0.1 Donoghue, Sorbo - A Prelude to Quantum Field Theory 2022

0.1.1 Problem 2.1 - Contractions

Using the commutation relation $[a_i, a_j^\dagger] = \delta_{ij}$ and $a_i|0\rangle = 0$

- 1-particle

$$\begin{aligned}
H|n\rangle &= \sum_{n'} \hbar\omega_{n'} a_{n'}^\dagger a_{n'} |n\rangle && (1)\\
&= \sum_{n'} \hbar\omega_{n'} a_{n'}^\dagger a_{n'} a_n^\dagger |0\rangle && (2)\\
&= \sum_{n'} \hbar\omega_{n'} a_{n'}^\dagger (a_n^\dagger a_{n'} + \delta_{n'n}) |0\rangle && (3)\\
&= \sum_{n'} \hbar\omega_{n'} a_{n'}^\dagger \delta_{n'n} |0\rangle && (4)\\
&= \hbar\omega_n |n\rangle && (5)
\end{aligned}$$

- 2-particle

$$\begin{aligned}
H|n_1 n_2\rangle &= \sum_{n'} \hbar\omega_{n'} a_{n'}^\dagger a_{n'} |n_1 n_2\rangle && (6)\\
&= \sum_{n'} \hbar\omega_{n'} a_{n'}^\dagger a_{n'} a_{n_1}^\dagger a_{n_2}^\dagger |0\rangle && (7)\\
&= \sum_{n'} \hbar\omega_{n'} a_{n'}^\dagger (a_{n_1}^\dagger a_{n'} + \delta_{n'n_1}) a_{n_2}^\dagger |0\rangle && (8)\\
&= \sum_{n'} \hbar\omega_{n'} a_{n'}^\dagger a_{n_1}^\dagger a_{n'} a_{n_2}^\dagger |0\rangle + \sum_{n'} \hbar\omega_{n'} a_{n'}^\dagger \delta_{n'n_1} a_{n_2}^\dagger |0\rangle && (9)\\
&= \sum_{n'} \hbar\omega_{n'} a_{n'}^\dagger a_{n_1}^\dagger (a_{n_2}^\dagger a_{n'} + \delta_{n_2 n'}) |0\rangle + \sum_{n'} \hbar\omega_{n'} a_{n'}^\dagger \delta_{n'n_1} a_{n_2}^\dagger |0\rangle && (10)\\
&= \sum_{n'} \hbar\omega_{n'} \delta_{n'n_2} |n_1 n_{n'}\rangle + \sum_{n'} \hbar\omega_{n'} \delta_{n'n_1} |n_2 n_{n'}\rangle && (11)\\
&= \hbar(\omega_{n_2} + \omega_{n_1}) |n_1 n_2\rangle && (12)\\
& && (13)
\end{aligned}$$

0.1.2 Problem 2.3 - Casimir effect - NOT DONE YET

(a)

$$\begin{aligned}
\langle 0|H|0\rangle &= \frac{1}{2}\sum_{n=-\infty}^{+\infty}\omega_n\langle 0|a_n a_n^\dagger + a_n^\dagger a_n|0\rangle && (14)\\
&= \frac{1}{2}\sum_{n=-\infty}^{+\infty}\omega_n\langle 0|(1+2a_n^\dagger a_n)|0\rangle && (15)\\
&= \sum_{n=-\infty}^{+\infty}\omega_n \underbrace{\langle 0|a_n^\dagger a_n|0\rangle}_{=0} + \frac{1}{2}\sum_{n=-\infty}^{\infty}\omega_n && (16)\\
&= \frac{2\pi}{2L}\sum_{n=-\infty}^{\infty}|n| && (17)\\
&= \frac{2\pi}{L}\sum_{n=0}^{\infty} n && (18)
\end{aligned}$$

$$F = -\frac{d\langle 0|H|0\rangle}{dL} = \frac{2\pi}{L^2}\sum_n n \tag{19}$$

(b) (i) Geometric series (start at $n=0$)

$$\sum_{n=0} ne^{-\epsilon n} = -\frac{d}{d\epsilon}\sum_n e^{-\epsilon n} = -\frac{d}{d\epsilon}\sum_n (e^{-\epsilon})^n = -\frac{d}{d\epsilon}\frac{1}{1-e^{-\epsilon}} = \frac{e^{-\epsilon}}{(1-e^{-\epsilon})^2} \tag{20}$$

But we can shift start to $n=1$ (should actually not change the result - as we just don't add n=0)

$$\sum_{n=1} ne^{-\epsilon n} = -\frac{d}{d\epsilon}\sum_n e^{-\epsilon n} = -\frac{d}{d\epsilon}\sum_n (e^{-\epsilon})^n = -\frac{d}{d\epsilon}\left(\frac{1}{1-e^{-\epsilon}} - 1\right) \tag{21}$$

(ii)

$$\sum_{n=1} n = \lim_{z\to -1}\sum_{n=1} n^{-z} = \zeta(-1) = -\frac{1}{12} \tag{22}$$

0.2 Schmueser - Feynman-Graphen und Eichtheorie

0.2.1 Problem 1.1

$$\sigma\cdot\mathbf{a} = \begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3\end{pmatrix} \tag{23}$$

$$\begin{aligned}
(\sigma\cdot\mathbf{a})(\sigma\cdot\mathbf{b}) &= \begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3\end{pmatrix}\begin{pmatrix} b_3 & b_1 - ib_2 \\ b_1 + ib_2 & -b_3\end{pmatrix} && (24)\\
&= \mathbf{a}\cdot\mathbf{b}\,1_2 + i\sigma\cdot(\mathbf{a}\times\mathbf{b}) && (25)
\end{aligned}$$

0.2.2 Problem 1.2

$$\mathbf{P}\times\mathbf{P} = (-i\hbar)^2\underbrace{(\nabla\times\nabla)}_{=0} + e^2\underbrace{(\mathbf{A}\times\mathbf{A})}_{=0} - i\hbar e(\nabla\times\mathbf{A} + \mathbf{A}\times\nabla) \tag{26}$$

$$= -i\hbar e(\nabla\times\mathbf{A} + \mathbf{A}\times\nabla) \tag{27}$$

$$= -i\hbar e\begin{pmatrix}\partial_y A_z - \partial_z A_y + A_y\partial_z - A_z\partial_y \\ ... \\ ...\end{pmatrix} \tag{28}$$

$$= -i\hbar e\begin{pmatrix}(\partial_y A_z + A_z\partial_y) - (\partial_z A_y + A_y\partial_z) + A_y\partial_z - A_z\partial_y \\ ... \\ ...\end{pmatrix} \tag{29}$$

$$= -i\hbar e\begin{pmatrix}\partial_y A_z - \partial_z A_y \\ ... \\ ...\end{pmatrix} \tag{30}$$

$$= -i\hbar e\mathbf{B} \tag{31}$$

and therefore

$$(\sigma\cdot\mathbf{P})(\sigma\cdot\mathbf{P}) = \mathbf{P}^2\, 1_2 + e\hbar\sigma\cdot\mathbf{B} \tag{32}$$

0.3 Lancaster, Blundell - Quantum Field Theory for the gifted amateur

Exercise 1.1 - Snell's law via Fermat's principle

The light travels from point A in medium 1 to point B in medium 2. We assume a vertical medium boundary at x_0 and that the light travels within a medium in the straight line. This makes y_0 the free parameter and the the travel time is given by

$$t = \frac{s_{A0}}{c/n_1} + \frac{s_{0B}}{c/n_2} \tag{33}$$

$$= \sqrt{\frac{(x_A - x_0)^2 + (y_A - y_0)^2}{c/n_1}} + \sqrt{\frac{(x_0 - x_B)^2 + (y_0 - y_B)^2}{c/n_2}} \tag{34}$$

The local extrema of the travel time is given by

$$0 = \frac{dt}{dy_0} \tag{35}$$

$$= \frac{y_A - y_0}{s_{A0}c/n_1} + \frac{y_0 - y_B}{s_{0B}c/n_2} \tag{36}$$

$$= \frac{\sin\alpha}{c/n_1} - \frac{\sin\beta}{c/n_2} \tag{37}$$

and therefore

$$n_1\sin\alpha = n_2\sin\beta. \tag{38}$$

Exercise 1.2 - Functional derivatives I

- $H[f] = \int G(x,y)f(y)dy$

$$\frac{\delta H[f]}{\delta f(z)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\int G(x,y)(f(y)+\epsilon\delta(z-y))dy - \int G(x,y)f(y)dy\right] \tag{39}$$
$$= \int G(x,y)\delta(z-y))dy \tag{40}$$
$$= G(x,z) \tag{41}$$

- $I[f] = \int_{-1}^{1} f(x)dx$

$$\frac{\delta^2 I[f^3]}{\delta f(x_0)\delta f(x_1)} = \frac{\delta}{\delta f(x_0)}\frac{\delta I[f^3]}{\delta f(x_1)} \tag{42}$$
$$= \frac{\delta}{\delta f(x_0)}\frac{\delta}{\delta f(x_1)}\int_{-1}^{1} f(x)^3 dx \tag{43}$$
$$= \frac{\delta}{\delta f(x_0)}\frac{1}{\epsilon}\int_{-1}^{1}(f(x)+\epsilon\delta(x_1-x))^3 - f(x)^3 dx \tag{44}$$
$$= \frac{\delta}{\delta f(x_0)}\frac{1}{\epsilon}\int_{-1}^{1}(f(x)^3 + 3\epsilon f(x)^2\delta(x_1-x) + \mathcal{O}(\epsilon^2) - f(x)^3 dx \tag{45}$$
$$= \frac{\delta}{\delta f(x_0)}\begin{cases} 3f(x_1)^2 & x_1 \in [-1,1] \\ 0 & \text{else}\end{cases} \tag{46}$$
$$= \begin{cases} 3\frac{1}{\epsilon}[(f(x_1)-\epsilon\delta(x_0-x_1))^2 - f(x_1)^2] & x_1 \in [-1,1] \\ 0 & \text{else}\end{cases} \tag{47}$$
$$= \begin{cases} 6f(x_1)\delta(x_0-x_1) & x_1 \in [-1,1] \\ 0 & \text{else}\end{cases} \tag{48}$$
$$\tag{49}$$

- $J[f] = \int \left(\frac{\partial f}{\partial y}\right)^2 dy$

$$\frac{\delta J[f]}{\delta f(x)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\int\left(\frac{\partial(f+\epsilon\delta(x-y))}{\partial y}\right)^2 dy - \int\left(\frac{\partial f}{\partial y}\right)^2 dy\right] \tag{50}$$
$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\int\left(\frac{\partial f}{\partial y} + \epsilon\frac{\partial\delta(x-y)}{\partial y}\right)^2 dy - \int\left(\frac{\partial f}{\partial y}\right)^2 dy\right] \tag{51}$$
$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\int\left(\frac{\partial f}{\partial y}\right)^2 + 2\epsilon\frac{\partial f}{\partial y}\frac{\partial\delta(x-y)}{\partial y} + \mathcal{O}(\epsilon^2) - \left(\frac{\partial f}{\partial y}\right)^2 dy\right] \tag{52}$$
$$= 2\int\frac{\partial f}{\partial y}\frac{\partial\delta(x-y)}{\partial y}dy \tag{53}$$
$$= \text{boundary terms} - 2\int\frac{\partial^2 f}{\partial y^2}\delta(x-y)dy \tag{54}$$
$$= -2\int\frac{\partial^2 f}{\partial x^2} \tag{55}$$

Exercise 1.3 - Functional derivatives II

- $$\begin{aligned}\frac{\delta G[f]}{\delta f(x)} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int g(y, f+\epsilon\delta(x-y)) - g(y,f)dy &(56)\\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int g(y,f) + \epsilon\frac{\partial g(y,f)}{\partial f}\delta(x-y)) - g(y,f)dy &(57)\\ &= \frac{\partial g(x,f)}{\partial f} &(58)\end{aligned}$$

- $$\begin{aligned}\frac{\delta H[f]}{\delta f(x)} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int g(y, f+\epsilon\delta(x-y), f'+\epsilon\partial_y\delta(x-y)) - g(y,f,f')dy &(59)\\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int g(y,f,f') + \epsilon\frac{\partial g(y,f,f')}{\partial f}\delta(x-y) + \epsilon\frac{\partial g(y,f,f')}{\partial f'}\partial_y\delta(x-y)) - g(y,f,f')dy &(60)\\ &= \int \frac{\partial g(y,f,f')}{\partial f}\delta(x-y) + \frac{\partial g(y,f,f')}{\partial f'}\partial_y\delta(x-y))dy &(61)\\ &= \frac{\partial g(x,f,f')}{\partial f} + \int \frac{\partial g(y,f,f')}{\partial f'}\partial_y\delta(x-y))dy &(62)\\ &= \frac{\partial g(x,f,f')}{\partial f} - \int \partial_y\frac{\partial g(y,f,f')}{\partial f'}\delta(x-y))dy &(63)\\ &= \frac{\partial g(x,f,f')}{\partial f} - \partial_x\frac{\partial g(x,f,f')}{\partial f'} &(64)\end{aligned}$$

- Same as above but two times integration by parts is needed. Therefore $(-1)^2 = 1$ giving the term a final + sign.

Exercise 1.4 - Functional derivatives III

- $$\begin{aligned}\frac{\delta\phi(x)}{\delta\phi(y)} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\phi(x) + \epsilon\delta(x-y) - \phi(x)\right) &(65)\\ &= \delta(x-y) &(66)\end{aligned}$$

- $$\begin{aligned}\frac{\delta\dot\phi(t)}{\delta\phi(t_0)} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\dot\phi(t) + \epsilon\partial_t\delta(t-t_0) - \dot\phi(t)\right) &(67)\\ &= \frac{d}{dt}\delta(t-t_0) &(68)\end{aligned}$$

Exercise 1.5 - Euler-Langrange equations for elastic medium

$$\mathcal{L} = T - V \tag{69}$$

$$\frac{\partial\mathcal{L}}{\partial\psi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right) = 0 \tag{70}$$

then

$$\frac{\partial \mathcal{L}}{\partial \psi} = 0 \tag{71}$$
$$\frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = \frac{\rho}{2} \int d^3x 2 \frac{\partial \psi}{\partial t} \tag{72}$$
$$\frac{\partial \mathcal{L}}{\partial(\partial_k \psi)} = -\frac{\mathcal{T}}{2} \int d^3x 2 \frac{\partial \psi}{\partial x^k} \tag{73}$$
$$\to -\left(\int d^3x [\rho \ddot{\psi} - \mathcal{T} \nabla^2 \psi] \right) = 0 \tag{74}$$
$$\to \frac{\rho}{\mathcal{T}} \ddot{\psi} = \nabla^2 \psi \tag{75}$$

Exercise 1.6 - Functional derivatives IV

$$\frac{\delta Z_0[J]}{\delta J(z_1)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \exp\left(-\frac{1}{2} \int d^4x d^4y (J(x) + \epsilon\delta(x - z_1))\Delta(x - y)(J(y) + \epsilon\delta(y - z_1)) \right) \tag{76}$$
$$- \exp\left(-\frac{1}{2} \int d^4x d^4y J(x)\Delta(x - y)J(y) \right) \tag{77}$$
$$= Z_0[J] \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\exp\left(-\frac{\epsilon}{2} \int d^4x d^4y\, J(x)\Delta(x - y)\delta(y - z_1) + \delta(x - z_1)\Delta(x - y)J(y) \right) - 1 \right) \tag{78}$$
$$= Z_0[J] \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(1 - \frac{\epsilon}{2} \int d^4x d^4y\, J(x)\Delta(x - y)\delta(y - z_1) + \delta(x - z_1)\Delta(x - y)J(y) - 1 \right) \tag{79}$$
$$= -\frac{1}{2} Z_0[J] \int d^4x d^4y\, J(x)\Delta(x - y)\delta(y - z_1) + \delta(x - z_1)\Delta(x - y)J(y) \tag{80}$$
$$= -\frac{1}{2} Z_0[J] \left(\int d^4x\, J(x)\Delta(x - z_1) + \int d^4y\, \Delta(z_1 - y)J(y) \right) \tag{81}$$
$$= -Z_0[J] \int d^4y\, \Delta(z_1 - y)J(y) \tag{82}$$

Exercise 2.1 - Commutators of creation and annihilation operators

With $[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar$

$$[\hat{a}, \hat{a}] = \frac{m\omega}{2\hbar} \left(\hat{x}\hat{x} + \frac{i}{m\omega}(\hat{x}\hat{p} + \hat{p}\hat{x}) + \frac{i^2}{m^2\omega^2}\hat{p}\hat{p} \right) - \frac{m\omega}{2\hbar} \left(\hat{x}\hat{x} + \frac{i}{m\omega}(\hat{x}\hat{p} + \hat{p}\hat{x}) + \frac{i^2}{m^2\omega^2}\hat{p}\hat{p} \right) \tag{83}$$
$$= 0 \tag{84}$$
$$[\hat{a}^\dagger, \hat{a}^\dagger] = ... = 0 \tag{85}$$

$$[\hat{a}, \hat{a}^\dagger] = \frac{m\omega}{2\hbar} \left(\hat{x}\hat{x} + \frac{i}{m\omega}(-\hat{x}\hat{p} + \hat{p}\hat{x}) - \frac{i^2}{m^2\omega^2}\hat{p}\hat{p} \right) - \frac{m\omega}{2\hbar} \left(\hat{x}\hat{x} + \frac{i}{m\omega}(\hat{x}\hat{p} - \hat{p}\hat{x}) - \frac{i^2}{m^2\omega^2}\hat{p}\hat{p} \right) \tag{86}$$
$$= \frac{m\omega}{2\hbar} \frac{i}{m\omega} 2(-\hat{x}\hat{p} + \hat{p}\hat{x}) \tag{87}$$
$$= \frac{i}{\hbar}(-\hat{p}\hat{x} - i\hbar + \hat{p}\hat{x}) \tag{88}$$
$$= 1 \tag{89}$$

Now the Hamiltonian

$$\hat{a}^\dagger\hat{a} = \frac{m\omega}{2\hbar}\left(\hat{x}\hat{x} + \frac{i}{m\omega}(\hat{x}\hat{p} - \hat{p}\hat{x}) - \frac{i^2}{m^2\omega^2}\hat{p}\hat{p}\right) \tag{90}$$

$$= \frac{m\omega}{2\hbar}\left(\hat{x}\hat{x} + \frac{i}{m\omega}i\hbar - \frac{i^2}{m^2\omega^2}\hat{p}\hat{p}\right) \tag{91}$$

$$= \frac{1}{2m\omega\hbar}\hat{p}^2 + \frac{m\omega}{2\hbar}\hat{x}^2 - \frac{1}{2} \tag{92}$$

$$\hat{a}^\dagger\hat{a} + \frac{1}{2} = \frac{1}{2m\omega\hbar}\hat{p}^2 + \frac{m\omega}{2\hbar}\hat{x}^2 \tag{93}$$

$$\hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) = \frac{1}{2m}\hat{p}^2 + \frac{m\omega^2}{2}\hat{x}^2 = \hat{H} \tag{94}$$

Exercise 2.2 - Perturbed harmonic oscillator

We see

$$a + a^\dagger = \sqrt{\frac{2m\omega}{\hbar}}x \tag{95}$$

$$(a + a^\dagger)^2 = \frac{2m\omega}{\hbar}x^2 \tag{96}$$

$$x^2 = \frac{\hbar}{2m\omega}(a + a^\dagger)^2 \tag{97}$$

$$x^4 = (a + a^\dagger)^2\frac{\hbar}{2m\omega}\cdot\frac{\hbar}{2m\omega}(a + a^\dagger)^2 \tag{98}$$

The first order energy perturbation is given by

$$E_n^{(1)} = \langle n|H_1|n\rangle \tag{99}$$

$$= \langle n|x^4|n\rangle \tag{100}$$

$$= \langle n|x^2 \cdot x^2|n\rangle. \tag{101}$$

By splitting H_1 the calculation gets a bit shorter. Using

$$a|n\rangle\sqrt{n}|n\rangle \qquad a^\dagger|n\rangle\sqrt{n+1}|n+1\rangle \tag{102}$$

we obtain

$$x^2|n\rangle = \frac{\hbar}{2m\omega}(a + a^\dagger)^2|n\rangle \tag{103}$$

$$= \frac{\hbar}{2m\omega}(aa^\dagger + a^\dagger a + (a^\dagger)^2 + a^2)|n\rangle \tag{104}$$

$$= \frac{\hbar}{2m\omega}\left((n+1)|n\rangle + n|n\rangle + \sqrt{n(n-1)}|n-2\rangle + \sqrt{(n+1)(n+2)}|n+2\rangle\right) \tag{105}$$

$$= \frac{\hbar}{2m\omega}\left((2n+1)|n\rangle + \sqrt{n(n-1)}|n-2\rangle + \sqrt{(n+1)(n+2)}|n+2\rangle\right) \tag{106}$$

$$\langle n|x^2 = (x^2|n\rangle)^\dagger \tag{107}$$

$$= \frac{\hbar}{2m\omega}\left((2n+1)|n\rangle + \sqrt{n(n-1)}|n-2\rangle + \sqrt{(n+1)(n+2)}|n+2\rangle\right) \tag{108}$$

Using the orthogonality of the unperturbed states (eigenstates of the Hamiltonian which is hermitian) we obtain

$$E_n^{(1)} = \langle n|x^2 \cdot x^2|n\rangle \tag{109}$$
$$= \frac{\hbar^2}{4m^2\omega^2}\left((2n+1)^2 + n(n-1) + (n+1)(n+2)\right) \tag{110}$$
$$= \frac{\hbar^2}{4m^2\omega^2}\left(4n^2 + 4n + 1 + n^2 - n + n^2 + 3n + 2\right) \tag{111}$$
$$= \frac{\hbar^2}{4m^2\omega^2}\left(6n^2 + 6n + 3\right) \tag{112}$$
$$= \frac{3}{4}\frac{\hbar^2}{m^2\omega^2}\left(2n^2 + 2n + 1\right) \tag{113}$$

which gives the desired result using $E_n = E_n^{(0)} + \lambda E_n^{(1)}$

Exercise 2.3 - ...

Odd notation $\tilde{x} = \hat{x}$

$$\hat{x}_j = \sqrt{\frac{\hbar}{2\omega_j m}}(\hat{a}_j + \hat{a}^\dagger_{-j}) \tag{114}$$
$$x_j = \frac{1}{\sqrt{N}}\sum_k \tilde{x}_k e^{ikja} \tag{115}$$
$$= \frac{1}{\sqrt{N}}\sqrt{\frac{\hbar}{m}}\sum_k \frac{1}{\sqrt{2\omega_k}}(\hat{a}_k + \hat{a}^\dagger_{-k})e^{ikja} \tag{116}$$
$$= \frac{1}{\sqrt{N}}\sqrt{\frac{\hbar}{m}}\sum_k \frac{1}{\sqrt{2\omega_k}}(\hat{a}_k e^{ikja} + \hat{a}^\dagger_k e^{-ikja}) \tag{117}$$

Exercise 2.4 - Wavefunction in space representation

$$\hat{a} = \sqrt{\frac{2\hbar}{m\omega}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right), \qquad \hat{a}|0\rangle = 0 \tag{118}$$
$$\rightarrow \sqrt{\frac{2\hbar}{m\omega}}\langle x|\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)|0\rangle = 0 \tag{119}$$
$$\rightarrow \sqrt{\frac{2\hbar}{m\omega}}\left(\langle x|\hat{x}|0\rangle + \frac{i}{m\omega}\langle x|\hat{p}|0\rangle\right) = 0 \tag{120}$$
$$\rightarrow \sqrt{\frac{2\hbar}{m\omega}}\left(x\langle x|0\rangle + \frac{i}{m\omega}(-i\hbar)\frac{d}{dx}\langle x|0\rangle\right) = 0 \tag{121}$$
$$\rightarrow \sqrt{\frac{2\hbar}{m\omega}}\left(x + \frac{\hbar}{m\omega}\frac{d}{dx}\right)\langle x|0\rangle = 0 \tag{122}$$

Now we can solve the ODE ($\psi_0(x) = \langle x|0\rangle$)

$$\left(x + \frac{\hbar}{m\omega}\frac{d}{dx}\right)\psi_0 = 0 \tag{123}$$

$$\int dx\, \psi_0' + \int dx\, \frac{m\omega}{\hbar} x\psi_0 = 0 \tag{124}$$

$$\frac{\psi_0'}{\psi_0} = -\frac{m\omega}{\hbar}x \tag{125}$$

$$\log\psi_0 = -\frac{m\omega}{2\hbar}x^2 + c \tag{126}$$

$$\psi_0 = Ce^{-m\omega x^2/2\hbar} \tag{127}$$

Normalization

$$\int dx\, \psi_0^*\, \psi_0 = 1 \tag{128}$$

$$C^*C\int dx\, e^{-m\omega x^2/\hbar} = 1 \tag{129}$$

$$|C|^2\sqrt{\frac{\pi\hbar}{m\omega}} = 1 \quad \rightarrow \quad C = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \tag{130}$$

Exercise 3.1 - Commutator Fourier Transformation

Bosons - commutator

$$\frac{1}{\mathcal{V}}\sum_{\mathbf{p},\mathbf{q}} e^{i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})}[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = \frac{1}{\mathcal{V}}\sum_{\mathbf{p},\mathbf{q}} e^{i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})}\delta_{\mathbf{pq}} \tag{131}$$

$$= \frac{1}{\mathcal{V}}\sum_{\mathbf{p}} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \tag{132}$$

$$= \frac{1}{L_xL_yL_z}\sum_{n_1=-N/2}^{N/2} e^{i\frac{2\pi n_1}{Na_x}(x_1-y_1)} \cdot \sum_{n_2=-N/2}^{N/2} e^{i\frac{2\pi n_2}{Na_y}(x_2-y_2)} \cdot \sum_{n_3=-N/2}^{N/2} e^{i\frac{2\pi n_3}{Na_z}(x_3-y_3)} \tag{133}$$

$$= \left(\frac{1}{L}\sum_{n=-N/2}^{N/2} e^{i\frac{2\pi n}{Na}(x-y)}\right)^3 \qquad \text{with } Na \equiv L \tag{134}$$

$$= \left(\frac{1}{L}\frac{Na}{2\pi}\sum_{p_n=-\pi/a}^{\pi/a} e^{ip_n(x-y)}\frac{2\pi}{Na}\right)^3 \qquad \text{with } \sum_{q_n} f(p_n)\Delta p = \int f(p)dp \tag{135}$$

$$= \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ip(x-y)}dp\right)^3 \qquad \text{with } N\to\infty,\, a\to 0 \tag{136}$$

$$= (\delta(x-y))^3 \tag{137}$$

$$= \delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{138}$$

with the discretization of the momentum-space $p_j = \left\{\frac{2\pi j}{Na}\right\}_{-N/2}^{N/2}$ and $\Delta p = \frac{2\pi}{Na}$.

Fermions - anticommutator

$$\{c_{\mathbf{p}}, c_{\mathbf{q}}^\dagger\} = \delta_{\mathbf{pq}} \tag{139}$$

yields same result.

Exercise 3.2 - Harmonic oscillator relations

With

$$[\hat{a}, \hat{a}^\dagger] = 1 \tag{140}$$
$$\hat{a}^\dagger \hat{a} = \hat{n} \tag{141}$$
$$\frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle = |n\rangle \tag{142}$$

Then

(a) $[\hat{a}, (\hat{a}^\dagger)^n]$

$$\begin{aligned}
\hat{a}(\hat{a}^\dagger)^n &= (aa^\dagger)(a^\dagger)^{n-1} && (143)\\
&= (a^\dagger a + 1)(a^\dagger)^{n-1} && (144)\\
&= a^\dagger a(a^\dagger)^{n-1} + (a^\dagger)^{n-1} && (145)\\
&= a^\dagger a a^\dagger (a^\dagger)^{n-2} + (a^\dagger)^{n-1} && (146)\\
&= a^\dagger(a^\dagger a + 1)(a^\dagger)^{n-2} + (a^\dagger)^{n-1} && (147)\\
&= (a^\dagger)^2 a(a^\dagger)^{n-2} + 2(a^\dagger)^{n-1} && (148)\\
&= ... && (149)\\
&= (a^\dagger)^n a + n(a^\dagger)^{n-1} && (150)\\
\rightarrow [\hat{a}, (\hat{a}^\dagger)^n] &= n(a^\dagger)^{n-1} && (151)
\end{aligned}$$

(b) $\langle 0|a^n(a^\dagger)^m|0\rangle$

If $n < m$ (similar for $n > m$) we get zero

$$\begin{aligned}
\langle 0|a^n(a^\dagger)^m|0\rangle &\sim \langle 1|a^{n-1}(a^\dagger)^{m-1}|1\rangle && (152)\\
&\sim \langle 2|a^{n-2}(a^\dagger)^{m-2}|2\rangle && (153)\\
&... && (154)\\
&\sim \langle k|(a^\dagger)^{m-k}|k\rangle && (155)\\
&= 0 \qquad (\langle k|a^\dagger = 0). && (156)
\end{aligned}$$

For $n = m$ we have with the definition

$$\begin{aligned}
\frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle &= |n\rangle && (157)\\
(a^\dagger)^n|0\rangle &= \sqrt{n!}|n\rangle && (158)\\
\langle 0|a^n(a^\dagger)^m|0\rangle &= \sqrt{n!}^2\langle n|n|\rangle && (159)\\
&= n! && (160)
\end{aligned}$$

Therefore $\langle 0|a^n(a^\dagger)^m|0\rangle = n!\delta_{nm}$

(c) $\langle m|a^\dagger|n\rangle$

$$\frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle = |n\rangle \tag{161}$$
$$a^\dagger\frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle = a^\dagger|n\rangle \tag{162}$$
$$\frac{1}{\sqrt{n+1}}a^\dagger\frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle = \frac{1}{\sqrt{n+1}}a^\dagger|n\rangle = |n+1\rangle \tag{163}$$

then

$$\langle m|a^\dagger|n\rangle = \sqrt{n+1}\langle m|n+1\rangle \tag{164}$$
$$= \sqrt{n+1}\delta_{m,n+1} \tag{165}$$

(d) $\langle m|a|n\rangle$

$$(\langle m|a)^\dagger = a^\dagger|m\rangle \tag{166}$$
$$= \sqrt{m+1}|m+1\rangle \tag{167}$$

then

$$\langle m|a|n\rangle = \sqrt{m+1}\delta_{m+1,n} \tag{168}$$
$$= \sqrt{n}\delta_{m+1,n} \tag{169}$$

Exercise 3.2 - 3d Harmonic oscillator

Rewriting the Hamiltonian

$$H = H_1 + H_2 + H_3 \tag{170}$$
$$H_i = \frac{p_i^2}{2m} + \frac{1}{2}m\omega^2 x_i^2 \tag{171}$$

the we can reutilise the know ladder operators

$$a_i = \sqrt{\frac{m\omega}{2\hbar}}\left(x_i + \frac{i}{m\omega}p_i\right) \tag{172}$$
$$a_i^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x_i - \frac{i}{m\omega}p_i\right) \tag{173}$$

and the Hamiltonian can be obviously written as the sum

$$H = \hbar\omega\sum_k\left(a_k^\dagger a_k + \frac{1}{2}\right). \tag{174}$$

With the classic definition $\vec{L} = \vec{x}\times\vec{p}$ we see (inverting a and $a^\dagger$ to get x and p)

$$L_i = \varepsilon_{ijk}x_j p_k \tag{175}$$
$$= -i\varepsilon_{ijk}\sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{\hbar m\omega}{2}}(a_j + a_j^\dagger)(a_k - a_k^\dagger) \tag{176}$$
$$= -\frac{i\hbar}{2}\varepsilon_{ijk}(a_j a_k + a_j^\dagger a_k - a_j a_k^\dagger - a_j^\dagger a_k^\dagger) \tag{177}$$
$$= -\frac{i\hbar}{2}\varepsilon_{ijk}(a_j^\dagger a_k - \delta_{jk} - a_k^\dagger a_j) \qquad [a_j, a_k^\dagger] = \delta_{jk},\ a_j|0\rangle = 0,\ \langle 0|a_k = 0 \tag{178}$$
$$= -\frac{i\hbar}{2}(\varepsilon_{ijk}a_j^\dagger a_k - \varepsilon_{ijk}\delta_{jk} - \varepsilon_{ijk}a_k^\dagger a_j) \tag{179}$$
$$= -\frac{i\hbar}{2}(\varepsilon_{ijk}a_j^\dagger a_k - \varepsilon_{ikk} - \varepsilon_{ikj}a_j^\dagger a_k) \qquad \text{reindexing} \tag{180}$$
$$= -\frac{i\hbar}{2}(\varepsilon_{ijk}a_j^\dagger a_k + \varepsilon_{ijk}a_j^\dagger a_k) \qquad \varepsilon_{ikk} = 0 \tag{181}$$
$$= -i\hbar\varepsilon_{ijk}a_j^\dagger a_k \tag{182}$$

Now the new commutation relations

$$
\begin{aligned}
[b_0, b_0^\dagger] &= [a_3, a_3^\dagger] = 1 = \delta_{00} && (183)\\
[b_0, b_1^\dagger] &= -\frac{1}{\sqrt{2}}(a_3(a_1^\dagger + ia_2^\dagger) - (a_1^\dagger + ia_2^\dagger)a_3) && (184)\\
&= -\frac{1}{\sqrt{2}}(a_3a_1^\dagger + ia_3a_2^\dagger - a_1^\dagger a_3 - ia_2^\dagger a_3) && (185)\\
&= -\frac{1}{\sqrt{2}}(\delta_{12} + i\delta_{23}) && (186)\\
&= 0 && (187)\\
[b_{-1}, b_1^\dagger] &= -\frac{1}{2}((a_1 - ia_2)(a_1^\dagger - ia_2^\dagger) - (a_1^\dagger - ia_2^\dagger)(a_1 - ia_2)) && (188)\\
&= -\frac{1}{2}(a_1a_1^\dagger - ia_2a_1^\dagger - ia_1a_2^\dagger - a_2a_2^\dagger - a_1^\dagger a_1 + ia_1^\dagger a_2 + ia_2^\dagger a_1 + a_2^\dagger a_2) && (189)\\
&= -\frac{1}{2}(1 - i\cdot 0 - i\cdot 0 - 1) && (190)\\
&= 0 && (191)\\
&= \delta_{-1,1} && (192)\\
&\ldots && (193)
\end{aligned}
$$

Now the Hamiltonian with

$$
\begin{aligned}
b_{-1}^\dagger b_{-1} + b_1^\dagger b_1 &= \frac{1}{2}(a_1^\dagger - ia_2^\dagger)(a_1 + ia_2) + \frac{1}{2}(a_1^\dagger + ia_2^\dagger)(a_1 - ia_2) && (194)\\
&= \frac{1}{2}(a_1^\dagger a_1 - ia_2^\dagger a_1 + ia_1^\dagger a_2 + a_2^\dagger a_2) + \frac{1}{2}(a_1^\dagger a_1 + ia_2^\dagger a_1 - ia_1^\dagger a_2 + a_2^\dagger a_2) && (195)\\
&= a_1^\dagger a_1 + a_2^\dagger a_2 && (196)
\end{aligned}
$$

and $b_0^\dagger b_0 = a_3^\dagger a_3$ we have $H = \hbar\omega \sum(1/2 + b_m^\dagger b_m)$. While

$$
\begin{aligned}
-b_{-1}^\dagger b_{-1} + b_1^\dagger b_1 &= -\frac{1}{2}(a_1^\dagger - ia_2^\dagger)(a_1 + ia_2) + \frac{1}{2}(a_1^\dagger + ia_2^\dagger)(a_1 - ia_2) && (197)\\
&= -\frac{1}{2}(a_1^\dagger a_1 - ia_2^\dagger a_1 + ia_1^\dagger a_2 + a_2^\dagger a_2) + \frac{1}{2}(a_1^\dagger a_1 + ia_2^\dagger a_1 - ia_1^\dagger a_2 + a_2^\dagger a_2) && (198)\\
&= ia_2^\dagger a_1 - ia_1^\dagger a_2 && (199)\\
&= -i(-a_2^\dagger a_1 + a_1^\dagger a_2) && (200)
\end{aligned}
$$

gives $L^3 = \hbar \sum_m m b_m^\dagger b_m$.

Exercise 5.1 - Time derivative of Lagrangian

With $\frac{\partial L}{\partial q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)$ we have

$$
\begin{aligned}
\frac{dL}{dt} &= \frac{\partial L}{\partial t} + \frac{\partial L}{\partial q}\dot q + \frac{\partial L}{\partial \dot q}\ddot q && (201)\\
&= \frac{\partial L}{\partial t} + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)\dot q + \frac{\partial L}{\partial \dot q}\ddot q && (202)\\
&= \frac{\partial L}{\partial t} + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\dot q\right) && (203)\\
&= \frac{\partial L}{\partial t} + \frac{d}{dt}(p\dot q) && (204)
\end{aligned}
$$

then

$$0 = \frac{\partial L}{\partial t} + \frac{d}{dt}\left(p\dot{q} - L\right) \tag{205}$$

and

$$\frac{\partial L}{\partial t} = -\frac{dH}{dt} \tag{206}$$

Exercise 5.3 - Commutator of Hermitian operators

In general we have

$$[A, B]^\dagger = (AB - BA)^\dagger \tag{207}$$
$$= B^\dagger A^\dagger - A^\dagger B^\dagger \tag{208}$$
$$= [B^\dagger, A^\dagger] \tag{209}$$
$$= -[A^\dagger, B^\dagger] \tag{210}$$

now using $A = A^\dagger$ and $B = B^\dagger$ we obtain

$$[A, B]^\dagger = -[A, B] \tag{211}$$

Exercise 5.4 - Relativistic free particle

Taylor series expansion of the square root gives

$$L = -mc^2\sqrt{1 - \frac{v^2}{c^2}} \tag{212}$$
$$\simeq -mc^2 - \frac{1}{2}mv^2 - \frac{3}{8}mv^2\frac{1}{c^2} + ... \tag{213}$$
$$\simeq -mc^2 - \frac{1}{2}mv^2 + ... \tag{214}$$

Conjugated momentum

$$p = \frac{\partial L}{\partial v} = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma mv \simeq mv \tag{215}$$

Lets solve for v to get exact expression for H

$$v = \frac{cp}{m^2c^2 + p^2} \tag{216}$$

Then

$$H = pv - L \tag{217}$$
$$= p\frac{cp}{m^2c^2 + p^2} + mc^2\sqrt{1 - \frac{v^2}{c^2}} \tag{218}$$
$$= c\frac{m^2c^2 + p^2}{\sqrt{p^2 + m^2c^2}} = \sqrt{m^2c^4 + p^2c^2} \tag{219}$$
$$\simeq mc^2 + \frac{mv^2}{2} \tag{220}$$

Exercise 5.6 - Relativistic free particle in EM field

Euler-Lagrange equations:

$$\frac{\partial L}{\partial x_i} = \frac{d}{dt}\frac{\partial L}{\partial v_i} \tag{221}$$

Definition of the EM potentials

$$\mathbf{E} = -\nabla V - \frac{d\mathbf{A}}{dt} \tag{222}$$
$$\mathbf{B} = \nabla \times \mathbf{A} \tag{223}$$

From Problem 5.4

$$\frac{d}{dt}\frac{\partial L}{\partial v_i} = \frac{d}{dt}(\gamma m v_i) + q\frac{d}{dt}A_i(x,t) \tag{224}$$

Lets proof the identity by calculating the single terms

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = [(\partial_k a_i)b_i + (\partial_k b_i)a_i]\mathbf{e}_k \tag{225}$$
$$(\mathbf{a}\cdot\nabla)\mathbf{b} = a_i(\partial_i b_k)\mathbf{e}_k \tag{226}$$
$$(\mathbf{b}\cdot\nabla)\mathbf{a} = b_i(\partial_i a_k)\mathbf{e}_k \tag{227}$$
$$\mathbf{b}\times(\nabla\times\mathbf{a}) = \epsilon_{kja}\epsilon_{bca}b_j(\partial_b a_c)\mathbf{e}_k \tag{228}$$
$$= (\delta_{kb}\delta_{jc} - \delta_{kc}\delta_{jb})b_j(\partial_b a_c)\mathbf{e}_k \tag{229}$$
$$= [b_c(\partial_k a_c) - b_c(\partial_c a_k)]\mathbf{e}_k \tag{230}$$
$$\mathbf{a}\times(\nabla\times\mathbf{b}) = [a_c(\partial_k b_c) - a_c(\partial_c b_k)]\mathbf{e}_k \tag{231}$$

by adding up we see

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{b}\times(\nabla\times\mathbf{a}) + \mathbf{a}\times(\nabla\times\mathbf{b}) \tag{232}$$

Now we calculate

$$\frac{\partial L}{\partial x_i} = q\frac{\partial}{\partial x_i}[\mathbf{A}\cdot\mathbf{v} - V] \tag{233}$$
$$= -q\partial_i V(x,t) + q[\nabla(\mathbf{A}(x,t)\cdot\mathbf{v})]_i \tag{234}$$
$$= -q[\nabla V]_i + q[(\mathbf{v}\cdot\nabla)\mathbf{A} + \mathbf{v}\times(\nabla\times\mathbf{A})]_i \tag{235}$$

then (combining all vector components)

$$\frac{d}{dt}(\gamma m\mathbf{v}) + q\frac{d}{dt}\mathbf{A} = q(\mathbf{v}\cdot\nabla)\mathbf{A} + q\mathbf{v}\times(\nabla\times\mathbf{A}) - q\nabla V \tag{236}$$
$$\frac{d}{dt}(\gamma m\mathbf{v}) = q\mathbf{v}\times(\nabla\times\mathbf{A}) - q\nabla V - q\left(\frac{d}{dt}\mathbf{A} - (\mathbf{v}\cdot\nabla)\mathbf{A}\right) \tag{237}$$
$$= q\mathbf{v}\times\mathbf{B} - q\nabla V - q\frac{\partial\mathbf{A}}{\partial t} \tag{238}$$
$$= q[\mathbf{v}\times\mathbf{B} + \mathbf{E}] \tag{239}$$

where we used

$$\frac{d\mathbf{A}}{dt} = \frac{\partial\mathbf{A}}{\partial t} + \frac{\partial\mathbf{A}}{\partial x_i}\frac{\partial x_i}{\partial t} \tag{240}$$
$$= \frac{\partial\mathbf{A}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{A} \tag{241}$$

Exercise 5.6 - Non-relativistic free particle in EM field

From Problem 5.4/5.5

$$p_i = \frac{\partial L}{\partial v_i} = \gamma m v_i + qA_i(x,t) \tag{242}$$
$$\mathbf{p} = \gamma m\mathbf{v} + q\mathbf{A} \tag{243}$$
$$\simeq m\mathbf{v} + q\mathbf{A} \tag{244}$$

also

$$\gamma m\mathbf{v} = \mathbf{p} - q\mathbf{A} \tag{245}$$
$$\mathbf{v} = \frac{\mathbf{p} - q\mathbf{A}}{\gamma m} \tag{246}$$
$$v^2 = \frac{(\mathbf{p} - q\mathbf{A})^2}{\gamma^2 m^2} \tag{247}$$
$$= \frac{(\mathbf{p} - q\mathbf{A})^2 c^2}{m^2c^2 + (\mathbf{p} - q\mathbf{A})^2} \tag{248}$$
$$\sqrt{1 - \frac{v^2}{c^2}} = \frac{mc}{\sqrt{(\mathbf{p} - q\mathbf{A})^2 + m^2c^2}} \tag{249}$$

then

$$E = H = \mathbf{p}\cdot\dot{\mathbf{q}} - L \tag{250}$$
$$= \mathbf{p}\cdot\mathbf{v} - L \tag{251}$$
$$= (\gamma m\mathbf{v})\cdot\mathbf{v} + q\mathbf{A}\cdot\mathbf{v} - \left(-\frac{mc^2}{\gamma} + q\mathbf{A}\cdot\mathbf{v} - qV\right) \tag{252}$$
$$= (\gamma m\mathbf{v})\cdot\mathbf{v} + \frac{mc^2}{\gamma} + qV \tag{253}$$
$$= (\mathbf{p} - q\mathbf{A})\cdot\mathbf{v} + \frac{mc^2}{\gamma} + qV \tag{254}$$
$$= (\mathbf{p} - q\mathbf{A})\cdot\frac{\mathbf{p} - q\mathbf{A}}{m\gamma} + \frac{mc^2}{\gamma} + qV \tag{255}$$
$$= \left(\frac{(\mathbf{p} - q\mathbf{A})^2}{m} + mc^2\right)\sqrt{1 - \frac{v^2}{c^2}} + qV \tag{256}$$
$$= \frac{1}{m}\left((\mathbf{p} - q\mathbf{A})^2 + m^2c^2\right)\frac{mc}{\sqrt{(\mathbf{p} - q\mathbf{A})^2 + m^2c^2}} + qV \tag{257}$$
$$= \sqrt{(\mathbf{p} - q\mathbf{A})^2c^2 + m^2c^4} + qV \tag{258}$$
$$= mc^2\sqrt{1 + \frac{(\mathbf{p} - q\mathbf{A})^2c^2}{m^2c^4}} + qV \tag{259}$$
$$\simeq mc^2\left(1 + \frac{(\mathbf{p} - q\mathbf{A})^2}{2m^2c^2} + \dots\right) + qV \tag{260}$$
$$\simeq mc^2 + \frac{(\mathbf{p} - q\mathbf{A})^2}{2m} + qV \tag{261}$$

Exercise 6.1 - Klein-Gordon

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi \tag{262}$$
$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \frac{1}{2} g^{\alpha\beta}(\partial_\alpha\phi)(\partial_\beta\phi) \tag{263}$$
$$= \frac{1}{2} g^{\alpha\beta}\left[\delta^\mu_\alpha(\partial_\beta\phi) + \delta^\mu_\beta(\partial_\alpha\phi)\right] \tag{264}$$
$$= g^{\alpha\mu}\partial_\alpha\phi \tag{265}$$
$$= \partial^\mu\phi \tag{266}$$
$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial_\mu\partial^\mu\phi \tag{267}$$

Euler Lagrange

$$(\partial_\mu\partial^\mu + m^2)\phi = 0 \tag{268}$$

Canonical momentum

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial^0\phi = \dot{\phi} \tag{269}$$

Hamiltonian

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} \tag{270}$$
$$= \pi^2 - \left(\frac{1}{2}\pi^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{m^2}{2}\phi^2\right) \tag{271}$$
$$= \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2 \tag{272}$$
$$\tag{273}$$

Exercise 7.1 - Klein-Gordon plus higher orders

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi - \sum_{n=1}(2n+2)\lambda_n\phi^{2n+1} \tag{274}$$
$$\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi)} = \frac{1}{2}\frac{\partial(\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi)}{\partial(\partial_\alpha\phi)} = \frac{1}{2}(\eta^{\mu\nu}\partial_\nu\phi\delta^\alpha_\mu + \eta^{\mu\nu}\partial_\mu\phi\delta^\alpha_\nu) \tag{275}$$
$$= \frac{1}{2}(\eta^{\alpha\nu}\partial_\nu\phi + \eta^{\mu\alpha}\partial_\mu\phi) = \partial^\alpha\phi \tag{276}$$
$$\rightarrow \partial_\alpha\partial^\alpha\phi + m^2\phi + \sum_{n=1}(2n+2)\lambda_n\phi^{2n+1} = 0 \tag{277}$$

Exercise 7.2 - Klein-Gordon plus source

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi + J(x) \tag{278}$$
$$\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi)} = \frac{1}{2}\frac{\partial(\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi)}{\partial(\partial_\alpha\phi)} = \frac{1}{2}(\eta^{\mu\nu}\partial_\nu\phi\delta^\alpha_\mu + \eta^{\mu\nu}\partial_\mu\phi\delta^\alpha_\nu) \tag{279}$$
$$= \frac{1}{2}(\eta^{\alpha\nu}\partial_\nu\phi + \eta^{\mu\alpha}\partial_\mu\phi) = \partial^\alpha\phi \tag{280}$$
$$\rightarrow \partial_\alpha\partial^\alpha\phi + m^2\phi - J(x) = 0 \tag{281}$$

Exercise 7.3 - Two interacting Klein-Gordon fields

$$\frac{\partial \mathcal{L}}{\partial \phi_i} = -m^2\phi_i - 2g(\phi_i^2 + \phi_k^2)2\phi_i \tag{282}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_i)} = \frac{1}{2}\frac{\partial(\eta^{\mu\nu}\partial_\mu\phi_i\partial_\nu\phi_i)}{\partial(\partial_\alpha\phi_i)} = \frac{1}{2}(\eta^{\mu\nu}\partial_\nu\phi_i\delta_\mu^\alpha + \eta^{\mu\nu}\partial_\mu\phi_i\delta_\nu^\alpha) \tag{283}$$

$$= \frac{1}{2}(\eta^{\alpha\nu}\partial_\nu\phi_i + \eta^{\mu\alpha}\partial_\mu\phi_i) = \partial^\alpha\phi_i \tag{284}$$

$$\to \partial_\alpha\partial^\alpha\phi_i + m^2\phi_i + 4g\phi_i(\phi_1^2 + \phi_2^2) = 0 \tag{285}$$

Exercise 7.4 - Klein-Gordon again

Same calculation as in 6.1

Exercise 8.1 - Time evolution operator - NOT DONE YET

With

$$U(t_2, t_1) = e^{-iH(t_2 - t_1)} \tag{286}$$

Then

(1) $U(t_1, t_1) = e^{-iH(t_1-t_1)} = e^0 = 1$

(2) $U(t_3, t_2)U(t_2, t_1) = e^{-iH(t_3-t_2)}e^{-iH(t_2-t_1)} = e^{-iH(t_3-t_2+t_2-t_1)} = e^{-iH(t_3-t_1)} = U(t_3, t_1)$

(3) $U(t_2, t_1)^{-1}$

(4)

(5)

Exercise 8.2 - Heisenberg equations of motions for ladder operators

With $[a_k, a_q^\dagger] = \delta_{kq}$ we have

$$\frac{d}{dt}a_k^\dagger = \frac{1}{i\hbar}[a_k^\dagger, H] = \frac{1}{i\hbar}\sum_n E_n[a_k^\dagger, a_n^\dagger a_n] = \frac{1}{i\hbar}\sum_n E_n(a_k^\dagger a_n^\dagger a_n - a_n^\dagger a_n a_k^\dagger) \tag{287}$$

$$= \frac{1}{i\hbar}E_k(a_k^\dagger a_k^\dagger a_k - a_k^\dagger a_k a_k^\dagger) = \frac{1}{i\hbar}E_k(a_k^\dagger a_k^\dagger a_k - a_k^\dagger(1 + a_k^\dagger a_k)) = -\frac{1}{i\hbar}E_k a_k^\dagger \tag{288}$$

then

$$a_k^\dagger = c \cdot e^{-E_k t/i\hbar} = a_k^\dagger(0) \cdot e^{-E_k t/i\hbar} \tag{289}$$

And similar

$$\frac{d}{dt}a_k = \frac{1}{i\hbar}[a_k, H] = \frac{1}{i\hbar}\sum_n E_n[a_k, a_n^\dagger a_n] = \frac{1}{i\hbar}\sum_n E_n(a_k a_n^\dagger a_n - a_n^\dagger a_n a_k) \tag{290}$$

$$= \frac{1}{i\hbar}E_k(a_k a_k^\dagger a_k - a_k^\dagger a_k a_k) = \frac{1}{i\hbar}E_k(a_k a_k^\dagger a_k - (a_k a_k^\dagger - 1)a_k) = \frac{1}{i\hbar}E_k a_k \tag{291}$$

then

$$a_k = c \cdot e^{E_k t/i\hbar} = a_k(0) \cdot e^{E_k t/i\hbar} \tag{292}$$

Exercise 10.1 - Commutator of field and energy momentum tensor

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \tag{293}$$

$$\begin{aligned}
[\phi(x), P^\alpha] &= \left[\phi(x), \int d^3y\, \pi(y)\partial^\alpha\phi(y) - \delta_{0\alpha}\mathcal{L}\right] && (294)\\
&= \int d^3y\ [\phi(x), \pi(y)\partial^\alpha\phi(y)] - [\phi(x), \delta_{0\alpha}\mathcal{L}] && (295)\\
&= \int d^3y\ [\phi(x)\pi(y)\partial^\alpha\phi(y) - \pi(y)\partial^\alpha\phi(y)\phi(x)] - [\phi(x), \delta_{0\alpha}\mathcal{L}] && (296)\\
&= \int d^3y \left[\underbrace{\phi(x)\pi(y)}_{=i\delta(x-y)+\pi(y)\phi(x)}\ \partial^\alpha\phi(y) - \pi(y)(\partial^\alpha(\phi(y)\phi(x)) - \phi(y)\ \underbrace{\partial^\alpha\phi(x)}_{=\frac{\partial}{\partial y^\alpha}\phi(x)=0}\ )\right] - [\phi(x), \delta_{0\alpha}\mathcal{L}] && (297)\\
&= i\partial^\alpha\phi(x) + \int d^3y\, \pi(y)\phi(x)\partial^\alpha\phi(y) - \pi(y)\underbrace{\partial^\alpha(\phi(x)\phi(y))}_{=\phi(x)\partial^\alpha\phi(y)} - \delta_{0\alpha}[\phi(x), \mathcal{L}] && (298)\\
&= i\partial^\alpha\phi(x) - \delta_{0\alpha}[\phi(x), \mathcal{L}] && (299)
\end{aligned}$$

Exercise 10.3 - Energy momentum tensor for scalar field

With

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 && (300)\\
\Pi^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} && (301)\\
&= \partial^\mu\phi && (302)\\
T^{\mu\nu} &= \Pi^\mu\partial^\nu\phi - g^{\mu\nu}\mathcal{L} && (303)\\
&= \partial^\mu\phi\partial^\nu\phi - \frac{1}{2}g^{\mu\nu}((\partial_\alpha\phi)^2 - m^2\phi^2) && (304)
\end{aligned}$$

Now

$$\begin{aligned}
\partial_\mu T^{\mu\nu} &= \Box\phi\partial^\nu\phi + \partial^\mu\phi\partial^\nu_\mu\phi - g^{\mu\nu}\left[(\partial_\alpha\phi)\partial_{\alpha\mu}\phi - m^2\phi\partial_\mu\phi\right] && (305)\\
&= (\Box\phi + m^2\phi)\partial^\nu\phi && (306)\\
&= 0 && (307)
\end{aligned}$$

then with $g^{00} = 1$ and $g^{0i} = 0$

$$\begin{aligned}
T^{00} &= (\partial^0\phi)^2 - \frac{1}{2}(\partial_\alpha\phi)^2 + \frac{1}{2}m^2\phi^2 && (308)\\
&= \frac{1}{2}(\partial^0\phi)^2 + \frac{1}{2}(\partial_k\phi)^2 + \frac{1}{2}m^2\phi^2 && (309)\\
&= \mathcal{H} && (310)\\
T^{0i} &= \partial^0\phi\partial^i\phi && (311)
\end{aligned}$$

and

$$\begin{aligned}
P^0 &= \int d^3x T^{00} = \int d^3x\mathcal{H} && (312)\\
P^k &= \int d^3x T^{0k} = \int d^3x\partial^0\phi\partial^i\phi && (313)
\end{aligned}$$

Exercise 11.1 - Commutator of field operators

$$[\hat{\phi}(x), \hat{\phi}(y)] = \left[\int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_p}}(\hat{a}_{\mathbf{p}} e^{-ipx} + \hat{a}^\dagger_{\mathbf{p}} e^{ipx}), \int \frac{d^3q}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_q}}(\hat{a}_{\mathbf{q}} e^{-iqy} + \hat{a}^\dagger_{\mathbf{q}} e^{iqy})\right] \quad (314)$$

$$= \iint \frac{d^3p}{(2\pi)^{3/2}} \frac{d^3q}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_q}} \frac{1}{\sqrt{2E_p}}([\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{q}}]e^{-i(xp+yq)} + [\hat{a}^\dagger_{\mathbf{p}}, \hat{a}_{\mathbf{q}}]e^{i(px-qy)} + [\hat{a}_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{q}}]e^{i(-px+qy)} + [\hat{a}^\dagger_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{q}}]e \quad (315)$$

$$= \iint \frac{d^3p}{(2\pi)^{3/2}} \frac{d^3q}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_q}} \frac{1}{\sqrt{2E_p}}(-\delta^{(3)}(\mathbf{p}-\mathbf{q})e^{i(px-qy)} + \delta^{(3)}(\mathbf{p}-\mathbf{q})e^{-i(px-qy)}) \quad (316)$$

$\delta^{(3)}(\mathbf{p}-\mathbf{q}) \to \mathbf{p} = \mathbf{q}, E_p = E_q$ meaning $p = q$

$$[\hat{\phi}(x), \hat{\phi}(y)] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p}(-e^{ip(x-y)} + e^{-ip(x-y)}) \quad (317)$$

$$(318)$$

0.4 van Baal - A Course in Field Theory

0.4.1 Problem 1. Violation of causality in 1+1 dimensions

(a) With $H^2 = m^2c^4 + p^2c^2$ and $p = -i\hbar\partial_x$

$$H\psi(x,t) = i\hbar\partial_t\psi(x,t) \quad (319)$$

$$H^2\psi(x,t) = -\hbar^2\partial_{tt}\psi(x,t) \quad (320)$$

$$\left(\partial_{xx} - \frac{1}{c^2}\partial_{tt} - \frac{m^2c^2}{\hbar^2}\right)\psi(x,t) = 0 \quad (321)$$

$$\left(\Box_x - \frac{m^2c^2}{\hbar^2}\right)\psi(x,t) = 0 \quad (322)$$

then we try the plane wave ansatz $\psi_k(x,t) = e^{-i(\omega_k t - kx)}$ and see

$$-k^2 + \frac{1}{c^2}\omega_k^2 - \frac{m^2c^2}{\hbar^2} = 0 \quad (323)$$

$$\to \omega_k^2 = k^2c^2 + \frac{m^2c^4}{\hbar^2} \to \omega_k = \sqrt{k^2c^2 + \frac{m^2c^4}{\hbar^2}}. \quad (324)$$

Therefore the general solution is a superposition

$$\psi(x,t) = \int dk f(k)e^{-i(\omega_k t - kx)} + g(k)e^{-i(-\omega_k t - kx)} \quad (325)$$

(b) Assume $\psi_0(x,t)$ is a solution then $\psi_0(x-y,t)$ is also a solution

$$\left(\Box_x - \frac{m^2c^2}{\hbar^2}\right)\psi_0(x,t) = 0 \quad (326)$$

$$\to \left(\Box_x - \frac{m^2c^2}{\hbar^2}\right)\psi_0(x-y,t) = 0 \quad (327)$$

then with $\psi(x,t) = \int dy\, f(y)\psi_0(x-y,t)$

$$\left(\Box_x - \frac{m^2c^2}{\hbar^2}\right)\psi(x,t) = \int dy\, f(y)\left(\Box_x - \frac{m^2c^2}{\hbar^2}\right)\psi_0(x-y,t) \quad (328)$$

$$= 0 \quad (329)$$

and

$$\psi(x,0) = \lim_{t\to 0}\int dy\, f(y)\psi_0(x-y,t) \tag{330}$$

$$= \int dy\, f(y)\delta(x-y) \tag{331}$$

$$= f(x) \tag{332}$$

Now we can use the time propagation operator

$$\psi_0(x,t) = e^{-iHt/\hbar}\psi(x,0) \tag{333}$$

$$= e^{-it\sqrt{p^2c^2+m^2c^4}/hbar}\delta(x) \tag{334}$$

$$= \frac{1}{2\pi\hbar}\int dp\, e^{-it\frac{mc^2}{\hbar}\sqrt{\frac{p^2}{m^2c^2}+1}}e^{ipx/\hbar} \tag{335}$$

and use $\cosh^2 u - \sinh^2 u = 1$ and

$$p = mc\sinh u \tag{336}$$

$$dp = mc\cosh u\; du \tag{337}$$

then

$$\psi_0(x,t) = \frac{mc}{2\pi\hbar}\int du\, e^{-it\frac{mc^2}{\hbar}\sqrt{\sinh^2 u+1}}e^{i\frac{mc}{\hbar}x\sinh u}\cosh u \tag{338}$$

$$= \frac{mc}{2\pi\hbar}\int du\, e^{-it\frac{mc^2}{\hbar}\cosh u}e^{i\frac{mc}{\hbar}x\sinh u}\cosh u \tag{339}$$

$$= \frac{mc}{2\pi\hbar}\int du\, e^{i\frac{mc}{\hbar}(x\sinh u - ct\cosh u)}\cosh u \tag{340}$$

$$= \frac{i}{2\pi c}\partial_t\int du\, e^{i\frac{mc}{\hbar}(x\sinh u - ct\cosh u)}. \tag{341}$$

Now we replace x, t by new coordinates v and z

$$x = \frac{\hbar}{mc}z\cosh v \tag{342}$$

$$ct = \frac{\hbar}{mc}z\sinh v \tag{343}$$

$$\to x^2 - c^2t^2 = \frac{\hbar^2}{m^2c^2}z^2 \tag{344}$$

then we obtain with $y = u - v$

$$\psi_0(x,t) = \frac{i}{2\pi c}\partial_t\int du\, e^{iz(\cosh v\sinh u - \sinh v\cosh u)} \tag{345}$$

$$= \frac{i}{2\pi c}\partial_t\int du\, e^{iz\sinh(u-v)} \tag{346}$$

$$= \frac{i}{2\pi c}\partial_t\int du\; [\cos(z\sinh(u-v)) + i\sin(z\sinh(u-v))] \tag{347}$$

$$= \frac{i}{2\pi c}\partial_t\int dy\; [\cos(z\sinh y) + i\sin(z\sinh y)] \tag{348}$$

$$= \frac{i}{2\pi c}\partial_t\int_{-\infty}^{\infty} dy\; \cos(z\sinh y) \tag{349}$$

$$= \frac{i}{\pi c}\partial_t\int_0^{\infty} dy\; \cos(z\sinh y) \tag{350}$$

(c)

(d)

0.5 Nastase - Introduction to Quantum Field Theory

0.5.1 Exercise 1.4 Scalar Dirac–Born–Infeld equations of motion

With

$$\begin{aligned}
\frac{\partial(\partial_\mu\phi)^2}{\partial_\nu\phi} &= \frac{\partial(\partial_\mu\phi\partial^\mu\phi)}{\partial(\partial_\nu\phi)} && (351)\\
&= \frac{\partial(\eta^{\mu\alpha}\partial_\mu\phi\partial_\alpha\phi)}{\partial(\partial_\nu\phi)} && (352)\\
&= \eta^{\mu\alpha}\frac{\partial(\partial_\mu\phi\partial_\alpha\phi)}{\partial(\partial_\nu\phi)} && (353)\\
&= \eta^{\mu\alpha}(\delta_{\mu\nu}\partial_\alpha\phi + \partial_\mu\phi\delta_{\alpha\nu}) && (354)\\
&= \eta^{\mu\alpha}\delta_{\mu\nu}\partial_\alpha\phi + \eta^{\mu\alpha}\delta_{\alpha\nu}\partial_\mu\phi && (355)\\
&= \delta^\alpha_\nu\partial_\alpha\phi + \delta^\mu_\nu\partial_\mu\phi && (356)\\
&= 2\partial_\nu\phi && (357)
\end{aligned}$$

we can calculate the parts for the Euler-Lagrange equations

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\phi} &= -\frac{1}{L^4}\frac{L^4\left[\frac{\partial g}{\partial\phi}(\partial_\mu\phi)^2 + 2m^2\phi\right]}{2\sqrt{1+L^4[g(\partial_\mu\phi)^2+m^2\phi^2]}} && (358)\\
&= -\frac{\left[\frac{\partial g}{\partial\phi}(\partial_\mu\phi)^2 + 2m^2\phi\right]}{2\sqrt{1+L^4[g(\partial_\mu\phi)^2+m^2\phi^2]}} && (359)\\
\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)} &= -\frac{1}{L^4}\frac{L^4\left[2g(\partial_\mu\phi)\delta^\mu_\nu\right]}{2\sqrt{1+L^4[g(\partial_\mu\phi)^2+m^2\phi^2]}} && (360)\\
&= -\frac{g(\partial_\nu\phi)}{\sqrt{1+L^4[g(\partial_\mu\phi)^2+m^2\phi^2]}} && (361)\\
\partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)} &= -\frac{g(\partial_\nu\partial_\nu\phi)\sqrt{1+L^4[g(\partial_\mu\phi)^2+m^2\phi^2]} - g(\partial_\nu\phi)\frac{L^4[2g(\partial_\mu\phi)(\partial_\nu\partial_\mu\phi)+2m^2\phi\partial_\nu\phi]}{2\sqrt{1+L^4[g(\partial_\mu\phi)^2+m^2\phi^2]}}}{1+L^4[g(\partial_\mu\phi)^2+m^2\phi^2]} && (362)
\end{aligned}$$

Multiplying the Euler-Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)} = 0 \qquad (363)$$

by $\sqrt{1+L^4[g(\partial_\mu\phi)^2+m^2\phi^2]}$ we obtain

$$-\frac{1}{2}\left[\frac{\partial g}{\partial\phi}(\partial_\mu\phi)^2 + 2m^2\phi\right] + g(\partial_\nu\partial_\nu\phi) + \frac{1}{2}g(\partial_\nu\phi)\frac{L^4[2g(\partial_\mu\phi)(\partial_\nu\partial_\mu\phi)+2m^2\phi\partial_\nu\phi]}{1+L^4[g(\partial_\mu\phi)^2+m^2\phi^2]} = 0 \qquad (364)$$

$$g(\Box\phi - m^2\phi) - \frac{1}{2}\frac{\partial g}{\partial\phi}(\partial_\mu\phi)^2 + gL^4\frac{g(\partial_\nu\phi)(\partial_\mu\phi)(\partial_\nu\partial_\mu\phi) + m^2\phi(\partial_\nu\phi)^2}{1+L^4[g(\partial_\mu\phi)^2+m^2\phi^2]} = 0 \qquad (365)$$

0.5.2 Exercise 2.1 Equations of motion for an anharmonic

With

$$
\begin{aligned}
p &= \frac{\partial L}{\partial \dot{q}} = \dot{q} && (366)\\
H &= p\dot{q} - L && (367)\\
&= p^2 - \frac{p^2}{2} + \frac{\lambda}{4!}q^4 && (368)\\
&= \frac{p^2}{2} + \frac{\lambda}{4!}q^4 && (369)\\
& && (370)
\end{aligned}
$$

then

$$
\begin{aligned}
\dot{p} &= -\frac{\partial H}{\partial q} = -\frac{\lambda}{3!}q^3 && (371)\\
\dot{q} &= \frac{\partial H}{\partial p} = p && (372)
\end{aligned}
$$

Phase space path integral

$$
\begin{aligned}
M(q', t'; q, t) &= \mathcal{D}p(t)\mathcal{D}q(t) \exp\left\{ i \int_t^{t'} dt[p(t)\dot{q}(t) - H(p(t), q(t))] \right\} && (373)\\
&= \mathcal{D}p(t)\mathcal{D}q(t) \exp\left\{ i \int_t^{t'} dt[p(t)\dot{q}(t) - \frac{p(t)^2}{2} - \frac{\lambda}{4!}q(t)^4] \right\} && (374)
\end{aligned}
$$

0.6 Aitchison, Hey - Gauge theory in particle physics, Volume I 5th ed

0.6.1 Problem 2.2 - Degrees of freedom of the EM field

- $3 \times E_k + 3 \times B_k$ - 4 Maxwell equations - gives 2 degrees of freedom

- $F^{\mu\nu}$ antisymmetric - 6 components

- $F_0 j = E_j$ and $F_{23} = -B_1, F_{12} = -B_3, F_{13} = +B_2$

- With $\partial_\mu F^{\mu\nu} = j^\nu$ we do some handwaving

$$
\begin{aligned}
\partial_\mu F^{\mu 0} &= j^0 \qquad \rightarrow \qquad \partial_k E^k = \rho && (375)\\
\partial_\mu F^{\mu k} &= j^k \qquad \rightarrow \qquad \partial_0 E^k + \partial_i B^k = j^k && (376)
\end{aligned}
$$

0.6.2 Problem 2.3 - Momentum operator identity

$$
\begin{aligned}
e^{iqf}\hat{p}e^{-iqf} &= -ie^{iqf}\partial_x e^{-iqf} && (377)\\
&= -ie^{iqf}\left(\partial_x e^{-iqf} + e^{-iqf}\partial_x\right) && (378)\\
&= -ie^{iqf}\left(\partial_x\left[1+(-iqf)+\frac{1}{2}(-iqf)^2+\frac{1}{6}(-iqf)^3+\frac{1}{24}(-iqf)^4+...\right]+e^{-iqf}\partial_x\right) && (379)\\
&= -ie^{iqf}\left(\partial_x[1+(-iqf)+\frac{1}{2}(-iqf)^2+\frac{1}{6}(-iqf)^3+\frac{1}{24}(-iqf)^4+...]+e^{-iqf}\partial_x\right) && (380)\\
&= -ie^{iqf}\left(0-iq[f'+\frac{1}{2}2(-iqf)f'+\frac{1}{6}3(-iqf)^2f'+\frac{1}{24}4(-iqf)^3f'+...]+e^{-iqf}\partial_x\right) && (381)\\
&= -ie^{iqf}\left(-iqe^{-iqf}f'+e^{-iqf}\partial_x\right) && (382)\\
&= -qf'-i\partial_x && (383)\\
&= -qf'+\hat{p} && (384)
\end{aligned}
$$

0.6.3 Problem 3.1 - Continuity equation

(a) With

$$-\triangle\psi + V\psi - i\partial_t\psi = 0 \quad (385)$$

then

$$
\begin{aligned}
&\to -\psi^*\triangle\psi + V\psi^*\psi - i\psi^*\partial_t\psi = 0 && (386)\\
&\to -\psi\triangle\psi^* + V\psi\psi^* + i\psi\partial_t\psi^* = 0 && (387)
\end{aligned}
$$

and

$$
\begin{aligned}
\psi^*\triangle\psi - \psi\triangle\psi^* + i(\psi^*\partial_t\psi + \psi\partial_t\psi^*) &= 0 && (388)\\
\nabla\frac{-\psi\nabla\psi^* + \psi^*\nabla\psi)}{i} + \partial_t(\psi^*\psi) &= 0 && (389)
\end{aligned}
$$

(b) Similarly

$$
\begin{aligned}
\psi(\partial_{tt}-\triangle)\psi^* - \psi^*(\partial_{tt}-\triangle)\psi &= 0 && (390)\\
\to \partial_t(\psi^*\partial_t\psi - \psi\partial_t\psi^*) + \nabla(\psi^*\nabla\psi - \psi\nabla\psi^*) &= 0 && (391)
\end{aligned}
$$

Then

$$
\begin{aligned}
\rho &= \psi^*\partial_t\psi - \psi\partial_t\psi^* && (392)\\
&= |N|^2(e^{ipx}(-ip^0)e^{-ipx} - e^{-ipx}(ip^0)e^{ipx}) && (393)\\
&= |N|^2(-2iE) && (394)\\
\vec{j} &= \psi^*\nabla\psi - \psi\nabla\psi^* && (395)\\
&= |N|^2(e^{ipx}(-i\vec{p})e^{-ipx} - e^{-ipx}(i\vec{p})e^{ipx}) && (396)\\
&= |N|^2(-2i\vec{p}) && (397)\\
&\to j^\mu = (-2i)|N|^2(E,\vec{p}) = (-2i)|N|^2p^\mu && (398)
\end{aligned}
$$

0.7 Mandl, Shaw - Quantum Field Theory 2e

0.7.1 Problem 1.1. Radiation field in a cube - NOT DONE YET

First checking orthogonality

$$a(a^\dagger)^n = (1 + a^\dagger a)(a^\dagger)^{n-1} \tag{399}$$
$$= (a^\dagger)^{n-1} + a^\dagger a(a^\dagger)^{n-1} \tag{400}$$
$$= (a^\dagger)^{n-1} + (a^\dagger)(1 + a^\dagger a)(a^\dagger)^{n-2} \tag{401}$$
$$= 2(a^\dagger)^{n-1} + (a^\dagger)^2 a^\dagger a(a^\dagger)^{n-2} \tag{402}$$
$$= n(a^\dagger)^{n-1} + (a^\dagger)^n a \tag{403}$$

then iteratively

$$a^2(a^\dagger)^n = n(n-1)(a^\dagger)^{n-2} + n(a^\dagger)^{n-1}a + (a^\dagger)^n a^2 \tag{404}$$
$$... \tag{405}$$
$$a^n(a^\dagger)^n = n! + ...a + ...a^2 + ... \tag{406}$$

so only the first term survives because of $a|0\rangle = 0$

$$\langle k|n\rangle = \langle 0|\frac{a^k}{\sqrt{k!}}\frac{(a^\dagger)^n}{\sqrt{n!}}0\rangle = \delta_{kn}. \tag{407}$$

(i)

$$\langle c|c\rangle = e^{|c|^2}\sum_{n,k}\frac{(c^*)^k c^n}{\sqrt{k!n!}}\underbrace{\langle k|n\rangle}_{\delta_{kn}} \tag{408}$$
$$= e^{-|c|^2}\sum_n \frac{|c|^{2n}}{n!} \tag{409}$$
$$= e^{-|c|^2}\sum_n \frac{(|c|^2)^n}{n!} \tag{410}$$
$$= e^{-|c|^2}e^{|c|^2} \tag{411}$$
$$= 1 \tag{412}$$

(ii) With

$$a_r(\mathbf{k})|...n_r(\mathbf{k})...\rangle = \sqrt{n_r(\mathbf{k})}|...n_r(\mathbf{k}) - 1...\rangle \tag{413}$$

then

$$a_r(\mathbf{k})|c\rangle = a_r(\mathbf{k})e^{|c|^2}\sum_{n=0}^{\infty}\frac{c^n}{\sqrt{n!}}|n\rangle \tag{414}$$
$$= e^{|c|^2}\sum_{n=0}^{\infty}\frac{c^n}{\sqrt{n!}}a_r(\mathbf{k})|n\rangle \tag{415}$$
$$= e^{|c|^2}\sum_{n=0}^{\infty}\frac{c^n}{\sqrt{n!}}\sqrt{n}|n-1\rangle \tag{416}$$
$$= c\, e^{|c|^2}\sum_{n=0}^{\infty}\frac{c^{n-1}}{\sqrt{n!}}\sqrt{n}|n-1\rangle \tag{417}$$
$$= x|c\rangle \tag{418}$$

(iii)

$$\begin{aligned}
\langle c|N|c\rangle &= \langle c|a^\dagger a|c\rangle && (419)\\
&= \langle c|c^* c|c\rangle && (420)\\
&= c^* c\langle c||c\rangle && (421)\\
&= |c|^2 && (422)
\end{aligned}$$

(iv)

$$\begin{aligned}
\langle c|N^2|c\rangle &= \langle c|a^\dagger a a^\dagger a|c\rangle && (423)\\
&= |c|^2\langle c|a a^\dagger|c\rangle && (424)\\
& && (425)
\end{aligned}$$

0.7.2 Problem 1.2. Lagrangian of point particle in EM potential - NOT DONE YET

(i)

$$\begin{aligned}
\frac{dL}{d\dot{\mathbf{x}}} &= m\dot{\mathbf{x}} + \frac{q}{c}\mathbf{A} && (426)\\
\frac{\partial}{\partial t}\frac{dL}{d\dot{\mathbf{x}}} &= m\ddot{\mathbf{x}} + \frac{q}{c}\dot{\mathbf{A}} && (427)\\
\frac{dL}{d\mathbf{x}} &= \frac{q}{c}\nabla(\mathbf{A}\cdot\dot{\mathbf{x}}) - q\nabla\phi && (428)\\
&= \frac{q}{c}\left[\mathbf{A}\times(\nabla\times\dot{\mathbf{x}}) + \dot{\mathbf{x}}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\dot{\mathbf{x}} + (\dot{\mathbf{x}}\cdot\nabla)\mathbf{A}\right] - q\nabla\phi && (429)\\
&= \frac{q}{c}\left[0 + \dot{\mathbf{x}}\times\mathbf{B} + 0 + (\dot{\mathbf{x}}\cdot\nabla)\mathbf{A}\right] - q\nabla\phi && (430)\\
\rightarrow m\ddot{\mathbf{x}} &= q\left(+\nabla\phi - \frac{1}{c}\frac{\partial}{\partial t}\dot{\mathbf{A}}\right) - \frac{q}{c}\dot{\mathbf{x}}\times\mathbf{B} - \frac{q}{c}(\dot{\mathbf{x}}\cdot\nabla)\mathbf{A} && (431)
\end{aligned}$$

(ii)

0.7.3 Problem 2.1 - NOT DONE YET

$$\begin{aligned}
\delta S &= \int d^4x\,\delta(\mathcal{L} + \partial_\alpha\Lambda^\alpha) && (432)\\
&= \int d^4x\,\delta\mathcal{L} + \delta\int d^3\sigma_\alpha\Lambda^\alpha && (433)\\
&= \int d^4x\,\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\phi_{,\beta}}\delta\phi_{,\beta} + \int d^3\sigma_\alpha\frac{\partial\Lambda^\alpha}{\partial\phi}\delta\phi && (434)\\
&= \int d^4x\,\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi - \frac{\partial}{\partial x^\beta}\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\beta}}\right)\delta\phi + \int d^4\frac{\partial}{\partial x^\beta}\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\beta}}\delta\phi\right) + \int d^3\sigma_\alpha\frac{\partial\Lambda^\alpha}{\partial\phi}\delta\phi && (435)\\
&= \int_\Omega d^4x\,\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi - \frac{\partial}{\partial x^\beta}\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\beta}}\right)\delta\phi + \int_{\partial\Omega} d^3\sigma_\beta\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\beta}}\delta\phi\right) + \int_{\partial\Omega} d^3\sigma_\alpha\frac{\partial\Lambda^\alpha}{\partial\phi}\delta\phi && (436)
\end{aligned}$$

as $\delta\phi$ vanishes on the boundary $\partial\Omega$ the $\Lambda^a lpha$ does not change the equation of motion.

0.8 Straumann - Relativistische Quantentheorie

0.8.1 Problem 1.11.1. Momentum and angular momentum of the radiation field

$$\mathbf{P} = \frac{1}{4\pi c}\int_V \mathbf{E}\times\mathbf{B}\, d^3x \tag{437}$$
$$\mathbf{J} = \frac{1}{4\pi c}\int_V [\mathbf{x}\times(\mathbf{E}\times\mathbf{B})]\, d^3x \tag{438}$$

In Coulomb gauge we have

$$\mathbf{E} = -\frac{1}{c}\partial_t\mathbf{A} = -\frac{1}{c}\dot{A}_l\mathbf{e}_l \tag{439}$$
$$\mathbf{B} = \nabla\times\mathbf{A} = \varepsilon_{ijk}(\partial_j A_k)\mathbf{e}_i \tag{440}$$
$$\mathbf{E}\times\mathbf{B} = -\frac{1}{c}\varepsilon_{nli}\mathbf{e}_n(\dot{A}_l\mathbf{e}_l)(\varepsilon_{ijk}(\partial_j A_k)\mathbf{e}_i) \tag{441}$$
$$= -\frac{1}{c}\varepsilon_{nli}\mathbf{e}_n\dot{A}_l\varepsilon_{ijk}(\partial_j A_k)\mathbf{e}_i\mathbf{e}_l \tag{442}$$
$$= -\frac{1}{c}\varepsilon_{nli}\mathbf{e}_n\dot{A}_l\varepsilon_{ijk}(\partial_j A_k)\delta_{il} \tag{443}$$
$$= -\frac{1}{c}\varepsilon_{nli}\varepsilon_{ijk}(\partial_j A_k)\dot{A}_i\mathbf{e}_n \tag{444}$$
$$= -\frac{1}{c}(\delta_{nj}\delta_{lk}-\delta_{nk}\delta_{lj})(\partial_j A_k)\dot{A}_l\mathbf{e}_n \tag{445}$$
$$= -\frac{1}{c}((\partial_j A_k)\dot{A}_k\mathbf{e}_j - (\partial_j A_k)\dot{A}_j\mathbf{e}_k) \tag{446}$$
$$= -\frac{1}{c}((\mathbf{e}_j\partial_j A_k)\dot{A}_k - \dot{A}_j(\partial_j A_k)\mathbf{e}_k) \tag{447}$$
$$= -\frac{1}{c}[\nabla(\mathbf{A}\cdot\dot{\mathbf{A}}) - (\dot{\mathbf{A}}\cdot\nabla)\mathbf{A}] \tag{448}$$

And from (1.44) and (1.33)

$$\mathbf{A}(x,t) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k},\lambda}\sqrt{\frac{2\pi\hbar c^3}{\omega_k}}\left[a_{\mathbf{k},\lambda}\boldsymbol{\varepsilon}(k,\lambda)e^{i\mathbf{k}\cdot\mathbf{x}} + a^*_{\mathbf{k},\lambda}\boldsymbol{\varepsilon}(k,\lambda)^* e^{-i\mathbf{k}\cdot\mathbf{x}}\right] \tag{449}$$
$$= \sum_{\mathbf{k},\lambda}\sqrt{\frac{2\pi\hbar c^3}{\omega_k}}\left[a_{\mathbf{k},\lambda}\mathbf{u}_{\mathbf{k},\lambda}(\mathbf{x}) + a^*_{\mathbf{k},\lambda}\mathbf{u}^*_{\mathbf{k},\lambda}(\mathbf{x})\right] \tag{450}$$
$$\tag{451}$$

0.8.2 Problem 4.5.1. Approximation for polarization potential

$$\Phi^{\mathrm{Pol}}(\mathbf{x}) = \frac{e}{(2\pi)^3}\int d^3k e^{i\mathbf{k}\cdot\mathbf{x}}\int_{4m^2}^{\infty} d\kappa^2\frac{\Pi(x^2)}{\kappa^2(\kappa^2+\mathbf{k}^2)} \tag{452}$$

0.9 Ramond - Field Theory - A modern primer

0.9.1 Problem 1.1 A

(i) With

$$\left(\frac{d(x+\delta x)}{dt}\right)^2 = \left(\frac{dx}{dt}+\delta\frac{dx}{dt}\right)\left(\frac{dx}{dt}+\delta\frac{dx}{dt}\right) \tag{453}$$

$$= \left(\frac{dx}{dt}\right)^2 + 2\frac{dx}{dt}\cdot\delta\frac{dx}{dt} + \left(\delta\frac{dx}{dt}\right)^2 \tag{454}$$

$$= \left(\frac{dx}{dt}\right)^2 + \frac{d}{dt}\left(\frac{dx}{dt}\delta x\right) - 2\frac{d^2x}{dt^2}\delta x + \left(\delta\frac{dx}{dt}\right)^2 \tag{455}$$

where we integrates the second term by parts. Now we can expand the action

$$S = \int dt\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 \tag{456}$$

$$S[x+\delta x] = \int dt\frac{1}{2}m\left(\frac{d(x+\delta x)}{dt}\right)^2 \tag{457}$$

$$\delta S = -\frac{1}{2}m\int_{t_1}^{t_2} dt\, 2\frac{dx}{dt}\frac{d\delta(x)}{dt} \tag{458}$$

$$= -\frac{1}{2}m\int_{t_1}^{t_2} dt\delta x\left(2\frac{d^2x}{dt^2}\right) + \frac{1}{2}m\frac{dx}{dt}\delta x\Big|_{t_1}^{t_2} \tag{459}$$

Assuming the equations of motion hold $\ddot{x}=0$ an forcing the surface term to vanish (we CAN'T force $\delta x = 0$) we have

$$\frac{d}{dt}\left(\frac{dx}{dt}\right) = 0 \tag{460}$$

(ii) We could assume a velocity dependent potential is considered

$$V = \sqrt{\dot{x}_1^2+\dot{x}_2^2+\dot{x}_3^2}\left(1-\cos\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{a}\right) \tag{461}$$

but then units would be off - so we assume v to be a constant. The

$$\delta S_V = \frac{\partial V}{\partial x_i}\delta x_i \tag{462}$$

$$= \frac{vx_i}{ar}\sin\frac{r}{a}\delta x_i \tag{463}$$

$$\to m\ddot{x}_i = \frac{vx_i}{ar}\sin\frac{r}{a}\delta x_i \tag{464}$$

Surface term

$$\left(\frac{\partial L}{\partial \dot{x}_i}\delta x_i\right)_{t_1}^{t_2} \tag{465}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = \frac{d}{dt}\frac{\partial L}{\partial p_i} = \frac{\partial L}{\partial x_i} = \frac{vx_i}{ar}\sin\frac{r}{a}\delta x_i \tag{466}$$

0.10 Maggiore - A modern introduction to quantum field theory

0.10.1 Problem 2.3 - Four vectors from Weyl spinors - NOT DONE YET!

1. First we consider $\xi_R^\dagger \sigma^\mu \psi_R$

   (a) now consider a boost in x-direction $\Lambda_R = e^{\eta \frac{\sigma^1}{2}}$

$$(\Lambda_R \xi_R)^\dagger \sigma^\mu (\Lambda_R \psi_R) = (e^{+\eta_1 \frac{\sigma^1}{2}} \xi_R)^\dagger \sigma^\mu (e^{+\eta_1 \frac{\sigma^1}{2}} \psi_R^\dagger) \tag{467}$$

$$\simeq \xi_R^\dagger \left(1 + \eta_1 \frac{\sigma^1}{2}\right)^\dagger \sigma^\mu \left(1 + \eta_1 \frac{\sigma^1}{2}\right) \psi_R \tag{468}$$

$$= \xi_R^\dagger \sigma^\mu \psi_R + \frac{1}{2} \eta_1 \xi_R^\dagger (\sigma^1 \sigma^\mu + \sigma^\mu \sigma^1) \psi_R \qquad (\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij} 1_{2\times 2}) \tag{469}$$

$$= \xi_R^\dagger \sigma^\mu \psi_R + \frac{1}{2} \eta_1 \xi_R^\dagger \begin{pmatrix} 2\sigma^1 \\ 2\sigma^0 \\ 0 \\ 0 \end{pmatrix} \psi_R \tag{470}$$

$$= \xi_R^\dagger \sigma^\mu \psi_R + \eta_1 \xi_R^\dagger \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sigma^0 \\ \sigma^1 \\ \sigma^2 \\ \sigma^3 \end{pmatrix} \psi_R \tag{471}$$

$$= \xi_R^\dagger \sigma^\mu \psi_R - i\eta_1 \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xi_R^\dagger \sigma^\mu \psi_R \tag{472}$$

$$= (1 - i\eta_1 K_1)^\nu{}_\mu \xi_R^\dagger \sigma^\mu \psi_R \tag{473}$$

$$\simeq \Lambda^\nu{}_\mu \, \xi_R^\dagger \sigma^\mu \psi_R \tag{474}$$

   (b) now consider a rotation in the z-plane ...similar calculation

2. Now we consider $\xi_L^\dagger \bar{\sigma}^\mu \psi_L$

   (a) now consider a boost in x-direction $\Lambda_R = e^{\eta \frac{\sigma^1}{2}}$ ...similar calculation

   (b) now consider a rotation in the z-plane ...similar calculation

0.11 Muenster - Von der Quantenfeldtheorie zum Standardmodell

0.11.1 Problem 2.1 - 1

(a) The Klein-Gordon equations is given by

$$\left(\partial_\mu \partial^\mu + \frac{m^2 c^2}{\hbar^2}\right) \varphi = 0 \tag{475}$$

$$\left(c^2 \partial_{tt} - \triangle + \frac{m^2 c^2}{\hbar^2}\right) \varphi = 0 \tag{476}$$

We make the ansatz

$$\varphi = \phi_1 + \phi_2 \tag{477}$$

$$\phi_1 = \frac{1}{2}\varphi - \alpha\partial_t\varphi \tag{478}$$

$$\phi_2 = \frac{1}{2}\varphi + \alpha\partial_t\varphi \tag{479}$$

Then we get expressions for the time derivatives

$$\phi_2 - \phi_1 = 2\alpha\partial_t\varphi \tag{480}$$

$$\to \partial_t\varphi = \frac{1}{2\alpha}(\phi_2 - \phi_1) \tag{481}$$

and

$$\partial_{tt}\varphi = c^2\left(\triangle - \frac{m^2c^2}{\hbar^2}\right)\varphi \tag{482}$$

$$= c^2\left(\triangle - \frac{m^2c^2}{\hbar^2}\right)(\phi_1 + \phi_2) \tag{483}$$

Therefore we get for $\phi_{1,2}$

$$\partial_t\phi_1 = \frac{1}{2}\partial_t\varphi - \alpha\partial_{tt}\varphi \tag{484}$$

$$= \frac{1}{2\alpha}(\phi_2 - \phi_1) - \alpha c^2\left(\triangle - \frac{m^2c^2}{\hbar^2}\right)(\phi_1 + \phi_2) \tag{485}$$

$$\partial_t\phi_2 = \frac{1}{2}\partial_t\varphi + \alpha\partial_{tt}\varphi \tag{486}$$

$$= \frac{1}{2\alpha}(\phi_2 - \phi_1) + \alpha c^2\left(\triangle - \frac{m^2c^2}{\hbar^2}\right)(\phi_1 + \phi_2) \tag{487}$$

which we can write in the form

$$i\hbar\partial_t\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = -i\hbar\alpha c^2\left(\triangle - \frac{m^2c^2}{\hbar^2}\right)\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} + \frac{i\hbar}{2\alpha}\begin{pmatrix}-1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \tag{488}$$

$$= i\hbar\begin{pmatrix}-\alpha c^2\left(\triangle - \frac{m^2c^2}{\hbar^2}\right) - \frac{1}{2\alpha} & -\alpha c^2\left(\triangle - \frac{m^2c^2}{\hbar^2}\right) + \frac{1}{2\alpha}\\ \alpha c^2\left(\triangle - \frac{m^2c^2}{\hbar^2}\right) - \frac{1}{2\alpha} & \alpha c^2\left(\triangle - \frac{m^2c^2}{\hbar^2}\right) + \frac{1}{2\alpha}\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \tag{489}$$

(b) Diagonalization gives

$$i\hbar\partial_t\phi = \hat{H}\phi \tag{490}$$

$$\to i\hbar\partial_t S^{-1}\phi = \underbrace{S^{-1}\hat{H}S}_{=h}S^{-1}\phi \tag{491}$$

$$\lambda_\pm = \pm\sqrt{2}c\hbar\sqrt{\Delta - \frac{m^2c^2}{\hbar^2}} \tag{492}$$

$$= \mp\sqrt{2}mc^2\sqrt{1 - \frac{\hbar^2}{m^2c^2}\triangle} \tag{493}$$

A semi-canonical choice for the parameter α is to make the $\triangle$ look like a momentum operator

$$i\hbar\alpha c^2 = -\frac{\hbar^2}{2m} \quad \to \quad \alpha = \frac{i\hbar}{2mc^2} \tag{494}$$

0.12 Peskin, Schroeder - An Introduction to Quantum Field Theory

0.12.1 Problem 2.1 - Maxwell equations

(a)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}F_{\mu\nu}F_{\alpha\beta} \tag{495}$$
$$= -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\alpha A_\beta - \partial_\beta A_\alpha) \tag{496}$$

With

$$\frac{\partial \mathcal{L}}{\partial A_\gamma} - \partial_\sigma \frac{\mathcal{L}}{\partial(\partial_\sigma A_\gamma)} = 0 \tag{497}$$

then

$$\frac{\mathcal{L}}{\partial(\partial_\sigma A_\gamma)} = -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\alpha A_\beta - \partial_\beta A_\alpha) - \frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\alpha A_\beta - \partial_\beta A_\alpha) \tag{498}$$
$$= -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}(\delta^\sigma_\mu\delta^\gamma_\nu - \delta^\sigma_\nu\delta^\gamma_\mu)(\partial_\alpha A_\beta - \partial_\beta A_\alpha) - \frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\alpha A_\beta - \partial_\beta A_\alpha) \tag{499}$$
$$= -\frac{1}{4}(\delta^{\alpha\sigma}\delta^{\beta\gamma} - \delta^{\beta\sigma}\delta^{\alpha\gamma})(\partial_\alpha A_\beta - \partial_\beta A_\alpha) - ... \tag{500}$$
$$= -\frac{1}{4}(\partial^\sigma A^\gamma - \partial^\gamma A^\sigma - \partial^\gamma A^\sigma + \partial^\sigma A^\gamma) - ... \tag{501}$$
$$= -\frac{1}{4}2F^{\sigma\gamma} - ... \tag{502}$$
$$= -F^{\sigma\gamma} \tag{503}$$

and therefore

$$\partial_\sigma F^{\sigma\gamma} = 0 \tag{504}$$

Rewriting into the common form

$$\gamma = 0 \quad \rightarrow \quad \partial_0 F^{00} + \sum_i \partial_i F^{i0} = 0 \tag{505}$$
$$\rightarrow \quad \sum_i \partial_i(-F^{0i}) = 0 \tag{506}$$
$$\rightarrow \quad \sum_i \partial_i E^i = 0 \tag{507}$$
$$\rightarrow \quad \nabla \cdot \mathbf{E} = 0 \tag{508}$$

$$\gamma = k \quad \rightarrow \quad \partial_0 F^{0k} + \sum_i \partial_i F^{ik} = 0 \tag{509}$$
$$\rightarrow \quad \partial_0(-E^k) + \sum_i \partial_i F^{ik} = 0 \tag{510}$$
$$\rightarrow \quad \partial_0(-E^k) + \sum_i \partial_i(-\epsilon_{ikm}B^m) = 0 \tag{511}$$
$$\rightarrow \quad \dot{\mathbf{E}} = \nabla \times \mathbf{B} \tag{512}$$

The other two equations come from

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{513}$$
$$\rightarrow \quad \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \tag{514}$$

(b) With the definition (2.17)

$$T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\lambda)}\partial_\nu A_\lambda - \mathcal{L}\delta^\mu_\nu \tag{515}$$
$$= -F^{\mu\lambda}\partial_\nu A_\lambda + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\delta^\mu_\nu \tag{516}$$

we rewrite

$$T^{\mu\nu} = -F^{\mu\lambda}\partial^\nu A_\lambda + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\eta^{\mu\nu} \tag{517}$$
$$\widehat{T}^{\mu\nu} = -F^{\mu\lambda}\partial^\nu A_\lambda + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\eta^{\mu\nu} + \partial_\lambda(F^{\mu\lambda}A^\nu) \tag{518}$$
$$= -F^{\mu\lambda}\partial^\nu A_\lambda + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\eta^{\mu\nu} + \underbrace{(\partial_\lambda F^{\mu\lambda})}_{=0\text{ (Maxwell)}} A^\nu + F^{\mu\lambda}(\partial_\lambda A^\nu) \tag{519}$$
$$= F^{\mu\lambda}F_\lambda^{\ \nu} + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\eta^{\mu\nu} \tag{520}$$
$$= F^{\mu\lambda}F_{\lambda\sigma}\eta^{\sigma\nu} + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\eta^{\mu\nu} \tag{521}$$
$$= F^{\uparrow\uparrow}F_{\downarrow\downarrow}\eta + \frac{1}{4}\mathrm{tr}(-F^{\uparrow\uparrow}F_{\downarrow\downarrow})\eta \tag{522}$$

and with

$$F_{\mu\nu} = F_{\downarrow\downarrow} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \qquad F^{\mu\nu} = F_{\uparrow\uparrow} = \eta F_{\downarrow\downarrow}\eta^T = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{523}$$

$$F_{\mu\nu}F^{\mu\nu} = -\mathrm{tr}(F_{\downarrow\downarrow}.F_{\uparrow\uparrow}) = 2(\mathbf{B}^2 - \mathbf{E}^2) \qquad F^{\mu\lambda}F_{\lambda\nu} = ... \tag{524}$$

we obtain

$$\widehat{T}^{\mu\nu} = \begin{pmatrix} \mathcal{E} & \mathbf{S} \\ \mathbf{S} & ... \end{pmatrix} \tag{525}$$

which looks symmetric.

0.12.2 Problem 2.2 - The complex scalar field

(a) Using $\partial_\mu\phi^*\partial^\mu\phi = \partial_\mu\phi^*\eta^{\mu\nu}\partial_\nu\phi = \partial^\mu\phi^*\partial_\mu\phi$ and $\partial^\mu = \eta^{\mu\nu}\partial_\nu = (\partial_0, -\partial_i)$ we find

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial\dot{\phi})} = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \partial^0\phi^* = \partial_0\phi^* = \dot{\phi}^* \tag{526}$$
$$\pi^* = \frac{\partial\mathcal{L}}{\partial(\partial\dot{\phi}^*)} = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi^*)} = \partial^0\phi = \partial_0\phi = \dot{\phi} \tag{527}$$

then

$$H = \int d^3x[\pi\dot{\phi} + \pi^*\dot{\phi}^* - \mathcal{L}] \tag{528}$$
$$= \int d^3x[\pi\pi^* + \pi^*\pi - \partial_\mu\phi^*\eta^{\mu\nu}\partial_\nu\phi + m^2\phi^*\phi] \tag{529}$$
$$= \int d^3x[\pi\pi^* + \pi^*\pi - (\underbrace{\dot{\phi}^*\dot{\phi}}_{=\pi\pi^*} - \nabla\phi^* \cdot \nabla\phi) + m^2\phi^*\phi] \tag{530}$$
$$= \int d^3x[\pi^*\pi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi] \tag{531}$$

Let's rewrite the Lagrangian with $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi \tag{532}$$
$$= \frac{1}{2}\partial_\mu(\phi_1 - i\phi_2)\partial^\mu(\phi_1 + i\phi_2) - \frac{1}{2}m^2(\phi_1 - i\phi_2)(\phi_1 + i\phi_2) \tag{533}$$
$$= \frac{1}{2}(\partial_\mu\phi_1\partial^\mu\phi_1 - m^2\phi_1^2) + i\frac{1}{2}(\partial_\mu\phi_2\partial^\mu\phi_2 - m^2\phi_2^2) \tag{534}$$

So we use the results for the scalar field

$$\phi_1(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_\mathbf{p}}}\left(a_\mathbf{p}e^{i\mathbf{p}\cdot\mathbf{x}} + a_\mathbf{p}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right) \tag{535}$$
$$\pi_1(\mathbf{x}) = -i\int \frac{d^3p}{(2\pi)^3\sqrt{2}}\sqrt{\omega_\mathbf{p}}\left(a_\mathbf{p}e^{i\mathbf{p}\cdot\mathbf{x}} + a_\mathbf{p}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right) \tag{536}$$
$$\phi_2(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_\mathbf{p}}}\left(b_\mathbf{p}e^{i\mathbf{p}\cdot\mathbf{x}} + b_\mathbf{p}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right) \tag{537}$$
$$\pi_2(\mathbf{x}) = -i\int \frac{d^3p}{(2\pi)^3\sqrt{2}}\sqrt{\omega_\mathbf{p}}\left(b_\mathbf{p}e^{i\mathbf{p}\cdot\mathbf{x}} + b_\mathbf{p}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right) \tag{538}$$
$$[a_\mathbf{p}, a_\mathbf{q}^\dagger] = (2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{q}) \tag{539}$$
$$[b_\mathbf{p}, b_\mathbf{q}^\dagger] = (2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{q}) \tag{540}$$

then

$$\phi(\mathbf{x}) = \frac{1}{\sqrt{2}}\int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_\mathbf{p}}}\left((a_\mathbf{p} + ib_\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}} + (a_\mathbf{p}^\dagger + ib_\mathbf{p}^\dagger)e^{-i\mathbf{p}\cdot\mathbf{x}}\right) \tag{541}$$
$$\equiv \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_\mathbf{p}}}\left(\alpha_\mathbf{p}e^{i\mathbf{p}\cdot\mathbf{x}} + \beta_\mathbf{p}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right) \tag{542}$$
$$\phi^\dagger(\mathbf{x}) = \frac{1}{\sqrt{2}\int \frac{d^3p}{(2\pi)^3}\sqrt{2\omega_\mathbf{p}}}\left((a_\mathbf{p}^\dagger - ib_\mathbf{p}^\dagger)e^{-i\mathbf{p}\cdot\mathbf{x}} + (a_\mathbf{p} - ib_\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}\right) \tag{543}$$
$$\equiv \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_\mathbf{p}}}\left(\alpha_\mathbf{p}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} + \beta_\mathbf{p}e^{i\mathbf{p}\cdot\mathbf{x}}\right) \tag{544}$$

With the new defines creation/annihilation operators

$$\alpha_\mathbf{p} = \frac{1}{\sqrt{2}}(a_\mathbf{p} + ib_\mathbf{p}) \quad \rightarrow \quad \alpha_\mathbf{p}^\dagger = \frac{1}{\sqrt{2}}(a_\mathbf{p}^\dagger - ib_\mathbf{p}^\dagger) \tag{545}$$
$$\beta_\mathbf{p} = \frac{1}{\sqrt{2}}(a_\mathbf{p} - ib_\mathbf{p}) \quad \rightarrow \quad \beta_\mathbf{p}^\dagger = \frac{1}{\sqrt{2}}(a_\mathbf{p}^\dagger + ib_\mathbf{p}^\dagger) \tag{546}$$

we can calculate their commutation relations (**assuming all the cross commutators between $a, a^\dagger$ and $b, b^\dagger$ are zero**)

$$[\alpha_{\mathbf{p}}, \alpha_{\mathbf{q}}] = \frac{1}{2}[a_{\mathbf{p}} + ib_{\mathbf{p}}, a_{\mathbf{q}} + ib_{\mathbf{q}}] \tag{547}$$
$$= \frac{1}{2}([a_{\mathbf{p}}, a_{\mathbf{q}}] + i[b_{\mathbf{p}}, a_{\mathbf{q}}] + i[a_{\mathbf{p}}, b_{\mathbf{q}}] - [b_{\mathbf{p}}, b_{\mathbf{q}}]) \tag{548}$$
$$= \frac{1}{2}i([b_{\mathbf{p}}, a_{\mathbf{q}}] + [a_{\mathbf{p}}, b_{\mathbf{q}}]) \tag{549}$$
$$= 0 \tag{550}$$

$$[\alpha^\dagger_{\mathbf{p}}, \alpha^\dagger_{\mathbf{q}}] = \frac{1}{2}([a^\dagger_{\mathbf{p}} - ib^\dagger_{\mathbf{p}}, a^\dagger_{\mathbf{q}} - ib^\dagger_{\mathbf{q}}]) \tag{551}$$
$$= \frac{1}{2}([a^\dagger_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] - i[b^\dagger_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] - i[a^\dagger_{\mathbf{p}}, b^\dagger_{\mathbf{q}}] - [b^\dagger_{\mathbf{p}}, b^\dagger_{\mathbf{q}}]) \tag{552}$$
$$= \frac{1}{2}(-i[b^\dagger_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] - i[a^\dagger_{\mathbf{p}}, b^\dagger_{\mathbf{q}}]) \tag{553}$$
$$= 0 \tag{554}$$

$$[\alpha_{\mathbf{p}}, \alpha^\dagger_{\mathbf{q}}] = \frac{1}{2}[a_{\mathbf{p}} + ib_{\mathbf{p}}, a^\dagger_{\mathbf{q}} - ib^\dagger_{\mathbf{q}}] \tag{555}$$
$$= \frac{1}{2}([a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] + i[b_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] - i[a_{\mathbf{p}}, b^\dagger_{\mathbf{q}}] + [b_{\mathbf{p}}, b^\dagger_{\mathbf{q}}]) \tag{556}$$
$$= (2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{q}) + i[b_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] - i[a_{\mathbf{p}}, b^\dagger_{\mathbf{q}}] \tag{557}$$
$$= (2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{q}) \tag{558}$$

$$[\beta_{\mathbf{p}}, \beta^\dagger_{\mathbf{q}}] = \frac{1}{2}[a_{\mathbf{p}} - ib_{\mathbf{p}}, a^\dagger_{\mathbf{q}} + ib^\dagger_{\mathbf{q}}] \tag{559}$$
$$= \frac{1}{2}([a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] - i[b_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] + i[a_{\mathbf{p}}, b^\dagger_{\mathbf{q}}] + [b_{\mathbf{p}}, b^\dagger_{\mathbf{q}}]) \tag{560}$$
$$= (2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{q}) \tag{561}$$

$$[\alpha_{\mathbf{p}}, \beta_{\mathbf{q}}] = \frac{1}{2}[a_{\mathbf{p}} + ib_{\mathbf{p}}, a_{\mathbf{q}} - ib_{\mathbf{q}}] \tag{562}$$
$$= \frac{1}{2}([a_{\mathbf{p}}, a_{\mathbf{q}}] + i[a_{\mathbf{p}}, b_{\mathbf{q}}] + i[b_{\mathbf{p}}, a_{\mathbf{q}}] - [b_{\mathbf{p}}, b_{\mathbf{q}}]) \tag{563}$$
$$= 0 \tag{564}$$

$$[\alpha_{\mathbf{p}}, \beta^\dagger_{\mathbf{q}}] = \frac{1}{2}[a_{\mathbf{p}} + ib_{\mathbf{p}}, a^\dagger_{\mathbf{q}} - ib^\dagger_{\mathbf{q}}] \tag{565}$$
$$= \frac{1}{2}([a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] + i[a_{\mathbf{p}}, b^\dagger_{\mathbf{q}}] + i[b_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] - [b_{\mathbf{p}}, b^\dagger_{\mathbf{q}}]) \tag{566}$$
$$= (2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{q}) - (2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{q}) \tag{567}$$
$$= 0 \tag{568}$$
$$[\alpha^\dagger_{\mathbf{p}}, \beta^\dagger_{\mathbf{q}}] = 0 \tag{569}$$

As the $\phi\mathbf{x}$ is in the Schroedinger picture there is not time dependency and we can not calculate $\pi(\mathbf{x})$ - therefore we need to transform to the Heisenberg picture. To make it simple we do this first for ϕ_1 and ϕ_2 using $p \cdot x = E_p t - \mathbf{p} \cdot \mathbf{x}$ and $p^2 = E_{\mathbf{p}}^2 - \mathbf{p}^2 = m^2$ (meaning $p^0 \equiv E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$

$$\phi_1(x) = e^{iHt}\phi(\mathbf{x})e^{-iHt} \tag{570}$$

$$= ... \tag{571}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}}(a_{\mathbf{p}}e^{-ipx} + a_{\mathbf{p}}^\dagger e^{ipx}) \tag{572}$$

$$\phi_2(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}}(b_{\mathbf{p}}e^{-ipx} + b_{\mathbf{p}}^\dagger e^{ipx}) \tag{573}$$

$$\tag{574}$$

Here we cheated a bit - we used the result from the scalar Lagrangian - meaning using the scalar Hamiltonian. Then

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} \left(\alpha_{\mathbf{p}}e^{-ipx} + \beta_{\mathbf{p}}^\dagger e^{ipx}\right) \tag{575}$$

$$\phi^\dagger(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} \left(\alpha_{\mathbf{p}}^\dagger e^{ipx} + \beta_{\mathbf{p}}e^{-ipx}\right) \tag{576}$$

and

$$\rightarrow \quad \pi^*(x) = \dot{\phi}(x) = i\int \frac{d^3p}{(2\pi)^3\sqrt{2}}\sqrt{E_{\mathbf{p}}}\left(-\alpha_{\mathbf{p}}e^{-ipx} + \beta_{\mathbf{p}}^\dagger e^{ipx}\right) \tag{577}$$

$$\rightarrow \quad \pi(x) = \dot{\phi}^\dagger(x) = i\int \frac{d^3p}{(2\pi)^3\sqrt{2}}\sqrt{E_{\mathbf{p}}}\left(\alpha_{\mathbf{p}}^\dagger e^{ipx} - \beta_{\mathbf{p}}e^{-ipx}\right) \tag{578}$$

The only non-vanishing commutator relations for field and momentum operators are

$$[\phi(\mathbf{x},t), \pi(\mathbf{y},t)] = i\int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}\int \frac{d^3q}{(2\pi)^3\sqrt{2}}\sqrt{E_{\mathbf{q}}}[\alpha_{\mathbf{p}}e^{-ipx} + \beta_{\mathbf{p}}^\dagger e^{ipx}, \alpha_{\mathbf{q}}^\dagger e^{iqy} - \beta_{\mathbf{q}}e^{-iqy}] \tag{579}$$

$$= i\int \frac{d^3p\, d^3q}{(2\pi)^6}\frac{1}{2}\sqrt{\frac{E_{\mathbf{q}}}{E_{\mathbf{p}}}}([\alpha_{\mathbf{p}}, \alpha_{\mathbf{q}}^\dagger]e^{-ipx+iqy} - [\beta_{\mathbf{p}}^\dagger, \beta_{\mathbf{q}}]e^{ipx-iqy}) \tag{580}$$

$$= i\int \frac{d^3p\, d^3q}{(2\pi)^6}\frac{1}{2}\sqrt{\frac{E_{\mathbf{q}}}{E_{\mathbf{p}}}}(e^{-ipx+iqy} + e^{ipx-iqy})(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) \tag{581}$$

$$= i\int \frac{d^3p}{(2\pi)^3}\frac{1}{2}(e^{-ip(x-y)} + e^{ip(x-y)}) \tag{582}$$

$$= i\delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{583}$$

$$[\phi^\dagger(\mathbf{x},t), \pi^\dagger(\mathbf{y},t)] = i\delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{584}$$

To calculate the Heisenberg equations of motion we start with

$$\nabla\phi(x) = i\int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}\mathbf{p}\left(\alpha_{\mathbf{p}}e^{-ipx} - \beta_{\mathbf{p}}^\dagger e^{ipx}\right) \tag{585}$$

$$\nabla\phi^\dagger(x) = i\int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}\mathbf{p}\left(-\alpha_{\mathbf{p}}^\dagger e^{ipx} + \beta_{\mathbf{p}}e^{-ipx}\right) \tag{586}$$

and then

$$i\dot{\phi}(x) = [\phi(x), H] = \left[\phi(x), \int d^3y(\pi^\dagger\pi + \nabla\phi^\dagger \cdot \nabla\phi + m^2\phi^\dagger\phi)\right] \tag{587}$$

$$= \int d^3y\pi^\dagger(y)[\phi(x), \pi(y)] \tag{588}$$

$$= i\pi^\dagger(x) \tag{589}$$

$$i\dot{\phi}^\dagger(x) = [\phi^\dagger(x), H] = \left[\phi(x), \int d^3y(\pi^\dagger\pi + \nabla\phi^\dagger \cdot \nabla\phi + m^2\phi^\dagger\phi)\right] \tag{590}$$

$$= \int d^3y[\phi^\dagger(x), \pi^\dagger(y)]\pi(y) \tag{591}$$

$$= i\pi(x) \tag{592}$$

and

$$i\dot{\pi}(x) = [\pi(x), H] = \left[\pi(x), \int d^3y(\pi^\dagger\pi + \nabla\phi^\dagger \cdot \nabla\phi + m^2\phi^\dagger\phi)\right] \tag{593}$$

$$= \left[\pi(x), \int d^3y(\pi^\dagger\pi - \triangle\phi^\dagger \cdot \phi + m^2\phi^\dagger\phi)\right] \tag{594}$$

$$= \int d^3y(-\triangle\phi^\dagger + m^2\phi^\dagger)[\pi(x), \phi(y)] \tag{595}$$

$$= i(\triangle_x - m^2)\phi^\dagger(x) \tag{596}$$

$$i\dot{\pi}^\dagger(x) = [\pi^\dagger(x), H] = \left[\pi^\dagger(x), \int d^3y(\pi^\dagger\pi + \nabla\phi^\dagger \cdot \nabla\phi + m^2\phi^\dagger\phi)\right] \tag{597}$$

$$= \left[\pi^\dagger(x), \int d^3y(\pi^\dagger\pi - \phi^\dagger \cdot \triangle\phi + m^2\phi^\dagger\phi)\right] \tag{598}$$

$$= \int d^3y[\pi^\dagger(x), \phi^\dagger(y)](-\triangle\phi + m^2\phi) \tag{599}$$

$$= i(\triangle_x - m^2)\phi(x) \tag{600}$$

resulting in

$$i\dot{\pi}(x) \quad \rightarrow \quad \ddot{\phi}^\dagger = (\triangle - m^2)\phi^\dagger \tag{601}$$

$$\rightarrow \quad (\Box + m^2)\phi^\dagger = 0 \tag{602}$$

$$i\dot{\pi}^\dagger(x) \quad \rightarrow \quad \ddot{\phi} = (\triangle - m^2)\phi \tag{603}$$

$$\rightarrow \quad (\Box + m^2)\phi = 0 \tag{604}$$

(b)

(c)

(d)

0.12.3 Problem 2.3 - Calculating $D(x-y)$

As we are calculation the vacuum expectation value we need to get the $a^\dagger$'s to the right and the a's to the left

$$\phi(x)\phi(y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}(a_{\mathbf{p}}e^{-ipx} + a^\dagger_{\mathbf{p}}e^{ipx}) \int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{q}}}}(a_{\mathbf{q}}e^{-iqy} + a^\dagger_{\mathbf{q}}e^{iqy}) \tag{605}$$

$$= \iint \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{q}}}}\frac{1}{\sqrt{2E_{\mathbf{p}}}}(a_{\mathbf{p}}e^{-ipx} + a^\dagger_{\mathbf{p}}e^{ipx})(a_{\mathbf{q}}e^{-iqy} + a^\dagger_{\mathbf{q}}e^{iqy}) \tag{606}$$

$$= \iint \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{q}}}}\frac{1}{\sqrt{2E_{\mathbf{p}}}}(a_{\mathbf{p}}a_{\mathbf{q}}e^{-ipx-iqy} + a^\dagger_{\mathbf{p}}a_{\mathbf{q}}e^{ipx-iqy} + a_{\mathbf{p}}a^\dagger_{\mathbf{q}}e^{-ipx+iqy} + a^\dagger_{\mathbf{p}}a^\dagger_{\mathbf{q}}e^{ipx+iqy}) \tag{607}$$

$$= \iint \frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{\sqrt{4E_{\mathbf{q}}E_{\mathbf{p}}}}(a_{\mathbf{p}}a_{\mathbf{q}}e^{-ipx-iqy} + (a_{\mathbf{q}}a^\dagger_{\mathbf{p}} - (2\pi)^3\delta(\mathbf{q}-\mathbf{p}))e^{ipx-iqy} + a_{\mathbf{p}}a^\dagger_{\mathbf{q}}e^{-ipx+iqy} + a^\dagger_{\mathbf{p}}a^\dagger_{\mathbf{q}}e^{ipx+iqy}) \tag{608}$$

then with $a^\dagger|0\rangle = 0$ and $\langle 0|a = 0$

$$\langle 0|\phi(x)\phi(y)|\rangle = \iint \frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{\sqrt{4E_{\mathbf{q}}E_{\mathbf{p}}}}(((\langle 0|a_{\mathbf{q}}a^\dagger_{\mathbf{p}}|0\rangle - \langle 0|0\rangle(2\pi)^3\delta(\mathbf{q}-\mathbf{p}))e^{ipx-iqy} + \langle 0|a_{\mathbf{p}}a^\dagger_{\mathbf{q}}|0\rangle e^{-ipx+iqy}) \tag{609}$$

$$= \iint \frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{\sqrt{4E_{\mathbf{q}}E_{\mathbf{p}}}}\left(\left(\frac{\langle\mathbf{q}|\mathbf{p}\rangle}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} - (2\pi)^3\delta(\mathbf{q}-\mathbf{p})\right)e^{ipx-iqy} + \frac{\langle\mathbf{p}|\mathbf{q}\rangle}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}e^{-ipx+iqy}\right) \tag{610}$$

$$= \iint \frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{\sqrt{4E_{\mathbf{q}}E_{\mathbf{p}}}}\left(\left(\frac{2E_{\mathbf{p}}(2\pi)^3\delta^{(3)}(\mathbf{q}-\mathbf{p})}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} - (2\pi)^3\delta(\mathbf{q}-\mathbf{p})\right)e^{ipx-iqy} + \frac{2E_{\mathbf{p}}(2\pi)^3\delta^{(3)}(\mathbf{q}-\mathbf{p})}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}e^{-i}\right. \tag{611}$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{4E^2_{\mathbf{p}}}}\left(\underbrace{\left(\frac{2E_{\mathbf{p}}}{\sqrt{4E^2_{\mathbf{p}}}} - 1\right)}_{=0}e^{ipx-ipy} + \frac{2E_{\mathbf{p}}}{\sqrt{4E^2_{\mathbf{p}}}}e^{-ipx+ipy}\right) \tag{612}$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ip(x-y)} \tag{613}$$

Now we can calculate with $x^0 - y^0 = 0$ and $\mathbf{x}-\mathbf{y} = \mathbf{r}$

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ip(x-y)} \tag{614}$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-i(E_p(x^0-y^0)-\mathbf{p}\cdot(\mathbf{x}-\mathbf{y}))} \tag{615}$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \tag{616}$$

transforming to spherical coordinates

$$D(x-y) = 2\pi \int_0^\infty \frac{p^2 dp}{(2\pi)^3} \frac{1}{2\sqrt{p^2+m^2}} \int \sin\theta \, e^{ipr\cos\theta} d\theta \tag{617}$$

$$= 2\pi \int_0^\infty \frac{p^2 dp}{(2\pi)^3} \frac{1}{2\sqrt{p^2+m^2}} \left[\frac{1}{(-ipr)} e^{ipr\cos\theta} \right]_0^\pi \tag{618}$$

$$= 2\pi \int_0^\infty \frac{p^2 dp}{(2\pi)^3} \frac{1}{2\sqrt{p^2+m^2}} \frac{1}{(-ipr)} (e^{-ipr} - e^{ipr}) \tag{619}$$

$$= \frac{i}{2(2\pi)^2 r} \int_0^\infty \frac{p\, dp}{\sqrt{p^2+m^2}} (e^{-ipr} - e^{ipr}) \tag{620}$$

$$= \frac{i}{2(2\pi)^2 r} \left(\int_0^\infty \frac{p\, dp}{\sqrt{p^2+m^2}} e^{-ipr} - \int_0^\infty \frac{p\, dp}{\sqrt{p^2+m^2}} e^{ipr} \right) \tag{621}$$

$$= \frac{i}{2(2\pi)^2 r} \left(\int_0^\infty \frac{p\, dp}{\sqrt{p^2+m^2}} e^{-ipr} - \int_0^{-\infty} \frac{(-p)\,(-dp)}{\sqrt{(-p)^2+m^2}} e^{i(-p)r} \right) \tag{622}$$

$$= \frac{i}{2(2\pi)^2 r} \int_{-\infty}^\infty \frac{p\, dp}{\sqrt{p^2+m^2}} e^{-ipr} \tag{623}$$

$$= \frac{-i}{2(2\pi)^2 r} \int_{-\infty}^\infty \frac{p\, dp}{\sqrt{p^2+m^2}} e^{ipr} \qquad (r \to -r) \tag{624}$$

Let's use contour integration (closing the contour above - $\lim_{p\to i\infty} e^{ipr} = e^{-\infty r} = 0$ so the upper half circle integral vanishes). Furthermore we see that the square root becomes zero at $\pm im$.

0.12.4 Problem 3.1 - Lorentz group - NOT FINISHED YET (c)

With the Lie algebra for the six generators ($J^{01}, J^{02}, J^{03}, J^{12}, J^{13}, J^{12}$ - three boosts and three rotations) are given by

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho}) \tag{625}$$

and

$$L^i = \frac{1}{2}\epsilon^{ijk} J^{jk}, \qquad K^i = J^{0i} \tag{626}$$

(a) We start with calculating $[L^a, L^b]$, $[K^a, K^b]$ and $[L^a, K^b]$. Using $g^{kl} = -\delta^{kl}$ where $k = 1, 2, 3$

$$[L^a, L^b] = \frac{1}{4}[\epsilon^{ajk} J^{jk}, \epsilon^{blm} J^{lm}] \tag{627}$$

$$= \frac{1}{4}\epsilon^{ajk}\epsilon^{blm}[J^{jk}, J^{lm}] \tag{628}$$

$$= \frac{i}{4}\epsilon^{ajk}\epsilon^{blm}(g^{kl} J^{jm} - g^{jl} J^{km} - g^{km} J^{jl} + g^{jm} J^{kl}) \tag{629}$$

$$= -\frac{i}{4}(\epsilon^{ajk}\epsilon^{bkm} J^{jm} - \epsilon^{ajk}\epsilon^{bjm} J^{km} - \epsilon^{ajk}\epsilon^{blk} J^{jl} + \epsilon^{ajk}\epsilon^{blj} J^{kl}) \tag{630}$$

$$= -\frac{i}{4}(-\epsilon^{ajk}\epsilon^{bmk} J^{jm} - \epsilon^{akj}\epsilon^{bmj} J^{km} - \epsilon^{ajk}\epsilon^{blk} J^{jl} - \epsilon^{akj}\epsilon^{blj} J^{kl}) \tag{631}$$

and use $\epsilon_{abk}\epsilon^{cdk} = \delta_a^c\delta_b^d - \delta_a^d\delta_b^c$

$$[L^a, L^b] = -\frac{i}{4}[-(\delta_{ab}\delta_{jm} - \delta_{am}\delta_{jb})J^{jm} - (\delta_{ab}\delta_{km} - \delta_{am}\delta_{kb})J^{km} - (\delta_{ab}\delta_{jl} - \delta_{al}\delta_{jb})J^{jl} - (\delta_{ab}\delta_{kl} - \delta_{al}\delta_{kb})J^{kl}) \tag{632}$$

$$= -\frac{i}{4}\left[-(\delta_{ab}J^{mm} - J^{ba}) - (\delta_{ab}J^{mm} - J^{ba}) - (\delta_{ab}J^{ll} - J^{ba}) - (\delta_{ab}J^{ll} - J^{ba})\right] \tag{633}$$

as the diagonal elements of J are zero the trace J^{mm} vanishes as well and we obtain

$$
\begin{aligned}
[L^a, L^b] &= -iJ^{ba} = iJ^{ab} = i\frac{1}{2}(J^{ab} - J^{ba}) && (634)\\
&= \frac{i}{2}(\delta_{am}\delta_{bn} - \delta_{an}\delta_{bm})J^{mn} && (635)\\
&= \frac{i}{2}\epsilon_{abk}\epsilon^{mnk}J^{mn} && (636)\\
&= \frac{i}{2}\epsilon_{abk}\epsilon^{mnk}J^{mn} && (637)\\
&= i\epsilon_{abk}\frac{1}{2}\epsilon^{mnk}J^{mn} && (638)\\
&= i\epsilon_{abk}\frac{1}{2}\epsilon^{kmn}J^{mn} && (639)\\
&= i\epsilon_{abk}L^k. && (640)
\end{aligned}
$$

Now with $a, b = 1, 2, 3$

$$
\begin{aligned}
[K^a, K^b] &= [J^{0a}, J^{0b}] && (641)\\
&= i(g^{a0}J^{0b} - g^{00}J^{ab} - g^{ab}J^{00} + g^{0b}J^{a0}) && (642)\\
&= i(0 \cdot J^{0b} - 1 \cdot J^{ab} - 0 \cdot J^{00} + 0 \cdot J^{a0}) && (643)\\
&= -iJ^{ab} && (644)\\
&= ... \qquad \text{(same as last calculation above)} && (645)\\
&= -i\epsilon_{abk}L^k && (646)
\end{aligned}
$$

And

$$
\begin{aligned}
[L^a, K^b] &= \frac{1}{2}\epsilon^{ajk}[J^{jk}, J^{0b}] && (647)\\
&= \frac{i}{2}\epsilon^{ajk}(g^{k0}J^{jb} - g^{j0}J^{kb} - g^{kb}J^{j0} + g^{jb}J^{k0}) && (648)\\
&= \frac{i}{2}\epsilon^{ajk}\left(0 \cdot J^{jb} - 0 \cdot J^{kb} - g^{kb} \cdot (-K^j) + g^{jb} \cdot (-K^k)\right) && (649)\\
&= \frac{i}{2}\left(+\epsilon^{ajb}(-1)K^j - \epsilon^{abk}(-1)K^k\right) && (650)\\
&= \frac{i}{2}\left(-\epsilon^{abj}(-1)K^j - \epsilon^{abk}(-1)K^k\right) && (651)\\
&= i\epsilon^{abj}K^j && (652)
\end{aligned}
$$

Now we can finally calculate

$$
\begin{aligned}
[J_+^a, J_+^b] &= \frac{1}{4}\left([L^a, L^b] + i[L^a, K^b] + i[K^a, L^b] + i^2[K^a, K^b]\right) && (653)\\
&= \frac{1}{4}\left(i\epsilon^{abk}L^k + i \cdot i\epsilon^{abj}K^j + i \cdot i\epsilon^{abj}K^j - (-1)i\epsilon^{abk}L^k\right) && (654)\\
&= \frac{1}{4}\left(i\epsilon^{abk}L^k - \epsilon^{abj}K^j - \epsilon^{abj}K^j + i\epsilon^{abk}L^k\right) && (655)\\
&= \frac{1}{2}i\epsilon^{abk}(L^k + iK^k) && (656)\\
&= i\epsilon^{abk}J_+^k && (657)
\end{aligned}
$$

and

$$[J_-^a, J_-^b] = \frac{1}{4}\left([L^a, L^b] - i[L^a, K^b] - i[K^a, L^b] + (-i)^2[K^a, K^b]\right) \quad (658)$$
$$= \frac{1}{4}\left(i\epsilon^{abk}L^k - i \cdot i\epsilon^{abj}K^j - i \cdot i\epsilon^{abj}K^j - (-1)i\epsilon^{abk}L^k\right) \quad (659)$$
$$= \frac{1}{4}\left(i\epsilon^{abk}L^k + \epsilon^{abj}K^j + \epsilon^{abj}K^j + i\epsilon^{abk}L^k\right) \quad (660)$$
$$= \frac{1}{2}i\left(\epsilon^{abk}L^k - i\epsilon^{abj}K^j\right) \quad (661)$$
$$= i\epsilon^{abk}\frac{1}{2}\left(L^k - iK^k\right) \quad (662)$$
$$= i\epsilon^{abk}J_-^k \quad (663)$$
$$[J_-^a, J_+^b] = \frac{1}{4}\left([L^a, L^b] + i[L^a, K^b] - i[K^a, L^b] - i^2[K^a, K^b]\right) \quad (664)$$
$$= \frac{1}{4}\left(i\epsilon^{abk}L^k - \epsilon^{abj}K^j + \epsilon^{abj}K^j - i\epsilon^{abk}L^k\right) \quad (665)$$
$$= 0 \quad (666)$$

(b) With inverse transformation

$$\mathbf{L} = \mathbf{J}_+ + \mathbf{J}_- \quad (667)$$
$$\mathbf{K} = -i(\mathbf{J}_+ - \mathbf{J}_-) \quad (668)$$

we write for the $(j_+, j_-) \equiv (j_+, 0) \otimes (0, j_-)$ representation with

$$j_+ = \frac{\dim_+ - 1}{2} \quad (669)$$
$$j_- = \frac{\dim_- - 1}{2} \quad (670)$$

we write more technically with $\mathbf{J}_{j_\pm}$ being $\dim_\pm$ dimensional representations

$$\mathbf{L}_{(j_+,j_-)} = \mathbf{J}_{j_+} \otimes 1_{\dim_-} + 1_{\dim_+} \otimes \mathbf{J}_{j_-} \quad (671)$$
$$\mathbf{K}_{(j_+,j_-)} = -i(\mathbf{J}_{j_+} \otimes 1_{\dim_-} - 1_{\dim_+} \otimes \mathbf{J}_{j_-}) \quad (672)$$

which means that $\mathbf{K}, \mathbf{L}_{(j_+,j_-)}$ are $(\dim_+ \times \dim_-)$ dimensional representation. Therefore we obtain ($\mathbf{J}_+ = \mathbf{J}_{1/2} = \boldsymbol{\sigma}/2$ and $\mathbf{J}_- = \mathbf{J}_0 = 0$)

$$\mathbf{L}_{(1/2,0)} = \mathbf{J}_{1/2} \otimes 1_1 + 1_1 \otimes 0 = +\frac{1}{2}\boldsymbol{\sigma} \quad (673)$$
$$\mathbf{K}_{(1/2,0)} = -i(\mathbf{J}_{1/2} \otimes 1_1 - 1_1 \otimes 0) = -i\frac{1}{2}\boldsymbol{\sigma} \quad (674)$$
$$\Rightarrow \psi_R \to (1 - \frac{i}{2}\boldsymbol{\theta\sigma} + \frac{i}{2}\boldsymbol{\beta\sigma})\psi_R \quad (675)$$

we obtain ($\mathbf{J}_+ = \mathbf{J}_0 = 0$ and $\mathbf{J}_- = \mathbf{J}_{1/2} = \boldsymbol{\sigma}/2$)

$$\mathbf{L}_{(0,1/2)} = +\frac{1}{2}\boldsymbol{\sigma} \quad (676)$$
$$\mathbf{K}_{(0,1/2)} = +i\frac{1}{2}\boldsymbol{\sigma} \quad (677)$$
$$\Rightarrow \psi_L \to (1 - \frac{i}{2}\boldsymbol{\theta\sigma} - \frac{i}{2}\boldsymbol{\beta\sigma})\psi_L \quad (678)$$

(c) With $\psi' = \psi_L^T \sigma^2$

$$\psi_L \to (1 - \frac{i}{2}\boldsymbol{\theta\sigma} - \frac{i}{2}\boldsymbol{\beta\sigma})\psi_L \tag{679}$$

we can transpose the equation

$$\psi_L^T \to \psi_L^T(1 - \frac{i}{2}\boldsymbol{\theta\sigma} - \frac{i}{2}\boldsymbol{\beta\sigma})^T \tag{680}$$

$$\to \psi_L^T(1 - \frac{i}{2}\boldsymbol{\theta\sigma}^T - \frac{i}{2}\boldsymbol{\beta\sigma}^T) \tag{681}$$

and substitude $\boldsymbol{\sigma}^T = -\sigma^2\boldsymbol{\sigma}\sigma^2$

$$\to \psi_L^T(\underbrace{\sigma^2\sigma^2}_{=1} + \frac{i}{2}\sigma^2\boldsymbol{\theta\sigma}\sigma^2 + \frac{i}{2}\sigma^2\boldsymbol{\beta\sigma}\sigma^2) \tag{682}$$

$$\to \underbrace{\psi_T^T\sigma^2}_{=\psi'}(\sigma^2 + \frac{i}{2}\boldsymbol{\theta\sigma}\sigma^2 + \frac{i}{2}\boldsymbol{\beta\sigma}\sigma^2) \tag{683}$$

$$\to \psi'(1 + \frac{i}{2}\boldsymbol{\theta\sigma} + \frac{i}{2}\boldsymbol{\beta\sigma})\sigma^2 \tag{684}$$

$$\underbrace{\psi_L^T\sigma^2}_{=\psi'} \to \psi'(1 + \frac{i}{2}\boldsymbol{\theta\sigma} + \frac{i}{2}\boldsymbol{\beta\sigma})\underbrace{\sigma^2\sigma^2}_{=1} \tag{685}$$

implying

$$\psi' \to \psi'(1 + \frac{i}{2}\boldsymbol{\theta\sigma} + \frac{i}{2}\boldsymbol{\beta\sigma}). \tag{686}$$

Now for $(j_+, j_-) = (1/2, 1/2) \equiv (1/2, 0) \otimes (0, 1/2)$ we have

$$j_+ = \frac{\dim_+ - 1}{2} \tag{687}$$

$$j_- = \frac{\dim_- - 1}{2} \tag{688}$$

$$\to \dim_+ = \dim_- = 2 \tag{689}$$

we can/should write ($\mathbf{J}_+ = \boldsymbol{\sigma}/2$ and $\mathbf{J}_- = \boldsymbol{\sigma}/2$) and the representation should be 4-dimensional (because $\dim_+ \times \dim_- = 2 \cdot 2 = 4$)

$$\mathbf{L}_{(1/2,1/2)} = \mathbf{J}_{1/2} \otimes 1_2 + 1_2 \otimes \mathbf{J}_{1/2} = +\frac{1}{2}(\boldsymbol{\sigma} \otimes 1_2 + 1_2 \otimes \boldsymbol{\sigma}) \tag{690}$$

$$\mathbf{K}_{(1/2,1/2)} = -i(\mathbf{J}_{1/2} \otimes 1_2 - 1_2 \otimes \mathbf{J}_{1/2}) = -\frac{i}{2}(\boldsymbol{\sigma} \otimes 1_2 - 1_2 \otimes \boldsymbol{\sigma}) \tag{691}$$

$$\Phi \to (1 - i\boldsymbol{\theta}\mathbf{L} - i\boldsymbol{\beta}\mathbf{K})\Phi \tag{692}$$

The suggested transformation (why?) for the $(1/2, 1/2)$ representation is then

$$(1 - \frac{i}{2}\boldsymbol{\theta\sigma} + \frac{i}{2}\boldsymbol{\beta\sigma})\left(\psi_R(\psi_L^T\sigma^2)\right)(1 + \frac{i}{2}\boldsymbol{\theta\sigma} + \frac{i}{2}\boldsymbol{\beta\sigma}) \tag{693}$$

$$= (1 - \frac{i}{2}\boldsymbol{\theta\sigma} + \frac{i}{2}\boldsymbol{\beta\sigma})\begin{pmatrix} V^0 + V^3 & V^1 - iV^2 \\ V^1 + iV^2 & V^0 - V^3 \end{pmatrix}(1 + \frac{i}{2}\boldsymbol{\theta\sigma} + \frac{i}{2}\boldsymbol{\beta\sigma}) \tag{694}$$

$$= (1 - \frac{i}{2}\boldsymbol{\theta\sigma} + \frac{i}{2}\boldsymbol{\beta\sigma})(V^\mu\bar{\sigma}_\mu)(1 + \frac{i}{2}\boldsymbol{\theta\sigma} + \frac{i}{2}\boldsymbol{\beta\sigma}) \tag{695}$$

$$\tag{696}$$

0.13 Schwartz - Quantum Field Theory and the Standard Model

0.13.1 Problem 2.2 - Special relativity and colliders

1. Quick special relativity recap

$$p'^{\mu} = \Lambda^{\mu}_{\nu} p^{\nu} \quad p^{\mu} p_{\mu} = m^2 c^2 \tag{697}$$

At rest

$$p^{\mu} p_{\mu} = (p^0)^2 - \vec{p}^2 = (p^0)^2 = m^2 c^2 \tag{698}$$

After Lorentz trafo in x direction

$$\Lambda = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{699}$$

$$p'^{\mu} = (\gamma p^0, -\beta\gamma p^0, 0, 0) \tag{700}$$

$$\equiv \left(\frac{E}{c}, \vec{p}\right) \tag{701}$$

with $p^{\mu} p_{\mu} = m^2 c^2$ we have $E^2/c^2 + \vec{p}^2 = m^2 c^2$.

Now we can solve the problem

$$\frac{E_{cm}}{2} = \sqrt{m_p^2 c^4 + p^2 c^2} \tag{702}$$

$$\to p = \frac{1}{c}\sqrt{\frac{E_{cm}^2}{4} - m_p^2 c^4} \equiv \beta\gamma m_p c \tag{703}$$

$$\to \frac{E_{cm}^2}{4} = m_p^2 c^4 (\beta^2\gamma^2 + 1) \tag{704}$$

$$\to \gamma = \frac{E_{cm}}{2 m_p c^2} \tag{705}$$

$$\to \beta = \sqrt{1 - \left(\frac{2 m_p c}{E_{cm}}\right)^2} \approx 1 - \frac{1}{2}\left(\frac{2 m_p c^2}{E_{cm}}\right)^2 \tag{706}$$

$$\to c - v = 2\left(\frac{m_p c^2}{E_{cm}}\right)^2 c = 2.69 \mathrm{m/s} \tag{707}$$

2. Using the velocity addition formula

$$\Delta v = \frac{2v}{1 + \frac{v^2}{c^2}} \approx c\left(1 - 2\left[\frac{m_p c^2}{E_{cm}}\right]^4\right) \tag{708}$$

0.13.2 Problem 2.3 - GZK bound

1. We are utilizing Plancks law

$$w_{\nu} d\nu = \frac{8\pi h \nu^3}{c^3} \frac{d\nu}{e^{h\nu/k_B T} - 1} \tag{709}$$

where the spectral energy density w_ν [J m^{-3} s] gives the spacial energy density per frequency interval $d\nu$. The total radiative energy density is then given by

$$\rho_{\text{rad}} = \frac{8\pi h}{c^3}\int_0 \infty \frac{\nu^3 d\nu}{e^{h\nu/k_B T}-1} \tag{710}$$

$$= \frac{8\pi h}{c^3}\cdot\frac{(\pi k_B T)^4}{15h^4} \tag{711}$$

$$= \frac{8\pi^5 k_B^4 T^4}{15h^3c^3} = 0.26\text{MeV/m}^3. \tag{712}$$

The photon density is given by

$$n_{\text{rad}} = \int_0^\infty \frac{w_\nu}{h\nu}d\nu \tag{713}$$

$$= \frac{8\pi}{c^3}\int \frac{\nu^2 d\nu}{e^{h\nu/k_B T}-1} \tag{714}$$

$$= \frac{8\pi}{c^3}\cdot\frac{2\zeta(3)k_B^3T^3}{h^3} \tag{715}$$

$$= \frac{16\pi\zeta(3)k_B^3T^3}{h^3c^3} = 416\text{cm}^{-3}. \tag{716}$$

The average photon energy is then given by

$$E_{\text{ph}} = \frac{\rho_{\text{rad}}}{n_{\text{rad}}} = \frac{\pi^4}{30\zeta(3)}k_B T = 0.63\text{meV} \tag{717}$$

$$\lambda_{\text{ph}} = \frac{hc}{E_{\text{ph}}} = 1.9\text{mm} \tag{718}$$

therefore it is called CM(icrowave)B. One obtains slightly other values if the peak of the Planck spectrum is used as definition of the average photon energy.

2. In the center-of-mass system the total momentum before and after the collision vanishes

$$\vec{p}^{\,cm}_{p^+} + \vec{p}^{\,cm}_{\gamma} = 0 = \vec{\hat{p}}^{cm}_{p^+} + \vec{\hat{p}}^{cm}_{\pi^0}. \tag{719}$$

which implies for (Lorentz-invariant) norm the systems 4-momentum $P^{cm} = p^{cm}_{p^+} + p^{cm}_{\pi^0}$

$$(P^{cm})^2 = (E^{cm}_{p^+} + E^{cm}_{\gamma})^2 - c^2(\vec{p}^{\,cm}_{p^+} + \vec{p}^{\,cm}_{\gamma})^2 \tag{720}$$

$$= (E^{cm}_{p^+} + E^{cm}_{\gamma})^2 \tag{721}$$

$$= (E^{cm})^2 \tag{722}$$

$$\stackrel{!}{=} (E_{p^+} + E_\gamma)^2 - c^2(\vec{p}_{p^+} + \vec{p}_\gamma)^2 \tag{723}$$

$$\stackrel{!}{=} (\hat{E}_{p^+} + \hat{E}_{\pi^0})^2 - c^2(\vec{\hat{p}}_{p^+} + \vec{\hat{p}}_{\pi^0})^2 \tag{724}$$

with $p^i = \hbar k^i = \hbar(\omega, \vec{k}) = \hbar(\omega, \frac{2\pi}{\lambda}\vec{e}_k) = h(\nu, \frac{\nu}{c}\vec{e}_k)$ and the values before

$$E_{p^+} = m_{p^+}c^2 + T_{p^+} \tag{725}$$

$$E_\gamma = h\nu \tag{726}$$

$$(\vec{p}_{p^+})^2 = \frac{1}{c^2}\left[(E_{p^+})^2 - (m_{p^+})^2c^4\right] \tag{727}$$

$$= \frac{T_{p^+}}{c^2}\left[T_{p^+} + 2m_{p^+}c^2\right] \tag{728}$$

$$(\vec{p}_\gamma)^2 = \frac{h^2\nu^2}{c^2} \tag{729}$$

At the threshold the π^0 is created without any kinetic energy. As the total momentum is vanishing the proton also needs to be at rest

$$(E_{p^+} + E_\gamma)^2 - c^2(\vec{p}_{p^+} + \vec{p}_\gamma)^2 = \left(m_{p^+}c^2 + m_{\pi^0}c^2\right)^2 \tag{730}$$
$$E_{p^+}^2 + 2E_{p^+}E_\gamma + E_\gamma^2 - c^2\left(\vec{p}_{p^+}^2 + \vec{p}_\gamma^2 - 2\vec{p}_{p^+}\cdot\vec{p}_\gamma\right) = \left(m_{p^+}c^2 + m_{\pi^0}c^2\right)^2 \tag{731}$$
$$m_{p^+}^2c^4 + 2E_{p^+}E_\gamma + 2c^2\vec{p}_{p^+}\cdot\vec{p}_\gamma = \left(m_{p^+}c^2 + m_{\pi^0}c^2\right)^2 \tag{732}$$
$$m_{p^+}^2c^4 + 2E_{p^+}E_\gamma + 2E_\gamma\sqrt{E_{p^+}^2 - m_{p^+}^2c^2}\cos\phi = \left(m_{p^+}c^2 + m_{\pi^0}c^2\right)^2 \tag{733}$$
$$E_{p^+}E_\gamma + E_\gamma\sqrt{E_{p^+}^2 - m_{p^+}^2c^2}\cos\phi = \left(m_{p^+} + \frac{m_{\pi^0}}{2}\right)m_{\pi^0}c^4 \tag{734}$$

Now we can square the equation and solve approximately assuming $E_\gamma \ll m_{p^+}c^2$

$$E_\gamma\sqrt{E_{p^+}^2 - m_{p^+}^2c^2}\cos\phi = \left(m_{p^+} + \frac{m_{\pi^0}}{2}\right)m_{\pi^0}c^4 - E_{p^+}E_\gamma \tag{735}$$
$$E_\gamma^2\left(E_{p^+}^2 - m_{p^+}^2c^2\right)\cos^2\phi = \left(m_{p^+} + \frac{m_{\pi^0}}{2}\right)^2 m_{\pi^0}^2c^8 + (E_{p^+}E_\gamma)^2 - 2E_{p^+}E_\gamma\left(m_{p^+} + \frac{m_{\pi^0}}{2}\right)m_{\pi^0}c^4 \tag{736}$$
$$-E_\gamma^2m_{p^+}^2c^2\cos^2\phi = \left(m_{p^+} + \frac{m_{\pi^0}}{2}\right)^2 m_{\pi^0}^2c^8 - 2E_{p^+}E_\gamma\left(m_{p^+} + \frac{m_{\pi^0}}{2}\right)m_{\pi^0}c^4 \tag{737}$$
$$E_{p^+} \approx \frac{\left(m_{p^+} + m_{\pi^0}/2\right)m_{\pi^0}c^4}{2E_\gamma} \tag{738}$$
$$= 10.8\cdot 10^{19}\text{eV} \tag{739}$$

3. By assumption the p^+ and the π^0 would rest in the CM system

$$(P^\mu)^{cm} = (p^\mu_{p^+})^{cm} + (p^\mu_{\pi^0})^{cm} \tag{740}$$
$$= \left([m_{p^+} + m_{\pi^0}]c^2, \vec{0}\right) \tag{741}$$
$$= \Lambda^\mu_\alpha\left[\hat{p}^\alpha_{p^+} + \hat{p}^\alpha_{\pi^0}\right] \tag{742}$$
$$= \Lambda^\mu_\alpha\left[p^\alpha_{p^+} + p^\alpha_\gamma\right] \tag{743}$$
$$\tag{744}$$

We can therefore calculate γ

$$\mu = 1: \quad 0 = \underbrace{\Lambda^1_0}_{-\gamma\beta}(E_{p^+} + E_\gamma) + \underbrace{\Lambda^1_1}_{\gamma}c(p^x_{p^+} + p^x_\gamma) \tag{745}$$
$$= -\gamma\beta(E_{p^+} + E_\gamma) + \gamma\left(\sqrt{E_{p^+}^2 - m_p^2c^4} + E_\gamma\right) \tag{746}$$
$$\to \beta = \frac{\sqrt{E_{p^+}^2 - m_p^2c^4} + E_\gamma}{E_{p^+} + E_\gamma} \approx \frac{\sqrt{E_{p^+}^2 - m_p^2c^4}}{E_{p^+}} \tag{747}$$
$$\to \gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{E_{p^+}}{m_{p^+}c^2} \tag{748}$$

which can be used to calculate the pion momentum

$$c\hat{p}_{\pi^0} = \Lambda^0_\mu(p^\mu_{\pi^0})^{cm} \tag{749}$$
$$= \Lambda^0_0(p^0_{\pi^0})^{cm} \tag{750}$$
$$= \gamma m_{\pi^0}c^2 \tag{751}$$
$$= E_{p^+}\frac{m_{\pi^0}}{m_{p^+}}. \tag{752}$$

The $p+$ energy after the collision is then given by

$$E_{p^+} + E_\gamma = \hat{E}_{p^+} + \hat{E}_{\pi^0} \tag{753}$$

$$\to \hat{E}_{p^+} = E_{p^+} + E_\gamma - \hat{E}_{\pi^0} \tag{754}$$

$$= E_{p^+} + E_\gamma - \sqrt{m_{\pi^0}^2 c^4 + \hat{\vec{p}}_{\pi^0} c^2} \tag{755}$$

$$= E_{p^+} + E_\gamma - \sqrt{m_{\pi^0}^2 c^4 + E_{p^+}^2 \frac{m_{\pi^0}^2}{m_{p^+}^2}} \tag{756}$$

$$= E_{p^+} + E_\gamma - m_{\pi^0} c^2 \sqrt{1 + \frac{E_{p^+}^2}{m_{p^+}^2 c^4}} \tag{757}$$

$$\approx E_{p^+} - m_{\pi^0} c^2 \frac{E_{p^+}}{m_{p^+} c^2} \tag{758}$$

$$= E_{p^+} \left(1 - \frac{m_{\pi^0}}{m_{p^+}}\right) \tag{759}$$

$$\approx 0.85 \cdot E_{p^+}. \tag{760}$$

0.13.3 Problem 2.5 - Compton scattering

1. the binding energy of outer(!!!) electrons is in the eV range while typical X-rays energies are in the keV range.

2. In the nonrelativistic case we have energy and momentum conservation

$$\frac{hc}{\lambda} = \frac{hc}{\lambda'} + \frac{1}{2} m_e v^2 \tag{761}$$

$$\frac{h}{\lambda} = \frac{h}{\lambda'} \cos\theta + m_e v \cos\phi \tag{762}$$

$$0 = \frac{h}{\lambda'} \sin\theta + m_e v \sin\phi \tag{763}$$

then we see

$$v = \sqrt{\frac{2hc}{m_e}\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)} = \sqrt{\frac{2hc}{m_e}\frac{\lambda' - \lambda}{\lambda\lambda'}} \tag{764}$$

and

$$\sin\phi = -\frac{h}{m_e v}\frac{1}{\lambda'} \sin\theta \tag{765}$$

$$\cos\phi = \frac{h}{m_e v}\frac{1}{\lambda'}\left(\frac{\lambda'}{\lambda} - \cos\theta\right) \tag{766}$$

$$\to 1 = \sin^2\phi + \cos^2\phi \tag{767}$$

$$= \frac{h^2}{m_e^2 v^2 \lambda'^2}\left(\sin^2\theta + \frac{\lambda'^2}{\lambda^2} - 2\frac{\lambda'}{\lambda}\cos\theta + \cos^2\theta\right) \tag{768}$$

$$= \frac{h^2}{m_e^2 v^2 \lambda'^2}\left(1 + \frac{\lambda'^2}{\lambda^2} - 2\frac{\lambda'}{\lambda}\cos\theta\right) \tag{769}$$

$$= \frac{h\lambda}{2m_e c \lambda'(\lambda' - \lambda)}\left(1 + \frac{\lambda'^2}{\lambda^2} - 2\frac{\lambda'}{\lambda}\cos\theta\right) \tag{770}$$

$$= \frac{h}{2m_e c(\lambda' - \lambda)}\left(\frac{\lambda}{\lambda'} + \frac{\lambda'}{\lambda} - 2\cos\theta\right) \tag{771}$$

$$\lambda' - \lambda \approx \frac{h}{m_e c}(1 - \cos\theta) \tag{772}$$

where we used $\lambda \approx \lambda'$.

3.

0.13.4 Problem 2.6 - Lorentz invariance

1. With $\omega_k = \sqrt{\vec{k}^2 + m^2}$

$$\int_{-\infty}^{\infty} dk^0 \delta(k^2 - m^2)\theta(k^0) = \int_{-\infty}^{\infty} dk^0 \delta({k^0}^2 - [\vec{k}^2 + m^2])\theta(k^0) \tag{773}$$

$$= \frac{\theta(\omega_k)}{2\omega_k} + \frac{\theta(-\omega_k)}{2\omega_k} \tag{774}$$

$$= \frac{1}{2\omega_k} \tag{775}$$

2. Under Lorentz transformations we have $k^2 - m^2 = 0$. For orthochronous transformation we have k^0...

3. Now we can put it all together

$$\int d^4k \delta(k^2 - m^2)\theta(k^0) = \int d^3k \int dk^0 \delta(k^2 - m^2)\theta(k^0) \tag{776}$$

$$= \int \frac{d^3k}{2\omega_k} \tag{777}$$

0.13.5 Problem 2.7 - Coherent states

1.

$$\partial_z \left(e^{-za^\dagger} a e^{-za^\dagger} \right) = -e^{-za^\dagger} a^\dagger a e^{-za^\dagger} + e^{-za^\dagger} a a^\dagger e^{-za^\dagger} \tag{778}$$

$$= e^{-za^\dagger} [a, a^\dagger] e^{-za^\dagger} \tag{779}$$

$$= 1 \tag{780}$$

2. Rolling the a through the $(a^\dagger)^k$ using the commutator $[a, a^\dagger] = 1$

$$a|z\rangle = a e^{za^\dagger} |0\rangle \tag{781}$$

$$= a \sum_{k=0} \frac{1}{k!} z^k (a^\dagger)^k |0\rangle \tag{782}$$

$$= a|0\rangle + \sum_{k=1} \frac{k}{k!} z^k (a^\dagger)^{k-1} |0\rangle \tag{783}$$

$$= z \sum_{n=0} \frac{1}{n!} z^n (a^\dagger)^n |0\rangle \tag{784}$$

$$= z|z\rangle \tag{785}$$

3. With $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ and using the $|z\rangle$ is an eigenstate of a we have

$$\langle n|z\rangle = \frac{1}{\sqrt{n!}} \langle 0|a^n|z\rangle = \frac{z^n}{\sqrt{n!}} \langle 0|z\rangle = \frac{z^n}{\sqrt{n!}} \langle 0|e^{za^\dagger}|0\rangle \tag{786}$$

$$= \frac{z^n}{\sqrt{n!}} \langle 0|1 + za^\dagger + \frac{1}{2} z^2 (a^\dagger)^2 + ...|0\rangle \tag{787}$$

$$= \frac{z^n}{\sqrt{n!}} \langle 0|0\rangle = \frac{z^n}{\sqrt{n!}} \tag{788}$$

where we used $\langle 0|a^\dagger = 0$.

4. With

$$a + a^\dagger = \sqrt{\frac{m\omega}{2}} 2q \quad \rightarrow \quad q = \frac{1}{\sqrt{2m\omega}}(a + a^\dagger) \tag{789}$$

$$a - a^\dagger = \sqrt{\frac{m\omega}{2}} 2\frac{ip}{m\omega} \quad \rightarrow \quad p = -i\frac{\sqrt{m\omega}}{\sqrt{2}}(a - a^\dagger) \tag{790}$$

and $a|z\rangle = z|z\rangle$ and $\langle z|a^\dagger = \bar{z}\langle z|$

$$\langle z|q|z\rangle = \frac{1}{\sqrt{2m\omega}}\langle z|a + a^\dagger|z\rangle = \frac{1}{\sqrt{2m\omega}}\langle z|z\rangle(z + \bar{z}) \tag{791}$$

$$\langle z|p|z\rangle = -i\frac{\sqrt{m\omega}}{\sqrt{2}}\langle z|a - a^\dagger|z\rangle = -i\frac{\sqrt{m\omega}}{\sqrt{2}}\langle z|z\rangle(z - \bar{z}) \tag{792}$$

$$\langle z|q^2|z\rangle = \frac{1}{2m\omega}\langle z|aa + \underbrace{aa^\dagger}_{=1+a^\dagger a} + a^\dagger a + a^\dagger a^\dagger|z\rangle \tag{793}$$

$$= \frac{1}{2m\omega}\langle z|z\rangle \left(z^2 + 1 + 2z\bar{z} + \bar{z}^2\right) \tag{794}$$

$$\langle z|p^2|z\rangle = -\frac{m\omega}{2}\langle z|aa - \underbrace{aa^\dagger}_{=1+a^\dagger a} - a^\dagger a + a^\dagger a^\dagger|z\rangle \tag{795}$$

$$= -\frac{m\omega}{2}\langle z|z\rangle \left(z^2 - 1 - 2z\bar{z} + \bar{z}^2\right) \tag{796}$$

Therefore

$$\Delta q^2 = \langle q^2\rangle - \langle q\rangle^2 \tag{797}$$

$$= \frac{1}{2m\omega}\left(z^2 + 1 + 2z\bar{z} + \bar{z}^2\right) - \left(\frac{1}{\sqrt{2m\omega}}(z + \bar{z})\right)^2 \tag{798}$$

$$= \frac{1}{2m\omega} \tag{799}$$

and

$$\Delta p^2 = \langle p^2\rangle - \langle p\rangle^2 \tag{800}$$

$$= -\frac{m\omega}{2}\left(z^2 - 1 - 2z\bar{z} + \bar{z}^2\right) - \left(-i\frac{\sqrt{m\omega}}{\sqrt{2}}(z - \bar{z})\right)^2 \tag{801}$$

$$= \frac{m\omega}{2} \tag{802}$$

which means

$$\Delta p \Delta q = \frac{1}{\sqrt{2m\omega}}\frac{\sqrt{m\omega}}{\sqrt{2}} = \frac{1}{2}. \tag{803}$$

5. At first let's construct the eigenstate $|w\rangle$ for a manually

$$a|w\rangle = c_w|w\rangle \tag{804}$$

Expanding the eigenstate with $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$

$$|w\rangle = \sum_n \alpha_n |n\rangle \tag{805}$$

$$a|w\rangle = \sum_n \alpha_n \sqrt{n}|n-1\rangle \stackrel{!}{=} c_w \sum_n \alpha_n |n\rangle = c_w|n\rangle \tag{806}$$

$$\rightarrow \alpha_n \sqrt{n} = c_w \alpha_{n-1} \tag{807}$$

$$\rightarrow \alpha_n = \frac{c_w}{\sqrt{n}} \alpha_{n-1} \tag{808}$$

$$|w\rangle = \sum_n \alpha_0 \frac{c_w^n}{\sqrt{n!}}|n\rangle = \alpha_0 \sum_n \frac{c_w^n}{n!}(a^\dagger)^n|0\rangle = \alpha_0 e^{c_w a^\dagger}|0\rangle \tag{809}$$

Now we do the same for $a^\dagger$

$$a^\dagger|v\rangle = c_v|v\rangle \tag{810}$$

Expanding the eigenstate

$$|v\rangle = \sum_n \beta_n |n\rangle \tag{811}$$

$$a^\dagger|v\rangle = \sum_n \beta_n \sqrt{n+1}|n+1\rangle \stackrel{!}{=} c_v \sum_n \beta_n |n\rangle = c_v|n\rangle \tag{812}$$

$$\rightarrow \beta_n \sqrt{n+1} = c_v \beta_{n+1} \tag{813}$$

$$\rightarrow \beta_{n+1} = \frac{\sqrt{n+1}}{c_v} \beta_n \tag{814}$$

$$|v\rangle = \sum_n \beta_0 \frac{\sqrt{n!}}{c_v^n}|n\rangle = \beta_0 \sum_n \frac{1}{c_v^n}(a^\dagger)^n|0\rangle \tag{815}$$

Now we calculate with $\langle 0|a^\dagger = 0$

$$\langle 0|a^\dagger|v\rangle = \beta_0 \sum_n \frac{1}{c_v^n}\langle 0|(a^\dagger)^{n+1}|0\rangle \tag{816}$$

$$= \beta_0 \frac{1}{c_v^0}\langle 0|a^\dagger|0\rangle \tag{817}$$

$$\tag{818}$$

0.13.6 Problem 3.1 - Higher order Lagrangian

With the principle of least action

$$\delta S = \delta \int \mathcal{L} d^4x = \int \delta\mathcal{L} d^4x \tag{819}$$

we calculate

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu\phi) + \frac{\partial \mathcal{L}}{\partial(\partial_\nu\partial_\mu \phi)}\delta(\partial_\nu\partial_\mu\phi) + ... \tag{820}$$

Now we can integrate each term

$$\delta\mathcal{L}_0 = \int \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi d^4x \tag{821}$$

$$\delta\mathcal{L}_1 = \int \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi)d^4x = \int \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu\delta\phi d^4x \tag{822}$$

$$= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\bigg|_{\partial\Omega} - \int \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi d^4x \tag{823}$$

$$\delta\mathcal{L}_2 = \int \frac{\partial\mathcal{L}}{\partial(\partial_\nu\partial_\mu\phi)}\delta(\partial_\nu\partial_\mu\phi)d^4x = \int \frac{\partial\mathcal{L}}{\partial(\partial_\nu\partial_\mu\phi)}\partial_\nu\delta\partial_\mu\phi d^4x \tag{824}$$

$$= \frac{\partial\mathcal{L}}{\partial(\partial_\nu\partial_\mu\phi)}\delta\partial_\mu\phi\bigg|_{\partial\Omega} - \int \partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu\partial_\mu\phi)}\delta\partial_\mu\phi d^4x \tag{825}$$

$$= \frac{\partial\mathcal{L}}{\partial(\partial_\nu\partial_\mu\phi)}\delta\partial_\mu\phi\bigg|_{\partial\Omega} - \partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu\partial_\mu\phi)}\delta\phi\bigg|_{\partial\Omega} + \int \partial_\mu\partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu\partial_\mu\phi)}\delta\phi d^4x \tag{826}$$

Requiring that all derivatives vanish at infinity we obtain

$$\delta S = \int d^4x\left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} + \partial_\mu\partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu\partial_\mu\phi)} - ...\right)\delta\phi \tag{827}$$

and therefore

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} + \partial_\mu\partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu\partial_\mu\phi)} - ... = 0 \tag{828}$$

0.13.7 Problem 3.5 - Spontaneous symmetry

$$\mathscr{L} = -\frac{1}{2}\phi\Box\phi + \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{829}$$

$$\frac{\partial\mathscr{L}}{\partial\phi} - \partial_\beta\frac{\partial\mathscr{L}}{\partial(\partial_\beta\phi)} + \partial_\mu\partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu\partial_\mu\phi)} = 0 \tag{830}$$

$$\to -\Box\phi + m^2\phi - \frac{\lambda}{3!}\phi^3 = 0 \tag{831}$$

and the Hamiltonian with $-\phi\Box\phi \sim (\partial_\mu\phi)(\partial^\mu\phi) = \eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} \tag{832}$$

$$= \dot{\phi} \tag{833}$$

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} \tag{834}$$

$$= (\dot{\phi})^2 - \mathcal{L} \tag{835}$$

$$= \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{836}$$

(a)

$$m^2\phi - \frac{\lambda}{3!}\phi^3 = 0 \tag{837}$$

$$(m^2 - \frac{\lambda}{3!}\phi^2)\phi = 0 \tag{838}$$

$$\phi_0 = 0 \quad \to \quad \mathcal{H}[\phi] = 0 \tag{839}$$

$$\phi_{1,2} = \pm\sqrt{\frac{3!}{\lambda}}m \quad \to \quad \mathcal{H}[\phi] = -\frac{3m^4}{2\lambda} \tag{840}$$

(b)

(c)

0.13.8 Problem 3.6 - Yukawa potential

(a) We slit the Lagranian in three parts

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}m^2 A_\mu^2 - A_\mu J_\mu \tag{841}$$
$$= \mathscr{L}_F + \mathscr{L}_m + \mathscr{L}_J \tag{842}$$

with the Euler Lagrange equations

$$\frac{\partial \mathscr{L}}{\partial A_\alpha} - \partial_\beta \frac{\partial \mathscr{L}}{\partial(\partial_\beta A_\alpha)} = 0 \tag{843}$$

with

$$\frac{\partial(\partial_\mu A_\nu)}{\partial(\partial_\beta A_\alpha)} = \delta_{\mu\beta}\delta_{\nu\alpha} \tag{844}$$

we can calculate

$$\frac{\partial \mathscr{L}_m}{\partial A_\alpha} - \partial_\beta \frac{\partial \mathscr{L}_m}{\partial(\partial_\beta A_\alpha)} = m^2 A_\alpha \tag{845}$$
$$\frac{\partial \mathscr{L}_J}{\partial A_\alpha} - \partial_\beta \frac{\partial \mathscr{L}_J}{\partial(\partial_\beta A_\alpha)} = -J_\alpha \tag{846}$$
$$\frac{\partial \mathscr{L}_F}{\partial A_\alpha} - \partial_\beta \frac{\partial \mathscr{L}_F}{\partial(\partial_\beta A_\alpha)} = -\frac{1}{4}\partial_\beta\left(-2F_{\mu\nu}(\delta_{\mu\beta}\delta_{\nu\alpha} - \delta_{\nu\beta}\delta_{\mu\alpha})\right) \tag{847}$$
$$= \frac{1}{4}\partial_\beta\left(2(F_{\beta\alpha} - F_{\alpha\beta})\right) \tag{848}$$
$$= \partial_\beta F_{\beta\alpha} \tag{849}$$
$$= \partial_\beta\partial_\beta A_\alpha - \partial_\beta\partial_\alpha A_\beta \tag{850}$$

to obtain (the Proca equation)

$$\Box A_\alpha - \partial_\beta\partial_\alpha A_\beta + m^2 A_\alpha - J_\alpha = 0. \tag{851}$$

Now we can calculate the divergence of the equations

$$\partial_\alpha\left(\Box A_\alpha - \partial_\beta\partial_\alpha A_\beta + m^2 A_\alpha - J_\alpha\right) = 0. \tag{852}$$
$$\Box\partial_\alpha A_\alpha - \partial_\alpha\partial_\alpha\partial_\beta A_\beta + m^2\partial_\alpha A_\alpha - \underbrace{\partial_\alpha J_\alpha}_{=0} = 0 \tag{853}$$

which implies $\partial_\alpha A_\alpha = 0$ and therefore

$$\Box A_\alpha + m^2 A_\alpha - J_\alpha = 0. \tag{854}$$

(b) For A_0 we have for a static potential

$$(\partial_{tt} - \triangle)A_0 + m^2 A_0 - e\delta(x) = 0 \tag{855}$$
$$-\triangle A_0 + m^2 A_0 - e\delta(x) = 0. \tag{856}$$

A Fourier transformation of the equation of motion yields

$$-(ik)^2 A_0(k) + m^2 A_0(k) - e = 0 \tag{857}$$
$$\to A_0(k) = \frac{e}{k^2 + m^2} \tag{858}$$

which we can now transform back

$$A_0 = \frac{e}{(2\pi)^3} \int d^3k \frac{e^{ikx}}{k^2 + m^2} \tag{859}$$

$$= \frac{e}{4\pi r} e^{-mr} \tag{860}$$

where we used the integral evaluation from Kachelriess Problem 3.5.

(c)

$$\lim_{m \to 0} \frac{e}{4\pi r} e^{-mr} = \frac{e}{4\pi r} \tag{861}$$

(d) Scaling down the Coulomb potential exponentially with a characteristic length of $1/m$.

(e)

(f) We can expand and the integrate each term by parts to move over the partial derivatives

$$\mathscr{L}_F = -\frac{1}{4} F_{\mu\nu}^2 \tag{862}$$

$$= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) \tag{863}$$

$$= -\frac{1}{4}\left(\partial_\mu A_\nu \partial_\mu A_\nu - \partial_\mu A_\nu \partial_\nu A_\mu - \partial_\nu A_\mu \partial_\mu A_\nu + \partial_\nu A_\mu \partial_\nu A_\mu\right) \tag{864}$$

$$= -\frac{1}{2}\left(\partial_\mu A_\nu \partial_\mu A_\nu - \partial_\mu A_\nu \partial_\nu A_\mu\right) \tag{865}$$

$$= -\frac{1}{2}\left(-A_\nu \partial_\mu \partial_\mu A_\nu + A_\nu \partial_\nu \partial_\mu A_\mu\right) \tag{866}$$

$$= \frac{1}{2}\left(A_\mu \Box A_\mu - A_\nu \partial_\nu \underbrace{\partial_\mu A_\mu}_{=0}\right) \tag{867}$$

$$= \frac{1}{2} A_\mu \Box A_\mu \tag{868}$$

We can plug this into the full Lagrangian (renaming the summation index)

$$\mathscr{L} = \frac{1}{2} A_\mu \Box A_\mu + \frac{1}{2} m^2 A_\mu^2 - A_\mu J_\mu \tag{869}$$

$$= \frac{1}{2} A_\mu \left(\Box + m^2\right) A_\mu - A_\mu J_\mu \tag{870}$$

then we calculate the derivatives for the Euler-Lagrange equations up to second order (see problem 3.1)

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = \frac{1}{2} \Box A_\mu + m^2 A_\mu - J_\mu \tag{871}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\mu)} = 0 \tag{872}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \partial_\alpha A_\mu)} = \frac{1}{2} A_\mu \tag{873}$$

and get

$$(\Box + m^2) A_\mu = J_\mu \tag{874}$$

0.13.9 Problem 3.7 - Perihelion shift of Mercury by dimensional analysis - NOT DONE YET

(a) Lets summarize the rules of dimensional analysis

varible	SI unit	equation	natural unit
c	m/s	-	1
$\hbar$	Js	-	1
Velocity	m/s	-	1
mass	kg	$E = mc^2$	E
frequency	1/s	$E = \hbar\omega$	E
time	s	$t = 2\pi/\omega$	E^{-1}
length	m	$s = ct$	E^{-1}
∂_μ	1/m	-	E
momentum	kg m/s	$E = p^2/2m$	E
action	Js	$S = Et$	1
$\mathcal{L}$	J/m^3	$S = \int d^4x\mathcal{L}$	E^4
energy density	J/m^3	$\rho = E/V$	E^4
$T^{\mu\nu}$	J/m^3	$\rho = E/V$	E^4

Now we can do a dimensions count for each term

$$\underbrace{\mathcal{L}}_{=4} = -\frac{1}{2}\underbrace{h\Box h}_{2\cdot[h]+2} + \underbrace{M_{\rm Pl}^a h^2\Box h}_{=a+3\cdot[h]+2} - \underbrace{M_{\rm Pl}^b hT}_{b+[h]+4} \tag{875}$$

$$\rightarrow \quad [h] = 1 \tag{876}$$

$$\rightarrow \quad a = -1 \tag{877}$$

$$\rightarrow \quad b = -1 \tag{878}$$

(b) Deriving the equations of motions: keeping in mind that the Lagrangian contains second order derivatives with implies and extra term in the Euler-Lagrange equations (see problem 3.1)

$$\mathcal{L} = -\frac{1}{2}h\Box h + \frac{1}{M_{\rm Pl}}h^2\Box h - \frac{1}{M_{\rm Pl}}hT \tag{879}$$

$$\frac{\partial\mathcal{L}}{\partial h} = -\frac{1}{2}\cdot\Box h + 2\frac{1}{M_{\rm Pl}}h\Box h - \frac{1}{M_{\rm Pl}}T \tag{880}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial h)} = 0 \tag{881}$$

$$\frac{\partial\mathcal{L}}{\partial(\Box h)} = -\frac{1}{2}h + \frac{1}{M_{\rm Pl}}h^2 \tag{882}$$

$$\rightarrow \Box h = \frac{1}{M_{\rm Pl}}\Box(h^2) + \frac{2}{M_{\rm Pl}}h\Box h - \frac{1}{M_{\rm Pl}}T \tag{883}$$

which show an extra term. Alternatively we can integrate the Lagrangian by parts (neglecting the boundary terms) and get

$$\mathcal{L} = \frac{1}{2}\partial h\partial h - \frac{1}{M_{\rm Pl}}\partial(h^2)\partial h - \frac{1}{M_{\rm Pl}}hT \tag{884}$$

$$\frac{\partial\mathcal{L}}{\partial h} = -\frac{1}{M_{\rm Pl}}T \tag{885}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial h)} = \Box h - \frac{1}{M_{\rm Pl}}\Box(h^2) \tag{886}$$

$$\rightarrow \Box h = \frac{1}{M_{\rm Pl}}\Box(h^2) - \frac{1}{M_{\rm Pl}}T \tag{887}$$

We now assume a solution of the form

$$h = h_0 + \frac{1}{M_{\rm Pl}} h_1 + \frac{1}{M_{\rm Pl}^2} h_2 + ... \tag{888}$$

$$\rightarrow h^2 = h_0^2 + \frac{1}{M_{\rm Pl}} 2h_0 h_1 + \frac{1}{M_{\rm Pl}^2}(2h_0 h_2 + h_1^2) + \frac{1}{M_{\rm Pl}^3}(2h_1 h_2 + 2h_0 h_3) + ... \tag{889}$$

and obtain (with the Coulomb solution 3.61 and 3.61)

$$k = 0: \quad \Box h_0 = 0 \quad \rightarrow \quad h_0 = 0 \tag{890}$$

$$k = 1: \quad \Box h_1 = \Box h_0^2 - m\delta^{(3)} \tag{891}$$

$$\Box h_1 = -m\delta^{(3)} \quad \rightarrow \quad h_1 = -\frac{m}{\Box}\delta^{(3)} = \frac{m}{\triangle}\delta^{(3)} = -\frac{m}{4\pi r} \tag{892}$$

$$k = 2: \quad \Box h_2 = 2\Box h_0 h_1 \quad \rightarrow \quad h_2 = 0 \tag{893}$$

$$k = 3: \quad \Box h_3 = \Box(2h_0 h_2 + h_1^2) \tag{894}$$

$$\Box h_3 = \Box(h_1^2) \quad \rightarrow \quad h_3 = h_1^2 = \frac{m^2}{16\pi^2 r^2} \tag{895}$$

and therefore

$$h = -\frac{m}{4\pi r}\frac{1}{M_{\rm Pl}} + \frac{m^2}{16\pi^2 r^2}\frac{1}{M_{\rm Pl}^3} \tag{896}$$

$$= -\frac{m}{4\pi r}\sqrt{G_N} + \frac{m^2}{16\pi^2 r^2}\sqrt{G_N^3} \tag{897}$$

(c) The Newton potential is actually given by (and additional power of $M_{\rm Pl}$ is missing and we are dropping the 4π)

$$V_N = h_1 \frac{1}{M_{\rm Pl}} \cdot \frac{1}{M_{\rm Pl}} = -\frac{G m_{\rm Sun}}{r} \tag{898}$$

the virial theorem implies $E_{\rm kin} \simeq E_{\rm pot}$ and therefore

$$\frac{1}{2} J\omega^2 \simeq \frac{G_N m_{\rm Sun} m_{\rm Mercury}}{R} \tag{899}$$

$$\frac{1}{2} m_{\rm Mercury} R^2 \omega^2 \simeq \frac{G_N m_{\rm Sun} m_{\rm Mercury}}{R} \tag{900}$$

$$\omega^2 \simeq \frac{G_N m_{\rm Sun}}{R^3} \tag{901}$$

(d)

(e)

(f)

(g)

0.13.10 Problem 3.9 - Photon polarizations

(a) Then using the results from problem 3.6 and the corrected sign in the Lagrangian we get

$$
\begin{aligned}
-\frac{1}{4}(F_{\mu\nu})^2 &= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) && (902)\\
&= -\frac{1}{4}\left(\partial_\mu A_\nu \partial_\mu A_\nu - \partial_\mu A_\nu \partial_\nu A_\mu - \partial_\nu A_\mu \partial_\mu A_\nu + \partial_\nu A_\mu \partial_\nu A_\mu\right) && (903)\\
&= -\frac{1}{2}\left(\partial_\mu A_\nu \partial_\mu A_\nu - \partial_\mu A_\nu \partial_\nu A_\mu\right) && (904)\\
&= -\frac{1}{2}\left(-A_\nu \partial_\mu \partial_\mu A_\nu + A_\nu \partial_\nu \partial_\mu A_\mu\right) && (905)\\
&= \frac{1}{2}\left(A_\mu \Box A_\mu - A_\nu \partial_\nu \underbrace{\partial_\mu A_\mu}_{=0}\right) && (906)\\
&= \frac{1}{2} A_\mu \Box A_\mu && (907)
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\mathcal{L} &= -\frac{1}{4}(F_{\mu\nu})^2 - J_\mu A_\mu && (908)\\
&= \frac{1}{2} A_\mu \Box A_\mu - J_\mu A_\mu && (909)\\
&= \frac{1}{2} A_\mu \Box A_\mu - (\Box A_\mu) A_\mu && (910)\\
&= -\frac{1}{2} A_\mu \Box A_\mu && (911)
\end{aligned}
$$

The equations of motion are $\Box A_\mu = J_\mu$ which can be written in momentum space as $k^2 A_\mu(k) = J_\mu(k)$. Now let's write the Lagranian in momentum space as well

$$
\begin{aligned}
\mathcal{L} &= \int d^4k e^{ikx} A_\mu(k) k^2 A_\mu(k) && (912)\\
&= \int d^4k e^{ikx} \frac{J_\mu(k)}{k^2} k^2 \frac{J_\mu(k)}{k^2} && (913)\\
&= \int d^4k e^{ikx} J_\mu(k) \frac{1}{k^2} J_\mu(k) && (914)
\end{aligned}
$$

(b) In momentum space charge conservation is given by

$$
\begin{aligned}
k_\mu J_\mu &= 0 && (915)\\
\omega J_0 - \kappa J_1 &= 0 && (916)\\
\rightarrow \quad J_1 &= \frac{\omega}{\kappa} J_0 && (917)
\end{aligned}
$$

(c)

$$
\begin{aligned}
\mathcal{L} &= \int d^4k e^{ikx} J_\mu(k) \frac{1}{k^2} J_\mu(k) && (918)\\
&\simeq \frac{J_0^2 - J_1^2 - J_2^2 - J_3^2}{\omega^2 - \kappa^2} && (919)\\
&\simeq \frac{J_0^2(1-\omega^2/\kappa^2)}{\omega^2 - \kappa^2} - \frac{J_2^2 + J_3^2}{\omega^2 - \kappa^2} && (920)\\
&\simeq -\frac{J_0^2}{\kappa^2} - \frac{J_2^2 + J_3^2}{\omega^2 - \kappa^2} && (921)\\
&\simeq \triangle J_0^2 - \Box (J_2^2 + J_3^2) && (922)
\end{aligned}
$$

(d) A time derivative in the Lagrangian results in a time derivative in time derivative in the equations of motion which means a time-evolution equation. There are two causally propagating degrees of freedom J_2 and J_3.

(e) Hmmmm .... calculate the two point field correlation functions and see if they vanish outside of the light cone.

0.13.11 Problem 3.10 - Graviton polarizations - NOT DONE YET

(a) With the higher order Euler-Lagrange equations from 3.1

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} + \partial_\mu\partial_\nu \frac{\partial\mathcal{L}}{\partial(\partial_\nu\partial_\mu\phi)} - ... = 0 \tag{923}$$

we obtain

$$-\frac{1}{2}\Box h_{\mu\nu} + \frac{1}{M_{\mathrm{Pl}}}T_{\mu\nu} - \frac{1}{2}\Box h_{\mu\nu} = 0 \tag{924}$$

$$\rightarrow \Box h_{\mu\nu} = \frac{1}{M_{\mathrm{Pl}}}T_{\mu\nu} \tag{925}$$

$$\rightarrow h_{\mu\nu} = \frac{1}{M_{\mathrm{Pl}}}\frac{1}{\Box}T_{\mu\nu} \tag{926}$$

and

$$\mathcal{L} = -\frac{1}{2}h_{\mu\nu}\Box h_{\mu\nu} + \frac{1}{M_{\mathrm{Pl}}}h_{\mu\nu}T_{\mu\nu} \tag{927}$$

$$= -\frac{1}{2}\frac{1}{M_{\mathrm{Pl}}^2}(\frac{1}{\Box}T_{\mu\nu})T_{\mu\nu} + \frac{1}{M_{\mathrm{Pl}}^2}(\frac{1}{\Box}T_{\mu\nu})T_{\mu\nu} \tag{928}$$

$$= \frac{1}{2}\frac{1}{M_{\mathrm{Pl}}^2}T_{\mu\nu}\frac{1}{\Box}T_{\mu\nu} \tag{929}$$

$$\simeq \frac{1}{2}\frac{1}{M_{\mathrm{Pl}}^2}T_{\mu\nu}\frac{1}{k^2}T_{\mu\nu} \tag{930}$$

$$\tag{931}$$

(b)

(c)

(d)

0.13.12 Problem 5.1 - 2 → 2 differential cross-section - NOT DONE YET

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 m_A}\left[E_B + E_f\left(1 - \frac{|\mathbf{p}_i|}{|\mathbf{p}_f|}\cos\theta\right)\right]^{-1}\frac{|\mathbf{p}_i|}{|\mathbf{p}_f|}|\mathcal{M}|^2 \tag{932}$$

with $E_B = \sqrt{|\mathbf{p}_f - \mathbf{p}_i|^2 + m_B^2}$ and $E_f = \sqrt{\mathbf{p}_f^2 + m_f^2}$ and $\theta = \angle(\mathbf{p}_i, \mathbf{p}_f)$

0.13.13 Problem 10.1 - Dirac eqation and angular momentum couplings - NOT DONE YET

(a) In the Weyl basis we have $(\gamma^0)^\dagger = \gamma^0, (\gamma^k)^\dagger = -\gamma^k$ and $(\gamma^0)^2 = 1_{4\times4}$ and $(\gamma^k)^2 = -1_{4\times4}$. Now we multiply the Dirac equation by γ^0

$$\gamma^0(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi = 0 \tag{933}$$

$$\gamma^0(i\hbar c\partial\!\!\!/ - eA\!\!\!/ - mc^2)\psi = 0 \tag{934}$$

and replace $p_i = -i\hbar\partial_i$

$$\rightarrow i\partial_t\psi = (\gamma^0\gamma^1(p_1 + eA_1) + \gamma^0\gamma^2(p_2 + eA_2) + \gamma^0\gamma^3(p_3 + eA_3) + eA_0 + \gamma^0 m)\psi \tag{935}$$
$$H_D = \gamma^0\gamma^1(p_1 + eA_1) + \gamma^0\gamma^2(p_2 + eA_2) + \gamma^0\gamma^3(p_3 + eA_3) + eA_0 + \gamma^0 m \tag{936}$$

(b) Multiplying $\gamma^0\gamma^k$ we see

$$H_D = \gamma^0\gamma^1(p_1 + eA_1) + \gamma^0\gamma^2(p_2 + eA_2) + \gamma^0\gamma^3(p_3 + eA_3) + eA_0 + \gamma^0 m \tag{937}$$
$$= \begin{pmatrix} 0 & \sigma_i(p_i + eA_i) \\ \sigma_i(p_i - eA_i) & 0 \end{pmatrix}\psi + \begin{pmatrix} 1_2 eA_0 & 0 \\ 0 & 1_2 eA_0 \end{pmatrix}\psi + \begin{pmatrix} 1_2 m & 0 \\ 0 & -1_2 m \end{pmatrix}\psi \tag{938}$$
$$= \begin{pmatrix} 1_2(eA_0 + m) & 1_2 m + \sigma_i(p_i + eA_i) \\ \sigma_i(p_i + eA_i) & 1_2(eA_0 - m) \end{pmatrix}\psi \tag{939}$$
$$= \begin{pmatrix} 1_2(eA_0 + mc^2) & \sigma_i(cp_i + eA_i) \\ \sigma_i(cp_i + eA_i) & 1_2(eA_0 - mc^2) \end{pmatrix}\psi \tag{940}$$

then

$$H_D - eA_0 = \begin{pmatrix} mc^2 1_2 & \sigma_i(cp_i + eA_i) \\ \sigma_i(cp_i + eA_i) & -mc^2 1_2 \end{pmatrix} \tag{941}$$
$$(H_D - eA_0)^2 = [m^2c^4 + c^2(\sigma\cdot\mathbf{p})^2 + e^2(\sigma\cdot\mathbf{A})^2 + ce(\sigma\cdot\mathbf{p})(\sigma\cdot\mathbf{A}) + ce(\sigma\cdot\mathbf{p})(\mathbf{A}\cdot\mathbf{p})]1_2 \tag{942}$$
$$\sqrt{(H_D - eA_0)^2} \simeq \Bigg(mc^2 + \frac{(\sigma\cdot\mathbf{p})^2}{2m} + e\frac{(\sigma\cdot\mathbf{A})(\sigma\cdot\mathbf{p}) + (\sigma\cdot\mathbf{p})(\sigma\cdot\mathbf{A})}{2mc} + \frac{(2mq(\sigma\cdot\mathbf{A}) - (\sigma\cdot\mathbf{p})^2)(2mq(\sigma\cdot\mathbf{A}) + (\sigma}{8m^3c^2} \tag{943}$$

Bohr Magneton $\mu_B = \frac{e\hbar}{2mc}$

Exercise 10.2 - Finite-dimensional irreducible representations of SU(2) - NOT DONE YET

(a) We calculate

$$[\tau^\pm, \tau_3] = [\tau_1, \tau_3] \pm i[\tau_2, \tau_3] \tag{944}$$
$$= i\epsilon_{132}\tau_2 \pm i(i\epsilon_{231}\tau_1) \tag{945}$$
$$= -i\tau_2 \mp \tau_1 \tag{946}$$
$$= \mp(\tau_1 \pm i\tau_2) \tag{947}$$
$$= \mp\tau^\pm \tag{948}$$

then

$$\tau^\pm\tau_3 V_j - \tau_3\tau^\pm V_3 = \mp\tau^\pm V_j \tag{949}$$
$$\rightarrow \tau^\pm\tau_3 V_j = \tau_3\tau^\pm V_3 \mp \tau^\pm V_j \tag{950}$$
$$\rightarrow \tau_3\tau^\pm V_3 = \lambda_j\tau^\pm V_j \pm \tau^\pm V_j \tag{951}$$
$$\rightarrow \tau_3(\tau^\pm V_3) = (\lambda_j \pm 1)(\tau^\pm V_j) \tag{952}$$

(b)

(c)

(d)

0.14 Srednicki - Quantum Field Theory

0.14.1 Problem 1.2 - Schroedinger equation

$$H = \int d^3x a^\dagger(x) \left(-\frac{\hbar^2}{2m}\triangle_x + V(x)\right) a(x) + \frac{1}{2}\int d^3x d^3y V(x-y) a^\dagger(x) a^\dagger(y) a(x) a(y) \quad (953)$$

$$|\psi, t\rangle = \int d^3x_1 ... d^3x_n \psi(x_1, ..., x_n; t) a^\dagger(x_1) ... a^\dagger(x_n)|0\rangle \quad (954)$$

1. Bosons: With the commutations relation and $a|0\rangle = 0$

$$a(x)a^\dagger(x_1)...a^\dagger(x_n)|0\rangle = \left(\delta^3(x-x_1) - a^\dagger(x_1)a(x)\right)...a^\dagger(x_n)|0\rangle \quad (955)$$

$$= \sum_{k=1}^{n}(-1)^{k-1}\delta^3(x-x_k)\underbrace{a^\dagger(x_1)...a^\dagger(x_n)}_{(n-1)\times a^\dagger}|0\rangle \quad (956)$$

and similar

$$a(y)a(x)a^\dagger(x_1)...a^\dagger(x_n)|0\rangle = \sum_{j\neq k}^{n}\delta^3(x-x_k)\delta^3(y-x_j)\underbrace{a^\dagger(x_1)...a^\dagger(x_n)}_{(n-2)\times a^\dagger}|0\rangle \quad (957)$$

we obtain

$$i\hbar\frac{\partial}{\partial t}|\psi, t\rangle = \int d^3x_1...d^3x_n \frac{\partial}{\partial t}\psi(x_1, ..., x_n; t) a^\dagger(x_1)...a^\dagger(x_n)|0\rangle \quad (958)$$

and

$$H|\psi, t\rangle = \sum_{k=1}^{n} a^\dagger(x_k)\left(-\frac{\hbar^2}{2m}\triangle_{x_k} + V(x_k)\right)\psi(x_1, ..., x_n; t)\underbrace{a^\dagger(x_1)...a^\dagger(x_n)}_{(n-1)\times a^\dagger}|0\rangle \quad (959)$$

$$+ \frac{1}{2}\sum_{j\neq k}^{n} V(x_k - x_j)\psi(x_1, ..., x_n; t) a^\dagger(x_k)a^\dagger(x_j)\underbrace{a^\dagger(x_1)...a^\dagger(x_n)}_{(n-2)\times a^\dagger}|0\rangle \quad (960)$$

2. Fermions:

0.14.2 Problem 1.3 - Commutator of the number operator

Preliminary calculations (we use the boson commutation relations)

$$a^\dagger(z)a(z)a^\dagger(x) = a^\dagger(z)(\delta(x-z) + a^\dagger(x)a(z)) \quad (961)$$

$$= a^\dagger(z)\delta^3(x-z) + a^\dagger(z)a^\dagger(x)a(z) \quad (962)$$

$$= a^\dagger(z)\delta^3(x-z) + a^\dagger(x)a^\dagger(z)a(z) \quad (963)$$

and

$$a(x)a^\dagger(z)a(z) = (\delta(x-z) + a^\dagger(z)a(x))a(z) \quad (964)$$

$$= \delta^3(x-z)a(z) + a^\dagger(z)a(x)a(z) \quad (965)$$

$$= \delta^3(x-z)a(z) + a^\dagger(z)a(z)a(x) \quad (966)$$

With

$$N = \int d^3z\, a^\dagger(z)a(z) \quad (967)$$

$$H = H_1 + H_{\text{int}} \quad (968)$$

$$= \int d^3x\, a^\dagger(x)\left(-\frac{\hbar^2}{2m}\triangle_x + U(x)\right)a(x) + \frac{1}{2}\int d^3x d^3y\, V(x-y)a^\dagger(x)a^\dagger(y)a(y)a(x) \quad (969)$$

We are calculating the commutator in two parts. We start with $[N, H_1]$

$$NH_1 = \int d^3xd^3z \left(a^\dagger(z)\delta^3(x-z) + a^\dagger(x)a^\dagger(z)a(z)\right)\left(-\frac{\hbar^2}{2m}\triangle_x + U(x)\right)a(x) \tag{970}$$

$$= \int d^3xa^\dagger(x)\left(-\frac{\hbar^2}{2m}\triangle_x + U(x)\right)a(x) + \int d^3xd^3za^\dagger(x)\left(-\frac{\hbar^2}{2m}\triangle_x + U(x)\right)a^\dagger(z)a(z)a(x) \tag{971}$$

and

$$H_1N = \int d^3x\, a^\dagger(x)\left(-\frac{\hbar^2}{2m}\triangle_x + U(x)\right)(\delta^3(x-z)a(z) + a^\dagger(z)a(z)a(x)) \tag{972}$$

$$= \int d^3xa^\dagger(x)\left(-\frac{\hbar^2}{2m}\triangle_x + U(x)\right)a(x) + \int d^3xd^3za^\dagger(x)\left(-\frac{\hbar^2}{2m}\triangle_x + U(x)\right)a^\dagger(z)a(z)a(x) \tag{973}$$

therefore $[N, H_1] = 0$. For the second part $[N, H_{\text{int}}]$ we calculate

$$a_z^\dagger a_z a_x^\dagger a_y^\dagger a_y a_x = a_z^\dagger(\delta_{zx}^3 + a_x^\dagger a_z)a_y^\dagger a_y a_x \tag{974}$$

$$= \delta_{zx}^3 a_z^\dagger a_y^\dagger a_y a_x + a_z^\dagger a_x^\dagger a_z a_y^\dagger a_y a_x \tag{975}$$

$$= \delta_{zx}^3 a_y^\dagger a_z^\dagger a_y a_x + a_z^\dagger a_x^\dagger(\delta_{zy}^3 + a_y^\dagger a_z)a_y a_x \tag{976}$$

$$= \delta_{zx}^3 a_y^\dagger a_z^\dagger a_y a_x + \delta_{zy}^3 a_z^\dagger a_x^\dagger a_y a_x + a_z^\dagger a_x^\dagger a_y^\dagger a_z a_y a_x \tag{977}$$

$$= \delta_{zx}^3 a_y^\dagger a_z^\dagger a_y a_x + \delta_{zy}^3 a_x^\dagger a_z^\dagger a_y a_x + a_x^\dagger a_y^\dagger a_z^\dagger a_z a_y a_x \tag{978}$$

$$\to a_y^\dagger a_x^\dagger a_y a_x + a_x^\dagger a_y^\dagger a_y a_x + a_x^\dagger a_y^\dagger a_z^\dagger a_z a_y a_x \tag{979}$$

and

$$a_x^\dagger a_y^\dagger a_y a_x a_z^\dagger a_z = a_x^\dagger a_y^\dagger a_y(\delta_{xz}^3 + a_z^\dagger a_x)a_z \tag{980}$$

$$= \delta_{xz}^3 a_x^\dagger a_y^\dagger a_y a_z + a_x^\dagger a_y^\dagger a_y a_z^\dagger a_x a_z \tag{981}$$

$$= \delta_{xz}^3 a_x^\dagger a_y^\dagger a_z a_y + a_x^\dagger a_y^\dagger(\delta_{zy}^3 + a_z^\dagger a_y)a_x a_z \tag{982}$$

$$= \delta_{xz}^3 a_x^\dagger a_y^\dagger a_z a_y + \delta_{zy}^3 a_x^\dagger a_y^\dagger a_x a_z + a_x^\dagger a_y^\dagger a_z^\dagger a_y a_x a_z \tag{983}$$

$$= \delta_{xz}^3 a_x^\dagger a_y^\dagger a_z a_y + \delta_{zy}^3 a_x^\dagger a_y^\dagger a_z a_x + a_x^\dagger a_y^\dagger a_z^\dagger a_z a_y a_x \tag{984}$$

$$\to a_x^\dagger a_y^\dagger a_x a_y + a_x^\dagger a_y^\dagger a_y a_x + a_x^\dagger a_y^\dagger a_z^\dagger a_z a_y a_x \tag{985}$$

We therefore see that the commutator vanishes as well.

0.14.3 Problem 2.1 - Infinitesimal LT

$$g_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = g_{\rho\sigma} \tag{986}$$

$$g_{\mu\nu}\left(\delta^\mu_\rho + \delta\omega^\mu_\rho\right)(\delta^\nu_\sigma + \delta\omega^\nu{}_\sigma) = g_{\rho\sigma} \tag{987}$$

$$g_{\mu\nu}\left(\delta^\mu{}_\rho\delta^\nu{}_\sigma + \delta^\nu{}_\sigma\cdot\delta\omega^\mu{}_\rho + \delta^\mu{}_\rho\cdot\delta\omega^\nu{}_\sigma + \mathcal{O}(\delta\omega^2)\right) = g_{\rho\sigma} \tag{988}$$

$$g_{\rho\sigma} + g_{\mu\sigma}\cdot\delta\omega^\mu{}_\rho + g_{\rho\nu}\cdot\delta\omega^\nu{}_\sigma = g_{\rho\sigma} \tag{989}$$

which implies

$$\delta\omega_{\sigma\rho} + \delta\omega_{\rho\sigma} = 0 \tag{990}$$

0.14.4 Problem 2.2 - Infinitesimal LT II

Important: each $M^{\mu\nu}$ is an operator and $\delta\omega$ is just a coefficient matrix so $\delta\omega_{\mu\nu}M^{\mu\nu}$ ist a weighted sum of operators.

$$U(\Lambda^{-1}\Lambda'\Lambda) = U(\Lambda^{-1})U(\Lambda')U(\Lambda) \tag{991}$$

$$U(\Lambda^{-1}(I+\delta\omega')\Lambda) = U(\Lambda^{-1})\left(I + \frac{i}{2\hbar}\delta\omega'_{\mu\nu}M^{\mu\nu}\right)U(\Lambda) \tag{992}$$

$$U(I+\Lambda^{-1}\delta\omega'\Lambda) = I + \frac{i}{2\hbar}\delta\omega'_{\mu\nu}U(\Lambda^{-1})M^{\mu\nu}U(\Lambda) \tag{993}$$

now we calculate recalling successive LT's $(\Lambda^{-1})^{\varepsilon}{}_{\gamma}\delta\omega'^{\gamma}{}_{\beta}\Lambda^{\beta}{}_{\alpha}x^{\alpha}$

$$\begin{aligned}
(\Lambda^{-1}\delta\omega'\Lambda)_{\rho\sigma} &= g_{\varepsilon\rho}(\Lambda^{-1})^{\varepsilon}{}_{\mu}\delta\omega'^{\mu}{}_{\nu}\Lambda^{\nu}{}_{\sigma} && (994)\\
&= g_{\varepsilon\rho}\Lambda_{\mu}{}^{\varepsilon}\delta\omega'^{\mu}{}_{\nu}\Lambda^{\nu}{}_{\sigma} && (995)\\
&= \delta\omega'_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} && (996)
\end{aligned}$$

now we can rewrite $U(I+\Lambda^{-1}\delta\omega'\Lambda)$ and therefore

$$\delta\omega'_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}M^{\rho\sigma} = \delta\omega'_{\mu\nu}U(\Lambda^{-1})M^{\mu\nu}U(\Lambda) \tag{997}$$

As all $\delta\omega'$ components are basically independent the equation must hold for each pair μ,ν.

0.14.5 Problem 2.3 - Commutators of LT generators I

LHS:

$$\begin{aligned}
U(\Lambda)^{-1}M^{\mu\nu}U(\Lambda) &\simeq \left(I - \frac{i}{2\hbar}\delta\omega_{\alpha\beta}M^{\alpha\beta}\right)M^{\mu\nu}\left(I + \frac{i}{2\hbar}\delta\omega_{\rho\sigma}M^{\rho\sigma}\right) && (998)\\
&\simeq M^{\mu\nu} - \frac{i}{2\hbar}\delta\omega_{\rho\sigma}(M^{\rho\sigma}M^{\mu\nu} - M^{\mu\nu}M^{\rho\sigma}) + \mathcal{O}(\delta\omega^2) && (999)\\
&= M^{\mu\nu} - \frac{i}{2\hbar}\delta\omega_{\rho\sigma}[M^{\rho\sigma}, M^{\mu\nu}] && (1000)\\
&= M^{\mu\nu} + \frac{i}{2\hbar}\delta\omega_{\rho\sigma}[M^{\mu\nu}, M^{\rho\sigma}] && (1001)
\end{aligned}$$

RHS:

$$\begin{aligned}
\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}M^{\rho\sigma} &\simeq \left(\delta^{\mu}{}_{\rho} + \delta\omega^{\mu}{}_{\rho}\right)\left(\delta^{\nu}{}_{\sigma} + \delta\omega^{\nu}{}_{\sigma}\right)M^{\rho\sigma} && (1002)\\
&\simeq M^{\mu\nu} + \delta^{\mu}_{\rho}\delta\omega^{\nu}{}_{\sigma}M^{\rho\sigma} + \delta^{\nu}{}_{\sigma}\delta\omega^{\mu}{}_{\rho}M^{\rho\sigma} && (1003)\\
&\simeq M^{\mu\nu} + \delta\omega^{\nu}{}_{\sigma}M^{\mu\sigma} + \delta\omega^{\mu}{}_{\rho}M^{\rho\nu} && (1004)\\
&\simeq M^{\mu\nu} + \delta\omega_{\alpha\sigma}g^{\alpha\nu}M^{\mu\sigma} + \delta\omega_{\alpha\rho}g^{\alpha\mu}M^{\rho\nu} && (1005)\\
&\simeq M^{\mu\nu} + \delta\omega_{\alpha\sigma}(g^{\alpha\nu}M^{\mu\sigma} + g^{\alpha\mu}M^{\sigma\nu}) && (1006)\\
&\simeq M^{\mu\nu} + \delta\omega_{\rho\sigma}(g^{\rho\nu}M^{\mu\sigma} + g^{\rho\mu}M^{\sigma\nu}) && (1007)\\
&\simeq M^{\mu\nu} + \frac{1}{2}\delta\omega_{\rho\sigma}\left(g^{\rho\nu}(M^{\mu\sigma} - M^{\sigma\mu}) + g^{\rho\mu}(M^{\sigma\nu} - M^{\nu\sigma})\right) && (1008)\\
&\simeq M^{\mu\nu} + \frac{1}{2}\delta\omega_{\rho\sigma}\left(g^{\rho\nu}M^{\mu\sigma} - g^{\nu\rho}M^{\sigma\mu} + g^{\rho\mu}M^{\sigma\nu} - g^{\mu\rho}M^{\nu\sigma}\right) && (1009)
\end{aligned}$$

Now we use the antisymmetry of M

$$\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}M^{\rho\sigma} \simeq M^{\mu\nu} + \frac{1}{2}\delta\omega_{\rho\sigma}\left(g^{\nu\rho}M^{\mu\sigma} - g^{\nu\rho}M^{\sigma\mu} + g^{\rho\mu}M^{\sigma\nu} - g^{\mu\rho}M^{\nu\sigma}\right) \tag{1010}$$

$$\simeq M^{\mu\nu} - \frac{1}{2}\delta\omega_{\rho\sigma}\left(-g^{\nu\rho}M^{\mu\sigma} + g^{\nu\rho}M^{\sigma\mu} - g^{\rho\mu}M^{\sigma\nu} + g^{\mu\rho}M^{\nu\sigma}\right) \tag{1011}$$

$$\simeq M^{\mu\nu} - \frac{1}{2}\delta\omega_{\rho\sigma}\left(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma} - g^{\rho\mu}M^{\sigma\nu} + g^{\nu\rho}M^{\sigma\mu}\right) \tag{1012}$$

$$\simeq M^{\mu\nu} - \frac{1}{2}\delta\omega_{\rho\sigma}\left(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma}\right) - \frac{1}{2}\underbrace{\delta\omega_{\rho\sigma}\left(-g^{\rho\mu}M^{\sigma\nu} + g^{\nu\rho}M^{\sigma\mu}\right)}_{=\underbrace{\delta\omega_{\sigma\rho}\left(-g^{\sigma\mu}M^{\rho\nu} + g^{\nu\sigma}M^{\rho\mu}\right)}_{=-\delta\omega_{\rho\sigma}\left(-g^{\mu\sigma}(-M^{\nu\rho})+g^{\nu\sigma}(-M^{\mu\rho})\right)}} \tag{1013}$$

$$\simeq M^{\mu\nu} - \frac{1}{2}\delta\omega_{\rho\sigma}\left(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma}\right) - \frac{1}{2}\delta\omega_{\rho\sigma}\left(-g^{\mu\sigma}M^{\nu\rho} + g^{\nu\sigma}M^{\mu\rho}\right) \tag{1014}$$

$$\simeq M^{\mu\nu} - \frac{1}{2}\delta\omega_{\rho\sigma}\left(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma} - g^{\mu\sigma}M^{\nu\rho} + g^{\nu\sigma}M^{\mu\rho}\right) \tag{1015}$$

As the components of $\delta\omega$ (besides the antisymmetry) are independent we get

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\hbar\left(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma} - g^{\mu\sigma}M^{\nu\rho} + g^{\nu\sigma}M^{\mu\rho}\right) \tag{1016}$$

0.14.6 Problem 2.4 - Commutators of LT generators II

Preliminary calculations

$$\epsilon_{ijk}J_k = \varepsilon_{ijk}\frac{1}{2}\varepsilon_{kab}M^{ab} \tag{1017}$$

$$= -\frac{1}{2}\varepsilon_{kij}\varepsilon_{kab}M^{ab} \tag{1018}$$

$$= -\frac{1}{2}\left(\delta_{ia}\delta_{jb} - \delta_{ja}\delta_{ib}\right)M^{ab} \tag{1019}$$

$$= -\frac{1}{2}\left(M^{ij} - M^{ji}\right) \tag{1020}$$

$$= -M^{ij} \tag{1021}$$

- With

$$J_1 = \frac{1}{2}(\varepsilon_{123}M^{23} + \varepsilon_{132}M^{32}) \tag{1022}$$

$$= \varepsilon_{123}M^{23} \tag{1023}$$

$$= M^{23} \tag{1024}$$

then

$$[J_1, J_3] = [M^{23}, M^{12}] \tag{1025}$$

$$= i\hbar\left(g^{21}M^{32} - g^{31}M^{22} - g^{22}M^{31} + g^{32}M^{21}\right) \tag{1026}$$

$$= -i\hbar g^{22}M^{31} \tag{1027}$$

$$= -i\hbar M^{31} \tag{1028}$$

$$= -i\hbar J_2 \tag{1029}$$

- analog ...

-

$$[K^i, K^j] = [M^{i0}, M^{j0}] \tag{1030}$$
$$= i\hbar \left(g^{ij}M^{00} - g^{0j}M^{i0} - g^{i0}M^{0j} + g^{00}M^{ij}\right) \tag{1031}$$
$$= i\hbar \left(-\delta^{ij}M^{00} + M^{ij}\right) \tag{1032}$$
$$= \begin{cases} i\hbar M^{ij} = -i\hbar\epsilon_{ijk}J_k & (i = j) \\ 0 & (i \neq j) \end{cases} \tag{1033}$$

where we used the result from the preliminary calculation in the last step.

0.14.7 Problem 2.7 - Translation operator

The obvious property $T(a)T(b) = T(a+b)$. Then

$$T(\delta a + \delta b) = T(\delta a)T(\delta b) \tag{1034}$$
$$= \left(1 - \frac{i}{\hbar}\delta a_\mu P^\mu\right)\left(1 - \frac{i}{\hbar}\delta b_\nu P^\nu\right) \tag{1035}$$
$$\simeq 1 - \frac{i}{\hbar}(\delta a_\mu + \delta b_\mu)P^\mu + \frac{1}{\hbar^2}\delta a_\mu \delta b_\mu P^\mu P^\nu \tag{1036}$$

and

$$T(\delta a + \delta b) = T(\delta b)T(\delta a) \tag{1037}$$
$$= \left(1 - \frac{i}{\hbar}\delta b_\nu P^\nu\right)\left(1 - \frac{i}{\hbar}\delta a_\mu P^\mu\right) \tag{1038}$$
$$\simeq 1 - \frac{i}{\hbar}(\delta a_\mu + \delta b_\mu)P^\mu + \frac{1}{\hbar^2}\delta a_\mu \delta b_\mu P^\nu P^\mu \tag{1039}$$

which implies $P^\mu P^\nu = P^\nu P^\mu$.

0.14.8 Problem 2.8 - Transformation of scalar field - NOT DONE YET

(a) We start with

$$U(\Lambda)^{-1}\varphi(x)U(\Lambda) = \varphi(\Lambda^{-1}x) \tag{1040}$$
$$\left(1 - \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right)\varphi(x)\left(1 + \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right) = \varphi([\delta^\mu_{\ \nu} - \delta\omega^\mu_{\ \nu}]x^\nu) \tag{1041}$$
$$\varphi(x) - \frac{i}{2\hbar}\delta\omega_{\mu\nu}[M^{\mu\nu}, \varphi(x)] = \varphi(x) - \delta\omega^\mu_{\ \nu}x^\nu\frac{\partial\varphi}{\partial x^\mu} \tag{1042}$$
$$= \varphi(x) - \delta\omega^\mu_{\ \nu}\frac{1}{2}\left(x^\nu\frac{\partial\varphi}{\partial x^\mu} - x^\mu\frac{\partial\varphi}{\partial x^\nu}\right) \tag{1043}$$
$$= \varphi(x) - \delta\omega_{\mu\nu}\frac{1}{2}\left(x^\nu\partial^\mu - x^\mu\partial^\nu\right)\varphi \tag{1044}$$

and therefore

$$[\varphi, M^{\mu\nu}] = \frac{\hbar}{i}(x^\mu\partial^\nu - x^\nu\partial^\mu)\varphi \tag{1045}$$

(b) (c) (d) (e) (f)

0.14.9 Problem 3.2 - Multiparticle eigenstates of the hamiltonian

With

$$|k_1...k_n\rangle = a^\dagger_{k_1}...a^\dagger_{k_n}|0\rangle \tag{1046}$$

$$H = \int \widetilde{dk}\ \omega_k a^\dagger_k a_k \tag{1047}$$

$$[a_k, a^\dagger_q] = \underbrace{(2\pi)^3 2\omega_k \delta^3(\vec{k}-\vec{q})}_{\delta_{kq}} \tag{1048}$$

we see that the expression which needs calculating is the creation and annihilation operators. The idea is to use the commutation relations to move the a_k to the right end to use $a_k|0\rangle$

$$\begin{aligned} a^\dagger_k a_k a^\dagger_{k_1}...a^\dagger_{k_n}|0\rangle &= a^\dagger_k(a^\dagger_{k_1}a_k + \delta_{kk_1})a^\dagger_{k_2}...a^\dagger_{k_n}|0\rangle && (1049)\\ &= \delta_{kk_1}a^\dagger_k a^\dagger_{k_2}...a^\dagger_{k_n}|0\rangle + a^\dagger_k a^\dagger_{k_1}a_k a^\dagger_{k_2}...a^\dagger_{k_n}|0\rangle && (1050)\\ &= ... && (1051)\\ &= \sum_j \delta_{kk_j}a^\dagger_k \underbrace{a^\dagger_{k_2}...a^\dagger_{k_n}}_{(n-1)\text{ times with } a_{k_j} \text{ missing}}|0\rangle + a^\dagger_k a^\dagger_{k_1}...a^\dagger_{k_n}\underbrace{a_k|0\rangle}_{=0}. && (1052)\end{aligned}$$

Therefore we obtain

$$\begin{aligned} H|k_1...k_n\rangle &= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\omega_k \sum_j \delta_{kk_j}a^\dagger_k a^\dagger_{k_2}...a^\dagger_{k_n}|0\rangle && (1053)\\ &= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\omega_k \sum_j (2\pi)^3 2\omega_k \delta^3(\vec{k}-\vec{k}_j)a^\dagger_k a^\dagger_{k_2}...a^\dagger_{k_n}|0\rangle && (1054)\\ &= \int d^3k\omega_k \sum_j \delta^3(\vec{k}-\vec{k}_j)a^\dagger_k a^\dagger_{k_2}...a^\dagger_{k_n}|0\rangle && (1055)\end{aligned}$$

which we can integrate obtaining the desired result

$$\begin{aligned} H|k_1...k_n\rangle &= \sum_j \omega_{k_j}a^\dagger_{k_j}a^\dagger_{k_2}...a^\dagger_{k_n}|0\rangle && (1056)\\ &= \left(\sum_j \omega_{k_j}\right)a^\dagger_{k_1}a^\dagger_{k_2}...a^\dagger_{k_n}|0\rangle && (1057)\\ &= \left(\sum_j \omega_{k_j}\right)|k_1...k_n\rangle. && (1058)\end{aligned}$$

0.14.10 Problem 3.4 - Heisenberg equations of motion for free field

(a) For the translation operator $T(a) = e^{-iP^\mu a_\mu}$ we expand in first order

$$\begin{aligned} T(a)^{-1}\varphi(a)T(a) &= \left(1-(-i)P^\mu a_\mu + \mathcal{O}(a^2)\right)\varphi(x)\left(1+(-i)P^\mu a_\mu + \mathcal{O}(a^2)\right) && (1059)\\ &= \left(1+iP^\mu a_\mu + \mathcal{O}(a^2)\right)\varphi(x)\left(1-iP^\mu a_\mu + \mathcal{O}(a^2)\right) && (1060)\\ &\simeq \varphi(x) + ia_\mu P^\mu\varphi(x) - ia_\mu\varphi(x)P^\mu && (1061)\\ &\simeq \varphi(x) + ia_\mu[P^\mu, \varphi(x)] && (1062)\end{aligned}$$

for the right hand right we get

$$\varphi(x-a) \simeq \varphi(x) - \partial^\mu\varphi(x)a_\mu \tag{1063}$$

and therefore

$$i[P^\mu, \varphi(x)] = -\partial^\mu \varphi(x) \tag{1064}$$

(b) With $\mu = 0$ and $\partial^0 = g_{0\nu}\partial_\nu = -\partial_0$ we have

$$\begin{aligned} i[H, \varphi(x)] &= -\partial^0 \varphi(x) = +\partial_0 \varphi(x) && (1065)\\ \rightarrow \quad \dot{\varphi}(x) &= i[H, \varphi(x)] && (1066) \end{aligned}$$

(c) We start with the hamiltonian (3.25)

$$H = \int d^3y \frac{1}{2}\Pi^2(y) + \frac{1}{2}(\nabla_y \varphi(y))^2 + \frac{1}{2}m^2\varphi(y)^2 - \Omega_0 \tag{1067}$$

- Obtaining $\dot{\varphi}(x) = i[H, \varphi(x)]$

  We need to calculate (setting $x^0 = y^0$ - why can we?)

$$\begin{aligned}
[\Pi^2(y), \varphi(x)] &= \Pi(y)\Pi(y)\varphi(x) - \varphi(x)\Pi(y)\Pi(y) && (1068)\\
&= \Pi(y)\Pi(y)\varphi(x) - \Pi(y)\varphi(x)\Pi(y) + \Pi(y)\varphi(x)\Pi(y) - \varphi(x)\Pi(y)\Pi(y) && (1069)\\
&= \Pi(y)[\Pi(y), \varphi(x)] + [\Pi(y), \varphi(x)]\Pi(y) && (1070)\\
&= 2\Pi(y)(-1)i\delta^3(\vec{y} - \vec{x}) && (1071)\\
[(\nabla_y \varphi(y))^2, \varphi(x)] &= \nabla_y\varphi(y)\nabla_y\varphi(y)\varphi(x) - \varphi(x)\nabla_y\varphi(y)\nabla_y\varphi(y) && (1072)\\
&= \nabla_y\varphi(y)[\nabla_y\varphi(y), \varphi(x)] + [\nabla_y\varphi(y), \varphi(x)]\nabla_y\varphi(y) && (1073)\\
&= \nabla_y\varphi(y)\nabla_y[\varphi(y), \varphi(x)] + \nabla_y[\varphi(y), \varphi(x)]\nabla_y\varphi(y) && (1074)\\
&= 0 && (1075)\\
[\varphi(y)^2, \varphi(x)] &= \varphi(y)\varphi(y)\varphi(x) - \varphi(x)\varphi(y)\varphi(y) && (1076)\\
&= \varphi(y)\varphi(y)\varphi(x) - \varphi(y)\varphi(x)\varphi(y) + \varphi(y)\varphi(x)\varphi(y) - \varphi(x)\varphi(y)\varphi(y) && (1077)\\
&= \varphi(y)[\varphi(y), \varphi(x)] + [\varphi(y), \varphi(x)]\varphi(y) && (1078)\\
&= 0 && (1079)
\end{aligned}$$

  then

$$\begin{aligned}
\int d^3y[\Pi^2(y), \varphi(x)] &= -2i\Pi(x) && (1080)\\
\int d^3y[(\nabla_y\varphi(y))^2, \varphi(x)] &= \int d^3y \nabla_y\varphi(y)[\nabla_y\varphi(y), \varphi(x)] + [\nabla_y\varphi(y), \varphi(x)]\nabla_y\varphi(y) && (1081)\\
&= 0 && (1082)\\
\int d^3y[\varphi(y)^2, \varphi(x)] &= 0 && (1083)
\end{aligned}$$

  and therefore

$$\begin{aligned}
\dot{\varphi}(x) &= i[H, \varphi(x)] && (1084)\\
&= i\frac{1}{2}(-2i)\Pi(x) && (1085)\\
&= \Pi(x) && (1086)
\end{aligned}$$

- Obtaining $\dot{\Pi}(x) = -i[H, \Pi(x)]$ (sign!?!)

Now we need to calculate - by using the results from above we can now shortcut a bit

$$[\Pi^2(y), \Pi(x)] = 0 \tag{1087}$$
$$[(\nabla_y\varphi(y))^2, \Pi(x)] = (\nabla_y\varphi(y))(\nabla_y\varphi(y))\Pi(x) - \Pi(x)(\nabla_y\varphi(y))(\nabla_y\varphi(y)) \tag{1088}$$
$$= (\nabla_y\varphi(y))[(\nabla_y\varphi(y)), \Pi(x)] - [\Pi(x), (\nabla_y\varphi(y))](\nabla_y\varphi(y)) \tag{1089}$$
$$= (\nabla_y\varphi(y))\nabla_y[\varphi(y), \Pi(x)] - (\nabla_y[\Pi(x), \varphi(y)])(\nabla_y\varphi(y)) \tag{1090}$$
$$= (\nabla_y\varphi(y))\nabla_y i\delta^3(\vec{x}-\vec{y}) - (\nabla_y(-i)\delta^3(\vec{x}-\vec{y}))(\nabla_y\varphi(y)) \tag{1091}$$
$$= 2i(\nabla_y\delta^3(\vec{x}-\vec{y}))(\nabla_y\varphi(y)) \tag{1092}$$
$$[\varphi(y)^2, \Pi(x)] = \varphi(y)\varphi(y)\Pi(x) - \Pi(x)\varphi(y)\varphi(y) \tag{1093}$$
$$= \varphi(y)\varphi(y)\Pi(x) - \varphi(y)\Pi(x)\varphi(y) + \varphi(y)\Pi(x)\varphi(y) - \Pi(x)\varphi(y)\varphi(y) \tag{1094}$$
$$= \varphi(y)[\varphi(y), \Pi(x)] + [\varphi(y), \Pi(x)]\varphi(y) \tag{1095}$$
$$= 2i\varphi(y)\delta^3(\vec{x}-\vec{y}) \tag{1096}$$

then

$$\int d^3y[\Pi^2(y), \Pi(x)] = 0 \tag{1097}$$
$$\int d^3y[(\nabla_y\varphi(y))^2, \Pi(x)] = 2i\int d^3y(\nabla_y\delta^3(\vec{x}-\vec{y}))(\nabla_y\varphi(y)) \tag{1098}$$
$$= -2i\int d^3y\delta^3(\vec{x}-\vec{y})(\nabla_y\nabla_y\varphi(y)) \tag{1099}$$
$$= -2i\triangle_x\varphi(x) \tag{1100}$$
$$\int d^3y[\varphi(y)^2, \Pi(x)] = 2i\varphi(x) \tag{1101}$$

and therefore

$$\dot{\Pi}(x) = -i[H, \Pi(x)] \tag{1102}$$
$$= -i\left(\frac{1}{2}(-2i)\triangle_x\varphi(x) + \frac{1}{2}m^2 2i\varphi(x)\right) \tag{1103}$$
$$= -i\left(-i\triangle_x\varphi(x) + m^2 i\varphi(x)\right) \tag{1104}$$
$$= -\triangle_x\varphi(x) + m^2\varphi(x) \tag{1105}$$

which finally leads to (with $\Box = \partial_{tt} - \triangle$)

$$\partial^0\partial_0\varphi(x) = \partial^0\Pi(x) \tag{1106}$$
$$= -\partial_0\Pi(x) \tag{1107}$$
$$= -(-\triangle_x\varphi(x) + m^2\varphi(x)) \tag{1108}$$
$$\to (\Box_x + m^2)\varphi(x) = 0 \tag{1109}$$

(d) With

$$\vec{P} \equiv -\int d^3x\Pi(x)\nabla_x\varphi(x) \tag{1110}$$

we have to calculate

$$[\vec{P}, \varphi(y)] = -\int d^3x[\Pi(x)\nabla_x\varphi(x), \varphi(y)]. \tag{1111}$$

Let's start with

$$
\begin{aligned}
[\Pi(x)\nabla_x\varphi(x),\varphi(y)] &= \Pi(x)\nabla_x\varphi(x)\varphi(y) - \varphi(y)\Pi(x)\nabla_x\varphi(x) && (1112)\\
&= \Pi(x)\nabla_x\varphi(x)\varphi(y) - (\Pi(x)\varphi(y) + i\delta^3(\vec{x}-\vec{y}))\nabla_x\varphi(x) && (1113)\\
&= \Pi(x)\nabla_x\varphi(x)\varphi(y) - \Pi(x)\varphi(y)\nabla_x\varphi(x) + i\delta^3(\vec{x}-\vec{y})\nabla_x\varphi(x) && (1114)\\
&= \Pi(x)\nabla_x(\varphi(x)\varphi(y)) - \Pi(x)\nabla_x(\varphi(y)\varphi(x)) + i\delta^3(\vec{x}-\vec{y})\nabla_x\varphi(x) && (1115)\\
&= \Pi(x)\nabla_x[\varphi(x),\varphi(y)] + i\delta^3(\vec{x}-\vec{y})\nabla_x\varphi(x) && (1116)\\
&= i\delta^3(\vec{x}-\vec{y})\nabla_x\varphi(x) && (1117)
\end{aligned}
$$

and then

$$
\begin{aligned}
[\vec{P},\varphi(y)] &= -i\int d^3x\delta^3(\vec{x}-\vec{y})\nabla_x\varphi(x) && (1118)\\
&= -i\nabla_y\varphi(y) && (1119)
\end{aligned}
$$

(e) With

$$
\begin{aligned}
\Pi(x) &= \dot{\varphi}(x) && (1120)\\
&= \int \frac{d^3k}{(2\pi)^3 2\omega_k}(-i\omega_k)(a_k e^{ikx} - a_k^\dagger e^{-ikx}) && (1121)\\
\nabla\varphi(x) &= \int \frac{d^3q}{(2\pi)^3 2\omega_k}(i\vec{q})(a_q e^{iqx} - a_q^\dagger e^{-iqx}) && (1122)\\
& && (1123)
\end{aligned}
$$

then

$$
\begin{aligned}
\vec{P} &= -\int d^3x\Pi(x)\nabla_x\varphi(x) && (1124)\\
&= -\iiint d^3x\frac{d^3k}{(2\pi)^3 2\omega_k}\frac{d^3q}{(2\pi)^3 2\omega_k}(-i\omega_k)(i\vec{q})(a_k e^{ikx} - a_k^\dagger e^{-ikx})(a_q e^{iqx} - a_q^\dagger e^{-iqx}) && (1125)\\
&= -\iiint d^3x\frac{d^3k}{(2\pi)^3 2}\frac{d^3q}{(2\pi)^3 2\omega_k}\vec{q}(a_k a_q e^{i(k+q)x} - a_k^\dagger a_q e^{-i(k-q)x} - a_k a_q^\dagger e^{i(k-q)x} + a_k^\dagger a_q^\dagger e^{-i(k+q)x}) && (1126)\\
& && (1127)
\end{aligned}
$$

now we can use the commutation relations and reindex

$$
= -\iiint d^3x\frac{d^3kd^3q}{4\omega_k(2\pi)^6}\vec{q}(a_k a_q e^{i(k+q)x} - a_k^\dagger a_q e^{-i(k-q)x} - (a_q^\dagger a_k + (2\pi)^3 2\omega_k\delta^3(\vec{k}-\vec{q}))e^{i(k-q)x} + a_k^\dagger a_q^\dagger e^{-i(k+q)x}) \quad (1128)
$$

$$
= -\iiint d^3x\frac{d^3kd^3q}{4\omega_k(2\pi)^6}\vec{q}(a_k a_q e^{i(k+q)x} + a_k^\dagger a_q^\dagger e^{-i(k+q)x}) + \iiint d^3x\frac{d^3kd^3q}{4\omega_k(2\pi)^6}\vec{q}2a_k^\dagger a_q e^{-i(k-q)x} \quad (1129)
$$

$$
+ \iiint d^3x\frac{d^3kd^3q}{4\omega_k(2\pi)^6}\vec{q}(2\pi)^3 2\omega_k\delta^3(\vec{k}-\vec{q})e^{i(k-q)x} \quad (1130)
$$

Now we can look at the integrals individually and use the asymmetry. The first

$$
\begin{aligned}
-\iiint d^3x\frac{d^3kd^3q}{4\omega_k(2\pi)^6}\vec{q}(a_k a_q e^{i(k+q)x} + a_k^\dagger a_q^\dagger e^{-i(k+q)x}) &= ... && (1131)\\
&= 0 && (1132)
\end{aligned}
$$

second

$$\iiint d^3x \frac{d^3k d^3q}{4\omega_k (2\pi)^6} \vec{q}(2\pi)^3 2\omega_k \delta^3(\vec{k}-\vec{q}) e^{i(k-q)x} = \iiint d^3x \frac{d^3k d^3q}{2(2\pi)^3} \vec{q}\delta^3(\vec{k}-\vec{q}) e^{i(k-q)x} \tag{1133}$$

$$= \iiint d^3x \frac{d^3k}{2(2\pi)^3} \vec{k} \tag{1134}$$

$$= 0 \tag{1135}$$

and third

$$\iiint d^3x \frac{d^3k d^3q}{4\omega_k (2\pi)^6} \vec{q} 2 a_k^\dagger a_q e^{-i(k-q)x} = \iint \frac{d^3k d^3q}{4\omega_k (2\pi)^6} \vec{q} 2 a_k^\dagger a_q \int d^3x \; e^{-i(k-q)x} \tag{1136}$$

$$= \iint \frac{d^3k d^3q}{4\omega_k (2\pi)^6} \vec{q} 2 a_k^\dagger a_q e^{-i(k-q)x} e^{-i(k^0-q^0)x^0} \int d^3x \; e^{-i(\vec{k}-\vec{q})\vec{x}} \tag{1137}$$

$$= \iint \frac{d^3k d^3q}{4\omega_k (2\pi)^6} \vec{q} 2 a_k^\dagger a_q e^{-i(k-q)x} e^{-i(k^0-q^0)x^0} (2\pi)^3 \delta^3(\vec{k}-\vec{q}) \tag{1138}$$

$$= \int \frac{d^3k}{2\omega_k (2\pi)^3} \vec{k} a_k^\dagger a_k \tag{1139}$$

$$= \int \widetilde{d^3k} \; \vec{k} a_k^\dagger a_k \tag{1140}$$

Therefore we obtain

$$\vec{P} = \int \frac{d^3k}{2\omega_k (2\pi)^3} \vec{k} a_k^\dagger a_k \tag{1141}$$

$$= \int \widetilde{d^3k} \; \vec{k} a_k^\dagger a_k \tag{1142}$$

0.14.11 Problem 3.5 - Complex scalar field

(a) Sloppy way - Calculating the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \varphi} = -m^2 \varphi^\dagger \tag{1143}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = -\partial^\mu \varphi^\dagger \tag{1144}$$

$$\to -m^2\varphi^\dagger + \partial_\mu \partial^\mu \varphi^\dagger = 0 \tag{1145}$$

$$\to (\partial_\mu \partial^\mu - m^2)\varphi^\dagger = 0 \tag{1146}$$

Bit more rigorous with

$$\frac{\delta \phi(x_1, t_1)}{\delta \phi(x_2, t_2)} = \delta(x_1 - x_2) \times \delta(t_1 - t_2) \tag{1147}$$

$$\frac{\delta \partial_\mu \phi(x)}{\delta \phi(y)} = \frac{\delta}{\delta \phi(y)} \lim_{\epsilon \to 0} \frac{\phi(x_1, x_\mu + \epsilon, ..., x_4) - \phi(x_1, x_2, x_3, x_4)}{\epsilon} \tag{1148}$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\delta(x_\mu + \epsilon - y_\mu) - \delta(x_\mu - y_\mu) \right) \times \delta(x_1 - y_1) \times ... \times \delta(x_4 - y_4) \tag{1149}$$

$$= \frac{\partial}{\partial x^\mu} \delta^4(x-y) \tag{1150}$$

we get

$$S[\varphi] = \int d^4x \left(-\partial^\mu \varphi^\dagger(x)\partial_\mu \varphi(x) - m^2 \varphi^\dagger(x)\varphi(x)\right) \tag{1151}$$

$$\frac{\delta S[\varphi]}{\delta \varphi(y)} = \int d^4x \left(-\partial^\mu \varphi^\dagger(x)\partial_\mu \delta^4(x-y) - m^2 \varphi^\dagger(y)\delta^4(x-y)\right) \tag{1152}$$

$$= \int d^4x \left(\partial_\mu \partial^\mu \varphi^\dagger(x)\delta^4(x-y) - m^2 \varphi^\dagger(x)\delta^4(x-y)\right) \tag{1153}$$

$$= (\Box_y - m^2)\varphi^\dagger(y) \tag{1154}$$

(b) With

$$\mathcal{L} = -\partial^0 \varphi^\dagger \partial_0 \varphi - \partial^a \varphi^\dagger \partial_a \varphi - m^2 \varphi^\dagger \varphi + \Omega_0 \tag{1155}$$

$$= \partial_0 \varphi^\dagger \partial_0 \varphi - \partial^a \varphi^\dagger \partial_a \varphi - m^2 \varphi^\dagger \varphi + \Omega_0 \tag{1156}$$

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \dot{\varphi}^\dagger \tag{1157}$$

$$\Pi^\dagger = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}^\dagger} = \dot{\varphi} \tag{1158}$$

$$\to \mathcal{H} = \Pi \dot{\varphi} + \Pi^\dagger \dot{\varphi}^\dagger - \mathcal{L} \tag{1159}$$

$$= \dot{\varphi}^\dagger \dot{\varphi} + \dot{\varphi}\dot{\varphi}^\dagger - \dot{\varphi}^\dagger \dot{\varphi} + (\nabla^a \varphi^\dagger)(\nabla_a \varphi) + m^2 \varphi^\dagger \varphi - \Omega_0 \tag{1160}$$

$$= \Pi^\dagger \Pi + (\nabla^a \varphi^\dagger)(\nabla_a \varphi) + m^2 \varphi^\dagger \varphi - \Omega_0 \tag{1161}$$

(c) Considering the plane wave solutions $e^{i\vec{k}\vec{x} \pm i\omega_k t}$ with

$$kx = g_{\mu\nu}k^\mu x^\nu = g_{00}k^0 x^0 + g_{ik}k^i x^k = -\omega_k t + \vec{k}\vec{x} \tag{1162}$$

we have

$$\varphi(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\ a_k e^{ikx} + b_k^\dagger e^{-ikx} \tag{1163}$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\ a_k e^{i\vec{k}\vec{x} - i\omega_k t} + b_k^\dagger e^{-i\vec{k}\vec{x} + i\omega_k t} \tag{1164}$$

$$e^{-iqx}\varphi(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\ a_k e^{i(k-q)x} + b_k^\dagger e^{-i\vec{k}\vec{x} + i\omega_k t} e^{-iqx} \tag{1165}$$

$$\int d^3x e^{-iqx}\varphi(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\ a_k \underbrace{\int d^3x e^{i(k-q)x}}_{(2\pi)^3 \delta^3(\vec{k}-\vec{q}) e^{-i(\omega_k - \omega_q)t}} + b_{-k} \underbrace{\int d^3x e^{i(\vec{k}-\vec{q})\vec{x}}}_{(2\pi)^3 \delta^3(\vec{k}-\vec{q})} e^{i(\omega_k + \omega_q)t} \tag{1166}$$

$$= \frac{1}{2\omega_q}\left(a_q + b_{-q}^\dagger e^{2i\omega_q t}\right) \tag{1167}$$

and

$$\partial_0 \varphi(\vec{x}, t) = \int \frac{d^3k\ (-i\omega_k)}{(2\pi)^3 2\omega_k}\ a_k e^{i\vec{k}\vec{x} - i\omega_k t} - b_k^\dagger e^{-i\vec{k}\vec{x} + i\omega_k t} \tag{1168}$$

$$\int d^3x e^{-iqx}\partial_0 \varphi(\vec{x}, t) = -\frac{i}{2}\left(a_q - b_{-q}^\dagger e^{2i\omega_q t}\right) \tag{1169}$$

adding both equations gives with $\partial_0 e^{-iqx} = \partial_0 e^{-i(-\omega_k t + \vec{k}\vec{x})} = -i\omega_q e^{-iqx}$ and $f \overleftrightarrow{\partial_\mu} g = f(\partial_\mu g) - (\partial_\mu f) g$

$$a_q = \omega_q \int d^3x e^{-iqx} \varphi(\vec{x}, t) + i \int d^3x e^{-iqx} \partial_0 \varphi(\vec{x}, t) \tag{1170}$$

$$= i \int d^3x e^{-iqx} (-i\omega_q + \partial_0) \varphi(\vec{x}, t) \tag{1171}$$

$$= i \int d^3x e^{-iqx} \overleftrightarrow{\partial_0} \varphi(\vec{x}, t) \tag{1172}$$

To get b_q we solve a second set of equations for $\varphi^\dagger$

$$\varphi(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\, a_k e^{ikx} + b_k^\dagger e^{-ikx} \tag{1173}$$

$$\to \varphi^\dagger(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\, a_k^\dagger e^{-ikx} + b_k e^{ikx} \tag{1174}$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\, b_k e^{ikx} + a_k^\dagger e^{-ikx} \tag{1175}$$

Now b_k takes the role of a_k and we can just copy the solution

$$b_q = \omega_q \int d^3x e^{-iqx} \varphi^\dagger(\vec{x}, t) + i \int d^3x e^{-iqx} \partial_0 \varphi^\dagger(\vec{x}, t) \tag{1176}$$

$$= i \int d^3x e^{-iqx} (-i\omega_q + \partial_0) \varphi^\dagger(\vec{x}, t) \tag{1177}$$

$$= i \int d^3x e^{-iqx} \overleftrightarrow{\partial_0} \varphi^\dagger(\vec{x}, t) \tag{1178}$$

(d) Starting with the observation

$$[A, B]^\dagger = (AB)^\dagger - (BA)^\dagger \tag{1179}$$

$$= B^\dagger A^\dagger - A^\dagger B^\dagger \tag{1180}$$

$$= [B^\dagger, A^\dagger] \tag{1181}$$

$$= -[A^\dagger, B^\dagger] \tag{1182}$$

therefore the relevant commutation relations for the fields are

$$[\varphi(\vec{x}, t), \varphi(\vec{y}, t)] = 0 \qquad \to \qquad [\varphi^\dagger(\vec{x}, t), \varphi^\dagger(\vec{y}, t)] = 0 \tag{1183}$$

$$[\varphi^\dagger(\vec{x}, t), \varphi(\vec{y}, t)] = 0 \tag{1184}$$

$$[\Pi(\vec{x}, t), \Pi(\vec{y}, t)] = 0 \qquad \to \qquad [\Pi^\dagger(\vec{x}, t), \Pi^\dagger(\vec{y}, t)] = 0 \tag{1185}$$

$$[\Pi^\dagger(\vec{x}, t), \Pi(\vec{y}, t)] = 0 \tag{1186}$$

$$[\varphi(\vec{x}, t), \Pi(\vec{y}, t)] = i\delta^3(\vec{x} - \vec{y}) \qquad \to \qquad [\varphi^\dagger(\vec{x}, t), \Pi^\dagger(\vec{y}, t)] = i\delta^3(\vec{x} - \vec{y}) \tag{1187}$$

$$[\varphi^\dagger(\vec{x}, t), \Pi(\vec{y}, t)] = 0 \qquad \to \qquad [\varphi(\vec{x}, t), \Pi^\dagger(\vec{y}, t)] = 0 \tag{1188}$$

with the previous results

$$a_q = i\int d^3x e^{-iqx}(-i\omega_q + \partial_0)\varphi(\vec{x},t) \tag{1189}$$

$$= i\int d^3x e^{-iqx}(-i\omega_q\varphi(\vec{x},t) + \Pi^\dagger(\vec{x},t)) \tag{1190}$$

$$a_q^\dagger = i\int d^3x e^{iqx}(i\omega_q\varphi^\dagger(\vec{x},t) + \Pi(\vec{x},t)) \tag{1191}$$

$$b_q = i\int d^3x e^{-iqx}(-i\omega_q + \partial_0)\varphi^\dagger(\vec{x},t) \tag{1192}$$

$$= i\int d^3x e^{-iqx}(-i\omega_q\varphi(\vec{x},t) + \Pi(\vec{x},t)) \tag{1193}$$

$$b_q^\dagger = i\int d^3x e^{iqx}(i\omega_q\varphi^\dagger(\vec{x},t) + \Pi^\dagger(\vec{x},t)) \tag{1194}$$

let's calculate each of the commutators

$$[a_k, a_q^\dagger] = \iint d^3x\, d^3y\, e^{-ikx}e^{iqy}\left(\omega_k\omega_q[\varphi_x, \varphi_y^\dagger] - i\omega_q[\varphi_x, \Pi_y] + i\omega_q[\Pi_x^\dagger, \varphi_y^\dagger] + [\Pi_x^\dagger, \Pi_y]\right) \tag{1195}$$

$$= \iint d^3x\, d^3y\, e^{-i(kx-qy)}\left(-i\omega_q[\varphi_x, \Pi_y] + i\omega_q[\Pi_x^\dagger, \varphi_y^\dagger]\right) \tag{1196}$$

$$= \iint d^3x\, d^3y\, e^{-i(kx-qy)}\left(-i\omega_q i\delta^3(\vec{x}-\vec{y}) + i\omega_q(-i)\delta^3(\vec{x}-\vec{y})\right) \tag{1197}$$

$$= (\omega_q + \omega_q)\iint d^3x\, e^{-i(k-q)x} \tag{1198}$$

$$= (\omega_q + \omega_q)\,(2\pi)^3\delta^3(\vec{k}-\vec{q}) \tag{1199}$$

$$= 2\omega_q(2\pi)^3\delta^3(\vec{k}-\vec{q}) \tag{1200}$$

and so on

$$[b_k, b_q^\dagger] = ... = 2\omega_q(2\pi)^3\delta^3(\vec{k}-\vec{q}) \tag{1201}$$

(e) Now

$$H = \int d^3x\, \Pi^\dagger\Pi + (\nabla^a\varphi^\dagger)(\nabla_a\varphi) + m^2\varphi^\dagger\varphi - \Omega_0 \tag{1202}$$

$$\Pi^\dagger\Pi = \dot{\varphi}\dot{\varphi}^\dagger \tag{1203}$$

$$= \int \widetilde{d^3k}\widetilde{d^3q}(i\omega_k)(i\omega_q)\left(a_k e^{ikx} - b_k^\dagger e^{-ikx}\right)\left(a_q^\dagger e^{-iqx} - b_q e^{iqx}\right) \tag{1204}$$

$$= \int \widetilde{d^3k}\widetilde{d^3q}(-\omega_k\omega_q)\left(a_k a_q^\dagger e^{-iqx}e^{ikx} - b_k^\dagger a_q^\dagger e^{-iqx}e^{-ikx} - a_k b_q e^{iqx}e^{ikx} + b_k^\dagger b_q e^{iqx}e^{-ikx}\right) \tag{1205}$$

$$= \int \widetilde{d^3k}\widetilde{d^3q}(-\omega_k\omega_q)\left([a_q^\dagger a_k - 2\omega_k(2\pi)^3\delta^3(\vec{k}-\vec{q})]e^{-i(q-k)x} - b_k^\dagger a_q^\dagger e^{-i(q+k)x} - a_k b_q e^{i(q+k)x} + b_k^\dagger b_q e^{i(q-k)x}\right) \tag{1206}$$

$$(\nabla^a\varphi^\dagger)(\nabla_a\varphi) = \int \widetilde{d^3k}\widetilde{d^3q}(k^a q_a)\left(-a_k^\dagger e^{-ikx} + b_k e^{ikx}\right)\left(a_q e^{iqx} - b_q^\dagger e^{-iqx}\right) \tag{1207}$$

$$= \int \widetilde{d^3k}\widetilde{d^3q}(k^a q_a)\left(-a_k^\dagger a_q e^{iqx}e^{-ikx} + b_k a_q e^{iqx}e^{ikx} + a_k^\dagger b_q^\dagger e^{-iqx}e^{-ikx} - b_k b_q^\dagger e^{-iqx}e^{ikx}\right) \tag{1208}$$

$$= \int \widetilde{d^3k}\widetilde{d^3q}(k^a q_a)\left(-a_k^\dagger a_q e^{i(q-k)x} + a_q b_k e^{i(q+k)x} + a_k^\dagger b_q^\dagger e^{-i(q+k)x} - [b_q^\dagger b_k - 2\omega_k(2\pi)^3\delta^3(\vec{k}-\vec{q})]e^{-i(q-k)} \right. \tag{1209}$$

$$\varphi^\dagger\varphi = \int \widetilde{d^3k}\widetilde{d^3q}\left(a_k^\dagger e^{-ikx} + b_k e^{ikx}\right)\left(a_q e^{iqx} + b_q^\dagger e^{-iqx}\right) \tag{1210}$$

$$= \int \widetilde{d^3k}\widetilde{d^3q}\left(a_k^\dagger a_q e^{iqx}e^{-ikx} + b_k a_q e^{iqx}e^{ikx} + a_k^\dagger b_q^\dagger e^{-iqx}e^{-ikx} + b_k b_q^\dagger e^{-iqx}e^{ikx}\right) \tag{1211}$$

$$= \int \widetilde{d^3k}\widetilde{d^3q}\left(a_k^\dagger a_q e^{i(q-k)x} + a_q b_k e^{i(q+k)x} + a_k^\dagger b_q^\dagger e^{-i(q+k)x} + [b_q^\dagger b_k - 2\omega_k(2\pi)^3\delta^3(\vec{k}-\vec{q})]e^{-i(q-k)x}\right) \tag{1212}$$

then

$$H_{a^\dagger a} = \int \widetilde{d^3k}\widetilde{d^3q}\int d^3x\left[(-\omega_k\omega_q)[a_q^\dagger a_k - 2\omega_k(2\pi)^3\delta^3(\vec{k}-\vec{q})]e^{-i(q-k)x}\right] \tag{1213}$$

$$+\int \widetilde{d^3k}\widetilde{d^3q}\int d^3x(k^a q_a)\left[-a_k^\dagger a_q e^{i(q-k)x}\right] + m^2 a_k^\dagger a_q e^{i(q-k)x} \tag{1214}$$

$$= \int \widetilde{d^3k}\widetilde{d^3q}\; a_k^\dagger a_q\left[-\omega_k\omega_q - k^a q_a + m^2\right]\int d^3x e^{i(q-k)x} \tag{1215}$$

$$-\int \widetilde{d^3k}\widetilde{d^3q}\;(-\omega_k\omega_q)2\omega_q(2\pi)^3\delta^3(\vec{q}-\vec{k})\int d^3x\; e^{i(q-k)x} \tag{1216}$$

$$= \int \widetilde{d^3k}\frac{d^3q}{(2\pi)^3 2\omega_q}\; a_k^\dagger a_q\left[-\omega_k\omega_q - k^a q_a + m^2\right](2\pi)^3\delta^3(\vec{q}-\vec{k})e^{-i(\omega_q-\omega_k)t} \tag{1217}$$

$$-\int \frac{d^3k}{(2\pi)^3 2\omega_k}\frac{1}{(2\pi)^3 2\omega_k}\;(-\omega_k^2)2\omega_k(2\pi)^3 e^{-i(\omega_k-\omega_k)t}\int d^3x \tag{1218}$$

$$= \int \widetilde{d^3k}\frac{1}{2\omega_k}\; a_k^\dagger a_k \underbrace{\left[-\omega_k^2 - \vec{k}^2 + m^2\right]}_{2\omega_k^2 !?!?!} + \frac{V}{2(2\pi)^3}\int d^3k\;\omega_k \tag{1219}$$

$$= \int \widetilde{d^3k}\omega_k\; a_k^\dagger a_k + \frac{V}{2(2\pi)^3}\int d^3k\;\omega_k \tag{1220}$$

and similar for $H_{b^\dagger b}, H_{ab}, H_{a^\dagger b^\dagger}$.

$$H = \int \widetilde{d^3k}\omega_k\;(a_k^\dagger a_k + b_k^\dagger b_k) + \frac{V}{2(2\pi)^3}\int d^3k\;\omega_k \tag{1221}$$

0.14.12 Problem 4.1 - Commutator non-hermitian field

With $t = t'$ and $|\vec{x} - \vec{x}'| = r$ we have

$$[\varphi^+(x), \varphi^-(x')]_\pm = \int \widetilde{dk} e^{ik(x-x')} \tag{1222}$$

$$= \int d^3k \frac{1}{(2\pi)^3 2\omega_k} e^{ik(x-x')} \tag{1223}$$

$$= \frac{1}{2 \cdot 8\pi^3} \int d^3k \frac{1}{\sqrt{|k|^2 + m^2}} e^{i[\vec{k}(\vec{x}-\vec{x}')]} \tag{1224}$$

$$= \frac{1}{16\pi^3} \int |k|^2 dk d\phi d\theta \sin\theta \frac{1}{\sqrt{|k|^2 + m^2}} e^{i|k|r\cos\theta} \tag{1225}$$

$$= \frac{2\pi}{16\pi^3} \int |k|^2 dk \underbrace{d\theta \sin\theta}_{-d\cos\theta} \frac{1}{\sqrt{|k|^2 + m^2}} e^{i|k|r\cos\theta} \tag{1226}$$

$$= \frac{2\pi}{16\pi^3} \int |k|^2 dk \frac{1}{\sqrt{|k|^2 + m^2}} \int_{-1}^{1} d\cos\theta e^{i|k|r\cos\theta} \tag{1227}$$

$$= \frac{2\pi}{16\pi^3} \int |k|^2 dk \frac{1}{\sqrt{|k|^2 + m^2}} 2 \frac{\sin(|k|r)}{|k|r} \tag{1228}$$

$$= \frac{1}{4\pi^2 r} \int_0^\infty dk \frac{|k| \sin(|k|r)}{\sqrt{|k|^2 + m^2}} \tag{1229}$$

With Gradshteyn, Ryzhik 7ed (8.486) - we find for the definition of the modified Bessel function K_1

$$\frac{d}{dz} K_0(z) = -K_1(z) \tag{1230}$$

and Gradshteyn, Ryzhik 7ed (3.754)

$$\int_0^\infty dx \frac{\cos(ax)}{\sqrt{\beta^2 + x^2}} = K_0(a\beta) \tag{1231}$$

therefore

$$\frac{d}{da} K_0(a\beta) = \int_0^\infty dx \frac{-x \sin(ax)}{\sqrt{\beta^2 + x^2}} \tag{1232}$$

$$= \beta K_0'(a\beta) \tag{1233}$$

$$= -\beta K_1(a\beta) \tag{1234}$$

$$\rightarrow K_1(a\beta) = \frac{1}{\beta} \int_0^\infty dx \frac{x \sin(ax)}{\sqrt{\beta^2 + x^2}} \tag{1235}$$

which we can use to finish the calculation

$$[\varphi^+(x), \varphi^-(x')]_\pm = \frac{1}{4\pi^2 r} m K_1(mr) \tag{1236}$$

From `https://dlmf.nist.gov/10.30` we get

$$\lim_{z\to 0} K_\nu(z) \sim \frac{1}{2}\Gamma(\nu) \left(\frac{1}{2}z\right)^{-\nu} \tag{1237}$$

$$\rightarrow \lim_{z\to 0} K_1(z) \sim \frac{1}{2} \left(\frac{1}{2}z\right)^{-1} = 1/z \tag{1238}$$

and therefore

$$[\varphi^+(x), \varphi^-(x')]_\pm = \frac{1}{4\pi^2 r^2}. \tag{1239}$$

0.14.13 Problem 5.1 - LSZ reduction for complex scalar field

From Exercise 3.5 we have

$$a_q = i\int d^3x e^{-iqx} \overleftrightarrow{\partial_0}\, \varphi(\vec{x},t) \tag{1240}$$

$$a_q^\dagger = -i\int d^3x e^{iqx} \overleftrightarrow{\partial_0}\, \varphi^\dagger(\vec{x},t) \tag{1241}$$

$$b_q = i\int d^3x e^{-iqx} \overleftrightarrow{\partial_0}\, \varphi^\dagger(\vec{x},t) \tag{1242}$$

$$b_q^\dagger = -i\int d^3x e^{iqx} \overleftrightarrow{\partial_0}\, \varphi(\vec{x},t) \tag{1243}$$

then

$$a_1^\dagger(+\infty) - a_1^\dagger(-\infty) = -i\int d^3k f_1(\vec{k})\int d^4x e^{ikx}(-\Box_x + m^2)\varphi^\dagger(x) \tag{1244}$$

rearranging leads to

$$a_1^\dagger(-\infty) = a_1^\dagger(+\infty) + i\int d^3k f_1(\vec{k})\int d^4x e^{ikx}(-\Box_x + m^2)\varphi^\dagger(x) \tag{1245}$$

$$a_1(+\infty) = a_1(-\infty) + i\int d^3k f_1(\vec{k})\int d^4x e^{-ikx}(-\Box_x + m^2)\varphi(x) \tag{1246}$$

$$b_1^\dagger(-\infty) = b_1^\dagger(+\infty) + i\int d^3k f_1(\vec{k})\int d^4x e^{ikx}(-\Box_x + m^2)\varphi^\dagger(x) \tag{1247}$$

$$b_1(+\infty) = b_1(-\infty) + i\int d^3k f_1(\vec{k})\int d^4x e^{-ikx}(-\Box_x + m^2)\varphi(x) \tag{1248}$$

then we get for a, b particle scattering with the time ordering operator T (Later time to the Left)

$$\begin{aligned}
\langle f|i\rangle &= \langle 0|a_{1'}(+\infty)b_{2'}(+\infty)a_1^\dagger(-\infty)b_2^\dagger(-\infty)|0\rangle && (1249)\\
&= \langle 0|Ta_{1'}(+\infty)b_{2'}(+\infty)a_1^\dagger(-\infty)b_2^\dagger(-\infty)|0\rangle && (1250)\\
&= \langle 0|T(a_{1'}(-\infty) + i\int)(b_{2'}(-\infty) + i\int)(a_1^\dagger(+\infty) + i\int)(b_2^\dagger(+\infty) + i\int)|0\rangle && (1251)\\
&= i^4\int d^4x_1' e^{-ik_1'x_1'}(-\Box_{x_1'} + m_a^2)\int d^4x_2' e^{-ik_2'x_2'}(-\Box_{x_2'} + m_b^2)\times && (1252)\\
&\quad \times\int d^4x_1 e^{-ik_1x_1}(-\Box_{x_1} + m_a^2)\int d^4x_2 e^{-ik_2x_2}(-\Box_{x_2} + m_b^2)\langle 0|\phi_{x_1'}\phi_{x_2'}\phi_{x_1}^\dagger\phi_{x_2}^\dagger|0\rangle && (1253)
\end{aligned}$$

0.14.14 Problem 6.1 - Path integral in quantum mechanics

(a) The transition amplitude $\langle q''|e^{-iH(t''-t')}|q'\rangle$ (particle to start at q', t' and ends at position q'' at time t'') can be written in the Heisenberg picture as

$$\begin{aligned}
\langle q''|e^{-iH(t''-t')}|q'\rangle &= \langle q''|e^{-iHt''}e^{iHt''}e^{-iH(t''-t')}e^{-iHt'}e^{iHt'}|q'\rangle && (1254)\\
&= \langle q'', t''|e^{iHt''}e^{iH(t''-t')}e^{-iHt'}|q', t'\rangle && (1255)\\
&= \langle q'', t''|q', t'\rangle. && (1256)
\end{aligned}$$

Now we can do the standard path integral derivation

$$\langle q'', t''|q', t'\rangle = \int \left(\prod_{j=1}^{N} dq_j\right) \langle q''|e^{-iH\delta t}|q_N\rangle\langle q_N|e^{-iH\delta t}|q_{N-1}\rangle \dots \langle q_1|e^{-iH\delta t}|q'\rangle \tag{1257}$$

$$= \int \left(\prod_{j=1}^{N} dq_j\right) \int \frac{dp_N}{2\pi} e^{-iH(p_N,q_N)\delta t} e^{ip_N(q'-q_N)} \dots \int \frac{dp'}{2\pi} e^{-iH(p',q')\delta t} e^{ip'(q_1-q')} \tag{1258}$$

$$= \int \left(\prod_{j=1}^{N} dq_j\right) \left(\prod_{k=0}^{N} \frac{dp_k}{2\pi} e^{ip_k(q_{k+1}-q_k)} e^{-iH(p_k,q_k)\delta t}\right) \quad (q_0 = q', q_{N+1} = q'') \tag{1259}$$

which under Weyl ordering (see Greiner, Reinhard - field quantization) has to be replaced by

$$\langle q'', t''|q', t'\rangle = \int \left(\prod_{j=1}^{N} dq_j\right) \left(\prod_{k=0}^{N} \frac{dp_k}{2\pi} e^{ip_k(q_{k+1}-q_k)} e^{-iH(p_k,\bar{q}_k)\delta t}\right) \quad \bar{q}_k = (q_{k+1} + q_k)/2 \tag{1260}$$

$$= \int \left(\prod_{j=1}^{N} dq_j\right) \left(\prod_{k=0}^{N} \frac{dp_k}{2\pi} e^{i[p_k\dot{q}_k - H(p_k,\bar{q}_k)]\delta t}\right) \quad \dot{q}_k = (q_{k+1} - q_k)/\delta t \tag{1261}$$

$$= \int \left(\prod_{j=1}^{N} dq_j\right) \left(\prod_{k=0}^{N} \frac{dp_k}{2\pi}\right) \left(e^{i\sum_{n=0}^{N}[p_n\dot{q}_n - H(p_n,\bar{q}_n)]\delta t}\right) \tag{1262}$$

$$= \int \mathcal{D}q\mathcal{D}p \exp\left[i\int_{t'}^{t''} dt\,(p(t)\dot{q}(t) - H(p(t), q(t)))\right] \tag{1263}$$

Let's now assume $H(p, q)$ has only a quadratic term in p which is independent of q meaning

$$H(p, q) = \frac{p^2}{2m} + V(q) \tag{1264}$$

then

$$\langle q'', t''|q', t'\rangle = \int \left(\prod_{j=1}^{N} dq_j\right) \left(\prod_{k=0}^{N} \frac{dp_k}{2\pi}\right) \left(e^{i\sum_{n=0}^{N}[p_n\dot{q}_n - \frac{1}{2m}p_n^2 - V(\bar{q}_n)]\delta t}\right) \tag{1265}$$

We can evaluate a single p-integral using

$$\int_{-\infty}^{\infty} dx e^{-ax^2+bx+c} = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}+c} \tag{1266}$$

and obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dp_k \left(e^{i[p_k\dot{q}_k - \frac{1}{2m}p_k^2 - V(\bar{q}_k)]\delta t}\right) = \frac{1}{2\pi} e^{-iV(\bar{q}_k)\delta t} \int dp_k \left(e^{i[p_k\dot{q}_k - \frac{1}{2m}p_k^2]\delta t}\right) \tag{1267}$$

$$= \frac{1}{2\pi} e^{-iV(\bar{q}_k)\delta t} \sqrt{\frac{\pi}{i\frac{\delta t}{2m}}} e^{\frac{-\dot{q}_k^2\delta t^2}{4\frac{i\delta t}{2m}}} \tag{1268}$$

$$= \frac{1}{2\pi}\sqrt{\frac{2\pi m}{i\delta t}} e^{i\left(\frac{m\dot{q}_k^2}{2} - V(\bar{q}_k)\right)\delta t} \tag{1269}$$

$$= \sqrt{\frac{m}{2\pi i\delta t}} e^{iL(\bar{q}_k,\dot{q}_k)\delta t}. \tag{1270}$$

As there are $N+1$ p-integrals we have

$$\mathcal{D}q = \left(\frac{m}{2\pi i\delta t}\right)^{(N+1)/2} \prod_{j=1}^{N} dq_j \tag{1271}$$

(b) We now assume $V(q) = 0$

$$\langle q'', t''|q', t'\rangle = \int \mathcal{D}q\, e^{i\int_{t'}^{t''} dt \frac{\dot q^2}{2m}} \tag{1272}$$

$$= \lim_{N\to\infty} \left(\frac{m}{2\pi i\delta t}\right)^{\frac{N+1}{2}} \left(\prod_{j=1}^{N}\int_{-\infty}^{\infty} dq_j\, e^{im\frac{(q_j-q_{j+1})^2}{2\delta t^2}\delta t}\right) e^{im\frac{(q'-q_1)^2}{2\delta t}} e^{im\frac{(q_N-q'')^2}{2\delta t}} \tag{1273}$$

$$= \lim_{N\to\infty} \left(\frac{m}{2\pi i\delta t}\right)^{\frac{N+1}{2}} \left(\prod_{j=3}^{N}\int_{-\infty}^{\infty} dq_j\, e^{im\frac{(q_j-q_{j+1})^2}{2\delta t}}\right) \int dq_2 e^{im\frac{(q_2-q_3)^2}{2\delta t}} \int dq_1 e^{im\frac{(q_1-q_2)^2}{2\delta t}} e^{im\frac{(q_0-q_1)^2}{2\delta t}} \tag{1274}$$

now we can simplify the q_1-integral

$$\int_{-\infty}^{\infty} dq_1\, e^{im\frac{(q_1-q_2)^2}{2\delta t}} e^{im\frac{(q_0-q_1)^2}{2\delta t}} = \int_{-\infty}^{\infty} dq_1\, e^{\frac{im}{2\delta t}(q_0^2-2q_0q_1+q_1^2+q_1^2-2q_1q_2+q_2^2)} \tag{1275}$$

$$= e^{\frac{im}{2\delta t}(q_0^2+q_2^2)} \int_{-\infty}^{\infty} dq_1\, e^{\frac{im}{\delta t}(q_1^2-q_1(q_2+q_0))} \tag{1276}$$

$$= e^{\frac{im}{2\delta t}(q_0^2+q_2^2)} \sqrt{\frac{\pi\delta t}{m}} e^{\frac{i}{4}\left(\pi-\frac{(q_2+q_0)^2 m}{\delta t}\right)} \tag{1277}$$

$$= e^{\frac{im}{4\delta t}(q_0-q_2)^2} \sqrt{\frac{\pi\delta t}{m}}\sqrt{i} \tag{1278}$$

$$= e^{\frac{im}{4\delta t}(q_0-q_2)^2} \sqrt{\frac{i\pi\delta t}{m}} \tag{1279}$$

now simplify the q_2-integral

$$\sqrt{\frac{i\pi\delta t}{m}} \int_{-\infty}^{\infty} dq_2 e^{\frac{im}{2\delta t}(q_2-q_3)^2} e^{\frac{im}{4\delta t}(q_0-q_2)^2} = \sqrt{\frac{i\pi\delta t}{m}} \int_{-\infty}^{\infty} dq_2 e^{\frac{im}{4\delta t}(2q_2^2-4q_3q_2+2q_3^2+q_0^2-2q_0q_2+q_2^2)} \tag{1280}$$

$$= \sqrt{\frac{i\pi\delta t}{m}} \int_{-\infty}^{\infty} dq_2 e^{\frac{im}{4\delta t}(3q_2^2-(4q_3+2q_0)q_2+2q_3^2+q_0^2)} \tag{1281}$$

$$= \sqrt{\frac{i\pi\delta t}{m}} e^{\frac{im}{4\delta t}(2q_3^2+q_0^2)} \int_{-\infty}^{\infty} dq_2 e^{\frac{im}{4\delta t}(3q_2^2-(4q_3+2q_0)q_2)} \tag{1282}$$

$$= \sqrt{\frac{i\pi\delta t}{m}} e^{\frac{im}{4\delta t}(2q_3^2+q_0^2)} \sqrt{\frac{\pi 4\delta t}{3m}} e^{\frac{i}{4}\left(\pi-\frac{(4q_3+2q_0)^2 m}{12\delta t}\right)} \tag{1283}$$

$$= \sqrt{\frac{i\pi\delta t}{m}} \sqrt{\frac{4i\pi\delta t}{3m}} e^{\frac{im}{6\delta t}(q_3-q_0)^2} \tag{1284}$$

then we can extend the results (without explicitly proving)

$$\langle q'', t''|q', t'\rangle = \lim_{N\to\infty} \left(\frac{m}{2\pi i\delta t}\right)^{\frac{N+1}{2}} \prod_{j=1}^{N} \sqrt{\frac{2i\pi\delta t}{m}\frac{j}{j+1}} \cdot e^{\frac{im}{2(j+1)\delta t}(q''-q')^2} \tag{1285}$$

$$= \lim_{N\to\infty} \sqrt{\frac{m}{2\pi i\delta t}} \sqrt{\frac{1}{N+1}} \cdot e^{\frac{im}{2(N+1)\delta t}(q_{N+1}-q_0)^2} \tag{1286}$$

$$= \sqrt{\frac{m}{2\pi i(t''-t')}} \cdot e^{\frac{im(q''-q')^2}{2(t''-t')}}. \tag{1287}$$

The exponent has the dimension $\text{kg} \cdot \text{m}^2/s$ which is the same as Js. So we just insert an $\hbar$

$$\langle q'', t''|q', t'\rangle = \sqrt{\frac{m}{2\pi i\hbar(t''-t')}} \cdot e^{\frac{im(q''-q')^2}{2\hbar(t''-t')}}. \tag{1288}$$

(c) Simple - with $H|k\rangle = \frac{k^2}{2m}|k\rangle$ we get

$$\langle q'', t''|q', t'\rangle = \langle q''|\exp(-iH(t''-t'))|q'\rangle \tag{1289}$$
$$= \int dp \int dk \langle q''|p\rangle\langle p|\exp(-iH(t''-t'))|k\rangle\langle k|q'\rangle \tag{1290}$$
$$= \int dp \int dk \frac{1}{\sqrt{2\pi}} e^{ipq'} \langle p|k\rangle \exp(-i\frac{k^2}{2m}(t''-t'))\frac{1}{\sqrt{2\pi}}e^{-ikq''} \tag{1291}$$
$$= \int dp \int dk \frac{1}{\sqrt{2\pi}} e^{ipq'} \exp(-i\frac{k^2}{2m}(t''-t'))\delta(k-p)\frac{1}{\sqrt{2\pi}}e^{-ikq''} \tag{1292}$$
$$= \frac{1}{2\pi}\int dp e^{ip(q'-q'')} \exp(-i\frac{p^2}{2m}(t''-t')) \tag{1293}$$
$$= \frac{1}{2\pi}\sqrt{-\frac{2m\pi}{t''-t'}} e^{\frac{i}{4}\left(\pi - \frac{-2m(q''-q')^2}{t''-t'}\right)} \tag{1294}$$
$$= \sqrt{-\frac{im}{2\pi(t''-t')}} e^{-\frac{i}{4}\frac{-2m(q''-q')^2}{t''-t'}} \tag{1295}$$
$$= \sqrt{\frac{m}{2\pi i(t''-t')}} e^{\frac{-im(q''-q')^2}{2(t''-t')}} \tag{1296}$$

which is the same as in (b).

0.14.15 Problem 7.1 - Oscillator Green's function I

$$G(t-t') = \int_{-\infty}^{+\infty} \frac{dE}{2\pi} \frac{e^{-iE(t-t')}}{-E^2+\omega^2-i\epsilon} \tag{1297}$$
$$= -\frac{1}{2\pi}\int_{-\infty}^{+\infty} dE \frac{e^{-iE(t-t')}}{E^2-\omega^2+i\epsilon} \tag{1298}$$

with

$$E^2-\omega^2+i\epsilon = (E+\sqrt{\omega^2-i\epsilon})(E-\sqrt{\omega^2-i\epsilon}) \tag{1299}$$
$$= \left(E+\omega\sqrt{1-\frac{i\epsilon}{\omega^2}}\right)\left(E-\omega\sqrt{1-\frac{i\epsilon}{\omega^2}}\right) \tag{1300}$$
$$\simeq \left(E+\omega-\frac{i\epsilon}{2\omega}\right)\left(E-\omega+\frac{i\epsilon}{2\omega^2}\right) \tag{1301}$$

we can simplify

$$G(\Delta t) = -\frac{1}{2\pi}\int_{-\infty}^{+\infty} dE e^{-iE\Delta t}\left(\frac{1}{E+\omega-\frac{i\epsilon}{2\omega}} + \frac{1}{E-\omega+\frac{i\epsilon}{2\omega}}\right) \tag{1302}$$
$$= -\frac{1}{2\pi}\frac{1}{2\left(\omega-\frac{i\epsilon}{2\omega}\right)}\int_{-\infty}^{+\infty} dE e^{-iE\Delta t}\left(-\frac{1}{E+\omega-\frac{i\epsilon}{2\omega}} + \frac{1}{E-\omega+\frac{i\epsilon}{2\omega}}\right) \tag{1303}$$

Integrating along the closed contour along the lower half plane (seeing that the exponential function makes the arc part vanish - for $\Delta t > 0$) and using the residual theorem (only one pole is inside) we get (with $\epsilon \to 0$)

$$G(\Delta t) = +\frac{1}{2\pi}\frac{1}{2\left(\omega - \frac{i\epsilon}{2\omega}\right)}(2\pi i)e^{-i(\omega - \frac{i\epsilon}{2\omega})\Delta t} \tag{1304}$$

$$= \frac{i}{2\omega}e^{-i\omega\Delta t} \tag{1305}$$

For $\Delta t < 0$ we integrate along the contour of the upper plane - combining both results we get

$$G(t) = \frac{i}{2\omega}e^{-i\omega|t|} \tag{1306}$$

0.14.16 Problem 7.2 - Oscillator Green's function II

We can rewrite the Greens function using the Heaviside theta function

$$|t| = (2\theta(t) - 1)t \tag{1307}$$

$$\frac{d}{dt}|t| = 2\theta'(t)t + (2\theta(t) - 1) \tag{1308}$$

$$= 2\underbrace{\delta(t)t}_{=0} + 2\theta(t) - 1 \tag{1309}$$

$$= 2\theta(t) - 1 \tag{1310}$$

and then differentiate and use $\theta'(t) = \delta(t)$

$$G(t) = \frac{i}{2\omega}e^{-i\omega(2\theta(t)-1)t} \tag{1311}$$

$$\partial_t G(t) = \frac{i}{2\omega}e^{-i\omega(2\theta(t)-1)t}(-i\omega)(2\theta(t) - 1)) \tag{1312}$$

$$= (-i\omega)G(t)\,(2\theta(t) - 1) \tag{1313}$$

$$\partial_{tt}G(t) = (-i\omega)\partial_t G(t)\,(2\theta(t) - 1) + (-2i\omega)G(t)\delta(t) \tag{1314}$$

$$= (-i\omega)^2 G(t)\,(2\theta(t) - 1)^2 + (-2i\omega)G(t)\delta(t) \tag{1315}$$

$$= -\omega^2 G(t) + e^{-i\omega|t|}\delta(t) \tag{1316}$$

where we used $(2\theta(t) - 1)^2 \equiv 1$

$$\left(\partial_{tt} + \omega^2\right)G(t) = \left(-\omega^2 + \omega^2\right)G(t) + \delta(t) = \delta(t) \tag{1317}$$

0.14.17 Problem 7.3 - Harmonic Oscillator - Heisenberg and Schroedinger picture

(a) With $\hbar = 1$ and

$$H = \frac{1}{2}P^2 + \frac{1}{2}m\omega^2 Q^2 \tag{1318}$$

$$[Q, P] = QP - PQ = i \tag{1319}$$

$$[Q, Q] = [P, P] = 0 \tag{1320}$$

we obtain for the commutators

$$
\begin{aligned}
[P^2, Q] &= P(PQ) - QP^2 && (1321)\\
&= P(QP - i) - QP^2 && (1322)\\
&= (PQ)P - Pi - QP^2 && (1323)\\
&= (QP - i)P - Pi - QP^2 && (1324)\\
&= -2Pi && (1325)\\
[Q^2, P] &= Q(QP) - PQ^2 && (1326)\\
&= Q(PQ + i) - PQ^2 && (1327)\\
&= (QP)Q + iQ - PQ^2 && (1328)\\
&= (PQ + i)Q + iQ - PQ^2 && (1329)\\
&= 2Qi && (1330)
\end{aligned}
$$

Then the Heisenberg equations are

$$
\begin{aligned}
\dot{Q}(t) &= i[H, Q(t)] = i\frac{1}{2m}[P^2(t), Q(t)] = \frac{1}{m}P(t) && (1331)\\
\dot{P}(t) &= i[H, P(t)] = i\frac{1}{2}m\omega^2[Q^2(t), P(t)] = -m\omega^2 Q(t) && (1332)\\
\to \ddot{Q}(t) &= \frac{1}{m}\dot{P}(t) = -\omega^2 Q(t) && (1333)
\end{aligned}
$$

with the solutions (initial conditions $Q(0) = Q, P(0) = P$)

$$
\begin{aligned}
Q(t) &= A\cos\omega t + B\sin\omega t \qquad \to A = Q, \quad \omega B = \frac{1}{m}P && (1334)\\
&= Q\cos\omega t + \frac{1}{\omega m}P\sin\omega t && (1335)\\
P(t) &= m\dot{Q}(t) && (1336)\\
&= -m\omega Q\sin\omega t + P\cos\omega t && (1337)
\end{aligned}
$$

(b) Using Diracs trick from QM (rewriting H in terms of a and $a^\dagger$)

$$
\begin{aligned}
a &= \sqrt{\frac{m\omega}{2}}(Q + \frac{i}{m\omega}P) && (1338)\\
a^\dagger &= \sqrt{\frac{m\omega}{2}}(Q - \frac{i}{m\omega}P) && (1339)
\end{aligned}
$$

we can invert the relation

$$
\begin{aligned}
Q &= \frac{1}{\sqrt{2m\omega}}(a^\dagger + a) && (1340)\\
P &= i\sqrt{\frac{m\omega}{2}}(a^\dagger - a) && (1341)
\end{aligned}
$$

and

$$Q(t) = Q\cos\omega t + \frac{1}{\omega m}P\sin\omega t \tag{1342}$$
$$= \frac{1}{\sqrt{2m\omega}}(a^\dagger + a)\cos\omega t + \frac{1}{\omega m}i\sqrt{\frac{m\omega}{2}}(a^\dagger - a)\sin\omega t \tag{1343}$$
$$= \frac{1}{\sqrt{2m\omega}}\left((a^\dagger + a)\cos\omega t + i(a^\dagger - a)\sin\omega t\right) \tag{1344}$$
$$= \frac{1}{\sqrt{2m\omega}}\left(a^\dagger(\cos\omega t + i\sin\omega t) + a(\cos\omega t - i\sin\omega t)\right) \tag{1345}$$
$$= \frac{1}{\sqrt{2m\omega}}\left(a^\dagger e^{i\omega t} + ae^{-i\omega t}\right) \tag{1346}$$
$$P(t) = i\sqrt{\frac{m\omega}{2}}\left(a^\dagger e^{i\omega t} - ae^{-i\omega t}\right) \tag{1347}$$
$$\tag{1348}$$

(c) Now with $t_1 < t_2$ and the time ordering operator (larger time to the left)

$$\langle 0|TQ(t_1)Q(t_2)|0\rangle = \frac{1}{2m\omega}\langle 0|T\left(a^\dagger e^{i\omega t_1} + ae^{-i\omega t_1}\right)\left(a^\dagger e^{i\omega t_2} + ae^{-i\omega t_2}\right)|0\rangle \tag{1349}$$
$$= \frac{1}{2m\omega}\langle 0|\left(a^\dagger e^{i\omega t_2} + ae^{-i\omega t_2}\right)\left(a^\dagger e^{i\omega t_1} + ae^{-i\omega t_1}\right)|0\rangle \tag{1350}$$
$$= \frac{1}{2m\omega}\langle 0|ae^{-i\omega t_2}a^\dagger e^{i\omega t_1}|0\rangle \tag{1351}$$

all other terms are vanishing because of $a|0\rangle = 0$ and $\langle 0|a^\dagger = 0$. Then

$$\langle 0|TQ(t_1)Q(t_2)|0\rangle = \frac{1}{2m\omega}e^{-i\omega(t_2-t_1)}\underbrace{\langle 0|aa^\dagger|0\rangle}_{=1} \tag{1352}$$
$$= \frac{1}{2m\omega}e^{-i\omega(t_2-t_1)} \tag{1353}$$
$$\equiv \frac{1}{i}G(t_2 - t_1) \tag{1354}$$

And now the next case with $t_1 > t_2 > t_3 > t_4$

$$\langle 0|TQ(t_1)Q(t_2)Q(t_3)Q(t_4)|0\rangle = \frac{1}{(2m\omega)^2}\cdots \tag{1355}$$

0.14.18 Problem 7.4 - Harmonic Oscillator with perturbation

As $f(t)$ is a real function we have $\tilde{f}(-E) = (\tilde{f}(E))^*$ then with (7.10)

$$\langle 0|0\rangle_f = \exp\left[\frac{i}{2}\int_{-\infty}^{+\infty}\frac{dE}{2\pi}\frac{\tilde{f}(E)\tilde{f}(-E)}{-E^2 + \omega^2 - i\epsilon}\right] \tag{1356}$$
$$= \exp\left[\frac{i}{2}\int_{-\infty}^{+\infty}\frac{dE}{2\pi}\frac{\tilde{f}(E)\tilde{f}(E)^*}{-E^2 + \omega^2 - i\epsilon}\right] \tag{1357}$$

But we actually need to calculate $|\langle 0|0\rangle_f|^2$ therefore we observe with

$$e^{iz} = e^{i(x+iy)} = e^{-y}e^{ix} = e^{-y}(\cos x + i\sin x) \tag{1358}$$
$$\rightarrow \quad (e^{iz})^* = e^{-y}(\cos x - i\sin x) = e^{-y-ix}e^{-i(x-iy)} = e^{-iz^*} \tag{1359}$$
$$\langle 0|0\rangle_f = e^{iA} \quad \rightarrow \quad |\langle 0|0\rangle_f|^2 = e^{iA}(e^{iA})^* = e^{iA}e^{-iA^*} = e^{i(A-A^*)} = e^{-2\Im A} \tag{1360}$$

Now we calculate the imaginary part of the integral

$$\Im\frac{1}{4\pi}\int_{-\infty}^{+\infty}dE\frac{\tilde{f}(E)\tilde{f}(E)^*}{-E^2+\omega^2-i\epsilon}=\frac{1}{4\pi}\int_{-\infty}^{+\infty}dE\Im\frac{\tilde{f}(E)\tilde{f}(E)^*}{-E^2+\omega^2-i\epsilon} \tag{1361}$$

$$=\frac{1}{4\pi}\int_{-\infty}^{+\infty}dE\tilde{f}(E)\tilde{f}(E)^*\Im\frac{1}{-E^2+\omega^2-i\epsilon} \tag{1362}$$

$$=\frac{1}{4\pi}\int_{-\infty}^{+\infty}dE\tilde{f}(E)\tilde{f}(E)^*\Im\frac{-E^2+\omega^2+i\epsilon}{(-E^2+\omega^2)^2+\epsilon^2} \tag{1363}$$

$$=\frac{1}{4\pi}\int_{-\infty}^{+\infty}dE\tilde{f}(E)\tilde{f}(E)^*\frac{\epsilon}{(-E^2+\omega^2)^2+\epsilon^2} \tag{1364}$$

$$\simeq\frac{1}{4\pi}\int_{-\infty}^{+\infty}dE\tilde{f}(E)\tilde{f}(E)^*\pi\delta(-E^2+\omega^2) \tag{1365}$$

$$\simeq\frac{1}{4\pi}\int_{-\infty}^{+\infty}dE\tilde{f}(E)\tilde{f}(E)^*\pi\delta((\omega+E)(\omega-E)) \tag{1366}$$

$$\simeq\frac{1}{4\cdot 2\omega}(\tilde{f}(\omega)\tilde{f}(\omega)^*+\tilde{f}(-\omega)\tilde{f}(-\omega)^*) \tag{1367}$$

$$\simeq\frac{1}{8\omega}(\tilde{f}(\omega)\tilde{f}(\omega)^*+\tilde{f}(\omega)^*\tilde{f}(\omega)) \tag{1368}$$

$$\simeq\frac{1}{4\omega}\tilde{f}(\omega)\tilde{f}(\omega)^* \tag{1369}$$

then

$$|\langle 0|0\rangle_f|^2=e^{-2\left(\frac{1}{4\omega}\right)\tilde{f}(\omega)\tilde{f}(\omega)^*} \tag{1370}$$

$$=e^{-\frac{1}{2\omega}\tilde{f}(\omega)\tilde{f}(\omega)^*} \tag{1371}$$

$$\tag{1372}$$

0.14.19 Problem 8.1 - Feynman propagator is Greens function Klein-Gordon equation

With

$$\Delta(x-x')=\frac{1}{(2\pi)^4}\int d^4k\frac{e^{ik(x-x')}}{k^2+m^2-i\epsilon} \tag{1373}$$

we have

$$(-\partial_x^2+m^2)\Delta(x-x')=\frac{1}{(2\pi)^4}\int d^4k(-i^2k^2+m^2)\frac{e^{ik(x-x')}}{k^2+m^2-i\epsilon} \tag{1374}$$

$$=\frac{1}{(2\pi)^4}\int d^4k\frac{k^2+m^2}{k^2+m^2-i\epsilon}e^{ik(x-x')} \tag{1375}$$

$$\simeq\frac{1}{(2\pi)^4}\int d^4ke^{ik(x-x')} \tag{1376}$$

$$=\delta^4(x-x') \tag{1377}$$

0.14.20 Problem 8.2 - Feynman propagator II

With $\widetilde{dk} = d^3k/((2\pi)^3 2\omega_k)$ and $\omega_k = \sqrt{\vec{k}^2 + m^2}$

$$\Delta(x - x') = \frac{1}{(2\pi)^4} \int d^4k \frac{e^{ik(x-x')}}{k^2 + m^2 - i\epsilon} \tag{1378}$$
$$= \frac{1}{(2\pi)^4} \int d^3k \int dk^0 e^{-ik^0(t-t')} \frac{e^{i\vec{k}(\vec{x}-\vec{x}')}}{-(k^0)^2 + \vec{k}^2 + m^2 - i\epsilon} \tag{1379}$$
$$= \frac{1}{(2\pi)^4} \int d^3k \, e^{i\vec{k}(\vec{x}-\vec{x}')} \int dE \frac{e^{-iE(t-t')}}{-E^2 + \vec{k}^2 + m^2 - i\epsilon} \tag{1380}$$
$$= \frac{1}{(2\pi)^4} \int d^3k \, e^{i\vec{k}(\vec{x}-\vec{x}')} 2\pi \frac{i}{2(\vec{k}^2 + m^2)} e^{-i(\vec{k}^2+m^2)|t-t'|} \tag{1381}$$

where we used exercise (7.1). Then

$$\Delta(x - x') = \frac{i}{(2\pi)^3} \int d^3k \, e^{i\vec{k}(\vec{x}-\vec{x}')} \frac{i}{2\omega_k} e^{-i\omega_k|t-t'|} \tag{1382}$$
$$= i \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{i\vec{k}(\vec{x}-\vec{x}')} e^{-i\omega_k|t-t'|} \tag{1383}$$
$$= i \int \widetilde{dk} \, e^{i\vec{k}(\vec{x}-\vec{x}')-i\omega_k|t-t'|} \tag{1384}$$
$$= i\theta(t - t') \int \widetilde{dk} \, e^{i\vec{k}(\vec{x}-\vec{x}')-i\omega_k(t-t')} + i\theta(t' - t) \int \widetilde{dk} \, e^{i\vec{k}(\vec{x}-\vec{x}')+i\omega_k(t-t')} \tag{1385}$$
$$= i\theta(t - t') \int \widetilde{dk} \, e^{ik(x-x')} + i\theta(t' - t) \int \widetilde{dk} \, e^{-i\vec{k}(\vec{x}-\vec{x}')+i\omega_k(t-t')} \tag{1386}$$
$$= i\theta(t - t') \int \widetilde{dk} \, e^{ik(x-x')} + i\theta(t' - t) \int \widetilde{dk} \, e^{-ik(x-x')} \tag{1387}$$
$$\tag{1388}$$

0.15 Coleman - Lectures of Sidney Coleman on quantum field theory

0.15.1 Problem 1.1 - Momentum space measure

Boost in z-direction

$$p_\mu = \Lambda^\nu{}_\mu p'_\nu \tag{1389}$$
$$\Lambda = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \tag{1390}$$

Combining everything using $dp_i \wedge dp_i = 0$

$$\rightarrow dp_x = dp'_x \tag{1391}$$
$$\rightarrow dp_y = dp'_y \tag{1392}$$
$$\rightarrow dp_z = -\gamma\beta\, dp'_0 + \gamma\, dp'_z \tag{1393}$$
$$= -\gamma\beta \left(\frac{\partial p'_0}{\partial p'_x} dp'_x + \frac{\partial p'_0}{\partial p'_y} dp'_y + \frac{\partial p'_0}{\partial p'_z} dp'_z \right) + \gamma\, dp'_z \tag{1394}$$
$$= -\gamma\beta \frac{1}{2\omega'_p} (2p'_x\, dp'_x + 2p'_y\, dp'_y + 2p'_z\, dp'_z) + \gamma\, dp'_z \tag{1395}$$
$$= -\gamma\beta \frac{p'_x\, dp'_x + p'_y\, dp'_y}{\omega'_p} + \gamma \left(1 - \frac{\beta}{\omega'_p} p'_z\right) dp'_z \tag{1396}$$

where we used $p'_0 = \omega'_p = \sqrt{m^2 - {p'_x}^2 - {p'_y}^2 - {p'_z}^2}$ and

$$\rightarrow \omega_p = p_0 \tag{1397}$$
$$= \gamma p'_0 - \gamma\beta\, p'_z \tag{1398}$$
$$= \gamma(\omega'_p - \beta\, p'_z) \tag{1399}$$

then

$$\frac{d^3p}{(2\pi)^3 2\omega_p} = \frac{dp_x\, dp_y\, dp_z}{(2\pi)^3 2\omega_p} \tag{1400}$$
$$= \frac{dp'_x\, dp'_y \gamma \left(1 - \frac{\beta}{\omega'_p} p'_z\right) dp'_z}{(2\pi)^3 2\gamma(\omega'_p - \beta\, p'_z)} \tag{1401}$$
$$= \frac{dp'_x\, dp'_y \gamma \left(1 - \frac{\beta}{\omega'_p} p'_z\right) dp'_z}{(2\pi)^3 2\omega'_p \gamma \left(1 - \frac{\beta}{\omega'_p}\, p'_z\right)} \tag{1402}$$
$$= \frac{dp'_x\, dp'_y\, dp'_z}{(2\pi)^3 2\omega'_p} \tag{1403}$$

0.16 Kachelriess - Quantum Fields - From the Hubble to the Planck scale

0.16.1 Problem 1.1 - Units

1. The fundamental constants are given by

$$k = 1.381 \cdot 10^{-23} \mathrm{m}^2 \mathrm{s}^{-2} \mathrm{kg}^1 \mathrm{K}^{-1} \tag{1404}$$
$$G = 6.674 \cdot 10^{-11} \mathrm{m}^3 \mathrm{s}^{-2} \mathrm{kg}^{-1} \tag{1405}$$
$$\hbar = 1.054 \cdot 10^{-34} \mathrm{m}^2 \mathrm{s}^{-1} \mathrm{kg}^1 \tag{1406}$$
$$c = 2.998 \cdot 10^{-8} \mathrm{m}^1 \mathrm{s}^{-1} \tag{1407}$$

A newly constructed Planck constant has the general form

$$X_P = c^{\alpha_c} \cdot G^{\alpha_G} \cdot \hbar^{\alpha_\hbar} \cdot k^{\alpha_k} \tag{1408}$$

and the dimension of X_P is given by $\mathrm{m}^{\beta_m} \mathrm{s}^{\beta_s} \mathrm{kg}^{\beta_{kg}} \mathrm{K}^{\beta_K}$ are determined by

$$\text{Meter} \quad \beta_m = 2\alpha_k + 3\alpha_G + 2\alpha_h + \alpha_c \tag{1409}$$
$$\text{Second} \quad \beta_s = -2\alpha_k - 2\alpha_G - \alpha_c - \alpha_h \tag{1410}$$
$$\text{Kilogram} \quad \beta_{kg} = \alpha_k - \alpha_G + \alpha_h \tag{1411}$$
$$\text{Kelvin} \quad \beta_K = -\alpha_k \tag{1412}$$

Solving the linear system gives

$$l_P = \sqrt{\frac{\hbar G}{c^3}} = 1.616 \cdot 10^{-35}\text{m} \tag{1413}$$
$$m_P = \sqrt{\frac{\hbar c}{G}} = 2.176 \cdot 10^{-8}\text{kg} \tag{1414}$$
$$t_P = \sqrt{\frac{\hbar G}{c^5}} = 5.391 \cdot 10^{-44}\text{s} \tag{1415}$$
$$T_P = \sqrt{\frac{\hbar c^5}{G k^2}} = 1.417 \cdot 10^{-32}\text{K} \tag{1416}$$
$$\tag{1417}$$

As the constants are made up from QM, SR and GR constants they indicate magnitudes at which a quantum theory of gravity is needed to make a sensible predictions.

2. We use the definition 1barn $= 10^{-28}\text{m}^2$

$$1\text{cm}^2 = 10^{-4}\text{m}^2 \tag{1418}$$
$$1\text{mbarn} = 10^{-31}\text{m}^2 \tag{1419}$$
$$= 10^{-27}\text{cm}^2 \tag{1420}$$

We also have $1\text{eV} = 1.602 \cdot 10^{-19}\text{As} \cdot 1\text{V} = 1.602 \cdot 10^{-19}\text{J}$

$$E = mc^2 \quad \to \quad 1\text{kg} \cdot c^2 = 8.987 \cdot 10^{16}\text{J} = 5.609 \cdot 10^{35}\text{eV} \tag{1421}$$
$$\to \quad 1\text{GeV} = 1.782 \cdot 10^{-27}\text{kg} \tag{1422}$$
$$E = \hbar\omega \quad \to \quad \frac{1}{1\text{s}} \cdot \hbar = 1.054 \cdot 10^{-34}\text{J} = 6.582 \cdot 10^{-16}\text{eV} \tag{1423}$$
$$\to \quad 1\text{GeV}^{-1} = 6.582 \cdot 10^{-25}\text{s} \tag{1424}$$
$$E = \frac{\hbar c}{\lambda} \quad \to \quad \frac{1}{1\text{m}} \cdot \hbar c = 3.161 \cdot 10^{-26}\text{J} = 1.973 \cdot 10^{-7}\text{eV} \tag{1425}$$
$$\to \quad 1\text{GeV}^{-1} = 1.973 \cdot 10^{-16}\text{m} \tag{1426}$$
$$E \sim pc \quad \to \quad 1\text{kgms}^{-1} \cdot c = 2.998 \cdot 10^{8}\text{J} = 1.871 \cdot 10^{27}\text{eV} \tag{1427}$$
$$\to \quad 1\text{GeV} = 5.344 \cdot 10^{-19}\text{kgms}^{-1} \tag{1428}$$

therefore

$$1\text{GeV}^{-2} = (1.973 \cdot 10^{-16}\text{m})^2 \tag{1429}$$
$$= 3.893 \cdot 10^{-32}\text{m}^2 \tag{1430}$$
$$= 0.389\text{mbarn} \tag{1431}$$

0.16.2 Problem 3.2 - Maxwell Lagrangian

1. First we observe that

$$F_{\mu\nu}F^{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \tag{1432}$$
$$= (\partial_\mu A_\nu)(\partial^\mu A^\nu) - (\partial_\mu A_\nu)(\partial^\nu A^\mu) - \underbrace{(\partial_\nu A_\mu)(\partial^\mu A^\nu)}_{=(\partial_\mu A_\nu)(\partial^\nu A^\mu)} + \underbrace{(\partial_\nu A_\mu)(\partial^\nu A^\mu)}_{=(\partial_\mu A_\nu)(\partial^\mu A^\nu)} \tag{1433}$$
$$= 2\left((\partial_\mu A_\nu)(\partial^\mu A^\nu) - (\partial_\mu A_\nu)(\partial^\nu A^\mu)\right) \tag{1434}$$
$$= 2(\partial_\mu A_\nu)F^{\mu\nu}. \tag{1435}$$

The variation is then given by

$$
\begin{aligned}
\delta\left(F_{\mu\nu}F^{\mu\nu}\right) &= 2\delta\left((\partial_\mu A_\nu)F^{\mu\nu}\right) && (1436)\\
&= 2\left[\delta\left(\partial_\mu A_\nu\right)F^{\mu\nu} + (\partial_\mu A_\nu)\delta F^{\mu\nu}\right] && (1437)\\
&= 2[\delta\left(\partial_\mu A_\nu\right)\underbrace{(\partial^\mu A^\nu - \partial^\nu A^\mu)}_{=F^{\mu\nu}} + (\partial_\mu A_\nu)\underbrace{(\delta(\partial^\mu A^\nu - \partial^\nu A^\mu))}_{\delta F^{\mu\nu}}] && (1438)\\
&= 2\left[\delta\left(\partial_\mu A_\nu\right)\partial^\mu A^\nu - \delta\left(\partial_\mu A_\nu\right)\partial^\nu A^\mu + (\partial_\mu A_\nu)\delta(\partial^\mu A^\nu) - (\partial_\mu A_\nu)\delta(\partial^\nu A^\mu)\right] && (1439)\\
&= 4\left[\delta\left(\partial_\mu A_\nu\right)\partial^\mu A^\nu - \delta\left(\partial_\mu A_\nu\right)\partial^\nu A^\mu\right] && (1440)\\
&= 4(\partial^\mu A^\nu - \partial^\nu A^\mu)\ \delta(\partial_\mu A_\nu) && (1441)\\
&= 4F^{\mu\nu}\ \delta(\partial_\mu A_\nu) && (1442)\\
&= 4F^{\mu\nu}\ \partial_\mu(\delta A_\nu) && (1443)
\end{aligned}
$$

We start with the source free Maxwell equations $\partial_\mu F^{\mu\nu} = 0$

$$
\begin{aligned}
0 &= \int_\Omega d^4x\ (\delta A_\nu)\partial_\mu F^{\mu\nu} && (1444)\\
&= F^{\mu\nu}(\delta A_\nu)|_{\partial\Omega} - \int_\Omega d^4x\ \underbrace{\partial_\mu(\delta A_\nu)F^{\mu\nu}}_{=\frac{1}{4}\delta(F_{\mu\nu}F^{\mu\nu})} && (1445)\\
&= \int_\Omega d^4x\ \delta\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) && (1446)
\end{aligned}
$$

and therefore $\mathscr{L}_{\text{ph}} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$.

2. So we see that the Lagrangian $\mathscr{L}_{\text{ph}} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = 2(\partial_\mu A_\nu)F^{\mu\nu}$ yields the inhomogeneous Maxwell equations

$$
\begin{aligned}
\frac{\partial \mathscr{L}_{\text{ph}}}{\partial A_\alpha} - \partial_\beta \frac{\partial \mathscr{L}_{\text{ph}}}{\partial(\partial_\beta A_\alpha)} &= 0 && (1447)\\
-\partial_\beta\left[(2\delta_{\alpha\mu}\delta_{\beta\nu}F^{\mu\nu} + 2(\partial_\mu A_\nu)(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\nu_\alpha\delta^\mu_\beta)\right] &= 0 && (1448)\\
-\partial_\beta\left[(2F^{\alpha\beta} + 2(\partial^\alpha A^\beta - \partial^\beta A^\alpha)\right] &= 0 && (1449)\\
\partial_\beta(F^{\alpha\beta}) &= 0 && (1450)
\end{aligned}
$$

but not the homogeneous ones. They are fulfilled trivially - by construction of $F^{\mu\nu}$.

3. The conjugated momentum is given by

$$
\begin{aligned}
\pi_\mu &= \frac{\partial \mathscr{L}_{\text{ph}}}{\partial \dot{A}^\mu} && (1451)\\
&= F_{0\mu} && (1452)
\end{aligned}
$$

0.16.3 Problem 3.3 - Dimension of ϕ

1. With $c = 1 = \hbar$ we see

$$
\begin{aligned}
E = mc^2 &\to E \sim M && (1453)\\
E = \hbar\omega &\to T \sim E^{-1} \sim M^{-1} && (1454)\\
s = ct &\to L \sim T \sim M^{-1} && (1455)
\end{aligned}
$$

As $\mathscr{L}$ is an action density we have

$$
\mathscr{L} \sim \frac{E \cdot T}{TL^3} \sim M \cdot M^{d-1} = M^d \tag{1456}
$$

From the explicit form of the scalar Lagrangian we derive

$$\mathscr{L} \sim \frac{[\phi^2]}{M^{-2}} = [\phi^2]M^{-2} \tag{1457}$$

and therefore $[\phi] = M^{(d-2)/2}$

2. Using the previous result we see

$$\lambda\phi^3: \qquad M^d \sim [\lambda]M^{3(d-2)/2} \to d = 6 \tag{1458}$$
$$\lambda\phi^4: \qquad M^d \sim [\lambda]M^{4(d-2)/2} \to d = 4 \tag{1459}$$

3. With

$$\mathscr{L} = \frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - \frac{1}{2}m^2\phi^2 + \lambda\phi^4 \tag{1460}$$
$$= \frac{1}{2}\eta^{\mu\nu}\left(\partial_\mu\frac{\tilde{\phi}}{\sqrt{\lambda}}\right)\left(\partial_\nu\frac{\tilde{\phi}}{\sqrt{\lambda}}\right) - \frac{1}{2}m^2\frac{\tilde{\phi}^2}{\lambda} + \lambda\frac{\tilde{\phi}^4}{\lambda^2} \tag{1461}$$
$$= \frac{1}{\lambda}\left[\frac{1}{2}\eta^{\mu\nu}(\partial_\mu\tilde{\phi})(\partial_\nu\tilde{\phi}) - \frac{1}{2}m^2\tilde{\phi}^2 + \tilde{\phi}^4\right] \tag{1462}$$

0.16.4 Problem 3.5 - Yukawa potential

Integration in spherical coordinates yields (with $x = kr$)

$$\int d^3k\frac{e^{-ik\cdot r}}{k^2+m^2} = 2\pi\int\frac{e^{-ikr\cos\theta}}{k^2+m^2}k^2\sin\theta d\theta\, dk \tag{1463}$$
$$= -2\pi\int\frac{e^{-ikr\cos\theta}}{k^2+m^2}k^2\; d(\cos\theta)\, dk \tag{1464}$$
$$= -2\pi\int\frac{k^2}{ikr}\frac{e^{-ikr\cos\theta}}{k^2+m^2}\bigg|_{-1}^{+1} dk \tag{1465}$$
$$= -2\pi\int\frac{k}{ir}\frac{e^{-ikr}-e^{+ikr}}{k^2+m^2}\, dk \tag{1466}$$
$$= \frac{4\pi}{r}\int_0^\infty\frac{k\sin kr}{k^2+m^2}\, dk \tag{1467}$$
$$= \frac{4\pi}{r^2}\int_0^\infty\frac{\frac{x}{r}\sin x}{\frac{x^2}{r^2}+m^2}\, dx \tag{1468}$$
$$= \frac{4\pi}{r}\int_0^\infty\frac{x\sin x}{x^2+m^2r^2}\, dx \tag{1469}$$
$$\tag{1470}$$

Now we use a small trick

$$= \frac{2\pi}{ir}\int_0^\infty \frac{x(e^{ix}-e^{-ix})}{x^2+m^2r^2}\,dx \tag{1471}$$

$$= \frac{2\pi}{ir}\left[\int_0^\infty \frac{xe^{ix}}{x^2+m^2r^2}\,dx - \int_0^\infty \frac{xe^{-ix}}{x^2+m^2r^2}\,dx\right] \tag{1472}$$

$$= \frac{2\pi}{ir}\left[\int_0^\infty \frac{xe^{ix}}{x^2+m^2r^2}\,dx - (-1)^3\int_{-\infty}^0 \frac{ye^{iy}}{y^2+m^2r^2}\,dy\right] \tag{1473}$$

$$= \frac{2\pi}{ir}\int_{-\infty}^\infty \frac{xe^{ix}}{x^2+m^2r^2}\,dx \tag{1474}$$

$$= \frac{2\pi}{ir}\int_{-\infty}^\infty \frac{xe^{ix}}{(x+imr)(x-imr)}\,dx \tag{1475}$$

$$= \frac{2\pi}{ir}\left(2\pi i\cdot \underbrace{\mathrm{Res}_{x=imr}}_{=\frac{imr\exp(i^2mr)}{2imr}} - \int_{\text{upper half circle}} ...\right) \tag{1476}$$

$$= \frac{2\pi^2}{r}e^{-mr} \tag{1477}$$

Therefore

$$\frac{1}{(2\pi)^3}\int d^3k\frac{e^{-ik\cdot r}}{k^2+m^2} = \frac{1}{4\pi r}e^{-mr} \tag{1478}$$

0.16.5 Problem 3.9 - ζ function regularization

1. Calculation the Taylor expansion (using L'Hopital's rule for the limits) we obtain

$$f(t) = \frac{t}{e^t-1} \tag{1479}$$

$$= \sum_k \left.\frac{d^kf}{dt^k}\right|_{t=0} t^k \tag{1480}$$

$$= 1-\frac{1}{2}t+\frac{1}{12}t^2-\frac{1}{12}t^4+... \tag{1481}$$

$$\stackrel{!}{=} B_0+B_1t+\frac{B_2}{2}t^2+\frac{B_3}{6}t^2+... \tag{1482}$$

$$\rightarrow B_n = \{1,-\frac{1}{2},\frac{1}{6},0,...\} \tag{1483}$$

2. Avoiding mathematical rigor we see after playing around for a while

$$\sum_{n=1}^{\infty} ne^{-an} = -\frac{d}{da}\sum_{n=1}^{\infty} e^{-an} \tag{1484}$$
$$= -\frac{d}{da}\sum_{n=1}^{\infty} \left(e^{-a}\right)^n \tag{1485}$$
$$= -\frac{d}{da}\frac{1}{1-e^{-a}} \tag{1486}$$
$$= -\frac{d}{da}\left(\frac{1}{a}\frac{a}{1-e^{-a}}\right) \tag{1487}$$
$$= -\frac{d}{da}\left(\frac{1}{a}f(t)\right) \tag{1488}$$
$$= -\frac{d}{da}\left(\frac{1}{a}\sum_{n=0}^{\infty}\frac{B_n}{n!}a^n\right) \tag{1489}$$
$$= -\frac{d}{da}\left(\frac{1}{a}\left[1-\frac{a}{2}+\frac{a^2}{12}-\frac{a^4}{720}+...\right]\right) \tag{1490}$$
$$= -\frac{d}{da}\left(\frac{1}{a}-\frac{1}{2}+\frac{a}{12}-\frac{a^3}{720}...\right) \tag{1491}$$
$$= \frac{1}{a^2}-\frac{1}{12}+\frac{a}{240}-... \tag{1492}$$
$$\overset{a\to 0}{\to} \frac{1}{a^2}-\frac{1}{12} \tag{1493}$$

3. Using the definition of the Riemann ζ function

$$\zeta(s) = \sum_{k=1}^{\infty}\frac{1}{k^s} \tag{1494}$$

0.16.6 Problem 4.1 - $Z[J]$ at order λ in ϕ^4 theory

Lets start at (4.6a) with $\mathscr{L}_I = -\lambda/4!\phi^4$

$$Z[J] = \exp\left[\mathrm{i}\int d^4x \mathscr{L}_I\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}\right)\right]\int \mathcal{D}\phi \exp\left[\mathrm{i}\int d^4x(\mathscr{L}_0 + J\phi)\right] \tag{1495}$$
$$= \exp\left[\mathrm{i}\int d^4x \mathscr{L}_I\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}\right)\right] Z_0[J] \tag{1496}$$
$$= \exp\left[-\frac{\mathrm{i}\lambda}{4!}\int d^4x\left(\frac{\delta^4}{\delta J(x)^4}\right)\right] Z_0[J] \tag{1497}$$
$$= Z_0[J] - \frac{\mathrm{i}\lambda}{4!}\int d^4x\left(\frac{\delta^4 Z_0[J]}{\delta J(x)^4}\right) + \dots . \tag{1498}$$

Using (4.7)

$$Z_0[J] = Z_0[0]\exp\left[-\frac{\mathrm{i}}{2}\int d^4y d^4z J(y)\Delta_F(y-z)J(z)\right] = Z_0[0]e^{iW_0[J]} \tag{1499}$$
$$W_0[J] = -\frac{1}{2}\int d^4y d^4z J(y)\Delta_F(y-z)J(z) \tag{1500}$$

we derive (4.10) in various steps

1. Calculating $\frac{\delta W_0[J]}{\delta J(x)}$

$$\frac{\delta W_0[J]}{\delta J(x)} = -\frac{1}{2}\lim_{\epsilon\to 0}\int d^4y d^4z \frac{\left(J(y)+\epsilon\delta^{(4)}(y-x)\right)\Delta_F(y-z)\left(J(z)+\epsilon\delta^{(4)}(z-x)\right) - W_0[J]}{\epsilon} \tag{1501}$$

$$= -\frac{1}{2}\int d^4y d^4z \left[\delta^{(4)}(y-x)\Delta_F(y-z)J(z) + J(y)\Delta_F(y-z)\delta^{(4)}(z-x)\right] \tag{1502}$$

$$= -\frac{1}{2}\int d^4z \Delta_F(x-z)J(z) - \frac{1}{2}\int d^4y J(y)\Delta_F(y-x) \tag{1503}$$

$$= -\int d^4y \Delta_F(y-x)J(y) \tag{1504}$$

where we used $\Delta_F(x) = \Delta_F(-x)$.

2. Calculating $\frac{\delta^2 W_0[J]}{\delta J(x)^2}$

$$\frac{\delta^2 W_0[J]}{\delta J(x)^2} = -\int d^4y \Delta_F(y-x)\frac{\delta J(y)}{\delta J(x)} \tag{1505}$$

$$= -\int d^4y \Delta_F(y-x)\delta(y-x) \tag{1506}$$

$$= -\Delta_F(0) \tag{1507}$$

3. Calculating $\delta F[J]/\delta J(x)$ for $F[J] = f\left(W_0[J]\right)$

$$\frac{\delta F[J]}{\delta J(x)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon} f(W_0[\phi(x)+\epsilon\delta(x-y)]) - f(W_0[\phi(x)]) \tag{1508}$$

$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon} f(W_0[\phi(x)] + \epsilon\frac{\delta W_0}{\delta\phi}) - f(W_0[\phi(x)]) \tag{1509}$$

$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon} f(W_0[\phi(x)]) + g'\epsilon\frac{\delta W_0}{\delta\phi} - f(W_0[\phi(x)]) \tag{1510}$$

$$= f'(W_0[J])\frac{\delta W_0}{\delta J} \tag{1511}$$

4. Calculating first derivative

$$\frac{\delta}{\mathrm{i}\delta J(x)}\exp\left(\mathrm{i}W_0[J]\right) = \frac{\delta W_0[J]}{\delta J(x)}\exp\left(\mathrm{i}W_0[J]\right) \tag{1512}$$

5. Calculating second derivative (using the functional derivative product rule)

$$\left(\frac{\delta}{\mathrm{i}\delta J(x)}\right)^2\exp\left(\mathrm{i}W_0[J]\right) = \left(\left(\frac{\delta W_0[J]}{\delta J(x)}\right)^2 + \frac{1}{i}\frac{\delta^2 W_0[J]}{\delta J(x)^2}\right)\exp\left(\mathrm{i}W_0[J]\right) \tag{1513}$$

6. Calculating third derivative

$$\left(\frac{\delta}{\mathrm{i}\delta J(x)}\right)^3\exp\left(\mathrm{i}W_0[J]\right) = \left(\left(\frac{\delta W_0[J]}{\delta J(x)}\right)^3 + \frac{3}{i}\frac{\delta^2 W_0[J]}{\delta J(x)^2}\frac{\delta W_0[J]}{\delta J(x)} + \frac{1}{i^2}\frac{\delta^3 W_0[J]}{\delta J(x)^3}\right)\exp\left(\mathrm{i}W_0[J]\right) \tag{1514}$$

7. Calculating fourth derivative

$$\left(\frac{\delta}{\mathrm{i}\delta J(x)}\right)^4 \exp\left(\mathrm{i}W_0[J]\right) = \left(\left(\frac{\delta W_0[J]}{\delta J(x)}\right)^4 + \frac{6}{i}\frac{\delta^2 W_0[J]}{\delta J(x)^2}\left(\frac{\delta W_0[J]}{\delta J(x)}\right)^2 + \frac{3}{i^2}\left(\frac{\delta^2 W_0[J]}{\delta J(x)^2}\right)^2 + \right.$$
$$\left. + \frac{4}{i^2}\frac{\delta W_0[J]}{\delta J(x)}\frac{\delta^3 W_0[J]}{\delta J(x)^3} + \frac{1}{i^3}\frac{\delta^4 W_0[J]}{\delta J(x)^4}\right)\exp\left(\mathrm{i}W_0[J]\right)$$
$$= \left(\left(\frac{\delta W_0[J]}{\delta J(x)}\right)^4 + \frac{6}{i}\frac{\delta^2 W_0[J]}{\delta J(x)^2}\left(\frac{\delta W_0[J]}{\delta J(x)}\right)^2 + \frac{3}{i^2}\left(\frac{\delta^2 W_0[J]}{\delta J(x)^2}\right)^2\right)\exp\left(\mathrm{i}W_0[J]\right)$$

8. Substituting the functional derivatives

$$\left(\frac{\delta}{\mathrm{i}\delta J(x)}\right)^4 \exp\left(\mathrm{i}W_0[J]\right) = \left[\left(\int d^4y \Delta_F(y-x)J(y)\right)^4 + 6i\Delta_F(0)\left(\int d^4y\Delta_F(y-x)J(y)\right)^2\right.$$
$$\left. + 3\left(i\Delta_F(0)\right)^2\right]\exp\left(\mathrm{i}W_0[J]\right)$$

0.16.7 Problem 19.1 - Dynamical stress tensor

Preliminaries

- The Laplace expansion of the determinate by row or column is given by

$$|g| = \sum_\kappa g_{\kappa\mu}G_{\kappa\mu} \quad \text{(no sum over } \mu!) \tag{1515}$$

  with the cofactor matrix $G_{\kappa\mu}$ (matrix of determinants of minors of g).

- The inverse matrix is given by

$$g^{\alpha\beta} = \frac{1}{|g|}G_{\alpha\beta} \tag{1516}$$

- Therefore we have

$$\frac{\partial |g|}{\delta g_{\alpha\beta}} = \frac{\partial\left(\sum_\kappa g_{\kappa\beta}G_{\kappa\beta}\right)}{\delta g_{\alpha\beta}} \tag{1517}$$
$$= \delta_{\kappa\alpha}G_{\kappa\beta} \tag{1518}$$
$$= G_{\alpha\beta} \tag{1519}$$
$$= |g|g^{\alpha\beta} \tag{1520}$$

Now we can calculate

$$\delta\sqrt{|g|} = \frac{\partial\sqrt{|g|}}{\delta g_{\mu\nu}}\delta g_{\mu\nu} = \frac{1}{2\sqrt{|g|}}\frac{\partial|g|}{\delta g_{\mu\nu}}\delta g_{\mu\nu} = \frac{1}{2}\sqrt{|g|}g^{\mu\nu}\delta g_{\mu\nu} \tag{1521}$$
$$\frac{\delta\sqrt{|g(x)|}}{\delta g_{\mu\nu}(y)} = \frac{1}{2}\sqrt{|g|}\delta(x-y) \tag{1522}$$

We now use the action and definition (7.49)

$$S_\mathrm{m} = \int d^4x\sqrt{|g|}\mathscr{L}_\mathrm{m} \tag{1523}$$
$$T^{\mu\nu} = \frac{2}{\sqrt{|g|}}\frac{\delta S_\mathrm{m}}{\delta g^{\mu\nu}} \tag{1524}$$
$$= \frac{2}{\sqrt{|g|}}\int d^4x\left[\frac{1}{2}\sqrt{|g|}g^{\mu\nu}\mathscr{L}_\mathrm{m} + \sqrt{|g|}\frac{\delta\mathscr{L}_\mathrm{m}}{\delta g_{\mu\nu}}\right] \tag{1525}$$

0.16.8 Problem 19.6 - Dirac-Schwarzschild

1. (19.13) - adding the bi-spinor index might be helpful for some readers, see (B.27)
2. (19.13) vs (B.27) naming of generators $J^{\mu\nu}$ vs $\sigma_{\mu\nu}/2$

The Dirac equation in curved space is obtained (from the covariance principle) by replacing all derivatives ∂_k with covariant tetrad derivatives $\mathscr{D}_k$

$$(i\hbar\gamma^k \mathscr{D}_k + mc)\psi = 0 \tag{1526}$$

Lets start with the Schwarzschild line element

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2(d\vartheta^2 + \sin^2\vartheta\, d\phi^2) \tag{1527}$$

$$= \eta_{mn} d\xi^m d\xi^n \tag{1528}$$

with

$$d\xi^0 = \left(1 - \frac{2M}{r}\right)^{1/2} dt, \quad d\xi^1 = \left(1 - \frac{2M}{r}\right)^{-1/2} dr, \quad d\xi^2 = r d\vartheta, \quad d\xi^3 = r\sin\vartheta d\phi. \tag{1529}$$

and the tetrad fields e^m_μ can then be derived via $d\xi^m = e^m_\mu(x) dx^\mu$.

0.16.9 Problem 23.1 - Conformal transformation

For a change of coordinates we find in general

$$x^\mu \mapsto \tilde{x}^\mu \tag{1530}$$

$$g_{\mu\nu}(x) \mapsto \tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(x) \tag{1531}$$

which for $x \mapsto \tilde{x} = e^\omega x$ results in (there might be a sign error in (18.1))

$$g_{\mu\nu}(x) \mapsto \tilde{g}_{\mu\nu}(\tilde{x}) = e^{-2\omega} g_{\alpha\beta}(x) \tag{1532}$$

while for a conformal transformation we have

$$g_{\mu\nu}(x) \mapsto \tilde{g}_{\mu\nu}(x) = \Omega^2 g_{\alpha\beta}(x) \tag{1533}$$

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \Omega^2 g_{\alpha\beta}(e^\omega x) \tag{1534}$$

0.16.10 Problem 23.2 - Conformal transformation properties

- Christoffel symbol:

$$\tilde{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x) = e^{2\omega(x)} g_{\mu\nu}(x) \tag{1535}$$

$$\tilde{g}_{\mu\nu,\alpha} = 2\Omega\Omega_{,\alpha} g_{\mu\nu} + \Omega^2 g_{\mu\nu,\alpha} \tag{1536}$$

$$= \Omega(2g_{\mu\nu}\Omega_{,\alpha} + \Omega g_{\mu\nu,\alpha}) \tag{1537}$$

and

$$\delta^\mu_{\ \nu} = \tilde{g}^{\mu\alpha}\tilde{g}_{\alpha\nu} = \tilde{g}^{\mu\alpha} g_{\alpha\nu}\Omega^2 \tag{1538}$$

$$\delta^\mu_{\ \nu} g^{\nu\beta} = \tilde{g}^{\mu\alpha} g_{\alpha\nu} g^{\nu\beta}\Omega^2 \tag{1539}$$

$$g^{\mu\beta} = \tilde{g}^{\mu\alpha}\delta^\beta_\alpha \Omega^2 \tag{1540}$$

$$\to \tilde{g}^{\mu\beta} = \Omega^{-2} g^{\mu\beta} \tag{1541}$$

we find by using $\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(g_{\alpha\mu,\beta} + g_{\beta\mu,\alpha} - g_{\alpha\beta,\mu} \right)$

$$\tilde{\Gamma}^{\mu}{}_{\alpha\beta} = \frac{1}{2}\tilde{g}^{\mu\nu} \left(\tilde{g}_{\alpha\nu,\beta} + \tilde{g}_{\beta\nu,\alpha} - \tilde{g}_{\alpha\beta,\nu} \right) \tag{1542}$$

$$= \frac{1}{2}\Omega^{-2} g^{\mu\nu} \left[\Omega(2g_{\alpha\nu}\Omega_{,\beta} + \Omega g_{\alpha\nu,\beta}) + \Omega(2g_{\beta\nu}\Omega_{,\alpha} + \Omega g_{\beta\nu,\alpha}) - \Omega(2g_{\alpha\beta}\Omega_{,\nu} + \Omega g_{\alpha\beta,\nu}) \right] \tag{1543}$$

$$= \Gamma^{\mu}{}_{\alpha\beta} + \Omega^{-1} g^{\mu\nu} \left[g_{\alpha\nu}\Omega_{,\beta} + g_{\beta\nu}\Omega_{,\alpha} - g_{\alpha\beta}\Omega_{,\nu} \right] \tag{1544}$$

$$= \Gamma^{\mu}{}_{\alpha\beta} + \Omega^{-1} \left[\delta^{\mu}_{\alpha}\Omega_{,\beta} + \delta^{\mu}_{\beta}\Omega_{,\alpha} - g^{\mu\nu} g_{\alpha\beta}\Omega_{,\nu} \right] \tag{1545}$$

- Ricci tensor: with

$$\Omega = e^{2\omega} \tag{1546}$$

$$\Omega^{-2}\Omega_{,\lambda} = e^{-4\omega} e^{2\omega} 2\omega_{,\lambda} \tag{1547}$$

$$= 2e^{-2\omega}\omega_{,\lambda} \tag{1548}$$

$$\Omega_{,\lambda\alpha} = \left(2e^{2\omega}\omega_{,\lambda} \right)_{,\alpha} \tag{1549}$$

$$= 4e^{2\omega}\omega_{,\lambda}\omega_{,\alpha} + 2e^{2\omega}\omega_{,\lambda\alpha} \tag{1550}$$

$$= 2e^{2\omega} \left(2\omega_{,\lambda}\omega_{,\alpha} + \omega_{,\lambda\alpha} \right) \tag{1551}$$

and

$$\partial_\lambda \tilde{\Gamma}^{\mu}{}_{\alpha\beta} = \partial_\lambda \Gamma^{\mu}{}_{\alpha\beta} - \Omega^{-2}\Omega_{,\lambda} \left[\delta^{\mu}_{\alpha}\Omega_{,\beta} + \delta^{\mu}_{\beta}\Omega_{,\alpha} - g^{\mu\nu} g_{\alpha\beta}\Omega_{,\nu} \right] + \Omega^{-1} \left[\delta^{\mu}_{\alpha}\Omega_{,\beta\lambda} + \delta^{\mu}_{\beta}\Omega_{,\alpha\lambda} - (g^{\mu\nu} g_{\alpha\beta}\Omega_{,\nu})_{,\lambda} \right] \tag{1552}$$

$$= \partial_\lambda \Gamma^{\mu}{}_{\alpha\beta} - 4\omega_{,\lambda} \left[\delta^{\mu}_{\alpha}\omega_{,\beta} + \delta^{\mu}_{\beta}\omega_{,\alpha} - g^{\mu\nu} g_{\alpha\beta}\omega_{,\nu} \right] + 2 \left[\delta^{\mu}_{\alpha} \left(2\omega_{,\beta}\omega_{,\lambda} + \omega_{,\beta\lambda} \right) + \delta^{\mu}_{\beta} \left(2\omega_{,\alpha}\omega_{,\lambda} + \omega_{,\alpha\lambda} \right) \right] \tag{1553}$$

$$- 2 \left[g^{\mu\nu}{}_{,\lambda} g_{\alpha\beta}\omega_{,\nu} + g^{\mu\nu} g_{\alpha\beta,\lambda}\omega_{,\nu} + g^{\mu\nu} g_{\alpha\beta}(2\omega_{,\nu}\omega_{,\lambda} + \omega_{,\nu\lambda}) \right] \tag{1554}$$

$$\tag{1555}$$

$$\partial_\rho \tilde{\Gamma}^{\rho}{}_{\mu\nu} = \partial_\rho \Gamma^{\rho}{}_{\mu\nu} - 4\omega_{,\rho} \left[\delta^{\rho}_{\mu}\omega_{,\nu} + \delta^{\rho}_{\nu}\omega_{,\mu} - g^{\rho\nu} g_{\mu\nu}\omega_{,\lambda} \right] + 2 \left[\delta^{\rho}_{\mu} \left(2\omega_{,\nu}\omega_{,\rho} + \omega_{,\nu\rho} \right) + \delta^{\rho}_{\nu} \left(2\omega_{,\mu}\omega_{,\rho} + \omega_{,\mu\rho} \right) \right] \tag{1556}$$

$$- 2 \left[g^{\rho\lambda}{}_{,\rho} g_{\mu\nu}\omega_{,\lambda} + g^{\rho\lambda} g_{\mu\nu,\rho}\omega_{,\lambda} + g^{\rho\lambda} g_{\mu\nu}(2\omega_{,\lambda}\omega_{,\rho} + \omega_{,\lambda\rho}) \right] \tag{1557}$$

$$= \partial_\rho \Gamma^{\rho}{}_{\mu\nu} - 4 \left[2\omega_{,\mu}\omega_{,\nu} - \omega_{,\rho} g^{\rho\nu} g_{\mu\nu}\omega_{,\lambda} \right] + 4 \left(2\omega_{,\nu}\omega_{,\mu} + \omega_{,\nu\mu} \right) \tag{1558}$$

$$- 2 \left[g^{\rho\lambda}{}_{,\rho} g_{\mu\nu}\omega_{,\lambda} + g^{\rho\lambda} g_{\mu\nu,\rho}\omega_{,\lambda} + g^{\rho\lambda} g_{\mu\nu}(2\omega_{,\lambda}\omega_{,\rho} + \omega_{,\lambda\rho}) \right] \tag{1559}$$

$$= \partial_\rho \Gamma^{\rho}{}_{\mu\nu} + 4g^{\rho\nu} g_{\mu\nu}\omega_{,\lambda}\omega_{,\rho} + 4\omega_{,\nu\mu} - 2 \left[g^{\rho\lambda}{}_{,\rho} g_{\mu\nu}\omega_{,\lambda} + g^{\rho\lambda} g_{\mu\nu,\rho}\omega_{,\lambda} + (2g^{\rho\lambda} g_{\mu\nu}\omega_{,\lambda}\omega_{,\rho} + g^{\rho\lambda} g_{\mu\nu}\omega_{,\lambda\rho}) \right] \tag{1560}$$

$$= \partial_\rho \Gamma^{\rho}{}_{\mu\nu} + 4\omega_{,\lambda}\omega_{,\mu} + 4\omega_{,\nu\mu} - 2 \left[g^{\rho\lambda}{}_{,\rho} g_{\mu\nu}\omega_{,\lambda} + g_{\mu\nu,\rho}\omega^{,\rho} + 2g_{\mu\nu}\omega^{,\rho}\omega_{,\rho} + g_{\mu\nu}\omega^{,\rho}{}_{\rho} \right] \tag{1561}$$

$$\partial_\nu \tilde{\Gamma}^{\rho}{}_{\mu\rho} = \partial_\nu \Gamma^{\rho}{}_{\mu\rho} - 4\omega_{,\nu} \left[\delta^{\rho}_{\mu}\omega_{,\rho} + \delta^{\rho}_{\rho}\omega_{,\mu} - g^{\rho\kappa} g_{\mu\rho}\omega_{,\kappa} \right] + 2 \left[\delta^{\rho}_{\mu} \left(2\omega_{,\rho}\omega_{,\nu} + \omega_{,\rho\nu} \right) + \delta^{\rho}_{\rho} \left(2\omega_{,\mu}\omega_{,\nu} + \omega_{,\mu\nu} \right) \right] \tag{1562}$$

$$- 2 \left[g^{\rho\kappa}{}_{,\nu} g_{\mu\rho}\omega_{,\kappa} + g^{\rho\kappa} g_{\mu\rho,\nu}\omega_{,\kappa} + g^{\rho\kappa} g_{\mu\rho}(2\omega_{,\kappa}\omega_{,\nu} + \omega_{,\kappa\nu}) \right] \tag{1563}$$

$$= \partial_\nu \Gamma^{\rho}{}_{\mu\rho} - 4 \left[(d+1)\omega_{,\mu}\omega_{,\nu} - \omega_{,\mu}\omega_{,\nu} \right] + 2(d+1) \left(2\omega_{,\mu}\omega_{,\nu} + \omega_{,\mu\nu} \right) \tag{1564}$$

$$- 2 \left[g^{\rho\kappa}{}_{,\nu} g_{\mu\rho}\omega_{,\kappa} + g^{\rho\kappa} g_{\mu\rho,\nu}\omega_{,\kappa} + \delta^{\kappa}_{\mu}(2\omega_{,\kappa}\omega_{,\nu} + \omega_{,\kappa\nu}) \right] \tag{1565}$$

$$= \partial_\nu \Gamma^{\rho}{}_{\mu\rho} + 4\omega_{,\mu}\omega_{,\nu} + 2(d+1)\omega_{,\mu\nu} - 2 \left[g^{\rho\kappa}{}_{,\nu} g_{\mu\rho}\omega_{,\kappa} + g^{\rho\kappa} g_{\mu\rho,\nu}\omega_{,\kappa} + (2\omega_{,\mu}\omega_{,\nu} + \omega_{,\mu\nu}) \right] \tag{1566}$$

$$= \partial_\nu \Gamma^{\rho}{}_{\mu\rho} + 2d \cdot \omega_{,\mu\nu} - 2 \left[g^{\rho\kappa}{}_{,\nu} g_{\mu\rho}\omega_{,\kappa} + g_{\mu\rho,\nu}\omega^{,\rho} \right] \tag{1567}$$

$$\tilde{\Gamma}^{\mu}{}_{\alpha\beta} = \Gamma^{\mu}{}_{\alpha\beta} + \Omega^{-1}\left[\delta^{\mu}_{\alpha}\Omega_{,\beta} + \delta^{\mu}_{\beta}\Omega_{,\alpha} - g^{\mu\nu}g_{\alpha\beta}\Omega_{,\nu}\right] \tag{1568}$$

$$\tag{1569}$$

$$\tilde{\Gamma}^{\rho}{}_{\mu\nu}\tilde{\Gamma}^{\sigma}{}_{\rho\sigma} = \left(\Gamma^{\rho}{}_{\mu\nu} + \Omega^{-1}\left[\delta^{\rho}_{\mu}\Omega_{,\nu} + \delta^{\rho}_{\nu}\Omega_{,\mu} - g^{\rho\lambda}g_{\mu\nu}\Omega_{,\lambda}\right]\right)\left(\Gamma^{\sigma}{}_{\rho\sigma} + d\cdot\Omega^{-1}\Omega_{,\rho}\right) \tag{1570}$$

$$= \Gamma^{\rho}{}_{\mu\nu}\Gamma^{\sigma}{}_{\rho\sigma} + \Gamma^{\rho}{}_{\mu\nu}d\cdot\Omega^{-1}\Omega_{,\rho} + \Gamma^{\sigma}{}_{\rho\sigma}\Omega^{-1}\left[\delta^{\rho}_{\mu}\Omega_{,\nu} + \delta^{\rho}_{\nu}\Omega_{,\mu} - g^{\rho\lambda}g_{\mu\nu}\Omega_{,\lambda}\right] \tag{1571}$$

$$+ d\cdot\Omega^{-2}\left[\delta^{\rho}_{\mu}\Omega_{,\nu} + \delta^{\rho}_{\nu}\Omega_{,\mu} - g^{\rho\lambda}g_{\mu\nu}\Omega_{,\lambda}\right]\Omega_{,\rho} \tag{1572}$$

$$\tilde{R}_{\mu\nu} = \tilde{R}^{\rho}{}_{\mu\rho\nu} \tag{1573}$$

$$= \partial_{\rho}\tilde{\Gamma}^{\rho}{}_{\mu\nu} - \partial_{\nu}\tilde{\Gamma}^{\rho}{}_{\mu\rho} + \tilde{\Gamma}^{\rho}{}_{\mu\nu}\tilde{\Gamma}^{\sigma}{}_{\rho\sigma} - \tilde{\Gamma}^{\sigma}{}_{\nu\rho}\tilde{\Gamma}^{\rho}{}_{\mu\sigma} \tag{1574}$$

- Curvature scalar

$$\tilde{R} = \tilde{g}^{\mu\nu}\tilde{R}_{\mu\nu} \tag{1575}$$

$$= \tilde{g}^{\mu\nu}\left[R_{\mu\nu} - g_{\mu\nu}\Box\omega - (d-2)\nabla_{\mu}\nabla_{\nu}\omega + (d-2)\nabla_{\mu}\omega\nabla_{\nu}\omega - (d-2)g_{\mu\nu}\nabla^{\lambda}\omega\nabla_{\lambda}\omega\right] \tag{1576}$$

$$= \Omega^{-2}\left[R - d\Box\omega - (d-2)\Box\omega + (d-2)\nabla^{\mu}\omega\nabla_{\mu}\omega - (d-2)d\nabla^{\lambda}\omega\nabla_{\lambda}\omega\right] \tag{1577}$$

$$= \Omega^{-2}\left[R - 2(d-1)\Box\omega - (d-2)(d-1)\nabla^{\lambda}\omega\nabla_{\lambda}\omega\right] \tag{1578}$$

$$\tag{1579}$$

0.16.11 Problem 23.6 - Reflection formula

$$\Gamma(z) = \int_0^{\infty} x^{z-1}e^{-x}dx \tag{1580}$$

0.16.12 Problem 23.7 - Unruh temperature

0.16.13 Problem 24.14 - Jeans length and the speed of sound

We start with the Euler equations

$$\frac{D\rho}{Dt} = -\rho\nabla\cdot\vec{u} \quad\rightarrow\quad \frac{\partial\rho}{\partial t} + \vec{u}\cdot(\nabla\rho) + \rho(\nabla\cdot\vec{u}) = 0 \tag{1581}$$

$$\frac{D\vec{u}}{Dt} = -\nabla\left(\frac{P}{\rho}\right) + \vec{g} \quad\rightarrow\quad \frac{\partial\vec{u}}{\partial t} + \vec{u}\cdot(\nabla\vec{u}) + \frac{\nabla P}{\rho} = \vec{g}. \tag{1582}$$

With the perturbation ansatz (small perturbation in a resting fluid)

$$\rho = \rho_0 + \varepsilon\rho_1(x,t) \tag{1583}$$

$$P = P_0 + \varepsilon P_1(x,t) \tag{1584}$$

$$\vec{u} = \varepsilon\vec{u}_1(x,t) \tag{1585}$$

and the Newton equation

$$\triangle\phi = 4\pi G\rho \quad\rightarrow\quad \nabla\cdot\vec{g}_1 = -4\pi G\rho_1 \tag{1586}$$

we obtain (with the EoS $P = w\rho$) in order ε

$$\frac{\partial \rho_1}{\partial t} + \rho_0(\nabla \cdot \vec{u}_1) = 0 \tag{1587}$$

$$\frac{\partial \vec{u}_1}{\partial t} + \underbrace{\frac{1}{\rho_0}\nabla P_1}_{=\frac{w}{\rho_0}\nabla\rho_1} = \vec{g}_1. \tag{1588}$$

Differentiating both (with respect to space and time) we obtain a wave equation

$$\frac{\partial^2 \rho_1}{\partial t^2} - w\triangle\rho_1 = 4\pi G\rho_0\rho_1 \tag{1589}$$

with the speed of sound $c_s^2 = w$. Inserting the wave ansatz $\rho_1 \sim \exp[i(\vec{k}\cdot\vec{x} - \omega t)]$ yields the dispersion relation

$$\omega^2 = c_s^2 k^2 - 4\pi G\rho_0. \tag{1590}$$

For wave numbers $k_J < \sqrt{4\pi G/c_s^2}$ the ω becomes complex which gives rise to exponentially growing modes. Therefore the Jeans length is given by

$$\lambda_J = \frac{2\pi}{k_J} = c_s\sqrt{\frac{\pi}{G\rho_0}} = \sqrt{\frac{\pi w}{G\rho_0}}. \tag{1591}$$

0.16.14 Problem 25.1 - Schwarzschild metric

The simplified vacuum Einstein equations are given by

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0 \tag{1592}$$

$$\rightarrow \quad R - \frac{1}{2}R\cdot 4 = 0 \quad \rightarrow \quad R = 0 \tag{1593}$$

$$R_{\mu\nu} = 0 \tag{1594}$$

Lets start with the metric ansatz (25.4)

$$g_{\mu\nu} = \mathrm{diag}(A(r), -B(r), -r^2, -r^2\sin^2\theta) \tag{1595}$$

$$g^{\mu\nu} = \mathrm{diag}(1/A(r), -1/B(r), -1/r^2, -1/r^2\sin^2\theta) \tag{1596}$$

The non-vanishing Chistoffel symbols are then

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2}g^{\mu\kappa}(g_{\kappa\lambda,\nu} + g_{\nu\kappa,\lambda} - g_{\nu\lambda,\kappa}) \tag{1597}$$

$$\Gamma^0_{01} = \frac{A'}{2A}, \quad \Gamma^1_{00} = \frac{A'}{2B} \quad \Gamma^1_{11} = \frac{B'}{2B} \quad \Gamma^1_{22} = -\frac{r}{B} \quad \Gamma^1_{33} = \frac{r\sin^2\theta}{B} \tag{1598}$$

$$\Gamma^2_{12} = 1/r \quad \Gamma^2_{12} = -\cos\theta\sin\theta \quad \Gamma^2_{12} = 1/r \quad \Gamma^2_{12} = \cot\theta \tag{1599}$$

The non-vanishing components of the Ricci tensor are

$$R_{00} = \frac{A'}{rB} - \frac{A'^2}{4AB} - \frac{A'B'}{4B^2} + \frac{A''}{2B} \tag{1600}$$

$$R_{11} = \frac{A'^2}{4A^2} + \frac{B'}{rB} + \frac{A'B'}{4AB} - \frac{A''}{2A} \tag{1601}$$

$$R_{22} = -\frac{1}{B} + 1 - \frac{rA'}{2AB} + \frac{rB''}{2B^2} \tag{1602}$$

$$R_{33} = R_{22}\sin^2\theta \tag{1603}$$

As there are only the two unknown functions A, B we only need two vacuum equations $R_{00} = 0$ and $R_{11} = 0$. Multiplying the first by B/A and leaving the second one untouched we obtain the system

$$\frac{A'}{rA} - \frac{A'^2}{4A^2} - \frac{A'B'}{4AB} + \frac{A''}{2A} = 0 \tag{1604}$$

$$\frac{B'}{rB} + \frac{A'^2}{4A^2} + \frac{A'B'}{4AB} - \frac{A''}{2A} = 0 \tag{1605}$$

Adding bot we get $B'/B = -A'/A$ which we can substitude into the first one obtaining

$$\frac{A'}{rA} + \frac{A''}{2A} = 0 \tag{1606}$$

$$\to A'(r) = \frac{c_1}{r^2} \tag{1607}$$

$$\to A(r) = c_2 - \frac{c_1}{r} \tag{1608}$$

now we can solve for $B(r)$

$$\frac{B'}{B} = -\frac{A'}{A} \tag{1609}$$

$$\to B(r) = \frac{c_3 r}{c_1 - rc_2} = \frac{-c_3}{c_2 - \frac{c_1}{r}} \tag{1610}$$

0.16.15 Problem 26.4 - Fixed points of (26.18)

We start with

$$\text{(F1)} \qquad H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V + \rho\right) \tag{1611}$$

$$\text{(F2)} \qquad \dot{H} = -4\pi G\left[\dot{\phi}^2 + (1+w_m)\rho\right] \tag{1612}$$

$$\text{(KG)} \qquad \ddot{\phi} = -3H\dot{\phi} - V_{,\phi}. \tag{1613}$$

Using $H = \dot{a}/a$, $N = \ln(a)$ and $\lambda = -V_{,\phi}/(\sqrt{8\pi G}V)$ we obtain for the time derivatives of x and y

$$\dot{V} = \frac{dV}{d\phi}\frac{d\phi}{dt} = V_{,\phi}\dot{\phi} \tag{1614}$$

$$x = \sqrt{\frac{4}{3}\pi G}\frac{\dot{\phi}}{H} \quad \to \quad \frac{dx}{dt} = \frac{dx}{dN}\frac{d\ln(a)}{dt} = \frac{dx}{dN}H = \sqrt{\frac{4}{3}\pi G}\frac{\ddot{\phi}H - \dot{\phi}\dot{H}}{H^2} \tag{1615}$$

$$y = \sqrt{\frac{8}{3}\pi G}\frac{\sqrt{V}}{H} \quad \to \quad \frac{dy}{dt} = \frac{dy}{dN}\frac{d\ln(a)}{dt} = \frac{dy}{dN}H = \sqrt{\frac{8}{3}\pi G}\frac{\frac{V_{,\phi}\dot{\phi}}{2\sqrt{V}} - \sqrt{V}\dot{H}}{H^2}. \tag{1616}$$

With the substitutions

$$\dot{H} = -4\pi G\left[\dot{\phi}^2 + (1+w_m)\rho\right] \tag{1617}$$

$$\ddot{\phi} = -3H\dot{\phi} - V_{,\phi} \tag{1618}$$

$$V_{,\phi} = -\sqrt{8\pi G}\lambda V \tag{1619}$$

$$\rho = \frac{3H^2}{8\pi G} - \frac{1}{2}\dot{\phi}^2 - V \tag{1620}$$

$$\dot{\phi} = xH/\sqrt{\frac{4}{3}\pi G} \tag{1621}$$

$$\sqrt{V} = yH/\sqrt{\frac{8}{3}\pi G} \tag{1622}$$

we obtain

$$\frac{dx}{dN} = -3x + \frac{\sqrt{6}}{2}\lambda y^2 + \frac{3}{2}x[(1-w_m)x^2 + (1+w_m)(1-y^2)] \tag{1623}$$

$$\frac{dy}{dN} = -\frac{\sqrt{6}}{2}\lambda xy + \frac{3}{2}y[(1-w_m)x^2 + (1+w_m)(1-y^2)]. \tag{1624}$$

To find the fix points of (26.17) we need to solve

$$-3x + \frac{\sqrt{6}}{2}\lambda y^2 + \frac{3}{2}x[(1-w_m)x^2 + (1+w_m)(1-y^2)] = 0 \tag{1625}$$

$$-\frac{\sqrt{6}}{2}\lambda xy + \frac{3}{2}y[(1-w_m)x^2 + (1+w_m)(1-y^2)] = 0. \tag{1626}$$

- An obvious solution is

$$x_0 = 0, y_0 = 0. \tag{1627}$$

- Two semi-obvious solutions can be found for $y = 0$ which solves the second equation and transforms the first to the quadratic equation $x^2 - 1 = 0$ which gives

$$x_1 = +1, y_1 = 0 \tag{1628}$$

$$x_2 = -1, y_2 = 0. \tag{1629}$$

- Substituting the square bracket of the second equation into the first and simplifying the second gives

$$-3x + \frac{\sqrt{6}}{2}\lambda(x^2 + y^2) = 0 \tag{1630}$$

$$-\frac{\sqrt{6}}{2}\lambda x + \frac{3}{2}[1 + 2x^2 - (x^2 + y^2) - w_m((x^2 + x^2) - 1)] = 0. \tag{1631}$$

  Now we can eliminate $x^2 + y^2$ and obtain a single quadratic equation in x

$$-\frac{\sqrt{6}}{2}\lambda x + \frac{3}{2}\left[1 + 2x^2 - \frac{\sqrt{6}}{\lambda}x - w_m\left(\frac{\sqrt{6}}{\lambda}x - 1\right)\right] = 0 \tag{1632}$$

  which can be simplified to

$$x^2 - \frac{3(1+w_m) + \lambda^2}{\sqrt{6}\lambda}x + \frac{1+w_m}{2} = 0. \tag{1633}$$

  This gives us two more solutions

$$x_3 = \frac{\lambda}{\sqrt{6}}, y_3 = \sqrt{1 - \frac{\lambda^2}{6}} \qquad (\lambda^2 < 6) \tag{1634}$$

$$x_4 = \sqrt{\frac{3}{2}}\frac{1+w_m}{\lambda}, y_4 = \sqrt{\frac{3}{2}}\frac{\sqrt{1-w_m^2}}{\lambda} \qquad (w_m^2 < 1). \tag{1635}$$

- Let's quickly check the stability of the fix points. The characteristic equation for the fix points of a 2d system is given by

$$\alpha^2 + a_1(x_i, y_i)\alpha + a_2(x_i, y_i) = 0 \tag{1636}$$

$$a_1(x_i, y_i) = -\left(\frac{df_x}{dx} + \frac{df_y}{dy}\right)_{x=x_i, y=y_i} \tag{1637}$$

$$a_2(x_i, y_i) = \left.\frac{df_x}{dx}\frac{df_y}{dy} - \frac{df_x}{dy}\frac{df_y}{dx}\right|_{x=x_i, y=y_i} \tag{1638}$$

  with the stability classification (assuming for EoS parameter $w_m^2 < 1$)

type	condition	fix point 0	fix point 1	fix point 2
saddle node	$a_2 < 0$	$-1 < w_m < 1$	$\lambda > \sqrt{6}$	$\lambda < -\sqrt{6}$
unstable node	$0 < a_2 < a_1^2/4$	-	$\lambda < \sqrt{6}$	$\lambda > -\sqrt{6}$
unstable spiral	$a_1^2/4 < a_2, a_1 < 0$	-	-	-
center	$0 < a_2, a_1 = 0$	-	-	-
stable spiral	$a_1^2/4 < a_2, a_1 > 0$	-	-	-
stable node	$0 < a_2 < a_1^2/4$	-	-	-

type	fix point 3	fix point 4
saddle node	$3(1+w_m) < \lambda^2 < 6$	-
unstable node	-	-
unstable spiral	-	-
center	-	-
stable spiral	-	$\lambda^2 > \frac{24(1+w_m)^2}{7+9w_m}$
stable node	$\lambda^2 < 3(1+w_m)$	$\lambda^2 < \frac{24(1+w_m)^2}{7+9w_m}$

0.16.16 Problem 26.5 - Tracker solution

Inserting the ansatz

$$\phi(t) = C(\alpha, n) M^{1+\nu} t^{\nu} \tag{1639}$$

into the ODE

$$\ddot{\phi} + \frac{3\alpha}{t}\dot{\phi} - \frac{M^{4+n}}{\phi^{n+1}} = 0 \tag{1640}$$

gives

$$CM^{1+\nu}\nu(\nu-1)t^{\nu-2} + CM^{1+\nu}\frac{3\alpha}{t}t^{\nu-1} - \frac{M^{4+n}}{C^{n+1}M^{(n+1)(1+\nu)}t^{\nu(n+1)}} = 0 \tag{1641}$$

$$CM^{1+\nu}\left[\nu(\nu-1)+3\alpha\right]t^{\nu-2} - \frac{M^{3-\nu(n+1)}}{C^{n+1}}t^{-\nu(n+1)} = 0 \tag{1642}$$

From equating coefficients and powers (in t) we obtain

$$\nu = \frac{2}{2+n} \tag{1643}$$

$$C(\alpha, n) = \left(\frac{(2+n)^2}{6\alpha(2+n)-2n}\right)^{\frac{1}{2+n}}. \tag{1644}$$

0.17 Veltman - Diagrammatica

0.17.1 Problem 1.1 - Matrix exponential

We compare

$$e^{\alpha} = 1 + \alpha + \frac{1}{2!}\alpha^2 + \frac{1}{3!}\alpha^3 + ... + \frac{1}{n!}\alpha^n + ... \tag{1645}$$

$$\left[1 + \frac{1}{n}\alpha\right]^n = \sum_k \binom{n}{k} \frac{1}{n^k}\alpha^k = \sum_k \frac{n!}{k!(n-k)!}\frac{1}{n^k}\alpha^k \tag{1646}$$

$$= \frac{n!}{0!(n-0)!}\frac{1}{n^0}\alpha^0 + \frac{n!}{1!(n-1)!}\frac{1}{n}\alpha^1 + \frac{n!}{2!(n-2)!}\frac{1}{n^2}\alpha^2 + \frac{n!}{3!(n-3)!}\frac{1}{n^3}\alpha^3 + ... \tag{1647}$$

$$= 1 + \alpha + \frac{1}{2!}\underbrace{\frac{n(n-1)}{n^2}}_{\to 1}\alpha^2 + \frac{1}{3!}\underbrace{\frac{n(n-1)(n-2)}{n^3}}_{\to 1}\alpha^3 + ... \tag{1648}$$

0.17.2 Problem 1.2 - Lorentz rotation

Calculating the matrix product in first order we obtain

$$RR^T = \begin{pmatrix} a^2+b^2+(g+1)^2 & a(h+1)+bc+d(g+1) & af+b(k+1)+e(g+1) & 0 \\ a(h+1)+bc+d(g+1) & c^2+d^2+(h+1)^2 & c(k+1)+de+f(h+1) & 0 \\ af+b(k+1)+e(g+1) & c(k+1)+de+f(h+1) & e^2+f^2+(k+1)^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1649}$$

$$\simeq \begin{pmatrix} 1+2g & a+d & b+e & 0 \\ a+d & 1+2h & cf & 0 \\ b+e & c+f & 1+2k & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1650}$$

This only becomes the identity for $g = h = k = 0$ as well as $a = -d$, $b = -e$ and $c = -f$.

0.18 Banks - Quantum Field Theory

0.18.1 Problem 2.2 - Time evolution operator in the Dirac picture

With the definitions

$$i\partial_t U_S = (H_0 + V)U_S \tag{1651}$$

$$U_D(t,t_0) = e^{iH_0 t} U_S(t,t_0) e^{-iH_0 t_0} \tag{1652}$$

we can start rewriting

$$\begin{aligned}
i\partial_t U_D(t,t_0) &= i\partial_t \left(e^{iH_0 t} U_S(t,t_0) e^{-iH_0 t_0}\right) && (1653)\\
&= i^2 H_0 \underbrace{e^{iH_0 t} U_S(t,t_0) e^{-iH_0 t_0}}_{=U_D} + e^{iH_0 t} i[\partial_t U_S(t,t_0)] e^{-iH_0 t_0} && (1654)\\
&= -H_0 U_D(t,t_0) + e^{iH_0 t} i[\partial_t U_S(t,t_0)] e^{-iH_0 t_0} && (1655)\\
&= -H_0 U_D(t,t_0) + e^{iH_0 t}(H_0 + V) U_S(t,t_0) e^{-iH_0 t_0} && (1656)\\
&= -H_0 U_D(t,t_0) + H_0 \underbrace{e^{iH_0 t} U_S(t,t_0) e^{-iH_0 t_0}}_{=U_D} + e^{iH_0 t} V U_S(t,t_0) e^{-iH_0 t_0} && (1657)\\
&= e^{iH_0 t} V U_S(t,t_0) e^{-iH_0 t_0} && (1658)\\
&= e^{iH_0 t} V \underbrace{e^{-iH_0 t} e^{iH_0 t}}_{=1} U_S(t,t_0) e^{-iH_0 t_0} && (1659)\\
&= e^{iH_0 t} V e^{-iH_0 t} U_D(t,t_0) && (1660)
\end{aligned}$$

0.19 Kugo - Eichtheorie

0.19.1 Problem 1.1

With $\Lambda^{\alpha}_{\ \mu} \approx \delta^{\alpha}_{\mu} + \epsilon^{\alpha}_{\ \mu}$ we obtain

$$
\begin{aligned}
g_{\mu\nu} &= \Lambda^{\alpha}_{\ \mu}\Lambda^{\beta}_{\ \nu} g_{\alpha\beta} && (1661)\\
&\simeq \left(\delta^{\alpha}_{\mu} + \epsilon^{\alpha}_{\ \mu}\right)\left(\delta^{\beta}_{\nu} + \epsilon^{\beta}_{\ \nu}\right) g_{\alpha\beta} && (1662)\\
&\simeq g_{\mu\nu} + \epsilon^{\alpha}_{\ \mu}\delta^{\beta}_{\nu} g_{\alpha\beta} + \epsilon^{\beta}_{\ \nu}\delta^{\alpha}_{\mu} g_{\alpha\beta} + \mathcal{O}(\epsilon^2) && (1663)\\
&\simeq g_{\mu\nu} + \epsilon_{\nu\mu} + \epsilon_{\mu\nu} + \mathcal{O}(\epsilon^2) && (1664)
\end{aligned}
$$

which means that ϵ is antisymmetric $\epsilon_{\nu\mu} = -\epsilon_{\mu\nu}$ and we can write

$$
\epsilon_{\nu\mu} = \frac{1}{2}\left(\epsilon_{\nu\mu} - \epsilon_{\mu\nu}\right). \quad (1665)
$$

The infinitesimal Poincare transformation can then be written as

$$
\begin{aligned}
x'^{\mu} &= \Lambda^{\mu}_{\ \alpha} x^{\alpha} + a^{\mu} && (1666)\\
&\simeq \left(\delta^{\mu}_{\alpha} + \epsilon^{\mu}_{\ \alpha}\right) x^{\alpha} + a^{\mu} && (1667)\\
&\simeq x^{\mu} + \epsilon^{\mu}_{\ \alpha} x^{\alpha} + a^{\mu}. && (1668)
\end{aligned}
$$

The inverted PT is then given by

$$
\begin{aligned}
x &= \Lambda^{-1}(x' - a) && (1669)\\
&= \Lambda^{-1}x' - \Lambda^{-1}a && (1670)\\
x^{\mu} &\simeq \left(\delta^{\mu}_{\alpha} - \epsilon^{\mu}_{\ \alpha}\right) x'^{\alpha} - \left(\delta^{\mu}_{\alpha} - \epsilon^{\mu}_{\ \alpha}\right) a^{\alpha} && (1671)\\
&\simeq x'^{\mu} - \epsilon^{\mu}_{\ \alpha} x'^{\alpha} - a^{\mu} + \mathcal{O}(\epsilon \cdot a) && (1672)
\end{aligned}
$$

Because of

$$
\begin{aligned}
\phi'(x') = \phi(x) \quad &\Leftrightarrow \quad \phi'(\Lambda x + a) = \phi(x) && (1673)\\
&\Leftrightarrow \quad \phi'(x) = \phi(\Lambda^{-1}(x - a)) && (1674)
\end{aligned}
$$

we can now calculate

$$
\begin{aligned}
\delta\phi(x) &\equiv \phi'(x) - \phi(x) && (1675)\\
&= \phi(\Lambda^{-1}(x - a)) - \phi(x) && (1676)\\
&\simeq \phi(x^{\mu} - \epsilon^{\mu}_{\ \alpha} x^{\alpha} - a^{\mu}) - \phi(x) && (1677)\\
&\simeq \phi(x) + \partial_{\mu}\phi(x) \cdot (-\epsilon^{\mu}_{\ \alpha} x^{\alpha} - a^{\mu}) - \phi(x) && (1678)\\
&\simeq -(a^{\mu} + \epsilon^{\mu}_{\ \alpha} x^{\alpha})\partial_{\mu}\phi(x) && (1679)\\
&\simeq -(a^{\mu} + \epsilon^{\mu\alpha} x_{\alpha})\partial_{\mu}\phi(x) && (1680)\\
&\simeq -\left(a^{\mu} + \frac{1}{2}\left(\epsilon^{\mu\alpha} - \epsilon^{\alpha\mu}\right) x_{\alpha}\right)\partial_{\mu}\phi(x) && (1681)\\
&\simeq -\left(a^{\mu}\partial_{\mu} + \frac{1}{2}\left(\epsilon^{\mu\alpha} x_{\alpha}\partial_{\mu} - \epsilon^{\alpha\mu} x_{\alpha}\partial_{\mu}\right)\right)\phi(x) && (1682)\\
&\simeq -\left(a^{\mu}\partial_{\mu} + \frac{1}{2}\left(\epsilon^{\mu\alpha} x_{\alpha}\partial_{\mu} - \epsilon^{\mu\alpha} x_{\mu}\partial_{\alpha}\right)\right)\phi(x) && (1683)\\
&\simeq i\left(a^{\mu} i\partial_{\mu} + \frac{1}{2}\epsilon^{\mu\alpha} i\left(x_{\alpha}\partial_{\mu} - x_{\mu}\partial_{\alpha}\right)\right)\phi(x) && (1684)\\
&\simeq i\left(a^{\mu} i\partial_{\mu} - \frac{1}{2}\epsilon^{\mu\alpha} i\left(x_{\mu}\partial_{\alpha} - x_{\alpha}\partial_{\mu}\right)\right)\phi(x) && (1685)\\
&\simeq i\left(a^{\mu} P_{\mu} - \frac{1}{2}\epsilon^{\mu\alpha} M_{\mu\alpha}\right)\phi(x) && (1686)
\end{aligned}
$$

Calculating the commutators

$$[P_\mu, P_\nu] = 0 \tag{1687}$$

$$\begin{aligned}
[M_{\mu\nu}, P_\rho] &= i^2(x_\mu\partial_\nu - x_\nu\partial_\mu)\partial_\rho - i^2\partial_\rho(x_\mu\partial_\nu - x_\nu\partial_\mu) && (1688)\\
&= -(x_\mu\partial_\nu - x_\nu\partial_\mu)\partial_\rho + \partial_\rho(x_\mu\partial_\nu - x_\nu\partial_\mu) && (1689)\\
&= -x_\mu\partial_\nu\partial_\rho + x_\nu\partial_\mu\partial_\rho + (\partial_\rho g_{\mu\alpha}x^\alpha)\partial_\nu + x_\mu\partial_\rho\partial_\nu - (\partial_\rho g_{\nu\alpha}x^\alpha)\partial_\mu - x_\nu\partial_\rho\partial_\mu && (1690)\\
&= (\partial_\rho g_{\mu\alpha}x^\alpha)\partial_\nu - (\partial_\rho g_{\nu\alpha}x^\alpha)\partial_\mu && (1691)\\
&= (g_{\mu\alpha}\partial_\rho x^\alpha)\partial_\nu - (g_{\nu\alpha}\partial_\rho x^\alpha)\partial_\mu && (1692)\\
&= (g_{\mu\alpha}\delta^\alpha_\rho)\partial_\nu - (g_{\nu\alpha}\delta^\alpha_\rho)\partial_\mu && (1693)\\
&= g_{\mu\rho}\partial_\nu - g_{\nu\rho}\partial_\mu && (1694)\\
&= -i(g_{\mu\rho}i\partial_\nu - g_{\nu\rho}i\partial_\mu) && (1695)\\
&= -i(g_{\mu\rho}P_\nu - g_{\nu\rho}P_\mu) && (1696)
\end{aligned}$$

$$[M_{\mu\nu}, M_{\rho,\sigma}] = \text{...painful} \tag{1697}$$

0.20 LeBellac - Quantum and Statistical Field Theory

0.20.1 Problem 1.1

Some simple geometry

$$\begin{aligned}
l &= 2a\cos\theta && (1698)\\
x &= l\sin\theta && (1699)\\
&= 2a\cos\theta\sin\theta && (1700)\\
h &= x\tan\theta && (1701)\\
&= 2a\sin^2\theta && (1702)
\end{aligned}$$

Then the potential is given by

$$\begin{aligned}
V(\phi) &= 2mga\sin^2\theta + \frac{1}{2}Ca^2(2\cos\theta - 1)^2 && (1703)\\
\frac{\partial V}{\partial\theta} &= 4mga\sin\theta\cos\theta - 2Ca^2(2\cos\theta - 1)\sin\theta && (1704)\\
&= 2a\sin\theta\,(2mg\cos\theta - Ca(2\cos\theta - 1)) && (1705)\\
&= 2a\sin\theta\,(2(mg - Ca)\cos\theta + Ca) && (1706)\\
&\to \theta_0 = 0 && (1707)\\
&\to \theta_{1,2} = \arccos\frac{Ca}{2(Ca - mg)} && (1708)
\end{aligned}$$

Stability

$$\begin{aligned}
\frac{\partial^2 V}{\partial\theta^2}(\theta_{1,2}) &= 2a(2mg - Ca) && (1709)\\
\frac{\partial^2 V}{\partial\theta^2}(\theta_0) &= 2a(2mg - Ca) && (1710)
\end{aligned}$$

0.21 de Witt - Dynamical theory of groups and fields

0.21.1 Problem 1 - Functional derivatives of actions

$$\delta F = \int dx \frac{\delta F[\phi]}{\delta\phi(x)} \cdot \delta\phi(x) \tag{1711}$$
$$= \int dx \frac{\delta F[\phi]}{\delta\phi(x)} \cdot \epsilon\delta(x-y) \tag{1712}$$
$$= \epsilon\frac{\delta F[\phi]}{\delta\phi(y)} \tag{1713}$$
$$= F[\phi + \epsilon\delta(x-y)] - F[\phi] \tag{1714}$$

which means

$$\frac{\delta F[\phi]}{\delta\phi(y)} = \lim_{\epsilon\to 0}\frac{F[\phi+\epsilon\delta(x-y)]-F[\phi]}{\epsilon} \tag{1715}$$
$$F[\phi+\epsilon\delta(x-y)] = F[\phi] + \epsilon\frac{\delta F[\phi]}{\delta\phi(y)} \tag{1716}$$
$$= F[\phi] + \epsilon\int dx \frac{\delta F[\phi]}{\delta\phi(x)} \cdot \delta(x-y) \tag{1717}$$

Now

(a) Neutral scalar meson

$$S = \int dx\ L(\varphi, \varphi_{,\mu}) \tag{1718}$$
$$= -\frac{1}{2}\int dx\ (\varphi_{,\mu}\varphi^{,\mu} + m^2\varphi^2) \tag{1719}$$
$$= -\frac{1}{2}\left(\int dx\ (\varphi_{,\mu}\varphi^{,\mu} + m^2\varphi^2)\right) \tag{1720}$$
$$= -\frac{1}{2}\left(\int dx\ \varphi_{,\mu}\varphi^{,\mu} + \int dx\ m^2\varphi^2)\right) \tag{1721}$$

Now we calculate the first part (all derivatives are with respect to x) neglecting $\mathcal{O}(\epsilon^2)$

$$\frac{\delta S_1[\varphi]}{\delta\varphi(y)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int dx\ g^{\mu\nu}(\varphi(x)+\epsilon\delta(x-y))_{,\mu}(\varphi(x)+\epsilon\delta(x-y))_{,\nu} - \int dx\ g^{\mu\nu}\varphi(x)_{,\mu}\varphi(x)_{,\nu}\right) \tag{1722}$$
$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int dx\ g^{\mu\nu}(\varphi(x)_{,\mu}+\epsilon\partial_\mu\delta(x-y))(\varphi(x)_{,\nu}+\epsilon\partial_\nu\delta(x-y)) - \int dx\ g^{\mu\nu}\varphi(x)_{,\mu}\varphi(x)_{,\nu}\right) \tag{1723}$$
$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int dx\ g^{\mu\nu}(\varphi_{,\mu}\varphi_{,\nu}+\epsilon\varphi_{,\nu}\partial_\mu\delta(x-y)+\epsilon\varphi_{,\mu}\partial_\nu\delta(x-y)) - \int dx\ g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}\right) \tag{1724}$$
$$= \int dx\, g^{\mu\nu}(\varphi_{,\nu}\partial_\mu\delta(x-y)+\varphi_{,\mu}\partial_\nu\delta(x-y)) \tag{1725}$$
$$= -\int dx\, g^{\mu\nu}(\varphi_{,\nu\mu}\delta(x-y)+\varphi_{,\mu\nu}\delta(x-y)) \tag{1726}$$
$$= -2\varphi^{,\mu}_{\ ,\mu}(y) \tag{1727}$$

$$\frac{\delta S_2[\varphi]}{\delta\varphi(y)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}m^2\left(\int dx\ (\varphi(x)+\epsilon\delta(x-y))(\varphi(x)+\epsilon\delta(x-y)) - \int dx\ g^{\mu\nu}\varphi(x)\varphi(x)\right) \tag{1728}$$

$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}m^2\left(\int dx\ (\varphi(x)\varphi(x)+\epsilon\delta(x-y)\varphi(x)+\epsilon\varphi(x)\delta(x-y)) - \int dx\ g^{\mu\nu}\varphi(x)\varphi(x)\right) \tag{1729}$$

$$= m^2\int dx\ (\delta(x-y)\varphi(x)+\varphi(x)\delta(x-y)) \tag{1730}$$

$$= 2m^2\varphi(y) \tag{1731}$$

and therefore

$$\frac{\delta S[\varphi]}{\delta\varphi(y)} = \varphi^{;\mu}_{,\mu}(y) - m^2\varphi(y) \tag{1732}$$

(b) Neutral vector meson

$$S_1 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = (\varphi_{\nu,\mu}-\varphi_{\mu,\nu})(\varphi^{\nu,\mu}-\varphi^{\mu,\nu}) \tag{1733}$$

$$\frac{\delta S_1[\varphi]}{\delta\varphi_\alpha(y)} = -\frac{1}{4}\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int dx\,[(\varphi_\nu(x)+\epsilon\delta^\alpha_\nu\delta(x-y))_{,\mu} - (\varphi_\mu(x)+\epsilon\delta^\alpha_\mu\delta(x-y))_{,\nu}] \tag{1734}$$

$$\cdot\,[(\varphi^\nu(x)+\epsilon\delta^{\alpha\nu}\delta(x-y))^{,\mu} - (\varphi^\mu(x)+\epsilon\delta^{\alpha\mu}\delta(x-y))^{,\nu}] - [\varphi_{\nu,\mu}-\varphi_{\mu,\nu}][\varphi^{\nu,\mu}-\varphi^{\mu,\nu}] \tag{1735}$$

$$= -\frac{1}{4}\int dx\,\big((\delta^\alpha_\nu\partial_\mu\delta(x-y)-\delta^\alpha_\mu\partial_\nu\delta(x-y))[\varphi^{\nu,\mu}-\varphi^{\mu,\nu}] \tag{1736}$$

$$+[\varphi_{\nu,\mu}-\varphi_{\mu,\nu}](\delta^{\nu\alpha}\partial^\mu\delta(x-y)-\delta^{\mu\alpha}\partial^\nu\delta(x-y))\big) \tag{1737}$$

$$= -\frac{1}{4}\int dx\,(\partial_\mu\delta(x-y))[\varphi^{\alpha,\mu}-\varphi^{\mu,\alpha}] - \partial_\nu\delta(x-y))[\varphi^{\nu,\alpha}-\varphi^{\alpha,\nu}] \tag{1738}$$

$$+[\varphi^\alpha_{,\mu}-\varphi_\mu^{;\alpha}]\partial^\mu\delta(x-y) - [\varphi_\nu^{;\alpha}-\varphi^\alpha_{,\nu}]\partial^\nu\delta(x-y)\big) \tag{1739}$$

$$= \frac{1}{4}\int dx\,\delta(x-y)\,\big([\varphi^{\alpha,\mu}_{,\mu}-\varphi^{\mu,\alpha}_{,\mu}] - [\varphi^{\nu,\alpha}_{,\nu}-\varphi^{\alpha,\nu}_{,\nu}] + [\varphi^{\alpha\ ,\mu}_{,\mu}-\varphi_\mu^{;\alpha\mu}] - [\varphi_\nu^{;\alpha\nu}-\varphi^{\alpha\ ,\nu}_{,\nu}]\big) \tag{1740}$$

$$= \frac{1}{4}\int dx\,\delta(x-y)\,\big(4\varphi^{\alpha,\mu}_{,\mu}-2\varphi^{\mu,\alpha}_{,\mu}-2\varphi_\mu^{;\alpha\mu}\big) \tag{1741}$$

$$= \varphi(y)^{\alpha,\mu}_{,\mu} - \varphi(y)^{\mu,\alpha}_{,\mu} \tag{1742}$$

and

$$S_2 = -\frac{m^2}{2}\varphi_\mu\varphi^\mu \tag{1743}$$

$$\frac{\delta S_2[\varphi]}{\delta\varphi_\alpha(y)} = -\frac{m^2}{2}\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int dx\,\big[(\varphi_\mu+\epsilon\delta^\alpha_\mu\delta(x-y))(\varphi^\mu+\epsilon\delta^{\mu\alpha}\delta(x-y)) - \varphi_\mu\varphi^\mu\big] \tag{1744}$$

$$= -\frac{m^2}{2}\int dx\,\big[\delta^\alpha_\mu\delta(x-y)\varphi^\mu + \varphi_\mu\delta^{\mu\alpha}\delta(x-y)\big] \tag{1745}$$

$$= -\frac{m^2}{2}\int dx\,[\delta(x-y)\varphi^\alpha + \varphi^\alpha\delta(x-y)] \tag{1746}$$

$$= -m^2\varphi^\alpha(y) \tag{1747}$$

therefore

$$\frac{\delta S[\varphi]}{\delta\varphi^\alpha(y)} = \varphi(y)^{\alpha,\mu}_{,\mu} - \varphi(y)^{\mu,\alpha}_{,\mu} - m^2\varphi^\alpha \tag{1748}$$

(c) Neutral tensor meson

(d) Two-level mass spectrum
Using results from (a)

$$S_2 = \frac{1}{2}\varphi_{,\mu}\varphi^{,\mu}\frac{m_1^2+m_2^2}{m_1^2-m_2^2} \tag{1749}$$

$$\frac{\delta S_2[\varphi]}{\delta\varphi(y)} = -\frac{m_1^2+m_2^2}{m_1^2-m_2^2}\varphi^{,\mu}{}_{,\mu} \tag{1750}$$

$$S_3 = \frac{1}{2}\varphi^2\frac{m_1^2m_2^2}{m_1^2-m_2^2} \tag{1751}$$

$$\frac{\delta S_3[\varphi]}{\delta\varphi(y)} = \frac{m_1^2m_2^2}{m_1^2-m_2^2}\varphi \tag{1752}$$

and

$$S_1 = \varphi^{,\mu\nu}\varphi_{,\mu\nu} \tag{1753}$$

$$\frac{\delta S_1[\varphi]}{\delta\varphi(y)} = ... \tag{1754}$$

$$= \int dx\,(\partial^{\mu\nu}\delta(x-y)\varphi_{,\mu\nu} + \varphi^{,\mu\nu}\partial_{\mu\nu}\delta(x-y)) \tag{1755}$$

$$= \int dx\,\big(\delta(x-y)\varphi_{,\mu\nu}{}^{,\mu\nu} + \varphi^{,\mu\nu}{}_{,\mu\nu}\delta(x-y)\big) \tag{1756}$$

$$= 2\varphi^{,\mu\nu}{}_{,\mu\nu}(y) \tag{1757}$$

Resulting in

$$\frac{\delta S[\varphi]}{\delta\varphi(y)} = \frac{1}{m_1^2-m_2^2}\varphi^{,\mu\nu}{}_{,\mu\nu}(y) - \frac{m_1^2+m_2^2}{m_1^2-m_2^2}\varphi^{,\mu}{}_{,\mu} + \frac{m_1^2m_2^2}{m_1^2-m_2^2}\varphi \tag{1758}$$

$$= \frac{1}{m_1^2-m_2^2}(\partial^\mu\partial_\mu - m_1^2)(\partial^\nu\partial_\nu - m_2^2)\varphi \tag{1759}$$

0.21.2 Problem 2 - More Lagrangians

(a) Notation is a bit odd - vector field φ^μ and scalar field φ

$$\frac{\partial L}{\partial\varphi^\beta} - \partial_\alpha\frac{\partial L}{\partial\varphi^\beta_{,\alpha}} = 0 \tag{1760}$$

$$\varphi_\beta - \frac{1}{2}\varphi_{,\beta} - \partial_\alpha\left(\frac{1}{2}\varphi\delta^{\mu\alpha}\delta_{\mu\beta}\right) = 0 \tag{1761}$$

$$\to \varphi_\beta - \varphi_{,\beta} = 0 \tag{1762}$$

$$\frac{\partial L}{\partial\varphi} - \partial_\alpha\frac{\partial L}{\partial\varphi_{,\alpha}} = 0 \tag{1763}$$

$$\frac{1}{2}\varphi^\mu{}_{,\mu} - m^2\varphi - \partial_\alpha\left(-\frac{1}{2}\varphi^\alpha\right) = 0 \tag{1764}$$

$$\to \varphi^\mu{}_{,\mu} - m^2\varphi = 0 \tag{1765}$$

now we can separate both equations of motion by

$$\varphi^\alpha - \varphi^{,\alpha} = 0 \quad \to \quad \varphi^\alpha{}_{,\alpha} - \varphi^{,\alpha}{}_{,\alpha} = 0 \tag{1766}$$

$$\varphi^\mu{}_{,\mu\alpha} - m^2\varphi_{,\alpha} = 0 \tag{1767}$$

and obtain

$$\varphi^{,\alpha}_{\ \ ,\alpha} - m^2\varphi = 0 \tag{1768}$$

$$\varphi^{\mu}_{\ \ ,\mu\alpha} - m^2\varphi_\alpha = 0 \qquad \text{or better} \qquad \varphi_{,\beta} = \varphi_\beta \tag{1769}$$

(b)

(c)

0.21.3 Problem 3 - Implied equations of motion

(a) Nothing to do

(b)

(c)

0.22 University of Cambridge: Tong - Quantum Field Theory Oct 2007

0.22.1 Problem 1.1 - Vibrating string

A string of length a, mass per unit length σ and under tension T is fixed at each end. The Lagrangian governing the time evolution of the transverse displacement $y(x,t)$ is

$$L = \int_0^a dx \left[\frac{\sigma}{2} \left(\frac{\partial y}{\partial t} \right)^2 - \frac{T}{2} \left(\frac{\partial y}{\partial x} \right)^2 \right] \tag{1770}$$

where x identifies position along the string from one end point. By expressing the displacement as a sine series Fourier expansion in the form

$$y(x,t) = \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a} \right) q_n(t) \tag{1771}$$

show that the Lagrangian becomes

$$L = \sum_{n=1}^{\infty} \left[\frac{\sigma}{2} \dot{q}_n^2 - \frac{T}{2} \left(\frac{n\pi}{a} \right)^2 q_n^2 \right] \tag{1772}$$

Derive the equations of motion. Hence show that the string is equivalent to an infinite set of decoupled harmonic oscillators with frequencies

$$\omega_n = \sqrt{\frac{T}{\sigma}} \left(\frac{n\pi}{a} \right) \tag{1773}$$

Using the orthogonality of $\sin mx, \cos mx$

$$\frac{\partial y}{\partial t} = \sqrt{\frac{2}{a}} \sum_{n=1} \sin\left(\frac{n\pi x}{a} \right) \dot{q}_n \tag{1774}$$

$$\left(\frac{\partial y}{\partial t} \right)^2 = \frac{2}{a} \left(\sum_n \sin\left(\frac{n\pi x}{a} \right) \dot{q}_n \right)^2 \tag{1775}$$

$$= \frac{2}{a} \sum_n \sin^2\left(\frac{n\pi x}{a} \right) \dot{q}_n^2 + \frac{2}{a} \sum_{n,m} 2 \sin\left(\frac{n\pi x}{a} \right) \sin\left(\frac{m\pi x}{a} \right) \dot{q}_n \dot{q}_m \tag{1776}$$

$$\int_0^a \left(\frac{\partial y}{\partial t} \right)^2 dx = \frac{2}{a} \dot{q}_n^2 \sum_n \frac{a}{2} \tag{1777}$$

$$\frac{\partial y}{\partial x} = \sqrt{\frac{2}{a}} \sum_{n=1} \cos\left(\frac{n\pi x}{a} \right) \frac{n\pi}{a} q_n \tag{1778}$$

$$\left(\frac{\partial y}{\partial x} \right)^2 = \frac{2}{a} \left(\sum_n \cos\left(\frac{n\pi x}{a} \right) \frac{n\pi}{a} q_n \right)^2 \tag{1779}$$

$$= \frac{2}{a} \sum_n \cos^2\left(\frac{n\pi x}{a} \right) \frac{n^2\pi^2}{a^2} q_n^2 + \frac{2}{a} \sum_{n,m} 2 \cos\left(\frac{n\pi x}{a} \right) \cos\left(\frac{m\pi x}{a} \right) \frac{nm\pi^2}{a^2} q_n q_m \tag{1780}$$

$$\int_0^a \left(\frac{\partial y}{\partial x} \right)^2 dx = \frac{2}{a} q_n^2 \sum_n \frac{a}{2} \left(\frac{n\pi}{a} \right)^2 \tag{1781}$$

Then we see

$$L = \sum_n \left[\frac{\sigma}{2} \dot{q}_n^2 - \frac{T}{2} \left(\frac{n\pi}{a} \right)^2 q_n^2 \right] \tag{1782}$$

and therefore

$$\frac{\partial L}{\partial q_n} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_n} = 0 \tag{1783}$$

$$-\frac{T}{2}\left(\frac{n\pi}{a}\right)^2 2q_n - \frac{d}{dt}\frac{\sigma}{2}2\dot{q}_n = 0 \tag{1784}$$

$$-T\left(\frac{n\pi}{a}\right)^2 q_n - \sigma\ddot{q}_n = 0 \tag{1785}$$

$$\ddot{q}_n + \frac{T}{\sigma}\left(\frac{n\pi}{a}\right)^2 q_n = 0 \tag{1786}$$

0.22.2 Problem 1.2 - Lorentz transformation of the Klein-Gordon equation

Show directly that if $\phi(x)$ satisfies the Klein-Gordon equation, then $\phi(\Lambda^{-1}x)$ also satisfies this equation for any Lorentz transformation Λ.

With $x' = \Lambda x$ or $(x = \Lambda^{-1}x')$ and

$$\phi(x) \to \phi'(x') \equiv \phi(x) = \phi(\Lambda^{-1}x') \tag{1787}$$

we need to calculate the first derivative

$$\partial'_\mu \phi'(x') = \partial'_\mu \phi(x) = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial}{\partial x^\alpha}\phi(x) \tag{1788}$$

$$= (\Lambda^{-1})^\alpha_{\ \beta}\delta^\beta_\mu \frac{\partial}{\partial x^\alpha}\phi(x) \tag{1789}$$

$$= (\Lambda^{-1})^\alpha_{\ \mu}\frac{\partial}{\partial x^\alpha}\phi(x) \tag{1790}$$

and the second derivative

$$\eta^{\mu\nu}\partial'_\nu\partial'_\mu\phi'(x') = \underbrace{\eta^{\mu\nu}(\Lambda^{-1})^\alpha_{\ \mu}(\Lambda^{-1})^\beta_{\ \nu}}_{=\eta^{\alpha\beta}}\partial_\beta\partial_\alpha\phi(x) \tag{1791}$$

$$= \eta^{\alpha\beta}\partial_\beta\partial_\alpha\phi(x) \tag{1792}$$

and therefore

$$(\partial'^\mu\partial_\mu + m^2)\phi'(x') = \partial'^\mu\partial_\mu\phi'(x') + m^2\phi'(x') \tag{1793}$$

$$= \partial^\mu\partial_\mu\phi(x) + m^2\phi'(x') \tag{1794}$$

$$= \partial^\mu\partial_\mu\phi(x) + m^2\phi(x) \tag{1795}$$

$$= 0 \tag{1796}$$

0.22.3 Problem 1.3 - Complex Klein-Gordon field

The motion of a complex field $\psi(x)$ is governed by the Lagrangian

$$\mathcal{L} = \partial_\mu\psi^*\partial^\mu\psi - m^2\psi^*\psi - \frac{\lambda}{2}(\psi^*\psi)^2 \tag{1797}$$

Write down the Euler-Lagrange field equations for this system. Verify that the Lagrangian is invariant under the infinitesimal transformation

$$\delta\psi = i\alpha\psi, \qquad \delta\psi^* = -i\alpha\psi^* \tag{1798}$$

Derive the Noether current associated with this transformation and verify explicitly that it is conserved using the field equations satisfied by ψ.

With

$$\mathcal{L} = \eta^{\mu\nu}\partial_\mu\psi^*\partial_\nu\psi - m^2\psi^*\psi - \frac{\lambda}{2}(\psi^*\psi)^2 \tag{1799}$$

$$\frac{\partial\mathcal{L}}{\partial\psi} = -m^2\psi^* - \lambda(\psi^*\psi)\psi^* \tag{1800}$$

$$\frac{\partial\mathcal{L}}{\partial\psi^*} = -m^2\psi - \lambda(\psi^*\psi)\psi \tag{1801}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\psi)} = \eta^{\mu\nu}\partial_\mu\psi^*\delta^\alpha_\nu = \eta^{\mu\alpha}\partial_\mu\psi^* = \partial^\alpha\psi^* \tag{1802}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\psi^*)} = \partial^\alpha\psi \tag{1803}$$

then we calculate the equation of motions

$$\partial_\alpha\partial^\alpha\psi^* + m^2\psi^* + \lambda(\psi^*\psi)\psi^* = 0 \tag{1804}$$

$$\partial_\alpha\partial^\alpha\psi + m^2\psi + \lambda(\psi^*\psi)\psi = 0 \tag{1805}$$

Infinitesimal variation of the Lagrangian

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\psi_a}\delta\psi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_a)}\overbrace{\delta(\partial_\mu\psi_a)}^{=\partial_\mu(\delta\psi_a)} \tag{1806}$$

$$= \left[\frac{\partial\mathcal{L}}{\partial\psi_a} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_a)}\right]\delta\psi_a + \partial_\mu\underbrace{\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_a)}\delta\psi_a\right)}_{=j^\mu} \tag{1807}$$

Lagrangian invariance - substitute infinitesimal trafo $\delta\psi, \delta\psi^*$

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\psi_a}\delta\psi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_a)}\overbrace{\delta(\partial_\mu\psi_a)}^{=\partial_\mu(\delta\psi_a)} \tag{1808}$$

$$= i\alpha\left[-m^2\underbrace{(\psi^*\psi - \psi\psi^*)}_{=0} - \lambda(\psi^*\psi)\underbrace{(\psi^*\psi - \psi\psi^*)}_{=0} + \underbrace{(\partial^\mu\psi^*)\partial_\mu\psi - (\partial^\mu\psi)\partial_\mu\psi^*}_{=0}\right] \tag{1809}$$

$$= 0 \tag{1810}$$

Noether current

$$\partial_\mu j^\mu = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_a)}\delta\psi_a\right) \tag{1811}$$

$$= \partial_\mu\left(\partial^\mu\psi^*\delta\psi + \partial^\mu\psi\delta\psi^*\right) \tag{1812}$$

$$= i\alpha\partial_\mu\left[(\partial^\mu\psi^*)\psi - (\partial^\mu\psi)\psi^*\right] \tag{1813}$$

$$= i\alpha\left[(\partial_\mu\partial^\mu\psi^*)\psi - (\partial_\mu\partial^\mu\psi)\psi^* + (\partial^\mu\psi^*)(\partial_\mu\psi) - (\partial^\mu\psi)(\partial_\mu\psi^*)\right] \tag{1814}$$

$$= i\alpha\left[(\partial_\mu\partial^\mu\psi^*)\psi - (\partial_\mu\partial^\mu\psi)\psi^*\right] \tag{1815}$$

$$= i\alpha\left[(m^2\psi^* + (\psi^*\psi)\psi^*)\psi - (m^2\psi + (\psi^*\psi)\psi)\psi^*\right] \tag{1816}$$

$$= 0 \tag{1817}$$

0.22.4 Problem 1.4 - Lagrangian for a triplet of real fields - NOT FINISHED

Verify that the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi_a \partial^\mu \phi_a - \frac{1}{2}m^2 \phi_a \phi_a \tag{1818}$$

for a triplet of real fields ϕ_a $(a = 1, 2, 3)$ is invariant under the infinitesimal SO(3) rotation by θ

$$\phi_a \to \phi_a + \theta \epsilon_{abc} n_b \phi_c \tag{1819}$$

where n_a is a unit vector. Compute the Noether current j_μ. Deduce that the three quantities

$$Q_a = \int d^3x \, \epsilon_{abc} \dot{\phi}_b \phi_c \tag{1820}$$

are all conserved and verify this directly using the field equations satisfied by ϕ_a.

$$\frac{\partial \mathcal{L}}{\partial \phi_a} = -m^2 \phi_a \tag{1821}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_a)} = \eta^{\mu\nu} \partial_\mu \phi_a \delta^\alpha_\nu = \eta^{\mu\alpha} \partial_\mu \phi_a = \partial^\alpha \phi_a \tag{1822}$$

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi_a} \delta \phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \overbrace{\delta(\partial_\mu \phi_a)}^{=\partial_\mu(\delta \phi_a)} \tag{1823}$$

$$= -m \phi_a \theta \epsilon_{abc} n_b \phi_c + (\partial^\mu \phi_a) \theta \epsilon_{abc} n_b \partial_\mu \phi_c \tag{1824}$$

$$= \theta[\epsilon_{abc} n_b (\partial^\mu \phi_a)(\partial_\mu \phi_c) - m \epsilon_{abc} n_b \phi_a \phi_c] \tag{1825}$$

$$= \theta[-n_b \epsilon_{bac} (\partial^\mu \phi_a)(\partial_\mu \phi_c) + m n_b \epsilon_{bac} \phi_a \phi_c] \tag{1826}$$

$$= \theta[-\vec{n} \cdot (\partial_\mu \phi \times \partial_\mu \phi) + m \vec{n} \cdot (\vec{\phi} \times \vec{\phi})] \tag{1827}$$

$$= 0 \tag{1828}$$

Noether current

$$j^\mu = \theta(\partial^\mu \phi_a) \epsilon_{abc} n_b \phi_c \tag{1829}$$

$$j^0 = -\theta n_b \epsilon_{bac} \phi_c \dot{\phi}_a \tag{1830}$$

0.22.5 Problem 1.5 - Lorentz transformation

A Lorentz transformation $x^\mu \to x'^\mu = \Lambda^\mu_\nu x^\nu$ is such that it preserves the Minkowski metric $\eta_\mu\nu$ meaning that $\eta_{\mu\nu} x^\mu x^\nu = \eta_{\mu\nu} x'^\mu x'^\nu$ for all x. Show that this implies that

$$\eta_{\mu\nu} = \eta_{\sigma\tau} \Lambda^\sigma_{\ \mu} \Lambda^\tau_{\ \nu}. \tag{1831}$$

Use this result to show that an infinitesimal transformation of the form

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu \tag{1832}$$

Write down the matrix form for ω^μ_ν that corresponds to a rotation through an in- finitesimal angle θ about the x^3-axis. Do the same for a boost along the x^1-axis by an infinitesimal velocity v.

$$\eta_{\mu\nu}x^\mu x^\nu = \eta_{\mu\nu}x'^\mu x'^\nu \tag{1833}$$
$$= \eta_{\sigma\tau}(\Lambda^\sigma_{\ \mu}x^\mu)(\Lambda^\tau_{\ \nu}x^\nu) \tag{1834}$$
$$= \eta_{\sigma\tau}\Lambda^\sigma_{\ \mu}\Lambda^\tau_{\ \nu}x^\mu x^\nu \tag{1835}$$
$$\to \eta_{\mu\nu} = \eta_{\sigma\tau}\Lambda^\sigma_{\ \mu}\Lambda^\tau_{\ \nu} \tag{1836}$$

then

$$\eta_{\mu\nu} = \eta_{\sigma\tau}\Lambda^\sigma_{\ \mu}\Lambda^\tau_{\ \nu} \tag{1837}$$
$$= \eta_{\sigma\tau}(\delta^\sigma_{\ \mu} + \omega^\sigma_{\ \mu})(\delta^\tau_{\ \nu} + \omega^\tau_{\ \nu}) \tag{1838}$$
$$= \eta_{\sigma\tau}\delta^\sigma_{\ \mu}\delta^\tau_{\ \nu} + \eta_{\sigma\tau}\delta^\sigma_{\ \mu}\omega^\tau_{\ \nu} + \eta_{\sigma\tau}\omega^\sigma_{\ \mu}\delta^\tau_{\ \nu} + \mathcal{O}(\omega^2) \tag{1839}$$
$$\simeq \eta_{\mu\nu} + \eta_{\mu\tau}\omega^\tau_{\ \nu} + \eta_{\sigma\nu}\omega^\sigma_{\ \mu} \tag{1840}$$
$$\simeq \eta_{\mu\nu} + \omega^{\mu\nu} + \omega^{\nu\mu} \tag{1841}$$
$$\to \omega^{\mu\nu} = -\omega^{\nu\mu} \tag{1842}$$

Rotation in the $x - y$ plane (t and z are undisturbed)

$$\omega^\mu_{\ \nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & -\epsilon & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1843}$$

Boost in the x direction (y and z are undisturbed)

$$\omega^\mu_{\ \nu} = \begin{pmatrix} 0 & \epsilon & 0 & 0 \\ \epsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1844}$$

Note that $\omega^\mu_{\ \nu}$ for the boost is symmetric and becomes antisymmetric when $\omega_{\alpha\nu} = \eta_{\alpha\mu}\omega^\mu_{\ \nu}$.

0.22.6 Problem 1.6 - Lorentz transformation of a scalar field - NOT FINISHED

Consider the infinitesimal form of the Lorentz transformation derived in the previous question: $x^\mu \to x^\mu + \omega^\mu_{\ \nu}x^\nu$. Show that a scalar field transforms as

$$\phi(x) \to \phi'(x) = \phi(x) - \omega^\mu_{\ \nu}x^\nu\partial_\mu\phi(x) \tag{1845}$$

and hence show that the variation of the Lagrangian density is a total derivative

$$\delta\mathcal{L} = -\partial_\mu(\omega^\mu_{\ \nu}x^\nu\mathcal{L}) \tag{1846}$$

Using Noether's theorem deduce the existence of the conserved current

$$j^\mu = -\omega^\rho_{\ \nu}[T^\mu_{\ \rho}x^\nu] \tag{1847}$$

The three conserved charges arising from spatial rotational invariance define the total angular momentum of the field. Show that these charges are given by,

$$Q_i = \epsilon_{ijk}\int d^3x\,(x^jT^{0k} - x^lT^{0j}) \tag{1848}$$

Derive the conserved charges arising from invariance under Lorentz boosts. Show that they imply

$$\frac{d}{dt} \in d^3x\,(x^i T^{00}) = \text{const} \tag{1849}$$

and interpret this equation.

For $x' = \Lambda x$ the transformation of the scalar field is given by

$$\begin{aligned}
\phi(x) \to \phi'(x') &\equiv \phi(x) && (1850)\\
&= \phi(\Lambda^{-1}x') && (1851)\\
&\simeq \phi(x') + \partial_\mu \phi(x')[(\Lambda^{-1})^\mu{}_\alpha x'^\alpha - x'^\mu] && (1852)\\
&= \phi(x') + \partial_\mu \phi(x')[(\delta^\mu_\alpha - \omega^\mu{}_\alpha)x'^\alpha - x'^\mu] && (1853)\\
&= \phi(x') - \partial_\mu \phi(x')\omega^\mu{}_\alpha x'^\alpha && (1854)
\end{aligned}$$

Checking the expression for the inverse Λ^{-1}

$$\begin{aligned}
\Lambda^{-1}\Lambda &= 1 && (1855)\\
(\Lambda^{-1})^\mu{}_\alpha \Lambda^\alpha_\nu &= (\delta^\mu_\alpha - \omega^\mu{}_\alpha)(\delta^\alpha_\nu + \omega^\alpha{}_\nu) && (1856)\\
&= \delta^\mu_\nu - \omega^\mu_\nu + \omega^\mu_\nu && (1857)\\
&= \delta^\mu_\nu && (1858)
\end{aligned}$$

0.22.7 Problem 1.7 - Energy momentum tensor field for Maxwell field - NOT DONE YET

Maxwell's Lagrangian for the electromagnetic field is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} \tag{1859}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_n u A_\mu$ and A_μ is the 4-vector potential. Show that $\mathcal{L}$ is invariant under gauge transformations

$$A_\mu \to A_\mu + \partial_\mu \xi \tag{1860}$$

where $\xi = \xi(x)$ is a scalar field with arbitrary (differentiable) dependence on x. Use Noether's theorem, and the spacetime translational invariance of the action, to construct the energy-momentum tensor $T^{\mu\nu}$ for the electromagnetic field. Show that the resulting object is neither symmetric nor gauge invariant. Consider a new tensor given by

$$\Theta^{\mu\nu} = T^{\mu\nu} - F^{\rho\mu}\partial_\rho A^\nu \tag{1861}$$

Show that this object also defines four conserved currents. Moreover, show that it is symmetric, gauge invariant and traceless.

Comment: $T^{\mu\nu}$ and $\Theta^{\mu\nu}$ are both equally good definitions of the energy-momentum tensor. However $\Theta^{\mu\nu}$ clearly has the nicer properties. Moreover, if you couple Maxwell's Lagrangian to general relativity then it is $\Theta^{\mu\nu}$ which appears in Einstein's equations.

- Checking invariance

$$\begin{aligned}
\mathcal{L}' &= -F'_{\mu\nu}F'^{\mu\nu} && (1862)\\
&= -(\partial_\mu[A_\nu + \partial_\nu\xi] - \partial_\nu[A_\mu + \partial_\mu\xi])(\partial^\mu[A^\nu + \partial^\nu\xi] - \partial^\nu[A^\mu + \partial^\mu\xi]) && (1863)\\
&= -(\partial_\mu A_\nu + \partial_\mu\partial_\nu\xi - \partial_\nu A_\mu - \partial_\nu\partial_\mu\xi)(\partial^\mu A^\nu + \partial^\mu\partial^\nu\xi - \partial^\mu\partial^\nu A^\mu - \partial^\nu\partial^\mu\xi]) && (1864)\\
&= -F_{\mu\nu}F^{\mu\nu} && (1865)\\
&= \mathcal{L} && (1866)
\end{aligned}$$

  so $\mathcal{L}$ is invariant.

- Noether theorem: the action being invariant under the transform

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \epsilon G_i(A(x)) \tag{1867}$$

means that $\mathcal{L}$ can only differ by a total divergence

$$\delta\mathcal{L} = \mathcal{L}(A', \partial A') - \mathcal{L}(A, \partial A) \tag{1868}$$

$$\overset{!}{=} \epsilon\partial_\mu X^\mu(A(x)) \tag{1869}$$

but

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial A_\mu}\delta A_\mu + \frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\mu)}\delta(\partial_\nu A_\mu) \tag{1870}$$

$$= \frac{\partial\mathcal{L}}{\partial A_\mu}\delta A_\mu + \frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\mu)}\partial_\nu(\delta A_\mu) \tag{1871}$$

$$= \frac{\partial\mathcal{L}}{\partial A_\mu}\delta A_\mu + \partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\mu)}(\delta A_\mu)\right) - \left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\mu)}\right)(\delta A_\mu) \tag{1872}$$

$$= \partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\mu)}(\delta A_\mu)\right) \tag{1873}$$

$$= \epsilon\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\mu)}\partial_\mu X^\mu\right) \tag{1874}$$

-
-

0.22.8 Problem 1.8 - Massive vector field

The Lagrangian density for a massive vector field C_μ is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 C_\mu C^\mu \tag{1875}$$

where $F_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\nu$. Derive the equations of motion and show that when $m \neq 0$they imply

$$\partial_\mu C^\mu = 0 \tag{1876}$$

Further show that C_0 can be eliminated completely in terms of the other fields by

$$\partial_i\partial^i C_0 + m^2 C_0 = \partial^i C_i \tag{1877}$$

Construct the canonical momenta Π_i conjugate to $C_i, i = 1, 2, 3$ and show that the canonical momentum conjugate to C_0 is vanishing. Construct the Hamiltonian density $\mathcal{H}$ in terms of C_0, C_i and Π_i. (Note: Do not be concerned that the canonical momentum for C_0 is vanishing. C_0 is non-dynamical — it is determined entirely in terms of the other fields using equation above).

With $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 C_\mu C^\mu$

$$\frac{\partial\mathcal{L}}{\partial C_\alpha} = m^2 C^\alpha \tag{1878}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\beta C_\alpha)} = -\frac{2}{4}(\delta^\beta_\mu\delta^\alpha_\nu - \delta^\beta_\nu\delta^\alpha_\mu)F^{\mu\nu} = -\frac{1}{2}(F^{\beta\alpha} - F^{\alpha\beta}) = F^{\alpha\beta} \tag{1879}$$

resulting in the equations of motion

$$-\partial_\beta F^{\alpha\beta} + m^2 C^\alpha = 0 \tag{1880}$$

$$-\partial_\beta(\partial^\alpha C^\beta - \partial^\beta C^\alpha) + m^2 C^\alpha = 0 \tag{1881}$$

$$-\partial^\alpha\partial_\beta C^\beta + \partial_\beta\partial^\beta C^\alpha + m^2 C^\alpha = 0 \tag{1882}$$

One more differentiation ∂_α and rearranging the differential operators we see

$$-\partial_\alpha\partial^\alpha\partial_\beta C^\beta + \partial_\beta\partial^\beta\partial_\alpha C^\alpha + m^2\partial_\alpha C^\alpha = 0 \tag{1883}$$
$$\to \partial_\alpha C^\alpha = 0 \tag{1884}$$
$$\to \partial_0 C^0 = \partial_i C^i \tag{1885}$$

Therefore the equations of motions simplify

$$\partial_\beta\partial^\beta C^\alpha + m^2 C^\alpha = 0 \tag{1886}$$
$$(\partial_0\partial^0 - \partial_i\partial^i)C^\alpha + m^2 C^\alpha = 0 \tag{1887}$$
$$\partial_0\partial^0 C^\alpha - \partial_i\partial^i C^\alpha + m^2 C^\alpha = 0 \tag{1888}$$

then for $\alpha = 0$

$$\partial^0 \underbrace{\partial_0 C^0}_{=\partial_i C^i} - \partial_i\partial^i C^0 + m^2 C^0 = 0 \tag{1889}$$
$$\partial_i\partial^i C^0 - m^2 C^0 = \partial_i \dot{C}^i \qquad \text{sign missing!?!.} \tag{1890}$$

which means C^0 can be calculated from C^i by solving the PDE. Now

$$\Pi_\mu = \frac{\partial\mathcal{L}}{\partial(\partial_0 C^\mu)} = F^{\mu 0} = \partial^\mu C^0 - \partial^0 C^\mu \tag{1891}$$
$$\Pi_0 = 0 \tag{1892}$$
$$\Pi_i = \partial^i C^0 - \partial^0 C^i \tag{1893}$$

then with $F^{00} = 0$

$$\mathcal{H} = \Pi_\mu \partial_0 C^\mu - \mathcal{L} \tag{1894}$$
$$= \Pi_i\partial_0 C^i + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 C_\mu C^\mu \tag{1895}$$
$$= \Pi_i(\partial_i C^0 - \Pi_i) + \frac{1}{4}F_{ij}F^{ij} + \frac{1}{4}F_{0j}F^{0j} + \frac{1}{4}F_{i0}F^{i0} - \frac{1}{2}m^2 C_\mu C^\mu \tag{1896}$$
$$= \Pi_i(\partial_i C^0 - \Pi_i) + \frac{1}{4}F_{ij}F^{ij} + \frac{1}{4}\Pi_j\Pi_j + \frac{1}{4}\Pi_i\Pi_i - \frac{1}{2}m^2 C_\mu C^\mu \tag{1897}$$
$$= -\frac{1}{2}\Pi_i\Pi_i + \Pi_i\partial_i C^0 + \frac{1}{4}F_{ij}F^{ij} - \frac{1}{2}m^2 C_\mu C^\mu \tag{1898}$$

0.22.9 Problem 1.9 - Scale invariance

A class of interesting theories are invariant under the scaling of all lengths by

$$x^\mu \to (x')^\mu = \lambda x^\mu \qquad \text{and} \qquad \phi(x) \to \phi'(x) = \lambda^{-D}\phi(\lambda^{-1}x) \tag{1899}$$

Here D is called the scaling dimension of the field. Consider the action for a real scalar field given by

$$S = \int d^4x\, \frac{1}{2}\partial_\mu\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - g\phi^g \tag{1900}$$

Find the scaling dimension D such that the derivative terms remain invariant. For what values of m and p is the scaling a symmetry of the theory. How do these conclusions change for a scalar field living in an $(n+1)$-dimensional spacetime instead of a $3+1$-dimensional spacetime?
In $3+1$ dimensions, use Noether's theorem to construct the conserved current D^μ associated to scaling invariance.

With $x' = \lambda x$ or $(x = \lambda^{-1}x')$ and

$$\phi(x) \to \phi'(x) = \lambda^{-D}\phi(\lambda^{-1}x) \tag{1901}$$

we need to calculate the first derivative

$$\partial'_\mu \phi'(x') = \partial'_\mu \phi'(\lambda x) = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial}{\partial x^\alpha}\lambda^{-D}\phi(x) \tag{1902}$$

$$= \lambda^{-D-1}\partial_\mu\phi(x) \tag{1903}$$

then

$$S = \int d^n x\, (\partial_\mu\phi(x))(\partial^\mu\phi(x)) + ... \tag{1904}$$

$$\to S' = \int d^n x'\, (\partial'_\mu\phi'(x'))(\partial'^\mu\phi'(x')) + ... \tag{1905}$$

$$= \int \lambda^{n+1} d^n x\, \lambda^{2(-D-1)}(\partial_\mu\phi(x))(\partial^\mu\phi(x)) - \frac{1}{2}m^2\lambda^{-2D}\phi^2 - g\lambda^{-pD}\phi^p \tag{1906}$$

$$\to \lambda^{n+1-2(D+1)} = 1 \tag{1907}$$

$$\to D = \frac{n-1}{2} \tag{1908}$$

It is a symmetry of the theory if

$$n+1-2D = 0 \quad \to \quad D = \frac{n+1}{2} \quad \to \quad m = 0 \tag{1909}$$

and

$$n+1-pD = 0 \quad \to \quad p = \frac{n+1}{D} \quad \to \quad p = 2\frac{n+1}{n-1}. \tag{1910}$$

The scale invariant Lagrangian in 3+1 is the given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - g\phi^4 \tag{1911}$$

Now calculating the Noether current for $n = 3$, $D = 1$ and $p = 4$

$$\delta\phi = \lambda^{-1}\phi(\lambda^{-1}x) - \phi(x) \tag{1912}$$

$$= \lambda^{-1}\left(\phi(x) + \partial_\alpha\phi(x)[\lambda^{-1}x^\alpha - x^\mu] + ...\right) - \phi(x) \tag{1913}$$

$$= (\lambda^{-1}-1)\phi(x) + \partial_\alpha x^\alpha\phi(x)(\lambda^{-1}-1) + ... \tag{1914}$$

$$= (\lambda^{-1}-1)(\phi(x) + x^\alpha\partial_\alpha\phi(x)) + ... \tag{1915}$$

$$= \frac{1-\lambda}{\lambda}(\phi(x) + x^\alpha\partial_\alpha\phi(x)) + ... \tag{1916}$$

$$= \frac{\lambda-1}{\lambda}(-\phi(x) - x^\alpha\partial_\alpha\phi(x)) + ... \tag{1917}$$

alternatively

$$\delta\phi = \lim_{\lambda\to 1}\frac{d\,\lambda^{-1}\phi(\lambda^{-1}x)}{d\lambda} \tag{1918}$$

$$= -\phi(x) - x^\alpha\partial_\alpha\phi(x) \tag{1919}$$

$$\delta\mathcal{L} = \lim_{\lambda\to 1}\frac{d\mathcal{L}(d\,\lambda^{-1}\phi(\lambda^{-1}x))}{d\lambda} \tag{1920}$$

$$= \lim_{\lambda\to 1}\frac{d}{d\lambda}\lambda^{-4}\mathcal{L} \tag{1921}$$

$$= \lim_{\lambda\to 1} -4\lambda^3\mathcal{L} - \partial_\mu\mathcal{L}\frac{\partial(\lambda^{-1}x^\mu)}{\partial\lambda} \tag{1922}$$

$$= -4\mathcal{L} - x^\mu\partial_\mu\mathcal{L} \tag{1923}$$

$$= \partial_\mu(x^\mu\mathcal{L}) \tag{1924}$$

then

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi - K^\mu \tag{1925}$$

$$= -\partial_\mu \phi(\phi(x) + x^\alpha \partial_\alpha \phi(x)) + x^\mu \mathcal{L} \tag{1926}$$

0.23 MIT: Liu - Relativistic Quantum Field Theory I - 2023 Spring

0.23.1 Problem 1.2 - Lorentz invariance of various δ-functions

(a) Using $px = \tilde{p}\tilde{x}$ is a Lorentz scalar and

$$\eta = \Lambda\eta\Lambda^T \tag{1927}$$
$$\det\eta = \det\Lambda\cdot\det\eta\cdot\det\Lambda^T \quad \rightarrow \quad 1 = (\det\Lambda)^2 \tag{1928}$$

we see when rewriting the single δ-functions by their Fourier representation

$$\delta^{(4)}(p) = \delta(p^0)\delta(p^1)\delta(p^2)\delta(p^3) \tag{1929}$$
$$= \frac{1}{2\pi}\int(-1)\cdot e^{-ip^0x^0}dx^0\cdot ...\cdot\frac{1}{2\pi}\int 1\cdot e^{ip^3x^3}dx^3 \tag{1930}$$
$$= -\frac{1}{2\pi}\iiiint d^4x\, e^{ipx} \tag{1931}$$
$$= -\frac{1}{2\pi}\iiiint \underbrace{|\det\Lambda^{-1}|^4}_{=1} d^4\tilde{x}\, e^{i\tilde{p}\tilde{x}} \tag{1932}$$
$$= \frac{1}{2\pi}\int(-1)\cdot e^{-i\tilde{p}^0\tilde{x}^0}d\tilde{x}^0\cdot ...\cdot\frac{1}{2\pi}\int 1\cdot e^{i\tilde{p}^3\tilde{x}^3}d\tilde{x}^3 \tag{1933}$$
$$= \delta^{(4)}(\tilde{p}) \tag{1934}$$

(b)

(c)

0.24 DESY Hamburg: Reuter - Quantum Mechanics II - WS 2020/21 HS

0.24.1 Problem 15 - Casimir Operators of Poincare algebra - NOT FINISHED

Casimir operators are by definition those that commute with the whole set of operators (the underlying algebra) and with each other, so they constitute the maximal set of simultaneously diagonalizable operators. Hence, all states can be expressed by their eigenvalues (quantum numbers). The Poincare algebra of space-time symmetries consists of the boost and rotation generators, $M^{\mu\nu}$, as well as the energy-momentum 4-vector, P^μ, being the generator of time-space translations:

$$[P^\mu, P^\nu] = 0 \tag{1935}$$
$$[M^{\mu\nu}, M^{\rho\sigma}] = -i(-g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma} + g^{\nu\sigma}M^{\mu\rho} + g^{\mu\rho}M^{\nu\sigma}) \tag{1936}$$
$$[P^\mu, M^{\rho\sigma}] = i(g^{\mu\rho}P^\sigma - g^{\nu\sigma}P^\rho) \tag{1937}$$

(The mathematicians would call this the semi-direct product of the rotation group on a space with Minkowski signature and the 4-dim. [Abelian] translation group, $\mathcal{P} \simeq \mathrm{SO}(1,3) \rtimes \mathbf{R}^4$.)
Proof that the energy-momentum squared, P^2 as well as the square of Pauli-Ljubarski vector W^2 are Casimir operators of the Poincare group, where W^μ is defined as

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_\nu M_{\rho\sigma} \tag{1938}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor in four dimension with $\epsilon^{0123} = +1$ For the proof that W^2 is a Casimir operator, first show that it transforms as a 4-vector under boosts and rotations, i.e.

$$[W_\mu, M^{\rho,\sigma}] = i(g^\rho_\mu W^\sigma - g^\sigma_\mu W^\rho) \tag{1939}$$

For this use the identity

$$g_{\sigma\lambda}\epsilon_{\mu\nu\alpha\beta} + g_{\sigma\mu}\eta_{\nu\alpha\beta\lambda} + g_{\sigma\nu}\eta_{\alpha\beta\lambda\mu} + g_{\sigma\alpha}\eta_{\beta\lambda\mu\nu} + g_{\sigma\beta}\eta_{\lambda\mu\nu\alpha} = 0 \tag{1940}$$

Why does it hold? What is the commutator $[W^\mu, W^\nu]$?

0.24.2 Problem 16 - Connection between the Lorentz group SO(1,3) and SU(2)×SU(2) - NOT FINISHED

We take the Lorentz generators $M^{\mu\nu}$ fulfilling the Lorentz algebra

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i(-g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma} + g^{\nu\sigma}M^{\mu\rho} + g^{\mu\rho}M^{\nu\sigma}) \tag{1941}$$

from which we define as in the lecture the new generators:

$$J^i = \frac{1}{2}\epsilon^{ijk}M^{jk}, \qquad K^i = M^{0i} \tag{1942}$$

(a) Show that J^i are Hermitian operators (${J^i}^\dagger = J^i$) and the K^i and anti-Hermitian operators (${K^i}^\dagger = -K^i$).

(b) Show that these operators fulfill the commutation relations

$$[J^i, J^j] = i\epsilon^{ijk}J^k \tag{1943}$$
$$[J^i, K^j] = i\epsilon^{ijk}K^k \tag{1944}$$
$$[K^i, K^j] = -i\epsilon^{ijk}J^k \tag{1945}$$

(c) Proof that the two linear combinations from these operators,

$$T^i_+ = \frac{1}{2}(J^i + iK^i), \qquad T^i_- = \frac{1}{2}(J^i - iK^i) \tag{1946}$$

constitute two mutually commuting SU(2) algebras, i.e.

$$[T^i_\pm, T^j_\pm] = i\epsilon^{ijk}T^k_\pm \tag{1947}$$
$$[T^i_+, T^j_-] = 0 \tag{1948}$$

0.24.3 Problem 17 - Clifford algebra as representation of Lorentz algebra

The Clifford algebra is an algebra of (in 4 space-time dimensions) 4 anticommuting generators (matrices in an explicit representation), Γ^μ, $\mu = 0, 1, 2, 3$ that are the generators of Lorentz transformations for the spinor or spin-1/2 representation of the Lorentz group. They fulfill the Clifford algebra (which for this special case is also called Dirac algebra):

$$\{\Gamma^\mu, \Gamma^\nu\} = 2g^{\mu\nu}\mathbf{1}_{4\times4} \tag{1949}$$

Show that the generators

$$\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \tag{1950}$$

constitute a representation of the Lorentz group (namely for spin 1/2), i.e. they fulfill

$$[\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}] = -i(-g^{\mu\sigma}\Sigma^{\nu\rho} - g^{\nu\rho}\Sigma^{\mu\sigma} + g^{\nu\sigma}\Sigma^{\mu\rho} + g^{\mu\rho}\Sigma^{\nu\sigma}). \tag{1951}$$

Let's calculate

$$\Sigma^{\mu\nu}\Sigma^{\rho\sigma} = -\frac{1}{16}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)(\gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma^\rho) \tag{1952}$$

$$= -\frac{1}{16}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma - \gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho - \gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma + \gamma^\nu\gamma^\mu\gamma^\sigma\gamma^\rho) \tag{1953}$$

switching indices

$$\to \Sigma^{\rho\sigma}\Sigma^{\mu\nu} = -\frac{1}{16}(\gamma^\rho \underbrace{\gamma^\sigma\gamma^\mu}_{2g^{\sigma\mu}-\gamma^\mu\gamma^\sigma} \gamma^\nu - \gamma^\rho\gamma^\sigma\gamma^\nu\gamma^\mu - \gamma^\sigma\gamma^\rho\gamma^\mu\gamma^\nu + \gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\mu) \tag{1954}$$

$$= -\frac{1}{16}(2g^{\sigma\mu}\gamma^\rho\gamma^\nu - \underbrace{\gamma^\rho\gamma^\mu}_{2g^{\rho\mu}-\gamma^\mu\gamma^\rho} \gamma^\sigma\gamma^\nu + ...) \tag{1955}$$

$$= -\frac{1}{16}(2g^{\sigma\mu}\gamma^\rho\gamma^\nu - 2g^{\rho\mu}\gamma^\sigma\gamma^\nu + \gamma^\mu\gamma^\rho \underbrace{\gamma^\sigma\gamma^\nu}_{2g^{\sigma\nu}-\gamma^\nu\gamma^\sigma} + ...) \tag{1956}$$

$$= -\frac{1}{16}(2g^{\sigma\mu}\gamma^\rho\gamma^\nu - 2g^{\rho\mu}\gamma^\sigma\gamma^\nu + 2g^{\sigma\nu}\gamma^\mu\gamma^\rho - \gamma^\mu \underbrace{\gamma^\rho\gamma^\nu}_{2g^{\rho\nu}-\gamma^\nu\gamma^\rho} \gamma^\sigma + ...) \tag{1957}$$

$$= -\frac{1}{16}(2g^{\sigma\mu}\gamma^\rho\gamma^\nu - 2g^{\rho\mu}\gamma^\sigma\gamma^\nu + 2g^{\sigma\nu}\gamma^\mu\gamma^\rho - 2g^{\rho\nu}\gamma^\mu\gamma^\sigma + \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) \tag{1958}$$

$$+ \frac{1}{16}(2g^{\sigma\nu}\gamma^\rho\gamma^\mu - 2g^{\rho\nu}\gamma^\sigma\gamma^\mu + 2g^{\sigma\mu}\gamma^\nu\gamma^\rho - 2g^{\rho\mu}\gamma^\nu\gamma^\sigma + \gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma) \tag{1959}$$

$$+ \frac{1}{16}(2g^{\rho\mu}\gamma^\sigma\gamma^\nu - 2g^{\sigma\mu}\gamma^\rho\gamma^\nu + 2g^{\rho\nu}\gamma^\mu\gamma^\sigma - 2g^{\sigma\nu}\gamma^\mu\gamma^\rho + \gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho) \tag{1960}$$

$$- \frac{1}{16}(2g^{\rho\nu}\gamma^\sigma\gamma^\mu - 2g^{\sigma\nu}\gamma^\rho\gamma^\mu + 2g^{\rho\mu}\gamma^\nu\gamma^\sigma - 2g^{\sigma\mu}\gamma^\nu\gamma^\rho + \gamma^\nu\gamma^\mu\gamma^\sigma\gamma^\rho) \tag{1961}$$

then

$$[\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}] = \frac{1}{8}(g^{\sigma\mu}\gamma^\rho\gamma^\nu - g^{\rho\mu}\gamma^\sigma\gamma^\nu + g^{\sigma\nu}\gamma^\mu\gamma^\rho - g^{\rho\nu}\gamma^\mu\gamma^\sigma) \tag{1962}$$

$$- \frac{1}{8}(g^{\sigma\nu}\gamma^\rho\gamma^\mu - g^{\rho\nu}\gamma^\sigma\gamma^\mu + g^{\sigma\mu}\gamma^\nu\gamma^\rho - g^{\rho\mu}\gamma^\nu\gamma^\sigma) \tag{1963}$$

$$- \frac{1}{8}(g^{\rho\mu}\gamma^\sigma\gamma^\nu - g^{\sigma\mu}\gamma^\rho\gamma^\nu + g^{\rho\nu}\gamma^\mu\gamma^\sigma - g^{\sigma\nu}\gamma^\mu\gamma^\rho) \tag{1964}$$

$$+ \frac{1}{8}(g^{\rho\nu}\gamma^\sigma\gamma^\mu - g^{\sigma\nu}\gamma^\rho\gamma^\mu + g^{\rho\mu}\gamma^\nu\gamma^\sigma - g^{\sigma\mu}\gamma^\nu\gamma^\rho) \tag{1965}$$

$$= \frac{1}{8}g^{\sigma\mu}2(\gamma^\rho\gamma^\nu - \gamma^\nu\gamma^\rho) + ... \tag{1966}$$

$$= -ig^{\sigma\mu}\frac{i}{4}(\gamma^\rho\gamma^\nu - \gamma^\nu\gamma^\rho) + ... \tag{1967}$$

$$= -i(-g^{\sigma\mu}\Sigma^{\nu\rho} + ...) \tag{1968}$$

$$\tag{1969}$$

0.25 HU Berlin: Klose - Quantum Field Theory I - WS 2024/25

0.25.1 Problem 0.1 - General solution of the free Klein-Gordon equation

In the lectures, we looked for particular solutions of the Klein-Gordon equation

$$\left(\Box + \frac{m^2c^2}{\hbar^2}\right)\Psi(t,x) = 0 \tag{1970}$$

with definite (positive) energy E. Now, find the general solution!

(a) Translate the partial differential equation for $\Psi(t,x)$ to an ordinary differential equation for the Fourier transformed function $\tilde{\Psi}(t,k)$ defined by

$$\tilde{\Psi}(t,\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3}\tilde{\Psi}(t,\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} \tag{1971}$$

(b) Find the general solution of the ordinary differential equation for $\tilde{\Psi}(t,k)$.

(c) Insert the result of (b) into the FT and massage the result into the form

$$\tilde{\Psi}(t,\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3}\left(A(\mathbf{k})e^{-i\omega_k t + i\mathbf{k}\cdot\mathbf{x}} + B(\mathbf{k})e^{i\omega_k t + i\mathbf{k}\cdot\mathbf{x}}\right) \tag{1972}$$

where $\omega_k > 0$, A_k and B_k are functions of k left for you to sort out.

(d) What restrictions are imposed by demanding that $\Psi(t,x)$ is real?

(e) Which integration constants correspond to left and which to right moving partial waves?

(a) Simple calculation

$$\int \frac{d^3k}{(2\pi)^3}\left[\frac{1}{c^2}\partial_{tt} - i^2(k_x^2+k_y^2+k_z^2) + \frac{m^2c^2}{\hbar^2}\right]\tilde{\Psi}(t,\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} = 0 \tag{1973}$$

$$\to \left[\partial_{tt} + c^2\mathbf{k}^2 + \frac{m^2c^4}{\hbar^2}\right]\tilde{\Psi}(t,\mathbf{k}) = 0 \tag{1974}$$

$$\to \left[\partial_{tt} + \omega_k^2\right]\tilde{\Psi}(t,\mathbf{k}) = 0 \tag{1975}$$

$$\omega_k = \sqrt{c^2\mathbf{k}^2 + \frac{m^2c^4}{\hbar^2}} \tag{1976}$$

(b) Try solution to harmonic osci

$$\tilde{\Psi}_{\mathbf{k}}(t) = Ae^{-i\omega_k t} + Be^{i\omega_k t} \tag{1977}$$

(c) Then

$$\tilde{\Psi}(t,\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3}\tilde{\Psi}(t,\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} \tag{1978}$$

$$= \int \frac{d^3k}{(2\pi)^3}\left(A(\mathbf{k})e^{-i\omega_k t + i\mathbf{k}\cdot\mathbf{x}} + B(\mathbf{k})e^{i\omega_k t + i\mathbf{k}\cdot\mathbf{x}}\right) \tag{1979}$$

(d) $B(\mathbf{k}) = A^*(\mathbf{k})$

(e)

0.25.2 Problem 1.1 - Continuum equations

The Schroedinger equation

$$-\frac{hbar^2}{2m}\partial_{xx}\phi + V(x)\phi = i\hbar\dot{\phi} \tag{1980}$$

and the Klein-Gordon equation

$$\left(\Box + \frac{m^2c^2}{\hbar^2}\right)\psi = \frac{V(x)^2}{\hbar^2c^2}\psi - \frac{2V(x)}{\hbar^2c^2}(i\hbar)\dot{\psi} \tag{1981}$$

for the wave-functions $\psi(t,x)$ of a non-relativistic and relativistic particle in a one-dimensional potential $V(x)$, respectively, imply the conservation of certain quantities whose densities $\rho(t,x)$ and currents $j(t,x)$ satisfy the continuity equation

$$\partial_t\rho(t,x) = -\partial_x j(t,x) \tag{1982}$$

(a) In the case of the Schrödinger equation, look up the expressions for ρ and j in terms of the Schrödinger wave-function $\phi(t,x)$ as given in the lecture notes! Show that they satisfy the continuity equation, if wave-function satisfies!

(b) Repeat part (a) for the case of the Klein-Gordon equation!

We are guessing the Lagrangian densities and use the standard machinery

$$\mathcal{L}_S = i\hbar\phi^*\dot{\phi} - \frac{\hbar^2}{2m}|\nabla\phi|^2 - V(x)|\phi|^2 \tag{1983}$$

$$\mathcal{L}_{KG} = \underbrace{\partial_\mu\psi^*\partial^\mu\psi}_{=\eta^{\mu\nu}\partial_\mu\psi^*\partial_\nu\psi} - \frac{m^2c^2}{\hbar^2}\psi^*\psi + \frac{1}{\hbar^2c^2}V(x)^2\psi^*\psi - \frac{2}{\hbar^2c^2}V(x)\psi^*(i\hbar)\dot{\psi}. \tag{1984}$$

As conserved currents are usually caused by symmetries (Noether theorem) - we should be able to derive density and currents - rather than guessing them.

So the tricky part is to guess the symmetry and as spacetime symmetries usually imply energy/momentum conservation we need to look for something else (but once found the current will be conserved).

(a) Deriving the equations of motion

$$\begin{aligned}
\frac{\partial\mathcal{L}_S}{\partial\phi} &= -V(x)\phi^* & \frac{\partial\mathcal{L}_S}{\partial\phi^*} &= i\hbar\dot{\phi} - V(x)\phi \\
\frac{\partial\mathcal{L}_S}{\partial(\partial_0\phi)} &= i\hbar\phi^* & \frac{\partial\mathcal{L}_S}{\partial(\partial_0\phi^*)} &= 0 \\
\partial_t\frac{\partial\mathcal{L}_S}{\partial(\partial_0\phi)} &= i\hbar\dot{\phi}^* & \partial_t\frac{\partial\mathcal{L}_S}{\partial(\partial_0\phi^*)} &= 0 \\
\frac{\partial\mathcal{L}_S}{\partial(\partial_k\phi)} &= -\frac{\hbar^2}{2m}\nabla\phi^* & \frac{\partial\mathcal{L}_S}{\partial(\partial_k\phi^*)} &= -\frac{\hbar^2}{2m}\nabla\phi \\
\partial_k\frac{\partial\mathcal{L}_S}{\partial(\partial_k\phi)} &= -\frac{\hbar^2}{2m}\triangle\phi^* & \partial_k\frac{\partial\mathcal{L}_S}{\partial(\partial_k\phi^*)} &= -\frac{\hbar^2}{2m}\triangle\phi
\end{aligned} \tag{1985}$$

therefore we obtain two equations of motion

$$\rightarrow i\hbar\phi^* - \frac{\hbar^2}{2m}\triangle\phi^* + V(x)\phi^* = 0 \tag{1986}$$

$$\rightarrow -i\hbar\phi - \frac{\hbar^2}{2m}\triangle\phi + V(x)\phi = 0 \tag{1987}$$

As ϕ and ϕ^* all terms appear as some kind of product we try a global gauge transformation of (same idea as for complex KG field)

$$\phi \to e^{i\varepsilon}\phi \simeq (1+i\varepsilon)\phi = \phi + i\varepsilon\phi \tag{1988}$$
$$\phi^* \to e^{-i\varepsilon}\phi^* \simeq (1-i\varepsilon)\phi^* = \phi^* - i\varepsilon\phi^* \tag{1989}$$

so $\delta\phi = i\varepsilon\phi$ and $\delta\phi^* = -i\varepsilon\phi^*$.

Then we look at the three terms of the Lagrangian separately

$$\phi^*\dot{\psi} \to (\phi^* - i\varepsilon\phi^*)(\dot{\phi} + i\varepsilon\dot{\phi}) = \phi^*\dot{\psi} + i\varepsilon(-\phi^*\dot{\phi} + \phi^*\dot{\phi}) + \mathcal{O}(\varepsilon^2) \tag{1990}$$
$$= \phi^*\dot{\phi} \tag{1991}$$
$$\to \delta(\phi^*\dot{\phi}) = 0 \tag{1992}$$

second term

$$(\nabla\phi^*)\nabla(\psi) \to (\nabla\phi^* - i\varepsilon\phi^*)\nabla(\phi + i\varepsilon\phi) = (\nabla\phi^*)(\nabla\phi) + i\varepsilon((\nabla\phi^*)(\nabla\phi) - (\nabla\phi^*)(\nabla\phi)) + \mathcal{O}(\varepsilon^2) \tag{1993}$$
$$= (\nabla\phi^*)(\nabla\phi) \tag{1994}$$
$$\to \delta((\nabla\phi^*)(\nabla\phi)) = 0 \tag{1995}$$

and third term

$$\phi^*\phi \to (\phi^* - i\varepsilon\phi^*)(\phi + i\varepsilon\phi) = \phi^*\psi + i\varepsilon(-\phi^*\phi + \phi^*\phi) + \mathcal{O}(\varepsilon^2) \tag{1996}$$
$$= \phi^*\phi \tag{1997}$$
$$\to \delta(\phi^*\phi) = 0. \tag{1998}$$

So we conclude that the Lagrangian is invariant under this transformation. Then

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\delta\phi^* \tag{1999}$$
$$j^0 = i\hbar\phi^*(i\varepsilon\phi) \tag{2000}$$
$$= -\varepsilon\hbar\phi^*\phi \tag{2001}$$
$$j^m = -\frac{\hbar^2}{2m}\left((\nabla\phi^*)(i\varepsilon\phi) + (\nabla\phi)(-i\varepsilon\phi^*)\right) \tag{2002}$$
$$= -\frac{i\hbar^2\varepsilon}{2m}\left((\nabla\phi^*)\phi - (\nabla\phi)\phi^*\right) \tag{2003}$$

and

$$Q = \int d^3x\, j^0 = -\varepsilon \int d^3x\, \phi^*\phi. \tag{2004}$$

(b) Deriving the equations of motion

$$\mathcal{L}_{KG} = \underbrace{\partial_\mu\psi^*\partial^\mu\psi}_{=\eta^{\mu\nu}\partial_\mu\psi^*\partial_\nu\psi} - \frac{m^2c^2}{\hbar^2}\psi^*\psi + \frac{1}{\hbar^2c^2}V(x)^2\psi^*\psi - \frac{2}{\hbar^2c^2}V(x)\psi^*(i\hbar)\dot{\psi} \tag{2005}$$
$$\frac{\partial\mathcal{L}_{KG}}{\partial\psi^*} = -\frac{m^2c^2}{\hbar^2}\psi + \frac{V(x)^2}{\hbar^2c^2}\psi - \frac{2V(x)}{\hbar^2c^2}(i\hbar)\dot{\psi} \tag{2006}$$
$$\frac{\partial\mathcal{L}_{KG}}{\partial(\partial_\alpha\psi^*)} = \partial^\mu\psi\delta_\mu^\alpha \tag{2007}$$
$$\partial_\alpha\frac{\partial\mathcal{L}_{KG}}{\partial(\partial_\alpha\psi^*)} = \partial_\alpha\partial^\alpha\psi \tag{2008}$$
$$\to \left(\Box + \frac{m^2c^2}{\hbar^2}\right)\psi - \frac{V(x)^2}{\hbar^2c^2}\psi + \frac{2V(x)}{\hbar^2c^2}(i\hbar)\dot{\psi} = 0 \tag{2009}$$

$$\mathcal{L}_{KG} = \underbrace{\partial_\mu\psi^*\partial^\mu\psi}_{=\eta^{\mu\nu}\partial_\mu\psi^*\partial_\nu\psi} - \frac{m^2c^2}{\hbar^2}\psi^*\psi + \frac{1}{\hbar^2c^2}V(x)^2\psi^*\psi - \frac{2}{\hbar^2c^2}V(x)\psi^*(i\hbar)\dot{\psi} \tag{2010}$$

$$\frac{\partial\mathcal{L}_{KG}}{\partial\psi} = -\frac{m^2c^2}{\hbar^2}\psi^* + \frac{V(x)^2}{\hbar^2c^2}\psi^* \tag{2011}$$

$$\frac{\partial\mathcal{L}_{KG}}{\partial(\partial_\alpha\psi)} = \partial^\mu\psi^*\delta^\alpha_\mu - \frac{2V(x)}{\hbar^2c^2}(i\hbar)\psi^*\delta^\alpha_0 \tag{2012}$$

$$\partial_\alpha\frac{\partial\mathcal{L}_{KG}}{\partial(\partial_\alpha\psi)} = \partial_\alpha\partial^\alpha\psi^* - \frac{2V(x)}{\hbar^2c^2}(i\hbar)\dot{\psi}^* \tag{2013}$$

$$\to \left(\Box + \frac{m^2c^2}{\hbar^2}\right)\psi^* - \frac{V(x)^2}{\hbar^2c^2}\psi^* - \frac{2V(x)}{\hbar^2c^2}(i\hbar)\dot{\psi}^* = 0 \tag{2014}$$

Similar as above

$$\psi \to e^{i\epsilon}\psi \simeq \psi + \underbrace{i\epsilon\psi}_{\delta\psi} \tag{2015}$$

$$\psi^* \to e^{-i\epsilon}\psi^* \simeq \psi^* - \underbrace{i\epsilon\psi^*}_{-\delta\psi^*} \tag{2016}$$

then

$$j^\alpha = \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\psi)}\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\psi^*)}\delta\psi^* \tag{2017}$$

$$= \left(\partial^\alpha\psi^* - \frac{2V(x)}{\hbar^2c^2}(i\hbar)\psi^*\delta^\alpha_0\right)i\epsilon\psi + (\partial^\alpha\psi)(-i\epsilon\psi^*) \tag{2018}$$

$$= i\epsilon\left(\psi\partial^\alpha\psi^* - \psi^*\partial^\alpha\psi\right) - i\epsilon\frac{2V(x)}{\hbar^2c^2}(i\hbar)\psi^*\psi\delta^\alpha_0 \tag{2019}$$

$$j^0 = i\epsilon\left(\psi\partial^0\psi^* - \psi^*\partial^0\psi\right) - i\epsilon\frac{2V(x)}{\hbar^2c^2}(i\hbar)\psi^*\psi \tag{2020}$$

$$= -\frac{2m\epsilon}{\hbar}\left[\frac{i\hbar}{2mc}\left(\psi^*\dot{\psi} - \psi\dot{\psi}^*\right) - \frac{V(x)}{mc^2}\psi^*\psi\right] \tag{2021}$$

$$j^k = i\epsilon\left(\psi\nabla\psi^* - \psi^*\nabla\psi\right) \tag{2022}$$

$$= -\frac{2m\epsilon}{\hbar}\frac{i\hbar}{2m}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) \tag{2023}$$

0.25.3 Problem 2.1 - (Bosonic) Harmonic oscillator

(a) Simple calculation

$$[a, a^\dagger] = \frac{m\omega}{2\hbar}\left([x,x] - \frac{i}{m\omega}[x,p] + \frac{i}{m\omega}[p,x] - \frac{i^2}{m\omega}[p,p]\right) \tag{2024}$$

$$= \frac{m\omega}{2\hbar}\left(-\frac{i}{m\omega}i\hbar + \frac{i}{m\omega}(-i\hbar)\right) \tag{2025}$$

$$= 1 \tag{2026}$$

Induction (assuming $[a, (a^\dagger)^k] = k(a^\dagger)^{k-1}$ is true)

$$
\begin{aligned}
[a, (a^\dagger)^{k+1}] &= a(a^\dagger)^{k+1} - (a^\dagger)^{k+1}a && (2027)\\
&= a(a^\dagger)^k a^\dagger - (a^\dagger)^{k+1}a && (2028)\\
&= (k(a^\dagger)^{k-1} + (a^\dagger)^k a)a^\dagger - (a^\dagger)^{k+1}a && (2029)\\
&= k(a^\dagger)^k + (a^\dagger)^k aa^\dagger - (a^\dagger)^{k+1}a && (2030)\\
&= k(a^\dagger)^k + (a^\dagger)^k(1 + a^\dagger a) - (a^\dagger)^{k+1}a && (2031)\\
&= k(a^\dagger)^k + (a^\dagger)^k + (a^\dagger)^{k+1}a - (a^\dagger)^{k+1}a && (2032)\\
&= (k+1)(a^\dagger)^k && (2033)
\end{aligned}
$$

(b) with $\langle\psi_0|a = 0$

$$
\begin{aligned}
\langle\psi_k|\psi_k\rangle &= \mathcal{N}_k^2\langle\psi_0|(a^\dagger)^k a^k|\psi_0\rangle && (2034)\\
&= \mathcal{N}_k^2\langle\psi_0|(a^\dagger)^k a a^{k-1}|\psi_0\rangle && (2035)\\
&= \mathcal{N}_k^2\langle\psi_0|(a(a^\dagger)^k - k(a^\dagger)^{k-1})a^{k-1}|\psi_0\rangle && (2036)\\
&= -k\mathcal{N}_k^2\langle\psi_0|(a^\dagger)^{k-1}a^{k-1}|\psi_0\rangle && (2037)\\
&= ... && (2038)\\
&= k!\mathcal{N}_k^2 && (2039)
\end{aligned}
$$

meaning $\mathcal{N}_k = 1/\sqrt{k!}$

(c)

$$
\begin{aligned}
|\psi_k\rangle &= \frac{1}{\sqrt{k!}}(a^\dagger)^k|\psi_0\rangle && (2040)\\
\rightarrow a^\dagger|\psi_k\rangle &= \frac{1}{\sqrt{k!}}(a^\dagger)^{k+1}|\psi_0\rangle && (2041)\\
\rightarrow \frac{1}{\sqrt{(k+1)!}}a^\dagger|\psi_k\rangle &= \frac{1}{\sqrt{(k+1)!}}(a^\dagger)^{k+1}|\psi_0\rangle && (2042)\\
&= |\psi_{k+1}\rangle && (2043)\\
\rightarrow a^\dagger|\psi_k\rangle &= \sqrt{k+1}|\psi_{k+1}\rangle && (2044)\\
& && (2045)
\end{aligned}
$$

0.25.4 Problem 2.2 - Second Quantization

(a) Simple calculation - getting all annihilation operators to the right hand side (to use $a_j|0\rangle = 0$)

$$\begin{aligned}
\langle 0|\hat{a}_k\hat{a}_l\hat{a}_m^\dagger\hat{a}_n^\dagger|0\rangle &= \langle 0|\hat{a}_k\hat{a}_l\hat{a}_m^\dagger\hat{a}_n^\dagger|0\rangle && (2046)\\
&= \langle 0|\hat{a}_k(\delta_{lm} + \hat{a}_m^\dagger\hat{a}_l)\hat{a}_n^\dagger|0\rangle && (2047)\\
&= \langle 0|\delta_{lm}\hat{a}_k\hat{a}_n^\dagger + \hat{a}_k\hat{a}_m^\dagger\hat{a}_l\hat{a}_n^\dagger|0\rangle && (2048)\\
&= \langle 0|\delta_{lm}(\delta_{kn} + \hat{a}_n^\dagger\hat{a}_k) + \hat{a}_k\hat{a}_m^\dagger(\delta_{ln} + \hat{a}_n^\dagger\hat{a}_l)|0\rangle && (2049)\\
&= \langle 0|\delta_{lm}\delta_{kn} + \delta_{lm}\hat{a}_n^\dagger\hat{a}_k + \delta_{ln}\hat{a}_k\hat{a}_m^\dagger + \hat{a}_k\hat{a}_m^\dagger\hat{a}_n^\dagger\hat{a}_l|0\rangle && (2050)\\
&= \langle 0|\delta_{lm}\delta_{kn} + \delta_{lm}\hat{a}_n^\dagger\hat{a}_k + \delta_{ln}(\delta_{km} + \hat{a}_m^\dagger\hat{a}_k) + (\delta_{km} + \hat{a}_m^\dagger\hat{a}_k)\hat{a}_n^\dagger\hat{a}_l|0\rangle && (2051)\\
&= \langle 0|\delta_{lm}\delta_{kn} + \delta_{lm}\hat{a}_n^\dagger\hat{a}_k + \delta_{ln}\delta_{km} + \delta_{ln}\hat{a}_m^\dagger\hat{a}_k + \delta_{km}\hat{a}_n^\dagger\hat{a}_l + \hat{a}_m^\dagger\hat{a}_k\hat{a}_n^\dagger\hat{a}_l|0\rangle && (2052)\\
&= \langle 0|\delta_{lm}\delta_{kn} + \delta_{lm}\hat{a}_n^\dagger\hat{a}_k + \delta_{ln}\delta_{km} + \delta_{ln}\hat{a}_m^\dagger\hat{a}_k + \delta_{km}\hat{a}_n^\dagger\hat{a}_l + \hat{a}_m^\dagger(\delta_{kn} + \hat{a}_n^\dagger\hat{a}_k)\hat{a}_l|0\rangle && (2053)\\
&= \langle 0|\delta_{lm}\delta_{kn} + \delta_{lm}\hat{a}_n^\dagger\hat{a}_k + \delta_{ln}\delta_{km} + \delta_{ln}\hat{a}_m^\dagger\hat{a}_k + \delta_{km}\hat{a}_n^\dagger\hat{a}_l + \delta_{kn}\hat{a}_m^\dagger\hat{a}_l + \hat{a}_m^\dagger\hat{a}_n^\dagger\hat{a}_k\hat{a}_l|0\rangle && (2054)\\
&= \langle 0|\delta_{lm}\delta_{kn} + \delta_{ln}\delta_{km}|0\rangle && (2055)\\
&= (\delta_{lm}\delta_{kn} + \delta_{ln}\delta_{km})\langle 0||0\rangle && (2056)\\
&= (\delta_{lm}\delta_{kn} + \delta_{ln}\delta_{km}) && (2057)
\end{aligned}$$

and also

$$\begin{aligned}
\langle 0|\hat{a}_l\hat{a}_m^\dagger|0\rangle &= \langle 0|\delta_{lm} - \hat{a}_m^\dagger\hat{a}_l|0\rangle && (2058)\\
&= \delta_{lm} && (2059)
\end{aligned}$$

(b) Notation is a bit unclear - we assume single excitations are meant $a_m^\dagger|0\rangle = |\psi_m\rangle$

$$\hat{A}|\psi_m\psi_n\rangle \equiv \sum_{kl} a_k^\dagger A_{kl} a_l |\psi_m\psi_n\rangle \tag{2060}$$
$$= \sum_{kl} a_k^\dagger A_{kl} a_l a_m^\dagger a_n^\dagger |0\rangle \tag{2061}$$
$$= \sum_{kl} a_k^\dagger A_{kl} (\delta_{lm} + a_m^\dagger a_l) a_n^\dagger |0\rangle \tag{2062}$$
$$= \sum_{kl} a_k^\dagger A_{kl} (\delta_{lm} a_n^\dagger + a_m^\dagger a_l a_n^\dagger) |0\rangle \tag{2063}$$
$$= \sum_{kl} a_k^\dagger A_{kl} (\delta_{lm} a_n^\dagger + \delta_{ln} a_m^\dagger + a_m^\dagger a_n^\dagger a_l) |0\rangle \tag{2064}$$
$$= \sum_{kl} a_k^\dagger A_{kl} (\delta_{lm} a_n^\dagger + \delta_{ln} a_m^\dagger) |0\rangle \tag{2065}$$
$$= \sum_{kl} a_k^\dagger A_{kl} \delta_{lm} a_n^\dagger |0\rangle + \sum_{kl} a_k^\dagger A_{kl} \delta_{ln} a_m^\dagger |0\rangle \tag{2066}$$
$$= \sum_{k} A_{km} a_k^\dagger a_n^\dagger |0\rangle + \sum_{k} A_{kn} a_k^\dagger a_m^\dagger |0\rangle \tag{2067}$$
$$= \sum_{k} A_{km} |\psi_k\psi_n\rangle + \sum_{k} A_{kn} |\psi_k\psi_m\rangle \tag{2068}$$
$$= \sum_{k} A_{km} |\psi_k\rangle \otimes |\psi_n\rangle + \sum_{k} A_{kn} |\psi_k\rangle \otimes |\psi_m\rangle \tag{2069}$$
$$= \sum_{k} |\psi_k\rangle A_{km} \otimes |\psi_n\rangle + \sum_{k} |\psi_k\rangle A_{kn} \otimes |\psi_m\rangle \tag{2070}$$
$$= \sum_{k} |\psi_k\rangle\langle\psi_k| A^{(1)} |\psi_m\rangle \otimes |\psi_n\rangle + \sum_{k} |\psi_k\rangle\langle\psi_k| A^{(1)} |\psi_n\rangle \otimes |\psi_m\rangle \tag{2071}$$
$$= A^{(1)}|\psi_m\rangle \otimes |\psi_n\rangle + A^{(1)}|\psi_n\rangle \otimes |\psi_m\rangle \tag{2072}$$
$$\tag{2073}$$

(c) Analog:

$$V_{\text{int}} = \sum_{klmn} a_k^\dagger a_l^\dagger \langle\psi_k\psi_l|V^{(2)}(x_1,x_2)|\psi_m\psi_n\rangle a_m a_n \tag{2074}$$

N identical BUT distinguishable particles

0.25.5 Problem 3.1 - Bogoliubov Transformation - (e) NOT FINISHED YET

(a) Simple calculation

$$[e^{i\varphi}\hat{a}, e^{i\varphi}\hat{a}] = e^{i2\varphi}[\hat{a},\hat{a}] = 0 \tag{2075}$$
$$[e^{-i\varphi}\hat{a}^\dagger, e^{-i\varphi}\hat{a}^\dagger] = e^{-i2\varphi}[\hat{a}^\dagger,\hat{a}^\dagger] = 0 \tag{2076}$$
$$[e^{i\varphi}\hat{a}, e^{-i\varphi}\hat{a}^\dagger] = [\hat{a},\hat{a}^\dagger] = 1 \tag{2077}$$
$$\tag{2078}$$

(b) Simple calculation

$$e^{i\varphi}\hat{a} = \alpha_{\text{real}} e^{i\vartheta}\hat{b} + \beta_{\text{real}} e^{-i\vartheta}\hat{b}^\dagger \tag{2079}$$
$$\rightarrow \hat{a} = \alpha_{\text{real}} e^{i(\vartheta-\varphi)}\hat{b} + \beta_{\text{real}} e^{-i(\vartheta+\varphi)}\hat{b}^\dagger \tag{2080}$$

single particle Hilbert space	$\mathcal{H}$
single particle basis	$\{\|\nu\rangle\}$
single particle state	$\|\varphi\rangle = \sum_\nu c_\nu \|\nu\rangle$
single particle wave function	$u_\nu(\mathbf{r}) = \langle \mathbf{r}\|\nu\rangle$
multi particle Hilbert space	$\mathcal{H}_N = \mathcal{H}^{\otimes N}$
multi particle (product) basis	$\|\nu_1,\nu_2,...,\nu_N\rangle = \|\nu_1\rangle \otimes ... \otimes \|\nu_N\rangle$
multi particle 1-operator	$1 = \sum_{\nu_1,...\nu_N} \|\nu_1,\nu_2,...,\nu_N\rangle\langle\nu_1,\nu_2,...,\nu_N\|$
general state	$\|\psi\rangle = \sum_{\nu_1,...,\nu_N} \|\nu_1,\nu_2,...,\nu_N\rangle\langle\nu_1,\nu_2,...,\nu_N\|\psi\rangle$
	$= \sum_{\nu_1,...,\nu_N} c^{(1)}_{\nu_1}...c^{(N)}_{\nu_N}\|\nu_1,\nu_2,...,\nu_N\rangle$
	$= \sum_{\nu_1} c^{(1)}_{\nu_1}\|\nu_1\rangle ... \sum_{\nu_N} c^{(N)}_{\nu_N}\|\nu_N\rangle$
product state	$\|\psi\rangle = \prod_n \|\varphi_n\rangle$
entangled (non-product) state	$\|\psi\rangle = (\|\uparrow\rangle\|\downarrow\rangle - \|\downarrow\rangle\|\uparrow\rangle)$
coefficients	$\langle\nu_1,\nu_2,...,\nu_N\|\psi\rangle = c^{(1)}_{\nu_1}...c^{(N)}_{\nu_N}$
multi particle scalar product	$\langle\nu'_1,\nu'_2,...,\nu'_N\|\nu_1,\nu_2,...,\nu_N\rangle = \prod_n\langle\nu'_n\|\nu_n\rangle$
position basis	$\|\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_N\rangle = \|\mathbf{r}_1\rangle \otimes ... \otimes \|\mathbf{r}_N\rangle$
multiparticle wavefunction	$\psi(\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_N) = \langle\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_N\|\psi\rangle$
	$= \langle\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_N\| \sum_{\nu_1,...\nu_N}\|\nu_1,\nu_2,...,\nu_N\rangle\langle\nu_1,\nu_2,...,\nu_N\|\psi\rangle$
	$= \sum_{\nu_1,...\nu_N}\langle\mathbf{r}_1\|\nu_1\rangle...\langle\mathbf{r}_N\|\nu_N\rangle\langle\nu_1,\nu_2,...,\nu_N\|\psi\rangle$
	$= \sum_{\nu_1,...\nu_N} c^{(1)}_{\nu_1}...c^{(N)}_{\nu_N}\langle\mathbf{r}_1\|\nu_1\rangle...\langle\mathbf{r}_N\|\nu_N\rangle$
(part. k at $\mathbf{r}_k$ in state ν_k)	$= \sum_{\nu_1,...\nu_N} c^{(1)}_{\nu_1}...c^{(N)}_{\nu_N} u_{\nu_1}(\mathbf{r}_1)...u_{\nu_N}(\mathbf{r}_N)$
probability	$\|\psi(\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_N)\|^2 d^d\mathbf{r}_1 d^d\mathbf{r}...d^d\mathbf{r}_N$
observable of single k-th particle	$A^{(k)}_N = 1^{(1)}\otimes...\otimes 1^{(k-1)}\otimes A^{(k)}\otimes 1^{(k+1)}\otimes 1^{(N)}$
matrix element	$\langle\nu'_1,\nu'_2,...,\nu'_N\|A^{(k)}_N\|\nu_1,\nu_2,...,\nu_N\rangle = \langle\nu'_1\|\nu_1\rangle...\langle\nu'_k\|A^{(k)}\|\nu_k\rangle...\langle\nu'_N\|\nu_N\rangle$
matrix element	$= \delta_{\nu'_1,\nu_1}...\langle\nu'_k\|A^{(k)}\|\nu_k\rangle...\delta_{\nu'_N,\nu_N}$

with the two degrees of freedom ϑ, φ we can create two independent phases. So by tweaking ϑ, φ we can keep α and β real

(c) Simple calculation

$$\hat{a} = \alpha\hat{b} + \beta\hat{b}^\dagger \tag{2081}$$

$$\hat{a}^\dagger = \beta\hat{b} + \alpha\hat{b}^\dagger \tag{2082}$$

$$\to \hat{b} = \frac{1}{\alpha^2-\beta^2}(\alpha\hat{a} - \beta\hat{a}^\dagger) \tag{2083}$$

$$\to \hat{b}^\dagger = \frac{1}{\alpha^2-\beta^2}(-\beta\hat{a} + \alpha\hat{a}^\dagger) \tag{2084}$$

then because $[\hat{b},\hat{b}] = 0$, $[\hat{b}^\dagger,\hat{b}^\dagger] = 0$ (each commutator commutes with itself) we need to select

$$\to [\hat{b},\hat{b}^\dagger] = \frac{1}{(\alpha^2-\beta^2)^2}(-\alpha\beta[\hat{a},\hat{a}] - \alpha\beta[\hat{a}^\dagger,\hat{a}^\dagger] + \alpha^2[\hat{a},\hat{a}^\dagger] + \beta^2[\hat{a}^\dagger,\hat{a}]) \tag{2085}$$

$$= \frac{1}{(\alpha^2-\beta^2)^2}(-\alpha\beta\cdot 0 - \alpha\beta\cdot 0 + (\alpha^2-\beta^2)) \tag{2086}$$

$$= \frac{1}{\alpha^2-\beta^2} \tag{2087}$$

$$\to \alpha^2 - \beta^2 = 1 \tag{2088}$$

which is trivially true for $\cosh\theta$ and $\sinh\theta$.

(d) With

$$\hat{a} = \hat{b}\cosh\theta + \hat{b}^\dagger\sinh\theta \tag{2089}$$

$$\hat{a}^\dagger = \hat{b}^\dagger\cosh\theta + \hat{b}\sinh\theta \tag{2090}$$

then

$$\to \hat{H} = \frac{\epsilon}{2}(\hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger) + \frac{\lambda}{2}(\hat{a}^\dagger \hat{a}^\dagger + \hat{a}\hat{a}) \tag{2091}$$

$$= \frac{\epsilon}{2}\left((\hat{b}^\dagger \cosh\theta + \hat{b}\sinh\theta)(\hat{b}\cosh\theta + \hat{b}^\dagger \sinh\theta) + (\hat{b}\cosh\theta + \hat{b}^\dagger \sinh\theta)(\hat{b}^\dagger \cosh\theta + \hat{b}\sinh\theta)\right) \tag{2092}$$

$$+ \frac{\lambda}{2}\left((\hat{b}^\dagger \cosh\theta + \hat{b}\sinh\theta)(\hat{b}^\dagger \cosh\theta + \hat{b}\sinh\theta) + (\hat{b}\cosh\theta + \hat{b}^\dagger \sinh\theta)(\hat{b}\cosh\theta + \hat{b}^\dagger \sinh\theta)\right) \tag{2093}$$

$$= \frac{\epsilon}{2}\left(\hat{b}^\dagger\hat{b}\cosh^2\theta + \hat{b}\hat{b}^\dagger \sinh^2\theta + \hat{b}\hat{b}\sinh\theta\cosh\theta + \hat{b}^\dagger\hat{b}^\dagger \sinh\theta\cosh\theta\right) \tag{2094}$$

$$+ \frac{\epsilon}{2}\left(\hat{b}^\dagger\hat{b}\sinh^2\theta + \hat{b}\hat{b}^\dagger \cosh^2\theta + \hat{b}\hat{b}\sinh\theta\cosh\theta + \hat{b}^\dagger\hat{b}^\dagger \sinh\theta\cosh\theta\right) \tag{2095}$$

$$+ \frac{\lambda}{2}\left(\hat{b}^\dagger\hat{b}\sinh\theta\cosh\theta + \hat{b}\hat{b}^\dagger \sinh\theta\cosh\theta + \hat{b}\hat{b}\sinh^2\theta + \hat{b}^\dagger\hat{b}^\dagger \cosh^2\theta\right) \tag{2096}$$

$$+ \frac{\lambda}{2}\left(\hat{b}^\dagger\hat{b}\sinh\theta\cosh\theta + \hat{b}\hat{b}^\dagger \sinh\theta\cosh\theta + \hat{b}\hat{b}\cosh^2\theta + \hat{b}^\dagger\hat{b}^\dagger \sinh^2\theta\right) \tag{2097}$$

and with $\cosh^2\theta + \sinh^2\theta = \cosh 2\theta$ and $2\sinh\theta\cosh\theta = \sinh 2\theta$

$$H = \frac{\epsilon}{2}\left((\hat{b}^\dagger\hat{b} + \hat{b}\hat{b}^\dagger)\cosh 2\theta + (\hat{b}\hat{b} + \hat{b}^\dagger\hat{b}^\dagger)\sinh 2\theta\right) \tag{2098}$$

$$+ \frac{\lambda}{2}\left((\hat{b}^\dagger\hat{b} + \hat{b}\hat{b}^\dagger)\sinh 2\theta + (\hat{b}\hat{b} + \hat{b}^\dagger\hat{b}^\dagger)\cosh 2\theta\right) \tag{2099}$$

$$= \frac{\epsilon}{2}\left((2\hat{b}^\dagger\hat{b} + 1)\cosh 2\theta + (\hat{b}\hat{b} + \hat{b}^\dagger\hat{b}^\dagger)\sinh 2\theta\right) \tag{2100}$$

$$+ \frac{\lambda}{2}\left((2\hat{b}^\dagger\hat{b} + 1)\sinh 2\theta + (\hat{b}\hat{b} + \hat{b}^\dagger\hat{b}^\dagger)\cosh 2\theta\right) \tag{2101}$$

$$= (2\hat{b}^\dagger\hat{b} + 1)\left(\frac{\epsilon}{2}\cosh 2\theta + \frac{\lambda}{2}\sinh 2\theta\right) + (\hat{b}\hat{b} + \hat{b}^\dagger\hat{b}^\dagger)\left(\frac{\lambda}{2}\cosh 2\theta + \frac{\epsilon}{2}\sinh 2\theta\right) \tag{2102}$$

$$= (2\hat{b}^\dagger\hat{b} + 1)\frac{\epsilon}{2}\left(\cosh 2\theta + \frac{\lambda}{\epsilon}\sinh 2\theta\right) + (\hat{b}\hat{b} + \hat{b}^\dagger\hat{b}^\dagger)\frac{\epsilon}{2}\left(\frac{\lambda}{\epsilon}\cosh 2\theta + \sinh 2\theta\right) \tag{2103}$$

then

$$\frac{\lambda}{\epsilon}\cosh 2\theta + \sinh 2\theta \overset{!}{=} 0 \tag{2104}$$

$$\to \frac{\lambda}{\epsilon} = -\frac{\sinh 2\theta}{\cosh 2\theta} = -\frac{\sinh 2\theta}{\sqrt{1 + \sinh^2 2\theta}} \tag{2105}$$

$$\to \sinh 2\theta = \pm\frac{\lambda}{\sqrt{\epsilon^2 - \lambda^2}} \tag{2106}$$

$$\to \cosh 2\theta = \sqrt{1 + \sinh^2 2\theta} = \mp\frac{\epsilon}{\sqrt{\epsilon^2 - \lambda^2}} \tag{2107}$$

$$H = \left(\hat{b}^\dagger\hat{b} + \frac{1}{2}\right)\frac{\epsilon}{\sqrt{\epsilon^2 - \lambda^2}}\left(\mp\epsilon \pm \frac{\lambda^2}{\epsilon}\right) \tag{2108}$$

$$= \left(\hat{b}^\dagger\hat{b} + \frac{1}{2}\right)\frac{\mp 1}{\sqrt{\epsilon^2 - \lambda^2}}\left(\epsilon^2 - \lambda^2\right) \tag{2109}$$

$$= \mp\left(\hat{b}^\dagger\hat{b} + \frac{1}{2}\right)\sqrt{\epsilon^2 - \lambda^2} \tag{2110}$$

$$E_0 = \frac{1}{2}\sqrt{\epsilon^2 - \lambda^2} \tag{2111}$$

$$E_1 = \sqrt{\epsilon^2 - \lambda^2} \tag{2112}$$

(e) ???

0.25.6 Problem 4.1 - Generators of Lorentz transformations - NOT FINISHED YET d, e, f

Let

$$(M^{\alpha\beta})^{\mu}_{\ \nu} = i(\eta^{\alpha\mu}\delta^{\beta}_{\nu} - \eta^{\beta\mu}\delta^{\alpha}_{\nu}) \tag{2113}$$

for $\alpha, \beta, \mu, \nu \in \{0, 1, 2, 3\}$ with the (inverse) metric $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$. The Indices

$$M^{\alpha\beta} = \begin{pmatrix} (M^{\alpha\beta})^0_{\ 0} & (M^{\alpha\beta})^0_{\ 1} & \cdots \\ (M^{\alpha\beta})^1_{\ 0} & (M^{\alpha\beta})^1_{\ 1} & \\ \vdots & & \ddots \end{pmatrix} \tag{2114}$$

and the indices α and β label the matrices.

(a) Show that $\Lambda = \exp[\frac{i}{2}\omega_{\alpha\beta}M^{\alpha\beta}]$ with summation convention for α and β satisfies

$$\Lambda^{\mu}_{\ \rho}\Lambda^{\nu}_{\ \sigma}\eta_{\mu\nu} = \eta_{\rho\sigma} \tag{2115}$$

to linear order in ω! This shows that the matrices $M^{\alpha\beta}$ are generators of the Lorentz group.

(b) How many of the $M^{\alpha\beta}$ are linearly independent? Pick a set of linearly independent matrices and write their components down explicitly! Relate them to the Lorentz group generators $J_x, J_y, J_z, K_x, K_y, K_z$ defined in the lecture!

(c) Show that in therms of $M^{\alpha\beta}$ the Lorentz algebra takes the compact form

$$[M^{\alpha\beta}, M^{\gamma\delta}] = i(\eta^{\alpha\delta}M^{\beta\gamma} + \eta^{\beta\gamma}M^{\alpha\delta} - \eta^{\beta\delta}M^{\alpha\gamma} - \eta^{\alpha\gamma}M^{\beta\delta}) \tag{2116}$$

(d) Assuming there are four matrices γ^{α} of arbitrary dimension d satisfying the so-called Dirac algebra

$$\{\gamma^{\alpha}, \gamma^{\beta}\} = 2\eta^{\alpha\beta}\mathbb{I}_d \tag{2117}$$

where the anticommutator is defined by $\{A, B\} = AB + BA$. Show that the matrices

$$S^{\alpha\beta} = \frac{i}{4}[\gamma^{\alpha}, \gamma^{\beta}] \tag{2118}$$

satisfy (2116) with M replaced by S! If you want to use

$$[A, BC] = [A, B]C + B[A, C] = \{A, B\}C - B\{A, C\} \tag{2119}$$

$$[AB, C] = A[B, C] + [A, C]B = A\{B, C\} - \{A, C\}B \tag{2120}$$

$$\tag{2121}$$

then you have to show that these identities are indeed true.

(e) Show that the four 4×4 matrices written in block form as

$$\gamma^0 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}, \qquad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \qquad k \in \{1, 2, 3\} \tag{2122}$$

where σ_k are the Pauli matrices, satisfy the Dirac algebra (2)! You may use their property

$$\sigma_i\sigma_j = i\epsilon_{ijk}\sigma_k + \delta_{ij}\mathbb{I}_2 \tag{2123}$$

(f) Write out explicitly the matrices $S^{\alpha\beta}$ for the indices you chose for $M^{\alpha\beta}$ in (b) and compare the two representations!

(a)

$$\Lambda^{\mu}_{\ \rho} \simeq \left(\delta^{\mu}_{\rho} + \frac{i}{2}\omega_{\alpha\beta}(M^{\alpha\beta})^{\mu}_{\ \rho} + ...\right) \tag{2124}$$

$$\simeq \left(\delta^{\mu}_{\rho} - \frac{1}{2}\omega_{\alpha\beta}(\eta^{\alpha\mu}\delta^{\beta}_{\rho} - \eta^{\beta\mu}\delta^{\alpha}_{\rho}) + ...\right) \tag{2125}$$

then

$$\Lambda^{\mu}_{\ \rho}\Lambda^{\nu}_{\ \sigma}\eta_{\mu\nu} \simeq \left(\delta^{\mu}_{\rho} - \frac{1}{2}\omega_{\alpha\beta}(\eta^{\alpha\mu}\delta^{\beta}_{\rho} - \eta^{\beta\mu}\delta^{\alpha}_{\rho}) + ...\right)\left(\delta^{\nu}_{\sigma} - \frac{1}{2}\omega_{\gamma\delta}(\eta^{\gamma\nu}\delta^{\delta}_{\sigma} - \eta^{\delta\nu}\delta^{\gamma}_{\sigma}) + ...\right)\eta_{\mu\nu} \tag{2126}$$

$$\simeq \delta^{\mu}_{\rho}\delta^{\nu}_{\sigma}\eta_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\delta^{\mu}_{\rho}\omega_{\gamma\delta}(\eta^{\gamma\nu}\delta^{\delta}_{\sigma} - \eta^{\delta\nu}\delta^{\gamma}_{\sigma}) - \frac{1}{2}\eta_{\mu\nu}\delta^{\nu}_{\sigma}\omega_{\alpha\beta}(\eta^{\alpha\mu}\delta^{\beta}_{\rho} - \eta^{\beta\mu}\delta^{\alpha}_{\rho}) + ... \tag{2127}$$

$$= \eta_{\rho\sigma} - \frac{1}{2}\eta_{\rho\nu}\omega_{\gamma\delta}(\eta^{\gamma\nu}\delta^{\delta}_{\sigma} - \eta^{\delta\nu}\delta^{\gamma}_{\sigma}) - \frac{1}{2}\eta_{\mu\sigma}\omega_{\alpha\beta}(\eta^{\alpha\mu}\delta^{\beta}_{\rho} - \eta^{\beta\mu}\delta^{\alpha}_{\rho}) + ... \tag{2128}$$

$$= \eta_{\rho\sigma} - \frac{1}{2}\omega_{\gamma\delta}(\delta^{\gamma}_{\rho}\delta^{\delta}_{\sigma} - \delta^{\delta}_{\rho}\delta^{\gamma}_{\sigma}) - \frac{1}{2}\omega_{\alpha\beta}(\delta^{\alpha}_{\sigma}\delta^{\beta}_{\rho} - \delta^{\beta}_{\sigma}\delta^{\alpha}_{\rho}) + ... \tag{2129}$$

$$= \eta_{\rho\sigma} - \frac{1}{2}(\omega_{\rho\sigma} - \omega_{\sigma\rho}) - \frac{1}{2}(\omega_{\sigma\rho} - \omega_{\rho\sigma}) + ... \tag{2130}$$

$$= \eta_{\rho\sigma} - \frac{1}{2}(\omega_{\rho\sigma} - \omega_{\sigma\rho} - \omega_{\sigma\rho} + \omega_{\rho\sigma}) + ... \tag{2131}$$

$$= \eta_{\rho\sigma} - (\omega_{\rho\sigma} - \omega_{\sigma\rho}) + ... \tag{2132}$$

$$= \eta_{\rho\sigma} \tag{2133}$$

(b) There are sixteen 4×4 matrices minus four $M^{\alpha\alpha} = 0$ which is gives twelve matrices of which six remain (because of $M^{\alpha\beta} = -M^{\beta\alpha}$).

So we pick M^{01}, M^{02}, M^{03} which are the boosts K_1, K_2, K_3

$$M^{01} = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2134}$$

And we pick M^{23}, M^{31}, M^{12} which are the boosts J_1, J_2, J_3

$$M^{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & +i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2135}$$

(c) Then with

$$(M^{\mu\nu})^{\rho}_{\ \sigma} = i(\eta^{\mu\rho}\delta^{\nu}_{\sigma} - \eta^{\nu\rho}\delta^{\mu}_{\sigma}) \tag{2136}$$

$$(M^{\lambda\kappa})^{\sigma}_{\ \tau} = i(\eta^{\lambda\sigma}\delta^{\kappa}_{\tau} - \eta^{\kappa\sigma}\delta^{\lambda}_{\tau}) \tag{2137}$$

we obtain

$$\rightarrow (M^{\mu\nu}M^{\lambda\kappa})^{\rho}_{\ \tau} = (M^{\mu\nu})^{\rho}_{\ \sigma}(M^{\lambda\kappa})^{\sigma}_{\ \tau} \tag{2138}$$

$$= i(\eta^{\mu\rho}\delta^{\nu}_{\sigma} - \eta^{\nu\rho}\delta^{\mu}_{\sigma})i(\eta^{\lambda\sigma}\delta^{\kappa}_{\tau} - \eta^{\kappa\sigma}\delta^{\lambda}_{\tau}) \tag{2139}$$

$$= -(\eta^{\mu\rho}\delta^{\nu}_{\sigma}\eta^{\lambda\sigma}\delta^{\kappa}_{\tau} - \eta^{\mu\rho}\delta^{\nu}_{\sigma}\eta^{\kappa\sigma}\delta^{\lambda}_{\tau} - \eta^{\nu\rho}\delta^{\mu}_{\sigma}\eta^{\lambda\sigma}\delta^{\kappa}_{\tau} + \eta^{\nu\rho}\delta^{\mu}_{\sigma}\eta^{\kappa\sigma}\delta^{\lambda}_{\tau}) \tag{2140}$$

$$= -(\eta^{\mu\rho}\eta^{\lambda\nu}\delta^{\kappa}_{\tau} - \eta^{\mu\rho}\eta^{\kappa\nu}\delta^{\lambda}_{\tau} - \eta^{\nu\rho}\eta^{\lambda\mu}\delta^{\kappa}_{\tau} + \eta^{\nu\rho}\eta^{\kappa\mu}\delta^{\lambda}_{\tau}) \tag{2141}$$

and

$$\to (M^{\lambda\kappa}M^{\mu\nu})^{\rho}_{\ \tau} = -(\eta^{\lambda\rho}\eta^{\mu\kappa}\delta^{\nu}_{\tau} - \eta^{\lambda\rho}\eta^{\nu\kappa}\delta^{\mu}_{\tau} - \eta^{\kappa\rho}\eta^{\mu\lambda}\delta^{\nu}_{\tau} + \eta^{\kappa\rho}\eta^{\nu\lambda}\delta^{\mu}_{\tau}) \tag{2142}$$

then we can collect

$$[M^{\mu\nu}, M^{\lambda\kappa}]^{\rho}_{\ \tau} = \eta^{\mu\kappa}(-\eta^{\nu\rho}\delta^{\lambda}_{\tau} + \eta^{\lambda\rho}\delta^{\nu}_{\tau}) + \eta^{\nu\lambda}... \tag{2143}$$
$$= i\eta^{\mu\kappa}i(\eta^{\nu\rho}\delta^{\lambda}_{\tau} - \eta^{\lambda\rho}\delta^{\nu}_{\tau}) + ... \tag{2144}$$
$$= i\eta^{\mu\kappa}M^{\nu\lambda} + ... \tag{2145}$$

0.25.7 Problem 5.1 - Weyl particle and parity

Let the field $\psi_L(x)$ be a solution of the left-chiral Weyl equation $\bar{\sigma}^{\mu}\partial_{\mu}\psi_L(x) = 0$ in a given frame of reference. Say another observer uses a mirrored coordinate system, i.e. spacetime coordinates $x'^{\mu} = P^{\mu}{}_{\nu}x^{\nu}$ with the parity transformation

$$P = \text{diag}(+1, -1, -1, -1). \tag{2146}$$

What equation does the reflected field $\psi^P(x') := \psi_L(P^{-1}x')$ satisfy? What does this fact imply about a universe that contains a Weyl particle?

With $\bar{\sigma}^{\mu} \equiv (1, -\vec{\sigma}) = (\sigma_0, -\sigma_1, -\sigma_2, -\sigma_3)$ we can write

$$\bar{\sigma}^{\mu}\partial_{\mu}\psi_L(x) \equiv (\sigma_0\partial_0 - \sigma_1\partial_1 - \sigma_2\partial_2 - \sigma_3\partial_3)\psi_L(x) = 0 \tag{2147}$$

There will be a minus sign in the spacial derivative $\partial'_{\mu} = -\partial_{\mu}$ for the new coordinates $x' = Px = (x^0, -\mathbf{x})$ so with $\sigma^{\mu} = (1, \vec{\sigma})$ we guess and show

$$\sigma\partial'_{\mu}\psi^P(x') = (\sigma_0\partial_0 + \sigma_1\partial'_1 + \sigma_2\partial'_2 + \sigma_3\partial'_3)\,\psi^P(x') \tag{2148}$$
$$= (\sigma_0\partial_0 + \sigma_1(-\partial_1) + \sigma_2(-\partial_2) + \sigma_3(-\partial_3))\,\psi^P(x^0, -\mathbf{x}) \tag{2149}$$
$$= (\sigma_0\partial_0 - \sigma_1\partial_1 - \sigma_2\partial_2 - \sigma_3\partial_3)\,\psi_L(x^0, \mathbf{x}) \tag{2150}$$
$$= \bar{\sigma}^{\mu}\partial_{\mu}\psi_L(x) \tag{2151}$$
$$= 0 \tag{2152}$$

0.25.8 Problem 5.3 - Weyl equation - NOT DONE YET b,c

For a given momentum vector $\mathbf{p}$, let $P_L = (1/2 - h)$ and $P_R = (1/2 + h)$ where $h = \frac{\vec{\sigma}}{2} \cdot \frac{\mathbf{p}}{|\mathbf{p}|}$ is the helicity of a spin $1/2$ particle

(a) Show that these matrices satisfy

$$P_L^2 = P_L, \quad P_R^2 = P_R, \quad P_L P_R = 0, \quad P_L + P_R = 1 \tag{2153}$$

What is the intuitive meaning of these properties?

(b) Find the general solutions of the Weyl equations in momentum space

$$P_R\tilde{\psi}_L(p) = 0, \qquad P_L\tilde{\psi}_R(p) = 0 \tag{2154}$$

(c) Write the coordinate space solutions $\psi_{L,R}(x)$ as Fourier integrals of the solutions in part (b) and how many parameters in your expressions parameterize the general solution!

Using the properties of the Pauli matrices

$$\sigma_i\sigma_i = \delta_{ij}\sigma_0 + i\epsilon_{ijk}\sigma_k \tag{2155}$$

(a) Then with $p = (\sqrt{m^2+\mathbf{p}^2}, \mathbf{p})$ and $\mathbf{p}\times\mathbf{p} = 0$ we can show

$$\begin{aligned}
(\sigma_k p^k)^2 &= (\sigma_k p^k)(\sigma_l p^l) && (2156)\\
&= \sigma_k\sigma_l p^k p^l && (2157)\\
&= p^k p^l(\delta_{kl}\sigma_0 + i\epsilon_{klm}\sigma_m) && (2158)\\
&= p^l p^l\sigma_0 + i\sigma_m\epsilon_{mkl}p^k p^l && (2159)\\
&= \mathbf{p}^2\sigma_0 + i\vec{\sigma}\cdot(\mathbf{p}\times\mathbf{p}) && (2160)\\
&= \mathbf{p}^2\sigma_0 && (2161)
\end{aligned}$$

and therefore

$$h^2 = \frac{(\vec{\sigma}\cdot\mathbf{p})^2}{4\mathbf{p}^2} = \frac{\sigma_0}{4}. \tag{2162}$$

Now we are done with the preparation and calculate ($\sigma_0^2 = \sigma_0 = 1_{2\times2}$)

$$\begin{aligned}
P_{L/R}^2 &= \left(\frac{1}{2}\sigma_0 \pm h\right)^2 && (2163)\\
&= \left(\frac{1}{4}\sigma_0^2 \pm \frac{1}{2}\sigma_0 h \pm \frac{1}{2}h\sigma_0 + h^2\right) && (2164)\\
&= \left(\frac{1}{4}\sigma_0 \pm \frac{1}{2}h \pm \frac{1}{2}h + \frac{\sigma_0}{4}\right) && (2165)\\
&= \left(\frac{1}{2}\sigma_0 \pm h\right) && (2166)\\
&= P_{L/R} && (2167)
\end{aligned}$$

$$\begin{aligned}
P_L P_R &= \left(\frac{1}{2}\sigma_0 - h\right)\left(\frac{1}{2}\sigma_0 + h\right) && (2168)\\
&= \frac{\sigma_0}{4} + \frac{1}{2}\sigma_0 h - \frac{1}{2}h\sigma_0 - h^2 && (2169)\\
&= 0 && (2170)\\
P_L + P_R &= \left(\frac{1}{2}\sigma_0 - h\right) + \left(\frac{1}{2}\sigma_0 + h\right) && (2171)\\
&= \sigma_0 = 1_{2\times2} && (2172)
\end{aligned}$$

this all implies that $P_{L/R}$ are projection operators.

(b) Starting with the Dirac equation and setting $m = 0$

$$\begin{aligned}
&(i\gamma^\mu\partial_\mu - m)\psi(x) = 0 && (2173)\\
\rightarrow &(i\gamma^\mu\partial_\mu)\psi(x) = 0 && (2174)\\
& && (2175)
\end{aligned}$$

(c)

0.25.9 Problem 7.2 - Coupled scalar and vector fields

The Lagrangian density for a complex scalar field interacting with the electromagnetic field is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial_\mu\phi + ieA_\mu\phi)^*(\partial^\mu\phi + ieA'^\mu\phi) + m^2\phi^*\phi \tag{2176}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and e is a constant.

(a) In the lecture we discussed the gauge transformation

$$A'_\mu(x) = A_\mu(x) + \partial_\mu \alpha(x) \tag{2177}$$

of the vector field. This transformation leaves $F_{\mu\nu}$ and thus the first term in $\mathcal{L}$ invariant. Find an appropriate, possibly complex $\chi(x)$ such that the entire Lagrange density is invariant under the gauge transformations if the scalar field transforms as $\phi'(x) = e^{\chi(x)}\phi(x)$!

(b) Derive the equations of motion for $A_\mu(x)$ and for $\phi(x)$!

(a) Double checking the first term

$$A'_\mu(x) = A_\mu(x) + \partial_\mu \alpha(x) \tag{2178}$$
$$\to F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu \tag{2179}$$
$$= \partial_\mu (A_\nu + \partial_\nu \alpha) - \partial_\nu (A_\mu + \partial_\mu \alpha) \tag{2180}$$
$$= \partial_\mu A_\nu - \partial_\nu A_\mu + \partial_\mu \partial_\nu \alpha - \partial_\nu \partial_\mu \alpha \tag{2181}$$
$$= \partial_\mu A_\nu - \partial_\nu A_\mu + \partial_\mu \partial_\nu \alpha - \partial_\mu \partial_\nu \alpha \tag{2182}$$
$$= \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2183}$$
$$= F_{\mu\nu} \tag{2184}$$
$$\to F'_{\mu\nu} F'^{\mu\nu} = F_{\mu\nu} F^{\mu\nu} \tag{2185}$$

For the second and third term we should not check each term individually but the sum. So preparing the second term

$$\partial_\mu \phi' + ieA'_\mu \phi' = \partial_\mu (e^{\chi(x)}\phi) + ie(A_\mu + \partial_\mu \alpha)(e^{\chi(x)}\phi) \tag{2186}$$
$$= e^{\chi(x)}\partial_\mu \phi + e^{\chi(x)}\phi \partial_\mu \chi + ieA_\mu e^{\chi(x)}\phi + ie(\partial_\mu \alpha) e^{\chi(x)}\phi \tag{2187}$$
$$= e^{\chi(x)}(\partial_\mu \phi + ieA_\mu \phi) + e^{\chi(x)}\phi(\partial_\mu \chi + ie(\partial_\mu \alpha)) \tag{2188}$$

then results for the second in

$$(\partial_\mu \phi' + ieA'_\mu \phi')^*(\partial^\mu \phi' + ieA'^\mu \phi') + m^2 \phi'^* \phi' \tag{2189}$$
$$= \left(e^{\chi(x)}(\partial_\mu \phi + ieA_\mu \phi) + e^{\chi(x)}\phi(\partial_\mu \chi + ie(\partial_\mu \alpha))\right)^* \left(e^{\chi(x)}(\partial^\mu \phi + ieA^\mu \phi) + e^{\chi(x)}\phi(\partial^\mu \chi + ie(\partial^\mu \alpha))\right) \tag{2190}$$
$$= \left((e^{\chi(x)})^*(\partial_\mu \phi^* - ieA_\mu \phi^*) + (e^{\chi(x)})^* \phi^*(\partial_\mu \chi^* - ie(\partial_\mu \alpha))\right) \left(e^{\chi(x)}(\partial^\mu \phi + ieA^\mu \phi) + e^{\chi(x)}\phi(\partial^\mu \chi + ie(\partial^\mu \alpha \tag{2191}$$
$$= (e^{\chi(x)})^* e^{\chi(x)} \left((\partial_\mu \phi^* - ieA_\mu \phi^*) + \phi^*(\partial_\mu \chi^* - ie(\partial_\mu \alpha))\right)\left((\partial^\mu \phi + ieA^\mu \phi) + \phi(\partial^\mu \chi + ie(\partial^\mu \alpha))\right) \tag{2192}$$
$$= (e^{\chi(x)})^* e^{\chi(x)} (\partial_\mu \phi + ieA_\mu \phi)^*(\partial^\mu \phi + ieA^\mu \phi) \tag{2193}$$
$$+ (e^{\chi(x)})^* e^{\chi(x)} \phi^* \phi (\partial_\mu \chi^* - ie(\partial_\mu \alpha))(\partial^\mu \chi + ie(\partial^\mu \alpha)) \tag{2194}$$
$$+ (e^{\chi(x)})^* e^{\chi(x)} \phi (\partial_\mu \phi^* - ieA_\mu \phi^*)(\partial^\mu \chi + ie(\partial^\mu \alpha)) \tag{2195}$$
$$+ (e^{\chi(x)})^* e^{\chi(x)} \phi^* (\partial^\mu \phi + ieA^\mu \phi)(\partial_\mu \chi^* - ie(\partial_\mu \alpha)) \tag{2196}$$

So the sum of term two and three gives

$$(\partial_\mu \phi' + ieA'_\mu \phi')^*(\partial^\mu \phi' + ieA'^\mu \phi') + m^2 \phi'^* \phi' \tag{2197}$$
$$= (e^{\chi(x)})^* e^{\chi(x)} (\partial_\mu \phi + ieA_\mu \phi)^*(\partial^\mu \phi + ieA^\mu \phi) \tag{2198}$$
$$+ (e^{\chi(x)})^* e^{\chi(x)} m^2 \phi^* \phi \tag{2199}$$
$$+ (e^{\chi(x)})^* e^{\chi(x)} \phi^* \phi (\partial_\mu \chi^* - ie(\partial_\mu \alpha))(\partial^\mu \chi + ie(\partial^\mu \alpha)) \tag{2200}$$
$$+ (e^{\chi(x)})^* e^{\chi(x)} \phi (\partial_\mu \phi^* - ieA_\mu \phi^*)(\partial^\mu \chi + ie(\partial^\mu \alpha)) \tag{2201}$$
$$+ (e^{\chi(x)})^* e^{\chi(x)} \phi^* (\partial^\mu \phi + ieA^\mu \phi)(\partial_\mu \chi^* - ie(\partial_\mu \alpha)) \tag{2202}$$

We see that we need $(e^{\chi(x)})^* e^{\chi(x)} = 1$ so $\chi(x) = i\xi(x)$ and therefore

$$(e^{\chi(x)})^* e^{\chi(x)} = (e^{i\xi(x)})^* e^{\xi(x)} = e^{-i\xi(x)} e^{\xi(x)} = 1 \tag{2203}$$

so

$$\begin{aligned}(\partial_\mu \phi' + ieA'_\mu \phi')^*(\partial^\mu \phi' + ieA'^\mu \phi') + m^2 \phi'^* \phi' &= (\partial_\mu \phi + ieA_\mu \phi)^*(\partial^\mu \phi + ieA^\mu \phi) + m^2 \phi'^* \phi \\ &\quad + \phi^* \phi(-i\partial_\mu \xi - ie(\partial_\mu \alpha))(i\partial^\mu \xi + ie(\partial^\mu \alpha)) \\ &\quad + \phi(\partial_\mu \phi^* - ieA_\mu \phi^*)(i\partial^\mu \xi + ie(\partial^\mu \alpha)) \\ &\quad + \phi^*(\partial^\mu \phi + ieA^\mu \phi)(-i\partial_\mu \xi - ie(\partial_\mu \alpha)) \\ &= (\partial_\mu \phi + ieA_\mu \phi)^*(\partial^\mu \phi + ieA^\mu \phi) + m^2 \phi'^* \phi \\ &\quad + \phi^* \phi(\partial_\mu \xi + e(\partial_\mu \alpha))(\partial^\mu \xi + e(\partial^\mu \alpha)) \\ &\quad + i\phi(\partial_\mu \phi^* - ieA_\mu \phi^*)(\partial^\mu \xi + e(\partial^\mu \alpha)) \\ &\quad - i\phi^*(\partial^\mu \phi + ieA^\mu \phi)(\partial_\mu \xi + e(\partial_\mu \alpha))\end{aligned}$$

with

$$\xi(x) = -ea(x) \tag{2204}$$

$$\rightarrow \phi'(x) = e^{-iea(x)} \tag{2205}$$

(b) Equations of motions for A

$$\frac{\partial \mathcal{L}}{\partial A_\alpha} = -ie\phi^*(\partial^\mu \phi + ieA^\mu \phi)\delta^\alpha_\mu + ie\phi(\partial^\mu + ieA^\mu \phi)^* \delta^\alpha_\mu \tag{2206}$$

$$= -ie\phi^*(\partial^\alpha \phi + ieA^\alpha \phi) + ie\phi(\partial^\alpha \phi + ieA^\mu \phi)^* \tag{2207}$$

$$= ie(-\phi^* \partial^\alpha \phi + \phi \partial^\alpha \phi^*) + 2e^2 A^\alpha \phi^* \phi \tag{2208}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\beta A_\alpha)} = -\frac{1}{4} 2(\delta^\beta_\mu \delta^\alpha_\nu - \delta^\beta_\nu \delta^\alpha_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \tag{2209}$$

$$= -\frac{1}{2}(\delta^\beta_\mu \delta^\alpha_\nu - \delta^\beta_\nu \delta^\alpha_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \tag{2210}$$

$$= -\frac{1}{2}(\partial^\beta A^\alpha - \partial^\alpha A^\beta - \partial^\alpha A^\beta + \partial^\beta A^\alpha) \tag{2211}$$

$$= -(\partial^\beta A^\alpha - \partial^\alpha A^\beta) \tag{2212}$$

$$\rightarrow -\partial_\beta \partial^\beta A^\alpha + \partial_\beta \partial^\alpha A^\beta - 2e^2 A^\alpha \phi^* \phi + ie(\phi^* \partial^\alpha \phi - \phi \partial^\alpha \phi^*) = 0 \tag{2213}$$

$$\rightarrow \partial_\beta \partial^\beta A^\alpha - \partial_\beta \partial^\alpha A^\beta = -2e^2 A^\alpha \phi^* \phi + ie(\phi^* \partial^\alpha \phi - \phi \partial^\alpha \phi^*) \tag{2214}$$

Equations of motions for ϕ

$$\frac{\partial \mathcal{L}}{\partial \phi^*} = -ieA_\mu(\partial^\mu \phi + ieA^\mu \phi) - m^2 \phi \tag{2215}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} = (\partial^\mu \phi + ieA^\mu \phi) \tag{2216}$$

$$\rightarrow \partial_\mu \partial^\mu \phi + ie\partial_\mu A^\mu \phi + ieA^\mu \partial_\mu \phi + ieA_\mu(\partial^\mu \phi + ieA^\mu \phi) + m^2 \phi = 0 \tag{2217}$$

$$\rightarrow (\partial_\mu \partial^\mu + m^2)\phi = -ie\partial_\mu A^\mu \phi - ieA^\mu \partial_\mu \phi - ieA_\mu(\partial^\mu \phi + ieA^\mu \phi) \tag{2218}$$

$$\rightarrow (\partial_\mu \partial^\mu + m^2)\phi = -ie\partial_\mu A^\mu \phi - 2ieA^\mu \partial_\mu \phi + e^2 A_\mu A^\mu \phi \tag{2219}$$

0.25.10 Problem 7.3 - Weyl spinor field

$$\mathcal{L} = \psi^\dagger i\bar{\sigma}^\mu \partial_\mu \psi + \frac{i}{2} m\psi^\dagger \sigma_2 \psi^* - \frac{i}{2} m\psi^T \sigma_2 \psi \tag{2220}$$

where $\psi^\dagger = (\psi^T)^*$. Initial thoughts about independent variables ψ and $\psi^\dagger$ - so practically we ignore the transpose and regard the complex conjugated ψ as different (same as in scalar case)

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \quad \rightarrow \quad \frac{\partial\psi}{\partial\psi} = 1 \tag{2221}$$

$$\psi^T = \begin{pmatrix}\psi_1 & \psi_2\end{pmatrix} \quad \rightarrow \quad \frac{\partial\psi^T}{\partial\psi} = 1 \tag{2222}$$

$$\psi^* = \begin{pmatrix}\psi_1^*\\ \psi_2^*\end{pmatrix} \quad \rightarrow \quad \frac{\partial\psi^*}{\partial\psi^\dagger} = 1 \tag{2223}$$

$$\psi^\dagger = \begin{pmatrix}\psi_1^* & \psi_2^*\end{pmatrix} \quad \rightarrow \quad \frac{\partial\psi^\dagger}{\partial\psi^\dagger} = 1 \tag{2224}$$

Variable $\psi^\dagger$

$$\frac{\partial\mathcal{L}}{\partial\psi^\dagger} = i\bar{\sigma}^\alpha\partial_\alpha\psi + \frac{i}{2}m\sigma_2\psi^* + \frac{i}{2}m\psi^\dagger\sigma_2???? \tag{2225}$$

$$\partial_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\psi^\dagger)} = 0 \tag{2226}$$

$$\rightarrow i\bar{\sigma}^\mu\partial_\mu\psi + \frac{i}{2}m\sigma_2\psi^* = 0 \tag{2227}$$

$$\rightarrow \bar{\sigma}^\mu\partial_\mu\psi + \frac{1}{2}m\sigma_2\psi^* = 0 \tag{2228}$$

Variable ψ

$$\frac{\partial\mathcal{L}}{\partial\psi} = -\frac{i}{2}m\psi^T\sigma_2 \tag{2229}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\psi)} = \psi^\dagger i\bar{\sigma}^\mu\delta_\mu^\alpha \tag{2230}$$

$$= \psi^\dagger i\bar{\sigma}^\alpha \tag{2231}$$

$$\partial_\alpha\frac{\partial\mathcal{L}}{\partial\partial_\alpha\psi)} = \partial_\alpha\psi^\dagger i\bar{\sigma}^\alpha \tag{2232}$$

$$\tag{2233}$$

0.26 ETH Zurich: Beisert - Quantum Field Theory I - 2019 HS

0.26.1 Problem 5.1 - Representations of the Lorentz algebras - NOT FINISHED

The Lie algebra $\mathfrak{so}(d,1)$ of the Lorentz group SO(d,1) for $(d+1)$-dimensional spacetime is given in terms of its generators $M^{\mu\nu}$

$$[M^{\mu\nu}, M^{\lambda\kappa}] = i(\eta^{\mu\kappa}M^{\nu\lambda} + \eta^{\nu\lambda}M^{\mu\kappa} - \eta^{\nu\kappa}M^{\mu\lambda} - \eta^{\mu\lambda}M^{\nu\kappa}) \tag{2234}$$

Any representation of the Lorentz algebra must satisfy the above commutation relations.

a) Show explicitly that the following generators $J^{\mu\nu}$ of the vector representation satisfy the Lie algebra

$$(J^{\mu\nu})^\rho{}_\sigma = i(\eta^{\mu\rho}\delta^\nu_\sigma - \eta^{\nu\rho}\delta^\mu_\sigma) \tag{2235}$$

b) Show explicitly that the following differential operators $L^{\mu\nu}$ satisfy the Lie algebra

$$L^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu) \tag{2236}$$

a) Then with

$$(J^{\mu\nu})^{\rho}_{\ \sigma} = i(\eta^{\mu\rho}\delta^{\nu}_{\sigma} - \eta^{\nu\rho}\delta^{\mu}_{\sigma}) \tag{2237}$$
$$(J^{\lambda\kappa})^{\sigma}_{\ \tau} = i(\eta^{\lambda\sigma}\delta^{\kappa}_{\tau} - \eta^{\kappa\sigma}\delta^{\lambda}_{\tau}) \tag{2238}$$

we obtain

$$\rightarrow (M^{\mu\nu}M^{\lambda\kappa})^{\rho}_{\ \tau} = (J^{\mu\nu})^{\rho}_{\ \sigma}(J^{\lambda\kappa})^{\sigma}_{\ \tau} \tag{2239}$$
$$= i(\eta^{\mu\rho}\delta^{\nu}_{\sigma} - \eta^{\nu\rho}\delta^{\mu}_{\sigma})i(\eta^{\lambda\sigma}\delta^{\kappa}_{\tau} - \eta^{\kappa\sigma}\delta^{\lambda}_{\tau}) \tag{2240}$$
$$= -(\eta^{\mu\rho}\delta^{\nu}_{\sigma}\eta^{\lambda\sigma}\delta^{\kappa}_{\tau} - \eta^{\mu\rho}\delta^{\nu}_{\sigma}\eta^{\kappa\sigma}\delta^{\lambda}_{\tau} - \eta^{\nu\rho}\delta^{\mu}_{\sigma}\eta^{\lambda\sigma}\delta^{\kappa}_{\tau} + \eta^{\nu\rho}\delta^{\mu}_{\sigma}\eta^{\kappa\sigma}\delta^{\lambda}_{\tau}) \tag{2241}$$
$$= -(\eta^{\mu\rho}\eta^{\lambda\nu}\delta^{\kappa}_{\tau} - \eta^{\mu\rho}\eta^{\kappa\nu}\delta^{\lambda}_{\tau} - \eta^{\nu\rho}\eta^{\lambda\mu}\delta^{\kappa}_{\tau} + \eta^{\nu\rho}\eta^{\kappa\mu}\delta^{\lambda}_{\tau}) \tag{2242}$$

and

$$\rightarrow (M^{\lambda\kappa}M^{\mu\nu})^{\rho}_{\ \tau} = -(\eta^{\lambda\rho}\eta^{\mu\kappa}\delta^{\nu}_{\tau} - \eta^{\lambda\rho}\eta^{\nu\kappa}\delta^{\mu}_{\tau} - \eta^{\kappa\rho}\eta^{\mu\lambda}\delta^{\nu}_{\tau} + \eta^{\kappa\rho}\eta^{\nu\lambda}\delta^{\mu}_{\tau}) \tag{2243}$$

then we can collect

$$[M^{\mu\nu}, M^{\lambda\kappa}]^{\rho}_{\ \tau} = \eta^{\mu\kappa}(-\eta^{\nu\rho}\delta^{\lambda}_{\tau} + \eta^{\lambda\rho}\delta^{\nu}_{\tau}) + \eta^{\nu\lambda}... \tag{2244}$$
$$= i\eta^{\mu\kappa}i(\eta^{\nu\rho}\delta^{\lambda}_{\tau} - \eta^{\lambda\rho}\delta^{\nu}_{\tau}) + ... \tag{2245}$$
$$= i\eta^{\mu\kappa}J^{\nu\lambda} + ... \tag{2246}$$

b) Using $\partial_\mu x^\nu = \delta^\nu_\mu$ and $\partial^\alpha x^\nu = \eta^{\alpha\mu}\partial_\mu x^\nu = \eta^{\alpha\nu}$

$$L^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu) \tag{2247}$$
$$L^{\lambda\kappa} = i(x^\lambda\partial^\kappa - x^\kappa\partial^\lambda) \tag{2248}$$
$$\rightarrow L^{\mu\nu}L^{\lambda\kappa} = i(x^\mu\partial^\nu - x^\nu\partial^\mu)i(x^\lambda\partial^\kappa - x^\kappa\partial^\lambda) \tag{2249}$$
$$= -x^\mu\partial^\nu(x^\lambda\partial^\kappa) + x^\mu\partial^\nu(x^\kappa\partial^\lambda) + x^\nu\partial^\mu(x^\lambda\partial^\kappa) - x^\nu\partial^\mu(x^\kappa\partial^\lambda) \tag{2250}$$
$$= -x^\mu x^\lambda\partial^\nu\partial^\kappa + x^\mu x^\kappa\partial^\nu\partial^\lambda + x^\nu x^\lambda\partial^\mu\partial^\kappa - x^\nu x^\kappa\partial^\mu\partial^\lambda \tag{2251}$$
$$- x^\mu\eta^{\lambda\nu}\partial^\kappa + x^\mu\eta^{\kappa\nu}\partial^\lambda + x^\nu\eta^{\lambda\mu}\partial^\kappa - x^\nu\eta^{\kappa\mu}\partial^\lambda \tag{2252}$$

and

$$\rightarrow L^{\lambda\kappa}L^{\mu\nu} = -x^\lambda x^\mu\partial^\kappa\partial^\nu + x^\lambda x^\nu\partial^\kappa\partial^\mu + x^\kappa x^\mu\partial^\lambda\partial^\nu - x^\nu x^\kappa\partial^\lambda\partial^\mu \tag{2253}$$
$$- x^\lambda\eta^{\mu\kappa}\partial^\nu + x^\lambda\eta^{\kappa\nu}\partial^\mu + x^\kappa\eta^{\lambda\mu}\partial^\nu - x^\kappa\eta^{\nu\lambda}\partial^\mu \tag{2254}$$

then

$$[L^{\mu\nu}, L^{\lambda\kappa}] = -x^\mu x^\lambda\partial^\nu\partial^\kappa + x^\mu x^\kappa\partial^\nu\partial^\lambda + x^\nu x^\lambda\partial^\mu\partial^\kappa - x^\nu x^\kappa\partial^\mu\partial^\lambda \tag{2255}$$
$$- x^\mu\eta^{\lambda\nu}\partial^\kappa + x^\mu\eta^{\kappa\nu}\partial^\lambda + x^\nu\eta^{\lambda\mu}\partial^\kappa - x^\nu\eta^{\kappa\mu}\partial^\lambda \tag{2256}$$
$$+ x^\lambda x^\mu\partial^\kappa\partial^\nu - x^\lambda x^\nu\partial^\kappa\partial^\mu - x^\kappa x^\mu\partial^\lambda\partial^\nu + x^\nu x^\kappa\partial^\lambda\partial^\mu \tag{2257}$$
$$+ x^\lambda\eta^{\mu\kappa}\partial^\nu - x^\lambda\eta^{\kappa\nu}\partial^\mu - x^\kappa\eta^{\lambda\mu}\partial^\nu + x^\kappa\eta^{\nu\lambda}\partial^\mu \tag{2258}$$
$$= -x^\mu\eta^{\lambda\nu}\partial^\kappa + x^\mu\eta^{\kappa\nu}\partial^\lambda + x^\nu\eta^{\lambda\mu}\partial^\kappa - x^\nu\eta^{\kappa\mu}\partial^\lambda \tag{2259}$$
$$+ x^\lambda\eta^{\mu\kappa}\partial^\nu - x^\lambda\eta^{\kappa\nu}\partial^\mu - x^\kappa\eta^{\lambda\mu}\partial^\nu + x^\kappa\eta^{\nu\lambda}\partial^\mu \tag{2260}$$
$$= i\eta^{\kappa\mu}i(x^\nu\partial^\lambda - x^\lambda\partial^\nu) + ... \tag{2261}$$
$$= i\eta^{\kappa\mu}L^{\nu\lambda} + ... \tag{2262}$$
$$\tag{2263}$$

0.26.2 Problem 5.2 - Properties of gamma-matrices - NOT COMPLETE YET

The gamma-matrices in D dimensional spacetime satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\text{id} \tag{2264}$$

Derive the following identities using this algebraic relation (rather than an explicit matrix representation).

a) Prove the following contraction identities

$$\gamma^\mu\gamma_\mu = D\,\text{id}, \tag{2265}$$
$$\gamma^\mu\gamma^\nu\gamma_\mu = -(D-2)\gamma^\nu, \tag{2266}$$
$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = (D-4)\gamma^\nu\gamma^\rho + 4\eta^{\nu\rho}\text{id}, \tag{2267}$$
$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -(D-4)\gamma^\nu\gamma^\rho\gamma^\sigma - 2\gamma^\sigma\gamma^\rho\gamma^\nu \tag{2268}$$

b) Show that a trace of an odd number n of gamma-matrices is zero for an even number of spacetime dimensions D

$$\text{tr}(\gamma^{\mu_1}...\gamma^{\mu_n}) = 0. \tag{2269}$$

Hint: Eliminate double indices, insert $\text{id} = (\gamma^\rho)^{-1}(\gamma^\rho)$ for some index value ρ (no summation convention implied), and use cyclicity of the trace.

c) Show the following trace identities

$$\text{tr}(\gamma^\mu\gamma^\nu) = \text{tr}(\text{id})\eta^{\mu\nu}, \tag{2270}$$
$$\text{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = \text{tr}(\text{id})(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}). \tag{2271}$$

a)
- Setting $\mu = \nu$ in anticommutator gives $\gamma^\mu\gamma^\mu = \eta^{\mu\mu}\text{id}$. Then

$$\gamma^\mu\gamma_\mu = \gamma^\mu\eta_{\mu\alpha}\gamma^\alpha = \gamma^\mu\eta_{\mu\mu}\gamma^\mu = \underbrace{(\eta_{\mu\mu}\eta^{\mu\mu})}_{=1^2+(-1)^2+(-1)^2+(-1)^2}\text{id} = D\,\text{id} \tag{2272}$$

  where we used that the metric has only diagonal entries with ± 1.
- Contracting anticommutator with γ_μ

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\text{id} \tag{2273}$$
$$\to \gamma^\mu\gamma^\nu\gamma_\mu + \gamma^\nu\gamma^\mu\gamma_\mu = 2\eta^{\mu\nu}\gamma_\mu \tag{2274}$$
$$\to \gamma^\mu\gamma^\nu\gamma_\mu + D\,\gamma^\nu = 2\gamma^\nu \tag{2275}$$
$$\to \gamma^\mu\gamma^\nu\gamma_\mu = -(D-2)\gamma^\nu \tag{2276}$$

- Contracting anticommutator with $\gamma^\rho\gamma_\mu$

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\text{id} \tag{2277}$$
$$\to \gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu + \gamma^\nu\underbrace{\gamma^\mu\gamma^\rho\gamma_\mu}_{=-(D-2)\gamma^\rho} = 2\eta^{\mu\nu}\gamma^\rho\gamma_\mu \tag{2278}$$
$$\to \gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu - (D-2)\gamma^\nu\gamma^\rho = 2\gamma^\rho\gamma^\nu \tag{2279}$$
$$\to \gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = (D-2)\gamma^\nu\gamma^\rho + 2\underbrace{\gamma^\rho\gamma^\nu}_{=2\eta^{\rho\nu}\,\text{id}-\gamma^\nu\gamma^\mu} \tag{2280}$$
$$\to \gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = (D-4)\gamma^\nu\gamma^\rho + 4\eta^{\rho\nu}\,\text{id} \tag{2281}$$

- Contracting anticommutator with $\gamma^\rho\gamma^\sigma\gamma_\mu$

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathrm{id} \tag{2282}$$
$$\rightarrow \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu + \gamma^\nu \underbrace{\gamma^\mu\gamma^\rho\gamma^\sigma\gamma_\mu}_{=(D-4)\gamma^\rho\gamma^\sigma+4\eta^{\rho\sigma}\,\mathrm{id}} = 2\eta^{\mu\nu}\gamma^\rho\gamma^\sigma\gamma_\mu \tag{2283}$$
$$\rightarrow \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu + \gamma^\nu\left[(D-4)\gamma^\rho\gamma^\sigma + 4\eta^{\rho\sigma}\,\mathrm{id}\right] = 2\gamma^\rho\gamma^\sigma\gamma^\nu \tag{2284}$$
$$\rightarrow \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -(D-4)\gamma^\nu\gamma^\rho\gamma^\sigma - 2\cdot \underbrace{2\eta^{\rho\sigma}\,\mathrm{id}}_{=\gamma^\rho\gamma^\sigma+\gamma^\sigma\gamma^\rho} \gamma^\nu + 2\gamma^\rho\gamma^\sigma\gamma^\nu \tag{2285}$$
$$\rightarrow \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -(D-4)\gamma^\nu\gamma^\rho\gamma^\sigma - 2(\gamma^\rho\gamma^\sigma + \gamma^\sigma\gamma^\rho)\gamma^\nu + 2\gamma^\rho\gamma^\sigma\gamma^\nu \tag{2286}$$
$$\rightarrow \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -(D-4)\gamma^\nu\gamma^\rho\gamma^\sigma - 2\gamma^\sigma\gamma^\rho\gamma^\nu \tag{2287}$$

b) Assumption n odd and D even. In D spacetime dimensions there are D gamma matrices

- Showing $\mathrm{tr}(\gamma^\alpha) = 0$
  Starting again with the anticommutator

$$\gamma^\mu\gamma^\mu + \gamma^\mu\gamma^\mu = 2\eta^{\mu\mu}\mathrm{id} \tag{2288}$$
$$\rightarrow (\gamma^\mu)^2 = \eta^{\mu\mu}\mathrm{id} \tag{2289}$$
$$\rightarrow \mathrm{id} = \frac{1}{\eta^{\mu\mu}}(\gamma^\mu)^2 \tag{2290}$$

  Now select $\alpha \neq \mu$

$$\mathrm{tr}(\gamma^\alpha) = \frac{1}{\eta^{\mu\mu}}\mathrm{tr}(\gamma^\alpha\gamma^\mu\gamma^\mu) \tag{2291}$$
$$= \frac{1}{\eta^{\mu\mu}}\mathrm{tr}([2\underbrace{\eta^{\alpha\mu}}_{=0}\,\mathrm{id} - \gamma^\mu\gamma^\alpha]\gamma^\mu) \tag{2292}$$
$$= -\frac{1}{\eta^{\mu\mu}}\mathrm{tr}(\gamma^\mu\gamma^\alpha\gamma^\mu) \tag{2293}$$
$$= -\frac{1}{\eta^{\mu\mu}}\mathrm{tr}(\gamma^\mu\gamma^\mu\gamma^\alpha) \tag{2294}$$
$$= -\mathrm{tr}(\gamma^\alpha) \tag{2295}$$

  Resulting in $\mathrm{tr}(\gamma^\alpha) = 0$
- Showing $\mathrm{tr}(\gamma^\alpha\gamma^\beta\gamma^\delta) = 0$
  (a) Two γ's are identical $\mathrm{tr}(\gamma^\alpha\gamma^\beta\gamma^\beta) = \eta^{\beta\beta}\mathrm{tr}(\gamma^\alpha) = 0$
  (b) All three are different $\mathrm{tr}(\gamma^\alpha\gamma^\beta\gamma^\delta) =$
- Showing $\mathrm{tr}(\gamma^{\mu_1}...\gamma^{\mu_n}) = 0$

c) • Taking the trace of the anticommutator gives

$$\mathrm{tr}(\gamma^\mu\gamma^\nu) + \underbrace{\mathrm{tr}(\gamma^\nu\gamma^\mu)}_{\mathrm{tr}(\gamma^\mu\gamma^\nu)} = 2\eta^{\mu\nu}\mathrm{tr}(\mathrm{id}) \tag{2296}$$
$$\rightarrow \mathrm{tr}(\gamma^\mu\gamma^\nu) = \eta^{\mu\nu}\mathrm{tr}(\mathrm{id}) \tag{2297}$$

- replacing $\gamma^\rho\gamma^\sigma$ and utilising the previous result

$$\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = \mathrm{tr}(\gamma^\mu\gamma^\nu[2\eta^{\sigma\rho}\,\mathrm{id} - \gamma^\sigma\gamma^\rho]) \tag{2298}$$
$$= 2\eta^{\sigma\rho}\mathrm{tr}(\gamma^\mu\gamma^\nu) - \mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho) \tag{2299}$$
$$= 2\eta^{\sigma\rho}\eta^{\mu\nu}\mathrm{tr}(\mathrm{id}) - \mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho) \tag{2300}$$

analog - replacing $\gamma^\nu\gamma^\sigma$

$$\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho) = 2\eta^{\nu\sigma}\eta^{\mu\rho}\mathrm{tr}(\mathrm{id}) - \mathrm{tr}(\gamma^\mu\gamma^\sigma\gamma^\nu\gamma^\rho) \tag{2301}$$

analog - replacing $\gamma^\mu\gamma^\sigma$

$$\mathrm{tr}(\gamma^\mu\gamma^\sigma\gamma^\nu\gamma^\rho) = 2\eta^{\mu\sigma}\eta^{\nu\rho}\mathrm{tr}(\mathrm{id}) - \mathrm{tr}(\gamma^\sigma\gamma^\mu\gamma^\nu\gamma^\rho) \tag{2302}$$

Substituting all recursively

$$\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 2\eta^{\sigma\rho}\eta^{\mu\nu}\mathrm{tr}(\mathrm{id}) - 2\eta^{\nu\sigma}\eta^{\mu\rho}\mathrm{tr}(\mathrm{id}) + \mathrm{tr}(\gamma^\mu\gamma^\sigma\gamma^\nu\gamma^\rho) \tag{2303}$$

$$= 2\eta^{\sigma\rho}\eta^{\mu\nu}\mathrm{tr}(\mathrm{id}) - 2\eta^{\nu\sigma}\eta^{\mu\rho}\mathrm{tr}(\mathrm{id}) + 2\eta^{\mu\sigma}\eta^{\nu\rho}\mathrm{tr}(\mathrm{id}) - \mathrm{tr}(\gamma^\sigma\gamma^\mu\gamma^\nu\gamma^\rho) \tag{2304}$$

$$\to 2\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 2\eta^{\sigma\rho}\eta^{\mu\nu}\mathrm{tr}(\mathrm{id}) - 2\eta^{\nu\sigma}\eta^{\mu\rho}\mathrm{tr}(\mathrm{id}) + \mathrm{tr}(\gamma^\mu\gamma^\sigma\gamma^\nu\gamma^\rho) \tag{2305}$$

$$= 2\eta^{\sigma\rho}\eta^{\mu\nu}\mathrm{tr}(\mathrm{id}) - 2\eta^{\nu\sigma}\eta^{\mu\rho}\mathrm{tr}(\mathrm{id}) + 2\eta^{\mu\sigma}\eta^{\nu\rho}\mathrm{tr}(\mathrm{id}) \tag{2306}$$

$$\to \mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = \mathrm{tr}(\mathrm{id})\left(\eta^{\sigma\rho}\eta^{\mu\nu} - \eta^{\nu\sigma}\eta^{\mu\rho} + \eta^{\mu\sigma}\eta^{\nu\rho}\right) \tag{2307}$$