0.1 Feynman, Hibbs - Quantum mechanics and path integrals 2ed

0.1.1 2.1

With $\dot{x} = 0$ and $\dot{x}$ =const we see

$$
\begin{aligned}
S &= \int_{t_a}^{t_b} L dt && (1)\\
&= \frac{m}{2} \int_{t_a}^{t_b} \dot{x}^2 dt && (2)\\
&= \frac{m}{2} \left[\dot{x}x|_{t_a}^{t_b} - \int_{t_a}^{t_b} x\ddot{x} dt \right] && (3)\\
&= \frac{m}{2} \frac{x_b - x_a}{t_b - t_b} (x_b - x_a) && (4)\\
&= \frac{m}{2} \frac{(x_b - x_a)^2}{t_b - t_b} && (5)
\end{aligned}
$$

0.1.2 2.2

With the solution of the equation of motion

$$
\begin{aligned}
\ddot{x} + \omega^2 x = 0 \quad &\rightarrow \quad x = x_0 \sin(\omega t + \varphi_0) = (x_0 \cos\varphi_0) \sin\omega t + (x_0 \sin\varphi_0) \cos\omega t && (6)\\
&\rightarrow \quad \dot{x} = (x_0 \omega \cos\varphi_0) \cos\omega t - (x_0 \omega \sin\varphi_0) \sin\omega t && (7)
\end{aligned}
$$

then with (x_a, x_b, t_a, t_b) we can solve for x_0 and φ_0

$$
\begin{aligned}
x_0 \cos\varphi_0 &= \frac{x_a \cos\omega t_b - x_b \cos\omega t_a}{\cos\omega t_b \sin\omega t_a - \cos\omega t_a \sin\omega t_b} && (8)\\
&= \frac{x_a \cos\omega t_b - x_b \cos\omega t_a}{\sin\omega(t_a - t_b)} && (9)\\
x_0 \sin\varphi_0 &= -\frac{x_a \frac{\sin\omega t_b}{\sin\omega t_a} - x_b \tan\omega t_a}{-\sin\omega t_b + \cos\omega t_b \tan\omega t_a} && (10)\\
&= \frac{x_b \sin\omega t_a - x_a \sin\omega t_b}{\sin\omega(t_a - t_b)} && (11)
\end{aligned}
$$

and therefore

$$
\begin{aligned}
v_a &= \frac{x_a \cos\omega t_b - x_b \cos\omega t_a}{\sin\omega(t_a - t_b)} \sin\omega t_a + \frac{x_b \sin\omega t_a - x_a \sin\omega t_b}{\sin\omega(t_a - t_b)} \sin\omega t_a && (12)\\
&= -\frac{1}{\sin\omega T} \left[(x_a \cos\omega t_b - x_b \cos\omega t_a) \sin\omega t_a + (x_b \sin\omega t_a - x_a \sin\omega t_b) \sin\omega t_a \right] && (13)\\
&= -\frac{1}{\sin\omega T} \left[x_a (\cos\omega t_b \sin\omega t_a - \sin\omega t_a \sin\omega t_b) + x_b (\sin^2\omega t_a - \cos\omega t_a \sin\omega t_a) \right] && (14)\\
v_b &= \frac{x_a \cos\omega t_b - x_b \cos\omega t_a}{\sin\omega(t_a - t_b)} \sin\omega t_b + \frac{x_b \sin\omega t_a - x_a \sin\omega t_b}{\sin\omega(t_a - t_b)} \sin\omega t_b && (15)\\
&= -\frac{1}{\sin\omega T} \left[x_a (\cos\omega t_b \sin\omega t_b - \sin^2\omega t_b) + x_b (\sin\omega t_a \sin\omega t_b - \cos\omega t_a \sin\omega t_b) \right] && (16)
\end{aligned}
$$

Now we can write

$$S = \int_{t_a}^{t_b} L dt \tag{17}$$
$$= \frac{m}{2} \int_{t_a}^{t_b} (\dot{x}^2 - \omega^2 x^2) dt \tag{18}$$
$$= \frac{m}{2} x_0^2 \omega^2 \int_{t_a}^{t_b} dt \left(\cos^2(\omega t + \varphi) - \sin^2(\omega t + \varphi)\right) \tag{19}$$
$$= \frac{m}{2} x_0^2 \omega^2 \int_{t_a}^{t_b} dt \cos(2[\omega t + \varphi]) \tag{20}$$
$$= \frac{m}{4} x_0^2 \omega \sin(2[\omega t + \varphi])|_{t_a}^{t_b} \tag{21}$$
$$= \frac{m}{2} x_0^2 \omega \sin(\omega t + \varphi) \cos(\omega t + \varphi)|_{t_a}^{t_b} \tag{22}$$
$$= \frac{m}{2} x \dot{x}|_{t_a}^{t_b} \tag{23}$$
$$= \frac{m}{2} (x_b v_b - x_a v_a) \tag{24}$$
$$= \frac{m\omega}{2 \sin \omega T} \left[(x_a^2 + x_b^2) \cos \omega T - 2 x_a x_b\right] \tag{25}$$

0.1.3 2.3

$$m\ddot{x} + f = 0 \qquad \to \qquad x(t) = -\frac{f}{2m} t^2 + v_a t + x_a \tag{26}$$

then

$$S = \int_{t_a}^{t_b} \frac{m}{2} \left(-\frac{f}{m} t\right)^2 - \frac{f^2}{2m} t^2 - f v_a t + f x_a \, dt \tag{27}$$
$$= \int_{t_a}^{t_b} -\frac{f^2}{m} t^2 - f v_a t + f x_a \, dt \tag{28}$$
$$= -\frac{f^2}{3m} (t_b^3 - t_a^3) - \frac{f v_a}{2} (t_b^2 - t_a^2) + f x_a (t_b - t_a) \tag{29}$$
$$= -\frac{f^2}{3m} (t_b^3 - t_a^3) - v_a m (x_b - v_a t_b - x_a - x_a + v_a t_a + x_a) + f x_a (t_b - t_a) \tag{30}$$
$$= -\frac{f^2}{3m} (t_b^3 - t_a^3) - v_a m (x_b - x_a) + v_a^2 m (t_b - t_a) + f x_a (t_b - t_a) \tag{31}$$
$$\tag{32}$$

0.2 Straumann - Quantenmechanik 2ed

0.2.1 2.1 - Spectral oscillator density

The vanishing electrical field in the surface requires for each standing wave

$$k_i = \frac{\pi}{L} n_i. \tag{33}$$

and

$$k^2 = k_x^2 + k_y^2 + k_z^2 \tag{34}$$
$$\Delta V = \frac{\pi^3}{L^3}. \tag{35}$$

With $k = 2\pi/\lambda = \omega/c$ we have $dk = \frac{d\omega}{c}$ and the volume of a sphere in k-space is given by

$$V(k) = \frac{4}{3}\pi k^3 \tag{36}$$

$$dV = 4\pi k^2 dk = 4\pi \frac{\omega^2}{c^2}\frac{d\omega}{c} = 4\pi(2\pi)^3\frac{\nu^2}{c^3}d\nu \tag{37}$$

The number of oscillator are then given by the number of points in the positive quadrant (all k_i positive) time two (polarization)

$$dN(\nu) = 2\frac{V(\nu)/8}{\Delta V} = L^3\frac{8\pi}{c^3}\nu^2 d\nu \tag{38}$$

0.2.2 2.2 - Energy variance of the harmonic oscillator

First we obtain an expression for T

$$E = \frac{h\nu}{e^{h\nu/kT} - 1} \quad \rightarrow \quad \frac{h\nu}{kT} = \ln\left(\frac{h\nu}{E} + 1\right) \tag{39}$$

which we can use in

$$\frac{dS}{dE} = \frac{1}{T} = \frac{k}{h\nu}\ln\left(\frac{h\nu}{E} + 1\right) \tag{40}$$

and take one more derivative

$$\frac{d^2S}{dE^2} = -\frac{k}{h\nu}\frac{\frac{h\nu}{E^2}}{\frac{h\nu}{E} + 1} \tag{41}$$

$$= -k\frac{1}{h\nu E + E^2}. \tag{42}$$

Now we see

$$\langle(\Delta E)^2\rangle = E^2 + Eh\nu. \tag{43}$$

0.2.3 3.6 - 1D molecular potential

With the given coordinate transformation we get for the single terms

$$e^{-\alpha x} = \frac{\alpha\hbar\xi}{2\sqrt{2mA}} \tag{44}$$

$$e^{-2\alpha x} = \frac{(\alpha\hbar\xi)^2}{8mA} \tag{45}$$

$$\frac{\partial}{\partial x} = \frac{\partial\xi}{\partial x}\frac{\partial}{\partial\xi} \tag{46}$$

$$= -\alpha\xi\frac{\partial}{\partial\xi} \tag{47}$$

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2\xi}{\partial x^2}\frac{\partial}{\partial\xi} + \left(\frac{\partial\xi}{\partial x}\right)^2\frac{\partial^2}{\partial\xi^2} \tag{48}$$

$$= \alpha^2\xi\frac{\partial}{\partial\xi} + (\alpha\xi)^2\frac{\partial^2}{\partial\xi^2} \tag{49}$$

and combined

$$-\frac{\hbar^2}{2m}\partial_{xx}\psi + A(e^{-2\alpha x} - 2e^{-\alpha x})\psi = E\psi \tag{50}$$

$$-\frac{\hbar^2}{2m}\left(\alpha^2\xi\frac{\partial}{\partial\xi} + (\alpha\xi)^2\frac{\partial^2}{\partial\xi^2}\right)\psi + A\left(\frac{(\alpha\hbar\xi)^2}{8mA} - 2\frac{\alpha\hbar\xi}{2\sqrt{2mA}}\right)\psi = E\psi \tag{51}$$

$$\left(\alpha^2\xi\frac{\partial}{\partial\xi} + (\alpha\xi)^2\frac{\partial^2}{\partial\xi^2}\right)\psi - \frac{2mA}{\hbar^2}\left(\frac{(\alpha\hbar\xi)^2}{8mA} - 2\frac{\alpha\hbar\xi}{2\sqrt{2mA}}\right)\psi = -\frac{2mE}{\hbar^2}\psi \tag{52}$$

$$\left(\frac{1}{\xi}\frac{\partial}{\partial\xi} + \frac{\partial^2}{\partial\xi^2}\right)\psi - \frac{2mA}{\alpha^2\xi^2\hbar^2}\left(\frac{(\alpha\hbar\xi)^2}{8mA} - 2\frac{\alpha\hbar\xi}{2\sqrt{2mA}}\right)\psi = -\frac{2mE}{\hbar^2\alpha^2\xi^2}\psi \tag{53}$$

$$\left(\frac{1}{\xi}\frac{\partial}{\partial\xi} + \frac{\partial^2}{\partial\xi^2}\right)\psi + \left(-\frac{1}{4} + \frac{\sqrt{2mA}}{\alpha\hbar\xi}\right)\psi = -\frac{2mE}{\hbar^2\alpha^2\xi^2}\psi \tag{54}$$

$$\left(\frac{1}{\xi}\frac{\partial}{\partial\xi} + \frac{\partial^2}{\partial\xi^2}\right)\psi + \left(-\frac{1}{4} + \frac{n+s+\frac{1}{2}}{\xi}\right)\psi = \frac{s^2}{\xi^2}\psi \tag{55}$$

$$\left(\frac{\partial^2}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial}{\partial\xi}\right)\psi + \left(-\frac{1}{4} + \frac{n+s+\frac{1}{2}}{\xi} - \frac{s^2}{\xi^2}\right)\psi = 0. \tag{56}$$

The units of ξ is $\sqrt{\mathrm{kg}\cdot\mathrm{J}}/\mathrm{m}^{-1}\mathrm{Js} = 1$ so ξ in dimensionless.

1. Case $\xi \gg 1$ ($x \to -\infty$) Dropping all $1/\xi$ terms

$$\psi'' - \frac{1}{4}\psi = 0 \quad \to \quad \psi = c_1 e^{\xi/2} + c_2 e^{-\xi/2} \tag{57}$$

2. Case $0 < \xi \ll 1$ ($x \to +\infty$) Ansatz $\psi \sim \xi^m$

$$m(m-1)\xi^{m-2} + m\xi^{m-2} - \frac{1}{4}\xi^m + \left(n+s+\frac{1}{2}\right)\xi^{m-1} - s^2\xi^{m-2} = 0 \tag{58}$$

$$\left[(m^2 - s^2) - \frac{1}{4}\xi^2 + \left(n+s+\frac{1}{2}\right)\xi\right]\xi^{m-2} = 0 \tag{59}$$

which for small ξ becomes

$$(m^2 - s^2)\xi^{m-2} = 0 \quad \to \quad \psi = \xi^{\pm s} \tag{60}$$

With the two asymptotics we can make a physically sensible ansatz for a full solutions $\psi = \xi^s e^{-\xi/2}u(\xi)$ which leads to

$$\xi u'' + (2s+1-\xi)u' + nu = 0 \tag{61}$$

To solve this equation we use the Sommerfeld polynomial method

$$u = \sum_k a_k\xi^k \quad \to \quad \sum_k k(k-1)a_k\xi^{k-1} + (2s+1)ka_k\xi^{k-1} - ka_k\xi^k + na_k\xi^k = 0 \tag{62}$$

$$\sum_k (k+1)ka_{k+1}\xi^k + (2s+1)(k+1)a_{k+1}\xi^k - ka_k\xi^k + na_k\xi^k = 0 \tag{63}$$

$$a_{k+1} = \frac{k-n}{(k+1)(2s+1+k)}a_k. \tag{64}$$

The requirement for the series to cut off (making u a finite order polynomial) is $n_k = k$. The energies of the bound states are therefore

$$E_k = -\frac{\alpha^2\hbar^2}{2m}s_k^2 \tag{65}$$
$$= -\frac{\alpha^2\hbar^2}{2m}\left[\frac{\sqrt{2mA}}{\alpha\hbar} - (k+1/2)\right]^2 \tag{66}$$
$$= -A\left[1 - \frac{\alpha\hbar}{\sqrt{2mA}}(k+1/2)\right]^2 \tag{67}$$

where the only valid k are the ones where E_k is in $[-A, 0]$.

0.3 Straumann - Relativistische Quantenmechanik

0.4 Schwabl - Advanced Quantum Mechanics

0.4.1 Exercise 3.1 Heisenberg model and Holstein-Primakoff transformation - NOT DONE YET

a)

$$\varphi \simeq 1 - \frac{1}{2}\left(\frac{\hat{a}_i^\dagger \hat{a}_i}{2S}\right) - \frac{1}{8}\left(\frac{\hat{a}_i^\dagger \hat{a}_i}{2S}\right)^2 - \frac{1}{16}\left(\frac{\hat{a}_i^\dagger \hat{a}_i}{2S}\right)^3 - \frac{5}{128}\left(\frac{\hat{a}_i^\dagger \hat{a}_i}{2S}\right)^4 \ldots \tag{68}$$

$$[\hat{n}_i, \hat{a}_i] = [\hat{a}_i^\dagger \hat{a}_i, \hat{a}_i] = -\hat{a}_i \tag{69}$$
$$[\hat{n}_i, \hat{a}_i^\dagger] = [\hat{a}_i^\dagger \hat{a}_i, \hat{a}_i^\dagger] = \hat{a}_i^\dagger \tag{70}$$

$$S_i^x = \sqrt{2S}(\varphi a_i + a_i^\dagger \varphi) \tag{71}$$
$$S_i^y = \sqrt{2S}(\varphi a_i - a_i^\dagger \varphi) \tag{72}$$
$$S_i^z = S - \hat{a}_i^\dagger \hat{a}_i \tag{73}$$

1^{th}-order

$$
\begin{aligned}
[S_i^x, S_i^y] &= 2S\left[a_i - \frac{1}{2\cdot 2S}n_i a_i + a_i^\dagger - \frac{1}{2\cdot 2S}a_i^\dagger n_i, a_i - \frac{1}{2\cdot 2S}n_i a_i - a_i^\dagger + \frac{1}{2\cdot 2S}a_i^\dagger n_i\right] && (74)\\
&= 2S\left[(a_i + a_i^\dagger) - \frac{1}{2\cdot 2S}(n_i a_i + a_i^\dagger n_i), (a_i - a_i^\dagger) - \frac{1}{2\cdot 2S}(n_i a_i - a_i^\dagger n_i)\right] && (75)\\
&= 2S([a_i^\dagger, a_i] - [a_i, a_i^\dagger]) - \frac{2S}{4S}([(a_i + a_i^\dagger), (n_i a_i + a_i^\dagger n_i)] + [(a_i - a_i^\dagger), (n_i a_i - a_i^\dagger n_i)]) && (76)\\
&= 2S([a_i^\dagger, a_i] - [a_i, a_i^\dagger]) - \frac{2S}{4S}([(a_i + a_i^\dagger), (a_i + a_i^\dagger)n_i - a_i] + [(a_i - a_i^\dagger), (a_i - a_i^\dagger)n_i - a_i]) && (77)\\
&= -4S - \frac{2S}{4S}([(a_i + a_i^\dagger), (a_i + a_i^\dagger)n_i] - [(a_i + a_i^\dagger), a_i] + [(a_i - a_i^\dagger), (a_i - a_i^\dagger)n_i] - [(a_i - a_i^\dagger), a_i]) && (78)\\
&= -4S - \frac{2S}{4S}([(a_i + a_i^\dagger), (a_i + a_i^\dagger)n_i] + [(a_i - a_i^\dagger), (a_i - a_i^\dagger)n_i]) && (79)\\
&= -4S - \frac{2S}{4S}(-(a_i + a_i^\dagger)[(a_i + a_i^\dagger), n_i] - (a_i - a_i^\dagger)[(a_i - a_i^\dagger), n_i]) && (80)\\
&= -4S - \frac{2S}{4S}((a_i + a_i^\dagger)[n_i, (a_i + a_i^\dagger)] + (a_i - a_i^\dagger)[n_i, (a_i - a_i^\dagger)]) && (81)\\
&= -4S - \frac{2S}{4S}((a_i + a_i^\dagger)(-a_i + a_i^\dagger) + (a_i - a_i^\dagger)(-a_i - a_i^\dagger)) && (82)\\
&= -4S - \frac{2S}{4S}((a_i + a_i^\dagger)(-a_i + a_i^\dagger) - (a_i - a_i^\dagger)(a_i + a_i^\dagger)) && (83)\\
&= -4S + \frac{4S}{4S}(a_i + a_i^\dagger)(a_i - a_i^\dagger) && (84)\\
&= -4S + \frac{4S}{4S}(a_i a_i + a_i^\dagger a_i - a_i^\dagger a_i - a_i^\dagger a_i^\dagger) && (85)\\
&= -4S + \frac{4S}{4S}(a_i a_i - a_i^\dagger a_i^\dagger - 1) && (86)\\
& && (87)
\end{aligned}
$$

b)

c)

0.5 Schwinger - Quantum Mechanics Symbolism of Atomic Measurements

0.5.1 2.1

Observe

$$
\begin{aligned}
\int_{-\infty}^{\infty}(\theta(x+a) + \theta(a-x))\, e^{ikx}dx &= \int_{-a}^{a} e^{ikx}dx && (88)\\
&= \frac{1}{ik}\left(e^{ika} - e^{-ika}\right) && (89)\\
&= 2a\frac{\sin ka}{ka} && (90)
\end{aligned}
$$

$$
\lim_{P\to\infty}\int_{-\infty}^{\infty}\frac{d\chi}{\pi}\frac{\sin\chi}{\chi}e^{ik\left(q'+\frac{\chi}{P}\right)} = \frac{1}{\pi}e^{ikq'}\lim_{P\to\infty}\int_{-\infty}^{\infty}d\chi\frac{\sin\chi}{\chi}e^{i\frac{k}{P}\chi} \quad (91)
$$

0.6 Weinberg - Quantum Mechanics 2nd edition

0.6.1 1.1

- The solution of for a free particle in the interval $-a < x < a$ is given by

$$\left[-\frac{\hbar^2}{2M}\frac{d^2}{dx^2} - E\right]\phi = 0 \tag{92}$$

$$\left[\frac{d^2}{dx^2} + \frac{2ME}{\hbar^2}\right]\phi = 0 \tag{93}$$

$$\to \phi = A\sin\left(\frac{\sqrt{2ME}}{\hbar}x\right) + B\cos\left(\frac{\sqrt{2ME}}{\hbar}x\right) \tag{94}$$

with the two boundary conditions

$$A\sin\left(\frac{\sqrt{2ME}}{\hbar}(-a)\right) + B\cos\left(\frac{\sqrt{2ME}}{\hbar}(-a)\right) = 0 \tag{95}$$

$$A\sin\left(\frac{\sqrt{2ME}}{\hbar}a\right) + B\cos\left(\frac{\sqrt{2ME}}{\hbar}a\right) = 0. \tag{96}$$

The possible energy eigenvalues are therefore

$$A = 0, \quad \frac{\sqrt{2ME_{2n+1}}}{\hbar}a = (2n+1)\frac{\pi}{2} \quad \to \quad E_{2n+1} = \frac{\pi^2\hbar^2}{8Ma^2}(2n+1)^2 \tag{97}$$

$$\to \quad \phi = \frac{1}{\sqrt{a}}\cos\left(x\frac{\pi}{2a}(2n+1)\right) \tag{98}$$

$$B = 0, \quad \frac{\sqrt{2ME_{2n}}}{\hbar}a = 2n\frac{\pi}{2} \quad \to \quad E_{2n} = \frac{\pi^2\hbar^2}{8Ma^2}(2n)^2 \tag{99}$$

$$\to \quad \phi = \frac{1}{\sqrt{a}}\sin\left(x\frac{\pi}{2a}(2n)\right) \tag{100}$$

where we calculated the normalization via

$$\int_{-a}^{a}\sin^2(kx)dx = \int_{-a}^{a}(1-\cos^2(kx))dx \tag{101}$$

$$= 2a - \int_{-a}^{a}\cos^2(kx)dx \quad \to \int_{-a}^{a}\sin^2(kx)dx = a. \tag{102}$$

- Lets first calculate the normalization

$$\int_{-a}^{a}(a^2-x^2)^2dx = a^4x - 2a^2\frac{x^3}{3} + \frac{x^5}{5}\bigg|_{-a}^{a} \tag{103}$$

$$= a^4(2a) - \frac{2}{3}a^2(16a^3) + \frac{1}{5}(64a^5) \tag{104}$$

$$= \left(2 - \frac{4}{3} + \frac{2}{5}\right)a^5 = \frac{16}{15}a^5 \tag{105}$$

and then obtain

$$\int_{-a}^{a}\frac{1}{\sqrt{\frac{16a^5}{15}}}\left(a^2-x^2\right)\frac{1}{\sqrt{a}}\cos\left(\frac{\pi x}{2a}\right)dx = \frac{8\sqrt{15}}{\pi^3} \tag{106}$$

0.6.2 1.2

- We can write the Hamiltonian as

$$H = \frac{\vec{P}^2}{2M} + \frac{M\omega_0^2}{2}\vec{X}^2 \tag{107}$$

$$= \sum_{k=1}^{3} \frac{p_k^2}{2M} + \frac{M\omega_0^2}{2}x_k^2 \tag{108}$$

the energy is therefore given by

$$E_{n_1,n_2,n_3} = \hbar\omega_0 \left(n_1 + n_2 + n_3 + \frac{3}{2}\right) \tag{109}$$

$$N_{n=n_1+n_2+n_3} = \sum_{k=0}^{n}(k+1) \tag{110}$$

$$= \frac{n(n+1)}{2} + n + 1 \tag{111}$$

$$= \frac{(n+1)(n+2)}{2} \tag{112}$$

- With (1.4.5), (1.4.15) and $\omega_{01} = \omega_0$ we have

$$\vec{x}]_{01} = e^{i\omega_0 t}\sqrt{\frac{\hbar}{2M\omega_0}} \tag{113}$$

$$A_{n=1}^{n=0} = \frac{4e^2\omega_0^3}{3c^3\hbar}\,|[\vec{x}]_{01}|^2 \tag{114}$$

$$= \frac{2e^2\omega_0^2}{3c^3M} \tag{115}$$

where with (1.4.15).

0.7 Hannabuss - An Introduction to Quantum Theory

0.7.1 Problem 12.2 - Harmonic oscillator with x^4 perturbation

With

$$[a, a^\dagger] = aa^\dagger - a^\dagger a = 1 \tag{116}$$

$$[a, x] = \sqrt{\frac{\hbar}{2m\omega}}[a, a + a^\dagger] = \sqrt{\frac{\hbar}{2m\omega}}\left([a, a] + [a, a^\dagger]\right) = \sqrt{\frac{\hbar}{2m\omega}}[a, a^\dagger] = \sqrt{\frac{\hbar}{2m\omega}} \tag{117}$$

$$[a^n, x] = ... = \sqrt{\frac{\hbar}{2m\omega}}[a^n, a^\dagger] = \sqrt{\frac{\hbar}{2m\omega}}(a^n a^\dagger - a^\dagger a^n) = \sqrt{\frac{\hbar}{2m\omega}}(a^n a^\dagger - (a^\dagger a)a^{n-1}) \tag{118}$$

$$= \sqrt{\frac{\hbar}{2m\omega}}(a^n a^\dagger - (aa^\dagger - 1)a^{n-1}) \tag{119}$$

$$= \sqrt{\frac{\hbar}{2m\omega}}(a^n a^\dagger + a^{n-1} - aa^\dagger a^{n-1}) \tag{120}$$

$$= \sqrt{\frac{\hbar}{2m\omega}}(a^n a^\dagger + a^{n-1} - a(aa^\dagger - 1)a^{n-2}) \tag{121}$$

$$= ... = \sqrt{\frac{\hbar}{2m\omega}}na^{n-1} \tag{122}$$

the first order energy perturbation can be written as

$$\Delta E_n^{(1)} = \langle \psi_n^{(0)} | H_1 | \psi_n^{(0)} \rangle \tag{123}$$

$$= \frac{1}{n!} \langle 0 | a^n x^4 (a^\dagger)^n | 0 \rangle \tag{124}$$

$$= \frac{1}{n!} \langle 0 | \left(x a^n + \sqrt{\frac{\hbar}{2m\omega}} n a^{n-1} \right) x^3 (a^\dagger)^n | 0 \rangle \tag{125}$$

$$= \frac{1}{n!} \langle 0 | x a^n x^3 (a^\dagger)^n | 0 \rangle + \frac{n}{n!} \sqrt{\frac{\hbar}{2m\omega}} \langle 0 | a^{n-1} x^3 (a^\dagger)^n | 0 \rangle \tag{126}$$

$$= \frac{1}{n!} \langle 0 | x \left(x a^n + \sqrt{\frac{\hbar}{2m\omega}} n a^{n-1} \right) x^2 (a^\dagger)^n | 0 \rangle + \frac{n}{n!} \sqrt{\frac{\hbar}{2m\omega}} \langle 0 | \left(x a^{n-1} + \sqrt{\frac{\hbar}{2m\omega}} (n-1) a^{n-2} \right) x^2 (a^\dagger)^n | 0 \rangle \tag{127}$$

$$= \frac{1}{n!} \langle 0 | x^2 a^n x^2 (a^\dagger)^n | 0 \rangle + 2 \frac{n}{n!} \sqrt{\frac{\hbar}{2m\omega}} \langle 0 | x a^{n-1} x^2 (a^\dagger)^n | 0 \rangle + \frac{n(n-1)}{n!} \sqrt{\frac{\hbar}{2m\omega}}^2 \langle 0 | a^{n-2} x^2 (a^\dagger)^n | 0 \rangle \tag{128}$$

$$\tag{129}$$

$$= \frac{1}{n!} \langle 0 | x^3 a^n x (a^\dagger)^n | 0 \rangle + 3 \frac{n}{n!} \sqrt{\frac{\hbar}{2m\omega}} \langle 0 | x^2 a^{n-1} x (a^\dagger)^n | 0 \rangle + 3 \frac{n(n-1)}{n!} \sqrt{\frac{\hbar}{2m\omega}}^2 \langle 0 | x a^{n-2} x (a^\dagger)^n | 0 \rangle \tag{130}$$

$$+ \frac{n(n-1)(n-2)}{n!} \sqrt{\frac{\hbar}{2m\omega}}^3 \langle 0 | a^{n-3} x (a^\dagger)^n | 0 \rangle \tag{131}$$

$$= \frac{1}{n!} \langle 0 | x^4 a^n (a^\dagger)^n | 0 \rangle + 4 \frac{n}{n!} \sqrt{\frac{\hbar}{2m\omega}} \langle 0 | x^3 a^{n-1} (a^\dagger)^n | 0 \rangle + 6 \frac{n(n-1)}{n!} \sqrt{\frac{\hbar}{2m\omega}}^2 \langle 0 | x^2 a^{n-2} (a^\dagger)^n | 0 \rangle \tag{132}$$

$$+ 4 \frac{n(n-1)(n-2)}{n!} \sqrt{\frac{\hbar}{2m\omega}}^3 \langle 0 | x a^{n-3} (a^\dagger)^n | 0 \rangle + \frac{n(n-1)(n-2)(n-3)}{n!} \sqrt{\frac{\hbar}{2m\omega}}^4 \langle 0 | a^{n-4} (a^\dagger)^n | 0 \rangle \tag{133}$$

$$= \langle 0 | x^4 | 0 \rangle + \frac{4n}{\sqrt{1!}} \sqrt{\frac{\hbar}{2m\omega}} \langle 0 | x^3 | 1 \rangle + \frac{6n(n-1)}{\sqrt{2!}} \sqrt{\frac{\hbar}{2m\omega}}^2 \langle 0 | x^2 | 2 \rangle \tag{134}$$

$$+ \frac{4n(n-1)(n-2)}{\sqrt{3!}} \sqrt{\frac{\hbar}{2m\omega}}^3 \langle 0 | x | 3 \rangle + \frac{n(n-1)(n-2)(n-3)}{\sqrt{4!}} \sqrt{\frac{\hbar}{2m\omega}}^4 \langle 0 | 4 \rangle \tag{135}$$

where we used $\frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle = |n\rangle$ and $\frac{\sqrt{k!}}{\sqrt{n!}} a^{n-k}|n\rangle = |k\rangle$. Using additionally information about the unperturbed solution

$$H_0(y) = 1 \tag{136}$$

$$H_1(y) = 2y \tag{137}$$

$$H_2(y) = 4y^2 - 2 \tag{138}$$

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n \left(\sqrt{\frac{m\omega}{\hbar}} x \right) e^{-m\omega x^2/2\hbar} \tag{139}$$

we can rewrite

$$x^2|0\rangle \simeq \sqrt{\frac{\hbar}{m\omega}}^2 \left(\sqrt{\frac{m\omega}{\hbar}}^2 x^2\right) \frac{1}{\sqrt{2^0 0!}} H_0(\sqrt{\frac{m\omega}{\hbar}}x) \tag{140}$$

$$= \sqrt{\frac{\hbar}{m\omega}}^2 \left(\frac{1}{4} H_2(\sqrt{\frac{m\omega}{\hbar}}x) + \frac{1}{2} H_0(\sqrt{\frac{m\omega}{\hbar}}x)\right) \underbrace{\frac{1}{\sqrt{2^0 0!}} H_0(\sqrt{\frac{m\omega}{\hbar}}x)}_{=1} \tag{141}$$

$$= \sqrt{\frac{\hbar}{m\omega}}^2 \left[\frac{\sqrt{2^2 2!}}{4} \frac{1}{\sqrt{2^2 2!}} H_2(\sqrt{\frac{m\omega}{\hbar}}x) + \frac{1}{2} \frac{1}{\sqrt{2^0 0!}} H_0(\sqrt{\frac{m\omega}{\hbar}}x)\right] \tag{142}$$

$$= \sqrt{\frac{\hbar}{m\omega}}^2 \left[\frac{\sqrt{2}}{2}|2\rangle + \frac{1}{2}|0\rangle\right] \tag{143}$$

and

$$x|1\rangle \simeq \sqrt{\frac{\hbar}{m\omega}} \left(\sqrt{\frac{m\omega}{\hbar}}x\right) \frac{1}{\sqrt{2^1 1!}} H_1(\sqrt{\frac{m\omega}{\hbar}}x) \tag{144}$$

$$= \sqrt{\frac{\hbar}{m\omega}} \frac{1}{\sqrt{2^1 1!}} \left(\frac{1}{2} H_2(\sqrt{\frac{m\omega}{\hbar}}x) + H_0(\sqrt{\frac{m\omega}{\hbar}}x)\right) \tag{145}$$

$$= \sqrt{\frac{\hbar}{m\omega}} \left(\frac{1}{2\sqrt{2}} \sqrt{2^2 2!} \frac{1}{\sqrt{2^2 2!}} H_2(\sqrt{\frac{m\omega}{\hbar}}x) + \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2^0 0!}} H_0(\sqrt{\frac{m\omega}{\hbar}}x)\right) \tag{146}$$

$$= \sqrt{\frac{\hbar}{m\omega}} \left(|2\rangle + \frac{1}{\sqrt{2}}|0\rangle\right) \tag{147}$$

then with $\langle m|n\rangle = \delta_{mn}$

$$\langle 0|x^4|0\rangle = \langle 0|x^2 \cdot x^2|0\rangle = \sqrt{\frac{\hbar}{m\omega}}^4 \left(\frac{2}{4} + \frac{1}{4}\right) = \frac{3}{4}\frac{\hbar^2}{m^2\omega^2} \tag{148}$$

$$\langle 0|x^3|1\rangle = \langle 0|x^2 \cdot x|1\rangle = \sqrt{\frac{\hbar}{m\omega}}^3 \left(\frac{\sqrt{2}}{2} + \frac{1}{2\sqrt{2}}\right) = \frac{3}{2\sqrt{2}}\frac{\hbar}{m\omega}\sqrt{\frac{\hbar}{m\omega}} \tag{149}$$

$$\langle 0|x^2|2\rangle = \langle 0|x^2 \cdot 1|2\rangle = \sqrt{\frac{\hbar}{m\omega}}^2 \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}\frac{\hbar}{m\omega} \tag{150}$$

$$\langle 0|x|3\rangle = 0 \tag{151}$$

$$\langle 0|4\rangle = 0 \tag{152}$$

we obtain

$$\Delta E_n^{(1)} = \frac{3}{4}\frac{\hbar^2}{m^2\omega^2} + 4n\sqrt{\frac{\hbar}{2m\omega}}\frac{3}{2\sqrt{2}}\frac{\hbar}{m\omega}\sqrt{\frac{\hbar}{m\omega}} + \frac{6n(n-1)}{\sqrt{2!}}\sqrt{\frac{\hbar}{2m\omega}}^2 \frac{\sqrt{2}}{2}\frac{\hbar}{m\omega} + 0 + 0 \tag{153}$$

$$= \frac{3\hbar^2}{4m^2\omega^2}\left(1 + 2n + 2n^2\right) \tag{154}$$

0.7.2 Problem 12.3 - Harmonic oscillator with other perturbations

(i) Calculating the first order energy correction using $x = \sqrt{\hbar/2m\omega}(a + a^\dagger)$

$$\begin{aligned}
\Delta E_n^{(1)} &= \langle \psi_n^{(0)} | x | \psi_n^{(0)} \rangle & (155)\\
&= \sqrt{\hbar/2m\omega} \langle \psi_n^{(0)} | a + a^\dagger | \psi_n^{(0)} \rangle & (156)\\
&= \sqrt{\hbar/2m\omega} \langle n | a + a^\dagger | n \rangle & (157)\\
&= \sqrt{\hbar/2m\omega} \left(\sqrt{n} \langle n-1 | n \rangle + \sqrt{n+1} \langle n | n+1 \rangle \right) & (158)\\
&= 0 & (159)
\end{aligned}$$

Calculating the second order energy correction

$$\begin{aligned}
\Delta E_n^{(2)} &= \sum_{k \neq n} \frac{|\langle k^{(0)} | x | n^{(0)} \rangle|^2}{E_n^{(0)} - E_k^{(0)}} & (160)\\
&= \sqrt{\hbar/2m\omega} \sum_{k \neq n} \frac{|\langle k^{(0)} | a + a^\dagger | n^{(0)} \rangle|^2}{(n-k)\hbar\omega} & (161)\\
&= \sqrt{\hbar/2m\omega} \sum_{k \neq n} \frac{|\sqrt{k} \langle (k-1)^{(0)} | n^{(0)} \rangle + \sqrt{n+1} \langle k^{(0)} | (n+1)^{(0)} \rangle|^2}{(n-k)\hbar\omega} & (162)\\
&= \sqrt{\hbar/2m\omega} \sum_{k \neq n} \frac{|\sqrt{k} \delta_{k-1,n} + \sqrt{n+1} \delta_{k,n+1}|^2}{(n-k)\hbar\omega} & (163)\\
&= \sqrt{\hbar/2m\omega} \sum_{k \neq n} \frac{k \delta_{k-1,n} + 2\sqrt{k(n+1)} \delta_{k-1,n} \delta_{k,n+1} + (n+1) \delta_{k,n+1}}{(n-k)\hbar\omega} & (164)\\
&= \sqrt{\frac{\hbar}{2m\omega}} \left(\frac{n+1}{[n-(n+1)]\hbar\omega} + \frac{2\sqrt{(n+1)(n+1)}}{[n-(n+1)]\hbar\omega} + \frac{n+1}{[n-(n+1)]\hbar\omega} \right) & (165)\\
&= \sqrt{\frac{1}{2m\hbar\omega^3}} \left(-(n+1) - 2(n+1) - (n+1) \right) & (166)\\
&= -4(n+1) \sqrt{\frac{1}{2m\hbar\omega^3}} & (167)
\end{aligned}$$

(ii)

0.8 Schwabl - Quantum Mechanics 4th ed

0.8.1 Problem 17.1 - 3d Harmonic oscillator

(a) Represent the 3d oscillator by three 1d oscillators

$$
\begin{aligned}
H &= \frac{\mathbf{p}^2}{2m} + \frac{m\omega^2}{2}\mathbf{x}^2 && (168)\\
&= \frac{p_x^2 + p_y^2 + p_z^2}{2m} + \frac{m\omega^2}{2}(x^2 + y^2 + z^2) && (169)\\
&= \sum_k^3 \frac{p_k^2}{2m} + \frac{m\omega^2}{2}x_k^2 && (170)\\
&= \hbar\omega \sum_k^3 \left(a_k^\dagger a_k + \frac{1}{2}\right) && (171)\\
&= \hbar\omega \sum_k^3 \left(n_k + \frac{1}{2}\right) && (172)\\
\to E &= \hbar\omega \left(n_x + n_y + n_y + \frac{3}{2}\right) && (173)
\end{aligned}
$$

level	1	2	3	4	...	N
energy	3/2	5/2	7/2	9/2	...	$3/2 + N$
multi	1	3	6	10	...	$N(N+1)/2$

The eigenfunctions are then

$$\psi(\mathbf{x}) = \psi_{n_x}(x)\psi_{n_y}(y)\psi_{n_z}(z) \tag{174}$$

(b)

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} - \frac{2m[V(r) - E]}{\hbar^2}\right)R(r) = 0 \tag{175}$$

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} + \frac{2mE}{\hbar^2} - \frac{m^2\omega^2}{\hbar^2}r^2\right)R(r) = 0 \tag{176}$$

For the asymptotics $r \to 0$ we set $R(r) = u(r)/r$ and obtain

$$\left(\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2}\right)u(r) = 0 \tag{177}$$

assuming $E - V(r)$ is small compared to the $1/r^2$. This gives

$$
\begin{aligned}
u(r) &= Ar^{l+1} + Br^{-l} && (178)\\
\to u(r) &= Ar^{l+1} && (179)
\end{aligned}
$$

We therefore guess the solution as $R(r) \sim r^l e^{-\alpha r^2}(a_0 + a_1 r + a_2 r^2 + ...) = r^l e^{-\alpha r^2} f(r)$ and substitute into the ODE obtaining a system of algebraic equations for the a_i and E. For the lowed energy levels we obtain

$$l=0 \quad R(r) = e^{-\frac{m\omega}{2\hbar}r^2} \quad \to \quad E = \frac{3}{2}\hbar\omega \tag{180}$$

$$R(r) = e^{-\frac{m\omega}{2\hbar}r^2}\left(1 - \frac{2m\omega r^2}{3\hbar}\right) \quad \to \quad E = \frac{7}{2}\hbar\omega \tag{181}$$

$$l=1 \quad R(r) = e^{-\frac{m\omega}{2\hbar}r^2} r \quad \to \quad E = \frac{5}{2}\hbar\omega \tag{182}$$

$$l=2 \quad R(r) = e^{-\frac{m\omega}{2\hbar}r^2} r^2 \quad \to \quad E = \frac{7}{2}\hbar\omega \tag{183}$$

Making the calculation more robust we insert a full series expansion $f(r) = \sum_k a_k r^k$ into the radial equation

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} + \frac{2mE}{\hbar^2} - \frac{m^2\omega^2}{\hbar^2}r^2\right)R(r) = 0$$

$$rf'' + 2(1+l-2\alpha r^2)f' - r\left(-\frac{2mE}{\hbar^2} + \alpha(3+2l-2\alpha r^2) + \frac{m^2\omega^2}{\hbar^2}\right)f = 0$$

$$f'' + 2\frac{1+l-2\alpha r^2}{r}f' - \left(-\frac{2mE}{\hbar^2} + \alpha(3+2l-2\alpha r^2) + \frac{m^2\omega^2}{\hbar^2}\right)f = 0$$

$$\sum_k \left[k(k-1)a_k + 2(1+l-2\alpha r^2)ka_k - \left(-\frac{2mE}{\hbar^2} + \alpha(3+2l-2\alpha r^2) + \frac{m^2\omega^2}{\hbar^2}\right)a_k r^2\right] r^{k-2} = 0$$

$$\sum_k \left[k(k-1)a_k + 2(1+l)ka_k - 2\alpha(k-2)a_{k-2} - \frac{m(m\omega^2-2E)}{\hbar^2}a_{k-2} + \alpha(3+2l)a_{k-2} - 2\alpha^2 r^2 a_k r^2\right] r^{k-2} = 0$$

0.8.2 Problem 17.2 - Delta-shell potential

With

$$y = r/a \tag{184}$$

$$\frac{d}{dr} = \frac{\partial y}{\partial r}\frac{d}{dy} = \frac{1}{a}\frac{d}{dy} \tag{185}$$

$$\frac{d^2}{dr^2} = \frac{d}{dr}\left(\frac{1}{a}\frac{d}{dy}\right) = \frac{1}{a^2}\frac{d}{dy} \tag{186}$$

we can rewrite

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} - \frac{2m[V(r)-E]}{\hbar^2}\right)R(r) = 0 \tag{187}$$

$$\left(\frac{1}{a^2}\frac{d^2}{dy^2} + \frac{2}{ya}\frac{1}{a}\frac{d}{dy} - \frac{l(l+1)}{y^2a^2} - \frac{2m}{\hbar^2}\left[-\lambda\frac{\hbar^2}{2m}\delta(r-a)\right] + \frac{2mE}{\hbar^2}\right)R(r) = 0 \tag{188}$$

$$\left(\frac{1}{a^2}\frac{d^2}{dy^2} + \frac{2}{ya}\frac{1}{a}\frac{d}{dy} - \frac{l(l+1)}{y^2a^2} + \lambda\delta(r-a) + \frac{2mE}{\hbar^2}\right)R(r) = 0 \tag{189}$$

$$\left(\frac{d^2}{dy^2} + \frac{2}{y}\frac{d}{dy} - \frac{l(l+1)}{y^2} + ga\delta(r-a) + \frac{2ma^2E}{\hbar^2}\right)R(r) = 0 \tag{190}$$

and see

$$y \neq 1 \quad \left(\frac{d^2}{dy^2} + \frac{2}{y}\frac{d}{dy} - \frac{l(l+1)}{y^2} + ak^2\right)R(y) = 0 \tag{191}$$

$$k^2 = g + \frac{2maE}{\hbar^2} \tag{192}$$

Independent solutions

$$R(y) = Aj_l(y\sqrt{ka}) + By_l(y\sqrt{ka}) \tag{193}$$

Here the requirements for the wavefunction

- regular at the origin with $R(r) \sim r^l$
- continuous (not differentiable) at $r = a$ (or $y = 1$)
- jump of the first derivative of ga
- exponential decay outside to ensure normalizability

and here a quick overview of the two functions and a special linear combination

$$\begin{aligned}
&j_l(x) = (-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\sin x}{x} && y_l(x) = -(-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\cos x}{x} && h_0^{(1)}(x) = j_l(ix) + iy_l(ix) \\
&j_0(x) = \frac{\sin x}{x} && y_0(x) = -\frac{\cos x}{x} && h_0^{(1)}(x) = -\frac{e^{-x}}{x} \\
&j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} && y_1(x) = -\frac{\cos x}{x} - \frac{\sin x}{x} && h_1^{(1)}(x) = i(1+x)\frac{e^{-x}}{x^2} \\
&J_2(x) = ... && y_l(x) = ... && h_2^{(1)}(x) = (x^2+3x+3)\frac{e^{-x}}{x^3}
\end{aligned}$$

We see that j_l is suitable for the inside and $h_l^{(1)}$ for the outside.

$$R(\rho) = \begin{cases} Aj_l(\rho) & r < a \\ Ch_l^{(1)}(\rho) & r > a \end{cases} \tag{194}$$

0.9 Shankar - Modern Quantum Mechanics 3rd ed

0.9.1 13.3.1 Pion rest energy

Remebering Yukawa potential and fixing units in the exponential

$$V(r) \sim \frac{e^{-mr}}{r} = \frac{e^{-\frac{mcr}{\hbar}}}{r} \tag{195}$$

Range is given by

$$\frac{m_\pi c d_\pi}{\hbar} \sim 1 \tag{196}$$

$$\rightarrow m_\pi = \frac{\hbar}{cd_\pi} = 200\text{MeV} \tag{197}$$

0.9.2 13.3.2 de Broglie wavelength

With $E = \frac{p^2}{2m} = qU$ we have

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mqU}} = 0.86\text{Å} \tag{198}$$

0.9.3 13.3.3 Balmer and Lyman lines in sun spectrum

$$E_2 - E_1 = \frac{1}{2}mc^2\alpha^2\left(\frac{1}{1^2} - \frac{1}{2^2}\right) \tag{199}$$
$$= \frac{3}{8}mc^2\alpha^2 \tag{200}$$
$$= 10.2\text{eV} \tag{201}$$
$$\frac{E_2 - E_1}{kT_{6,000K}} = \frac{10.2}{20\frac{1}{40}} = 20.4 \qquad \to \frac{P(n=2)}{P(n=1)} = 5.5 \cdot 10^{-9} \tag{202}$$
$$\frac{E_2 - E_1}{kT_{100,000K}} = \frac{10.2}{333\frac{1}{40}} = 1.2 \qquad \to \frac{P(n=2)}{P(n=1)} = 1.2 \tag{203}$$

0.9.4 13.3.4 Energy levels of multi-electron atoms - NOT DONE YET

We always remember

$$E_n = \frac{1}{2}mc^2\frac{(\alpha Z)^2}{n^2} \tag{204}$$

Justification - Virial theorem $E_{kin} \sim E_{pot}$

$$E_n = \langle n|H|n\rangle \sim \langle n|V_C|n\rangle \sim \langle n|\frac{Ze^2}{r}|n\rangle \tag{205}$$

0.10 Zettili - Quantum Mechanics - Concepts and Applications 2nd ed

0.11 Banks - Quantum Mechanics

0.11.1 Exercise 13.1 - Cubic and Quartic perturbed harmonic oscillator

We split the Hamiltonian and see

$$H = H_0 + a\left(X^3 + \frac{b}{a}X^4\right) \tag{206}$$
$$E_n^{(0)} = \hbar\omega\left(n + \frac{1}{2}\right) \tag{207}$$
$$|n^{(0)}\rangle = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) \tag{208}$$

then

$$E_n = E_n^{(0)} + a\langle n^{(0)}|X^3 + \frac{b}{a}X^4|n^{(0)}\rangle + a^2\sum_{k\neq n}\frac{|\langle k^{(0)}|X^3 + \frac{b}{a}X^4|n^{(0)}\rangle|^2}{E_n^{(0)} - E_k^{(0)}} \tag{209}$$

we can use the identities for the Hermite polynomials

$$xH_n(x) = nH_{n-1}(x) + \frac{1}{2}H_{n+1}(x) \tag{210}$$

$$x^2H_n(x) = n(n-1)H_{n-2}(x) + \frac{2n+1}{2}H_n(x) + \frac{1}{4}H_{n+2}(x) \tag{211}$$

$$x^3H_n(x) = n(n-1)(n-2)H_{n-3}(x) + \left(\frac{n(n-1)}{2} + \frac{(2n+1)n}{2}\right)H_{n-1} + \left(\frac{2n+1}{4} + \frac{n+2}{4}\right)H_{n+1}(x) + \frac{1}{8}H_{n+3}(x) \tag{212}$$

$$= n(n-1)(n-2)H_{n-3}(x) + \frac{3n^2}{2}H_{n-1}(x) + 3\frac{n+1}{4}H_{n+1}(x) + \frac{1}{8}H_{n+3}(x) \tag{213}$$

$$x^4H_n(x) = n(n-1)(n-2)(n-3)H_{n-4}(x) + (2n^2-3n+1)nH_{n-2}(x) + \frac{3}{4}(2n^2+2n+1)H_n(x) \tag{214}$$

$$+\frac{1}{4}(2n+3)H_{n+2}(x) + \frac{1}{16}H_{n+4}(x) \tag{215}$$

and see

$$x^3|n^{(0)}\rangle = x^3\frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-\frac{m\omega x^2}{2\hbar}}H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) \tag{216}$$

$$= \left(\sqrt{\frac{\hbar}{m\omega}}x\right)^3\frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-\frac{m\omega x^2}{2\hbar}}\left(\sqrt{\frac{m\omega}{\hbar}}x\right)^3H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) \tag{217}$$

$$= \left(\sqrt{\frac{\hbar}{m\omega}}\right)^3\frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-\frac{m\omega x^2}{2\hbar}}\left[n(n-1)(n-2)H_{n-3}(\sqrt{\frac{m\omega}{\hbar}}x)\right. \tag{218}$$

$$\left.+\frac{3n^2}{2}H_{n-1}(\sqrt{\frac{m\omega}{\hbar}}x) + 3\frac{n+1}{4}H_{n+1}(\sqrt{\frac{m\omega}{\hbar}}x) + \frac{1}{8}H_{n+3}(\sqrt{\frac{m\omega}{\hbar}}x)\right] \tag{219}$$

$$= \left(\sqrt{\frac{\hbar}{m\omega}}\right)^3\left[n(n-1)(n-2)\frac{\sqrt{2^{n-3}(n-3)!}}{\sqrt{2^n n!}}|(n-3)^{(0)}\rangle + \frac{3n^2}{2}\frac{\sqrt{2^{n-1}(n-1)!}}{\sqrt{2^n n!}}|(n-1)^{(0)}\rangle\right. \tag{220}$$

$$\left.+\ \frac{3n+1}{4}\frac{\sqrt{2^{n+1}(n+1)!}}{\sqrt{2^n n!}}|(n-1)^{(0)}\rangle + \frac{1}{8}\frac{\sqrt{2^{n+1}(n+1)!}}{\sqrt{2^n n!}}|(n+1)^{(0)}\rangle\right] \tag{221}$$

$$x^4|n^{(0)}\rangle = \frac{\hbar^2}{m^2\omega^2}\left[n(n-1)(n-2)(n-3)\frac{\sqrt{2^{n-4}(n-4)!}}{\sqrt{2^n n!}}|(n-4)^{(0)}\rangle + (2n^2-3n+1)n\frac{\sqrt{2^{n-2}(n-2)!}}{\sqrt{2^n n!}}|(n-2)^{(0)}\rangle\right. \tag{222}$$

$$\left.+\ \frac{3}{4}(2n^2+2n+1)\||n^{(0)}\rangle + \frac{1}{4}(2n+3)\frac{\sqrt{2^{n+2}(n+2)!}}{\sqrt{2^n n!}}|(n+2)^{(0)}\rangle + \frac{1}{16}\frac{\sqrt{2^{n+4}(n+4)!}}{\sqrt{2^n n!}}|(n+4)^{(0)}\rangle\right] \tag{223}$$

Then

$$\langle n^{(0)}|X^3|n^{(0)}\rangle = 0 \tag{224}$$

$$\langle n^{(0)}|X^4|n^{(0)}\rangle = \frac{3}{4}[2n(n+1)+1]\frac{\hbar^2}{m^2\omega^2} \tag{225}$$

and the first order corrections are given by

$$a\langle n^{(0)}|X^3 + \frac{b}{a}X^4|n^{(0)}\rangle = a\langle n^{(0)}|X^3|n^{(0)}\rangle + a\langle n^{(0)}|\frac{b}{a}X^4|n^{(0)}\rangle \tag{226}$$

$$= \frac{3}{4}[2n(n+1)+1]\frac{\hbar^2}{m^2\omega^2}b \tag{227}$$

Also

$$\langle n^{(0)}|X^3|(n-3)^{(0)}\rangle = n(n-1)(n-2)\frac{\sqrt{2^{n-3}(n-3)!}}{\sqrt{2^n n!}}\left(\frac{\hbar}{m\omega}\right)^{3/2} \tag{228}$$

$$= \frac{\sqrt{n(n-1)(n-2)}}{\sqrt{8}}\left(\frac{\hbar}{m\omega}\right)^{3/2} \tag{229}$$

$$\langle n^{(0)}|X^3|(n-1)^{(0)}\rangle = \frac{3n^2}{2}\frac{\sqrt{2^{n-1}(n-1)!}}{\sqrt{2^n n!}}\left(\frac{\hbar}{m\omega}\right)^{3/2} \tag{230}$$

$$= \frac{3n^{3/2}}{\sqrt{8}}\left(\frac{\hbar}{m\omega}\right)^{3/2} \tag{231}$$

$$\langle n^{(0)}|X^3|(n+1)^{(0)}\rangle = \frac{3}{4}(n+1)\frac{\sqrt{2^{n+1}(n+1)!}}{\sqrt{2^n n!}}\left(\frac{\hbar}{m\omega}\right)^{3/2} \tag{232}$$

$$= \frac{3}{\sqrt{8}}(n+1)^{3/2}\left(\frac{\hbar}{m\omega}\right)^{3/2} \tag{233}$$

$$\langle n^{(0)}|X^3|(n+3)^{(0)}\rangle = \frac{1}{8}\frac{\sqrt{2^{n+3}(n+3)!}}{\sqrt{2^n n!}}\left(\frac{\hbar}{m\omega}\right)^{3/2} \tag{234}$$

$$= \frac{1}{\sqrt{8}}\sqrt{(n+1)(n+2)(n+3)}\left(\frac{\hbar}{m\omega}\right)^{3/2} \tag{235}$$

$$\langle n^{(0)}|X^4|(n-4)^{(0)}\rangle = n(n-1)(n-2)(n-3)\frac{\sqrt{2^{n-4}(n-4)!}}{\sqrt{2^n n!}}\left(\frac{\hbar}{m\omega}\right)^{2} \tag{236}$$

$$= \frac{1}{4}\sqrt{n(n-1)(n-2)(n-3)}\left(\frac{\hbar}{m\omega}\right)^{2} \tag{237}$$

$$\langle n^{(0)}|X^4|(n-2)^{(0)}\rangle = (2n^2-3n+1)n\frac{\sqrt{2^{n-2}(n-2)!}}{\sqrt{2^n n!}}\left(\frac{\hbar}{m\omega}\right)^{2} \tag{238}$$

$$= \frac{1}{2}(2n-1)\sqrt{n(n-1)}\left(\frac{\hbar}{m\omega}\right)^{2} \tag{239}$$

$$\langle n^{(0)}|X^4|(n)^{(0)}\rangle = \frac{3}{4}(2n^2+2n+1)\left(\frac{\hbar}{m\omega}\right)^{2} \tag{240}$$

$$\langle n^{(0)}|X^4|(n+2)^{(0)}\rangle = \frac{1}{4}(2n+3)\frac{\sqrt{2^{n+2}(n+2)!}}{\sqrt{2^n n!}}\left(\frac{\hbar}{m\omega}\right)^{2} \tag{241}$$

$$= \frac{1}{2}(2n+3)\sqrt{(n+1)(n+2)}\left(\frac{\hbar}{m\omega}\right)^{2} \tag{242}$$

$$\langle n^{(0)}|X^4|(n+4)^{(0)}\rangle = \frac{1}{16}\frac{\sqrt{2^{n+4}(n+4)!}}{\sqrt{2^n n!}}\left(\frac{\hbar}{m\omega}\right)^{2} \tag{243}$$

$$= \frac{1}{4}\sqrt{(n+1)(n+2)(n+3)(n+4)}\left(\frac{\hbar}{m\omega}\right)^{2} \tag{244}$$

and the second order corrections are given by

$$a^2\sum_{k\neq n}\frac{|\langle k^{(0)}|X^3+\frac{b}{a}X^4|n^{(0)}\rangle|^2}{E_n^{(0)}-E_k^{(0)}} = a^2\sum_{k\neq n}\frac{|\langle n^{(0)}|X^3+\frac{b}{a}X^4|k^{(0)}\rangle|^2}{(n-k)\hbar\omega} \tag{245}$$

$$= a^2\sum_{k\neq n}\frac{|\langle n^{(0)}|X^3|k^{(0)}\rangle+\frac{b}{a}\langle n^{(0)}|X^4|k^{(0)}\rangle|^2}{(n-k)\hbar\omega} \tag{246}$$

because X^3 and X^4 terms do not mix AND terms like $n-4$ vanish for $n=1,2,3$ we can write

$$E_n^{(2)} = \sum_{k\in\{n-4,\ldots,n+4\}} \frac{a^2|\langle n^{(0)}|X^3|k^{(0)}\rangle|^2 + b^2|\langle n^{(0)}|X^4|k^{(0)}\rangle|^2}{(n-k)\hbar\omega} \tag{247}$$

$$= -a^2\frac{1}{8}\frac{\hbar^2}{m^3\omega^4}(30n^2+30n+11) - b^2\frac{1}{16}\frac{\hbar^3}{m^4\omega^5}(68n^3+102n^2+118n+42) \tag{248}$$

0.11.2 Exercise 13.2 - Quartic perturbed harmonic oscillator

Substituting all into the Schroedinger equation

$$-\frac{\hbar^2}{2m}\psi'' - \frac{m\omega^2}{2}x^2\psi + bx^4\psi = E\psi \tag{249}$$

$$\sum_{k=0} b^k \left(-\frac{\hbar^2}{2m}\left[P_k''(x) - \frac{2m\omega}{\hbar}xP_k'(x) + \frac{m^2\omega^2}{\hbar^2}x^2P_k(x) - \frac{m\omega}{\hbar}P_k(x)\right] + \frac{m\omega^2}{2}x^2P_k(x) + bx^4P_k(x)\right)e^{-\frac{m\omega}{2\hbar}x^2} \tag{250}$$

$$= \sum_{k=0} b^kE_k \cdot \sum_{l=0} b^lP_l(x)e^{-\frac{m\omega}{2\hbar}x^2} \tag{251}$$

with $E_0 = \frac{1}{2}\hbar\omega$ (the book value of $\hbar\omega$ seems wrong). Now we can sort by powers of b

Zeroth order - using $E_0 = \hbar\omega/2$

$$b^0: \quad -\frac{\hbar^2}{2m}\left[P_0''(x) - \frac{2m\omega}{\hbar}xP_0'(x) + \frac{m^2\omega^2}{\hbar^2}x^2P_0(x) - \frac{m\omega}{\hbar}P_0(x)\right] + \frac{m\omega^2}{2}x^2P_0(x) = E_0P_0(x) \tag{252}$$

$$P_0''(x) - \frac{2m\omega}{\hbar}xP_0'(x) - \frac{m}{\hbar}\left(\omega - \frac{2E_0}{\hbar}\right)P_0(x) = 0 \tag{253}$$

$$\to P_0(x) = 1 \tag{254}$$

First order - using $E_0 = \hbar\omega/2$ and $P_0(x) = 1$

$$b^1: \quad -\frac{\hbar^2}{2m}\left[P_1''(x) - \frac{2m\omega}{\hbar}xP_1'(x) + \frac{m^2\omega^2}{\hbar^2}x^2P_1(x) - \frac{m\omega}{\hbar}P_1(x)\right] + \frac{m\omega^2}{2}x^2P_1(x) + x^4P_0(x) \tag{255}$$

$$= E_0P_1(x) + E_1P_0(x) \tag{256}$$

$$P_1''(x) - \frac{2m\omega}{\hbar}xP_1'(x) - \frac{2m}{\hbar^2}x^4 + \frac{m\omega}{\hbar}P_1(x) + \frac{2mE_1}{\hbar^2} = 0 \tag{257}$$

$$\to P_1(x) = -\frac{1}{4\hbar\omega}x^4 - \frac{3}{4m\omega^2}x^2 + c_1 \tag{258}$$

$$\to E_1(x) = \frac{3\hbar^2}{4m^2\omega^2} \tag{259}$$

Second order - using $E_0 = \hbar\omega/2$, $E_1(x) = \frac{3\hbar^2}{4m^2\omega^2}$ and $P_0(x) = 1$, $P_1(x) = -\frac{1}{4\hbar\omega}x^4 - \frac{3}{4m\omega^2}x^2 + c_1$

$$b^2 : \quad -\frac{\hbar^2}{2m}\left[P_2''(x) - \frac{2m\omega}{\hbar}xP_2'(x) + \frac{m^2\omega^2}{\hbar^2}x^2P_2(x) - \frac{m\omega}{\hbar}P_2(x)\right] + \frac{m\omega^2}{2}x^2P_2(x) + x^4P_1(x) \tag{260}$$

$$= E_0P_2(x) + E_1P_1(x) + E_2P_0(x) \tag{261}$$

$$P_2''(x) - \frac{2m\omega}{\hbar}xP_2'(x) - \frac{2m}{\hbar^2}x^4P_1(x) + \frac{m\omega}{\hbar}P_1(x) + \frac{2mE_1}{\hbar^2} = 0 \tag{262}$$

$$\to P_2(x) = \frac{1}{32\hbar^2\omega^2}x^8 + \frac{13}{48m\omega^3\hbar}x^6 + \frac{31\hbar - 8m^2\omega^3c_0}{32m^2\omega^4\hbar}x^4 + \frac{3(7\hbar - 2m^2\omega^3c_0)}{8m^3\omega^5}x^2 + c_2 \tag{263}$$

$$\to E_2(x) = -\frac{21\hbar^3}{8m^4\omega^5} \tag{264}$$

Then

$$E = \frac{1}{2}\hbar\omega + \frac{3\hbar^2}{4m^2\omega^2}b - \frac{21\hbar^3}{8m^4\omega^5}b^2 + ... \tag{265}$$

0.11.3 Exercise 13.3 - Normal matrix

A normal matrix A has the property $A^\dagger A = AA^\dagger$

0.12 Sakurai, Napolitano - Modern Quantum Mechanics 3rd ed

0.12.1 5.1 - Harmonic oscillator with linear perturbation

The Hamiltonians are given by

$$\hat{H}_0 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega_o^2x^2 \tag{266}$$

$$\hat{H}_1 = bx \tag{267}$$

We remember

$$\phi_0(x) = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4} e^{-m\omega_0x^2/2\hbar} \tag{268}$$

$$E_0 = \frac{1}{2}\hbar\omega_0 \tag{269}$$

$$\phi_n(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4} e^{-m\omega_0x^2/2\hbar}H_n\left(\sqrt{\frac{m\omega_0}{\hbar}}x\right) \tag{270}$$

$$E_n = \hbar\omega_0\left(n + \frac{1}{2}\right) \tag{271}$$

1. Time independent perturbation theory gives

$$\Delta E_n^{(1)} = \langle n^{(0)}|\hat{H}_1|n^{(0)}\rangle \tag{272}$$

$$\Delta E_0^{(1)} = \langle 0^{(0)}|\hat{H}_1|0^{(0)}\rangle = 0 \tag{273}$$

The first order energy shift vanishes because of the wave function is even and H_1 is odd. For the first order perturbation of the wave function we observe

$$H_1(x) = 2xH_0(x) \quad \to \quad \hat{H}_1|0^{(0)}\rangle = \frac{b}{2}\sqrt{2}\sqrt{\frac{\hbar}{m\omega_0}}|1^{(0)}\rangle \tag{274}$$

$$\langle m^{(0)}|n^{(0)}\rangle = \delta_{nm} \tag{275}$$

Now we can calculate

$$|n^{(1)}\rangle = \sum_{k\neq n} \frac{\langle k^{(0)}|\hat{H}_1|n^{(0)}\rangle}{E_n^{(0)} - E_k^{(0)}}|k^{(0)}\rangle \tag{276}$$

$$|0^{(1)}\rangle = \frac{\langle 0^{(0)}|\hat{H}_1|1^{(0)}\rangle}{E_0^{(0)} - E_1^{(0)}}|1^{(0)}\rangle \tag{277}$$

$$= -\frac{1}{\hbar\omega_0} b\sqrt{\frac{\hbar}{2m\omega_0}}|1^{(0)}\rangle \tag{278}$$

$$= -b\sqrt{\frac{1}{2m\hbar\omega_0^3}}|1^{(0)}\rangle \tag{279}$$

Second order enegy perturbation

$$\Delta E_n^{(2)} = \langle n^{(0)}|\hat{H}_1|n^{(1)}\rangle = \sum_{k\neq n} \frac{|\langle k^{(0)}|\hat{H}_1|n^{(0)}\rangle|^2}{E_n^{(0)} - E_k^{(0)}} \tag{280}$$

$$\Delta E_0^{(2)} = \langle 0^{(0)}|\hat{H}_1|0^{(1)}\rangle \tag{281}$$

$$= b\sqrt{\frac{\hbar}{2m\omega_0}}\langle 1^{(0)}|0^{(1)}\rangle \tag{282}$$

$$= b\sqrt{\frac{\hbar}{2m\omega_0}}\langle 1^{(0)}| \left(-b\sqrt{\frac{1}{2m\hbar\omega_0^3}}\right) |1^{(0)}\rangle \tag{283}$$

$$= -b^2\frac{1}{2m\omega_0^2} \tag{284}$$

2. The linear perturbation does not change the shape of the potential - only shifts the minimum

$$V(x) = \frac{m\omega_0^2}{2}x^2 + bx = \frac{m\omega_0^2}{2}\left(x + \frac{b}{m\omega_0^2}\right)^2 - \frac{b^2}{2m\omega_0^2} \tag{285}$$

$$\Delta E^{(\infty)} = -\frac{b^2}{2m\omega_0^2} \tag{286}$$

So the second order gives the exact result - interesting to see if higher orders would all vanish or give oscillating contributions.

0.12.2 5.2 - Potential well with linear slope

We will treat the slope as a perturbation with

$$\hat{H}_1 = \frac{V}{L}x \tag{287}$$

Therefore the unperturbed wave functions are given by

$$\phi_n = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L} \qquad E_n = \frac{\pi^2\hbar^2}{2mL^2}n^2 \tag{288}$$

Then

$$\Delta E_n^{(1)} = \langle n^{(0)}|\hat{H}_1|n^{(0)}\rangle \tag{289}$$
$$= \frac{V}{L}\frac{2}{L}\int_0^L x\sin^2\frac{n\pi x}{L}dx \tag{290}$$
$$= \frac{2V}{L^2}\int_0^L x\sin^2\frac{n\pi x}{L}dx \tag{291}$$
$$= \frac{2V}{L^2}\int_0^L x\left(1-\cos^2\frac{n\pi x}{L}\right)dx \tag{292}$$
$$= \frac{2V}{L^2}\frac{L^2}{2} - \Delta E_n^{(1)} \tag{293}$$

meaning $\Delta E_n^{(1)} = V/2$.

0.12.3 5.3 - Relativistic perturbation

We can approximate the kinetic energy by

$$E = \sqrt{m^2c^4 + p^2c^2} \tag{294}$$
$$\approx mc^2 + \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + \frac{p^6}{16m^5c^4} + \cdots \tag{295}$$
$$= mc^2 + \frac{mc^2}{2}\frac{p^2}{m^2c^2} - \frac{mc^2}{8}\frac{p^4}{m^4c^4} + \cdots \tag{296}$$
$$= mc^2\left(1 + \frac{1}{2}\beta^2 - \frac{1}{8}\beta^4 + \cdots\right) \tag{297}$$

so

$$\hat{H}_0 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2 \tag{298}$$
$$\hat{H}_1 = -\frac{1}{8m^3c^2}p^4 = -\frac{\hbar^4}{8m^3c^2}\frac{d^4}{dx^4} \tag{299}$$

and we remember

$$\phi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar} \tag{300}$$
$$E_0 = \frac{1}{2}\hbar\omega_0 \tag{301}$$

then

$$\Delta E_0^{(1)} = \langle 0^{(0)}|\hat{H}_1|0^{(0)}\rangle \tag{302}$$
$$= -\frac{\hbar^4}{8m^3c^2}\int_{-\infty}^{\infty}\phi_0(x)^*\frac{d^4}{d^4x}\phi_0(x)dx \tag{303}$$
$$= -\frac{3\hbar^2\omega^2}{32mc^2} \tag{304}$$

0.12.4 5.4 - Diatomic atomic rotor - NOT DONE YET

Hamiltonian of the problem is given by

$$H = \frac{L^2}{2I} \quad \rightarrow \quad \hat{H} = -\frac{\hbar^2}{2I}\frac{d^2}{d\varphi^2} \tag{305}$$

with the unperturbed solutions

$$\phi_n^{(0)} = Ce^{in\phi} \qquad E_n^{(0)} = \frac{\hbar^2 n^2}{2I} \tag{306}$$

where only E_0 is non-degenerate (all other are double degenerated). For the perturbation we use the Hamiltonian

$$\hat{H}_1 = Ed\cos\varphi \tag{307}$$

Hmmm....

0.12.5 5.6 - Two dimensional potential well

As the problem separates

$$\left(\hat{H}_x + \hat{H}_y\right)\phi_x\phi_y = (E_x + E_y)\phi_x\phi_y \tag{308}$$

$$\phi_y\hat{H}_x\phi_x + \phi_x\hat{H}_y\phi_y = (E_x + E_y)\phi_x\phi_y \tag{309}$$

$$\frac{\hat{H}_x\phi_x}{\phi_x} + \frac{\hat{H}_y\phi_y}{\phi_y} = (E_x + E_y) \tag{310}$$

the wave function can be written as a product of the 1-dimensional wave functions

$$\phi_{n_x,n_y} = \sqrt{\frac{2}{L}}\sqrt{\frac{2}{L}}\sin\left(\frac{n_x\pi}{L}x\right)\sin\left(\frac{n_y\pi}{L}y\right) \tag{311}$$

$$E_{n_x,n_y} = \frac{\pi^2\hbar^2}{2mL^2}(n_x^2 + n_y^2) \tag{312}$$

So

$$\phi_{1,1} \quad \rightarrow \quad E_{1,1} = 2\frac{\pi^2\hbar^2}{2mL^2} \tag{313}$$

$$\phi_{2,1},\phi_{1,2} \quad \rightarrow \quad E_{2,1} = 5\frac{\pi^2\hbar^2}{2mL^2} \tag{314}$$

$$\phi_{2,2} \quad \rightarrow \quad E_{1,1} = 8\frac{\pi^2\hbar^2}{2mL^2} \tag{315}$$

for the non-degenerated levels $E_{1,1}$ and $E_{2,2}$ we get

$$\Delta E_{1,1}^{(1)} = \langle 1, 1^{(0)}|\hat{H}_1|1, 1^{(0)}\rangle \tag{316}$$

$$= \frac{1}{4}\lambda L^2 \tag{317}$$

$$\Delta E_{2,2}^{(1)} = \langle 2, 2^{(0)}|\hat{H}_1|2, 2^{(0)}\rangle \tag{318}$$

$$= \frac{1}{4}\lambda L^2 \tag{319}$$

and for the degenerated levels $E_{1,2}/E_{2,1}$ we get

$$H = \begin{pmatrix} \langle 1, 2^{(0)}|\hat{H}_1|1, 2^{(0)}\rangle & \langle 1, 2^{(0)}|\hat{H}_1|2, 1^{(0)}\rangle \\ \langle 2, 1^{(0)}|\hat{H}_1|1, 2^{(0)}\rangle & \langle 2, 1^{(0)}|\hat{H}_1|2, 1^{(0)}\rangle \end{pmatrix} \tag{320}$$

with

$$H_{aa} = \langle 1, 2^{(0)}|\hat{H}_1|1, 2^{(0)}\rangle = \frac{\lambda L^2}{4} \tag{321}$$

$$H_{ab} = \langle 1, 2^{(0)}|\hat{H}_1|2, 1^{(0)}\rangle = \frac{256\lambda L^2}{81\pi^4} \tag{322}$$

$$H_{bb} = \langle 2, 1^{(0)}|\hat{H}_1|2, 1^{(0)}\rangle = \frac{\lambda L^2}{4} \tag{323}$$

and $\hat{H}_1 = \lambda xy$ Diagonalising the matrix H gives the perturbation

$$\Delta E^{(1)}_{12,21} = \frac{\lambda L^2}{4} - \frac{256\lambda L^2}{81\pi^4} \tag{324}$$
$$\Delta E^{(1)}_{12,21} = \frac{\lambda L^2}{4} + \frac{256\lambda L^2}{81\pi^4} \tag{325}$$
$$\tag{326}$$

0.12.6 5.8 - Quadratically perturbed harmonic oscillator

$$\hat{H}_1 = \epsilon\frac{1}{2}m\omega^2 x^2 \tag{327}$$

$$H_0(x) = 1 \tag{328}$$
$$H_2(x) = 4x^2 - 2 \quad \rightarrow \quad x^2 = \frac{H_2}{4} + \frac{1}{2} \tag{329}$$

0.12.7 5.13 - Two-dimensional infinite square well - NOT DONE YET

a. Separation ansatz

$$\left[-\frac{\hbar^2}{2m}(\partial_{xx} + \partial_{yy}) - E_{kl}\right]\psi_k(x)\psi_l(y) = 0 \tag{330}$$
$$\frac{1}{\psi_k(x)}\left(-\frac{\hbar^2}{2m}\partial_{xx}\right)\psi_k(x) = E_{kl} = \frac{1}{\psi_l(y)}\left(-\frac{\hbar^2}{2m}\partial_{yy}\right)\psi_l(y) \tag{331}$$

giving with boundary condition $\psi = 0$

$$\psi_{kl}(x,y) = \sqrt{\frac{2}{a}}\sqrt{\frac{2}{a}}\sin(\frac{k\pi}{a}x)\sin(\frac{k\pi}{a}y) \tag{332}$$
$$E_{kl} = \frac{\pi^2\hbar^2}{2ma^2}(k^2 + l^2) \tag{333}$$

Then the three lowest energy eigenstates are

$$E_{11} = 2 \cdot \frac{\pi^2\hbar^2}{2ma^2} \tag{334}$$
$$E_{21} = 5 \cdot \frac{\pi^2\hbar^2}{2ma^2} \tag{335}$$
$$E_{12} = 5 \cdot \frac{\pi^2\hbar^2}{2ma^2} \tag{336}$$

b.

$$E^{(1)}_{11} = \lambda\langle 1,1|xy|1,1\rangle = \lambda\frac{a^2}{4} \tag{337}$$
$$E^{(2)}_{11} = \lambda^2 \tag{338}$$

0.12.8 5.42 - Triton beta decay - NOT DONE YET

a. With the generic $1s$ wave function

$$\psi_{10} = \frac{1}{\sqrt{4\pi}}\sqrt{\frac{1}{2}\left(\frac{2Z}{a_\mu}\right)^3} e^{-Zr/a_\mu} \tag{339}$$

$$a_\mu = \frac{1}{\mu} a_0 \tag{340}$$

$$\mu = \frac{mM}{m+M} \tag{341}$$

we get with the initial state ($Z = 1$, $M = 3m1837$) and the final state ($Z = 2$, $M = 3m1837$) then

$$(i|f) = 4\pi \int_0^\infty r^2 \psi_i \psi_f = \frac{16\sqrt{2}}{27} \tag{342}$$

so the probability is 512/729.

b.

0.12.9 8.1 - Natural units

1. Proton Mass

$$E_p = m_p c^2/e = 0.937\text{GeV} \tag{343}$$

2. With $\Delta p \cdot \Delta x \geq \hbar/2$ and $E = \sqrt{m^2c^4 + p^2c^2} \approx pc$

$$E = \Delta pc/e = 98.6\text{MeV} \tag{344}$$

Alternatively we have $E = \frac{\hbar c}{e \cdot dx}$ meaning $1\text{fm} = \frac{1}{197.3\text{MeV}}$ and therefore

$$E = \frac{\hbar}{2 \cdot \Delta x} c = 197.3/2\text{MeV} \tag{345}$$

3. Solving for α, β, γ

$$M_P = G^\alpha c^\beta \hbar^\gamma \tag{346}$$

$$= \left(\frac{\text{Nm}^2}{\text{kg}^2}\right)^\alpha \left(\frac{\text{m}}{\text{s}}\right)^\beta (\text{Js})^\gamma \tag{347}$$

$$= \sqrt{\frac{\hbar c}{G}} \tag{348}$$

$$E_P = \sqrt{\frac{\hbar c}{G}} c^2 \frac{1}{e} = 1.22 \cdot 10^{19}\text{GeV} \tag{349}$$

0.12.10 8.2 - Minkowski Metric

The definition implies that $\eta_{\lambda\nu}$ is the inverse of $\eta^{\lambda\nu}$ - simple calculation shows that they are identical. Now we can calculate

$$\eta^{\mu\lambda}\eta^{\nu\sigma}\eta_{\lambda\sigma} = \eta^{\nu\sigma}\delta^\mu_\sigma \tag{350}$$

$$= \eta^{\nu\mu} \tag{351}$$

and

$$a^\mu b_\mu = a_\alpha \eta^{\alpha\mu} b^\beta \eta_{\beta\mu} = a_\alpha b^\beta \delta^\alpha_\beta = a_\alpha b^\alpha \tag{352}$$

0.13 Bethe, Jackiw - Intermediate Quantum Mechanics

0.13.1 1.1 - Atomic units

Set $\hbar = e = m_e = 1$ and $a_B = \frac{4\pi\varepsilon_0\hbar^2}{m_e e^2} = 1$ then $4\pi\varepsilon_0 = 1$ and therefore $\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} = 1/c$

1. energy: $E_{1s} = \frac{1}{2} m_e c^2 \alpha^2$ therefore 1 a.u. $= 2 \times 13.6\text{eV}$

2. momentum: $p = m_e c$ therefore 1 a.u. $= 2 \cdot 10^{-31}\text{kg} \times 3 \cdot 10^8 \text{m/s}^2 = 2.73 \cdot 10^{-22}\text{J}$

3. angular momentum: $L = \hbar$ therefore 1 a.u. $= 1.04 \cdot 10^{-34}\text{Js}$

0.13.2 1.7 - Hydrogen atom with finite nucleus

The field of a uniform sphere of charge Q can be found by Gauss law

$$E_r = \frac{1}{4\pi\epsilon_0} \cdot \begin{cases} Q/a^3 \cdot r & r < R \\ Q/r^2 & r > R \end{cases} \tag{353}$$

The potential is then given by

$$\phi = \frac{1}{4\pi\epsilon_0} \cdot \begin{cases} Q/2R\left(3 - \frac{r^2}{R^2}\right) & r < R \\ Q/r & r > R \end{cases} \tag{354}$$

Treating this as a perturbation problem the energy shift can be calculated via the perturbation Hamiltonian (switching the electrostatic energy within the finite nucleus)

$$\begin{aligned} H_1 &= (q\phi_{\text{finite}} - q\phi_{\text{point}})\theta(R - r) && (355)\\ &= -e\left(\phi_{\text{finite}} - \phi_{\text{point}}\right)\theta(R - r) && (356)\\ &= -\frac{e}{4\pi\epsilon_0}\left(\frac{Ze}{2R}\left[3 - \frac{r^2}{R^2}\right] - \frac{Ze}{r}\right)\theta(R - r) && (357)\\ &= -\frac{Ze^2}{4\pi\epsilon_0}\left(\frac{1}{2R}\left[3 - \frac{r^2}{R^2}\right] - \frac{1}{r}\right)\theta(R - r) && (358) \end{aligned}$$

with $R = r_0 A^{1/3}$. With the radial wavefunction (in Mathematica notation)

$$R_{nl}(r) = \frac{2}{n^2}\sqrt{\frac{(n-l-1)!Z^3}{(n+l)!a_B^3}}\left(\frac{2Zr}{na_B}\right)^l e^{-Zr/na_B} L_{n-l-1}^{2l+1}(\frac{2Zr}{na_B}) \tag{359}$$

we can do a series expansion at $r = 0$ and use the first term (as nucleus is small)

$$\begin{aligned} &R_{10}^2 \simeq 4Z^3 && (360)\\ &R_{20}^2 \simeq \frac{1}{2}Z^3 \qquad R_{21}^2 \simeq \frac{1}{24}Z^5 r^2 && (361)\\ &R_{30}^2 \simeq \frac{4}{27}Z^3 \qquad R_{31}^2 \simeq \frac{32}{2187}Z^5 r^2 \qquad R_{32}^2 \simeq \frac{8}{98415}Z^7 r^4 && (362)\\ &R_{40}^2 \simeq \frac{1}{16}Z^3 \qquad R_{41}^2 \simeq \frac{5}{768}Z^5 r^2 \qquad R_{42}^2 \simeq \frac{1}{20400}Z^7 r^4 \qquad R_{43}^2 \simeq \frac{1}{20643840}Z^9 r^6 && (363) \end{aligned}$$

then

$$\Delta E_{nl} = \int_0^R r^2 R_{nl}(r)^2 H_1(r) \tag{364}$$

$$\Delta E_{10} = -\frac{2}{5} r_0^2 A^{2/3} Z^4 \tag{365}$$

$$\Delta E_{20} = -\frac{1}{20} r_0^2 A^{2/3} Z^4 \qquad \Delta E_{21} = -\frac{1}{1120} r_0^4 A^{4/3} Z^6 \tag{366}$$

$$\Delta E_{30} = -\frac{2}{135} r_0^2 A^{2/3} Z^4 \qquad \Delta E_{31} = -\frac{8}{25515} r_0^4 A^{4/3} Z^6 \qquad \Delta E_{32} = -\frac{4}{6200145} r_0^6 A^2 Z^8 \tag{367}$$

$$\Delta E_{40} = -\frac{1}{160} r_0^2 A^{2/3} Z^4 \qquad \Delta E_{41} = -\frac{1}{7168} r_0^4 A^{4/3} Z^6 \qquad \Delta E_{42} = -\frac{1}{2580480} r_0^6 A^2 Z^8 \qquad \Delta E_{43} = -\frac{1}{5449973760} r_0^8 A^8 \tag{368}$$

and

$$\Delta E_{2p\to 1s} = \left(-\frac{1}{1120} r_0^4 A^{4/3} Z^6\right) - \left(-\frac{2}{5} r_0^2 A^{2/3} Z^4\right) \tag{369}$$

$$\Delta E_{H:\,2p\to 1s} = 2.05593 \cdot 10^{-10} = 0.000045\mathrm{cm}^{-1} \tag{370}$$

$$\Delta E_{Pb:\,2p\to 1s} = 0.003981 = 873.8\mathrm{cm}^{-1} \tag{371}$$

0.13.3 1.9 - Exponential potential

The Schroedinger equation is given by

$$-\frac{1}{2}\triangle_r \psi + V\psi = E\psi \tag{372}$$

$$-\frac{1}{2r^2}\partial_r(r^2\partial_r\psi) - \frac{a^2}{8} e^{-r/2r_0}\psi = E\psi \tag{373}$$

$$-\frac{1}{2}\left(\frac{2}{r}\psi' + \psi''\right) - \frac{a^2}{8} e^{-r/2r_0}\psi = E\psi \tag{374}$$

Ansatz $\psi(r) = u(r)/r$

$$-\frac{1}{2}\left(\frac{2}{r}\frac{u'r - u}{r^2} + \frac{(u''r + u' - u')r^2 - 2r(u'r - u)}{r^4}\right) - \frac{a^2}{8} e^{-r/2r_0}\frac{u}{r} = E\frac{u}{r} \tag{375}$$

$$-\frac{u'}{r^2} + \frac{u}{r^3} - \frac{u''}{2r} + \frac{2u'}{r^2} - \frac{u}{r^3} - \frac{a^2}{8} e^{-r/2r_0}\frac{u}{r} = E\frac{u}{r} \tag{376}$$

$$-\frac{u''}{2r} + \frac{u'}{r^2} - \frac{a^2}{8} e^{-r/2r_0}\frac{u}{r} = E\frac{u}{r} \tag{377}$$

$$\tag{378}$$

Stepwise calculation for the verification of the solution

$$r^2\partial_r\psi = u'r - u \tag{379}$$

$$= \frac{1}{2}\left[J_{n-1}(.) - J_{n+1}(.)\right] a r_0 e^{-\frac{r}{2r_0}} \frac{-1}{2r_0} r - J_n(.) \tag{380}$$

$$= -\frac{1}{4}\left[J_{n-1}(.) - J_{n+1}(.)\right] a r e^{-\frac{r}{2r_0}} - J_n(.) \tag{381}$$

$$= -\frac{1}{4}\left[J_{n-1}(.) - \left(\frac{2n}{a r_0 e^{-r/2r_0}} J_n(.) - J_{n-1}(.)\right)\right] a r e^{-\frac{r}{2r_0}} - J_n(.) \tag{382}$$

$$= -\frac{1}{4}\left[2J_{n-1}(.) - \frac{2n}{a r_0 e^{-r/2r_0}} J_n(.)\right] a r e^{-\frac{r}{2r_0}} - J_n(.) \tag{383}$$

$$= -\frac{1}{2} J_{n-1}(.) a r e^{-\frac{r}{2r_0}} + \left(\frac{nr}{2r_0} - 1\right) J_n(.) \tag{384}$$

$$\frac{1}{r^2}\partial_r(r^2\partial_r\psi) = -\frac{1}{2}\left(J_{n-1} - J_{n+1}\right)a^2\frac{r_0}{r} - \frac{1}{2}J_{n-1}(.)\frac{a}{r^2}e^{-\frac{r}{2r_0}} - \frac{1}{2}J_{n-1}(.)\frac{-a}{2rr_0}e^{-\frac{r}{2r_0}} \tag{385}$$

$$+\frac{n}{2r_0r^2}J_n(.) + \left(\frac{nr}{2r_0} - 1\right)\frac{1}{2r^2}(J_{n-1}(.) - J_{n+1}(.))ar_0e^{-\frac{r}{2r_0}}\frac{-1}{2r_0} \tag{386}$$

$$= \left(J_{n-1} - J_{n+1}\right)\left[-\frac{a^2r_0}{2r} - \left(\frac{nr}{2r_0} - 1\right)\frac{ar_0}{4r^2r_0}\right]e^{-\frac{r}{2r_0}} \tag{387}$$

0.14 Gottfried, Tung - Quantum Mechanics: Fundamentals, 2nd ed

0.15 Merzbacher - Quantum Mechanics, 3rd ed

0.16 Jackson - A Course in Quantum Mechanics

0.16.1 1.1 Lorentzian wave package

(a)

$$\psi(x,0) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dp\, e^{ixp/\hbar}\sqrt{\frac{2}{\pi\hbar}}\frac{\alpha^{3/2}}{(p-p_0)^2+\alpha^2} \tag{388}$$

$$= \frac{\alpha}{\pi\hbar}\int_{-\infty}^{\infty} dp\, e^{ixp/\hbar}\frac{\alpha^{1/2}}{(p-p_0)^2+\alpha^2} \tag{389}$$

$$= \frac{\alpha^{3/2}}{\pi\hbar}e^{ixp_0/\hbar}\int_{-\infty}^{\infty} d\hat{p}\, e^{ix\hat{p}/\hbar}\frac{1}{\hat{p}^2+\alpha^2} \tag{390}$$

$$= \frac{\alpha^{3/2}}{\pi\hbar}\frac{i}{2\alpha}e^{ixp_0/\hbar}\int_{-\infty}^{\infty} d\hat{p}\, e^{ix\hat{p}/\hbar}\left(\frac{1}{\hat{p}+i\alpha} - \frac{1}{\hat{p}-i\alpha}\right) \tag{391}$$

Close loop above for $x > 0$ (half loop integral vanishes) and below for $x < 0$

$$\psi(x>0,0) = \frac{i\sqrt{\alpha}}{2\pi\hbar}e^{ixp_0/\hbar}\cdot 2\pi i\,\mathrm{Res}(i\alpha, I_2) \tag{392}$$

$$= -\frac{\sqrt{\alpha}}{\hbar}e^{ixp_0/\hbar}\cdot\left(-e^{ix(i\alpha)/\hbar}\right) \tag{393}$$

$$= \frac{\sqrt{\alpha}}{\hbar}e^{-x\alpha/\hbar}\cdot e^{ixp_0/\hbar} \tag{394}$$

$$\to \psi(x,0) = \frac{\sqrt{\alpha}}{\hbar}e^{-|x|\alpha/\hbar}\cdot e^{ixp_0/\hbar} \tag{395}$$

α is the width of the package in momentum space. $1/\alpha$ is the package width in space.

(b)

$$\int \phi^*\phi\, dp = \frac{1}{\hbar} \tag{396}$$

$$\int \phi^* p\phi\, dp = \frac{p_0}{\hbar} \quad\to\quad \langle p\rangle = p_0 \tag{397}$$

$$\int \phi^* p^2\phi\, dp = \frac{p_0^2+\alpha^2}{\hbar} \quad\to\quad \langle p^2\rangle = p_0^2+\alpha^2 \tag{398}$$

$$\int \psi^* \psi \, dp = \frac{1}{\hbar} \tag{399}$$

$$\int \psi^* x \psi \, dp = 0 \quad \rightarrow \quad \langle x \rangle = 0 \tag{400}$$

$$\int \psi^* x^2 \psi \, dp = \frac{\hbar}{2\alpha^2} \quad \rightarrow \quad \langle x^2 \rangle = \frac{\hbar^2}{2\alpha^2} \tag{401}$$

(c)

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \frac{\hbar}{\sqrt{2}\alpha} \tag{402}$$

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \alpha \tag{403}$$

$$\Delta x \cdot \Delta p = \frac{\hbar}{\sqrt{2}} > \hbar/2 \tag{404}$$

0.16.2 1.2 1D Box with vanishing walls

With

$$\psi(x,t) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right) e^{-i\frac{\pi^2 \hbar}{2mL^2}t} \tag{405}$$

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \sqrt{\frac{2}{L}} \int_0^L dx \, \psi(x,0) \, e^{ipx/\hbar} \tag{406}$$

$$= \frac{1}{\sqrt{\pi\hbar L}} \int_0^L dx \, \sin\left(\frac{\pi x}{L}\right) e^{ipx/\hbar} \tag{407}$$

Two times integration by parted gives

$$\phi(p) = \frac{1}{\sqrt{\pi\hbar L}} \frac{\pi L \hbar^2 \left(1 + e^{ipL/\hbar}\right)}{\hbar^2 \pi^2 - p^2 L^2} \tag{408}$$

Now we can use the Schroedinger equation with $\phi(p,t) = \phi(p) f(t)$

$$i\hbar \partial_t \phi(p,t) = \frac{p^2}{2m} \phi(p,t) \tag{409}$$

$$\rightarrow i\hbar \partial_t f(t) = \frac{p^2}{2m} f(t) \tag{410}$$

$$\rightarrow f(t) = c \, e^{-i\frac{p^2}{2\hbar m}t} \tag{411}$$

$$\rightarrow \phi(p,t) = \frac{1}{\sqrt{\pi\hbar L}} \frac{\pi L \hbar^2 \left(1 + e^{ipL/\hbar}\right)}{\hbar^2 \pi^2 - p^2 L^2} e^{-i\frac{p^2}{2\hbar m}t} \tag{412}$$

$$\rightarrow \rho(p,t) = \phi^* \phi = \frac{2\hbar^4 L^2 \pi^2}{(p^2 L^2 + \hbar^2 \pi^2)^2} \left(1 + \cos\frac{pL}{\hbar}\right) \tag{413}$$

0.16.3 1.3 Protonium

$$\mu_x = \frac{m_p m_x}{m_p + m_x} \tag{414}$$

$$E_n^{(x)} = -\frac{\mu_x c^2}{2}\alpha^2 \frac{1}{n^2} \tag{415}$$

$$r_B^{(x)} = \frac{\hbar c}{\alpha\, \mu_x c^2} \tag{416}$$

$$\frac{dP}{dt} = \Gamma_{f\to i} = \frac{2\pi}{\hbar}|\langle f|H'|i\rangle|^2 \rho(E_f) \tag{417}$$

$$\sim \frac{1}{\tau} \tag{418}$$

(a) We have now with $m_p = 938.272$MeV, $\alpha = 1/137$ and $c = 1$

x	m_x	μ_x	r_B	E_{1s}	E_{2p}	$\Delta E_{2p/1s}$
e^-	511.0keV	510.7keV	$5.3 \cdot 10^{-11}$m	13.6eV	3.4eV	10.2eV
μ^-	105.7MeV	95.0MeV	$2.8 \cdot 10^{-13}$m	2,530eV	632eV	1,898eV
π^-	139.6MeV	121.5MeV	$2.2 \cdot 10^{-13}$m	3,237eV	809eV	2,428eV
K^-	493.6MeV	299.3MeV	$9.0 \cdot 10^{-14}$m	8,616eV	2,154eV	6,462eV
$\bar{\mathrm{p}}$	938.3MeV	469.1MeV	$5.8 \cdot 10^{-14}$m	12,498eV	3,123eV	9,373eV
Σ^-	1197.4MeV	526.1MeV	$5.1 \cdot 10^{-14}$m	14,019eV	3,505eV	10,510eV
Ξ^-	1321.7MeV	548.7MeV	$4.9 \cdot 10^{-14}$m	14,618eV	3,654eV	10,963eV

(b) Different values for hydrogen can be found $\tau_H = 1.76 \cdot 10^{-9}$s and $\Gamma(2p \to 1s) = 6.2 \cdot 10^8\mathrm{s}^{-1}$. Full valuation of Fermis golden rule gives

$$\tau = \left(\frac{3}{2}\right)^8 \frac{r_B}{c\alpha^4} \tag{419}$$

$$\tau \sim r_B \sim \frac{1}{\mu} \tag{420}$$

$$\tau_{p\bar{p}} = \frac{\mu_H}{\mu_{p\bar{p}}}\tau_H = 1.73 \cdot 10^{-12}\mathrm{s} \tag{421}$$

Rational: dipole matrix element scales with r_B (smaller object means smaller dipole)

$$\langle f|H'|i\rangle \sim \langle f|\mathbf{x}|i\rangle \sim r_B \tag{422}$$

$$|\langle f|H'|i\rangle|^2 \sim r_B^2 \tag{423}$$

$$\rho \sim \frac{1}{\Delta E_{s/p}} \sim \frac{1}{\mu} \sim r_B \tag{424}$$

$$\Gamma \sim r_B^3 \tag{425}$$

Hmmmmm ....

0.16.4 2.5 Unitary operators

Unitary: $U^\dagger = U^{-1}$ meaning $U^\dagger U = 1$. We see $(U^n)^\dagger = (U...U)^\dagger = (U^\dagger...U^\dagger) = (U^\dagger)^n$

- $U_1 = e^{iK}$: Lets start with

$$U_1^\dagger = \left(e^{iK}\right)^\dagger = \left(\sum_n \frac{1}{n!}(iK)^n\right)^\dagger = \sum_n \frac{1}{n!}\left((iK)^n\right)^\dagger = \sum_n \frac{(-i)^n}{n!}(K^\dagger)^n = \sum_n \frac{1}{n!}(-iK^\dagger)^n \tag{426}$$

$$= e^{-iK^\dagger} \qquad \text{with } K = K^\dagger \tag{427}$$

$$= e^{-iK} \tag{428}$$

Now with $[K, K] = 0$ (meaning we can Taylor-expand each exponential and flip term by term so $e^X e^Y = e^{X+Y}$ if $[X, Y] = 0$)

$$U_1^\dagger U_1 = e^{-iK} e^{iK} = e^{-iK+iK} = e^0 = 1 \tag{429}$$

- $U_2 = (1 + iK)(1 - iK)^{-1}$

$$\begin{aligned} U_2^\dagger &= \left((1 + iK)(1 - iK)^{-1}\right)^\dagger && (430)\\ &= \left((1 - iK)^{-1}\right)^\dagger (1 + iK)^\dagger \qquad \text{with } K = K^\dagger && (431)\\ &= \left((1 - iK)^{-1}\right)^\dagger (1 - iK) && (432) \end{aligned}$$

then

$$\begin{aligned} U_2^\dagger U_2 &= \left((1 - iK)^{-1}\right)^\dagger (1 - iK)(1 + iK)(1 - iK)^{-1} && (433)\\ &= \left((1 - iK)^{-1}\right)^\dagger (1 - iK + iK + K^2)(1 - iK)^{-1} && (434)\\ &= \left((1 - iK)^{-1}\right)^\dagger (1 + iK)(1 - iK)(1 - iK)^{-1} && (435)\\ &= \left((1 - iK)^{-1}\right)^\dagger (1 + iK) \qquad \text{with } B^\dagger A^\dagger = (AB)^\dagger && (436)\\ &= \left((1 + iK)^\dagger (1 - iK)^{-1}\right)^\dagger \qquad \text{with } K = K^\dagger && (437)\\ &= \left((1 - iK)(1 - iK)^{-1}\right)^\dagger && (438)\\ &= 1^\dagger = 1 && (439) \end{aligned}$$

- $U_2' = (1 - iK)^{-1}(1 + iK)$. Assume $U_2' = U_2$ then

$$\begin{aligned} 1 &= (U_2')^{-1} U_2' && (440)\\ &= U_2^{-1} U_2' && (441)\\ &= U_2^\dagger U_2' && (442)\\ &= \underbrace{\left((1 - iK)^{-1}\right)^\dagger (1 - iK)}_{U_2^\dagger} \underbrace{(1 - iK)^{-1}(1 + iK)}_{U_2'} && (443)\\ &= \left((1 - iK)^{-1}\right)^\dagger (1 + iK) && (444)\\ &= \left((1 + iK)^\dagger (1 - iK)^{-1}\right)^\dagger && (445)\\ &= \left((1 - iK)(1 - iK)^{-1}\right)^\dagger && (446)\\ &= 1^\dagger = 1 && (447) \end{aligned}$$

0.17 Basdevant - The Quantum machanics solver 3rd ed.

0.17.1 Exercise 8.1 - Neutrino Oscillations in Vacuum - NOT DONE YET

1.

$$\begin{aligned}
\Delta &= E_2 - E_1 &(448)\\
&= \sqrt{p^2c^2 + m_2^2c^4} - \sqrt{p^2c^2 + m_1^2c^4} &(449)\\
&= pc\sqrt{1 + \frac{m_2^2c^4}{p^2c^2}} - pc\sqrt{1 + \frac{m_1^2c^4}{p^2c^2}} &(450)\\
&\simeq pc\left(1 + \frac{m_2^2c^4}{2p^2c^2} - 1 - \frac{m_1^2c^4}{2p^2c^2}\right) &(451)\\
&= \frac{c^3}{2p}(m_2^2 - m_1^2) &(452)
\end{aligned}$$

2.

$$\begin{aligned}
\Delta(2\times 10^5\text{eV}/c) &= 2\times 10^{-10}\text{eV} &(453)\\
\Delta(8\times 10^6\text{eV}/c) &= 5\times 10^{-12}\text{eV} &(454)
\end{aligned}$$

3. (a)

$$\begin{aligned}
|\nu_e(t)\rangle &= \cos\theta\, e^{-iE_1t/\hbar}|\nu_1\rangle + \sin\theta\, e^{-iE_2t/\hbar}|\nu_2\rangle &(455)\\
&= \cos\theta\, e^{-iE_1t/\hbar}|\nu_1\rangle + \sin\theta\, e^{-i(E_1+\Delta)t/\hbar}|\nu_2\rangle &(456)\\
&= e^{-iE_1t/\hbar}\left(\cos\theta|\nu_1\rangle + \sin\theta\, e^{-i\Delta t/\hbar}|\nu_2\rangle\right) &(457)\\
& &(458)
\end{aligned}$$

(b)

$$\begin{aligned}
\langle\nu_e|\nu_e(t)\rangle &= (\langle\nu_1|\cos\theta + \langle\nu_2|\sin\theta)e^{-iE_1t/\hbar}\left(\cos\theta|\nu_1\rangle + \sin\theta\, e^{-i\Delta t/\hbar}|\nu_2\rangle\right) &(459)\\
&= e^{-iE_1t/\hbar}(\cos^2\theta + \sin^2\theta\, e^{-i\Delta t/\hbar}) &(460)\\
|\langle\nu_e|\nu_e(t)\rangle|^2 &= |\cos^2\theta + \sin^2\theta\, e^{-i\Delta t/\hbar}|^2 &(461)\\
&= (\cos^2\theta + \sin^2\theta\, e^{-i\Delta t/\hbar})(\cos^2\theta + \sin^2\theta\, e^{+i\Delta t/\hbar}) &(462)\\
&= \cos^4\theta + \sin^4\theta + 2\sin^2\theta\cos^2\theta\cos\Delta t/\hbar &(463)
\end{aligned}$$

4. (a) We check

$$\langle\nu_e|\nu_e(0)\rangle = \cos^4\theta + \sin^4\theta + 2\sin^2\theta\cos^2\theta = (\cos^2\theta + \sin^2\theta)^2 = 1 \tag{464}$$

then

$$\begin{aligned}
\frac{\Delta\cdot t}{\hbar} &= \Delta\frac{d_{ES}}{\hbar c} = 1.52\cdot 10^8 &(465)\\
N &= \frac{\Delta\cdot t}{2\pi\hbar} = 2.42\cdot 10^7 &(466)
\end{aligned}$$

(b)