1982}$$

and therefore

$$\omega^2 = \omega_0^2 + \frac{k}{m_1}(1 - B/A) \tag{1983}$$

$$\omega^2 = \omega_0^2 + \frac{k}{m_2}(1 - A/B) \tag{1984}$$

both expressions give the same valuers for ω if

$$A/B = 1 \quad \rightarrow \quad \omega = \omega_0 \tag{1985}$$

$$A/B = -m_2/m_1 \quad \rightarrow \quad \omega = \sqrt{\omega_0 + \frac{k}{m_1} + \frac{k}{m_2}} \tag{1986}$$

15.2 Jackson - Classical Electrodynamics

Exercise 1.3 Charge densities and the Dirac delta function

$$\rho_a = \frac{Q}{4\pi R^2}\delta(r - R) \quad \rightarrow \quad \int \rho_a d^3r = 4\pi \frac{Q}{4\pi R^2}\int_0^\infty \delta(r - R) r^2\, dr \tag{1987}$$

$$= Q \tag{1988}$$

$$\rho_b = \frac{\lambda}{2\pi b}\delta(r - b) \quad \rightarrow \quad \int \rho_b d^3r = \frac{\lambda}{2\pi b}2\pi \int_0^L dz \int_0^\infty \delta(r - b) r\, dr \tag{1989}$$

$$= \lambda L \tag{1990}$$

$$\rho_c = \frac{Q}{\pi R^2}\theta(R - r)\delta(z) \quad \rightarrow \quad \int \rho_c d^3r = \frac{Q}{\pi R^2}2\pi \int dz \int_0^\infty \theta(r - R) r\ dr \tag{1991}$$

$$= \frac{Q}{\pi R^2}2\pi \int dz \int_0^R r\ dr \tag{1992}$$

$$= \frac{Q}{\pi R^2}2\pi \frac{R^2}{2} = Q \tag{1993}$$

Now we got curvilinear coordinates so we need an additional $1/r$ scaling

$$\rho_d = \frac{Q}{\pi R^2 r}\theta(R - r)\delta(\vartheta - \pi/2) \quad \rightarrow \quad \int \rho_d d^3r = \frac{Q}{\pi R^2}2\pi \int_0^\infty \frac{r^2}{r}\theta(R - r)\int_0^\pi \delta(\vartheta - \pi/2)\sin\vartheta\ d\vartheta \tag{1994}$$

$$= \frac{Q}{\pi R^2}2\pi \int_0^R r \int_0^\pi \delta(\vartheta - \pi/2)\sin\vartheta\ d\vartheta \tag{1995}$$

$$= \frac{Q}{\pi R^2}2\pi \frac{R^2}{2}\sin\pi/2 = Q \tag{1996}$$

Exercise 1.4 Charged spheres

We can utilize the Gauss theorem

$$\oint_S \vec{E}\cdot\vec{n}dA = \frac{1}{\epsilon_0}\int_V \rho(x)d^3x \tag{1997}$$
$$4\pi r^2 E_r = \frac{q_r}{\epsilon_0} \tag{1998}$$
$$E_r = \frac{q_r}{4\pi\epsilon_0 r^2} \tag{1999}$$

assuming a radial electrical field.

- Conducting sphere

$$\rho_{\text{cond}} = Q\delta(r-a) \tag{2000}$$
$$E_r = \frac{1}{4\pi\epsilon_0}\cdot\begin{cases} 0 & r<a \\ Q/r^2 & r>a \end{cases} \tag{2001}$$

- Uniform sphere

$$\rho_{\text{hom}} = Q\theta(a-r) \tag{2002}$$
$$E_r = \frac{1}{4\pi\epsilon_0}\cdot\begin{cases} Qa^3 r & r<a \\ Q/r^2 & r>a \end{cases} \tag{2003}$$

- Nonuniform sphere

$$\rho_{\text{inhom}} = Q\frac{n+3}{a^{n+3}}r^n \quad (r<a) \tag{2004}$$
$$E_r = \frac{1}{4\pi\epsilon_0}\cdot\begin{cases} Qa^{n+3}r^{n+1} & r<a \\ Q/r^2 & r>a \end{cases} \tag{2005}$$

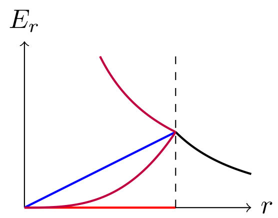


Figure 5: Jackson problem (1.4)

Exercise 1.5 Charge density of hydrogen atom

With the potential

$$\Phi = \frac{q}{4\pi\epsilon_0}\frac{e^{-\alpha r}}{r}\left(1+\frac{\alpha r}{2}\right) \tag{2006}$$

we calculate for $r>0$

$$\rho_1 = -\epsilon_0\triangle\Phi \tag{2007}$$
$$= -\epsilon_0\frac{1}{r^2}\partial_r(r^2\partial_r\Phi) \tag{2008}$$
$$= -\frac{q}{4\pi}e^{-\alpha r}\frac{\alpha^3}{2} \tag{2009}$$
$$= -\frac{q}{\pi a_0^3}e^{-2r/a_0} \tag{2010}$$

For $r = 0$ we have

$$\Phi(r \to 0) = \frac{q}{4\pi\epsilon_0 r} \tag{2011}$$

$$\to \quad \rho_0 = q\delta(r) \tag{2012}$$

Therefore

$$\rho = \rho_0 + \rho_1 \tag{2013}$$

$$= q\left(\delta^{(3)}(r) - \frac{1}{\pi a_0^3} e^{-2r/a_0}\right) \tag{2014}$$

Calculating the total charge

$$Q_0 = q\int d^3 r\delta(r) = q \tag{2015}$$

$$Q_1 = 4\pi\int_0^\infty r^2\rho_1 dr \tag{2016}$$

$$= -\frac{4\pi q}{\pi a_0^3}\int_0^\infty r^2 e^{-2r/a_0} dr \tag{2017}$$

$$= -\frac{4\pi q}{\pi a_0^3}\frac{a_0^3}{8}\int_0^\infty z^2 e^{-z} dz \tag{2018}$$

$$= -\frac{4\pi q}{\pi a_0^3}\frac{a_0^3}{8}\Gamma(3) \tag{2019}$$

$$= -q \tag{2020}$$

Exercise 1.6 Simple capacitors

(a) Assuming only front and back surfaces contribute

$$2E_x A = \frac{Q}{\epsilon_0} \tag{2021}$$

$$\to \quad E_x = \frac{Q}{2\epsilon_0 A} \tag{2022}$$

$$\to \quad \phi = -\frac{Q}{2\epsilon_0 A}x \tag{2023}$$

$$\to \quad \phi_{\text{tot}}(x) = -\frac{Q}{2\epsilon_0 A}x - \frac{-Q}{2\epsilon_0 A}(d - x) \tag{2024}$$

$$= -\frac{Q}{2\epsilon_0 A}(x - (d - x)) \tag{2025}$$

$$= -\frac{Q}{2\epsilon_0 A}(2x - d) \tag{2026}$$

$$\to \quad C = \frac{Q}{\Delta\phi} = \frac{Q}{-\frac{Q}{2\epsilon_0 A}(-d - d)} \tag{2027}$$

$$= \epsilon_0\frac{A}{d} \tag{2028}$$

(b) The outer sphere does not contribute to the total potential as it is field free

$$4\pi r^2 E_r = \frac{Q}{\epsilon_0} \tag{2029}$$

$$\rightarrow \quad E_r = \frac{Q}{4\pi\epsilon_0 r^2} \tag{2030}$$

$$\rightarrow \quad \phi = \frac{Q}{4\pi\epsilon_0 r} \tag{2031}$$

$$\rightarrow \quad \phi_{\text{tot}} = \frac{Q}{4\pi\epsilon_0 r} \quad (a < r < b) \tag{2032}$$

$$\rightarrow \quad C = \frac{Q}{\Delta\phi} = \frac{Q}{\frac{Q}{4\pi\epsilon_0 b} - \frac{Q}{4\pi\epsilon_0 a}} \tag{2033}$$

$$= \epsilon_0 \frac{4\pi ab}{b-a} \tag{2034}$$

(c)

$$2\pi r L E_r = \frac{Q}{\epsilon_0} \tag{2035}$$

$$\rightarrow \quad E_r = \frac{Q}{2\pi r L\epsilon_0} \tag{2036}$$

$$\rightarrow \quad \phi = -\frac{Q}{2\pi L\epsilon_0} \log r \tag{2037}$$

$$\rightarrow \quad \phi_{\text{tot}} = -\frac{Q}{2\pi L\epsilon_0} \log r \quad (a < r < b) \tag{2038}$$

$$\rightarrow \quad C = \frac{Q}{\Delta\phi} = \frac{Q}{-\frac{Q}{2\pi L\epsilon_0}\log b + \frac{Q}{2\pi L\epsilon_0}\log a} \tag{2039}$$

$$= \frac{2\pi L\epsilon_0}{\log a/b} \tag{2040}$$

(d) ...

Exercise 1.7 Capacity of two parallel cylinders

Gauss law for one cylinder

$$\oint_S \vec{E} \cdot \vec{n} dA = \frac{1}{\epsilon_0} \int_V \rho(x) d^3x \tag{2041}$$

$$2\pi r L E_r = \frac{\rho_1 L}{\epsilon_0} \tag{2042}$$

$$E_r = \frac{\rho}{2\pi\epsilon_0 r} \tag{2043}$$

$$\phi = -\frac{\rho}{2\pi\epsilon_0} \ln r \tag{2044}$$

For $d \gg a_{1,2}$ the potential of one cylinder on the surface of the second cylinder is constant - which means that the potential can be approximated by the sum of the potential of both cylinders (no need to make it complicated)

$$\phi(\vec{r}) = \phi_1 + \phi_2 \tag{2045}$$
$$= -\frac{\rho_1}{2\pi\epsilon_0}\ln|\vec{r}| - \frac{\rho_2}{2\pi\epsilon_0}\ln|\vec{r}-\vec{d}| \tag{2046}$$
$$= -\frac{\rho}{2\pi\epsilon_0}\ln|\vec{r}| + \frac{\rho}{2\pi\epsilon_0}\ln|\vec{r}-\vec{d}| \tag{2047}$$
$$= -\frac{\rho}{2\pi\epsilon_0}\left(\ln|\vec{r}| - \ln|\vec{r}-\vec{d}|\right) \tag{2048}$$
$$= -\frac{\rho}{2\pi\epsilon_0}\ln\frac{|\vec{r}|}{|\vec{r}-\vec{d}|} \tag{2049}$$
$$= -\frac{\rho}{\pi\epsilon_0}\ln\sqrt{\frac{|\vec{r}|}{|\vec{r}-\vec{d}|}} \tag{2050}$$

Then the potential difference between to surfaces is given by (with $\vec{n} = \vec{d}/d$ and $\rho = \rho_1 = -\rho_2$)

$$\Delta\phi = \phi(a_1\vec{n}) - \phi((d-a_2)\vec{n}) \tag{2051}$$
$$= -\frac{\rho}{\pi\epsilon_0}\left(\ln\sqrt{\frac{a_1}{d-a_1}} - \ln\sqrt{\frac{d-a_2}{a_2}}\right) \tag{2052}$$
$$= \frac{\rho}{\pi\epsilon_0}\left(\ln\sqrt{\frac{d-a_1}{a_1}} + \ln\sqrt{\frac{d-a_2}{a_2}}\right) \tag{2053}$$
$$\simeq \frac{\rho}{\pi\epsilon_0}\left(\ln\sqrt{\frac{d}{a_1}} + \ln\sqrt{\frac{d}{a_2}}\right) \tag{2054}$$
$$\simeq \frac{\rho}{\pi\epsilon_0}\ln\frac{d}{\sqrt{a_1a_2}} \tag{2055}$$

With $C = Q/U$ we have

$$C = \frac{\rho L}{\Delta\phi} = \frac{\pi\epsilon_0 L}{\ln\frac{d}{\sqrt{a_1a_2}}} \tag{2056}$$

which is the desired result. The numbers are 0.49mm, 1.47mm and 4.92mm.

Exercise 1.8 Energy of capacitors

$$W = \frac{1}{2}\int\rho(x)\phi(x)d^3x = -\frac{\epsilon_0}{2}\int\phi\Delta\phi d^3x = \frac{\epsilon_0}{2}\int(\nabla\phi)^2d^3x = \frac{\epsilon_0}{2}\int|\vec{E}|^2d^3x \tag{2057}$$

(a) With $\vec{E}_{\text{tot}} = -\nabla\phi_{\text{tot}}$ and $Q = C\cdot U$

$$W_{\text{plate}} = \frac{\epsilon_0}{2}\cdot\left(\frac{Q}{\epsilon_0 A}\right)^2\cdot(Ad) = \frac{Q^2d}{2\epsilon_0 A} \tag{2058}$$
$$= \frac{U^2d}{2\epsilon_0 A}\left(\frac{\epsilon_0 A}{d}\right)^2 = \frac{\epsilon_0 AU^2}{2d} \tag{2059}$$

$$W_{\text{sphere}} = \frac{\epsilon_0}{2}4\pi\int_a^b r^2\frac{Q^2}{16\pi^2\epsilon_0^2r^4}dr = \frac{Q^2}{8\pi\epsilon_0}\left(\frac{1}{b}-\frac{1}{a}\right) \tag{2060}$$
$$= \frac{U^2}{8\pi\epsilon_0}\left(\frac{a-b}{ab}\right)\cdot\left(\epsilon_0\frac{4\pi ab}{b-a}\right)^2 = 2\pi\epsilon_0 U^2\frac{ab}{b-a} \tag{2061}$$

$$W_{\text{cylinder}} = \frac{\epsilon_0}{2} 2\pi L \int_a^b \left(\frac{Q}{2\pi\epsilon_0 Lr} \right)^2 r\, dr = \frac{Q^2}{4\pi\epsilon_0 L} \log \frac{b}{a} \tag{2062}$$

$$= \frac{U^2}{4\pi\epsilon_0 L} \log \frac{b}{a} \left(\frac{2\pi\epsilon_0 L}{\log b/a} \right)^2 = \frac{\pi\epsilon_0 L U^2}{\log b/a} \tag{2063}$$

(b)

$$w_{\text{plate}} = \text{const} \tag{2064}$$

$$w_{\text{sphere}} \sim r^{-4} \tag{2065}$$

$$w_{\text{cylinder}} \sim r^{-2} \tag{2066}$$

Exercise 12.1 Lagrangian of point charge

1. With $U^\alpha = \frac{dx_\alpha}{ds}$

$$L = -\frac{mU_\alpha U^\alpha}{2} - \frac{q}{c} U_\alpha A^\alpha \tag{2067}$$

$$\frac{\partial L}{\partial x_\beta} = -\frac{q}{c} U_\alpha \frac{\partial A^\alpha}{\partial x_\beta} \tag{2068}$$

$$\frac{\partial L}{\partial U_\beta} = -mU^\beta - \frac{q}{c} A^\beta \tag{2069}$$

$$-m\frac{d}{ds}\left(\frac{dU^\beta}{ds}\right) - \frac{q}{c}\frac{dA^\beta}{ds} + \frac{q}{c} U_\alpha \frac{\partial A^\alpha}{\partial x_\beta} = 0 \tag{2070}$$

$$m\frac{d^2 x^\beta}{ds^2} + \frac{q}{c}\frac{dA^\beta}{ds} - \frac{q}{c}\frac{dx_\alpha}{ds}\frac{\partial A^\alpha}{\partial x_\beta} = 0 \tag{2071}$$

$$m\frac{d^2 x^\beta}{ds^2} + \frac{q}{c}\left(\frac{\partial A^\beta}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial s}\right) - \frac{q}{c}\frac{dx_\alpha}{ds}\frac{\partial A^\alpha}{\partial x_\beta} = 0 \tag{2072}$$

$$m\frac{d^2 x^\beta}{ds^2} + \frac{q}{c}\frac{\partial x^\alpha}{\partial s}\left(\frac{\partial A^\beta}{\partial x^\alpha} - \frac{\partial A^\alpha}{\partial x_\beta}\right) = 0 \tag{2073}$$

$$m\frac{d^2 x^\beta}{ds^2} + \frac{q}{c}\frac{\partial x^\alpha}{\partial s} F^{\alpha\beta} = 0 \tag{2074}$$

2. Bit of a odd sign convention for the canonical momentum

$$P^\beta = -\frac{\partial L}{\partial U_\beta} = mU^\beta + \frac{q}{c}A^\beta \quad \to \quad U^\beta = \frac{1}{m}\left(P^\beta - \frac{q}{c}A^\beta\right) \tag{2075}$$

$$H = P^\alpha U_\alpha + L \tag{2076}$$

$$= P^\alpha \frac{1}{m}\left(P_\alpha - \frac{q}{c}A_\alpha\right) - \frac{m}{2}\frac{1}{m}\left(P_\alpha - \frac{q}{c}A_\alpha\right)\frac{1}{m}\left(P_\alpha - \frac{q}{c}A_\alpha\right) - \frac{q}{c}\frac{1}{m}\left(P_\alpha - \frac{q}{c}A_\alpha\right)A^\alpha \tag{2077}$$

$$= \frac{1}{2m}\left(P^\alpha - \frac{q}{c}A^\alpha\right)\left(P_\alpha - \frac{q}{c}A_\alpha\right) \tag{2078}$$

In space-time coordinates we can write

$$H = \frac{1}{2m}\left((p_0)^2 - \vec{p}^2 + \frac{q^2}{c^2}[\phi^2 - \vec{A}^2] + \frac{2q}{c}[\vec{p}\cdot\vec{A} - p^0\phi]\right) \tag{2079}$$

$$= \frac{1}{2m}\left((\gamma mc)^2 - (\gamma m\vec{v})^2 + \frac{q^2}{c^2}[\phi^2 - \vec{A}^2] + \frac{2q}{c}[\gamma m\vec{v}\cdot\vec{A} - \gamma mc\phi]\right) \tag{2080}$$

$$= \frac{\gamma^2 mc^2}{2}\left(1 - \frac{\vec{v}^2}{c^2}\right) + \frac{q^2}{2mc^2}[\phi^2 - \vec{A}^2] + q\gamma[\frac{1}{c}\vec{v}\cdot\vec{A} - \phi] \tag{2081}$$

$$= \frac{mc^2}{2} + \frac{q^2}{2mc^2}[\phi^2 - \vec{A}^2] + q\gamma[\frac{1}{c}\vec{v}\cdot\vec{A} - \phi] \tag{2082}$$

15.3 Schwinger - Classical Electrodynamics

Exercise 31.1 Potentials of moving point charge

$$w = z - vt \to \frac{\partial}{\partial z} = \frac{\partial w}{\partial z}\frac{\partial}{\partial w} \tag{2083}$$

$$\to \frac{\partial^2}{\partial z^2} = \frac{\partial^2 w}{\partial z^2}\frac{\partial}{\partial w} + \left(\frac{\partial w}{\partial z}\right)^2 \frac{\partial^2}{\partial w^2} = \frac{\partial^2}{\partial w^2} \tag{2084}$$

$$\to \frac{\partial^2}{\partial t^2} = \frac{\partial^2 w}{\partial t^2}\frac{\partial}{\partial w} + \left(\frac{\partial w}{\partial t}\right)^2 \frac{\partial^2}{\partial w^2} = v^2\frac{\partial^2}{\partial w^2} \tag{2085}$$

then

$$\triangle - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial w^2} - \frac{v^2}{c^2}\frac{\partial^2}{\partial w^2} \tag{2086}$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(1 - \frac{v^2}{c^2}\right)\frac{\partial^2}{\partial w^2} \tag{2087}$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial u^2} \tag{2088}$$

with with $u = w/\sqrt{1 - v^2/c^2}$. The wave equation can then be rewritten

$$-\Box\phi = 4\pi\rho \tag{2089}$$

$$= 4\pi e\delta(x)\delta(y)\delta(z - vt) \tag{2090}$$

$$-\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial u^2}\right)\phi = 4\pi e\delta(x)\delta(y)\delta\left(\sqrt{1 - \frac{v^2}{c^2}}u\right) \tag{2091}$$

$$= \frac{4\pi}{\sqrt{1 - \frac{v^2}{c^2}}}e\delta(x)\delta(y)\delta\left(u\right) \tag{2092}$$

Using the Green function of the Coulomb equation (13.3) we obtain

$$\phi = \frac{e}{\sqrt{1 - \frac{v^2}{c^2}}\sqrt{u^2 + x^2 + y^2}} \tag{2093}$$

$$= \frac{e}{\sqrt{w^2 + (1 - \frac{v^2}{c^2})(x^2 + y^2)}} \tag{2094}$$

$$= \frac{e}{\sqrt{(z - vt)^2 + (1 - \frac{v^2}{c^2})(x^2 + y^2)}} \tag{2095}$$

For the vector potential we can calculate similarly

$$-\Box\vec{A} = 4\pi\frac{\vec{j}}{c} \tag{2096}$$

$$-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial u^2}\right)\vec{A} = 4\pi e\frac{\vec{v}}{c}\delta(x)\delta(y)\delta\left(\sqrt{1-\frac{v^2}{c^2}}u\right) \tag{2097}$$

$$= \frac{4\pi}{\sqrt{1-\frac{v^2}{c^2}}}e\frac{\vec{v}}{c}\delta(x)\delta(y)\delta\left(u\right) \tag{2098}$$

which gives $\vec{A} = \vec{v}/c\phi$.

Exercise 31.2 Fields of moving point charge

$$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} \tag{2099}$$

$$= \frac{e}{2}\left((z-vt)^2+(1-\frac{v^2}{c^2})(x^2+y^2)\right)^{-3/2}[(1-\frac{v^2}{c^2})2x,(1-\frac{v^2}{c^2})2y,2(z-vt)(1-\frac{v^2}{c^2})] \tag{2100}$$

$$= e(1-\frac{v^2}{c^2})\left((z-vt)^2+(1-\frac{v^2}{c^2})(x^2+y^2)\right)^{-3/2}[x,y,(z-vt)] \tag{2101}$$

$$\vec{B} = \nabla\times\vec{A} \tag{2102}$$

$$= -e\frac{v}{c}(1-\frac{v^2}{c^2})\left((z-vt)^2+(1-\frac{v^2}{c^2})(x^2+y^2)\right)^{-3/2}[y,x,0] \tag{2103}$$

Exercise 31.4 Wave equation for fields

With

$$\nabla\times\vec{B} = \frac{1}{c}\frac{\partial}{\partial t}\vec{E}+\frac{4\pi}{c}\vec{j}_e \tag{2104}$$

$$\nabla\cdot\vec{E} = 4\pi\rho_e \tag{2105}$$

$$-\nabla\times\vec{E} = \frac{1}{c}\frac{\partial}{\partial t}\vec{B}+\frac{4\pi}{c}\vec{j}_m \tag{2106}$$

$$\nabla\cdot\vec{B} = 4\pi\rho_m \tag{2107}$$

we obtain

$$\nabla\times\nabla\times\vec{B} = \nabla(\nabla\cdot\vec{B})-\triangle\vec{B} \tag{2108}$$

$$= 4\pi\nabla\rho_m-\triangle\vec{B} \tag{2109}$$

$$= \frac{1}{c}\frac{\partial}{\partial t}\nabla\times\vec{E}+\frac{4\pi}{c}\nabla\times\vec{j}_e \tag{2110}$$

$$= -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{B}-\frac{4\pi}{c^2}\frac{\partial}{\partial t}\vec{j}_m+\frac{4\pi}{c}\nabla\times\vec{j}_e \tag{2111}$$

$$\to -\Box\vec{B} = -4\pi\nabla\rho_m+\frac{4\pi}{c}(\nabla\times\vec{j}_e-\frac{1}{c}\frac{\partial}{\partial t}\vec{j}_m) \tag{2112}$$

$$\nabla \times \nabla \times \vec{E} = \nabla(\nabla \cdot \vec{E}) - \triangle \vec{E} \tag{2113}$$

$$= 4\pi \nabla \rho_e - \triangle \vec{E} \tag{2114}$$

$$= -\frac{1}{c}\frac{\partial}{\partial t}\nabla \times \vec{B} - \frac{4\pi}{c}\nabla \times \vec{j}_m \tag{2115}$$

$$= -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} - \frac{4\pi}{c^2}\frac{\partial}{\partial t}\vec{j}_e - \frac{4\pi}{c}\nabla \times \vec{j}_m \tag{2116}$$

$$\rightarrow -\Box \vec{E} = -4\pi \nabla \rho_e + \frac{4\pi}{c}(\nabla \times \vec{j}_m - \frac{1}{c}\frac{\partial}{\partial t}\vec{j}_e) \tag{2117}$$

Exercise 31.5 Lienard-Wiechert potentials

$$\phi(\vec{r}, t) = \int d\vec{r}' dt' \frac{\delta(\frac{1}{c}|\vec{r} - \vec{r}'| - (t - t'))}{|\vec{r} - \vec{r}'|}\rho(\vec{r}', t') \tag{2118}$$

$$= \int d\vec{r}' dt' \frac{\delta(t' - t + \frac{1}{c}|\vec{r} - \vec{r}'|)}{|\vec{r} - \vec{r}'|} e\delta(\vec{r}' - \vec{r}_B(t')) \tag{2119}$$

$$= e\int d\vec{r}' \frac{1}{|\vec{r} - \vec{r}'|}\delta(\vec{r}' - \vec{r}_B(t - \frac{1}{c}|\vec{r} - \vec{r}'|)) \tag{2120}$$

$$\delta(\vec{r}' - \vec{r}_B(t - \frac{1}{c}|\vec{r} - \vec{r}'|)) = \delta(f(\vec{r}')) \tag{2121}$$

$$= \sum \frac{\delta(\vec{r}')}{|f'(\vec{r}')|} \tag{2122}$$

$$= e\int d\vec{r}' \frac{1}{|\vec{r} - \vec{r}'|}\sum \tag{2123}$$

Exercise 38.1 Total radiated power

We observe

$$\frac{\lambda}{(1 + \lambda\beta)^4} = -\frac{1}{\beta(1 + \lambda\beta)^4} + \frac{1}{\beta(1 + \lambda\beta)^3}. \tag{2124}$$

Then

$$f(\lambda) = \frac{2}{(1 + \lambda\beta)^3}\left(-\frac{\beta^2}{2} + \frac{\lambda\beta}{8}\frac{\beta^2 - 1}{1 + \lambda\beta}\right) \tag{2125}$$

$$= -\beta^2 \frac{1}{(1 + \lambda\beta)^3} + \frac{\beta(\beta^2 - 1)}{4}\frac{\lambda}{(1 + \lambda\beta)^4} \tag{2126}$$

$$= \left(-\beta^2 + \frac{\beta^2 - 1}{4}\right)\frac{1}{(1 + \lambda\beta)^3} - \frac{(\beta^2 - 1)}{4}\frac{1}{(1 + \lambda\beta)^4} \tag{2127}$$

$$\int_{-1}^{1} f(\lambda) d\lambda = -\frac{1 + 3\lambda^2}{4} \tag{2128}$$

15.4 Smythe - Static and Dynamic Electricity

Exercise 1.1 Two coaxial rings and a point charge

Total charge of an axial ringlike charge distribution

$$Q = \int \rho_0(\varphi')\delta(z' - 0)\delta(r' - a) d\varphi' dz' dr' \tag{2129}$$

$$= 2\pi a \rho_0 \tag{2130}$$

which means that the 1-dimensional charge density is $\rho_0 = Q/2\pi a$. The axial potential of a single ring is then

$$\phi(z) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho_0\delta(z'-0)\delta(r'-a)}{\sqrt{a^2+z^2}} r d\varphi' dz' dr' \tag{2131}$$
$$= \frac{1}{4\pi\varepsilon_0} 2\pi a\rho_0 \frac{1}{\sqrt{a^2+z^2}} \tag{2132}$$
$$= \frac{Q}{4\pi\varepsilon_0}\frac{1}{\sqrt{a^2+z^2}} \tag{2133}$$

therefore we get for the energies

$$W_1 = \frac{qQ_1}{4\pi\varepsilon_0}\frac{1}{a} + \frac{qQ_2}{4\pi\varepsilon_0}\frac{1}{\sqrt{a^2+b^2}} \tag{2134}$$
$$W_2 = \frac{qQ_1}{4\pi\varepsilon_0}\frac{1}{\sqrt{a^2+b^2}} + \frac{qQ_2}{4\pi\varepsilon_0}\frac{1}{a} \tag{2135}$$

solving the linear system for the charges $Q_{1,2}$ we obtain

$$Q_1 = \frac{4\pi\varepsilon_0}{qb^2}\sqrt{a^2+b^2}\left(\sqrt{a^2+b^2}W_1 - aW_2\right) \tag{2136}$$
$$Q_2 = \frac{4\pi\varepsilon_0}{qb^2}\sqrt{a^2+b^2}\left(-aW_1 + \sqrt{a^2+b^2}W_2\right). \tag{2137}$$

Exercise 1.3 Flux of two point charges through circle

For the flux we have

$$N \equiv \int \vec{E}\cdot d\vec{A} \tag{2138}$$
$$= \int E\cos(\vec{E},\vec{n})\; dA \tag{2139}$$
$$= 2\pi\int \frac{q}{4\pi\varepsilon_0(a^2+r^2)}\frac{a}{\sqrt{a^2+r^2}} rdr - 2\pi\int \frac{Q}{4\pi\varepsilon_0(a^2+r^2)}\frac{a}{\sqrt{a^2+r^2}} rdr \tag{2140}$$
$$= \frac{2\pi a}{4\pi\varepsilon_0}(q-Q)\int_0^a \frac{1}{(a^2+r^2)^{3/2}} rdr \tag{2141}$$
$$= \frac{1}{4\varepsilon_0}(q-Q)\left(2-\sqrt{2}\right) \tag{2142}$$

therefore

$$Q = q - \frac{4N\varepsilon_0}{2-\sqrt{2}}. \tag{2143}$$

Exercise 1.4 Concentric charged rings

The axial potential of a single ring is with radius a and charge $Q = 2\pi a\rho_0$ is

$$\phi(x) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho_0\delta(z'-0)\delta(r'-a)}{\sqrt{a^2+x^2}} r d\varphi' dz' dr' \tag{2144}$$
$$= \frac{1}{4\pi\varepsilon_0} 2\pi a\rho_0 \frac{1}{\sqrt{a^2+x^2}} \tag{2145}$$
$$= \frac{Q}{4\pi\varepsilon_0}\frac{1}{\sqrt{a^2+x^2}} \tag{2146}$$

The total potential and the resulting electrical field is therefore

$$\phi(x) = -\frac{Q}{4\pi\varepsilon_0}\frac{1}{\sqrt{a_1^2+x^2}} + \frac{\sqrt{27}Q}{4\pi\varepsilon_0}\frac{1}{\sqrt{a_2^2+x^2}} \tag{2147}$$

$$E_x = -\frac{\partial\phi}{\partial x} \tag{2148}$$

$$= \frac{Qx}{4\pi\varepsilon_0}\left(-\frac{1}{(a_2^2+x^2)^{3/2}} + \frac{\sqrt{27}}{(a_2^2+x^2)^{3/2}}\right) \tag{2149}$$

which only vanishes for

$$x = 0, \pm\sqrt{\frac{-3a_1^2 + a2^2}{2}}. \tag{2150}$$

Due to the radial symmetry the other field components at this points vanish too.

Exercise 12.1 Linear quadrupole

$$\beta = \omega\sqrt{\mu\epsilon} \tag{2151}$$

$$q_{zz}^{(2)} = a^2 q \sin\omega t \tag{2152}$$

$$8\pi\epsilon\vec{Z}_{zz} = a^2 q \sin\omega t\left(\frac{\beta}{r} - \frac{j}{r^2}\right)(\vec{r}_1\cos\theta - \vec{\theta}\sin\theta)\cos\theta e^{-j\beta r} \tag{2153}$$

$$\tag{2154}$$

15.5 Thorne, Blandford - Modern Classical Physics

Exercise 1.1 Practice: Energy Change for Charged Particle

With $E = p^2/2m$ and (1.7c) we obtain

$$\frac{dE}{dt} = \frac{d}{dt}\frac{p^2}{2m} = \frac{2\vec{p}\cdot d\vec{p}/dt}{2m} \tag{2155}$$

$$= \frac{q}{m}\vec{p}\cdot(\vec{E} + \vec{v}\times\vec{B}) \tag{2156}$$

$$= q\vec{v}\cdot(\vec{E} + \vec{v}\times\vec{B}) \tag{2157}$$

$$= q\vec{v}\cdot\vec{E}. \tag{2158}$$

As $\vec{v}\times\vec{B}$ is orthogonal to $\vec{v}$ (and $\vec{B}$) the scalar product $\vec{v}\cdot(\vec{v}\times\vec{B})$ vanishes.

Exercise 1.2 Practice: Particle Moving in a Circular Orbit

(a) With

$$\frac{d\vec{n}}{ds} = \frac{\vec{n}' - \vec{n}}{R\cdot d\phi} = \frac{\vec{v}' - \vec{v}}{vR\cdot d\phi} \tag{2159}$$

we can calculate the norm

$$\left|\frac{d\vec{n}}{ds}\right| = \frac{\sqrt{v^2+v^2-2v^2\cos(d\phi)}}{vR\cdot d\phi} = \frac{v\sqrt{1-\cos(d\phi)}}{vR\cdot d\phi} = \frac{v\sqrt{2[1-\cos(d\phi)]}}{vR\cdot d\phi} \tag{2160}$$

$$\approx \frac{vd\phi}{vR\cdot d\phi} = \frac{1}{R} \tag{2161}$$

and the scalar product

$$\frac{d\vec{n}}{ds} \cdot \vec{n} = \frac{\vec{n}' \cdot \vec{n} - \vec{n} \cdot \vec{n}}{R \cdot d\phi} = \frac{n^2 \cos(d\phi) - n^2}{vR \cdot d\phi} \tag{2162}$$

$$\approx \frac{(1 - d\phi^2/2) - 1}{vR \cdot d\phi} = \frac{d\phi}{2vR} \tag{2163}$$

which vanished for $d\phi \to 0$ and therefore implies that $d\vec{n}$ is orthogonal to $\vec{n}$ (and therefore points to the center).

(b) From (a) we know

$$\vec{R} = R^2 \frac{d\vec{n}}{ds} = R^2 \frac{d\vec{v}}{v \cdot ds} = R^2 \frac{d\vec{v}}{v \cdot ds} = \frac{R^2}{v} \frac{d\vec{v}}{dt} \frac{dt}{ds} = \left(\frac{R}{v}\right)^2 \vec{a} \tag{2164}$$

Taking the absolute value we have

$$R = \frac{R^2}{v^2} a \quad \rightarrow \quad R = \frac{v^2}{a} \tag{2165}$$

and therefore

$$\vec{R} = \frac{R^2}{v^2} \vec{a} = \frac{v^4}{v^2 a^2} \vec{a} = \left(\frac{v}{a}\right)^2 \vec{a}. \tag{2166}$$

Exercise 1.3 Derivation: Component Manipulation Rules

1. (1.9g I) - using (1.9b), (1.9a) and (1.9c)

$$\mathbf{A} \cdot \mathbf{B} = (A_j \mathbf{e}_j) \cdot (B_k \mathbf{e}_k) = A_j B_k \mathbf{e}_j \cdot \mathbf{e}_k = A_j B_k \delta_{jk} = A_j B_j \tag{2167}$$

2. (1.9g II) - using (1.9d) and (1.5a)

$$\mathbf{T} = T_{ijk} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \tag{2168}$$

$$\mathbf{T}(\mathbf{A}, \mathbf{B}, \mathbf{C}) = T_{ijk} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k (\mathbf{A}, \mathbf{B}, \mathbf{C}) \tag{2169}$$

$$= T_{ijk} (\mathbf{A} \cdot \mathbf{e}_i)(\mathbf{B} \cdot \mathbf{e}_j)(\mathbf{C} \cdot \mathbf{e}_k) \tag{2170}$$

$$= T_{ijk} A_i B_j C_k \tag{2171}$$

3. (1.9h) - using (1.9d), (1.6b), (1.9a) and (1.5a)

$$\mathbf{R} = R_{abcd} \mathbf{e}_a \otimes \mathbf{e}_b \otimes \mathbf{e}_c \otimes \mathbf{e}_d \tag{2172}$$

$$\text{1\&3 contraction}(\mathbf{R}) = R_{abcd} (\mathbf{e}_a \cdot \mathbf{e}_c) \mathbf{e}_b \otimes \mathbf{e}_d \tag{2173}$$

$$= R_{abcd} \delta_{ac} \mathbf{e}_b \otimes \mathbf{e}_d \tag{2174}$$

$$= R_{abad} \mathbf{e}_b \otimes \mathbf{e}_d \tag{2175}$$

$$\text{components of [1\&3 contraction}(\mathbf{R})] = R_{abad} \mathbf{e}_b \otimes \mathbf{e}_d (\mathbf{e}_j, \mathbf{e}_k) \tag{2176}$$

$$= R_{abad} (\mathbf{e}_b \cdot \mathbf{e}_j)(\mathbf{e}_d \cdot \mathbf{e}_k) \tag{2177}$$

$$= R_{abad} \delta_{bj} \delta_{dk} \tag{2178}$$

$$= R_{ajak} \tag{2179}$$

Exercise 1.4 Example and Practice: Numerics of Component Manipulations

$$\mathbf{C} = \mathbf{S}(\mathbf{A}, \mathbf{B}, \_) \tag{2180}$$

$$= S_{ijk} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k (\mathbf{A}, \mathbf{B}, \_) \tag{2181}$$

$$= S_{ijk} (\mathbf{A} \cdot \mathbf{e}_i)(\mathbf{B} \cdot \mathbf{e}_j) \mathbf{e}_k \tag{2182}$$

$$= S_{ijk} A_i B_j \mathbf{e}_k \tag{2183}$$

$$\begin{aligned}C_k &= S_{11k}A_1B_1 + S_{12k}A_1B_2 && (2184)\\ C_1 &= 0, \quad C_2 = 0, \quad C_3 = S_{123}A_1B_2 = 15 && (2185)\end{aligned}$$

$$\begin{aligned}\mathbf{D} &= \mathbf{S}(\mathbf{A}, \_, \mathbf{B}) && (2186)\\ &= S_{ijk}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k(\mathbf{A}, \_, \mathbf{B}) && (2187)\\ &= S_{ijk}(\mathbf{A}\cdot\mathbf{e}_i)(\mathbf{B}\cdot\mathbf{e}_k)\mathbf{e}_j && (2188)\\ &= S_{ijk}A_iB_k\mathbf{e}_j && (2189)\end{aligned}$$

$$D_j = S_{1j1}A_1B_1 + S_{1j2}A_1B_2 = 0 \qquad (2190)$$

$$\begin{aligned}\mathbf{W} &= \mathbf{A}\otimes\mathbf{B} && (2191)\\ &= (A_i\mathbf{e}_i)\otimes(B_j\mathbf{e}_j) && (2192)\\ &= A_iB_j\mathbf{e}_i\otimes\mathbf{e}_j && (2193)\end{aligned}$$

$$W_{11} = 12, \quad W_{12} = 15, \qquad (2194)$$

Exercise 1.5 Practice: Meaning of Slot-Naming Index Notation

(a) Somewhat guessing

$$\begin{aligned}A_iB_{jk} &\to A_iB_{jk}\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k && (2195)\\ &= (A_i\mathbf{e}_i)\otimes(B_{jk}\mathbf{e}_j\otimes\mathbf{e}_k) && (2196)\\ &= A(\_)\otimes B(\_,\_) && (2197)\end{aligned}$$

$$\begin{aligned}A_iB_{ji} &\to A_iB_{ji}\mathbf{e}_j && (2198)\\ &= (\mathbf{A}\cdot\mathbf{e}_i)B_{ji}\mathbf{e}_j && (2199)\\ &= B_{ji}\mathbf{e}_j\otimes\mathbf{e}_i(\_,\mathbf{A}) && (2200)\\ &= \mathbf{B}(\_,\mathbf{A}) && (2201)\end{aligned}$$

$$S_{ijk} = S_{kji} \to ... \qquad (2202)$$

$$A_iB_i = AiB_jg_{ij} \to \mathbf{A}\cdot\mathbf{B} = \mathbf{g}(\mathbf{A},\mathbf{B}) \qquad (2203)$$

(b) Applying the standard machinery

$$\begin{aligned}\mathbf{T}(\_,\_,\mathbf{A}) &= T_{ijk}\mathbf{e}_i\otimes\mathbf{e}_j(\mathbf{A}\cdot\mathbf{e}_k) && (2204)\\ &= T_{ijk}A_k\mathbf{e}_i\otimes\mathbf{e}_j && (2205)\\ &\to T_{ijk}A_k && (2206)\end{aligned}$$

$$\begin{aligned}\mathbf{S}(\mathbf{B},\_) &= S_{ab}(\mathbf{B}\cdot\mathbf{e}_a)\mathbf{e}_b && (2207)\\ &= S_{ab}B_a\mathbf{e}_b && (2208)\\ \mathbf{T}(\_,\mathbf{S}(\mathbf{B},\_),\_) &= T_{ijk}\mathbf{e}_i\otimes\mathbf{e}_k(S_{ab}B_a\mathbf{e}_b\cdot\mathbf{e}_j) && (2209)\\ &= T_{ijk}\mathbf{e}_i\otimes\mathbf{e}_k(S_{ab}B_a\delta_{bj}) && (2210)\\ &= T_{ijk}S_{aj}B_a\mathbf{e}_i\otimes\mathbf{e}_k && (2211)\\ &\to T_{ijk}S_{aj} && (2212)\end{aligned}$$

Exercise 1.6 Example and Practice: Rotation in x-y Plane

(a)

(b)

(c)

(d)

Exercise 1.7 Derivation: Properties of the Levi-Civita Tensor

Exercise 1.8 Example and Practice: Vectorial Identities for the Cross Product and Curl

(a)

(b)

(c)

Exercise 1.9 Example and Practice: Levi-Civita Tensor in 2-Dimensional Euclidean Space

(a)

(b)

Exercise 1.10 Derivation and Practice: Volume Elements in Cartesian Coordinates

Exercise 1.11 Example and Practice: Integral of a Vector Field over a Sphere

(a)

(b)

(c)

(d)

Exercise 1.12 Example: Faraday's Law of Induction

Exercise 1.13 Example: Equations of Motion for a Perfect Fluid

(a)

(b)

(c)

(d)

(e)

Exercise 1.14 Problem: Electromagnetic Stress Tensor

(a)

(b)

Exercise 1.15 Practice: Geometrized Units

(a) $t_P = \sqrt{G\hbar} \quad \rightarrow \quad \sqrt{\frac{G\hbar}{c^5}} = 5.39 \cdot 10^{-44}\text{s} \equiv 1.61 \cdot 10^{-35}\text{m}$

(b) $E = 2mc^2$

(c)

(d)

(e) $1\text{m} \equiv 3.33 \cdot 10^{-9}\text{s}$ and $1\text{yr} \equiv 9.45 \cdot 10^{15}\text{m}$

Exercise 3.3 Practice and Example: Regimes of Particulate and Wave - Like Behavior

(a) The Schwarzschild radius of the BH is

$$R_S = \frac{2GM}{c^2} = 44,466\text{m} \tag{2213}$$

which gives a disk radius of $R = 7R_S = 311\text{km}$. With

$$F_{\text{Earth}} = \frac{dP}{dA} = \frac{dW}{dA\,dt} = \frac{dN \cdot E_{ph}c}{dA \cdot dl} = \left(\frac{dN}{d\mathcal{V}_x}\right)_{\text{Earth}} \cdot E_{\text{ph}}c \tag{2214}$$

$$\left(\frac{dN}{d\mathcal{V}_x}\right)_{\text{Earth}} = \frac{F_{\text{Earth}}}{cE_{\text{ph}}} = 0.00104\text{m}^{-3} \tag{2215}$$

$$F_{\text{CX1}} = \frac{r^2}{R^2} F_{\text{Earth}} \tag{2216}$$

$$\left(\frac{dN}{d\mathcal{V}_x}\right)_{\text{CX1}} = \frac{F_{\text{CX1}}}{cE_{\text{ph}}} = \frac{r^2}{R^2}\frac{F_{\text{Earth}}}{cE_{\text{ph}}} = 3.72 \cdot 10^{25}\text{m}^{-3} \tag{2217}$$

The momentum of the photons is $p = E/c$.

The mean occupation number is then

$$\eta = \frac{h^3}{g_s}\mathcal{N} = \frac{h^3}{g_s}\frac{dN}{d\mathcal{V}_x d\mathcal{V}_p} = \tag{2218}$$

Exercise 7.1 Practice: Group and Phase Velocities

With the definition of phase and group velocities

$$\vec{v}_{ph} = \frac{\omega}{k}\frac{\vec{k}}{k} \tag{2219}$$

$$\vec{v}_g = \nabla_k \omega \tag{2220}$$

$$\omega_1(\vec{k}) = C|\vec{k}| \tag{2221}$$

$$\rightarrow \vec{v}_{ph} = \frac{C|\vec{k}|}{k}\frac{\vec{k}}{k} = C\frac{\vec{k}}{k} \tag{2222}$$

$$\rightarrow \vec{v}_g = C\frac{2\vec{k}}{2\sqrt{k^2}} = C\frac{\vec{k}}{k} \tag{2223}$$

$$\omega_2(\vec{k}) = \sqrt{g|\vec{k}|} \tag{2224}$$

$$\rightarrow \vec{v}_{ph} = \frac{\sqrt{g|\vec{k}|}}{k}\frac{\vec{k}}{k} = \sqrt{\frac{g}{k}}\frac{\vec{k}}{k} \tag{2225}$$

$$\rightarrow \vec{v}_g = \sqrt{g}\frac{1}{2\sqrt{|\vec{k}|}}\frac{\vec{k}}{k} = \frac{1}{2}\sqrt{\frac{g}{k}}\frac{\vec{k}}{k} \tag{2226}$$

$$\omega_3(\vec{k}) = \sqrt{\frac{D}{\Lambda}}\vec{k}^2 \tag{2227}$$

$$\rightarrow \vec{v}_{ph} = \sqrt{\frac{D}{\Lambda}}\frac{\vec{k}^2}{k}\frac{\vec{k}}{k} = \sqrt{\frac{D}{\Lambda}}k\frac{\vec{k}}{k} \tag{2228}$$

$$\rightarrow \vec{v}_g = \sqrt{\frac{D}{\Lambda}}2\vec{k} = 2\sqrt{\frac{D}{\Lambda}}k\frac{\vec{k}}{k} \tag{2229}$$

$$\omega_4(\vec{k}) = \vec{a}\cdot\vec{k} \tag{2230}$$

$$\rightarrow \vec{v}_{ph} = \frac{\vec{a}\cdot\vec{k}}{k}\frac{\vec{k}}{k} = \left(\vec{a}\cdot\frac{\vec{k}}{k}\right)\frac{\vec{k}}{k} \tag{2231}$$

$$\rightarrow \vec{v}_g = \vec{a} \tag{2232}$$

Exercise 7.2 Example: Gaussian Wave Packet and Its Dispersion

(a) Taylor expansion of the dispersion relation gives

$$\omega = \Omega(k) = \omega(k_0) + \left.\frac{\partial\omega(k)}{\partial k}\right|_{k=k_0}(k-k_0) + \frac{1}{2}\left.\frac{\partial^2\omega(k)}{\partial k^2}\right|_{k=k_0}(k-k_0)^2 \tag{2233}$$

$$= \omega(k_0) + V_g|_{k=k_0}(k-k_0) + \frac{1}{2}\left.\frac{\partial V_g(k)}{\partial k}\right|_{k=k_0}(k-k_0)^2. \tag{2234}$$

(b) The wave packet can then be written as

$$\psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk A(k) e^{i\alpha(k)} e^{i(kx-\omega t)} \tag{2235}$$

$$= \frac{C}{2\pi}\int_{-\infty}^{\infty} dk e^{-\frac{(k-k_0)^2}{2\Delta k^2}} e^{i[\alpha_0 - x_0(k-k0)]} e^{i(kx-[\omega_0+V_g(k-k_0)+\frac{1}{2}V_g'(k-k_0)^2]t)} \tag{2236}$$

$$= \frac{C}{2\pi}\int_{-\infty}^{\infty} dk e^{-\frac{(k-k_0)^2}{2\Delta k^2}} e^{i(\alpha_0+k_0x-\omega_0t-(V_gt-x+x_0)(k-k_0)-\frac{1}{2}V_g't(k-k_0)^2)} \tag{2237}$$

$$= \frac{C}{2\pi}e^{i(\alpha_0+k_0x-\omega_0t)}\int_{-\infty}^{\infty} dk e^{-i(V_gt-x+x_0)(k-k_0)} e^{-\frac{1}{2}(k-k_0)^2\left(\frac{1}{\Delta k^2}+iV_g't\right)} \tag{2238}$$

$$= \frac{C}{2\pi}e^{i(\alpha_0+k_0x-\omega_0t)}\int_{-\infty}^{\infty} d\kappa e^{i(x-x_0-V_gt)\kappa} e^{-\frac{1}{2}\kappa^2\left(\frac{1}{\Delta k^2}+iV_g't\right)} \tag{2239}$$

$$\tag{2240}$$

(c) With

$$\int_{-\infty}^{\infty} dy e^{-(a+ic)y^2} e^{-iby} = \sqrt{\frac{\pi}{a^2+c^2}}\sqrt{a-ic}\, e^{-\frac{b^2}{4(a^2+c^2)}(a-ic)} \qquad a>0, a,b,c\in\mathbb{R} \tag{2241}$$

and the substitutions $a = \frac{1}{2\,\Delta k^2}, c = \frac{V_g' t}{2}$ and

$$a^2+c^2 = \frac{1}{4\,\Delta k^2}\frac{1}{\Delta k^2}\left(1+[V_g'(\Delta k)^2 t]^2\right) \tag{2242}$$

$$= \frac{1}{4\,\Delta k^2} L^2 \tag{2243}$$

$$= \frac{a}{2} L^2 \tag{2244}$$

we obtain

$$\psi(x,t) = \frac{C}{2\pi} e^{i(\alpha_0+k_0x-\omega_0 t)} \sqrt{\frac{\pi}{aL^2}}\sqrt{a-ic}\, e^{-\frac{ab^2}{4(a^2+c^2)}} e^{-\frac{(-ic)b^2}{4(a^2+c^2)}} \tag{2245}$$

$$= \frac{C}{2\pi} e^{i(\alpha_0+k_0x-\omega_0 t)} e^{\frac{2icb^2}{4aL^2}} \sqrt{\frac{\pi}{aL^2}}\sqrt{a-ic}\, e^{-\frac{(x-x_0-V_g t)^2}{2L^2}} \tag{2246}$$

and therefore (with $|\sqrt{a-ic}| = \sqrt{|a-ic|} = \sqrt{\sqrt{aL^2}} = a^{1/4}\sqrt{L}$)

$$|\psi(x,t)| = \frac{C}{2\pi}\sqrt{\frac{\pi}{aL^2}} a^{1/4}\sqrt{L}\, e^{-\frac{(x-x_0-V_g t)^2}{2L^2}} \tag{2247}$$

$$= \frac{C}{2\pi}\sqrt{\frac{\pi}{\sqrt{a}L}}\, e^{-\frac{(x-x_0-V_g t)^2}{2L^2}} \tag{2248}$$

$$= \frac{C}{2}\sqrt{\frac{1}{\pi\sqrt{a}}\frac{1}{\sqrt{L}}}\, e^{-\frac{(x-x_0-V_g t)^2}{2L^2}}. \tag{2249}$$

(d) At $t=0$ the packets width in position space is $L = 1/\Delta k$ while the width in momentum space is Δk which means the product is $\Delta x \cdot \Delta k = 1$.

(e) With the group velocity

$$V_g = \frac{1}{2}\sqrt{\frac{g}{k_0}} \tag{2250}$$

$$V_g' = \frac{\partial V_g}{\partial k}|_{k=k_0} = -\frac{1}{4}\sqrt{\frac{g}{k_0^3}} \tag{2251}$$

the width of the package is proportional to

$$L = \frac{1}{\Delta k}\sqrt{1+\left(V_g'(\Delta k)^2 t\right)^2} \tag{2252}$$

$$\to T_D = \frac{\sqrt{3}}{V_g'(\Delta k)^2} \tag{2253}$$

$$\to T_D = \frac{4}{\Delta k^2}\sqrt{\frac{3k_0^3}{g}}. \tag{2254}$$

The condition for the spread limitation is

$$S_{\text{HI-CA}} \leq V_g \cdot T_D \tag{2255}$$

$$= \frac{1}{2}\sqrt{\frac{g}{k_0}}\frac{4}{\Delta k^2}\sqrt{\frac{3k_0^3}{g}} \tag{2256}$$

$$= 2\sqrt{3}\frac{k_0}{\Delta k^2} \tag{2257}$$

Exercise 7.3 Derivation and Example: Amplitude Propagation for Dispersionless Waves Expressed as Constancy of Something along a Ray

(a)

(b)

(c)

(d)

Exercise 7.4 Example: Energy Density and Flux, and Adiabatic Invariant, or a Dispersionless Wave

(a) For a generic Lagrangian density $\mathcal{L}$ we find

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial x_i}\right)}\delta\left(\frac{\partial\psi}{\partial x_i}\right) \tag{2258}$$

$$= \frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial x_i}\right)}\frac{\partial}{\partial x_i}(\delta\psi) \tag{2259}$$

$$\rightarrow \delta\int\mathcal{L}d^4x = \int\delta\mathcal{L}d^4x \tag{2260}$$

$$= \int\left[\frac{\partial\mathcal{L}}{\partial\psi} - \frac{\partial}{\partial x_i}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial x_i}\right)}\right)\right]\delta\psi \tag{2261}$$

$$\rightarrow 0 = \frac{\partial\mathcal{L}}{\partial\psi} - \frac{\partial}{\partial x_i}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial x_i}\right)}\right) \tag{2262}$$

the general Euler-Lagrange equation. For the given density we can calculate the derivatives

$$\mathcal{L} = W\left[\frac{1}{2}\left(\frac{\partial\psi}{\partial t}\right)^2 - \frac{1}{2}C^2(\nabla\psi)^2\right] \tag{2263}$$

$$\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial t}\right)} = W\frac{\partial\psi}{\partial t} \tag{2264}$$

$$\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial x_i}\right)} = -WC^2\frac{\partial\psi}{\partial x_i} \tag{2265}$$

and obtain

$$\frac{\partial}{\partial t}\left(W\frac{\partial\psi}{\partial t}\right) - \frac{\partial}{\partial x_i}\left(WC^2\frac{\partial\psi}{\partial x_i}\right) = 0. \tag{2266}$$

(b) Using the definitions we obtain

$$\frac{\partial U}{\partial t} = \frac{\partial^2\psi}{\partial t^2}\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial t)} + \frac{\partial\psi}{\partial t}\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial t)}\right) - \frac{\partial\mathcal{L}}{\partial t} \tag{2267}$$

$$\frac{\partial F_j}{\partial x_j} = \frac{\partial^2\psi}{\partial t\partial x_j}\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_j)} + \frac{\partial\psi}{\partial t}\frac{\partial}{\partial x_j}\left(\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_j)}\right) \tag{2268}$$

and therefore

$$\frac{\partial U}{\partial t}+\frac{\partial F_j}{\partial x_j}=\frac{\partial^2\psi}{\partial t^2}\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial t)}+\frac{\partial\psi}{\partial t}\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial t)}\right)-\frac{\partial\mathcal{L}}{\partial t}+\frac{\partial^2\psi}{\partial t\partial x_j}\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_j)}+\frac{\partial\psi}{\partial t}\frac{\partial}{\partial x_j}\left(\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_j)}\right) \tag{2269}$$

$$=\frac{\partial^2\psi}{\partial t^2}\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial t)}+\frac{\partial\psi}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\psi}\right)-\frac{\partial\mathcal{L}}{\partial t}+\frac{\partial^2\psi}{\partial t\partial x_j}\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_j)} \tag{2270}$$

$$=\frac{\partial\psi}{\partial t}\left(-\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial t)}\right)+\frac{\partial\psi}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\psi}\right)-\frac{\partial\mathcal{L}}{\partial t}+\frac{\partial\psi}{\partial t}\left(-\frac{\partial}{\partial x_i}\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_j)}\right) \tag{2271}$$

$$=\frac{\partial\psi}{\partial t}\left(-\frac{\partial\mathcal{L}}{\partial\psi}\right)+\frac{\partial\psi}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\psi}\right)-\frac{\partial\mathcal{L}}{\partial t} \tag{2272}$$

$$=-\frac{\partial\mathcal{L}}{\partial t} \tag{2273}$$

(c) Substituting $\mathcal{L}$ into the definitions yields

$$U=\frac{\partial\psi}{\partial t}\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial t)}-\mathcal{L} \tag{2274}$$

$$=W\left(\frac{\partial\psi}{\partial t}\right)^2-\mathcal{L} \tag{2275}$$

$$=W\left[\frac{1}{2}\left(\frac{\partial\psi}{\partial t}\right)^2+\frac{1}{2}C^2\left(\nabla\psi\right)^2\right] \tag{2276}$$

$$F_j=\frac{\partial\psi}{\partial t}\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_j)} \tag{2277}$$

$$=-\frac{\partial\psi}{\partial t}WC^2\frac{\partial\psi}{\partial x_j}. \tag{2278}$$

(d) The momentum density is given by

$$\pi=\frac{\partial\mathcal{L}}{\partial\frac{\partial\phi}{\partial t}}=W\frac{\partial\phi}{\partial t} \tag{2279}$$

$$\Pi=\int\pi d^3x=\int W\frac{\partial\phi}{\partial t}d^3x \tag{2280}$$

$$J=\int_0^{\omega/2\pi}Ldt=\int_0^{\omega/2\pi}\int\mathcal{L}d^3xdt \tag{2281}$$

$$= \tag{2282}$$

Exercise 8.1 Practice: Convolutions and Fourier Transforms

(a) With $f_1(x) = e^{-\frac{x^2}{2\sigma^2}}$ and $f_2(x) = e^{-\frac{x}{h}}\theta(x)$ we obtain

$$\begin{aligned}
F_1(k) &= \int_{-\infty}^{\infty} f_1(x) e^{-ikx} dx && (2283)\\
&= \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} e^{-ikx} dx && (2284)\\
&= e^{-\frac{k^2\sigma^2}{2}} \int_{-\infty}^{\infty} e^{-\left(\frac{x}{\sqrt{2}\sigma}+\frac{ik\sigma}{\sqrt{2}}\right)^2} dx && (2285)\\
&= e^{-\frac{k^2\sigma^2}{2}} \sqrt{2\sigma^2} \int_{-\infty}^{\infty} e^{-y^2} dy && (2286)\\
&= \sqrt{2\pi\sigma^2} e^{-\frac{\sigma^2 k^2}{2}} && (2287)\\
F_2(k) &= \int_{-\infty}^{\infty} f_2(x) e^{-ikx} dx && (2288)\\
&= \int_{0}^{\infty} e^{-\frac{x}{h}} e^{-ikx} dx && (2289)\\
&= -\frac{1}{h} e^{-\frac{x}{h}} e^{-ikx}\Big|_0^{\infty} - \int_0^{\infty} \left(-\frac{1}{h}\right) e^{-\frac{x}{h}} \frac{1}{(-ik)} e^{-ikx} dx && (2290)\\
&= \frac{1}{h} - \frac{1}{ihk} \int_0^{\infty} e^{-\frac{x}{h}} e^{-ikx} dx && (2291)\\
&= ... && (2292)\\
&= \frac{1}{\frac{1}{h} + ik} && (2293)
\end{aligned}$$

(b)

(c)

$$\begin{aligned}
f_1 \otimes f_2 &= \int_{-\infty}^{\infty} f_2(y-x) f_1(x) dx && (2294)\\
&= \int_{-\infty}^{\infty} e^{-\frac{y-x}{h}} \theta(y-x) e^{-\frac{x^2}{2\sigma^2}} dx && (2295)\\
&= \int_{-\infty}^{y} e^{-\frac{y-x}{h}} e^{-\frac{x^2}{2\sigma^2}} dx && (2296)\\
&= ... && (2297)
\end{aligned}$$

Not done yet

Exercise 11.9 Derivation: Sag in a Cantilever

(a) For a cantilever with Young's modulus E, density ρ, width w and height h the weight per length is given by

$$W = \rho g w h \qquad (2298)$$

and

$$\begin{aligned}
D \equiv E \int z^2 \, dydz &= Ew \, \frac{z^3}{3}\Big|_{-h/2}^{h/2} && (2299)\\
&= Ew \frac{h^3}{3} \frac{2}{8} = \frac{1}{12} Ewh^3. && (2300)
\end{aligned}$$

We now solve

$$\frac{d^4\eta}{dx^4} = \frac{W}{D} \tag{2301}$$
$$= \frac{12\rho g}{Eh^2} \tag{2302}$$

with $\eta(0) = 0$, $\eta'(0 = 0)$, $\eta''(l) = 0$ and $\eta'''(l) = 0$ and obtain

$$\eta'''(x) = \frac{W}{D}(x + c_3) \tag{2303}$$
$$\eta''(x) = \frac{W}{D}\left(\frac{x^2}{2} + c_3 x + c_2\right) \tag{2304}$$
$$\eta'(x) = \frac{W}{D}\left(\frac{x^3}{3} + c_3\frac{x^2}{2} + c_2 x + c_1\right) \tag{2305}$$
$$\eta(x) = \frac{W}{D}\left(\frac{x^4}{24} + c_3\frac{x^3}{6} + c_2\frac{x^2}{2} + c_1 x + c_0\right) \tag{2306}$$

using the boundary conditions we see

$$\eta(0) = 0 \quad \to \quad c_0 = 0 \tag{2307}$$
$$\eta'(0) = 0 \quad \to \quad c_1 = 0 \tag{2308}$$
$$\eta'''(l) = 0 \quad \to \quad c_3 = -l \tag{2309}$$
$$\eta''(l) = 0 \quad \to \quad c_2 = \frac{l^2}{2} \tag{2310}$$

and therefore

$$\eta(x) = \frac{W}{D}\left(\frac{1}{24}x^4 - \frac{l}{6}x^3 + \frac{l^2}{4}x^2\right) \tag{2311}$$
$$\eta(l) = \frac{W}{D}\frac{l^4}{8} = \frac{3\rho g l^4}{2Eh^2} \tag{2312}$$

(b) Now we need to solve

$$\frac{d^4\eta}{dx^4} = \frac{1}{D}W(x). \tag{2313}$$

The solution for the special case in (a) was $\eta \sim Wx^4 \sim \int Wz^3\,dz$ so we try the ansatz

$$\eta(x) = \frac{1}{6D}\int_0^x (x-z)^3 W(z)dz \tag{2314}$$

Calculating the 4th derivative we see that our ansatz is correct.

Exercise 13.1 Example: Earth's Atmosphere

(a) With $PV = Nk_BT$, $\rho = \frac{\mu m_p N}{V}$ and assuming $g = \text{const}$ we obtain

$$\nabla P = \rho \mathbf{g} \tag{2315}$$
$$\frac{dP}{dz} = -\rho g \tag{2316}$$
$$= -\frac{\mu m_p N}{V}g \tag{2317}$$
$$= -\mu m_p g\frac{P}{k_BT} \tag{2318}$$
$$\tag{2319}$$

which can be solved by

$$\frac{dP}{P} = -\frac{\mu m_p g}{k_B T} \tag{2320}$$

$$P(z) = P_0 \exp\left(-\frac{\mu m_p g}{k_B T} z\right). \tag{2321}$$

With $\mu = 0.2 \cdot 2 \cdot 16 + 0.8 \cdot 2 \cdot 14+$ ($20\%O_2/80\%N_2$) and $T = 220$K we have

$$H = 6,400\text{m} \tag{2322}$$

$$P(16\text{km}) = 0.083\text{bar} \tag{2323}$$

$$\frac{P(35\text{km})}{P(16\text{km})} = 0.052 \tag{2324}$$

(b) The isentropic condition $P \sim \rho^\gamma$ acts as an additional condition on top of the equations of state. It can be rewritten as

$$P\rho^{-\gamma} = \text{const} \tag{2325}$$

$$PV^\gamma = \text{const} \tag{2326}$$

$$P\left(\frac{T}{P}\right)^\gamma = \text{const} \tag{2327}$$

$$TP^{\frac{1-\gamma}{\gamma}} = \text{const}. \tag{2328}$$

Differentiating the last equation gives

$$\frac{dT}{dz}P^{\frac{1-\gamma}{\gamma}} + \left(\frac{1-\gamma}{\gamma}\right)P^{\frac{1-2\gamma}{\gamma}}\frac{dP}{dz}T = 0 \tag{2329}$$

$$\to \frac{dT}{dz} = -\left(\frac{1-\gamma}{\gamma}\right)\frac{T}{P}\frac{dP}{dz} \tag{2330}$$

Inserting the

$$\frac{dP}{dz} = -\mu m_p g \frac{P}{k_B T} \tag{2331}$$

which we calculated in (a) we obtain

$$\frac{dT}{dz} = \left(\frac{1-\gamma}{\gamma}\right)\frac{\mu m_p g}{k_B}. \tag{2332}$$

With this we calculate a lapse rate of 9.76K km^{-1}.

Exercise 13.2 Practise: Weight in Vacuum

$$F_b = \rho_{\text{air}} g V_{\text{body}} \tag{2333}$$

$$= \rho_{\text{air}} g \frac{m_{\text{body}}}{\rho_{\text{body}}} \tag{2334}$$

$$= 1\text{N} \tag{2335}$$

where we used a mass of 100kg and $\rho_{\text{air}}/\rho_{\text{body}} = 0.001$.

Exercise 13.4 Example: Polytropes — The Power of Dimensionless Variables

(a) From

$$\frac{dP}{dr} = -\rho\frac{Gm}{r^2} \quad \rightarrow \quad m = -\frac{r^2}{G\rho}\frac{dP}{dr} \tag{2336}$$

$$\frac{dm}{dr} = 4\pi\rho r^2 \tag{2337}$$

we obtain by differentiation

$$\frac{d^2P}{dr^2} = -G\frac{(\frac{d\rho}{dr}m + \rho\frac{dm}{dr})r^2 - 2r\rho m}{r^4} \tag{2338}$$

$$= -\frac{G}{r^4}\left(\left[\frac{d\rho}{dr}m + \rho\frac{dm}{dr}\right]r^2 - 2r\rho m\right) \tag{2339}$$

$$= -\frac{G}{r^4}\left(\left[-\frac{r^2}{G\rho}\frac{dP}{dr}\frac{d\rho}{dr} + \rho 4\pi\rho r^2\right]r^2 + \frac{r^2}{G\rho}\frac{dP}{dr}2r\rho\right) \tag{2340}$$

$$= \left(\frac{1}{\rho}\frac{d\rho}{dr} - \frac{2}{r}\right)\frac{dP}{dr} - 4\pi G\rho^2 \tag{2341}$$

(b) With the polytropic equation of state $P = K\rho^{1+1/n}$ we find for the derivatives of P

$$\frac{dP}{dr} = K\left(1 + \frac{1}{n}\right)\rho^{1/n}\frac{d\rho}{dr} \tag{2342}$$

$$\frac{d^2P}{dr^2} = K\left(1 + \frac{1}{n}\right)\rho^{1/n}\left[\frac{1}{n}\rho^{-1}\left(\frac{d\rho}{dr}\right)^2 + \frac{d^2\rho}{dr^2}\right] \tag{2343}$$

and therefore

$$\frac{1}{n}\rho^{-1}\left(\frac{d\rho}{dr}\right)^2 + \frac{d^2\rho}{dr^2} = \left(\frac{1}{\rho}\frac{d\rho}{dr} - \frac{2}{r}\right)\frac{d\rho}{dr} - \frac{n}{1+n}\frac{4\pi G\rho^{2-1/n}}{K} \tag{2344}$$

$$\frac{d^2\rho}{dr^2} = \left(\frac{n-1}{n}\frac{1}{\rho}\frac{d\rho}{dr} - \frac{2}{r}\right)\frac{d\rho}{dr} - \frac{n}{1+n}\frac{4\pi G}{K}\rho^{2-1/n} \tag{2345}$$

$$\frac{d^2\rho}{dr^2} = \frac{n-1}{n}\frac{1}{\rho}\left(\frac{d\rho}{dr}\right)^2 - \frac{2}{r}\frac{d\rho}{dr} - \frac{n}{1+n}\frac{4\pi G}{K}\rho^{2-1/n}. \tag{2346}$$

(c) With

$$\rho(r) = \rho_c\theta^\alpha(r) \tag{2347}$$

$$\frac{d\rho}{dr} = \rho_c\alpha\theta^{\alpha-1}\frac{d\theta}{dr} \tag{2348}$$

$$\left(\frac{d\rho}{dr}\right)^2 = \rho_c^2\alpha^2\theta^{2(\alpha-1)}\left(\frac{d\theta}{dr}\right)^2 \tag{2349}$$

$$\frac{d^2\rho}{dr^2} = \rho_c\alpha(\alpha-1)\theta^{\alpha-2}\left(\frac{d\theta}{dr}\right)^2 + \rho_c\alpha\theta^{\alpha-1}\frac{d^2\theta}{dr^2} \tag{2350}$$

we can rewrite the differential equation as

$$\rho_c\alpha(\alpha-1)\theta^{\alpha-2}\left(\frac{d\theta}{dr}\right)^2 + \rho_c\alpha\theta^{\alpha-1}\frac{d^2\theta}{dr^2} \tag{2351}$$

$$= \frac{n-1}{n}\frac{1}{\rho_c\theta^\alpha}\rho_c^2\alpha^2\theta^{2(\alpha-1)}\left(\frac{d\theta}{dr}\right)^2 - \frac{2}{r}\rho_c\alpha\theta^{\alpha-1}\frac{d\theta}{dr} - \frac{n}{1+n}\frac{4\pi G}{K}\rho_c^{2-1/n}\theta^{\alpha(2-1/n)} \tag{2352}$$

and see that for $n = \alpha$ the $(d\theta/dr)^2$ terms and the left and right side cancel out.

(d) With $n = \alpha$ the simplified equation is given by

$$\frac{d^2\theta}{dr^2} + \frac{2}{r}\frac{d\theta}{dr} + \frac{4\pi G\rho_c^{1-1/n}}{(n+1)K}\theta^n = 0 \tag{2353}$$

$$\frac{d^2\theta}{dr^2} + \frac{2}{r}\frac{d\theta}{dr} + \frac{4\pi G}{(n+1)K\rho_c^{1/n-1}}\theta^n = 0 \tag{2354}$$

(e) With

$$r = a\xi \tag{2355}$$

$$\frac{d\theta}{dr} = \frac{d\theta}{d\xi}\frac{d\xi}{dr} = \frac{1}{a}\frac{d\theta}{d\xi} \tag{2356}$$

$$\frac{d^2\theta}{dr^2} = \frac{1}{a^2}\frac{d^2\theta}{d\xi^2} \tag{2357}$$

we obtain

$$\frac{d^2\theta}{d\xi^2} + \frac{2}{\xi}\frac{d\theta}{d\xi} + a^2\frac{4\pi G}{(n+1)K\rho_c^{1/n-1}}\theta^n = 0 \tag{2358}$$

which for $a^{-2} = \frac{4\pi G}{(n+1)K\rho_c^{1/n-1}}$ gives the Lane-Emden equation in standard form

$$\frac{d^2\theta}{d\xi^2} + \frac{2}{\xi}\frac{d\theta}{d\xi} + \theta^n = 0 \tag{2359}$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n. \tag{2360}$$

(f)
- $\theta(\xi = 0) = 1$

$$\rightarrow \quad \rho(r = 0) = \rho_c \tag{2361}$$

- $\theta'(\xi = 0) = 0$

$$\rightarrow \quad \frac{dP}{dr} = K\left(\frac{n+1}{n}\right)\rho^{1/n}\frac{d\rho}{dr} \tag{2362}$$

$$= K\left(\frac{n+1}{n}\right)(\rho_c^{1/n}\theta)\frac{d(\rho_c\theta^n)}{d\xi}\frac{d\xi}{dr} \tag{2363}$$

$$= K\left(\frac{n+1}{n}\right)\rho_c^{1+1/n}\theta n\theta^{n-1}\frac{d\theta}{d\xi}\frac{d\xi}{dr} \tag{2364}$$

$$= K(n+1)\rho_c^{1+1/n}\theta^n\frac{d\theta}{d\xi}\frac{1}{a} \tag{2365}$$

$$\rightarrow \left.\frac{dP}{dr}\right|_{r=0} = 0 \tag{2366}$$

(g) The mass integral can be rewritten by using the Lane-Emden equation

$$M = 4\pi\int_0^R \rho(r)r^2 dr \tag{2367}$$

$$= 4\pi\rho_c a^3\int_0^{\xi_1}\theta^n\xi^2 d\xi \tag{2368}$$

$$= -4\pi\rho_c a^3\int_0^{\xi_1}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)d\xi \tag{2369}$$

$$= -4\pi\rho_c a^3\left[\xi^2\frac{d\theta}{d\xi}\right]_0^{\xi_1} \tag{2370}$$

$$= -4\pi\rho_c a^3\xi_1^2\theta'(\xi_1). \tag{2371}$$

For the radius R we find

$$R = a\xi_1 \tag{2372}$$

$$= \left[\frac{(n+1)K\rho_c^{1/n-1}}{4\pi G}\right]^{\frac{1}{2}} \xi_1 \tag{2373}$$

$$= \left[\frac{(n+1)K}{4\pi G}\right]^{\frac{1}{2}} \rho_c^{(1-n)/2n}\xi_1 \tag{2374}$$

$$\tag{2375}$$

$$\xi_1 = \left[\frac{(n+1)K}{4\pi G}\right]^{-\frac{1}{2}} \rho_c^{(n-1)/2n}R. \tag{2376}$$

Furthermore we can write (somewhat arbitrarily)

$$\xi_1^2 = \xi_1^2 1^{\frac{3-n}{1-n}} = \xi_1^2 \left(\frac{R}{a\xi_1}\right)^{\frac{3-n}{1-n}} \tag{2377}$$

$$= \xi_1^{2-\frac{3-n}{1-n}} \left(\frac{1}{a}\right)^{\frac{3-n}{1-n}} R^{\frac{3-n}{1-n}} \tag{2378}$$

$$= \xi_1^{\frac{n+1}{n-1}} a^{\frac{3-n}{n-1}} R^{\frac{3-n}{1-n}} \tag{2379}$$

which results in

$$M = 4\pi\rho_c a^3 \cdot \xi_1^{\frac{n+1}{n-1}} a^{\frac{3-n}{n-1}} R^{\frac{3-n}{1-n}} \theta'(\xi_1) \tag{2380}$$

$$= 4\pi\rho_c a^{2n/(n-1)} \cdot \xi_1^{\frac{n+1}{n-1}} R^{\frac{3-n}{1-n}} \theta'(\xi_1) \tag{2381}$$

$$= 4\pi\rho_c \left[\frac{(n+1)K\rho_c^{1/n-1}}{4\pi G}\right]^{n/(n-1)} \cdot \xi_1^{\frac{n+1}{n-1}} R^{\frac{3-n}{1-n}} \theta'(\xi_1) \tag{2382}$$

$$= 4\pi R^{\frac{3-n}{1-n}} \left[\frac{(n+1)K}{4\pi G}\right]^{n/(n-1)} \cdot \xi_1^{\frac{n+1}{n-1}} \theta'(\xi_1) \tag{2383}$$

which is an expression without ρ_c.

(h) For $n = 1$ we have

$$R = \left[\frac{K}{2\pi G}\right]^{1/2} \xi_1 \tag{2384}$$

which means R is independent of mass and central pressure and therefore constant for all objects. So we conclude $R_S = R_J$.

For $n = 1$ have have $\theta(\xi) = \sin\xi/\xi$ and find

$$\theta' = \frac{\xi\cos\xi - \sin\xi}{\xi^2} \tag{2385}$$

$$\xi_1 = \pi \tag{2386}$$

$$\theta'(\xi_1) = -1/\pi. \tag{2387}$$

Therefore

$$R = \pi \left[\frac{K}{2\pi G}\right]^{1/2} \tag{2388}$$

$$M = 4\pi^2 \left[\frac{K}{2\pi G}\right]^{3/2} \rho_c = 4\pi^2 \left[\frac{R}{\pi}\right]^3 \rho_c \tag{2389}$$

$$= \frac{4R^3}{\pi}\rho_c \tag{2390}$$

$$\to \rho_c = \frac{\pi M}{4R^3} = \frac{\pi}{3}\frac{\pi M}{4\frac{\pi}{3}R^3} = \frac{\pi^2}{3}\rho_{\text{avg}} \tag{2391}$$

which gives $\rho_{c,J} = 4.6 \cdot 10^{12}\text{kg/m}^3$ and $\rho_{c,S} = 1.3 \cdot 10^{12}\text{kg/m}^3$.

Exercise 13.5 Example: Shape of a Constant-Density, Spinning Planet

(a) The gravitational potential is given by the integral of the mass distribution $\rho(\vec{r})$

$$\Phi(\vec{r}) = -G \int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3\vec{r}' \tag{2392}$$

$$= -2\pi G \int \frac{\rho(\vec{r}')}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}} r'^2 \sin\theta d\theta dr' \tag{2393}$$

$$= -\frac{2\pi G\rho}{r} \int_0^R \frac{1}{\sqrt{1 + (r'/r)^2 - 2r'/r\cos\theta}} d(\cos\theta) r'^2 dr' \tag{2394}$$

$$= -\frac{2\pi G\rho}{r} \int_0^R r' \left(r + r' - \sqrt{(r - r')^2}\right) dr'. \tag{2395}$$

For $\vec{r}$ inside the mass distribution the integral needs to be split

$$\Phi(\vec{r}) = -\frac{2\pi G\rho}{r} \left[\int_0^r r'\,(2r')\,dr' + \int_r^R r'\,(2r)\,dr'\right] \tag{2396}$$

$$= -\frac{4\pi G\rho}{r} \left[\int_0^r r'^2 dr' + r\int_r^R r' dr'\right] \tag{2397}$$

$$= -\frac{4\pi G\rho}{r} \left[\frac{r^3}{3} + r\frac{1}{2}(R^2 - r^2)\right] \tag{2398}$$

$$= -4\pi G\rho \left[\frac{r^2}{3} + \frac{1}{2}(R^2 - r^2)\right] \tag{2399}$$

$$= -4\pi G\rho \left[-\frac{r^2}{6} + \frac{R^2}{2}\right] \tag{2400}$$

$$= \frac{2\pi G\rho}{3}\left[r^2 - 3R^2\right] \tag{2401}$$

(b) For the centrifugal force and potential we find

$$F_{\rm cen} = m\frac{v^2}{\varpi} = m\Omega^2\varpi \tag{2402}$$
$$\Phi_{\rm cen} = -\frac{1}{2}(\Omega\varpi)^2 \tag{2403}$$
$$= -\frac{1}{2}(\Omega r\cos\theta)^2 \tag{2404}$$
$$= -\frac{1}{2}(\Omega r\sin\theta')^2 \tag{2405}$$
$$= -\frac{1}{2}(\vec{\Omega}\times\vec{r})^2 \tag{2406}$$

we results in

$$\Phi = \frac{2\pi G\rho r^2}{3} - 2\pi G\rho R^2 - \frac{1}{2}(\Omega r\cos\theta)^2 \tag{2407}$$
$$= \frac{2\pi G\rho r^2}{3} - 2\pi G\rho R^2 - \frac{1}{2}\Omega^2 r^2\cos^2\theta \tag{2408}$$
$$= \frac{2\pi G\rho r^2}{3} - 2\pi G\rho R^2 - \frac{\Omega^2}{3}r^2\frac{1}{2}3\cos^2\theta \tag{2409}$$
$$= \frac{2\pi G\rho r^2}{3} - 2\pi G\rho R^2 - \frac{\Omega^2 r^2}{3} - \frac{\Omega^2}{3}r^2\frac{1}{2}(3\cos^2\theta - 1) \tag{2410}$$
$$= \frac{2\pi G\rho r^2}{3} - 2\pi G\rho R^2 - \frac{\Omega^2 r^2}{3} - \frac{\Omega^2}{3}r^2 P_2(\cos\theta) \tag{2411}$$
$$\tag{2412}$$

(c)

(d)

(e)

Exercise 13.7 Problem: A Hole in My Bucket

Applying the Bernoulli equation

$$\frac{1}{2}\rho v^2 + \rho g h = \text{const} \tag{2413}$$

to the hole and the water surface we get

$$\frac{1}{2}\rho v_{\rm hole}^2 = \rho g h + \frac{1}{2}\rho v_{\rm surf}^2. \tag{2414}$$

The change in volume is given by (neglecting limitations from the Hagen-Poiseulle equation but having a hole significantly smaller than the bucket surface - with hole has the same size the assumption of a static pressure does not make sense)

$$\frac{dV}{dt} = A_{\rm bucket}v_{\rm surf} = A_{\rm hole}v_{\rm hole} = A_{\rm bucket}\frac{dh}{dt} \tag{2415}$$
$$\rightarrow \quad v_{\rm hole} = \frac{A_{\rm bucket}}{A_{\rm hole}}\frac{dh}{dt} \tag{2416}$$
$$\rightarrow \quad v_{\rm surf} = \frac{dh}{dt}. \tag{2417}$$

With this the Bernoulli equation turns into

$$\frac{1}{2}\left(\frac{A_{\text{bucket}}}{A_{\text{hole}}}\right)^2\left(\frac{dh}{dt}\right)^2 = gh + \frac{1}{2}\left(\frac{dh}{dt}\right)^2 \tag{2418}$$

$$\left(\frac{dh}{dt}\right)^2 = \frac{2g}{\left(\frac{A_{\text{bucket}}}{A_{\text{hole}}}\right)^2 - 1}h \tag{2419}$$

$$= \frac{2A_{\text{hole}}^2 g}{A_{\text{bucket}}^2 - A_{\text{hole}}^2}h \tag{2420}$$

$$\tag{2421}$$

which can be solved by

$$\frac{dh}{\sqrt{h}} = -\sqrt{\frac{2A_{\text{hole}}^2 g}{A_{\text{bucket}}^2 - A_{\text{hole}}^2}}dt \tag{2422}$$

$$2\sqrt{h} = -\sqrt{\frac{2A_{\text{hole}}^2 g}{A_{\text{bucket}}^2 - A_{\text{hole}}^2}} \cdot t + 2\sqrt{H_0} \tag{2423}$$

$$h(t) = \left(\sqrt{H_0} - \sqrt{\frac{A_{\text{hole}}^2 g}{2(A_{\text{bucket}}^2 - A_{\text{hole}}^2)}} \cdot t\right)^2 \tag{2424}$$

$$\approx \left(\sqrt{H_0} - \frac{A_{\text{hole}}}{A_{\text{surface}}}\sqrt{\frac{g}{2}} \cdot t\right)^2 \tag{2425}$$

For the time T to empty the bucket we solve for $h(T) = 0$ and obtain

$$T = \sqrt{\frac{2H_0}{g}}\frac{A_{\text{bucket}}}{A_{\text{hole}}}. \tag{2426}$$

in the case of small holes or buckets with thick walls D the Hagen-Poiseulle should be taken into account.

Exercise 14.1 Practice: Constant-Angular-Momentum Flow - Relative Motion of Fluid Elements

Taylor expansion of the components of the velocity field gives

$$v_j(\mathbf{x} + \boldsymbol{\xi}) = v_j(\mathbf{x}) + \left(\left.\frac{\partial v_j(\mathbf{y})}{\partial y_i}\right|_{y=x} \xi_i\right). \tag{2427}$$

For the vector we can then write

$$\mathbf{v}(\mathbf{x} + \boldsymbol{\xi}) = \mathbf{v}(\mathbf{x}) + \left(\left.\frac{\partial v_j(\mathbf{y})}{\partial y_i}\right|_{y=x} \xi_i\right)\mathbf{e}_j \tag{2428}$$

$$\nabla_{\boldsymbol{\xi}}\mathbf{v} \equiv \mathbf{v}(\mathbf{x} + \boldsymbol{\xi}) - \mathbf{v}(\mathbf{x}) \tag{2429}$$

$$= \boldsymbol{\xi} \cdot \nabla\mathbf{v} \tag{2430}$$

For the constant-angular-momentum flow we have

$$\mathbf{v} = \frac{1}{\varpi^2}\mathbf{j} \times \mathbf{x} \tag{2431}$$

$$= \frac{1}{\varpi^2}j\varpi\mathbf{e}_\phi = \frac{j}{\varpi}\mathbf{e}_\phi. \tag{2432}$$

The only non-vanishing component of $\nabla \mathbf{v}$ is

$$\frac{\partial v_\phi}{\partial \varpi} = -\frac{j}{\varpi^2}. \tag{2433}$$

- tangential: $\boldsymbol{\xi} = \varpi\, d\phi \mathbf{e}_\phi = d\epsilon\, \mathbf{e}_\phi$ Not done yet
- radial: $\boldsymbol{\xi} = d\varpi\, \mathbf{e}_\varpi = d\epsilon\, \mathbf{e}_\varpi$ Not done yet

Exercise 14.2 Practice: Vorticity and Incompressibility

Vorticity: $\boldsymbol{\omega} = \nabla \times \mathbf{v}$. Compressibility: $\rho = \text{const}$

$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0 \quad \rightarrow \quad \nabla \cdot \mathbf{v} = 0 \tag{2434}$$

(a)

$$\nabla \times \mathbf{v} = \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \mathbf{e}_z = 0 \cdot \mathbf{e}_z \tag{2435}$$

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 2y \tag{2436}$$

(b)

$$\nabla \times \mathbf{v} = -2y \cdot \mathbf{e}_z \tag{2437}$$

$$\nabla \cdot \mathbf{v} = 0 \tag{2438}$$

(c)

$$\nabla \times \mathbf{v} = \frac{1}{\varpi} \left(\frac{\partial (\varpi v_\phi)}{\partial \varpi} - \frac{\partial v_\varpi}{\partial \phi} \right) \mathbf{e}_z = 2 \cdot \mathbf{e}_z \tag{2439}$$

$$\nabla \cdot \mathbf{v} = \frac{1}{\varpi} \left(\frac{\partial (\varpi v_\varpi)}{\partial \varpi} + \frac{\partial v_\phi}{\partial \phi} \right) = 0 \tag{2440}$$

(d)

$$\nabla \times \mathbf{v} = 0 \cdot \mathbf{e}_z \tag{2441}$$

$$\nabla \cdot \mathbf{v} = 0 \tag{2442}$$

Exercise 16.9 Example: Breaking of a Dam

The PDEs

$$h_t + h v_x + v h_x = 0 \tag{2443}$$

$$v_t + v v_x + g h_x = 0 \tag{2444}$$

can be written as

$$A u_t + B u_x = 0 \tag{2445}$$

$$u = \begin{pmatrix} h \\ v \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} v & h \\ g & v \end{pmatrix} \tag{2446}$$

Now

$$vh_t + vhv_x + v^2 h_x = 0 \tag{2447}$$
$$hv_t + hvv_x + ghh_x = 0 \tag{2448}$$
$$\rightarrow (hv)_t = hv_t + vh_t \tag{2449}$$
$$= -2vhv_x - v^2 h_x - ghh_x \tag{2450}$$
$$= -\partial_x \left(h \left[v^2 + \frac{1}{2} gh \right] \right) \tag{2451}$$

Not done yet

15.6 Walter - Astronautics

Problem 1.1 - Balloon Propulsion

For the mass flow rate we have

$$\dot{m} = \rho \dot{V} \approx \rho A_t v_t \stackrel{!}{=} \frac{\rho V}{T} \quad \rightarrow \quad v_t = \frac{V}{A_t T} = 20\text{m/s} \tag{2452}$$

and the speed of sound in a diatomic gas ($f = 5$, $\rho_0 = 1.225\text{kg/m}^3$, $P_0 = 101.3 \cdot 10^3 Pa$) is

$$c = \sqrt{\kappa \frac{p}{\rho}} = \sqrt{\frac{f+2}{f} \frac{P}{\rho}} = 340\text{m/s} \tag{2453}$$

which justifies $v_t \ll c$. Newtons second law gives for the momentum thrust

$$F_e = \frac{dp}{dt} = \dot{m} v_t = \frac{\rho V}{T} \frac{V}{A_t T} = \frac{\rho}{A_t} \left(\frac{V}{T} \right)^2 = 0.0258\text{N} \tag{2454}$$

From the Bernoulli equation we can obtain the pressure difference

$$P = P_0 + \frac{\rho}{2} v_t^2 \quad \rightarrow \quad P - P_0 = \frac{\rho}{2} v_t^2 \tag{2455}$$

and can then calculate the pressure thrust

$$F_p = A_t (P - P_0) = \frac{A_t \rho}{2} v_t^2 = \frac{\rho V^2}{2 A_t T^2} = 0.0129\text{N} \tag{2456}$$

and see $F_e = 2F_p$.

Problem 1.2 - Nozzle Exit Area of an SSME

For the total thrust we have in vacuum and at sea level we have

$$F_{\text{SL}} = A_t (P - P_0) + \dot{m} v_t \tag{2457}$$
$$F_{\text{V}} = A_t (P - 0) + \dot{m} v_t \tag{2458}$$

which implies with $P_0 = 101.3\text{Pa}$

$$A_t = \frac{F_{\text{V}} - F_{\text{SL}}}{P_0} = 4.55\text{m}^2 \tag{2459}$$

Problem 1.3 - Proof of $\eta_{\text{VDF}} \leq 1$

$$\langle \nu_e \rangle_\mu = \frac{\int_0^{\pi/2} \nu_e(\theta) \cdot \mu(\theta) \sin\theta \, d\theta}{\int_0^{\pi/2} \mu(\theta) \sin\theta \, d\theta} \tag{2460}$$

$$\langle \nu_e \rangle_\mu^2 \leq \langle \nu_e^2 \rangle_\mu \tag{2461}$$

Not done yet

Problem 4.1 - Gas Velocity-Pressure Relation in a Nozzle

- Using the ideal gas equation $pV = NkT$ we have for a adiabatic process

$$pV^\kappa = p\left(\frac{NkT}{p}\right)^\kappa \tag{2462}$$
$$= p^{1-\kappa}T^\kappa \tag{2463}$$
$$= \text{const} \tag{2464}$$
$$\rightarrow \quad p^{\frac{1-\kappa}{\kappa}}T = p_0^{\frac{1-\kappa}{\kappa}}T_0 \tag{2465}$$

and with $pV = nRT$ we obtain more conservation laws for adiabatic processes

$$\rho = \frac{m}{V} = \frac{nM_p}{V} = \frac{M_p p}{RT} \quad \rightarrow \quad p = \frac{R}{M_p}\rho T \tag{2466}$$
$$(\rho T)^{\frac{1-\kappa}{\kappa}}T = \text{const} \tag{2467}$$
$$\rho^{1-\kappa}T = \text{const} \tag{2468}$$

as well as

$$\rho^{1-\kappa}T = \text{const} \tag{2469}$$
$$\rho^{1-\kappa}\left(\frac{p}{\rho}\right) = \text{const} \tag{2470}$$
$$\rho^{-\kappa}p = \text{const} \tag{2471}$$
$$\rho p^{-\frac{1}{\kappa}} = \text{const} \tag{2472}$$

We obtain with $\kappa = \frac{2+n}{n}$ for the energy conversion efficiency

$$\eta = 1 - \frac{T}{T_0} = 1 - \left(\frac{p}{p_0}\right)^{\frac{\kappa-1}{\kappa}} = 1 - \left(\frac{\rho}{\rho_0}\right)^{\kappa-1} \tag{2473}$$
$$= 1 - \left(\frac{p}{p_0}\right)^{\frac{2}{n+2}} = 1 - \left(\frac{\rho}{\rho_0}\right)^{\frac{2}{n}} \tag{2474}$$

- Energy conservation along the engine axis gives

$$\frac{1}{2}m_p v_0^2 + m_p c_p T_0 = \frac{1}{2}m_p v^2 + m_p c_p T. \tag{2475}$$

with $v_0 = 0$ we obtain. for the gas flow velocity

$$v^2 = 2c_p(T_0 - T) = 2c_p T_0 \eta = 2\left(\frac{\kappa}{\kappa-1}\frac{R}{M_p}\right)T_0\eta \tag{2476}$$

which is called the St. Venant-Wantzel equation. Differentiating yields

$$2v\frac{dv}{dp} = 2\left(\frac{\kappa}{\kappa-1}\frac{R}{M_p}\right)T_0\frac{d\eta}{dp} \tag{2477}$$

- The continuity equation is given by

$$\dot{m}_p = \rho v A \quad \rightarrow \quad v = \frac{\dot{m}_p}{A\rho} \tag{2478}$$

Now we can combine all parts

$$\frac{dv}{dp} = \frac{1}{v}\left(\frac{\kappa}{\kappa-1}\frac{R}{M_p}\right)T_0\frac{d\eta}{dp} \tag{2479}$$

$$= -\frac{A\rho}{\dot{m}_p}\left(\frac{\kappa}{\kappa-1}\frac{R}{M_p}\right)T_0\frac{\kappa-1}{\kappa}\left(\frac{p}{p_0}\right)^{-\frac{1}{\kappa}}\frac{1}{p_0} \tag{2480}$$

$$= -\frac{A\rho}{\dot{m}_p}\left(\frac{\kappa}{\kappa-1}\frac{R}{M_p}\right)\frac{\kappa-1}{\kappa}\left(\frac{p}{p_0}\right)^{-\frac{1}{\kappa}}\frac{M_p}{R\rho_0} \tag{2481}$$

$$= -\frac{A}{\dot{m}_p}\left(\frac{p}{p_0}\right)^{-\frac{1}{\kappa}}\frac{\rho}{\rho_0} \tag{2482}$$

$$= -\frac{A}{\dot{m}_p} \tag{2483}$$

and obtain

$$dv = -\frac{A}{\dot{m}_p}dp. \tag{2484}$$

Problem 4.2 - Approximation of the Infinite-Expansion Coefficient

$$C_\infty \equiv (n+2)\sqrt{\frac{n^n}{(n+1)^{n+1}}} \tag{2485}$$

$$= \frac{n+2}{\sqrt{n+1}}\left(\frac{n}{n+1}\right)^{n/2} \tag{2486}$$

$$= \frac{n+2}{\sqrt{n+1}}\left(1+\frac{1}{n}\right)^{-n/2} \tag{2487}$$

$$= \frac{n+2}{\sqrt{n+1}}\frac{4096}{6561}\left[1+\frac{1}{18}\left(1+\log\frac{2^{27}}{3^{18}}\right)(n-8)+O\left(n^2\right)\right] \tag{2488}$$

$$\approx 0.624\frac{n+2}{\sqrt{n+1}} \tag{2489}$$

Problem 7.1 - Solutions of Poisson's Equation

$$\triangle U = 4\pi\gamma\rho \tag{2490}$$

$$\triangle_x G(x) = 4\pi\gamma\delta(x) \tag{2491}$$

$$G(x) = \frac{1}{\sqrt{2\pi}}\int d^n y\, g(y)e^{-ixy} \tag{2492}$$

$$\triangle_x G(x) = \frac{1}{\sqrt{2\pi}}\int d^n y\, g(y)(-iy)^2 e^{-ixy} \equiv 4\pi\gamma\delta(x) \tag{2493}$$

$$\rightarrow g(y) = -\frac{e^{-i..}}{y^2} \tag{2494}$$

16 Simulations of Cosmic Structure Formation - MAUCA 2018

Exercise 2

1. Summary of Friedmann equations We start with the total energy density

$$\rho = \frac{3H_0^2}{8\pi G}\left[\Omega_\Lambda + \Omega_m\left(\frac{a_0}{a}\right)^3 + \Omega_r\left(\frac{a_0}{a}\right)^4\right] \tag{2495}$$

Using the Friedman equation we get

$$\dot{a}^2 + K = \frac{8\pi G\rho a^2}{3} \tag{2496}$$

$$\dot{a}^2 - \Omega_K a_0^2 H_0^2 = a^2 H_0^2\left[\Omega_\Lambda + \Omega_m\left(\frac{a_0}{a}\right)^3 + \Omega_r\left(\frac{a_0}{a}\right)^4\right] \tag{2497}$$

$$\dot{a}^2 = a^2 H_0^2\left[\Omega_\Lambda + \Omega_m\left(\frac{a_0}{a}\right)^3 + \Omega_r\left(\frac{a_0}{a}\right)^4 + \Omega_K\left(\frac{a_0}{a}\right)^2\right] \tag{2498}$$

where we used $\Omega_K = -K/(a_0H_0)^2$. With $a(t_0) = a_0$ and $H_0 \equiv \frac{\dot{a}(t_0)}{a(t_0)}$ we find a constraint on the Ω parameters

$$\Omega_\Lambda + \Omega_m + \Omega_r + \Omega_K = 1 \tag{2499}$$

Then with $x = a/a_0$

$$\dot{x}^2 = x^2 H_0^2\left[\Omega_\Lambda + \Omega_m x^{-3} + \Omega_r x^{-4} + \Omega_K x^{-2}\right] \tag{2500}$$

$$\frac{dx}{dt} = H_0\sqrt{\Omega_\Lambda x^2 + \Omega_m x^{-1} + \Omega_r x^{-2} + \Omega_K} \tag{2501}$$

$$H_0 dt = \frac{dx}{\sqrt{\Omega_\Lambda x^2 + \Omega_m x^{-1} + \Omega_r x^{-2} + \Omega_K}} \tag{2502}$$

2. Solutions

$$\dot{x} = H_0\sqrt{\Omega_\Lambda x^2 + \Omega_m x^{-1} + \Omega_r x^{-2} + \Omega_K} \tag{2503}$$

   - $k = 0, \Omega_\Lambda = 1, \Omega_m = \Omega_r = 0$

$$\dot{x} = H_0 x \quad \rightarrow \quad x = e^{H_0 t} \tag{2504}$$

   - $k = 0, \Omega_m = 1, \Omega_\Lambda = \Omega_r = 0$

$$\dot{x} = H_0 x^{-1/2} \quad \rightarrow \quad x = \left(1 + \frac{3}{2}H_0 t\right)^{2/3} \tag{2505}$$

   - $k = 0, \Omega_r = 1, \Omega_\Lambda = \Omega_m = 0$

$$\dot{x} = H_0 x^{-1} \quad \rightarrow \quad x = 1 + H_0 t \tag{2506}$$

17 Doodling

Fundamental ingredients for a quantum theory are a set of states $\{|\psi\rangle\}$ and operators $\{\mathcal{O}\}$. The time development is governed by a Hamilton operator

$$i\hbar\partial_t|\psi\rangle = H|\psi\rangle \tag{2507}$$

Lets assume that momentum eigenstates are simultaneously eigenstates of H then a simple relativistic theory looks like

$$H|\vec{p}\rangle = E_{\vec{p}}|\vec{p}\rangle \tag{2508}$$

$$E_{\vec{p}} = +\sqrt{\vec{p}^2c^2 + m^2c^4} \tag{2509}$$

The time evolution of the wave function is given by

$$\psi(\vec{p},t) = e^{-iE_{\vec{p}}t}\psi(\vec{p},0) \tag{2510}$$

$$\psi(\vec{x},t) = \int d^3\vec{p}\, e^{i\vec{p}\vec{x}}\psi(\vec{p},t) \tag{2511}$$

$$= \int d^3\vec{p}\, e^{-i(E_{\vec{p}}t-\vec{p}\vec{x})}\psi(\vec{p},0) \tag{2512}$$

$$= \frac{1}{(2\pi)^3}\int d^3\vec{p}\, e^{-i(E_{\vec{p}}t-\vec{p}\vec{x})}\int d^3\vec{y}e^{-i\vec{p}\vec{y}}\psi(\vec{y},0) \tag{2513}$$

$$= \int d^3\vec{y}\left[\frac{1}{(2\pi)^3}\int d^3\vec{p}\, e^{-i(E_{\vec{p}}t-\vec{p}(\vec{x}-\vec{y}))}\right]\psi(\vec{y},0) \tag{2514}$$

$$\psi(\vec{x},t) = \int d^3\vec{y}\, G(\vec{x}-\vec{y},t)\psi(\vec{y},0) \tag{2515}$$

Causality of the theory is guaranteed if the commutator of two operators/observables (associated with points x and y in space time) commute if the points are space-like separated

$$|x-y| < 0 \quad \rightarrow \quad [\mathcal{O}_i,\mathcal{O}_j] = 0. \tag{2516}$$

Localizing a particle in a small region L means

$$p \sim \frac{\hbar}{L} \tag{2517}$$

$$E = \sqrt{m^2c^4 + p^2c^2} = pc\sqrt{1+\frac{m^2c^2}{p^2}} \tag{2518}$$

The L at which the momentum contribution becomes comparable to the rest energy of the particle

$$mc^2 = pc = \frac{\hbar c}{L} \quad \rightarrow \quad L_c = \frac{\hbar}{mc} \tag{2519}$$

is called Compton wavelength at which a relativistic theory is required and creation of particles and antiparticles appears.

This is therefore the method of choice to produce particles. A collision of two particles localizes a large amount of energy in a small region - creating particles

$$p\bar{p} \rightarrow X\bar{X} + ... \tag{2520}$$

Important general principles

- CPT invariance
- Spin-statistic theorem
- Interactions of particles with higher spin rather quite constrainted
  1. for lower spins $s = 0, 1/2$ the only restrictions are locality and Lorentz invariance
  2. the constrains are so restrictive that there are no relativistic quantum particle with $s > 2$

18 Finance stuff for Aki

Stochastic ODE for Geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{2521}$$

Solving it via Ito's Lemma gives

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right) \tag{2522}$$

The transition probability for the price going from S_0 at time $t = 0$ to S_t at time t (with fixed σ and μ) is given by (only stating the result)

$$f(S_t, t, S_0, t; \mu, \sigma) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma\sqrt{t} S_t} \exp\left(-\frac{\left[\log \frac{S_t}{S_0} - \left(\mu - \frac{1}{2}\sigma^2\right) t\right]^2}{2\sigma^2 t}\right) \tag{2523}$$

You numbers are

- $S_{max} = 0.9\ S_0$ price down 10%
- $\sigma = 0.24$
- $\mu = 0.05$ maybe 5% discount rate
- $t = 1/12$ meaning one month

so market being down 10% means integrate over the tail of the probability density

$$p = \int_0^{0.9 S_0} f(S_t, t, S_0, t; \mu, \sigma) dS_t \tag{2524}$$
$$= 0.061 \tag{2525}$$
$$= 6.1\% \tag{2526}$$

19 Companion for Dyson QFT book

1. Calculating 2.1 (9)

$$\frac{\partial\psi}{\partial t} = -\sum_k c\alpha^k \frac{\partial\psi}{\partial x_k} - i\frac{mc^2}{\hbar}\beta\psi \tag{2527}$$

$$\frac{\partial\psi^*}{\partial t} = -\sum_k c\frac{\partial\psi^*}{\partial x_k}\alpha^{k*} + i\frac{mc^2}{\hbar}\psi^*\beta^* \tag{2528}$$

then

$$\frac{\partial\rho}{\partial t} = \frac{\partial\psi*}{\partial t}\psi + \psi^*\frac{\partial\psi}{\partial t} \tag{2529}$$

$$= -\sum_k c\left(\frac{\partial\psi^*}{\partial x_k}\alpha^{k*}\psi + \psi^*\alpha^k\frac{\partial\psi}{\partial x_k}\right) + i\frac{mc^2}{\hbar}(\psi^*\beta^*\psi - \psi^*\beta\psi) \tag{2530}$$

$$\overset{\beta=\beta^*}{=} -\sum_k c\left(\frac{\partial\psi^*}{\partial x_k}\alpha^{k*}\psi + \psi^*\alpha^k\frac{\partial\psi}{\partial x_k}\right) \tag{2531}$$

$$\overset{\alpha=\alpha^*}{=} -\sum_k c\left(\frac{\partial\psi^*}{\partial x_k}\alpha^{k}\psi + \psi^*\alpha^k\frac{\partial\psi}{\partial x_k}\right) \tag{2532}$$

$$= -c\partial_k(\psi^*\alpha^k\psi) \tag{2533}$$

so $j_k = \psi^*\alpha^k\psi$.

20 Companion for Banks QFT book

1. Obtaining (1.2)

$$
\begin{aligned}
p &= (\omega_p, \vec{p}) \\
\langle \vec{p} | \vec{q} \rangle &= N_p^2 \cdot \delta^3(\vec{p} - \vec{q}) \\
\mathbb{I} &= C \int d^3\vec{p} |\vec{p}\rangle \langle \vec{p}| \quad \to |q\rangle = C \int d^3\vec{p} |p\rangle \langle p|q\rangle \\
|\vec{y}\rangle &= C \int d^3\vec{p} |\vec{p}\rangle \langle \vec{p}|\vec{y}\rangle = C \int d^3\vec{p} |\vec{p}\rangle e^{-i\vec{p}\cdot\vec{y}} \\
H|\vec{p}\rangle &= \omega_p |\vec{p}\rangle \\
A_{\mathrm{AE}} &= \int d^4x d^4y J_{\mathrm{A}}(x) J_{\mathrm{B}}(y) \cdot C^2 \int d^3\vec{p} \int d^3\vec{q} \langle \vec{p}| e^{-H(x^0 - y^0)} |\vec{q}\rangle e^{i\vec{q}\cdot\vec{x}} e^{-i\vec{p}\cdot\vec{y}} \\
&= \int d^4x d^4y J_{\mathrm{A}}(x) J_{\mathrm{B}}(y) \cdot C^2 \int d^3\vec{p} \int d^3\vec{q} \langle \vec{p}| e^{-\omega_q(x^0 - y^0)} |\vec{q}\rangle e^{i(\vec{q}\cdot\vec{x} - \vec{p}\cdot\vec{y})} \\
&= \int d^4x d^4y J_{\mathrm{A}}(x) J_{\mathrm{B}}(y) \cdot C^2 \int d^3\vec{p} \int d^3\vec{q} \langle \vec{p}|\vec{q}\rangle e^{-\omega_q(x^0 - y^0)} e^{i(\vec{q}\cdot\vec{x} - \vec{p}\cdot\vec{y})} \\
&= \int d^4x d^4y J_{\mathrm{A}}(x) J_{\mathrm{B}}(y) \cdot C^2 \int d^3\vec{p} N_p^2 e^{-\omega_p(x^0 - y^0)} e^{i\vec{p}(\vec{x} - \cdot\vec{y})} \\
&= \int d^4x d^4y J_{\mathrm{A}}(x) J_{\mathrm{B}}(y) \cdot C^2 \int d^3\vec{p} N_p^2 e^{-ip(x-y))}
\end{aligned}
$$

2.

21 Some stuff for later

1. For a Quantum field theory on a Riemann sphere with $g : S^2 \to G$ consider the action

$$\mathcal{S}_0 = \frac{1}{4\lambda^2} \int_{S^2} d^2 z \, \mathrm{tr}(g^{-1}\partial_\mu g g^{-1} \partial^\mu g) \tag{2534}$$

   then $g^{-1}\partial_\mu g$ defines and element of the Lie algebra and $g^{-1}dg$ is the pullback of the Maurer-Cartan form to S^2 under the map defined by g.

2. Thirring Model, Thirring-Wess Model, CM-Sommerfeld Model

3. Volume measure under Lorentz trafo $x^\mu \to x'^\mu = \Lambda^\mu_\nu x^\nu$

$$d^4x = dx^0 dx^1 dx^2 dx^3 \tag{2535}$$
$$d^4x' = \Lambda^\mu_0 dx^0 \, \Lambda^\nu_1 dx^1 \, \Lambda^\sigma_2 dx^2 \, \Lambda^\rho_1 dx^3 \tag{2536}$$

   vs

$$d^4x = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \tag{2537}$$

4.
   - Baez review octonions HTTPS://ARXIV.ORG/ABS/MATH/0105155V4
   - Complex quaternions, octonions HTTPS://ARXIV.ORG/ABS/1611.09182
   - Conway, Smith - On quaternions and octonions

22 Representations CheatSheet

22.1 Preliminaries

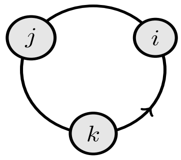
Definition 22.1. Number spaces $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$

- A **complex number** is an objects of the form $a + bi$ with $a, b \in \mathbb{R}$ and

$$i^2 = -1. \tag{2538}$$

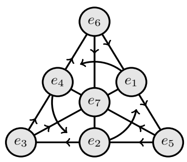
- A **quaternion** is an objects of the form $a + bi + cj + dk$ with $a, b, c, d \in \mathbb{R}$ and

$$i^2 = j^2 = k^2 = ijk = -1. \tag{2539}$$



- An **octonion** is an objects of the form $a + bi + cj + dk + el + fm + gn + ho$ with $a, \ldots, h \in \mathbb{R}$ and $e_0 = 1, e_1 = i, \ldots, e_7 = o$

$$e_i e_j = \begin{cases} e_j, & \text{if } i = 0 \\ e_i, & \text{if } j = 0 \\ -\delta_{ij} e_0 + \varepsilon_{ijk} e_k & \text{otherwise} \end{cases} \tag{2540}$$



Remark 22.1. *$\mathbb{C}$ forms a field, $\mathbb{H}$ forms a non-commutative ring*

Definition 22.2. The **conjugates** are defined by

$$\begin{aligned} \bar{z} &= a - bi && (2541) \\ \bar{q} &= a - bi - cj - dk && (2542) \\ &= -\frac{1}{2}\left[q + iqi + jqj + kqk\right] && (2543) \\ \bar{x} &= a - bi - cj - dk - el - fm - gn - ho && (2544) \\ &= -\frac{1}{6}\left[x + (ix)i + (jq)j + (kq)k) + (le)l + (mf)m + (ng)n + (oh)o\right] && (2545) \end{aligned}$$

22.2 Groups theory

Definition 22.3. For a subgroup H of a group G **a left-coset** of the subgroup H in G is defined as the set formed by a distinct $g \in G$

$$gH = \{gh : \forall h \in H\} \tag{2546}$$

G/H denotes the set of left cosets $\{gH : g \in G\}$ of H in G (called coset-space).

Definition 22.4. A subgroup N of a group G is called **normal subgroup (Normalteiler)** $N \lhd G$ if it is invariant under conjugation by members of G. Meaning

$$gng^{-1} \in N \quad \forall g \in G \tag{2547}$$
$$gN = Ng \quad \forall g \in G \tag{2548}$$
$$gNg^{-1} = N \quad \forall g \in G \tag{2549}$$

Definition 22.5. A **simple group** is a nontrivial group whose only normal subgroups are the trivial group and the group itself.

Definition 22.6. Let $(G, \circ)$ and $(K, *)$ be two groups with elements $g_a \in G$ and $k_i \in K$. The **direct product** is a group $(G \otimes K, \star)$ with elements (g_a, k_i) and the multiplication rule

$$(g_a, k_i) \star (g_b, k_j) = (g_a \circ g_b, k_i * k_j). \tag{2550}$$

Theorem 22.1. Every finite simple group is isomorphic to one of the following groups:

1. Z_p cyclic group of prime order
2. A_n alternating group of degree $n > 4$
3. groups of Lie type (names derived from Lie algebras with $q = p^k, m \in \mathbb{N}$
   - $A_n(q)$ Special projective linear group
   - $B_n(q), n > 1$ Commutator subgroup of $SO(2n + 1)$
   - $C_n(q), n > 2$ projective symplectic group
   - $D_n(q), n > 1$ Commutator subgroup of $SO(2n)$
   - $E_6(q), E_7(q), E_8(q), F_4(q), G_2(q)$ Chevalley group
   - ${}^2A_n(q^2), n > 1$ Special unitary group $SU(n)$
   - ${}^2B_2(2^{2m+1}))$ Suzuki Groups $Sz(2^{2m+1})$
   - ${}^2D_n(q^2),{}^3 D_4(q^3),{}^2 E_6(q^2)$ Steinberg group
   - ${}^2F_4(2^{2m+1}),{}^2 G_2(2^{2m+1})$ Ree group
4. 26 sporadic groups
   - Mathieu groups $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}$
   - Janko groups J_1, J_2, J_3, J_4
   - Conway groups Co_1, Co_2, Co_3
   - Fischer groups Fi_{22}, Fi_{23}, F_{3+}
   - Higman–Sims group HS
   - McLaughlin group McL
   - Held group F_7
   - Rudvalis group Ru
   - Suzuki group F_{3-}
   - O'Nan group $O'N$
   - Harada–Norton group F_5
   - Lyons group Ly
   - Thompson group F_3
   - Baby Monster group F_2
   - Fischer–Griess Monster group F_1

Figure 6: Periodic table of finite simple groups

5. ${}^2F_4(2)'$ Tits group (order $2^11 \cdot 3^3 \cdot 5^2 \cdot 13 = 17,971,200$)
   - sometimes called the 27th sporadic group - but belongs for $m = 0$ to the family ${}^2F_4(2^{2m+1})'$ of commutator subgroups of ${}^2F_4(2^{2m+1})$

Definition 22.7. Exceptional Lie groups

- G_2 (order 14)
- F_4 (order 52)
- E_6 (order 78)
- E_7 (order 133)
- E_8 (order 248)

Theorem 22.2. (Frobenius theorem, Hurwitz theorem) Any real finite-dimensional normed division algebra over the reals must be

- isomorphic to $\mathbb{R}$ or $\mathbb{C}$ if unitary and commutative (equivalently: associative and commutative)
- isomorphic to the quaternions $\mathbb{H}$ if noncommutative but associative
- isomorphic to the octonions $\mathbb{O}$ if non-associative but alternative.

Remark 22.2. *Projective spaces*

- $\mathfrak{so}(n+1)$ *is infinitesimal isometry of the real projective spaces* $\mathbb{RP}^n$
- $\mathfrak{su}(n+1)$ *is infinitesimal isometry of the complex projective spaces* $\mathbb{CP}^n$
- $\mathfrak{sp}(n+1)$ *is infinitesimal isometry of the quaternionic projective spaces* $\mathbb{HP}^n$
- *octonionic projective line* $\mathbb{OP}^1$ *reproduces* $\mathfrak{so}(8)$ *(already accomodated by* $\mathbb{RP}^7$*)*
- *Cayley projective plane* $\mathbb{OP}^2$ *reproduces* $\mathfrak{f}_4$*)*
- $\mathbb{OP}^n$ *for* $n > 2$ *gives nothing due to non-associativity of* $\mathbb{O}$

Remark 22.3. *Freudenthal-Rosenfeld-Tits magic square of Lie algebras*

$\mathbb{A}_1/\mathbb{A}_2$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{so}(3)$	$\mathfrak{su}(3)$	$\mathfrak{sp}(3)$	$\mathfrak{f}_4$
$\mathbb{C}$	$\mathfrak{su}(3)$	$\mathfrak{su}(3) \otimes \mathfrak{su}(3)$	$\mathfrak{su}(6)$	$\mathfrak{e}_6$
$\mathbb{H}$	$\mathfrak{sp}(3)$	$\mathfrak{su}(6)$	$\mathfrak{so}(12)$	$\mathfrak{e}_7$
$\mathbb{O}$	$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$

(2551)

22.3 Representation theory

Definition 22.8. A **representation** of a group $G = (\{g_i\}, \circ)$ is a mapping $g \mapsto D(g)$ of the elements $g \in G$ onto a set of linear operators with

1. $D(e) = \mathbb{I}$
2. $D(g_1)D(g_2) = D(g_1 \circ g_2)$.

This obviously implies $D(g^{-1}) = D(g)^{-1}$.

Remark 22.4. *A bit more formal - let G a group and V be a $\mathbb{K}$-vector space then a linear representation is a group homomorphism with $D : G \to \mathrm{GL}(V) \stackrel{!}{=} \mathrm{Aut}(V)$. V is then called representation space with* $\dim V$ *being the dimension of the representation and* $D(g) \in \mathrm{GL}(V)$

Definition 22.9. An **equivalent representation** D' of a representation D is defined by

$$D(g) \to D'(g) = S^{-1}D(g)S \qquad \forall g \in G \tag{2552}$$

Definition 22.10. A representation D is called **unitary representation** if

$$D(g)^\dagger = D(g)^{-1} \qquad \forall g \in G \tag{2553}$$

Remark 22.5. *For a unitary representation $D(g)^\dagger D(g) = \mathbb{I}$ an equivalent representation $D'(g) = S^{-1}D(g)S$ is only unitary*

$$\begin{aligned} D'(g)^\dagger D'(g) &= \left(S^{-1}D(g)S\right)^\dagger S^{-1}D(g)S &(2554)\\ &= S^\dagger D(g)^\dagger (S^{-1})^\dagger S^{-1}D(g)S &(2555)\\ &= S^\dagger D(g)^\dagger (S^\dagger)^{-1} S^{-1}D(g)S &(2556)\\ &= S^\dagger D(g)^\dagger (SS^\dagger)^{-1} D(g)S &(2557)\end{aligned}$$

iff S is unitary itself $SS^\dagger = \mathbb{I}$

$$D'(g)^\dagger D'(g) = S^{-1}D(g)^\dagger D(g)S = S^{-1}S = \mathbb{I}. \tag{2558}$$

Definition 22.11. A representation is called a **reducible representation** if V has an invariant subspace meaning that the action of any $D(g)$ on any vector of the subspace V_P is still in the subspace. If the projection operator $P : V \to V_P$ projects to this subspace then

$$PD(g)P = D(g)P \qquad \forall g \in G \tag{2559}$$

Remark 22.6. $\forall |v\rangle \in V$ *we have $P|v\rangle \in V_P$. If the subspace is invariant then any group action can not move it outside $D(g)P|v\rangle \in V_P$. But this means projecting it again would not change anything $PD(g)P|v\rangle = D(g)P|v\rangle$*

Definition 22.12. A representation is called an **irreducible representation** if it is not reducible.

Definition 22.13. A representation is called a **completely reducible representation** if it is equivalent to a representation whose matrix elements have the form

$$D(g) = \begin{pmatrix} D_1(g) & 0 & \cdots \\ 0 & D_2(g) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \tag{2560}$$

where all $D_j(g)$ are irreducible. Representation D is is said to be the direct sum of subrepresentation D_j

$$D = D_1(g) \oplus D_2(g) \oplus \cdots \tag{2561}$$

Definition 22.14. For a group of order n the n-dimensional representation D defined by

$$\begin{aligned} g_k &\to |e_k\rangle &(2562)\\ D(g_j)|e_k\rangle &\stackrel{!}{=} |e_m\rangle \qquad \text{with } g_j \circ g_k = g_m \to |e_m\rangle &(2563)\end{aligned}$$

(where $\{|e_i\rangle\}$ is the ordinary n-dimensional cartesian basis) is called the **regular representation**. The matrices are then constructed by

$$[D(g_j)]_{ik} = \langle e_i|D(g_j)|e_k\rangle = \langle e_i|e_m\rangle. \tag{2564}$$

Theorem 22.3. Every representation of a finite group is equivalent to a unitary representation.

Theorem 22.4. Every representation of a finite group is complete reducible.

Definition 22.15. Given two representations D_1 and D_2 acting on V_1 and V_2, an intertwiner between D_1 and D_2 is a linear operator $F : D_1 \to D_2$ which "commutes with G" in the sense that

$$FD_1(g) = D_2(g)F \quad \forall g \in G. \tag{2565}$$

23 Lie groups/algebras

Linear representation

$$g \to \tag{2566}$$

Remark 23.1. *Killing classification of simple Lie groups*

- *SO(2n), SO(2n+1) - Lie algebra:* $J^T = -J$ *(skew-hermitian, trace free matrices* $GL(n, \mathbb{R})$
- *SU(n) - Lie algebra:* $J^\dagger = -J$ *(skew-hermitian, trace free matrices in* $GL(n, \mathbb{C})$
- *Sp(2n) - Lie algebra:* $J^\dagger = -J$ *(skew-hermitian matrices in* $GL(n, \mathbb{H})$

24 Example representations

24.1 Cyclic group Z_2

$$\begin{array}{c||cc} Z_2 & e & p \\ \hline\hline e & e & p \\ p & p & e \end{array} \tag{2567}$$

1d

$$D'(e) = 1, \quad D'(p) = -1 \tag{2568}$$

24.2 Cyclic group Z_3

$$\begin{array}{c||ccc} Z_3 & e & a & b \\ \hline\hline e & e & a & b \\ a & a & b & e \\ b & b & e & a \end{array} \tag{2569}$$

1d

$$D'(e) = 1, \quad D'(a) = e^{i\frac{2\pi}{3}}, \quad D'(b) = e^{i\frac{4\pi}{3}} \tag{2570}$$

3d - regular representation

$$|e\rangle = (1,0,0)^T, \quad |a\rangle = (0,1,0)^T, \quad |b\rangle = (0,0,1)^T \tag{2571}$$

$$D(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad D(b) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \tag{2572}$$

24.3 Dihedral group $D_2 = Z_2 \otimes Z_2$

We construct the group table by utilizing the direct product rule

$$(g_a, k_i) \star (g_b, k_j) = (g_a \circ g_b, k_i * k_j) \tag{2573}$$

which simplifies to

$$(g_a, g_i) \star (g_b, g_j) = (g_a \circ g_b, g_i \circ g_j). \tag{2574}$$

$$\begin{array}{c||cccc} D_2 & (e,e) & (e,P) & (P,e) & (P,P) \\ \hline\hline (e,e) & (e,e) & (e,P) & (P,e) & (P,P) \\ (e,P) & (e,P) & (e,e) & (P,P) & (P,e) \\ (P,e) & (P,e) & (P,P) & (e,e) & (e,P) \\ (P,P) & (P,P) & (P,e) & (e,P) & (e,e) \end{array} \tag{2575}$$

24.4 Cyclic group Z_4

$$D'(e) = 1, \quad D'(a) = e^{i\frac{1\pi}{4}}, \quad D'(b) = e^{i\frac{2\pi}{4}}, \quad D'(c) = e^{i\frac{3\pi}{4}} \tag{2576}$$

24.5 Group S_3

$$\begin{array}{c||cccccc} S_3 & e & a_1 & a_2 & a_3 & a_4 & a_5 \\ \hline\hline e & e & a_1 & a_2 & a_3 & a_4 & a_5 \\ a_1 & a_1 & a_2 & e & a_5 & a_3 & a_4 \\ a_2 & a_2 & e & a_1 & a_4 & a_5 & a_3 \\ a_1 & a_3 & a_4 & a_5 & e & a_1 & a_2 \\ a_1 & a_4 & a_5 & a_3 & a_2 & e & a_1 \\ a_1 & a_5 & a_3 & a_4 & a_1 & a_2 & e \end{array} \tag{2577}$$

$$a_1 = (1,2,3), \quad a_2 = (3,2,1), \quad a_3 = (1,2), \quad a_4 = (2,3), \quad a_5 = (3,1) \tag{2578}$$

2d

$$D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D(a_1) = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}, \quad D(a_2) = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}, \tag{2579}$$

$$D(a_3) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D(a_4) = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}, \quad D(a_5) = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix} \tag{2580}$$

25 Fun with names

- Gordon vs Gordan
  - Paul Gordan (1837-1912) - Clebsch-Gordan decomposition
  - Walter Gordon (1893-1939) - Klein-Gordon equation
- Lorentz vs Lorenz
  - Hendrik Lorentz (1853-1928) - Lorentz transformation, Lorentz force
  - Ludvig Lorenz (1829-1891) - Lorenz gauge
- Hertz vs Hertz
  - Heinrich Hertz (1857-1894) - Hertzian dipole antenna
  - Gustav Hertz (1887-1975) - Franck-Hertz experiment
- Bragg vs Bragg
  - William Henry Bragg (1862-1942) - Bragg equation
  - William Lawrence Bragg (1890-1971) - Bragg equation
- Klein vs Klein
  - Oskar Klein (1894-1977) - Klein-Gordon equation, Kaluza-Klein theory
  - Felix Klein (1849-1925) - Klein bottle
- Euler vs Euler
  - Hans Heinrich Euler (1909-1941) - Euler–Heisenberg Lagrangian
  - Leonhard Euler (1707-1783) - Euler's formula
- Weyl vs Weil
  - Hermann Weyl (1885-1955) - Weyl spinor, Weyl group
  - Andre Weil (1906-1998) - Weil group, Chern–Weil homomorphism
- Jordan vs Jordan vs Jordan
  - Camille Jordan (1838-1922) - Jordan normal, Jordan-Hoelder theorem
  - Wilhelm Jordan (1842-1899) - Gauss-Jordan elimination
  - Pascual Jordan (1902-1980) - Jordan algebra, Jordan Wigner transformation
- Kac vs Kac
  - Victor Kac (1943-...) - Kac–Moody algebra
  - Mark Kac (1904-1984) - Feynman–Kac formula