4 Quantum Field Theory II – Exercise sheet 2 (2025-06-11)

4.1 Exercise 1 - Dimensional Regularization in QED

We consider the 1-loop vacuum polarization discussed in the lecture, for which we found

$$\Pi_2^{\mu\nu} = -4ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{2k^\mu k^\nu - \eta^{\mu\nu}(k^2 - p\cdot k + m^2)}{[(k-p)^2 + m^2][k^2+m^2]} \tag{1}$$

Our goal is to compute this 1-loop integral in dimensional regularization.

1. Use the Feynman parameter trick to write the integral with a denominator that is a complete square.
2. Prove
$$\int \frac{d^Dk}{(2\pi)^D} \frac{k^{2a}}{(k^2-\Delta)^b} = i(-1)^{a-b} \frac{1}{(4\pi)^{D/2}} \frac{1}{\Delta^{b-a-\frac{D}{2}}} \frac{\Gamma\left(a+\frac{D}{2}\right)\Gamma\left(b-a-\frac{D}{2}\right)}{\Gamma(b)\,\Gamma\left(\frac{D}{2}\right)} \tag{2}$$
   where Γ is the Euler gamma function, and write out the special cases $a=0, b=2$ and $a=1, b=2$.
3. Compute (1) in dimensional regularization setting $D = 4-\epsilon$. Give the result in the limit $p^2 \gg m^2$.

1. 1.) With the observation

$$\frac{1}{AB} = \int_0^1 \frac{1}{[At + B(1-t)]^2} dt \tag{224}$$

we can write

$$\begin{aligned}
\to \frac{2k^\mu k^\nu - \eta^{\mu\nu}(k^2 - p\cdot k + m^2)}{[(k-p)^2+m^2][k^2+m^2]} &= \int_0^1 dt \frac{2k^\mu k^\nu - \eta^{\mu\nu}(k^2 - p\cdot k + m^2)}{([(k-p)^2+m^2]t + [k^2+m^2](1-t))^2} &(225)\\
&= \int_0^1 dt \frac{2k^\mu k^\nu - \eta^{\mu\nu}(k^2 - p\cdot k + m^2)}{([k^2 - 2kp + p^2 + m^2]t + [k^2+m^2](1-t))^2} &(226)\\
&= \int_0^1 dt \frac{2k^\mu k^\nu - \eta^{\mu\nu}(k^2 - p\cdot k + m^2)}{(k^2 - 2kpt + p^2t + m^2)^2} &(227)\\
&= \int_0^1 dt \frac{2k^\mu k^\nu - \eta^{\mu\nu}(k^2 - p\cdot k + m^2)}{([k-pt]^2 - p^2t^2 + p^2t + m^2)^2} &(228)\\
&= \int_0^1 dt \frac{2k^\mu k^\nu - \eta^{\mu\nu}(k^2 - p\cdot k + m^2)}{([k-pt]^2 + p^2t(1-t) + m^2)^2} &(229)
\end{aligned}$$

then with $q^\mu = k^\mu - p^\mu t$ and $\Delta = -(p^2t(1-t) + m^2) = p^2t(t-1) - m^2$

$$\begin{aligned}
\to \frac{2k^\mu k^\nu - \eta^{\mu\nu}(k^2 - p\cdot k + m^2)}{[(k-p)^2+m^2][k^2+m^2]} &= \int_0^1 dt \frac{2(q^\mu + p^\mu t)(q^\nu + p^\nu t) - \eta^{\mu\nu}[(q+pt)^2 - p(q+pt) + m^2]}{(q^2-\Delta)^2} &(230)\\
= \int_0^1 dt \frac{2(q^\mu q^\nu + (p^\mu q^\nu + p^\nu q^\mu)t + p^\mu p^\nu t^2) - \eta^{\mu\nu}[q^2 + 2q\cdot pt + p^2t^2 - p\cdot q - p^2t + m^2]}{(q^2-\Delta)^2} &&(231)\\
= \int_0^1 dt \frac{2(q^\mu q^\nu + (p^\mu q^\nu + p^\nu q^\mu)t + p^\mu p^\nu t^2) - \eta^{\mu\nu}[q^2 + q\cdot p(2t-1) + p^2(t-1)t + m^2]}{(q^2-\Delta)^2} &&(232)
\end{aligned}$$

we have with $d^4q = d^4k$ (the momentum shift does not change the integral measure)

$$\begin{aligned}
\Pi_2^{\mu\nu} &= -4ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{2k^\mu k^\nu - \eta^{\mu\nu}(k^2 - p\cdot k + m^2)}{[(k-p)^2+m^2][k^2+m^2]} &(233)\\
&= -4ie^2 \int_0^1 dt \int \frac{d^4q}{(2\pi)^4} \frac{2(q^\mu q^\nu + (p^\mu q^\nu + p^\nu q^\mu)t + p^\mu p^\nu t^2) - \eta^{\mu\nu}[q^2 + q\cdot p(2t-1) + p^2(t-1)t + m^2]}{(q^2-\Delta)^2} &(234)
\end{aligned}$$

Since the denominator is rotationally symmetric in q so the linear terms are vanishing (see substitution $q \to -q$: $d^4q\, q f(q^2) = 0$)

$$\begin{aligned}
\Pi_2^{\mu\nu} &= -4ie^2 \int_0^1 dt \int \frac{d^4q}{(2\pi)^4} \frac{2(q^\mu q^\nu + \cancel{(p^\mu q^\nu + p^\nu q^\mu)}t + p^\mu p^\nu t^2) - \eta^{\mu\nu}[q^2 + \cancel{q\cdot p}(2t-1) + p^2(t-1)t + m^2]}{(q^2-\Delta)^2} &(235)\\
&= -4ie^2 \int_0^1 dt \int \frac{d^4q}{(2\pi)^4} \frac{2(q^\mu q^\nu + p^\mu p^\nu t^2) - \eta^{\mu\nu}[q^2 + p^2(t-1)t + m^2]}{(q^2-\Delta)^2} &(236)
\end{aligned}$$

now we can split-off the q^2-part of the integrand (for the dimensional regularization it is important to leave the $q^\mu q^\nu$-part untouched for now)

$$\Pi_2^{\mu\nu} = -4ie^2 \int_0^1 dt \int \frac{d^4q}{(2\pi)^4} \frac{2p^\mu p^\nu t^2 + 2q^\mu q^\nu - \eta^{\mu\nu}[q^2 + p^2(t-1)t + m^2]}{(q^2-\Delta)^2} \tag{237}$$

$$= -4ie^2 \int_0^1 dt \int \frac{d^4q}{(2\pi)^4} \frac{2p^\mu p^\nu t^2 - \eta^{\mu\nu}[p^2(t-1)t + m^2]}{(q^2-\Delta)^2} - 4ie^2 \int_0^1 dt \int \frac{d^4q}{(2\pi)^4} \frac{2q^\mu q^\nu - \eta^{\mu\nu} q^2}{(q^2-\Delta)^2} \tag{238}$$

Now we can split-off the elementary t-integration (marking the $p^\mu p^\nu$ part in red)

$$\Pi_2^{\mu\nu} = -4ie^2(2p^\mu p^\nu) \int_0^1 dt\, t^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2-\Delta)^2} - 4ie^2\eta^{\mu\nu} \int_0^1 dt\, (p^2(1-t)t - m^2) \int \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2-\Delta)^2} \tag{239}$$

$$-4ie^2 \int_0^1 dt \int \frac{d^4q}{(2\pi)^4} \frac{2q^\mu q^\nu - \eta^{\mu\nu} q^2}{(q^2-\Delta)^2} \tag{240}$$

2.) The surface and volume of D-dimensional unit sphere are given by $S_{D-1} = D \cdot V_D = \frac{2\pi^{D/2}}{\Gamma(\frac{D}{2})}$ (from the old the standard trick converting of D-dimensional Gauss integral spherical coordinates and recognizing the Gamma function in the radial integration).

$$\int \frac{d^Dk}{(2\pi)^D} \frac{k^{2a}}{(k^2-\Delta)^b} = \frac{1}{(2\pi)^D} \int d^{D-1}\Omega \int_0^\infty dk\, k^{D-1} \frac{k^{2a}}{(k^2-\Delta)^b} \tag{241}$$

$$= \frac{1}{(2\pi)^D} \frac{1}{(-\Delta)^b} \frac{2\pi^{D/2}}{\Gamma(\frac{D}{2})} \int_0^\infty dk \frac{k^{2a+D-1}}{(1-k^2/\Delta)^b} \tag{242}$$

$$= \frac{1}{2^{D-1}\pi^{D/2}} \frac{1}{(-1)^b \Delta^b} \frac{1}{\Gamma(\frac{D}{2})} \int_0^\infty dk \frac{k^{2a+D-1}}{(1-k^2/\Delta)^b} \tag{243}$$

$$\overset{q^2=k^2/\Delta}{=} \frac{1}{2^{D-1}\pi^{D/2}} \frac{1}{(-1)^b \Delta^b} \frac{1}{\Gamma(\frac{D}{2})} \Delta^{(2a+D-1)/2+1/2} \int_0^\infty dq \frac{q^{2a+D-1}}{(1-q^2)^b} \tag{244}$$

$$= \frac{1}{2^{D-1}\pi^{D/2}} \frac{1}{(-1)^b \Delta^{b-a-D/2}} \frac{1}{\Gamma(\frac{D}{2})} \int_0^\infty dq \frac{q^{2a+D-1}}{(1-q^2)^b} \tag{245}$$

$$\overset{q=iy}{=} \frac{1}{2^{D-1}\pi^{D/2}} \frac{1}{\Delta^{b-a-D/2}} \frac{1}{\Gamma(\frac{D}{2})} (-1)^a (-1)^{-b} i \int_0^\infty dy \frac{y^{2a+D-1}}{(1+y^2)^b} \tag{246}$$

$$= \frac{1}{2^{D-1}\pi^{D/2}} \frac{1}{\Delta^{b-a-D/2}} \frac{1}{\Gamma(\frac{D}{2})} (-1)^a (-1)^{-b} i \frac{\Gamma(a+\frac{D}{2})\Gamma(b-a-\frac{D}{2})}{2\Gamma(b)} \tag{247}$$

$$= \frac{1}{(4\pi)^{D/2}} \frac{1}{\Delta^{b-a-D/2}} (-1)^{a-b} i \frac{\Gamma(a+\frac{D}{2})\Gamma(b-a-\frac{D}{2})}{\Gamma(b)\Gamma(\frac{D}{2})} \tag{248}$$

Using $\Gamma(2) = 1$ and $\Gamma\left(1+\frac{D}{2}\right) = \frac{D}{2}\Gamma\left(\frac{D}{2}\right)$

- Special case $a = 0, b = 2$

$$\int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2-\Delta)^2} = \frac{i}{(4\pi)^{D/2}} \frac{1}{\Delta^{2-\frac{D}{2}}} \Gamma\left(2-\frac{D}{2}\right) \tag{249}$$

- Special case $a = 1, b = 2$

$$\int \frac{d^Dk}{(2\pi)^D} \frac{k^2}{(k^2-\Delta)^2} = -\frac{i}{(4\pi)^{D/2}} \frac{1}{\Delta^{1-\frac{D}{2}}} \frac{\Gamma\left(1+\frac{D}{2}\right)\Gamma\left(1-\frac{D}{2}\right)}{\Gamma\left(\frac{D}{2}\right)} \tag{250}$$

$$= -\frac{i}{(4\pi)^{D/2}} \frac{1}{\Delta^{1-\frac{D}{2}}} \frac{D}{2} \Gamma\left(1-\frac{D}{2}\right) \tag{251}$$

3.) With

$$\int \frac{d^{4-\epsilon}k}{(2\pi)^{4-\epsilon}} \frac{1}{(k^2-\Delta)^2} = \frac{i}{(4\pi)^{2-\epsilon/2}} \frac{1}{\Delta^{2-\frac{4-\epsilon}{2}}} \Gamma\left(2-\frac{4-\epsilon}{2}\right) \tag{252}$$

$$= \frac{i}{(4\pi)^{2-\epsilon/2}} \frac{1}{\Delta^{\frac{\epsilon}{2}}} \Gamma\left(\frac{\epsilon}{2}\right) \tag{253}$$

$$\int \frac{d^{4-\epsilon}k}{(2\pi)^{4-\epsilon}} \frac{k^2}{(k^2-\Delta^2)^2} = -\frac{i}{(4\pi)^{2-\epsilon/2}} \frac{1}{\Delta^{1-\frac{4-\epsilon}{2}}} \left(\frac{4-\epsilon}{2}\right) \Gamma\left(1-\frac{4-\epsilon}{2}\right) \tag{254}$$

$$= -\frac{i}{(4\pi)^{2-\epsilon/2}} \frac{1}{\Delta^{\frac{\epsilon}{2}-1}} \left(2-\frac{\epsilon}{2}\right) \Gamma\left(\frac{\epsilon}{2}-1\right) \tag{255}$$

we can simplify with $q^\mu q^\nu = \frac{q^2}{D}\eta^{\mu\nu}$ and write the integrals in dimensional regularization and introducing a mass-dimension parameter μ and $D = 4 - \epsilon$

$$\Pi_2^{\mu\nu} = -4ie^2(2p^\mu p^\nu)\int_0^1 dt\, t^2 \int \frac{d^4q}{(2\pi)^4}\frac{1}{(q^2-\Delta)^2} - 4ie^2\eta^{\mu\nu}\int_0^1 dt\,(p^2(1-t)t - m^2)\int \frac{d^4q}{(2\pi)^4}\frac{1}{(q^2-\Delta)^2} \tag{256}$$

$$-4i\left(\frac{2}{D}-1\right)e^2\eta^{\mu\nu}\int_0^1 dt \int \frac{d^4q}{(2\pi)^4}\frac{q^2}{(q^2-\Delta)^2} \tag{257}$$

$$= -4ie^2\mu^\epsilon(2p^\mu p^\nu)\frac{i}{(4\pi)^{2-\epsilon/2}}\Gamma\left(\frac{\epsilon}{2}\right)\int_0^1 dt\,\frac{t^2}{\Delta^{\frac{\epsilon}{2}}} - 4ie^2\mu^\epsilon\eta^{\mu\nu}\frac{i}{(4\pi)^{2-\epsilon/2}}\Gamma\left(\frac{\epsilon}{2}\right)\int_0^1 dt\,\frac{p^2(1-t)t-m^2}{\Delta^{\frac{\epsilon}{2}}} \tag{258}$$

$$-4ie^2\mu^\epsilon\left(\frac{2}{4-\epsilon}-1\right)\eta^{\mu\nu}\frac{-i}{(4\pi)^{2-\epsilon/2}}\left(2-\frac{\epsilon}{2}\right)\Gamma\left(\frac{\epsilon}{2}-1\right)\int_0^1 dt\frac{1}{\Delta^{\frac{\epsilon}{2}-1}} \tag{259}$$

$$= (p^\mu p^\nu)\frac{8e^2\mu^\epsilon}{(4\pi)^{2-\epsilon/2}}\Gamma\left(\frac{\epsilon}{2}\right)\int_0^1 dt\,\frac{t^2}{\Delta^{\frac{\epsilon}{2}}} + \eta^{\mu\nu}\frac{4e^2\mu^\epsilon}{(4\pi)^{2-\epsilon/2}}\Gamma\left(\frac{\epsilon}{2}\right)\int_0^1 dt\,\frac{p^2(1-t)t-m^2}{\Delta^{\frac{\epsilon}{2}}} \tag{260}$$

$$-\eta^{\mu\nu}\frac{4e^2\mu^\epsilon}{(4\pi)^{2-\epsilon/2}}\left(\frac{\epsilon-2}{2}\right)\Gamma\left(\frac{\epsilon}{2}-1\right)\int_0^1 dt\frac{1}{\Delta^{\frac{\epsilon}{2}-1}} \tag{261}$$

For the limit we use the following series expansions

$$\mu^\epsilon = 1 + \log(\mu)\epsilon + \frac{1}{2}\log^2(\mu)\epsilon^2 + \ldots \tag{262}$$

$$\frac{1}{(4\pi)^{2-\epsilon/2}} = \frac{1}{(4\pi)^2}\left(1 + \frac{1}{2}\log(4\pi)\epsilon + \frac{1}{8}\log^2(4\pi)\epsilon^2 + \ldots\right) \tag{263}$$

$$\Gamma\left(\frac{\epsilon}{2}\right) = \frac{2}{\epsilon} - \gamma_{EM} + \frac{1}{24}(6\gamma_{EM}^2 + \pi^2)\epsilon + \frac{1}{24}[-\gamma_{EM}^3 - \gamma_{EM}\frac{\pi^2}{2} + \psi^{(2)}(1)]\epsilon^2 + \ldots \tag{264}$$

$$\Gamma\left(\frac{\epsilon}{2}-1\right) = -\frac{2}{\epsilon} + (\gamma_{EM}-1) + \frac{1}{24}\left(-12 + 12\gamma_{EM} - 6\gamma_{EM}^2 - \pi^2\right)\epsilon + \ldots \tag{265}$$

$$\frac{1}{\Delta^{\epsilon/2}} = 1 - \frac{1}{2}\log(\Delta)\epsilon + \frac{1}{8}\log^2(\Delta)\epsilon^2 + \ldots \tag{266}$$

which gives combined

$$\frac{\mu^\epsilon}{(4\pi)^{-\epsilon/2}}\Gamma\left(\frac{\epsilon}{2}\right) = \frac{2}{\epsilon} + [2\log(\mu) - \gamma_{EM} + \log(4\pi)] + O(\epsilon) \tag{267}$$

$$= \frac{2}{\epsilon} + [\log(\mu^2) + \log(e^{-\gamma_{EM}}) + \log(4\pi)] + O(\epsilon) \tag{268}$$

$$= \frac{2}{\epsilon} + \log(4\pi\mu^2 e^{-\gamma_{EM}}) + O(\epsilon) \tag{269}$$

$$\frac{\mu^\epsilon}{(4\pi)^{-\epsilon/2}}\Gamma\left(\frac{\epsilon}{2}-1\right) = -\frac{2}{\epsilon} + (-2\log(\mu) + \gamma_{EM} - 1 - \log(4\pi)) + O(\epsilon) \tag{270}$$

$$= -\frac{2}{\epsilon} + (-\log(\mu^2) - \log(e^{-\gamma_{EM}}) - 1 - \log(4\pi)) + O(\epsilon) \tag{271}$$

$$= -\frac{2}{\epsilon} - 1 - \log(4\pi\mu^2 e^{-\gamma_{EM}}) + O(\epsilon) \tag{272}$$

then with $m^2 = p^2 t(t-1) - \Delta$

$$\Pi_2^{\mu\nu} = (p^\mu p^\nu)\frac{8e^2}{(4\pi)^2}\left[\frac{2}{\epsilon} + \log(4\pi\mu^2 e^{-\gamma_{EM}})\right]\int_0^1 dt\, t^2\left(1 - \frac{1}{2}\log(\Delta)\epsilon + ...\right) \tag{273}$$

$$+ \eta^{\mu\nu}\frac{4e^2}{(4\pi)^2}\left[\frac{2}{\epsilon} + \log(4\pi\mu^2 e^{-\gamma_{EM}}) + ...\right]\int_0^1 dt\,[p^2(1-t)t - m^2]\left(1 - \frac{1}{2}\log(\Delta)\epsilon + ...\right) \tag{274}$$

$$- \eta^{\mu\nu}\frac{4e^2}{(4\pi)^2}(-1)\left[-\frac{2}{\epsilon} - 1 - \log(4\pi\mu^2 e^{-\gamma_{EM}}) + ...\right]\int_0^1 dt\,\Delta\left(1 - \frac{1}{2}\log^2(\Delta)\epsilon + ...\right) \tag{275}$$

$$= (p^\mu p^\nu)\frac{8e^2}{(4\pi)^2}\left[\frac{2}{\epsilon} + \log(4\pi\mu^2 e^{-\gamma_{EM}})\right]\int_0^1 dt\, t^2\left(1 - \frac{1}{2}\log(\Delta)\epsilon + ...\right) \tag{276}$$

$$+ \eta^{\mu\nu}\frac{4e^2}{(4\pi)^2}\left[\frac{2}{\epsilon} + \log(4\pi\mu^2 e^{-\gamma_{EM}}) + ...\right]\int_0^1 dt\,[2p^2(1-t)t + \Delta]\left(1 - \frac{1}{2}\log(\Delta)\epsilon + ...\right) \tag{277}$$

$$- \eta^{\mu\nu}\frac{4e^2}{(4\pi)^2}\left[\frac{2}{\epsilon} + \log(4\pi\mu^2 e^{-\gamma_{EM}}) + 1 + ...\right]\int_0^1 dt\,\Delta\left(1 - \frac{1}{2}\log^2(\Delta)\epsilon + ...\right) \tag{278}$$

$$= (p^\mu p^\nu)\frac{8e^2}{(4\pi)^2}\left[\frac{2}{\epsilon} + \log(4\pi\mu^2 e^{-\gamma_{EM}})\right]\int_0^1 dt\, t^2\left(1 - \frac{1}{2}\log(\Delta)\epsilon + ...\right) \tag{279}$$

$$+ \eta^{\mu\nu}\frac{4e^2}{(4\pi)^2}\left[\frac{2}{\epsilon} + \log(4\pi\mu^2 e^{-\gamma_{EM}}) + ...\right]\int_0^1 dt\,[2p^2(1-t)t + \Delta]\left(1 - \frac{1}{2}\log(\Delta)\epsilon + ...\right) \tag{280}$$

$$- \eta^{\mu\nu}\frac{4e^2}{(4\pi)^2}\left[\frac{2}{\epsilon} + \log(4\pi\mu^2 e^{-\gamma_{EM}}) + 1 + ...\right]\int_0^1 dt\,\Delta\left(1 - \frac{1}{2}\log^2(\Delta)\epsilon + ...\right) \tag{281}$$

Then taking $\epsilon \to 0$

$$\Pi_2^{\mu\nu} = (p^\mu p^\nu)\frac{8e^2}{(4\pi)^2}\int_0^1 dt\, t^2\left(1 - \frac{1}{2}\log(\Delta)\epsilon + ...\right)\left[\frac{2}{\epsilon} + \log(4\pi\mu^2 e^{-\gamma_{EM}})\right] \tag{282}$$

$$+ \eta^{\mu\nu}p^2\frac{8e^2}{(4\pi)^2}\int_0^1 dt\,(1-t)t\left(1 - \frac{1}{2}\log(\Delta)\epsilon + ...\right)\left[\frac{2}{\epsilon} + \log(4\pi\mu^2 e^{-\gamma_{EM}}) + ...\right] \tag{283}$$

$$- \eta^{\mu\nu}\frac{4e^2}{(4\pi)^2}\int_0^1 dt\,\Delta\left(1 - \frac{1}{2}\log^2(\Delta)\epsilon + ...\right) \tag{284}$$

$$= (p^\mu p^\nu)\frac{8e^2}{(4\pi)^2}\int_0^1 dt\, t^2\left[\frac{2}{\epsilon} + \log\left(\frac{4\pi\mu^2 e^{-\gamma_{EM}}}{\Delta}\right)\right] \tag{285}$$

$$+ \eta^{\mu\nu}p^2\frac{8e^2}{(4\pi)^2}\int_0^1 dt\,(1-t)t\left[\frac{2}{\epsilon} + \log\left(\frac{4\pi\mu^2 e^{-\gamma_{EM}}}{\Delta}\right)\right] \tag{286}$$

$$- \eta^{\mu\nu}\frac{4e^2}{(4\pi)^2}\int_0^1 dt\,\Delta \tag{287}$$

Leaving the $p^\mu p^\nu$ term as is we can t-integrate the other using $p^2 \gg m^2$ and $\int_0^1 dt\,(1-t)t\log[-(t-1)t] = -5/18$

$$\Pi_2^{\mu\nu} = (p^\mu p^\nu)\frac{8e^2}{(4\pi)^2}\int_0^1 dt\, t^2\left[\frac{2}{\epsilon} + \log\left(\frac{4\pi\mu^2 e^{-\gamma_{EM}}}{\Delta}\right)\right] + \eta^{\mu\nu}p^2\frac{8e^2}{(4\pi)^2}\int_0^1 dt\,(1-t)t\left[\frac{2}{\epsilon} + \log\left(\frac{4\pi\mu^2 e^{-\gamma_{EM}}}{p^2 t(t-1) - \cancel{m^2}}\right)\right] \tag{288}$$

$$= (p^\mu p^\nu)\frac{8e^2}{(4\pi)^2}\int_0^1 dt\, t^2\left[\frac{2}{\epsilon} + \log\left(\frac{4\pi\mu^2 e^{-\gamma_{EM}}}{\Delta}\right)\right] + \eta^{\mu\nu}p^2\frac{8e^2}{(4\pi)^2}\left[\frac{1}{6}\frac{2}{\epsilon} + \frac{1}{6}\log\left(\frac{4\pi\mu^2 e^{-\gamma_{EM}}}{-p^2}\right) + \frac{5}{18}\right] \tag{289}$$

$$= (p^\mu p^\nu)\frac{8e^2}{(4\pi)^2}\int_0^1 dt\, t^2\left[\frac{2}{\epsilon} + \log\left(\frac{4\pi\mu^2 e^{-\gamma_{EM}}}{\Delta}\right)\right] + \eta^{\mu\nu}p^2\frac{e^2}{12\pi^2}\left[\frac{2}{\epsilon} + \log\left(\frac{4\pi\mu^2 e^{-\gamma_{EM}}}{-p^2}\right) + \frac{5}{3}\right] \tag{290}$$

4.2 Exercise 2:* Ward identity

In the integral (1) we neglected contributions proportional to $p^\mu p^\nu$. Compute these missing contributions and verify the Ward identities.

Applying the the same substitution from above $q^\mu = k^\mu - p^\mu t$ and $\Delta = -(p^2 t(1-t) + m^2) = p^2 t(t-1) - m^2$ with the missing part (writing everything now a bit more condensed - using the results from above like cancellation of linear q-terms)

$$-4ie^2\int\frac{d^4k}{(2\pi)^4}\frac{-k^\mu p^\nu-k^\nu p^\mu}{[(k-p)^2+m^2][k^2+m^2]}=-4ie^2\int_0^1 dt\int\frac{d^4q}{(2\pi)^4}\frac{-(q^\mu+tp^\mu)p^\nu-(q^\nu+tp^\nu)p^\mu}{(q^2-\Delta)^2} \tag{291}$$

$$=-4ie^2\int_0^1 dt\int\frac{d^4q}{(2\pi)^4}\frac{-\cancel{q^\mu p^\nu}-tp^\mu p^\nu-\cancel{q^\nu p^\mu}-tp^\nu p^\mu}{(q^2-\Delta)^2} \tag{292}$$

$$=-4ie^2\int_0^1 dt\int\frac{d^4q}{(2\pi)^4}\frac{-2tp^\mu p^\nu}{(q^2-\Delta)^2} \tag{293}$$

Now we can combine this with the red term (we use the from of the first appearance)

$$-4ie^2(2p^\mu p^\nu)\int_0^1 dt\,t^2\int\frac{d^4q}{(2\pi)^4}\frac{1}{(q^2-\Delta)^2}-4ie^2\int_0^1 dt\int\frac{d^4q}{(2\pi)^4}\frac{-2tp^\mu p^\nu}{(q^2-\Delta)^2} \tag{294}$$

$$=-4ie^2(2p^\mu p^\nu)\int_0^1 dt\,t(t-1)\int\frac{d^4q}{(2\pi)^4}\frac{1}{(q^2-\Delta)^2} \tag{295}$$

Here we see that the $p^\mu p^\nu$ t^2-term from the problem above is joined by an identical t-term - so we can reuse the calculation result from above and obtain for the $p^\mu p^\nu$ contribution

$$(p^\mu p^\nu)\frac{8e^2}{(4\pi)^2}\int_0^1 dt\,t(t-1)\left[\frac{2}{\epsilon}+\log\left(\frac{4\pi\mu^2e^{-\gamma_{EM}}}{\Delta}\right)\right]=(p^\mu p^\nu)\frac{e^2}{12\pi^2}\left[\frac{2}{\epsilon}+\log\left(\frac{4\pi\mu^2e^{-\gamma_{EM}}}{-p^2}\right)+\frac{5}{3}\right] \tag{296}$$

which implies that adding the missing contribution added to $\Pi_2^{\mu\nu}$ gives

$$\Pi_2'^{\mu\nu}=\Pi_2^{\mu\nu}+\text{neglected }(p^\mu p^\nu)=(-p^\mu p^\nu+\eta^{\mu\nu}p^2)\frac{e^2}{12\pi^2}\left[\frac{2}{\epsilon}+\log\left(\frac{4\pi\mu^2e^{-\gamma_{EM}}}{-p^2}\right)+\frac{5}{3}\right]. \tag{297}$$

Now we can see

$$p_\mu\Pi_2'^{\mu\nu}\sim p_\mu(-p^\mu p^\nu+\eta^{\mu\nu}p^2) \tag{298}$$

$$\sim -p^2p^\nu+p^\nu p^2 \tag{299}$$

$$=0 \tag{300}$$

So we proved the Ward identity for this case.