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Advanced Topics in Gravity – Exercise sheet 6 - 2025-07-09

Exercise 1 - Klein-Gordon inner product

Prove, by considering two different Cauchy surfaces Σ_1 and Σ_2, that the Klein-Gordon inner product

$$(\phi_1, \phi_2) = -i \int_\Sigma d^3x \, \sqrt{h} \, n^\mu \big(\phi_1 D_\mu \phi_2^* - \phi_2^* D_\mu \phi_1\big) \tag{1}$$

is independent of the choice of the Cauchy surface.

With KG in curved space

$$(D_\mu D^\mu + m^2)\phi = 0 \tag{2}$$

Observation (in analogy to 2nd order ODE's define a Wronskian)

$$\begin{aligned}
W_\mu &\equiv (D_\mu \phi_1)\phi_2^* - \phi_1(D_\mu \phi_2^*) && (3)\\
D^\mu W_\mu &= (D^\mu D_\mu \phi_1)\phi_2^* + \cancel{(D_\mu \phi_1)(D^\mu \phi_2^*)} - \cancel{(D^\mu \phi_1)(D_\mu \phi_2^*)} - \phi_1(D^\mu D_\mu \phi_2^*) && (4)\\
&= (D^\mu D_\mu \phi_1)\phi_2^* - \phi_1(D^\mu D_\mu \phi_2^*) && (5)\\
&= -m^2 \phi_1 \phi_2^* - \phi_1(-m^2 \phi_2^*) && (6)\\
&= 0 && (7)
\end{aligned}$$

With the covariant divergence theorem

$$\int_{\partial\Omega} T^\mu \sqrt{-g} dS_\mu = \int_\Omega D_\mu T^\mu \sqrt{-g} d\Omega \tag{8}$$

we define a 4-volume Ω between the two Cauchy Surfaces $\Sigma_{1,2}$ and use $T = W$. Then the (taking into account the orientation of top and bottom)

$$\begin{aligned}
0 &= \int_\Omega \underbrace{D_\mu W^\mu}_{=0} \sqrt{-g} d\Omega = \int_{\partial\Omega} W^\mu \sqrt{-g} dS_\mu && (9)\\
&= \int_{\Sigma_1} d^3x \, \sqrt{h} \, n^\mu \big(\phi_1 D_\mu \phi_2^* - \phi_2^* D_\mu \phi_1\big) + \int_{\Sigma_2} d^3x \, \sqrt{h} \, (-n^\mu) \big(\phi_1 D_\mu \phi_2^* - \phi_2^* D_\mu \phi_1\big) && (10)
\end{aligned}$$

resulting in

$$\int_{\Sigma_1} d^3x \, \sqrt{h} \, n^\mu \big(\phi_1 D_\mu \phi_2^* - \phi_2^* D_\mu \phi_1\big) = \int_{\Sigma_2} d^3x \, \sqrt{h} \, n^\mu \big(\phi_1 D_\mu \phi_2^* - \phi_2^* D_\mu \phi_1\big) \tag{11}$$

Exercise 2 - Bogoliubov transformation

We have seen that any field solution to the Klein-Gordon equation of motion can be expanded in terms of two bases:

$$\varphi = \sum_i (a_i f_i + a_i^\dagger f_i^*) = \sum_i (b_i g_i + b_i^\dagger g_i^*), \tag{2.1}$$

where the basis modes are defined in different stationary regions and are normalized with respect to the Klein-Gordon inner product.

Thus, the operators in the expansions satisfy

$$[a_k, a_{k'}^\dagger] = \delta(k - k'), \qquad [b_p, b_{p'}^\dagger] = \delta(p - p'). \tag{2.2}$$

The modes in the basis $\{g_i, g_i^*\}$ can be expressed in terms of the basis $\{f_j, f_j^*\}$ as

$$g_i = \sum_j (A_{ij} f_j + B_{ij} f_j^*), \tag{2.3}$$

$$g_i^* = \sum_j (B_{ij}^* f_j + A_{ij}^* f_j^*). \tag{2.4}$$

This relation between the two bases is called a *Bogoliubov transformation*, and it can also be written in matrix form as

$$\begin{pmatrix} g \\ g^* \end{pmatrix} = \begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix} \begin{pmatrix} f \\ f^* \end{pmatrix}. \tag{2.5}$$

Note: The basis is normalized in such a way that

$$(\alpha f, \beta g) = \alpha \beta^* (f, g),$$

where α, β are any Bogoliubov coefficients and f, g are any set of modes from the bases.

1. From the condition $(g_i, g_j) = \delta_{ij}$, find the relation:

$$AA^\dagger - BB^\dagger = 1. \tag{2.6}$$

2. From the condition $(g_i, g_j^*) = 0$, find the relation:

$$AB^{\mathrm{t}} - BA^{\mathrm{t}} = 0. \tag{2.7}$$

3. See how the previous relations (2.6) and (2.7) allow us to write the inverse matrix for the transformation.

4. By writing the field expansion in a matrix form as

$$\varphi = \begin{pmatrix} b & b^\dagger \end{pmatrix} \begin{pmatrix} g \\ g^* \end{pmatrix} = \begin{pmatrix} a & a^\dagger \end{pmatrix} \begin{pmatrix} f \\ f^* \end{pmatrix}, \tag{2.8}$$

and using the Bogoliubov transformations, write in a matrix form the relation between creation/annihilation operators with respect to the two different bases, i.e., find M' such that

$$\begin{pmatrix} b \\ b^\dagger \end{pmatrix} = M' \begin{pmatrix} a \\ a^\dagger \end{pmatrix}. \tag{2.9}$$

1.

$$
\begin{aligned}
(g_i, g_j) &= \left(\sum_k (A_{ik} f_k + B_{ik} f_k^*), \sum_l (A_{jl} f_l + B_{jl} f_l^*) \right) && (12) \\
&= \sum_{k,l} (A_{ik} f_k + B_{ik} f_k^*, A_{jl} f_l + B_{jl} f_l^*) && (13) \\
&= \sum_{k,l} A_{ik} A_{jl}^* \underbrace{(f_k, f_l)}_{=\delta_{kl}} + \sum_{k,l} A_{ik} B_{jl}^* \underbrace{(f_k, f_l^*)}_{=0} + \sum_{k,l} B_{ik} A_{jl}^* \underbrace{(f_k^*, f_l)}_{=0} + \sum_{k,l} B_{ik} B_{jl}^* \underbrace{(f_k^*, f_l^*)}_{=-\delta_{kl}} && (14) \\
&= \sum_k A_{ik} A_{jk}^* - B_{ik} B_{jk}^* && (15) \\
&\overset{!}{=} \delta_{ij} \quad \to AA^\dagger - BB^\dagger = 1 && (16)
\end{aligned}
$$

2.

$$
\begin{aligned}
(g_i, g_j^*) &= \left(\sum_k (A_{ik} f_k + B_{ik} f_k^*), \sum_l (A_{jl} f_l + B_{jl} f_l^*)^* \right) && (17) \\
&= \sum_{k,l} \left(A_{ik} f_k + B_{ik} f_k^*, A_{jl}^* f_l^* + B_{jl}^* f_l\right) && (18) \\
&= \sum_{k,l} A_{ik} A_{jl} \underbrace{(f_k, f_l^*)}_{=0} + \sum_{k,l} A_{ik} B_{jl} \underbrace{(f_k, f_l)}_{=\delta_{kl}} + \sum_{k,l} B_{ik} A_{jl} \underbrace{(f_k^*, f_l^*)}_{=-\delta_{kl}} + \sum_{k,l} B_{ik} B_{jl} \underbrace{(f_k^*, f_l)}_{=0} && (19) \\
&= \sum_k A_{ik} B_{jk} - B_{ik} A_{jk} && (20) \\
&= 0 \quad \to AB^t - BA^t = 0 && (21)
\end{aligned}
$$

3. We use the proved equations above and extend them by

$$
\begin{aligned}
AA^\dagger - BB^\dagger = 1 \quad &\to \quad A^* A^t - B^* B^t = 1 && (22) \\
AB^t - BA^t = 0 \quad &\to \quad A^* B^\dagger - B^* A^\dagger = 0 && (23)
\end{aligned}
$$

using $A^\dagger \equiv (A^*)^t \equiv (A^t)^*$. Using the four identities we can guess the inverse

$$
M^{-1} = \begin{pmatrix} A^\dagger & -B^t \\ -B^\dagger & A^t \end{pmatrix}. \tag{24}
$$

Checking the result

$$
\begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix} \begin{pmatrix} A^\dagger & -B^t \\ -B^\dagger & A^t \end{pmatrix} = \begin{pmatrix} AA^\dagger - BB^\dagger & -AB^t + BA^t \\ B^* A^\dagger - A^* B^\dagger & -B^* B^t + A^* A^t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{25}
$$

4. With $\vec{a}^T \vec{f} = \phi = \vec{b}^T \vec{g}$ we see

$$
\vec{a}^T \vec{f} = \phi = \vec{b}^T \vec{g} = (M' \vec{a})^T \vec{g} = \vec{a}^T M'^T \vec{g} = \vec{a}^T M'^T M \vec{f} \tag{26}
$$

meaning $M'^T M = 1 \to M'^T = M^{-1}$ so

$$
M' = (M^{-1})^T = \begin{pmatrix} A^\dagger & -B^\dagger \\ -B^t & A^t \end{pmatrix}. \tag{27}
$$