Solutions

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Chapter 1

Relativistic Quantum Field Theory 1B WS2022/23

1.1 Sheet 1 — Exercise 1 (Lagrangian and Hamiltonian formalism, constrained systems)

Good exposition: HANSON, REGGE, TEITELBOIM - Constrained Hamiltonian Systems - Accademia Nazionale dei Lincei (1976)

a) Free non-interacting particles

Euler-Lagrange Eqn:

$$\ddot{q}_1 = 0, \qquad \ddot{q}_1 = 0 \tag{1.1}$$

Solutions $q_1(0), q_2(0), \dot{q}_1(0), \dot{q}_2(0) \in \mathbb{R}$:

$$q_1(t) = v_1 t + s_1, \qquad q_2(t) = v_2 t + s_2 \tag{1.2}$$

Momenta:

$$p_1 = \frac{\partial L}{\partial q_1} = m_1 \dot{q}_1, \qquad p_2 = m_1 \dot{q}_1 \tag{1.3}$$

$$\rightarrow \dot{q}_1 = \frac{p_1}{m_1}, \qquad \dot{q}_1 = \frac{p_1}{m_1} \tag{1.4}$$

Also $\frac{dp_1}{dt} = 0 = \frac{dp_2}{dt}$ so p_1, p_2 are conserved quantities.

Hamiltonian

$$H = p_1 \dot{q}_1 + p_2 \dot{q}_2 - L \tag{1.5}$$

$$= \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} \tag{1.6}$$

Necessary and sufficient condition for a constraint is $\det M = 0$ which is not the case

$$M_{ij} = \frac{\partial p_i}{\partial \dot{q}_j} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \tag{1.7}$$

$$\det M = m_1 m_2 \tag{1.8}$$

b) Free particle $q_1' = q_1 - q_0$

$$L = \frac{m_1}{2}(\dot{q}_1 - \dot{q}_0)^2 \tag{1.9}$$

$$= \frac{m_1}{2}\dot{q}_1^2 + \frac{m_1}{2}\dot{q}_0^2 - m\dot{q}_1\dot{q}_0 \tag{1.10}$$

Invarianz trafo (gives free particle + total time derivative)

$$= L' + \frac{d\Lambda(q_0, q_1, t)}{dt} \tag{1.11}$$

$$= L' + \frac{\partial\Lambda}{\partial q_1}\frac{\partial q_1}{\partial t} + \frac{\partial\Lambda}{\partial t} \tag{1.12}$$

$$\rightarrow \frac{\partial\Lambda}{\partial q_1} = -m\dot{q}_0, \qquad \frac{\partial\Lambda}{\partial t} = \frac{1}{2}m\dot{q}_0^2 \tag{1.13}$$

which implies

$$\ddot{q}_0 = 0 \tag{1.14}$$

$$\rightarrow q_0 = \alpha + \beta t \tag{1.15}$$

$$\rightarrow \Lambda = -m\beta q_1 + \frac{1}{2}m\beta^2 t \tag{1.16}$$

Euler-Lagrange equations

$$\ddot{q}_1 - \ddot{q}_0 = 0, \qquad \ddot{q}_0 - \ddot{q}_1 = 0 \tag{1.17}$$

$$\rightarrow \frac{\partial^2}{\partial t^2}(q_1 - q_0) = 0 \tag{1.18}$$

Solution

$$q_1(t) - q_0(t) = vt + s \tag{1.19}$$

$$\rightarrow q_1(t) = v_0 t + s_0 + \lambda(t) \tag{1.20}$$

$$\rightarrow q_0(t) = v_1 t + s_1 - \lambda(t) \tag{1.21}$$

Momenta

$$p_1 = m_1(\dot{q}_1 - \dot{q}_0), \qquad p_0 = m_1(\dot{q}_0 - \dot{q}_1) = -p_1, \tag{1.22}$$

$$p_0 + p_1 = 0 \qquad \text{(constraint)} \tag{1.23}$$

$$\rightarrow p_1 - p_0 = 2m_1(\dot{q}_1 - \dot{q}_0) \tag{1.24}$$

$$\rightarrow p_1 = m_1(\dot{q}_1 - \dot{q}_0) \tag{1.25}$$

Also $\frac{dp_1}{dt} = 0 = \frac{dp_0}{dt}$ so p_1, p_0 are conserved quantities.
Hamiltonian (conjugated momenta are not independent)

$$H = p_1\dot{q}_1 + p_0\dot{q}_0 - L \tag{1.26}$$

$$= p_1(\dot{q}_1 - \dot{q}_0) - \frac{1}{2}m(\dot{q}_1 - \dot{q}_0)^2 \tag{1.27}$$

$$= \frac{p_1^2}{m_1} - \frac{m_1}{2}\frac{p_1^2}{m_1^2} \tag{1.28}$$

$$= \frac{p_1^2}{2m_1} \tag{1.29}$$

$$M_{ij} = \frac{\partial p_i}{\partial \dot{q}_j} = \begin{pmatrix} m_1 & -m_1 \\ -m_1 & m_1 \end{pmatrix} \tag{1.30}$$

$$\det M = 0 \tag{1.31}$$

c) Euler-Lagrange equation

$$m_1\ddot{q}_1 + \dot{q}_B = 0, \qquad \dot{q}_1 - \frac{q_B}{m_2} = 0 \tag{1.32}$$

$$\rightarrow (m_1 + m_2)\ddot{q}_1 = 0, \qquad q_B = m_2\dot{q}_1 \tag{1.33}$$

Solution

$$q_1(t) = \alpha t + \beta \tag{1.34}$$
$$q_B(t) = \alpha m_2 \tag{1.35}$$

Momenta

$$p_B = 0 \qquad \text{(constraint 1)} \tag{1.36}$$
$$p_1 = m_1\dot{q}_1 + q_B \tag{1.37}$$
$$\rightarrow p_1 = m_1\alpha + q_B = \frac{m_1}{m_2}q_B + q_B \qquad \text{(using equations of motion, constraint 2)} \tag{1.38}$$

Also $\frac{dp_1}{dt} = 0$ so p_1 is a conserved quantity.
Hamiltonian (only one canonical momentum)

$$H = p_1\dot{q}_1 + p_B\dot{q}_B - L \tag{1.39}$$
$$= p_1\frac{p_1 - q_B}{m_1} - \frac{m_1}{2}\left(\frac{p_1 - q_B}{m_1}\right)^2 + \frac{1}{2m_2}q_B^2 - q_B\frac{p_1 - q_B}{m_1} \tag{1.40}$$
$$= \frac{p_1^2}{2m_1} - \frac{p_1 q_B}{m_1} + \frac{m_1 + m_2}{2m_1 m_2}q_B^2 \tag{1.41}$$
$$= \frac{p_1^2}{2m_1} + \left(\frac{m_1 + m_2}{2m_2}q_B - p_1\right)\frac{q_B}{m_1} \tag{1.42}$$
$$\tag{1.43}$$

$$M_{ij} = \frac{\partial p_i}{\partial \dot{q}_j} = \begin{pmatrix} m_1 & 0 \\ 0 & 0 \end{pmatrix} \tag{1.44}$$
$$\det M = 0 \tag{1.45}$$

1.2 Sheet 1 — Exercise 2 (Theory of relativity, notation)

a) 4-vectors transforming under LT as

$$y' = \Lambda y \tag{1.46}$$

More mathematical: 4-vector is an element of a four-dimensional vector space considered as a representation space of the standard $(1/2, 1/2)$ of the Lorentz group.

$$x^\mu \equiv (t, \mathbf{x})^T \qquad \text{with} \quad \eta_{\alpha\beta}dx^\alpha dx^\beta = dx^\beta dx_\beta = ds^2 = dx'^\mu dx'_\mu = \eta_{\mu\nu}\Lambda^\mu{}_\alpha dx^\alpha \Lambda^\nu{}_\beta dx^\beta \tag{1.47}$$
$$\rightarrow \quad \eta_{\alpha\beta} = \eta_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta \tag{1.48}$$

with

$$\eta_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1.49}$$

and the inverse $(\eta_{\alpha\beta})^{-1} \sim \eta^{\beta\gamma}$ defined by $\eta_{\alpha\beta}\eta^{\beta\gamma} = \delta_\alpha^\gamma \equiv \eta_\alpha^\gamma$

$$\eta^{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1.50}$$

then

$$
\begin{aligned}
x_\nu &= \eta_{\nu\mu} x^\mu = (t, -\mathbf{x}) && (1.51)\\
u^\mu &\equiv \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt}\frac{dt}{d\tau} = (1, \mathbf{v})\gamma && (1.52)\\
&\quad\text{with} \qquad (u)^2 = u^\mu u_\mu = u^\mu(\eta_{\mu\nu}u^\nu) = \eta_{\mu\nu}u^\mu u^\nu = 1 && (1.53)\\
p^\mu &\equiv m u^\mu = (\sqrt{m^2 + \mathbf{p}^2}, \mathbf{p}) \qquad \text{with} \qquad p^\mu p_\mu = m^2 && (1.54)\\
p_\mu &= (\sqrt{m^2 + \mathbf{p}^2}, -\mathbf{p}) && (1.55)\\
&\to px = p \cdot x = p_\mu x^\mu = \eta_{\mu\nu} p^\nu x^\mu = p^0 x^0 - \mathbf{p}\cdot\mathbf{x} = p_0 x^0 - \mathbf{p}\cdot\mathbf{x} && (1.56)\\
\partial_\mu &= \frac{\partial}{\partial x^\mu} \qquad \text{with} \qquad \partial_\mu \partial^\mu = \Box && (1.57)\\
A^\mu &= (\Phi, \mathbf{A}) && (1.58)\\
F^{\mu\nu} &= \eta^{\mu\alpha}\eta^{\nu\beta}(\partial_\alpha A\beta - \partial_\beta A\alpha) && (1.59)\\
j^\mu &= (\rho, \mathbf{j}) && (1.60)
\end{aligned}
$$

Notation:

- abstract 4-vector a can be represented by 4 components a^μ

$$
\begin{aligned}
a &= a^0 \mathbf{e_0} + a^1 \mathbf{e_1} + a^0 \mathbf{e_1} + a^1 \mathbf{e_1} && (1.61)\\
&\simeq a^\mu && (1.62)
\end{aligned}
$$

- Metric η is a bilinear from that takes two 4-vectors and maps them to the real numbers - if positive definite it will define a scalar product

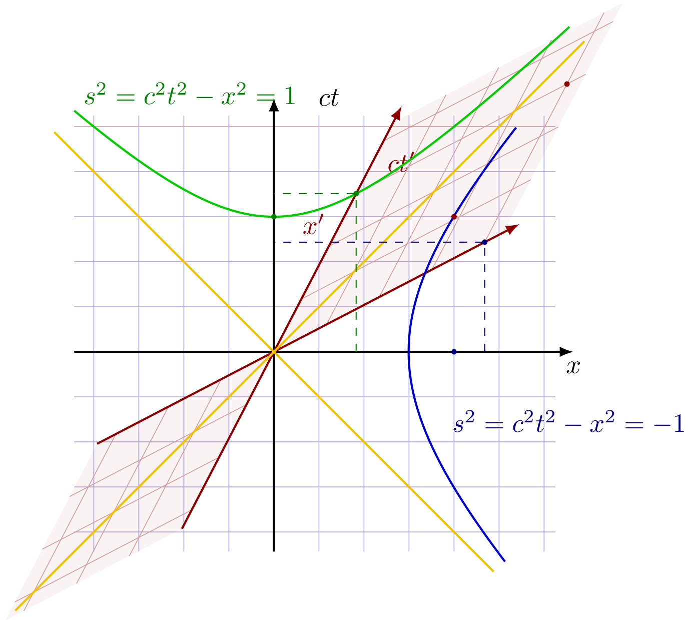
$$
\begin{aligned}
ab &= \eta(a, b) && (1.63)\\
&= \eta(a^\mu \mathbf{e}_\mu, b^\nu \mathbf{e}_\nu) && (1.64)\\
&= a^\mu b^\nu \eta(\mathbf{e}_\mu, \mathbf{e}_\nu) && (1.65)\\
&= a^\mu b^\nu \eta_{\mu\nu} && (1.66)\\
&= a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3 && (1.67)\\
&= a^\mu b_\mu && (1.68)\\
&= a^0 b_0 + a^1 b_1 + a^2 b_2 + a^3 b_3 && (1.69)
\end{aligned}
$$

  similar to standard linear algebra - where we understand the

$$
\begin{aligned}
(u, v) &= (u^\mu \mathbf{e}_\mu, v^\nu \mathbf{e}_\nu) && (1.70)\\
&= u^\mu (\mathbf{e}_\mu, \mathbf{e}_\nu) v^\nu && (1.71)\\
&= u^T G v && (1.72)
\end{aligned}
$$

- $b_\mu = \eta_{\mu\nu} b^\nu$ are the components of the 1-form $\tilde{b}$ associated to the 4-vector b

b) Good summary of all effects: Sexl, Urbantke - Relativitaet, Gruppen, Teilchen. Obviously all effects can be explained by a proper Minkowski diagram (important: unit scaling in the different inertial systems are given green and blue hyperbola)



1.3 Sheet 1 — Exercise 3 (Localization of relativistic particles)

a) Pair production - locate particle smaller then the Compton wavelenth $\lambda_C = \hbar/mc$. Back of the envelope estimate

$$E = \sqrt{m^2c^4 + p^2c^2} \qquad \rightarrow \Delta p \geq mc \tag{1.73}$$

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2} \qquad \rightarrow \Delta x \simeq \frac{\hbar}{2mc} \tag{1.74}$$

- Top quark $7 \cdot 10^{-18}$m
- Proton $1.3 \cdot 10^{-13}$m
- Electron $2.4 \cdot 10^{-12}$m
- Hydrogen $1.3 \cdot 10^{-13}$m

b)
- Top quark Δx **kind of similar** to physical size (10^{-19}m)
- Proton Δx **larger** than physical size ($0.8 \cdot 10^{-15}$m)
- Electron Δx **larger** than physical size of 0m (point)

c)

$$\Delta p \geq \frac{\hbar}{2\Delta x} \tag{1.75}$$

$$E_{\text{part}} \geq \sqrt{m_{\text{part}}^2 c^4 + \frac{\hbar^2 c^2}{4\Delta x^2}} \simeq \frac{\hbar c}{2\Delta x} \tag{1.76}$$

1.4 Sheet 2 — Exercise 1 (Schroedinger field quantization)

a) Canonical variables ψ, ψ^*

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\psi^* \tag{1.77}$$

$$\pi^* = \frac{\partial \mathcal{L}}{\partial \dot{\psi}^*} = 0 \qquad \text{(constraint)} \tag{1.78}$$

Then

$$\mathcal{H} = \pi\dot{\psi} + \pi^*\dot{\psi}^* - \mathcal{L} \tag{1.79}$$

$$= \frac{1}{2m}(\nabla\psi^*)(\nabla\psi) + V\psi^*\psi \tag{1.80}$$

$$= -\frac{i}{2m}(\nabla\pi)(\nabla\psi) - iV\pi\psi \tag{1.81}$$

$$H = -i\int d^3x \left(\frac{1}{2m}(\nabla_x\pi(x))(\nabla_x\psi(x)) + V\pi(x)\psi(x)\right) \tag{1.82}$$

$$= -i\int d^3x \left(-\frac{1}{2m}\pi(x)(\triangle_x\psi(x)) + V\pi(x)\psi(x)\right) \tag{1.83}$$

Using the standard commutators

$$[\psi(x), \pi(y)] = i\delta^{(3)}(x-y), \qquad [\psi(x), \psi(y)] = 0, \qquad [\pi(x), \pi(y)] = 0 \tag{1.84}$$

Heisenberg equations: when we integrate by parts to move the $\nabla, \triangle$ we assume well behaving boundaries

$$i\partial_t\psi(y) = [H, \psi(y)] \tag{1.85}$$

$$= i\int d^3x \frac{1}{2m}\pi(x)(\triangle_x\psi(x))\psi(y) - \frac{1}{2m}\psi(y)\pi(x)(\triangle_x\psi(x)) - V(x)\pi(x)\psi(x)\psi(y) + V(x)\psi(y)\pi(x)\psi(x) \tag{1.86}$$

$$= i\int d^3x \frac{1}{2m}\pi(x)(\triangle_x\psi(x))\psi(y) - \frac{1}{2m}(\pi(x)\psi(y) - i\delta^{(3)}(x-y))(\triangle_x\psi(x)) \tag{1.87}$$

$$- V(x)\pi(x)\psi(x)\psi(y) + V(x)(\pi(x)\psi(y) - i\delta^{(3)}(x-y))\psi(x) \tag{1.88}$$

$$= i\int d^3x - \frac{1}{2m}(-i\delta^{(3)}(x-y))(\triangle_x\psi(x)) - V(-i\delta^{(3)}(x-y))\psi(x) \tag{1.89}$$

$$= -\int d^3x \frac{1}{2m}\delta^{(3)}(x-y)(\triangle_x\psi(x)) - V(x)\delta^{(3)}(x-y)\psi(x) \tag{1.90}$$

$$= -\frac{1}{2m}\triangle_y\psi(y) + V(y)\psi(y) \tag{1.91}$$

which looks like the one particle Schroedinger equation.

$$i\partial_t \pi(y) = [H, \pi(y)] \tag{1.92}$$

$$= i\int d^3x \frac{1}{2m}\left(\pi(x)\triangle_x\psi(x)\pi(y) - \pi(y)\pi(x)\triangle_x\psi(x)\right) - V\pi(x)i\delta^{(3)}(x-y) \tag{1.93}$$

$$= i\int d^3x \frac{1}{2m}\Big(\pi(x)\triangle_x(\pi(y)\psi(x) + i\delta^{(3)}(x-y)) - \pi(y)\pi(x)\triangle_x\psi(x)\Big) - V\pi(x)i\delta^{(3)}(x-y) \tag{1.94}$$

$$= i\int d^3x \frac{1}{2m}i\pi(x)\triangle_x\delta^{(3)}(x-y) - \frac{1}{2m}\pi(x)\pi(y)\triangle_x\psi(x) - V\pi(x)i\delta^{(3)}(x-y) \tag{1.95}$$

$$= i\int d^3x \frac{1}{2m}i\pi(x)\triangle_x\delta^{(3)}(x-y) - \frac{1}{2m}\pi(x)\triangle_x\pi(y)\psi(x) - V\pi(x)i\delta^{(3)}(x-y) \tag{1.96}$$

$$= i\int d^3x \frac{1}{2m}i\triangle_x\pi(x)\delta^{(3)}(x-y) - V\pi(x)i\delta^{(3)}(x-y) \tag{1.97}$$

$$= -\frac{1}{2m}\triangle_y\pi(y) + V(y)\pi(y) \tag{1.98}$$

which with $\pi = i\psi^*$ gives the complex conjugated SG

$$-i\partial_t\psi^* = -\frac{1}{2m}\triangle\psi^* + V\psi^* \tag{1.99}$$

b) With

$$H = -i\int d^3x \left(-\frac{1}{2m}\pi(x)(\triangle_x\psi(x)) + V(x)\pi(x)\psi(x)\right) \tag{1.100}$$

and the usual definition of the field-theory Poisson bracket

$$\{H, \psi(x)\} = \int d^3y \left(\frac{\partial H}{\partial\psi(y)}\frac{\partial\psi(x)}{\partial\pi(y)} - \frac{\partial H}{\partial\pi(y)}\frac{\partial\psi(x)}{\partial\psi(y)}\right) + \left(\frac{\partial H}{\partial\psi^*(y)}\frac{\partial\psi(x)}{\partial\pi^*(y)} - \frac{\partial H}{\partial\pi^*(y)}\frac{\partial\psi(x)}{\partial\psi^*(y)}\right) \tag{1.101}$$

$$= -i\int d^3y \int d^3x(-1)\left(-\frac{1}{2m}\triangle_x\psi(x) + V(x)\psi(x)\right)\delta^{(3)}(x-y)\cdot\delta^{(3)}(x-y) \tag{1.102}$$

$$= i\int d^3y \left(-\frac{1}{2m}\triangle_y\psi(y) + V(y)\psi(y)\right)\delta^{(3)}(x-y) \tag{1.103}$$

$$= i\left(-\frac{1}{2m}\triangle_y\psi(y) + V(y)\psi(y)\right) \tag{1.104}$$

and with $\dot{\psi} = \{H, \psi(x)\}$ we recover the Schroedinger equation.
Just out of curiosity we calculate two more Poisson brackets ($\pi = i\psi^*$)

$$\{\psi(x), \psi(y)\} = \int d^3z \left(\frac{\partial\psi(x)}{\partial\psi(z)}\frac{\partial\psi(y)}{\partial\pi(z)} - \frac{\partial\psi(x)}{\partial\pi(z)}\frac{\partial\psi(y)}{\partial\psi(z)}\right) + \left(\frac{\partial\psi(x)}{\partial\psi^*(z)}\frac{\partial\psi(y)}{\partial\pi^*(z)} - \frac{\partial\psi(x)}{\partial\pi^*(z)}\frac{\partial\psi(y)}{\partial\psi^*(z)}\right) \tag{1.105}$$

$$= 0 \tag{1.106}$$

$$\{\psi(x), \pi(y)\} = \int d^3z \left(\frac{\partial\psi(x)}{\partial\psi(z)}\frac{\partial\pi(y)}{\partial\pi(z)} - \frac{\partial\psi(x)}{\partial\pi(z)}\frac{\partial\pi(y)}{\partial\psi(z)}\right) + \left(\frac{\partial\psi(x)}{\partial\psi^*(z)}\frac{\partial\pi(y)}{\partial\pi^*(z)} - \frac{\partial\psi(x)}{\partial\pi^*(z)}\frac{\partial\pi(y)}{\partial\psi^*(z)}\right) \tag{1.107}$$

$$= \int d^3z\,\delta^{(3)}(x-z)\delta^{(3)}(y-z) \tag{1.108}$$

$$= \delta^{(3)}(x-z) \tag{1.109}$$

c) As ψ and ψ^* all terms appear as some kind of product we try a global gauge transformation of (same idea as for complex KG field)

$$\psi \to e^{i\varepsilon}\psi \simeq (1+i\varepsilon)\psi = \psi + i\varepsilon\psi \tag{1.110}$$
$$\psi^* \to e^{-i\varepsilon}\psi^* \simeq (1-i\varepsilon)\psi^* = \psi^* - i\varepsilon\psi^* \tag{1.111}$$

so $\delta\psi = i\varepsilon\psi$ and $\delta\psi^* = -i\varepsilon\psi^*$.

Then we look at the three terms of the Lagrangian separately

$$\psi^*\dot{\psi} \to (\psi^* - i\varepsilon\psi^*)(\dot{\psi} + i\varepsilon\dot{\psi}) = \psi^*\dot{\psi} + i\varepsilon(-\psi^*\dot{\psi} + \psi^*\dot{\psi}) + \mathcal{O}(\varepsilon^2) \tag{1.112}$$
$$= \psi^*\dot{\psi} \tag{1.113}$$
$$\to \delta(\psi^*\dot{\psi}) = 0 \tag{1.114}$$

second term

$$(\nabla\psi^*)\nabla(\psi) \to (\nabla\psi^* - i\varepsilon\psi^*)\nabla(\psi + i\varepsilon\psi) = (\nabla\psi^*)(\nabla\psi) + i\varepsilon((\nabla\psi^*)(\nabla\psi) - (\nabla\psi^*)(\nabla\psi)) + \mathcal{O}(\varepsilon^2) \tag{1.115}$$
$$= (\nabla\psi^*)(\nabla\psi) \tag{1.116}$$
$$\to \delta((\nabla\psi^*)(\nabla\psi)) = 0 \tag{1.117}$$

and third term

$$\psi^*\psi \to (\psi^* - i\varepsilon\psi^*)(\psi + i\varepsilon\psi) = \psi^*\psi + i\varepsilon(-\psi^*\psi + \psi^*\psi) + \mathcal{O}(\varepsilon^2) \tag{1.118}$$
$$= \psi^*\psi \tag{1.119}$$
$$\to \delta(\psi^*\psi) = 0. \tag{1.120}$$

So we conclude that the Lagrangian is invariant under this transformation. Then

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^*)}\delta\psi^* \tag{1.121}$$
$$j^0 = i\psi^*(i\varepsilon\psi) \tag{1.122}$$
$$= -\varepsilon\psi^*\psi \tag{1.123}$$
$$j^m = -\frac{1}{2m}\left((\nabla\psi^*)(i\varepsilon\psi) + (\nabla\psi)(-i\varepsilon\psi^*)\right) \tag{1.124}$$
$$= -\frac{i\varepsilon}{2m}\left((\nabla\psi^*)\psi - (\nabla\psi)\psi^*\right) \tag{1.125}$$

and

$$Q = \int d^3x\, j^0 = -\varepsilon\int d^3x\, \psi^*\psi. \tag{1.126}$$

The charge operator becomes then (we are cheating a bit because we no idea about the operator ordering)

$$\hat{Q} = -\varepsilon\int d^3x\, \hat{\psi}^\dagger(x)\hat{\psi}(x) = -\varepsilon\int d^3x\int\frac{d^3p}{(2\pi)^{3/2}}e^{-ipx}a_p^\dagger\int\frac{d^3q}{(2\pi)^{3/2}}e^{iqx}a_q \tag{1.127}$$
$$= \varepsilon\int\frac{d^3p}{(2\pi)^{3/2}}\int\frac{d^3q}{(2\pi)^{3/2}}a_p^\dagger a_q\int d^3x\, e^{i(q-p)x} \tag{1.128}$$
$$= \varepsilon\int\frac{d^3p}{(2\pi)^{3/2}}\int\frac{d^3q}{(2\pi)^{3/2}}a_p^\dagger a_q\,(2\pi)^3\delta^{(3)}(q-p) \tag{1.129}$$
$$= \varepsilon\int d^3p\, a_p^\dagger a_q \tag{1.130}$$

d) As we recover the non-relativistic Schroedinger theory there are no anti-particles - so there is only one charge amount associated with a particle - so particle and charge conservation are identical.

   In a relativistic theory with particles always come with anti-particles. In case the particles have charge q then the anti-particles have charge $-q$. If a particle - anti-particle pair is created - the total charge will be conserved but the particle number is not.

e) The canonical approach starting with a field theory is to calculate the energy momentum tensor and derive the momentum from the T^{0k} components. So we start with the definition (with metric signature diag $\eta = (1, -1, -1, -1)$)

$$
\begin{aligned}
T^{\mu\nu} &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}(\partial^\nu \psi) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^*)}(\partial^\nu \psi^*) - \mathcal{L}\eta^{\mu\nu} \qquad (1.131)\\
&= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}(\eta^{\nu\alpha}\partial_\alpha \psi) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^*)}(\eta^{\nu\alpha}\partial_\alpha \psi^*) - \mathcal{L}\eta^{\mu\nu} \qquad (1.132)
\end{aligned}
$$

To sense check we calculate the T^{00} component first

$$
\begin{aligned}
T^{00} &= i\psi^*\dot{\psi} + 0 - i\psi^*\dot{\psi} + \frac{1}{2m}|\nabla_x \psi(x)|^2 + V(x)|\psi(x)|^2 \qquad (1.133)\\
&= \frac{1}{2m}|\nabla_x \psi(x)|^2 + V(x)|\psi(x)|^2 \qquad (1.134)
\end{aligned}
$$

and see that we recover the Hamiltonian density from (a). So we continue

$$
\begin{aligned}
T^{0k} &= i\psi^*(-\partial_k \psi) \qquad (1.135)\\
P^k &= -i\int d^3x\ \psi^*(\partial_k \psi) \qquad (1.136)
\end{aligned}
$$

The (guessing the operator ordering)

$$
\begin{aligned}
\hat{P}^k &= -i\int d^3x \int \frac{d^3p}{(2\pi)^{3/2}} e^{-ipx} a_p^\dagger \int \frac{d^3q}{(2\pi)^{3/2}} (\partial_k e^{iqx}) a_q \qquad (1.137)\\
&= -i\int \frac{d^3p}{(2\pi)^{3/2}} \int \frac{d^3q}{(2\pi)^{3/2}} (iq_k) a_p^\dagger a_q \int d^3x\, e^{i(q-p)x} \qquad (1.138)\\
&= \int \frac{d^3p}{(2\pi)^{3/2}} \int \frac{d^3q}{(2\pi)^{3/2}} q_k\, a_p^\dagger a_q (2\pi)^3 \delta^{(3)}(q-p) \qquad (1.139)\\
&= \int d^3p\, p_k\, a_p^\dagger a_q \qquad (1.140)
\end{aligned}
$$

1.5 Sheet 3 — Exercise 1 (Poincare representations on fields)

a) The representation is $f'(x') = f'(\Lambda x + a) = f(x)$.

Now lets check the result for Alice with the definitions

$$f'(\Lambda x + a) = f(x) \tag{1.141}$$
$$f'(x) = f(\Lambda^{-1}(x-a)) \tag{1.142}$$
$$U(\Lambda, a) f(x) = f'(x) \tag{1.143}$$
$$= f(\Lambda^{-1}(x-a)) \tag{1.144}$$
$$= f(\Lambda^{-1}x - \Lambda^{-1}a) \tag{1.145}$$

we calculate (evaluating left operator first!!!)

$$U(\Lambda_2, a_2)U(\Lambda_1, a_1)f(x) = U(\Lambda_2, a_2)(U(\Lambda_1, a_1)f(x)) \tag{1.146}$$
$$= (U(\Lambda_1, a_1)f)(\Lambda_2^{-1}(x-a_2)) \tag{1.147}$$
$$= f(\Lambda_1^{-1}((\Lambda_2^{-1}x - \Lambda_2^{-1}a_2) - a_1)) \tag{1.148}$$
$$= f(\Lambda_1^{-1}\Lambda_2^{-1}x - \Lambda_1^{-1}\Lambda_2^{-1}a_2 - \Lambda_1^{-1}a_1) \tag{1.149}$$

and

$$U(\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2)f(x) = f((\Lambda_2\Lambda_1)^{-1}(x - \Lambda_2 a_1 - a_2)) \tag{1.150}$$
$$= f(\Lambda_1^{-1}\Lambda_2^{-1}(x - \Lambda_2 a_1 - a_2)) \tag{1.151}$$
$$= f(\Lambda_1^{-1}\Lambda_2^{-1}x - \Lambda_1^{-1}a_1 - \Lambda_1^{-1}\Lambda_2^{-1}a_2) \tag{1.152}$$

I think the confusing point here is the notation of the order of the operations - $U(\Lambda_2, a_2)U(\Lambda_1, a_1)$ acting on $f(x)$. We used

$$x' = \Lambda_1 x + a_1 \tag{1.153}$$
$$x'' = \Lambda_2(\Lambda_1 x + a_1) + a_2 \tag{1.154}$$
$$= \Lambda_2\Lambda_1 x + \Lambda_2 a_1 + a_2 \tag{1.155}$$
$$\rightarrow (\Lambda_2, a_2) \circ (\Lambda_1, a_1) = (\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2) \tag{1.156}$$
$$\rightarrow U(\Lambda_2, a_2)U(\Lambda_1, a_1) = U((\Lambda_2, a_2) \circ (\Lambda_1, a_1)) \tag{1.157}$$
$$= U(\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2) \tag{1.158}$$

b) Poincare transformation for spacetime-independent objects

$$U(\Lambda, a) = e^{ia_\mu P^\mu} e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}} \tag{1.159}$$
$$\simeq 1 + ia_\mu P^\mu - \frac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma} \tag{1.160}$$

and for spacetime-dependent objects (generators are now operators) - so we have infinite dimensional representations

$$U(\delta + \omega, a) = 1 + ia_\mu \hat{P}^\mu - \frac{i}{2}\omega_{\rho\sigma}\hat{J}^{\rho\sigma} \tag{1.161}$$

We see that there are two sets of generators: Lorentz transformations and translations

With $\Lambda^\alpha_{\ \mu} \approx \delta^\alpha_\mu + \omega^\alpha_{\ \mu}$ we obtain for the infinitesimal Poincare transformation

$$x'^\mu = \Lambda^\mu_{\ \alpha}x^\alpha + a^\mu \tag{1.162}$$
$$\simeq (\delta^\mu_\alpha + \omega^\mu_{\ \alpha})\, x^\alpha + a^\mu \tag{1.163}$$
$$\simeq x^\mu + \omega^\mu_{\ \alpha}x^\alpha + a^\mu. \tag{1.164}$$

The inverted PT is then given by

$$x = \Lambda^{-1}(x' - a) \tag{1.165}$$
$$= \Lambda^{-1}x' - \Lambda^{-1}a \tag{1.166}$$
$$x^\mu \simeq (\delta^\mu_\alpha - \omega^\mu{}_\alpha)\, x'^\alpha - (\delta^\mu_\alpha - \omega^\mu{}_\alpha)\, a^\alpha \tag{1.167}$$
$$\simeq x'^\mu - \omega^\mu{}_\alpha x'^\omega - a^\mu + \mathcal{O}(\epsilon \cdot a) \tag{1.168}$$

Because of

$$\phi'(x') = \left(1 + i a_\mu \hat{P}^\mu - \frac{i}{2}\omega_{\rho\sigma}\hat{J}^{\rho\sigma}\right)\phi(x) \tag{1.169}$$

and

$$\phi'(x') = \phi(x) \quad \Leftrightarrow \quad \phi'(\Lambda x + a) = \phi(x) \tag{1.170}$$
$$\Leftrightarrow \quad \phi'(x) = \phi(\Lambda^{-1}(x - a)) \tag{1.171}$$

we can now calculate

$$\delta\phi(x) \equiv \phi'(x) - \phi(x) \tag{1.172}$$
$$= \phi(\Lambda^{-1}(x - a)) - \phi(x) \tag{1.173}$$
$$\simeq \phi(x^\mu - \omega^\mu{}_\alpha x^\alpha - a^\mu) - \phi(x) \tag{1.174}$$
$$\simeq \phi(x) + \partial_\mu\phi(x) \cdot (-\omega^\mu{}_\alpha x^\alpha - a^\mu) - \phi(x) \tag{1.175}$$
$$\simeq -(a^\mu + \omega^\mu{}_\alpha x^\alpha)\partial_\mu\phi(x) \tag{1.176}$$
$$\simeq -(a^\mu + \omega^{\mu\alpha} x_\alpha)\partial_\mu\phi(x) \tag{1.177}$$
$$\simeq -\left(a^\mu + \frac{1}{2}\left(\omega^{\mu\alpha} - \omega^{\alpha\mu}\right) x_\alpha\right)\partial_\mu\phi(x) \tag{1.178}$$
$$\simeq -\left(a^\mu\partial_\mu + \frac{1}{2}\left(\omega^{\mu\alpha}x_\alpha\partial_\mu - \omega^{\alpha\mu}x_\alpha\partial_\mu\right)\right)\phi(x) \tag{1.179}$$
$$\simeq -\left(a^\mu\partial_\mu + \frac{1}{2}\left(\omega^{\mu\alpha}x_\alpha\partial_\mu - \omega^{\mu\alpha}x_\mu\partial_\alpha\right)\right)\phi(x) \tag{1.180}$$
$$\simeq i\left(a^\mu i\partial_\mu + \frac{1}{2}\omega^{\mu\alpha} i\left(x_\alpha\partial_\mu - x_\mu\partial_\alpha\right)\right)\phi(x) \tag{1.181}$$
$$\simeq i\left(a^\mu i\partial_\mu - \frac{1}{2}\omega^{\mu\alpha} i\left(x_\mu\partial_\alpha - x_\alpha\partial_\mu\right)\right)\phi(x) \tag{1.182}$$
$$\simeq i\left(a^\mu\hat{P}_\mu - \frac{1}{2}\omega^{\mu\alpha}\hat{J}_{\mu\alpha}\right)\phi(x) \tag{1.183}$$

So the scalar field representation is given by

$$\hat{P}_\mu = i\partial_\mu \tag{1.184}$$
$$\hat{J}_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) \tag{1.185}$$

1.6 Sheet 3 — Exercise 2 (Poincare representations on quantum fields)

Adam looks right

$$U(\Lambda_1, a_1)^{-1}\psi_A(\Lambda_1 x + a_1)U(\Lambda_1, a_1) = D_{AB}(\Lambda_1)\psi_B(x) \tag{1.186}$$
$$U(\Lambda_1, a_1)^{-1}\psi_A(x)U(\Lambda_1, a_1) = D_{AB}(\Lambda_1)\psi_B(\Lambda_1^{-1}(x - a_1)) \tag{1.187}$$

The combined transformation ($\sim \Lambda_1\Lambda_2$) looks like

$$U(\Lambda_1\Lambda_2, \Lambda_1 a_2 + a_1)^{-1}\psi_A(\Lambda_1 x + a_1)U(\Lambda_1\Lambda_2, \Lambda_1 a_2 + a_1) = D_{AC}(\Lambda_1\Lambda_2)\psi_C(\Lambda_2^{-1}(x - a_1)) \tag{1.188}$$

With $U((\Lambda_1, a_1)(\Lambda_2, a_2))^{-1} = U(\Lambda_2, a_2)^{-1}U(\Lambda_1, a_1)^{-1}$ we calculate

$$U(\Lambda_2, a_2)^{-1}\underbrace{U(\Lambda_1, a_1)^{-1}\psi_A(\Lambda_1 x + a_1)U(\Lambda_1, a_1)}_{=D_{AB}(\Lambda_1)\psi_B(x)}U(\Lambda_2, a_2) = D_{AB}(\Lambda_1)U(\Lambda_2, a_2)^{-1}\psi_B(x)U(\Lambda_2, a_2) \tag{1.189}$$

$$= D_{AB}(\Lambda_1)D_{BC}(\Lambda_2)\psi_C(\Lambda_2^{-1}(x - a_2)) \tag{1.190}$$

which completes the proof.

1.7 Sheet 3 — Exercise 3 (Pauli-Lubanski operator)

a) Using the definition of the angular momentum $J_i = \frac{1}{2}\epsilon_{ijk}J^{jk}$

$$W_\mu = \frac{1}{2}\epsilon_{\mu 0\rho\sigma}P^0 J^{\rho\sigma} \qquad \rightarrow \qquad W = (0, \mathbf{W}) \tag{1.191}$$

$$W_0 = 0 \tag{1.192}$$

$$W_1 = \frac{m}{2}\left(\epsilon_{1023}J^{23} + \epsilon_{1032}J^{32}\right) = \frac{m}{2}\left(-J^{23} + J^{32}\right) = mJ_1 \tag{1.193}$$

$$W_2 = \frac{m}{2}\left(\epsilon_{2031}J^{31} + \epsilon_{2013}J^{13}\right) = \frac{m}{2}\left(-J^{13} + J^{31}\right) = mJ_2 \tag{1.194}$$

$$W_3 = \frac{m}{2}\left(\epsilon_{3012}J^{12} + \epsilon_{3021}J^{21}\right) = \frac{m}{2}\left(J^{12} - J^{21}\right) = mJ_3 \tag{1.195}$$

b) Using the result from a) and the definition of the boosts $K_i = J^{i0}$

$$W_\mu = \frac{1}{2}\epsilon_{\mu 0\rho\sigma}P^0 J^{\rho\sigma} + \frac{1}{2}\epsilon_{\mu 3\rho\sigma}P^3 J^{\rho\sigma} \tag{1.196}$$

$$W_0 = 0 + \frac{k}{2}(\epsilon_{0312}J^{12} + \epsilon_{0321}J^{21}) = \frac{k}{2}(J^{12} - J^{21}) = kJ_3 \tag{1.197}$$

$$W_1 = kJ_1 + \frac{k}{2}(\epsilon_{1320}J^{20} + \epsilon_{1302}J^{02}) = kJ_1 + \frac{k}{2}(J^{20} - J^{02}) = k(J_1 + K_2) \tag{1.198}$$

$$W_2 = kJ_2 + \frac{k}{2}(\epsilon_{2310}J^{10} + \epsilon_{2301}J^{01}) = kJ_2 + \frac{k}{2}(-J^{10} + J^{01}) = k(J_2 - K_1) \tag{1.199}$$

$$W_3 = kJ_3 + 0 = kJ_3 \tag{1.200}$$

c) With the Lorentz algebra

$$[J^i, J^j] = i\epsilon_{ijk}J^k \tag{1.201}$$

$$[J^i, K^j] = i\epsilon_{ijk}K^k \tag{1.202}$$

$$[K^i, K^j] = -i\epsilon_{ijk}J^k \tag{1.203}$$

$$\tag{1.204}$$

we calculate

$$
\begin{aligned}
[A, B] = [W_2, W_1] &= k^2[J_2 - K_1, J_1 + K_2] && (1.205)\\
&= k^2([J_2, J_1] - [K_1, J_1] + [J_2, K_2] - [K_1, K_2]) && (1.206)\\
&= k^2((-iJ_3) - 0 + 0 - (-iJ_3)) && (1.207)\\
&= 0 && (1.208)\\
[J_z, A] = [J_3, W_2] &= k[J_3, (J_2 - K_1)] && (1.209)\\
&= k[J_3, J_2] - [J_3, K_1)] && (1.210)\\
&= k(-iJ_1 - (iK_2)) && (1.211)\\
&= -ik(J_1 + K_2) && (1.212)\\
&= -iW_1 && (1.213)
\end{aligned}
$$

1.8 Sheet 4 — Exercise 1 (Noether Theorem)

a) With

$$x'^{\rho} = \Lambda^{\rho}{}_{\sigma} x^{\sigma} + a^{\rho} \tag{1.214}$$
$$= (\delta^{\rho}{}_{\sigma} + \omega^{\rho}{}_{\sigma}) x^{\sigma} + a^{\rho} \tag{1.215}$$
$$D(\Lambda)_{AB} \simeq 1 - \frac{i}{2}\omega_{\rho\sigma}(S^{\rho\sigma})_{AB} \tag{1.216}$$
$$\phi_A(x) \to \phi'_A(x) = D_{AB}(\Lambda)\phi_B(\Lambda^{-1}(x-a)) \tag{1.217}$$

then

$$\delta\phi_A(x) \equiv D_{AB}(\Lambda)\phi_B(\Lambda^{-1}(x-a)) - \phi_B(x) \tag{1.218}$$
$$\simeq \left[1 - \frac{i}{2}\omega_{\rho\sigma}(S^{\rho\sigma})_{AB}\right]\phi_B\left((1-\omega^{\rho}{}_{\sigma})(x^{\sigma} - a^{\sigma})\right) - \phi_B(x) \tag{1.219}$$
$$\simeq \left[1 - \frac{i}{2}\omega_{\rho\sigma}(S^{\rho\sigma})_{AB}\right]\phi_B\left(x^{\rho} - a^{\rho} - \omega^{\rho}{}_{\sigma}x^{\sigma}\right) - \phi_B(x) \tag{1.220}$$
$$\simeq \left[1 - \frac{i}{2}\omega_{\rho\sigma}(S^{\rho\sigma})_{AB}\right]\left[\phi_B(x) + \partial_{\rho}\phi_B(x)\cdot(a^{\sigma} + \omega^{\rho}{}_{\sigma}x^{\sigma})\right] - \phi_B(x) \tag{1.221}$$
$$\simeq \phi_B(x) - \frac{i}{2}\omega_{\rho\sigma}(S^{\rho\sigma})_{AB}\phi_B(x) + (a^{\sigma} + \omega^{\rho}{}_{\sigma}x^{\sigma})\cdot\partial_{\rho}\phi_B(x) - \phi_B(x) \tag{1.222}$$
$$\simeq \left(+ia_{\mu}P^{\mu} - \frac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma}\right)\phi_B(x) - \frac{i}{2}\omega_{\rho\sigma}(S^{\rho\sigma})_{AB}\phi_B(x) \tag{1.223}$$

were we used the results from Problem 3.1b in the last step.

b) Noether theorem

(i) Infinitesimal field transformation (same space-time point)

$$\phi_i(x) \to \phi'_i(x) = \phi_i(x) + \delta\phi_i(x) \tag{1.224}$$

(ii) Assume symmetry of the action (Lagrangian can differ by a 4-divergence)

$$S \to S \tag{1.225}$$
$$\mathcal{L} \to \mathcal{L} + \partial_{\rho}X^{\rho} \tag{1.226}$$
$$\tag{1.227}$$

(iii) alternatively

$$\mathcal{L} \to \mathcal{L}(\phi_i + \delta\phi, \partial_{\rho}\phi_i + \delta\partial_{\rho}\phi_i) \tag{1.228}$$
$$\to \mathcal{L}(\phi_i, \partial_{\rho}\phi_i) + \frac{\partial\mathcal{L}}{\partial\phi_i}\delta\phi_i + \frac{\partial\mathcal{L}}{\partial(\partial_{\rho}\phi_i)}\delta(\partial_{\rho}\phi_i) \tag{1.229}$$
$$\to \mathcal{L}(\phi_i, \partial_{\rho}\phi_i) + \partial_{\rho}\left(\frac{\partial\mathcal{L}}{\partial(\partial_{\rho}\phi_i)}\right)\delta\phi_i + \frac{\partial\mathcal{L}}{\partial(\partial_{\rho}\phi_i)}\partial_{\rho}(\delta\phi_i) \tag{1.230}$$
$$\to \mathcal{L}(\phi_i, \partial_{\rho}\phi_i) + \partial_{\rho}\left(\frac{\partial\mathcal{L}}{\partial(\partial_{\rho}\phi_i)}\delta\phi_i\right) \tag{1.231}$$

(iv) compare terms to get Noether currents

$$j^{\rho} = \frac{\partial\mathcal{L}}{\partial(\partial_{\rho}\phi_i)}\delta\phi_i - X^{\rho} \quad \to \quad \partial_{\rho}j^{\rho} = 0 \tag{1.232}$$

(v) conserved quantity

$$Q = \int d^3x \; j^0 \tag{1.233}$$
$$= \int d^3x \; \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_i)} \delta\phi_i - X^0 \quad \rightarrow \quad \frac{d}{dt} Q = 0 \tag{1.234}$$

Now application of Poincare symmetry to scalar field

$$x'^\mu = \Lambda^\mu_{\ \nu} x^\nu + a^\mu \tag{1.235}$$
$$\rightarrow x'^\mu \simeq (\delta^\mu_\nu + \omega^\mu_{\ \nu}) x^\nu + a^\mu \tag{1.236}$$
$$\rightarrow x'^\mu \simeq x^\mu + \omega^\mu_{\ \nu} x^\nu + a^\mu \tag{1.237}$$
$$\phi'_i(x') = \phi_i(x) \tag{1.238}$$
$$\rightarrow \phi'_i(x) = \phi(\Lambda^{-1}(x-a)) \tag{1.239}$$
$$\rightarrow \phi'_i(x) \simeq \phi_i((1-\omega)(x-a)) \tag{1.240}$$
$$\rightarrow \phi'_i(x) \simeq \phi_i(x - \omega x - a)) \tag{1.241}$$
$$\rightarrow \phi'_i(x) \simeq \phi_i(x) - (\omega^\mu_{\ \nu} x^\nu) \partial_\mu \phi_i(x) - a^\mu(\partial_\mu \phi_i(x)) \tag{1.242}$$
$$\rightarrow \delta\phi_i = -(\omega^\mu_{\ \nu} x^\nu) \partial_\mu \phi_i(x) - a^\mu(\partial_\mu \phi_i(x)) \tag{1.243}$$
$$\mathcal{L}'(\Lambda x + a) = \mathcal{L}(x) \tag{1.244}$$
$$\rightarrow \delta\mathcal{L} = -(\omega^\mu_{\ \nu} x^\nu) \partial_\mu \mathcal{L} - a^\mu(\partial_\mu \mathcal{L}) \tag{1.245}$$
$$\rightarrow \delta\mathcal{L} = \partial_\mu(-\omega^\mu_{\ \nu} x^\nu \mathcal{L} - a^\mu \mathcal{L}) \tag{1.246}$$
$$= -\omega^\mu_{\ \nu}(\partial_\mu x^\nu)\mathcal{L} - \omega^\mu_{\ \nu} x^\nu \partial_\mu \mathcal{L} - a^\mu \partial_\mu \mathcal{L} \tag{1.247}$$
$$= -\omega^\mu_{\ \nu} \delta^\nu_\mu \mathcal{L} - \omega^\mu_{\ \nu} x^\nu \partial_\mu \mathcal{L} - a^\mu \partial_\mu \mathcal{L} \tag{1.248}$$
$$= -\underbrace{\omega^\mu_{\ \mu}}_{=0} \mathcal{L} - \omega^\mu_{\ \nu} x^\nu \partial_\mu \mathcal{L} - a^\mu \partial_\mu \mathcal{L} \tag{1.249}$$
$$\rightarrow X^\mu = -\omega^\mu_{\ \nu} x^\nu \mathcal{L} - a^\mu \mathcal{L} \tag{1.250}$$
$$\tag{1.251}$$

then

$$j^\rho = \frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi_i)} \delta\phi_i - X^\rho \tag{1.252}$$
$$= -\frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi_i)} (\partial_\mu \phi_i)(\omega^\mu_{\ \nu} x^\nu + a^\mu) + \omega^\rho_{\ \nu} x^\nu \mathcal{L} + a^\rho \mathcal{L} \tag{1.253}$$
$$= \left[\frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi_i)} (\partial^\mu \phi_i) x^\nu - \eta^{\rho\mu} x^\nu \mathcal{L}\right] \omega_{\mu\nu} + \left[\frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi_i)} (\partial_\mu \phi_i) - \eta^\mu_{\ \rho} \mathcal{L}\right] (-a^\rho) \tag{1.254}$$

and we define the Energy momentum tensor

$$T^{\rho\mu} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi_i)} (\partial^\mu \phi_i) - \eta^{\mu\rho} \mathcal{L} \tag{1.255}$$

and the Angular momentum density

$$\mathcal{M}^{\rho\mu\nu} \equiv -\frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi_i)} (\partial^\mu \phi_i) x^\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi_i)} (\partial^\nu \phi_i) x^\mu + \eta^{\rho\mu} x^\nu \mathcal{L} - \eta^{\rho\nu} x^\mu \mathcal{L} \tag{1.256}$$
$$= T^{\rho\nu} x^\mu - T^{\rho\mu} x^\nu \tag{1.257}$$

c) Conserved quantities

$$P^\mu = \int d^3x T^{0\mu} \tag{1.258}$$

$$= \int d^3x \frac{\partial\mathcal{L}}{\partial\dot{\phi}_i}(\partial^\mu\phi_i) - \eta^{\mu 0}\mathcal{L} \tag{1.259}$$

$$J^{\mu\nu} = \int d^3x \mathcal{M}^{0\mu\nu} \tag{1.260}$$

$$= \int d^3x (T^{0\nu}x^\mu - T^{0\mu}x^\nu) \tag{1.261}$$

1.9 Sheet 4 — Exercise 2 (Quantization of spin 0 field)

a) We select the scalar Klein-Gordon field with $\phi = \phi(\mathbf{x}, t)$

$$\mathcal{L}(\phi, \partial\phi) = \frac{1}{2}g^{\mu\nu}\partial_\nu\phi\partial_\mu\phi - \frac{m^2}{2}\phi^2 \tag{1.262}$$

Euler Lagrange

$$0 = \partial_\rho\left(\frac{\partial\mathcal{L}}{\partial(\partial_\rho\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} \tag{1.263}$$

$$\to \frac{1}{2}\partial_\rho g^{\mu\nu}(\partial_\mu\phi\delta^\rho_\nu + \partial_\nu\phi\delta^\rho_\mu) + m^2\phi = 0 \tag{1.264}$$

$$\to \frac{1}{2}\partial_\rho(\partial^\nu\phi\delta^\rho_\nu + \partial^\mu\phi\delta^\rho_\mu) + m^2\phi = 0 \tag{1.265}$$

$$\to \partial_\rho\partial^\rho\phi + m^2\phi = 0 \tag{1.266}$$

Conjugated momentum

$$\pi(\mathbf{x}, t) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} \tag{1.267}$$

$$\to \pi = \frac{1}{2}g^{00}(2\dot{\phi}) \tag{1.268}$$

$$\to \pi = \dot{\phi} \tag{1.269}$$

Hamilton density

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} \tag{1.270}$$

$$\to \mathcal{H} = \pi^2 - \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2 \tag{1.271}$$

$$\to \mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2 \tag{1.272}$$

Hamiltonian

$$H = \int d^3x \; \frac{1}{2}\pi(x)^2 + \frac{1}{2}(\nabla\phi(x))^2 + \frac{m^2}{2}\phi(x)^2 \tag{1.273}$$

$$= \int d^3x \; \frac{1}{2}\pi(x)^2 - \phi(x)\triangle\phi(x) + \frac{m^2}{2}\phi(x)^2 \tag{1.274}$$

Poisson brackets I

$$\{\phi(\mathbf{x},t),\phi(\mathbf{y},t)\} = \int d^3z \left(\frac{\partial\phi(\mathbf{x},t)}{\partial\phi(\mathbf{z},t)}\frac{\partial\phi(\mathbf{y},t)}{\partial\pi(\mathbf{z},t)} - \frac{\partial\phi(\mathbf{x},t)}{\partial\pi(\mathbf{z},t)}\frac{\partial\phi(\mathbf{y},t)}{\partial\phi(\mathbf{z},t)}\right) \tag{1.275}$$

$$= 0 \tag{1.276}$$

$$\{\phi(\mathbf{x},t),\pi(\mathbf{y},t)\} = \int d^3z \left(\frac{\partial\phi(\mathbf{x},t)}{\partial\phi(\mathbf{z},t)}\frac{\partial\pi(\mathbf{y},t)}{\partial\pi(\mathbf{z},t)} - \frac{\partial\phi(\mathbf{x},t)}{\partial\pi(\mathbf{z},t)}\frac{\partial\pi(\mathbf{y},t)}{\partial\phi(\mathbf{z},t)}\right) \tag{1.277}$$

$$= \delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{1.278}$$

$$\{\pi(\mathbf{x},t),\pi(\mathbf{y},t)\} = \int d^3z \left(\frac{\partial\pi(\mathbf{x},t)}{\partial\phi(\mathbf{z},t)}\frac{\partial\pi(\mathbf{y},t)}{\partial\pi(\mathbf{z},t)} - \frac{\partial\pi(\mathbf{x},t)}{\partial\pi(\mathbf{z},t)}\frac{\partial\pi(\mathbf{y},t)}{\partial\phi(\mathbf{z},t)}\right) \tag{1.279}$$

$$= 0 \tag{1.280}$$

Poisson brackets II

$$\{H,\phi(\mathbf{y},t)\} = \int d^3z \left(\frac{\partial H}{\partial\phi(\mathbf{z},t)}\frac{\partial\phi(\mathbf{y},t)}{\partial\pi(\mathbf{z},t)} - \frac{\partial H}{\partial\pi(\mathbf{z},t)}\frac{\partial\phi(\mathbf{y},t)}{\partial\phi(\mathbf{z},t)}\right) \tag{1.281}$$

$$= -\int d^3z \frac{\partial H}{\partial\pi(\mathbf{z},t)}\frac{\partial\phi(\mathbf{y},t)}{\partial\phi(\mathbf{z},t)} \tag{1.282}$$

$$= -\int d^3z \int d^3x \pi(\mathbf{x},t)\delta^{(3)}(\mathbf{x}-\mathbf{z})\delta^{(3)}(\mathbf{y}-\mathbf{z}) \tag{1.283}$$

$$= -\pi(\mathbf{y},t) \tag{1.284}$$

$$\{H,\pi(\mathbf{y},t)\} = m^2\phi(\mathbf{y},t) - \triangle\phi(\mathbf{y}) \tag{1.285}$$

Equations of motion

$$\dot\phi(\mathbf{y},t) = -\{H,\phi\} \qquad \to \qquad \dot\phi(\mathbf{y},t) = \pi(\mathbf{y},t) \tag{1.286}$$

$$\dot\pi(\mathbf{y},t) = -\{H,\phi\} \qquad \to \qquad \dot\pi(\mathbf{y},t) = -m^2\phi(\mathbf{y},t) + \triangle\phi(\mathbf{y}) \tag{1.287}$$

$$\to \ddot\phi(\mathbf{y},t) + \triangle\phi(\mathbf{y}) - m^2\phi(\mathbf{y},t) = 0 \tag{1.288}$$

$$\to \Box\phi(\mathbf{y}) + m^2\phi(\mathbf{y},t) = 0 \tag{1.289}$$

Quantization (obtained from Poisson brackets 1)

$$[\hat\phi(\mathbf{x},t),\hat\phi(\mathbf{y},t)] = 0 \tag{1.290}$$

$$[\hat\phi(\mathbf{x},t),\hat\pi(\mathbf{y},t)] = i\delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{1.291}$$

$$[\hat\pi(\mathbf{x},t),\hat\pi(\mathbf{y},t)] = 0 \tag{1.292}$$

$$\mathcal{H} = \mathcal{H}(\hat\phi,\hat\pi) \tag{1.293}$$

Time evolution in the Heisenberg picture (calculated from $\hat{\mathcal{H}}, \hat\phi$ and $\hat\pi$)

$$\dot{\hat\phi}(x) = i[\hat H,\hat\phi(x)] = \hat\pi(x) \tag{1.294}$$

$$\dot{\hat\pi}(x) = i[\hat H,\hat\pi(x)] = \triangle\hat\phi(x) - m^2\hat\phi(x) \tag{1.295}$$

Equations of motion (operator identity)

$$(\Box + m^2)\hat\phi(x) = 0 \tag{1.296}$$

This equation gives us an ansatz for the (free) field operators (Heisenberg picture)

$$\hat\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}(\hat a_{\mathbf{p}}e^{-ipx} + \hat a^\dagger_{\mathbf{p}}e^{ipx}) \tag{1.297}$$

$$\hat\phi(\mathbf{x},t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}(\hat a_{\mathbf{p}}e^{-i(E_pt-\mathbf{p}\cdot\mathbf{x})} + \hat a^\dagger_{\mathbf{p}}e^{i(E_pt-\mathbf{p}\cdot\mathbf{x})}) \tag{1.298}$$

With the definition of the field operators and their commutators we can calculate the commutators of the ladder operators

$$[a_{\mathbf{p}}, a_{\mathbf{q}}] = 0 \tag{1.299}$$

$$[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = (2\pi)^3 2E_{\mathbf{p}}\delta^{(3)}(\mathbf{p}-\mathbf{q}) \tag{1.300}$$

$$[a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}^\dagger] = 0 \tag{1.301}$$

Hamiltonian

$$\hat{H} = \frac{1}{2}\int d^3x\ \hat{\pi}(x)^2 + (\nabla\hat{\phi}(x))^2 + m^2\hat{\phi}(x)^2 \tag{1.302}$$

With

$$\int dx\ e^{ix(p-q)} = 2\pi\delta(p-q) \tag{1.303}$$

and $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2+m^2} = E_{-\mathbf{p}}$ we can do a brute force calculation

$$\hat{H}_3 = \frac{m^2}{2}\int d^3x\ \hat{\phi}(x)^2 \tag{1.304}$$

$$= \frac{m^2}{2}\int d^3x\ \left(\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}(\hat{a}_{\mathbf{p}}e^{-i(E_pt-\mathbf{p}\cdot\mathbf{x})} + \hat{a}_{\mathbf{p}}^\dagger e^{i(E_pt-\mathbf{p}\cdot\mathbf{x})})\right) \tag{1.305}$$

$$\cdot\left(\int \frac{d^3q}{(2\pi)^3}\frac{1}{2E_{\mathbf{q}}}(\hat{a}_{\mathbf{q}}e^{-i(E_qt-\mathbf{q}\cdot\mathbf{x})} + \hat{a}_{\mathbf{q}}^\dagger e^{i(E_qt-\mathbf{q}\cdot\mathbf{x})})\right) \tag{1.306}$$

$$= \frac{m^2}{8(2\pi)^6}\iint d^3pd^3q\frac{1}{E_{\mathbf{p}}E_{\mathbf{q}}}\int dx(e^{-i(E_p+E_q)t}\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{q}}e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} \tag{1.307}$$

$$+ e^{-i(E_p-E_q)t}\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{q}}^\dagger e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} \tag{1.308}$$

$$+ e^{-i(-E_p+E_q)t}\hat{a}_{\mathbf{p}}^\dagger\hat{a}_{\mathbf{q}}e^{i(-\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} \tag{1.309}$$

$$+ e^{-i(E_p+E_q)t}\hat{a}_{\mathbf{p}}^\dagger\hat{a}_{\mathbf{q}}^\dagger e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}) \tag{1.310}$$

$$= \frac{m^2}{8(2\pi)^3}\iint d^3pd^3q\frac{1}{E_{\mathbf{p}}E_{\mathbf{q}}}(e^{-i(E_p+E_q)t}\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{q}}\delta^{(3)}(-\mathbf{p}-\mathbf{q}) \tag{1.311}$$

$$+ e^{-i(E_p-E_q)t}\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{q}}^\dagger\delta^{(3)}(\mathbf{p}-\mathbf{q}) \tag{1.312}$$

$$+ e^{-i(-E_p+E_q)t}\hat{a}_{\mathbf{p}}^\dagger\hat{a}_{\mathbf{q}}\delta^{(3)}(-\mathbf{p}+\mathbf{q}) \tag{1.313}$$

$$+ e^{-i(E_p+E_q)t}\hat{a}_{\mathbf{p}}^\dagger\hat{a}_{\mathbf{q}}^\dagger\delta^{(3)}(\mathbf{p}+\mathbf{q})) \tag{1.314}$$

$$= \frac{m^2}{8(2\pi)^3}\int d^3p\frac{1}{E_{\mathbf{p}}^2}(e^{-i(E_p+E_{-p})t}\hat{a}_{\mathbf{p}}\hat{a}_{-\mathbf{p}} \tag{1.315}$$

$$+ e^{-i(E_p-E_p)t}\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{p}}^\dagger \tag{1.316}$$

$$+ e^{-i(-E_p+E_p)t}\hat{a}_{\mathbf{p}}^\dagger\hat{a}_{\mathbf{p}} \tag{1.317}$$

$$+ e^{-i(E_p+E_{-p})t}\hat{a}_{\mathbf{p}}^\dagger\hat{a}_{-\mathbf{p}}^\dagger) \tag{1.318}$$

$$= \frac{m^2}{8(2\pi)^3}\int d^3p\frac{1}{E_{\mathbf{p}}^2}(e^{-2iE_pt}\hat{a}_{\mathbf{p}}\hat{a}_{-\mathbf{p}} + \hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{p}}^\dagger + \hat{a}_{\mathbf{p}}^\dagger\hat{a}_{\mathbf{p}} + e^{-2iE_pt}\hat{a}_{\mathbf{p}}^\dagger\hat{a}_{-\mathbf{p}}^\dagger) \tag{1.319}$$

$$\tag{1.320}$$

Now we can repeat the calculation and only pick up a additional scalar product of the 3-momenta

$$\hat{H}_2 = \frac{1}{2}\int d^3x\ (\nabla\hat{\phi}(x))^2 \tag{1.321}$$

$$= \frac{1}{8(2\pi)^3}\iint d^3p d^3q \frac{1}{E_{\mathbf{p}}E_{\mathbf{q}}}(e^{-i(E_p+E_q)t}\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{q}}\delta^{(3)}(-\mathbf{p}-\mathbf{q})(i\mathbf{p})(i\mathbf{q}) \tag{1.322}$$

$$+ e^{-i(E_p-E_q)t}\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\delta^{(3)}(\mathbf{p}-\mathbf{q})(i\mathbf{p})(-i\mathbf{q}) \tag{1.323}$$

$$+ e^{-i(-E_p+E_q)t}\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{q}}\delta^{(3)}(-\mathbf{p}+\mathbf{q})(-i\mathbf{p})(i\mathbf{q}) \tag{1.324}$$

$$+ e^{-i(E_p+E_q)t}\hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\delta^{(3)}(\mathbf{p}+\mathbf{q})(-i\mathbf{p})(-i\mathbf{q})) \tag{1.325}$$

$$= \frac{1}{8(2\pi)^3}\int d^3p \frac{\mathbf{p}^2}{E^2_{\mathbf{p}}}(e^{-2iE_pt}\hat{a}_{\mathbf{p}}\hat{a}_{-\mathbf{p}} + \hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} + e^{-2iE_pt}\hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{-\mathbf{p}}) \tag{1.326}$$

and similarly

$$\hat{H}_1 = \frac{1}{2}\int d^3x\ \hat{\pi}(x)^2 \tag{1.327}$$

$$= \frac{1}{2}\int d^3x\ \dot{\hat{\phi}}(x)^2 \tag{1.328}$$

$$= \frac{1}{8(2\pi)^3}\iint d^3p d^3q \frac{1}{E_{\mathbf{p}}E_{\mathbf{q}}}(e^{-i(E_p+E_q)t}\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{q}}\delta^{(3)}(-\mathbf{p}-\mathbf{q})(-iE_{\mathbf{p}})(-iE_{\mathbf{q}}) \tag{1.329}$$

$$+ e^{-i(E_p-E_q)t}\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\delta^{(3)}(\mathbf{p}-\mathbf{q})(-iE_{\mathbf{p}})(iE_{\mathbf{q}}) \tag{1.330}$$

$$+ e^{-i(-E_p+E_q)t}\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{q}}\delta^{(3)}(-\mathbf{p}+\mathbf{q})(iE_{\mathbf{p}})(-iE_{\mathbf{q}}) \tag{1.331}$$

$$+ e^{-i(E_p+E_q)t}\hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\delta^{(3)}(\mathbf{p}+\mathbf{q})(iE_{\mathbf{p}})(iE_{\mathbf{q}})) \tag{1.332}$$

$$= \frac{1}{8(2\pi)^3}\int d^3p \frac{E^2_{\mathbf{p}}}{E^2_{\mathbf{p}}}(-e^{-2iE_pt}\hat{a}_{\mathbf{p}}\hat{a}_{-\mathbf{p}} + \hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} - e^{-2iE_pt}\hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{-\mathbf{p}}) \tag{1.333}$$

$$= \frac{1}{8(2\pi)^3}\int d^3p(-e^{-2iE_pt}\hat{a}_{\mathbf{p}}\hat{a}_{-\mathbf{p}} + \hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} - e^{-2iE_pt}\hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{-\mathbf{p}}) \tag{1.334}$$

Now we can sum it all up

$$\hat{H}_2 + \hat{H}_3 = \frac{1}{8(2\pi)^3}\int d^3p \frac{\mathbf{p}^2+m^2}{E^2_{\mathbf{p}}}(e^{-2iE_pt}\hat{a}_{\mathbf{p}}\hat{a}_{-\mathbf{p}} + \hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} + e^{-2iE_pt}\hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{-\mathbf{p}}) \tag{1.335}$$

$$= \frac{1}{8(2\pi)^3}\int d^3p(e^{-2iE_pt}\hat{a}_{\mathbf{p}}\hat{a}_{-\mathbf{p}} + \hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} + e^{-2iE_pt}\hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{-\mathbf{p}}) \tag{1.336}$$

and finally

$$\hat{H} = \hat{H}_1 + \hat{H}_2 + \hat{H}_3 \tag{1.337}$$

$$= \frac{1}{4(2\pi)^3}\int d^3p(\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}) \tag{1.338}$$

$$= \frac{1}{4}\int \frac{d^3p}{(2\pi)^3}\frac{2E_{\mathbf{p}}}{2E_{\mathbf{p}}}(\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}) \tag{1.339}$$

$$= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}}E_{\mathbf{p}}(\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}) \tag{1.340}$$

$$= \frac{1}{2}\int d^3\tilde{p}\ E_{\mathbf{p}}(\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}) \tag{1.341}$$

and with $[a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] = (2\pi)^3 2E_{\mathbf{p}} \delta^{(3)}(\mathbf{p}-\mathbf{q})$

$$\hat{H} = \int d^3\tilde{p}\; (E_{\mathbf{p}} \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}} + (2\pi)^3 E_{\mathbf{p}} \delta^{(3)}(0)) \tag{1.342}$$

$$= \int d^3\tilde{p}\; E_{\mathbf{p}} \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}} + \frac{1}{2}\int d^3 p \delta^{(3)}(0) \tag{1.343}$$

The calculation of the commutator is now simple

$$[\hat{H}, \hat{a}^\dagger_{\mathbf{q}}] = \int d^3\tilde{p}\; E_{\mathbf{p}} [\hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{q}}] \tag{1.344}$$

$$= \int d^3\tilde{p}\; E_{\mathbf{p}} (\hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}} \hat{a}^\dagger_{\mathbf{q}} - \hat{a}^\dagger_{\mathbf{q}} \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}}) \tag{1.345}$$

$$= \int d^3\tilde{p}\; E_{\mathbf{p}} (\hat{a}^\dagger_{\mathbf{p}} \hat{a}^\dagger_{\mathbf{q}} \hat{a}_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}} (2\pi)^3 2E_{\mathbf{p}} \delta^{(3)}(\mathbf{p}-\mathbf{q}) - \hat{a}^\dagger_{\mathbf{q}} \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}}) \tag{1.346}$$

$$= \int d^3\tilde{p}\; E_{\mathbf{p}} \hat{a}^\dagger_{\mathbf{p}} (2\pi)^3 2E_{\mathbf{p}} \delta^{(3)}(\mathbf{p}-\mathbf{q}) \tag{1.347}$$

$$= \int d^3 p\; E_{\mathbf{p}} \hat{a}^\dagger_{\mathbf{p}} \delta^{(3)}(\mathbf{p}-\mathbf{q}) \tag{1.348}$$

$$= E_{\mathbf{q}} \hat{a}^\dagger_{\mathbf{q}} \tag{1.349}$$

1.10 Sheet 4 — Exercise 3 (Spin-statistics connection of spin 0 field - NOT DONE YET)

1.11 Sheet 4 — Exercise 4 (Spinors)

- Poincare transformation

$$\bar{\Psi} \equiv \Psi^\dagger \gamma^0 \tag{1.350}$$

$$\Psi \to S(\Lambda)\Psi \qquad \text{Dirac spinor} \tag{1.351}$$

$$\bar{\Psi} \to \bar{\Psi} S^{-1}(\Lambda) \tag{1.352}$$

$$x' = \Lambda x + a \simeq (1+\omega)x + \epsilon \tag{1.353}$$

$$\Psi'(\Lambda x + a) = S(\Lambda)\Psi(x) \qquad \text{Dirac spinor field} \tag{1.354}$$

$$\to \Psi'(x) = \left(1 - \frac{i}{2}\omega_{\rho\sigma} S^{\rho\sigma}\right) \Psi(x - \omega x - \epsilon) \tag{1.355}$$

$$= \left(1 - \frac{i}{2}\omega_{\rho\sigma} S^{\rho\sigma}\right) (\Psi(x) - \epsilon^\mu \partial_\mu \Psi(x) - \omega_{\mu\nu} x^\mu \partial^\nu \Psi(x)) \tag{1.356}$$

$$= \Psi(x) - \frac{i}{2}\left(\omega_{\rho\sigma} S^{\rho\sigma}\Psi(x) + 2i\omega_{\mu\nu} x^\mu \partial^\nu \Psi(x)\right) - \epsilon^\mu \partial_\mu \Psi(x) \tag{1.357}$$

$$= \Psi(x) - \frac{i}{2}\omega_{\rho\sigma}\left(S^{\rho\sigma} + L^{\rho\sigma}\right)\Psi(x) - \epsilon^\mu \partial_\mu \Psi(x) \qquad \text{with} \qquad L^{\rho\sigma} = i(x^\rho \partial^\sigma - x^\sigma \partial^\rho) \tag{1.358}$$

- Noether theorem

$$\phi_i(x) \to \phi'_i(x) = \phi_i(x) + \delta\phi_i(x) \tag{1.359}$$

$$j^\rho = \frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi_i)} \delta\phi_i - X^\rho \quad \to \quad \partial_\rho j^\rho = 0 \tag{1.360}$$

then we use Poincare invariance

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\Psi \tag{1.361}$$

$$x' = \Lambda x + a \simeq x + \omega x + \epsilon \tag{1.362}$$

Implied spinor field change

$$\Psi'(\Lambda x + a) = S(\Lambda)\Psi(x) \qquad \rightarrow \qquad \Psi'(x) \simeq S(\Lambda)\Psi(x - \omega x - \epsilon) \tag{1.363}$$

$$\delta\Psi \equiv \Psi'(x) - \Psi(x) \tag{1.364}$$

$$= -\frac{i}{2}\omega_{\rho\sigma}\left(S^{\rho\sigma} + L^{\rho\sigma}\right)\Psi(x) - \epsilon^{\mu}\partial_{\mu}\Psi(x) \tag{1.365}$$

Implied Langrangian (scalar) change

$$\mathcal{L}'(\Lambda x + a) = \mathcal{L}(x) \qquad \rightarrow \qquad \mathcal{L}'(x) = \mathcal{L}(x - \omega x - \epsilon) \tag{1.366}$$

$$\delta\mathcal{L}(x) \equiv \mathcal{L}'(x) - \mathcal{L}(x) \tag{1.367}$$

$$= -(\omega^{\mu}_{\ \nu}x^{\nu})\partial_{\mu}\mathcal{L} - \epsilon^{\mu}(\partial_{\mu}\mathcal{L}) \tag{1.368}$$

$$= \partial_{\mu}(-\omega^{\mu}_{\ \nu}x^{\nu}\mathcal{L} - \epsilon^{\mu}\mathcal{L}) \tag{1.369}$$

$$= -\omega^{\mu}_{\ \nu}(\partial_{\mu}x^{\nu})\mathcal{L} - \omega^{\mu}_{\ \nu}x^{\nu}\partial_{\mu}\mathcal{L} - \epsilon^{\mu}\partial_{\mu}\mathcal{L} \tag{1.370}$$

$$= -\omega^{\mu}_{\ \nu}\delta^{\nu}_{\mu}\mathcal{L} - \omega^{\mu}_{\ \nu}x^{\nu}\partial_{\mu}\mathcal{L} - \epsilon^{\mu}\partial_{\mu}\mathcal{L} \tag{1.371}$$

$$= -\underbrace{\omega^{\mu}_{\ \mu}}_{=0}\mathcal{L} - \omega^{\mu}_{\ \nu}x^{\nu}\partial_{\mu}\mathcal{L} - \epsilon^{\mu}\partial_{\mu}\mathcal{L} \tag{1.372}$$

$$\rightarrow X^{\mu} = -\omega^{\mu}_{\ \nu}x^{\nu}\mathcal{L} - \epsilon^{\mu}\mathcal{L} \tag{1.373}$$

Now we can calculate

$$\frac{\partial\mathcal{L}}{\partial(\partial_{\rho}\Psi)} = \bar{\Psi}\gamma^{\rho} \tag{1.374}$$

$$\delta\Psi = -\frac{i}{2}\omega_{\mu\nu}\left(S^{\mu\nu} + L^{\mu\nu}\right)\Psi(x) - \epsilon^{\mu}\partial_{\mu}\Psi(x) \tag{1.375}$$

$$X^{\rho} = -\omega^{\mu}_{\ \nu}x^{\nu}\mathcal{L} - \epsilon^{\mu}\mathcal{L} \tag{1.376}$$

$$\rightarrow j^{\mu} = \bar{\Psi}i\gamma^{\rho}\left(\epsilon^{\mu}\partial_{\mu}\Psi - \frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu} + L^{\mu\nu})\Psi\right) + \epsilon^{\rho}\mathcal{L} + \omega^{\rho\sigma}x_{\sigma}\mathcal{L} \tag{1.377}$$

Conservation law

$$0 = \partial_{\rho}\left[\bar{\Psi}i\gamma^{\rho}\left(\epsilon_{\mu}\partial^{\mu}\Psi - \frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu} + L^{\mu\nu})\Psi\right) + \epsilon^{\rho}\mathcal{L} + \omega^{\rho\sigma}x_{\sigma}\mathcal{L}\right] \tag{1.378}$$

Translational invariance ($\sim \epsilon^{\mu}$ coeff)

$$T^{\rho}_{\ \mu} = \bar{\Psi}i\gamma^{\rho}\partial_{\mu}\Psi - g^{\rho}_{\ \mu}\mathcal{L} \tag{1.379}$$

Lorenz invariance ($\sim \omega^{\mu\nu}/2$ coeff)

$$\mathcal{M}^{\rho}_{\ \mu\nu} = \bar{\Psi}\gamma^{\rho}(S^{\mu\nu} + L^{\mu\nu})\Psi + (g^{\rho}_{\ \mu}x_{\nu} - g^{\rho}_{\ \nu}x_{\mu})\mathcal{L} \tag{1.380}$$

1.12 Sheet 5 — Exercise 1 (Spin 1/2 quantization)

Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.381}$$

$$\rightarrow [\sigma_j, \sigma_k] = 2i\epsilon_{jkl}\sigma_l \tag{1.382}$$

$$\rightarrow \{\sigma_j, \sigma_k\} = 2\delta_{jk} \tag{1.383}$$

General Dirac definitions

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}1_{n\times n} \qquad \text{Dirac algebra} \tag{1.384}$$

$$\rightarrow S^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu] \qquad n\text{-dimensional rep. of Lorentz algebra because ...} \tag{1.385}$$

$$\rightarrow [S^{\mu\nu}, S^{\rho\sigma}] = i(g^{\nu\rho}S^{\mu\sigma} - g^{\mu\rho}S^{\nu\sigma} - g^{\nu\sigma}S^{\mu\rho} + g^{\mu\sigma}S^{\nu\rho}) \tag{1.386}$$

1. **Weyl/chiral basis/representation** - for 4d-Minkowski space SO(1,3) - needs 4 γ matrices which are coincidentally 4×4 matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1_{2\times2} \\ 1_{2\times2} & 0 \end{pmatrix} \qquad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} \tag{1.387}$$

$$\rightarrow S^{0k} \equiv \frac{i}{4}[\gamma^0, \gamma^k] = -\frac{i}{2}\begin{pmatrix} \sigma_k & 0 \\ 0 & -\sigma_k \end{pmatrix} \tag{1.388}$$

$$\rightarrow S^{jk} \equiv \frac{i}{4}[\gamma^j, \gamma^k] = \frac{1}{2}\epsilon^{jkl}\begin{pmatrix} \sigma_l & 0 \\ 0 & \sigma_l \end{pmatrix} \tag{1.389}$$

we see that in the Weyl basis $(\gamma^0)^\dagger = \gamma^0, (\gamma^k)^\dagger = -\gamma^k$ and $(\gamma^0)^2 = 1_{4\times4}$ and $(\gamma^k)^2 = -1_{4\times4}$.

2. **Dirac basis/representation**

$$\gamma^0 = \begin{pmatrix} 1_{2\times2} & 0 \\ 0 & -1_{2\times2} \end{pmatrix} \qquad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} \tag{1.390}$$

we see that in the Dirac basis $(\gamma^0)^\dagger = \gamma^0, (\gamma^k)^\dagger = -\gamma^k$ and $(\gamma^0)^2 = 1_{4\times4}$ and $(\gamma^k)^2 = -1_{4\times4}$.

For the **Weyl** and the **Dirac** representation we can can show (using the hermiticity relations)

$$\gamma^0 \underbrace{(\gamma^0)\gamma^0}_{=1_{2\times2}} = \gamma^0 = (\gamma^0)^\dagger \tag{1.391}$$

$$\gamma^0 \underbrace{(\gamma^k)\gamma^0}_{=-\gamma^0\gamma^k} = \underbrace{\gamma^0\gamma^0}_{=1_{2\times2}}\gamma^k = -\gamma^k = (\gamma^k)^\dagger \tag{1.392}$$

$$\rightarrow \gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger \tag{1.393}$$

With

$$[\gamma^\mu, S^{\rho\sigma}] = \frac{i}{4}(\gamma^\mu(\gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma^\rho) - (\gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma^\rho)\gamma^\mu) \tag{1.394}$$

$$= \frac{i}{4}(\gamma^\mu\gamma^\rho\gamma^\sigma - \gamma^\mu\gamma^\sigma\gamma^\rho - \gamma^\rho\gamma^\sigma\gamma^\mu + \gamma^\sigma\gamma^\rho\gamma^\mu) \tag{1.395}$$

$$= \frac{i}{4}((2g^{\mu\rho}\gamma^\sigma - \gamma^\rho\gamma^\mu\gamma^\sigma) - (2g^{\mu\sigma}\gamma^\rho - \gamma^\sigma\gamma^\mu\gamma^\rho) - (2g^{\mu\sigma}\gamma^\rho - \gamma^\rho\gamma^\mu\gamma^\sigma) + (2g^{\mu\rho}\gamma^\sigma - \gamma^\sigma\gamma^\mu\gamma^\rho)) \tag{1.396}$$

$$= \frac{i}{4}(2g^{\mu\rho}\gamma^\sigma - 2g^{\mu\sigma}\gamma^\rho - 2g^{\mu\sigma}\gamma^\rho + 2g^{\mu\rho}\gamma^\sigma) \tag{1.397}$$

$$= i(g^{\mu\rho}\gamma^\sigma - g^{\mu\sigma}\gamma^\rho) \tag{1.398}$$

$$= (i[g^{\rho\mu}\delta^\sigma_\nu - g^{\rho\nu}\delta^\sigma_\mu])\gamma^\nu \tag{1.399}$$

$$= (J^{\rho\sigma})^\mu{}_\nu\gamma^\nu \tag{1.400}$$

$$\rightarrow [\gamma^\mu, S^{\rho\sigma}] = (J^{\rho\sigma})^\mu{}_\nu\gamma^\nu \tag{1.401}$$

we obtain

$$
\begin{aligned}
S(\Lambda) &\simeq 1_{4\times 4} + \frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma} &(1.402)\\
S^{-1}(\Lambda)\gamma^\mu S(\Lambda) &\simeq (1_{4\times 4} - \frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma})\gamma^\mu(1_{4\times 4} + \frac{i}{2}\omega_{\lambda\kappa}S^{\lambda\kappa}) &(1.403)\\
&\simeq \gamma^\mu + \frac{i}{2}[\gamma^\mu, \omega_{\rho\sigma}S^{\rho\sigma}] &(1.404)\\
&= \gamma^\mu + \frac{i}{2}\omega_{\rho\sigma}[\gamma^\mu, S^{\rho\sigma}] &(1.405)\\
&= (1 + \frac{i}{2}\omega_{\sigma\rho}J^{\sigma\rho})\gamma^\mu &(1.406)\\
&\simeq \Lambda^\mu{}_\nu\gamma^\nu &(1.407)\\
&\to S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu &(1.408)
\end{aligned}
$$

$$
\begin{aligned}
\gamma^0 S^{\rho\sigma}\gamma^0 &= \frac{i}{4}(\gamma^0\gamma^\rho\gamma^\sigma\gamma^0 - \gamma^0\gamma^\sigma\gamma^\rho\gamma^0) &(1.409)\\
&= \ldots &(1.410)\\
&= -\frac{i}{4}((\gamma^\rho)^\dagger(\gamma^\sigma)^\dagger - (\gamma^\sigma)^\dagger(\gamma^\rho)^\dagger) &(1.411)\\
&= (S^{\rho\sigma})^\dagger &(1.412)
\end{aligned}
$$

And

$$
\begin{aligned}
\bar\Psi &\equiv \Psi^\dagger\gamma^0 &(1.413)\\
\Psi &\to S(\Lambda)\Psi \qquad \text{Dirac spinor} &(1.414)\\
\bar\Psi &\to \bar\Psi S^{-1}(\Lambda) &(1.415)\\
x' &= \Lambda x + a \simeq (1+\omega)x + \epsilon &(1.416)\\
\Psi'(\Lambda x + a) &= S(\Lambda)\Psi(x) \qquad \text{Dirac spinor field} &(1.417)\\
\to \Psi'(x) &= \left(1 - \frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}\right)\Psi(x - \omega x - \epsilon) &(1.418)\\
&= \left(1 - \frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}\right)(\Psi(x) - \epsilon^\mu\partial_\mu\Psi(x) - \omega_{\mu\nu}x^\mu\partial^\nu\Psi(x)) &(1.419)\\
&= \Psi(x) - \frac{i}{2}\left(\omega_{\rho\sigma}S^{\rho\sigma}\Psi(x) + 2i\omega_{\mu\nu}x^\mu\partial^\nu\Psi(x)\right) - \epsilon^\mu\partial_\mu\Psi(x) &(1.420)\\
&= \Psi(x) - \frac{i}{2}\omega_{\rho\sigma}\left(S^{\rho\sigma} + L^{\rho\sigma}\right)\Psi(x) - \epsilon^\mu\partial_\mu\Psi(x) \qquad \text{with} \qquad L^{\rho\sigma} = i(x^\rho\partial^\sigma - x^\sigma\partial^\rho) &(1.421)
\end{aligned}
$$

Now we do the exercises using the formulas above

-
$$
\begin{aligned}
\bar\Psi_1\gamma^\mu\Psi_2 &\to \bar\Psi\underbrace{S^{-1}(\Lambda)\gamma^\mu S(\Lambda)}_{=\Lambda^\mu{}_\nu\gamma^\nu}\Psi_2 &(1.422)\\
&= \bar\Psi\Lambda^\mu{}_\nu\gamma^\nu\Psi_2 &(1.423)\\
&= \Lambda^\mu{}_\nu(\bar\Psi\gamma^\nu\Psi_2) &(1.424)
\end{aligned}
$$

- First we summarize (important $\gamma^\mu\partial_\mu = \gamma^0\partial_0 + \gamma^1\partial_1 + \gamma^2\partial_2 + \gamma^3\partial_3$)

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi \tag{1.425}$$
$$\rightarrow \pi = \frac{\partial\mathcal{L}}{\partial\dot{\Psi}} = \bar{\Psi}i\gamma^0 \qquad \rightarrow \qquad \bar{\Psi} = -i\pi\gamma^0 \tag{1.426}$$
$$\mathcal{H} = \pi\dot{\Psi} + \bar{\pi}\dot{\bar{\Psi}} - \mathcal{L} \tag{1.427}$$
$$= \bar{\Psi}i\gamma^0\dot{\Psi} - \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi \tag{1.428}$$
$$= \bar{\Psi}(-i\gamma^k\partial_k + m)\Psi \tag{1.429}$$
$$= -i\pi\gamma^0(-i\gamma^k\partial_k + m)\Psi \tag{1.430}$$

$$[\hat{\Psi}(\mathbf{x},t), \hat{\pi}(\mathbf{y},t)]_\pm = i\delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{1.431}$$
$$\rightarrow \hat{\Psi}(\mathbf{x},t)\hat{\pi}(\mathbf{y},t) = \mp\hat{\pi}(\mathbf{y},t)\hat{\Psi}(\mathbf{x},t) + i\delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{1.432}$$

Using the anticommutator for the fields

$$\rightarrow \dot{\hat{\Psi}}(x) = i[\hat{H}, \hat{\Psi}(x)] \tag{1.433}$$
$$= i\int d^3y \left(-i\pi(y)\gamma^0(-i\gamma^k\partial_k + m)\Psi(y)\Psi(x)\right) \tag{1.434}$$
$$- i\int d^3y \left(-i\Psi(x)\pi(y)\gamma^0(-i\gamma^k\partial_k + m)\Psi(y)\right) \tag{1.435}$$
$$= i\int d^3y \left(-i\pi(y)\gamma^0(-i\gamma^k\partial_k + m)\Psi(y)\Psi(x)\right) \tag{1.436}$$
$$- i\int d^3y \left(-i\pi(y)\Psi(x)\gamma^0(-i\gamma^k\partial_k + m)\Psi(y)\right) \tag{1.437}$$
$$+ i\int d^3y\, i(i\delta^{(3)}(\mathbf{x}-\mathbf{y}))\gamma^0(-i\gamma^k\partial_k + m)\Psi(y) \tag{1.438}$$
$$= i^3\int d^3y\, \delta^{(3)}(\mathbf{x}-\mathbf{y})\gamma^0\left(-i\gamma^k\partial_k + m\right)\hat{\Psi}(y) \tag{1.439}$$
$$= -i\gamma^0\left(-i\gamma^k\partial_k + m\right)\hat{\Psi}(x) \tag{1.440}$$

- With

$$\not{p}u = mu \tag{1.441}$$
$$\not{p}v = -mv \tag{1.442}$$
$$\hat{\Psi}(x) = \int d\tilde{p}\sum_{s=\pm1/2}\left(e^{-ipx}u(p,s)a(p,s) + e^{ipx}v(p,s)b^\dagger(p,s)\right) \tag{1.443}$$

and the commutation relations

$$[a(p,s), a(p's')]_\pm = 0 \tag{1.444}$$
$$[a(p,s), b(p's')]_\pm = 0 \tag{1.445}$$
$$[b(p,s), b(p's')]_\pm = 0 \tag{1.446}$$
$$[a(p,s), b^\dagger(p's')]_\pm = 0 \tag{1.447}$$
$$[a(p,s), a^\dagger(p's')]_\pm = (2\pi)^3 2p^0\delta(\mathbf{p}-\mathbf{p}')\delta^{(3)}_{ss'} \tag{1.448}$$
$$[b^\dagger(p,s), b(p's')]_\pm = (2\pi)^3 2p^0\delta(\mathbf{p}-\mathbf{p}')\delta^{(3)}_{ss'} \tag{1.449}$$

as well as

$$\sum_s u(p,s)\bar{u}(p,s) \equiv \sum_s u(p,s)u^\dagger(p,s)\gamma^0 = \not{p} + m \tag{1.450}$$

$$\to \sum_s u(p,s)u^\dagger(p,s) = (\not{p} + m)\gamma^0 \tag{1.451}$$

$$\sum_s v(p,s)\bar{v}(p,s) \equiv \sum_s v(p,s)v^\dagger(p,s)\gamma^0 = \not{p} - m \tag{1.452}$$

$$\to \sum_s v(p,s)v^\dagger(p,s) = (\not{p} - m)\gamma^0 \tag{1.453}$$

We can calculate ($x^0 = y^0$)

$$[\hat{\psi}(x), \hat{\psi}^\dagger(y)]_\pm \tag{1.454}$$

$$= \int d\tilde{p} \int d\tilde{p}' \sum_{s,s'} \left[e^{-ipx}u(p,s)a(p,s) + e^{ipx}v(p,s)b^\dagger(p,s), e^{ip'y}a^\dagger(p',s')u^\dagger(p',s') + e^{-ip'y}b(p',s')v^\dagger(p',s') \right]_\pm \tag{1.455}$$

$$= \int d\tilde{p} \int d\tilde{p}' \sum_{s,s'} e^{-i(px-p'y)}u(p,s)u^\dagger(p',s')[a(p,s), a^\dagger(p',s')]_\pm \tag{1.456}$$

$$+ e^{-i(px+p'y)}u(p,s)v^\dagger(p',s')[a(p,s), b(p',s')]_\pm \tag{1.457}$$

$$+ e^{i(px+p'y)}v(p,s)u^\dagger(p',s')[b^\dagger(p,s), a^\dagger(p',s')]_\pm \tag{1.458}$$

$$+ e^{i(px-p'y)}v(p,s)v^\dagger(p',s')[b^\dagger(p,s), b(p',s')]_\pm \tag{1.459}$$

$$\overset{\text{comm}}{=} \int d\tilde{p} \int d\tilde{p}' \sum_{s,s'} (e^{-i(px-p'y)}u(p,s)u^\dagger(p',s') + e^{i(px-p'y)}v(p,s)v^\dagger(p',s'))(2\pi)^3 2p^0 \delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta_{ss'} \tag{1.460}$$

$$= \int d\tilde{p} \sum_s \, (e^{-ip(x-y)}u(p,s)u^\dagger(p,s) + e^{ip(x-y)}v(p,s)v^\dagger(p,s)) \tag{1.461}$$

$$= \int d\tilde{p} \, (e^{-ip(x-y)}(\not{p}+m)\gamma^0 + e^{ip(x-y)}(\not{p}-m)\gamma^0) \tag{1.462}$$

$$= \int d\tilde{p} \, (e^{-i(p^0(x^0-y^0)-\mathbf{p}(\mathbf{x}-\mathbf{y}))}(\not{p}+m)\gamma^0 + e^{i(p^0(x^0-y^0)-\mathbf{p}(\mathbf{x}-\mathbf{y}))}(\not{p}-m)\gamma^0) \tag{1.463}$$

$$\overset{(x^0=y^0)}{=} \int d\tilde{p} \; e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})}(\not{p}+m)\gamma^0 + \int d\tilde{p} \; e^{-i\mathbf{p}(\mathbf{x}-\mathbf{y})}(\not{p}-m)\gamma^0 \tag{1.464}$$

$$\overset{\mathbf{p}\to-\mathbf{p}}{=} \int d\tilde{p} \; e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})}(p^0\gamma_0 + p^k\gamma_k + m)\gamma^0 + \int d\tilde{p} \; e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})}(p^0\gamma_0 - p^k\gamma_k - m)\gamma^0 \tag{1.465}$$

$$\overset{(x^0=y^0)}{=} \int d\tilde{p} \; e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})}(\not{p}+m)\gamma^0 + \int d\tilde{p} \; e^{-i\mathbf{p}(\mathbf{x}-\mathbf{y})}(\not{p}-m)\gamma^0 \tag{1.466}$$

$$= \int \frac{d^3p}{(2\pi)^3 2p^0} \; e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})} 2p^0(\gamma_0)^2 \tag{1.467}$$

$$= \int \frac{d^3p}{(2\pi)^3} \; e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})} \tag{1.468}$$

$$= \delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{1.469}$$

- With

$$\hat{H} = \int d\tilde{p} \, p^0 \sum_s (a^\dagger(p,s)a(p,s) - b(p,s)b^\dagger(p,s)) \tag{1.470}$$

we calculate

$$[\hat{H}, a^\dagger(p', s')] \tag{1.471}$$

$$= \int d\tilde{p}\, p^0 \sum_s ([a^\dagger(p,s)a(p,s), a^\dagger(p',s')] - [b(p,s)b^\dagger(p,s), a^\dagger(p',s')]) \tag{1.472}$$

$$= \int d\tilde{p}\, p^0 \sum_s ([a^\dagger_{p,s}a_{p,s}, a^\dagger_{p',s'}] - [b_{p,s}b^\dagger_{p,s}, a^\dagger_{p',s'}]) \tag{1.473}$$

$$= \int d\tilde{p}\, p^0 \sum_s (a^\dagger_{p,s} \underbrace{a_{p,s}a^\dagger_{p',s'}}_{=a^\dagger_{p',s'}a_{p,s}+(2\pi)^3 2p^0\delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta_{ss'}} - a^\dagger_{p',s'}a^\dagger_{p,s}a_{p,s} - b_{p,s}b^\dagger_{p,s}a^\dagger_{p',s'} + a^\dagger_{p',s'}b_{p,s}b^\dagger_{p,s}) \tag{1.474}$$

$$= \int d\tilde{p}\, p^0 \sum_s (a^\dagger_{p,s}a^\dagger_{p',s'}a_{p,s} + a^\dagger_{p,s}(2\pi)^3 2p^0\delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta_{ss'} - a^\dagger_{p',s'}a^\dagger_{p,s}a_{p,s} - a^\dagger_{p',s'}b_{p,s}b^\dagger_{p,s} + a^\dagger_{p',s'}b_{p,s}b^\dagger_{p,s}) \tag{1.475}$$

$$= \int \frac{d^3p}{(2\pi)^3 2p^0}\, p^0 \sum_s a^\dagger_{p,s}(2\pi)^3 2p^0\delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta_{ss'} \tag{1.476}$$

$$= p'^0 a^\dagger_{p',s'} \tag{1.477}$$

and identically

$$[\hat{H}, b^\dagger(p', s')] \tag{1.478}$$

$$= \int d\tilde{p}\, p^0 \sum_s ([a^\dagger(p,s)a(p,s), b^\dagger(p',s')] - [b(p,s)b^\dagger(p,s), b^\dagger(p',s')]) \tag{1.479}$$

$$= \int d\tilde{p}\, p^0 \sum_s ([a^\dagger_{p,s}a_{p,s}, b^\dagger_{p',s'}] - [b_{p,s}b^\dagger_{p,s}, b^\dagger_{p',s'}]) \tag{1.480}$$

$$= \int d\tilde{p}\, p^0 \sum_s (a^\dagger_{p,s}a_{p,s}b^\dagger_{p',s'} - b^\dagger_{p',s'}a^\dagger_{p,s}a_{p,s} - \underbrace{b_{p,s}b^\dagger_{p',s'}}_{=b^\dagger_{p',s'}b_{p,s}+(2\pi)^3 2p^0\delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta_{ss'}} b^\dagger_{p,s} + b^\dagger_{p',s'}b_{p,s}b^\dagger_{p,s}) \tag{1.481}$$

$$= \int d\tilde{p}\, p^0 \sum_s (-\underbrace{b_{p,s}b^\dagger_{p',s'}}_{=b^\dagger_{p',s'}b_{p,s}-(2\pi)^3 2p^0\delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta_{ss'}} b^\dagger_{p,s} + b^\dagger_{p',s'}b_{p,s}b^\dagger_{p,s}) \tag{1.482}$$

$$= \int d\tilde{p}\, p^0 \sum_s (2\pi)^3 2p^0\delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta_{ss'}b^\dagger_{p,s} \tag{1.483}$$

$$= p'^0 b^\dagger_{p',s'} \tag{1.484}$$

1.13 Sheet 6 — Exercise 1 (Negative norm and probabilities)

Assuming there exists a ψ with

$$\langle\psi|\psi\rangle < 0 \tag{1.485}$$

Now assume the existence of a hermitian operator $\hat{A}$ with a discrete spectrum

$$\hat{A}|a_k\rangle = a_k|a_k\rangle \tag{1.486}$$

with

$$1 = \sum_k |a_k\rangle\langle a_k| \tag{1.487}$$

Now we calculate

$$\begin{aligned} 1|\psi\rangle &= \sum_k |a_k\rangle\langle a_k|\psi\rangle && (1.488)\\ \rightarrow \langle\psi|\psi\rangle &= \sum_k \langle\psi|a_k\rangle\langle a_k|\psi\rangle && (1.489)\\ \rightarrow \langle\psi|\psi\rangle &= \sum_k p_k && (1.490) \end{aligned}$$

As $\langle\psi|\psi\rangle < 0$ by assumption the sum of probabilities on the right hand side must contain negative terms p_k.

1.14 Sheet 6 — Exercise 2 (Polarization vectors for $m > 0$ I)

With $p = (\sqrt{m^2 + \mathbf{p}^2}, \mathbf{p})$

$$\begin{aligned} 0 &= p_\mu a^\mu_{\mathbf{p}} && (1.491)\\ &= p_\mu \sum_{\lambda=1,2,3} \underbrace{\epsilon^\mu(p,\lambda)}_{\text{3 lin. indep. 4-comp. vectors,}} \underbrace{a(p,\lambda)}_{\text{3 operators}} && (1.492)\\ &\rightarrow p_\mu \epsilon^\mu(p,\lambda) \qquad \lambda = 1,2,3 && (1.493) \end{aligned}$$

then

$$\begin{aligned} \epsilon^\mu(p,1) &= (0, \vec{\epsilon}_1) && (1.494)\\ \epsilon^\mu(p,2) &= (0, \vec{\epsilon}_2) && (1.495)\\ \epsilon^\mu(p,3) &= \left(|\mathbf{p}|, \frac{E\mathbf{p}}{|\mathbf{p}|}\right)\frac{1}{m} && (1.496) \end{aligned}$$

with $\vec{\epsilon}_i \cdot \mathbf{p} = 0$ and $\vec{\epsilon}_i \cdot \vec{\epsilon}_j = \delta_{ij}$.

The obvious choice for this fourth vector would be p^μ/m

(a) rest frame $p = (m, 0, 0, 0)$

$$\begin{aligned} \epsilon^\mu(p,0) &= (1,0,0,0) = \frac{p}{m} && (1.497)\\ \epsilon^\mu(p,1) &= (0,1,0,0) && (1.498)\\ \epsilon^\mu(p,2) &= (0,0,1,0) && (1.499)\\ \epsilon^\mu(p,3) &= (0,0,0,1) && (1.500) \end{aligned}$$

(b) Moving in z direction $p = (\sqrt{m^2 + p_z^2}, 0, 0, p_z)$

$$\epsilon^\mu(p,0) = (\sqrt{m^2 + p_z^2}, 0, 0, p_z)\frac{1}{m} = \frac{p}{m} \tag{1.501}$$
$$\epsilon^\mu(p,1) = (0,1,0,0) \tag{1.502}$$
$$\epsilon^\mu(p,2) = (0,0,1,0) \tag{1.503}$$
$$\epsilon^\mu(p,3) = (p_z, 0, 0, \sqrt{m^2 + p_z^2})\frac{1}{m} \tag{1.504}$$

(c) rest frame $p = (\sqrt{m^2 + \mathbf{p}^2}, p_x, p_y, p_z)$

1.15 Sheet 6 — Exercise 3 (Polarization vectors for $m > 0$ II)

Boosting a resting particle of mass m in x direction

$$p = \begin{pmatrix} \cosh\beta_x & \sinh\beta_x & 0 & 0 \\ \sinh\beta_x & \cosh\beta_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{1.505}$$
$$= \begin{pmatrix} \frac{m}{\sqrt{1-v_x^2}} \\ \frac{mv_x}{\sqrt{1-v_x^2}} \\ 0 \\ 0 \end{pmatrix} \tag{1.506}$$
$$= (\sqrt{m^2 + p_x^2}, p_x, 0, 0)^T \tag{1.507}$$

then the three polarization vectors are given by

$$\epsilon^\mu(p,1) = (0,1,0,0) \tag{1.508}$$
$$\epsilon^\mu(p,2) = (0,0,1,0) \tag{1.509}$$
$$\epsilon^\mu(p,3) = (p_x, \sqrt{m^2 + p_x^2}, 0, 0)\frac{1}{m} \tag{1.510}$$

The boost in z-direction is given by

$$\Lambda^\nu{}_\mu = \begin{pmatrix} \cosh\beta_z & 0 & 0 & \sinh\beta_z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\beta_z & 0 & 0 & \cosh\beta_z \end{pmatrix} \tag{1.511}$$

Then

$$(\Lambda p)^\mu = \Lambda^\nu{}_\mu p^\mu \tag{1.512}$$
$$= \left(\frac{m}{\sqrt{1-v_x^2}\sqrt{1-v_x^2}}, \frac{mv_x}{\sqrt{1-v_x^2}}, 0, \frac{mv_z}{\sqrt{1-v_x^2}\sqrt{1-v_x^2}} \right) \tag{1.513}$$
$$\equiv (E, p_x, p_y, p_z) \tag{1.514}$$

and

$$(\Lambda\epsilon_{\lambda=1})^{\mu} = \Lambda^{\nu}{}_{\mu}\epsilon^{\mu}_{\lambda=1} \tag{1.515}$$
$$= (0,1,0,0) \tag{1.516}$$
$$(\Lambda\epsilon_{\lambda=2})^{\mu} = \Lambda^{\nu}{}_{\mu}\epsilon^{\mu}_{\lambda=2} \tag{1.517}$$
$$= (0,0,1,0) \tag{1.518}$$
$$(\Lambda\epsilon_{\lambda=3})^{\mu} = \Lambda^{\nu}{}_{\mu}\epsilon^{\mu}_{\lambda=3} \tag{1.519}$$
$$= \left(\frac{p_x}{\sqrt{1-v_z^2}}, \sqrt{m^2+p_x^2}, 0, \frac{p_x v_z}{1-v_z^2}\right) \tag{1.520}$$
$$= \left(\frac{m v_x}{\sqrt{1-v_x^2}\sqrt{1-v_z^2}}, \frac{m}{\sqrt{1-v_x^2}}, 0, \frac{m v_x v_z}{\sqrt{1-v_x^2}\sqrt{1-v_z^2}}\right) \tag{1.521}$$

and therefore

$$\epsilon^{\mu}_{\lambda=3}(\Lambda p) = \left(|\mathbf{p}|, \frac{E\mathbf{p}}{|\mathbf{p}|}\right)\frac{1}{m} \tag{1.522}$$
$$= (\frac{\sqrt{1-(1-v_x^2)(1-v_z^2)}}{\sqrt{1-v_x^2}\sqrt{1-v_z^2}}, \frac{-v_x}{\sqrt{1-v_x^2}\sqrt{1-(1-v_x^2)(1-v_z^2)}}, 0, \frac{-v_z}{\sqrt{1-v_x^2}\sqrt{1-v_z^2}\sqrt{1-(1-v_x^2)(1-v_z^2)}}) \tag{1.523}$$
$$\epsilon^{\mu}_{\lambda=1}(\Lambda p) = (0,0,1,0) \tag{1.524}$$
$$\epsilon^{\mu}_{\lambda=2}(\Lambda p) = (0,...) \tag{1.525}$$

1.16 Sheet 7 — Exercise 1 (Polarization vectors for $m = 0$ I)

1.17 Sheet 7 — Exercise 2 (Polarization vectors for $m = 0$ II)

1.18 Sheet 7 — Exercise 3 (Physical states and gauge transformations)

1.19 Sheet 8 — Exercise 1 (Short exercise — propagators)

With the definition

$$T\psi(x)\bar{\psi}(y) = \theta(x^0 - y^0)(\psi(x)\bar{\psi}(y)) - \theta(y^0 - x^0)(\bar{\psi}(y)\psi(x)) \tag{1.526}$$

we see

$$(i\gamma^\mu\partial_\mu - m)\langle 0|T\psi(x)\bar{\psi}(y)|0\rangle = \tag{1.527}$$

1.20 Sheet 8 — Exercise 2 (Final long project — free fields)

The Lagrange density is given by three parts

$$\mathcal{L} = \mathcal{L}_{\text{GI}} + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{FP}} \tag{1.528}$$

the gauge invariant part, the gauge fixing and the Faddeev-Popov ghost term

$$\mathcal{L}_{\text{GI}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^\mu\partial_\mu\psi, \qquad \mathcal{L}_{\text{GF}} = B\cdot\partial^\mu A_\mu + \frac{\xi}{2}B^2, \qquad \mathcal{L}_{\text{FP}} = \partial^\mu\bar{c}\partial_\mu c \tag{1.529}$$

(a) • Equations of motions for B

$$\frac{\partial\mathcal{L}}{\partial B} = \partial^\mu A_\mu + \xi B, \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\mu B)} = 0 \tag{1.530}$$

$$\to \pi_B = \frac{\partial\mathcal{L}}{\partial\dot{B}} = 0 \tag{1.531}$$

$$\to \quad \partial^\mu A_\mu + \xi B = 0 \tag{1.532}$$

• Equations of motions for c

$$\frac{\partial\mathcal{L}}{\partial c} = 0, \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\mu c)} = \partial^\mu\bar{c} \tag{1.533}$$

$$\frac{\partial\mathcal{L}}{\partial\bar{c}} = 0, \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{c})} = \partial^\mu c \tag{1.534}$$

$$\to \pi_c = \partial^0\bar{c} \tag{1.535}$$

$$\to \pi_{\bar{c}} = \partial^0 c \tag{1.536}$$

$$\to \quad \partial_\mu\partial^\mu\bar{c} = 0 \tag{1.537}$$

$$\to \quad \partial_\mu\partial^\mu c = 0 \tag{1.538}$$

• Equations of motions for A

$$\frac{\partial\mathcal{L}}{\partial A_\mu} = 0, \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\mu)} = F^{\mu\nu} + B\delta^{\mu\nu} \tag{1.539}$$

$$\to (\pi_A)_\mu = F^{\mu 0} \tag{1.540}$$

$$\to (\pi_A)_0 = B \tag{1.541}$$

$$\to \quad \partial_\mu F^{\mu\nu} + \partial^\nu B = 0 \tag{1.542}$$

- Equations of motions for $\psi, \bar{\psi}$

$$\frac{\partial \mathcal{L}}{\partial \psi} = 0, \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = i\gamma^\mu \bar{\psi} \tag{1.543}$$

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = i\gamma^\mu \partial_\mu \psi, \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} = 0 \tag{1.544}$$

$$\rightarrow \pi_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\gamma^0 \bar{\psi}(x) = i\psi^\dagger(x) \tag{1.545}$$

$$\rightarrow \bar{\pi}_{\bar{\psi}} = 0 \tag{1.546}$$

$$\rightarrow \quad (i\gamma^\mu \partial_\mu)\psi(x) = 0 \tag{1.547}$$

$$\rightarrow \quad (i\gamma^\mu \partial_\mu)\bar{\psi}(x) = 0 \tag{1.548}$$

(b)

(c)

(d)

(e)

(f)

(g)

(h)

1.21 Sheet 11 — Exercise 3 (Integrals in D dimensions)

a) Let's start with

$$\pi^{D/2} \equiv \left(\int e^{-k^2} dk \right)^D \tag{1.549}$$
$$= \int e^{-k_1^2} dk_1 ... \int e^{-k_D^2} dk_D \tag{1.550}$$
$$= \int e^{-k_1^2 - ..k_D^2} dk_1 ... dk_D \tag{1.551}$$
$$= \int_0^\infty \int_{\partial S_D} e^{-k^2} k^{D-1} dk d\Omega_D \tag{1.552}$$
$$= \int_0^\infty e^{-k^2} k^{D-1} dk \cdot \int_{\partial S_D} d\Omega_D \tag{1.553}$$
$$= \int_0^\infty e^{-t} t^{(D-1)/2} \frac{dt}{2\sqrt{t}} \cdot \int_{\partial S_D} d\Omega_D \tag{1.554}$$
$$= \frac{1}{2} \int_0^\infty e^{-t} t^{(D/2-1)} dt \cdot \int_{\partial S_D} d\Omega_D \tag{1.555}$$
$$= \frac{1}{2} \Gamma(D/2) \cdot \int_{\partial S_D} d\Omega_D \tag{1.556}$$

using $t = k^2$ and therefore $dt/dk = 2k = 2\sqrt{t}$ and $dk = \frac{dt}{2\sqrt{t}} = \frac{dt}{2k}$. The we obtain

$$\int_{\partial S_D} d\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)} \tag{1.557}$$

b)

$$\int_0^\infty dr \frac{r^{D/2-1}}{(r+a)^n} = \frac{1}{a^n} \int_0^\infty dr \frac{r^{D/2-1}}{(r/a+1)^n} \tag{1.558}$$
$$= \frac{1}{a^n} a^{D/2-1} a \int_0^\infty \frac{dr}{a} \frac{(r/a)^{D/2-1}}{(r/a+1)^n} \tag{1.559}$$
$$= \frac{1}{a^n} a^{D/2-1} a \int_0^\infty \frac{dr}{a} \frac{(r/a)^{D/2-1}}{(r/a+1)^n} \tag{1.560}$$
$$= a^{D/2-n} \int_0^\infty dy \frac{y^{D/2-1}}{(y+1)^n} \tag{1.561}$$
$$= a^{D/2-n} B(D/2, n-D/2) \tag{1.562}$$
$$= a^{D/2-n} \frac{\Gamma(D/2)\Gamma(n-D/2)}{\Gamma(n)} \tag{1.563}$$

Chapter 2

Relativistic Quantum Field Theory II SS2023

2.1 Sheet 1 — Exercise 1 (Convergence of perturbative expansions

Observations:

- Not defined for $\mathrm{Re}(g) < 0$ because integrand diverges for $x \to \pm\infty$
- Easy to see $I(g = 0) = \sqrt{\pi}$ so we try an asymptotic expansion

$$I(g) = \int_{-\infty}^{\infty} e^{-x^2 - gx^4} dx \tag{2.1}$$

$$= \int_{-\infty}^{\infty} e^{-x^2} e^{-gx^4} dx \tag{2.2}$$

$$\simeq \int_{-\infty}^{\infty} e^{-x^2} \left(\sum_{k=0} (-1)^k \frac{g^k x^{4k}}{k!} \right) dx \tag{2.3}$$

$$\simeq \sum_{k=0} g^k \frac{(-1)^k}{k!} \int_{-\infty}^{\infty} e^{-x^2} x^{4k} dx \tag{2.4}$$

$$\simeq \sum_{k=0} g^k \frac{(-1)^k}{k!} \Gamma\left(2k + \frac{1}{2}\right) \tag{2.5}$$

$$\simeq \sqrt{\pi} \left(1 - \frac{3}{4} g + \frac{105}{32} g^2 - \frac{3465}{128} g^3 + ... \right) \tag{2.6}$$

Chapter 3

Effective Field Theory and Renormalization Group SS2024

3.1 Sheet 1 — Exercise 1 (Feynman Diagram)

a.) We examine all low-level diagrams

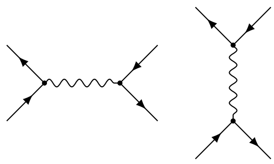


Figure 3.1: α^2 Scattering (t-channel) and Annihilation (s-channel) - no muons

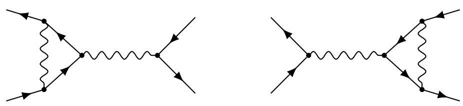


Figure 3.2: α^4 Scattering - no muons

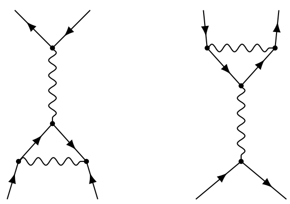


Figure 3.3: α^4 Annihilation - no muons

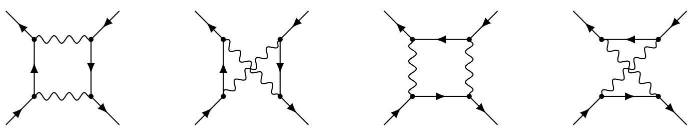


Figure 3.4: α^4 Box - no muons

b.)

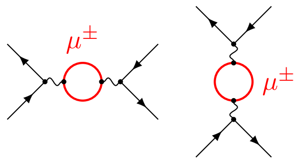


Figure 3.5: α^4 1-Loop

c.)

3.2 Open questions

- https://www.youtube.com/playlist?list=PLtPAv05VUDZrfcGZBoJqREm7XReqP6mPV

1. Klein-Gordon Hamiltonian - integration by parts of $(\nabla\phi)^2$ to get $\phi\triangle\phi$ - how do we know that the $\triangle\phi$ does not contain another hidden ϕ
2. Canonical quantization: classical field $\phi(\mathbf{x}, t)$ to Heisenberg picture?
3. guessing vs calculating Poisson brackets
4. Quantization

$$P^\mu = \int d^3x \frac{\partial\mathcal{L}}{\partial\dot{\phi}_i}(\partial^\mu\phi_i) - \eta^{\mu 0}\mathcal{L} \quad \rightarrow \quad \hat{P}^\mu = \hat{P}^\mu(a, a^\dagger) \tag{3.1}$$

$$J^{\mu\nu} = \int d^3x (T^{0\nu}x^\mu - T^{0\mu}x^\nu) \quad \rightarrow \quad \hat{J}^{\mu\nu} = \hat{J}^{\mu\nu}(a, a^\dagger) \tag{3.2}$$

   what do I get for $U(\Lambda, \epsilon) = e^{-\frac{i}{2}\omega_{\mu\nu}\hat{J}^{\mu\nu} - i\epsilon_\mu \hat{P}^\mu}$

5. $\hat{P}^\mu = \hat{P}^\mu(a, a^\dagger)$ vs $\hat{P}^\mu = -i\partial_\mu$
6. 2.2.5 - 3.) Transformation law for scalar field operator $U^\dagger(\Lambda, a)\hat{\phi}(\Lambda x + a)U \dagger (\Lambda, a) = \hat{\phi}(x)$ only valid for free KG field? (because we used the free KG commutation relation)
7. Which Dirac $\gamma, S^{\mu\nu}, \Psi$ formulas are valid in general, or in special dimensions or only for special representations - is there a good overview?
8. Meaning/implications of Hermitisity of $\mathcal{L}$
9. Meaning of $\mathcal{L} = (\partial_\mu\bar{\Psi})(\partial^\mu\Psi)$
10. commutations relations between a,b and u,v?

Chapter 4

Summaries

4.1 Representation Theory - Definitions

- For a **Lie group** $G = \{g\}$ the elements depend in a continuous and differentiable way on a set of real parameters θ^a

$$g(\theta = 0) = e \tag{4.1}$$

- A **group representation** R maps each group element onto a linear operator D_R defined on a linear space (base space)

$$g \to D_R(g) \tag{4.2}$$

with

  - $D_R(e) = 1$
  - $D_R(g_1)D_R(g_2) = D_R(g_1 g_2)$

In case the base space is of dimension n then a group element is represented by a $n \times n$ matrix and for an element of the base space $\phi = (\phi^1, .., \phi^n)$ we have

$$\phi^i \to (D_R(g))^i{}_j \phi^j \tag{4.3}$$

- **Irreducible representation** (irrep) ...
- **Generators** of the group T

$$D_R(\theta) \simeq 1 + i\theta^a T^a_R \tag{4.4}$$

$$\to T^a_R = -i \frac{\partial D_R}{\partial \theta}\bigg|_{\theta=0} \tag{4.5}$$

$$\to D_R(g(\theta)) = e^{i\theta_a T^a_R} \tag{4.6}$$

Must maintain group property

$$D_R(g_1) = e^{i\alpha_a T^a_R}, \qquad D_R(g_2) = e^{i\beta_b T^b_R} \tag{4.7}$$

$$\to D_R(g_1)D_R(g_2) = D_R(g_1 g_2) \tag{4.8}$$

$$\to e^{i\alpha_a T^a_R} e^{i\beta_b T^b_R} = e^{i\delta_c T^c_R} \tag{4.9}$$

- **Lie algebra** (independent of representation) - for matrix representation the Lie bracket is just the commutator

$$[T^a, T^b] = i f^{ab}_c T^c \tag{4.10}$$

- **Casimir operator** an operator which is NOT part of the Lie algebra but commutes with all generators

4.2 Representations - fact summary

- Most relevant groups (and associated Lie algebras in physics are
    1. SO(3) - Spacial rotations in three dimensions
    2. SU(2) - Angular momentum in quantum mechanics
    3. SU(3) - Light quark flavour symmetry, colors in QCD
    4. SL(2,$\mathbb{C}$) - Lorentz group
    5. $\mathrm{SO}(1,3)^+$ - (vector rep. of Lorentz group)
    6. $\mathrm{ISO}(1,3) \sim \mathbb{R}^{1,3} \times \mathrm{O(1,3)}$ - Poincare transformations
    7. $\mathrm{Sp}(2n, \mathbb{R})$ - Hamiltonian systems
- **SU(2)**: To get a systematic overview (of representations which are relevant in physics) it is best to start with group SU(2) - because most of the following can be derived from here
    - the $j = 1/2$ (2-dimensional defining) representation of the group follows from 2d geometry
    $$\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \qquad \text{with } |\alpha|^2 + |\beta|^2 = 1 \tag{4.11}$$
    - from this we derive the 2d representation of generators (of the Lie algebra $\mathfrak{su}(2)$) which is given by $L_k = \sigma_k/2$ (Pauli matrices)
    - Lie algebra can then be calculated for this representation (which holds for any representation)
    $$[L_i, L_j] = i\epsilon_{ijk} L_k \tag{4.12}$$
    - one Casimir operator: $L^2 = L_1^2 + L_2^2 + L_3^2$ with $[L^2, L_k] = 0$
    - exactly one irrep for each $j = 0, 1/2, 1, 3/2, 2, \ldots$ (dimension $n = 2j + 1$) - construction:
        1. start with $n = 2j + 1$ dimensional orthogonal euclidean basis $|jm_j\rangle$ $(-j \leq m_j \leq j)$
        2. action $L_\pm = L_x \pm iL_y$ and L^2 on them (using generic properties of the operators)
        3. calculate all matrix elements $\langle jm'|L_\pm|jm\rangle$ and $\langle jm'|L^2|jm\rangle$ - obtaining the representation of $L_\pm$ and L^2
        4. then calculate matrix representation of L_k
    - for spin-j irreps of the Lie algebra the $2j+1$-dimensional representation space is spanned by $\{|j, -j\rangle, ..., |j, +j\rangle\}$
        * **Tensor representations of the Lie algebra** $\mathfrak{su}(2)$ for $j = 0, 1, 2, 3, \ldots$ (with odd dimensions $1, 3, 4, \cdots$) the associated group-irreps are 2π periodic
        * **Spinor representations of the Lie algebra** $\mathfrak{su}(2)$ for $j = 1/2, 3/2, 5/2, \ldots$ (with even dimensions $2, 4, 6, \cdots$) we have
        $$D_R(g_{\theta=2\pi}) = e^{iL_k \cdot 2\pi} = -1 \tag{4.13}$$
        meaning the associated group-irreps are not 2π but 4π periodic
    - Clebsch-Gordon decomposition of tensor product of representations
    $$D_{j_1} \otimes D_{j_2} = D_{|j_1 - j_2|} \oplus \ldots \oplus D_{j_1 + j_2} \tag{4.14}$$
    $$\to D_{1/2} \otimes D_{1/2} = D_0 \oplus D_1 \qquad (\mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{3}) \tag{4.15}$$
    $$\to D_{1/2} \otimes D_{1/2} \otimes D_{1/2} = (D_0 \otimes D_{1/2}) \oplus (D_1 \otimes D_{1/2}) = D_{1/2} \otimes D_{1/2} \otimes D_{3/2} \tag{4.16}$$
    $$\to D_{3/2} \otimes D_{3/2} = D_0 \oplus D_1 \oplus D_2 \oplus D_3 \qquad (\mathbf{4} \otimes \mathbf{4} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{5} \oplus \mathbf{7}) \tag{4.17}$$

- **SO(3)**
  - the 3-dimensional (defining) representation of the group follows from 3d geometry
  - three generators of the $\mathfrak{so}(3)$ follow from derivatives of the three basis rotation matrices
  - same Lie algebra as SU(2)

$$[L_i, L_j] = i\epsilon_{ijk} L_k \tag{4.18}$$

    (so both groups look similar near the 1-element)
  - one Casimir Operator: $L^2 = L_x^2 + L_y^2 + L_z^2$
  - SU(2) is the universal covering group of SO(3)
  - Irreps
    * Lie algebra $\mathfrak{so}(3)$ has same tensor and spinor irreps as $\mathfrak{su}(2)$
    * Lie group SO(3) shares only tensor irreps as of SU(2) - as spinor irreps are 4π periodic
- **SO(1,3) or SL(2,$\mathbb{C}$) - Lorentz group**
  - 4-dimensional defining representation ($x'^\mu = \Lambda^\mu{}_\nu x^\nu$) follows from $g_{\mu\nu} = \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu g_{\rho\sigma}$ - rotations four dimensional $1+3$ coordinates (signature $+,-,-,-$) therefore SO(1,3)
  - side note: relation with SL(2,$\mathbb{C}$)

$$x^\mu \to X \equiv \sigma_\mu x^\mu = \begin{pmatrix} x^0 + x^3 & -x^1 - ix^2 \\ -x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \tag{4.19}$$

$$\to \det X = (x^0)^2 - \mathbf{x}^2 \tag{4.20}$$

    then a Lorentz transformation can be represented by $A(\Lambda)$ with

$$X \xrightarrow{\Lambda} X' = A(\Lambda) X A^\dagger(\Lambda) \tag{4.21}$$

$$\to \det X' = \det A \det X \det A^\dagger \tag{4.22}$$

    so A is really a Lorentz transformation if $\det A \det A^\dagger = |\det A|^2 = 1$ meaning $A(\Lambda) \in$ SL(2,$\mathbb{C}$) - (ignoring the overall phase of A)
  - open question - can the matrix X be expressed as a complex 2-vector or spinor
  - infinitesimal Lorentz transformation $x^\mu \simeq [\delta^\mu_\nu - \frac{i}{2}(\omega_{\rho\sigma} J^{\rho\sigma})^\mu{}_\nu] x^\nu$ can be found by derivative with respect to the six parameters $\omega_{\mu\nu} = -\omega_{\nu\mu}$ and the 6 generators can be written in the 4-dimensional representation as

$$(J^{\mu\nu})^\rho{}_\sigma = i(g^{\mu\rho}\delta^\nu_\sigma - g^{\nu\rho}\delta^\mu_\sigma) \tag{4.23}$$

  - from this we can obtain the Lie algebra by just calculating the commutators
  - so there are three forms
    1. we obtain form this directly

$$[J^{\mu\nu}, J^{\rho\sigma}] = i f^{\mu\nu\rho\sigma}{}_{\alpha\beta} J^{\alpha\beta} \tag{4.24}$$

$$= -i(g^{\mu\rho}J^{\nu\sigma} - g^{\mu\sigma}J^{\nu\rho} + g^{\nu\rho}J^{\mu\sigma} - g^{\nu\sigma}J^{\mu\rho}) \tag{4.25}$$

    2. rearranging 3 boosts $K_i = J^{0i}$ and 3 rotations $J_i = \frac{1}{2}\epsilon_{ijk}J^{jk}$ into two vectors $\mathbf{J}, \mathbf{K}$ with algebra

$$[J_i, J_j] = i\epsilon_{ijk} J_k \tag{4.26}$$

$$[K_i, K_j] = -i\epsilon_{ijk} J_k \tag{4.27}$$

$$[J_i, K_j] = i\epsilon_{ijk} K_k \tag{4.28}$$

$$\to \Lambda = \exp[\boldsymbol{\theta} \cdot \mathbf{J} - \boldsymbol{\eta} \cdot \mathbf{K}] \tag{4.29}$$

3. rearranging again $\mathbf{J}^{\pm} = \frac{\mathbf{J} \pm i\mathbf{K}}{2}$ gives

$$[J^{+,i}, J^{+,j}] = i\epsilon^{ijk} J^{+,k} \tag{4.30}$$
$$[J^{-,i}, J^{-,j}] = i\epsilon^{ijk} J^{-,k} \tag{4.31}$$
$$[J^{+,i}, J^{-,j}] = 0 \tag{4.32}$$

two copies of $\mathfrak{su}(2)$ which commute between themselves

- $\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{su}(2) \otimes \mathfrak{su}(2)$ but
  * SU(2)⊗SU(2) is the universal covering group of SO(4)
  * SO(3,1) is the universal covering group of SL(2,ℂ)
- two Casimir operators
  * $C_1 = \frac{1}{2} J_{\mu\nu} J^{\mu\nu} = \vec{J}^2 - \vec{K}^2 = \vec{J^+}^2 + \vec{J^-}^2$
  * $C_2 = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} J^{\mu\nu} J^{\rho\sigma} = \vec{J} \cdot \vec{K} = i(\vec{J^+}^2 - \vec{J^-}^2)$
- Using the $\mathfrak{su}(2)$ irreps - the Lorentz algebra irreps can be labeled

$$(j_-, j_+) = (j_-, 0) \otimes (0, j_+) \tag{4.33}$$

and because $\mathbf{J} = \mathbf{J}^+ + \mathbf{J}^-$ we have states a spins between $|j_- - j_+|, ..., j_- + j_+$

  * **Scalar representation** $(0,0)$ the spin 0 irrep - acting on $(2 \cdot 0 + 1)(2 \cdot 0 + 1) = 1$ dimensional objects - scalars

$$J_S^{\mu\nu} = 0 \quad \rightarrow \quad \Lambda^\mu_{\ \nu} = 1 \tag{4.34}$$
$$\rightarrow \quad \phi \rightarrow 1 \cdot \phi \tag{4.35}$$

  * **Weyl spinor representation** fundamental spinorial representations - there are two distinct spin 1/2 irreps - acting on $(2 \cdot 0 + 1)(2 \cdot 1 + 1) = 2$ dimensional objects - Weyl spinors
    · **Left-handed Weyl spinor** $(1/2, 0)$ - so $\mathbf{J}^- = \boldsymbol{\sigma}/2$ and $\mathbf{J}^+ = 0$ then $\mathbf{J} = \boldsymbol{\sigma}/2$ and $\mathbf{K} = +i\boldsymbol{\sigma}/2$

$$J_L^{\mu\nu} = S^{\mu\nu}, \quad S^{ij} = \frac{1}{2}\epsilon^{ijk}\sigma_k, S^{0i} = -\frac{i}{2}\sigma^i \quad \rightarrow \quad e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}} = \Lambda_L \tag{4.36}$$
$$\psi_L \rightarrow \Lambda_L \psi_L = \exp\left[(-i\boldsymbol{\theta} - \boldsymbol{\eta}) \cdot \frac{\boldsymbol{\sigma}}{2}\right] \psi_L \tag{4.37}$$

    · **Right-handed Weyl spinor** $(0, 1/2)$ - so $\mathbf{J}^- = 0$ and $\mathbf{J}^+ = \boldsymbol{\sigma}/2$ then $\mathbf{J} = \boldsymbol{\sigma}/2$ and $\mathbf{K} = -i\boldsymbol{\sigma}/2$

$$J_R^{\mu\nu} = S^{\mu\nu}, \quad S^{ij} = \frac{1}{2}\epsilon^{ijk}\sigma_k, S^{0i} = +\frac{i}{2}\sigma^i \quad \rightarrow \quad e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}} = \Lambda_R \tag{4.38}$$
$$\psi_R \rightarrow \Lambda_R \psi_R = \exp\left[(-i\boldsymbol{\theta} + \boldsymbol{\eta}) \cdot \frac{\boldsymbol{\sigma}}{2}\right] \psi_R \tag{4.39}$$

we also see that $\sigma_2 \psi_L^*$ transforms like a right handed spinor and define the charge conjugate of a Weyl spinor as $\psi_L^c = i\sigma_2\psi_L^*$ and $\psi_R^c = -i\sigma_2\psi_R^*$

  * **4-vector representation** $(1/2, 1/2)$ spin 0,1 irrep acting on $(2 \cdot 1/2 + 1)(2 \cdot 1/2 + 1) = 4$ dimensional objects - 4-vectors

$$(J_V^{\mu\nu})^\rho_{\ \sigma} = i(g^{\mu\rho}\delta^\nu_\sigma - g^{\nu\rho}\delta^\mu_\sigma) \quad \rightarrow \quad (e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}})^\rho_{\ \sigma} = \Lambda^\rho_{\ \sigma} \tag{4.40}$$

with $\sigma^\mu = (1, \sigma^k)$ and $\bar{\sigma}^\mu = (1, -\sigma^k)$ we see that $\xi_R^\dagger \sigma^\mu \psi_R$ and $\xi_L^\dagger \bar{\sigma}^\mu \psi_L$ are (transform like) 4-vectors

* **Dirac spinor representation** $(1/2, 0) \oplus (0, 1/2)$ reducible representation action on $2+2=4$ dimensional objects - Dirac spinors

$$S^{ij} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix}, \qquad S^{0i} = -\frac{1}{2}\begin{pmatrix}\sigma^i & 0\\ 0 & -\sigma^i\end{pmatrix} \tag{4.41}$$

$$J_D^{\mu\nu} = S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \quad \to \quad e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}} = \Lambda_D \tag{4.42}$$

$$\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} \to \Lambda_D \begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} \tag{4.43}$$

* **Infinite-dimensional Orbital representation** the L^{ij} are the classical generators of angular momentum

$$L^{\mu\nu} = x^\mu\partial_\nu - x^\nu\partial_\mu \tag{4.44}$$

– Field - with $\left(e^{-\frac{i}{2}\omega_{\mu\nu}J_V^{\mu\nu}}\right)^\rho_{\ \sigma} x^\sigma \simeq \left(1 - \frac{i}{2}\omega_{\mu\nu}i(g^{\mu\rho}\delta^\nu_\sigma - g^{\nu\rho}\delta^\mu_\sigma)\right)x^\sigma = x^\sigma + \omega^\sigma_{\ \mu}x^\mu$

$$\Phi_a \to M_{ab}(\Lambda)\Phi_b \qquad \Rightarrow \qquad \Phi_a(x) \to \Phi'_a(x) = M_{ab}(\Lambda)\Phi_b(\Lambda^{-1}x) \tag{4.45}$$

$$L^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu) \qquad \Rightarrow \qquad e^{-\frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}}\Phi(x) = \left(1 - \frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}\right)\Phi(x) \tag{4.46}$$

$$= \Phi(x) + (\partial^\nu\Phi)\cdot\omega_{\mu\nu}x^\mu \tag{4.47}$$

$$= \Phi(x + \omega_\mu^{\ \nu}x^\mu) \tag{4.48}$$

$$= \Phi(\Lambda^{-1}x) \tag{4.49}$$

then with $L^{\mu\nu} + J^{\mu\nu}$

$$\Phi_a(x) \to \Phi'_a(x) = \left(e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}\right)_{ab} e^{-\frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}}\Phi_b(x) \tag{4.50}$$

$$= \left(e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}\right)_{ab}\Phi_b(\Lambda^{-1}x) \tag{4.51}$$

- **Poincare group**
  - Lie algebra (Lorentz, just translation, Lorentz/translations)

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i(g^{\mu\rho}M^{\nu\sigma} - g^{\mu\sigma}M^{\nu\rho} + g^{\nu\rho}M^{\mu\sigma} - g^{\nu\sigma}M^{\mu\rho}) \tag{4.52}$$

$$[P^\mu, P^\nu] = 0 \tag{4.53}$$

$$[P^\mu, M^{\rho\sigma}] = i(g^{\mu\rho}P^\sigma - g^{\mu\sigma}P^\rho) \tag{4.54}$$

  - One Casimir operator: $L^2 = L_1^2 + L_2^2 + L_3^2$ with $[L^2, L_k] = 0$
    1. $\mathcal{M}^2 = P_\mu P^\mu$ squared mass
    2. $\mathcal{W}^2 = W_\mu W^\mu$ with Pauli-Lubaniski vector $W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}J^{\nu\rho}P^\sigma$

4.3 Spacetime Transformations

4.3.1 Lorentz Transformations

Exam question: *Whats the defining property of a Lorentz transformation?*

- Answer: It is a transformation on the spacetime coordinates $x'^\mu = \Lambda^\mu_{\ \nu}x^\nu$ which leaves the line element $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$ invariant.
- Physical substance: Speed of light c is the same in each inertial system.

- Expressed mathematically: $\eta_{\mu\nu} = \Lambda^\alpha_{\ \mu}\Lambda^\beta_{\ \nu}\eta_{\alpha\beta}$

The common Lorentz transformation law of a 4-vector

$$V'^\mu = \Lambda^\mu_{\ \nu}V^\nu \tag{4.55}$$

provides us naturally (each transformation is associated with a 4×4 matrix - obeying the definition of a representation) with a 4-dimensional representation of the Lorentz group meaning

$$D_{\text{4-dim}}(\Lambda) = \Lambda^\mu_{\ \sigma}. \tag{4.56}$$

Now consider an infinitesimal Lorentz transformation

$$\Lambda^\mu_{\ \nu} \simeq \delta^\mu_{\ \nu} + \omega^\mu_{\ \nu} \qquad (\omega_{\mu\nu} = -\omega_{\nu\mu}) \tag{4.57}$$

$$\to x'^\mu = \Lambda^\mu_{\ \nu}x^\nu \simeq x^\mu + \omega^\mu_{\ \nu}x^\nu \tag{4.58}$$

with

$$\omega^\mu_{\ \nu} = \begin{pmatrix} 0 & \eta_1 & \eta_2 & \eta_3 \\ \eta_1 & 0 & -\theta_3 & \theta_2 \\ \eta_2 & \theta_3 & 0 & -\theta_1 \\ \eta_3 & -\theta_2 & \theta_1 & 0 \end{pmatrix} \tag{4.59}$$

The antisymmetry of ω implies that there are only 6 independent parameters (3 infinitesimal boosts η_i and 3 infinitesimal rotations, i.e. θ_1 rotation in the $2-3$ plane). *It would be actually more consistent to write $d\eta$ and $d\theta$.*

Now we can do a technical step - splitting the ω

$$\omega^\rho_{\ \sigma} \to -\frac{i}{2}(\omega_{\mu\nu}J^{\mu\nu})^\rho_{\ \sigma} \simeq \eta_1 \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \eta_2 .... + \theta_{23} .... \tag{4.60}$$

which means we can write an infinitesimal trafo as

$$D(d\Lambda)^\rho_{\ \sigma} \simeq \delta^\rho_\sigma - \frac{i}{2}\omega_{\mu\nu}\cdot(J_R^{\mu\nu})^\rho_{\ \sigma} \tag{4.61}$$

$$= \delta^\rho_\sigma - i(\omega_{01}J_R^{01} + \omega_{02}J_R^{02} + \omega_{03}J_R^{03} + \omega_{12}J_R^{12} + \omega_{23}J_R^{23} + \omega_{13}J_R^{13}) \tag{4.62}$$

$$= \delta^\rho_\sigma - i(\eta_1 J_R^{01} + \eta_2 J_R^{02} + \eta_3 J_R^{03} + \theta_3 J_R^{12} + \theta_1 J_R^{23} + \theta_2 J_R^{13}) \tag{4.63}$$

where (in our special example) the $J_R^{\mu\nu}$ are 4×4 matrices which can we read off from the shape of $\omega^\mu_{\ \nu}$. In our example of the 4-dimensional (defining) representation we can write the so called generators $J^{\mu\nu}$ explicitly

$$J^{01}_{\text{4-dim}} = \begin{pmatrix} 0 & +i & 0 & 0 \\ +i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad J^{23}_{\text{4-dim}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +i \\ 0 & 0 & -i & 0 \end{pmatrix} \tag{4.64}$$

or shorter as $(J^{\mu\nu}_{\text{4-dim}})^\rho_{\ \sigma} = i(\eta^{\mu\rho}\delta^\nu_\sigma - \eta^{\nu\rho}\delta^\mu_\sigma)$ - which is the 4-dimensional representation of the generators. The associated representations of the (infinitesimal) transformations are given by

$$D_{\text{4-dim}}(d\Lambda^{01}) = \begin{pmatrix} 1 & \eta_1 & 0 & 0 \\ \eta_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad D_{\text{4-dim}}(d\Lambda^{23}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \theta_1 \\ 0 & 0 & -\theta_1 & 1. \end{pmatrix}. \tag{4.65}$$

To be consistent - they should (when promoted to finite transformations) be identical with the defining 4-dimensional representation of the Lorentz group mentioned above.

$$e^{\delta^\rho_\sigma - \frac{i}{2}\omega_{\mu\nu}(J^{\mu\nu}_{\text{4-dim}})^\rho{}_\sigma} \equiv D_{\text{4-dim}}(\Lambda) = \Lambda^\rho{}_\sigma \tag{4.66}$$

We can verify that the $(J^{\mu\nu}_{\text{4-dim}})^\rho{}_\sigma$ are indeed correct by using them to perform infinitesimal Lorentz transformations (spacetime 4-vectors) and comparing with expected result

$$\begin{aligned} x'^\rho &= D_{\text{4-dim}}(\Lambda)^\rho{}_\sigma x^\sigma && (4.67)\\ &= [\delta^\rho_\sigma - i\omega i(\eta^{\mu\rho}\delta^\nu_\sigma - \eta^{\nu\rho}\delta^\mu_\sigma)]x^\sigma && (4.68)\\ &= x^\rho + \omega(\eta^{\mu\rho}x^\nu - \eta^{\nu\rho}x^\mu) && (4.69)\end{aligned}$$

so

$$\begin{aligned} &D_{\text{4-dim}}(\Lambda^{01}) && \to \delta x^\mu = (+\eta_1 x, +\eta_1 t, 0, 0) && (4.70)\\ &D_{\text{4-dim}}(\Lambda^{23}) && \to \delta x^\mu = (0, -\theta_1 y, +\theta_1 x, 0) && (4.71)\end{aligned}$$

which is consistent with the expected result for an infinitesimal boost and a rotation.

Keep in mind that depending of the representation the $J^{\mu\nu}_R$ can have arbitrary dimension n.

Alternatively we can write

$$\begin{aligned} K^i \equiv J^{i0}, \qquad J^i \equiv \frac{1}{2}\epsilon^{ijk}J^{jk} \quad (J^{jk} = \epsilon^{jki}J^i) \qquad &\to \qquad \Lambda^\rho{}_\sigma = e^{i\eta\cdot\mathbf{K} - i\theta\cdot\mathbf{J}} && (4.72)\\ &\to \qquad [\hat{J}_i, \hat{J}_j] = i\epsilon_{ijk}\hat{J}_k && (4.73)\\ &\qquad\quad [\hat{J}_i, \hat{K}_j] = i\epsilon_{ijk}\hat{K}_k && (4.74)\\ &\qquad\quad [\hat{K}_i, \hat{K}_j] = -i\epsilon_{ijk}\hat{J}_k && (4.75)\end{aligned}$$

with the Casimir operators $C_1 = \frac{1}{2}J^{\mu\nu}J_{\mu\nu} = \mathbf{J}^2 - \mathbf{K}^2$ and $C_1 = \frac{1}{2}\tilde{J}^{\mu\nu}J_{\mu\nu} = \mathbf{J}\cdot\mathbf{K}$ where the dual $\tilde{J}_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta}J^{\alpha\beta}$

or even

$$\begin{aligned} \mathbf{A} = \frac{1}{2}(\mathbf{K} - i\mathbf{J}) \qquad \mathbf{B} = \frac{1}{2}(\mathbf{K} + i\mathbf{J}) \qquad &\to \qquad \Lambda^\rho{}_\sigma = e^{i\eta\cdot(\mathbf{A}+\mathbf{B}) - \theta\cdot(\mathbf{B}-\mathbf{A})} && (4.76)\\ &\qquad\qquad = e^{(i\eta+\theta)\cdot\mathbf{A} + (i\eta-\theta)\cdot\mathbf{B}} && (4.77)\\ &\to \qquad [\hat{A}_i, \hat{A}_j] = i\epsilon_{ijk}\hat{A}_k && (4.78)\\ &\qquad\quad [\hat{B}_i, \hat{B}_j] = i\epsilon_{ijk}\hat{B}_k && (4.79)\\ &\qquad\quad [\hat{A}_i, \hat{B}_j] = 0 && (4.80)\end{aligned}$$

Here we obtained to two copies of an $SU(2)$ algebra, with Casimir operators $\mathbf{A}^2 = \frac{1}{4}(\mathbf{K}^2 - \mathbf{J}^2) - 2i\mathbf{K}\cdot\mathbf{J}$ and $\mathbf{B}^2 = \frac{1}{4}(\mathbf{K}^2 - \mathbf{J}^2) + 2i\mathbf{K}\cdot\mathbf{J}$

4.3.2 Poincare transformations I

For a poincare trafo we can translate first and then rotate

$$\begin{aligned} x'^\mu &= x^\mu + a^\mu && (4.81)\\ x''^\mu &= (\delta^\mu_\nu + \omega^\mu_\nu)(x^\mu + a^\mu) && (4.82)\\ &= x^\mu + (\omega^\mu_\nu x^\nu) + (a^\mu + \omega^\mu_\nu a^\nu) && (4.83)\end{aligned}$$

or first rotate and then translate

$$\begin{aligned} x'^\mu &= x^\mu + \omega^\mu_\nu x^\nu && (4.84)\\ x''^\mu &= x^\mu + (\omega^\mu_\nu x^\nu) + (a^\mu) && (4.85)\end{aligned}$$

The second is commonly considered to be a Poincare trafo.

4.3.3 Poincare transformations II

Coordinates

With the defining representation $\Lambda^\mu_{\ \nu} = \left(e^{-\frac{i}{2}\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta}}\right)^\mu_{\ \nu} \simeq \delta^\mu_\nu - \frac{i}{2}(\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta})^\mu_{\ \nu} = \delta^\mu_\nu + \frac{1}{2}(\omega^\mu_{\ \beta}\delta^\beta_\nu - \omega_\alpha^{\ \mu}\delta^\alpha_\nu) = \delta^\mu_\nu + \omega^\mu_{\ \nu}$ using the representation of the Lie algebra $(\mathcal{J}^{\alpha\beta})^\mu_{\ \nu} = i(g^{\alpha\mu}\delta^\beta_\nu - \delta^\alpha_\nu g^{\beta\mu})$

$$x' = \Lambda x + a \simeq x + \omega x + \epsilon \tag{4.86}$$
$$\delta x^\alpha \equiv x'^\alpha - x^\alpha \tag{4.87}$$
$$= \omega^\alpha_{\ \beta} x^\beta + \epsilon^\alpha \tag{4.88}$$

In general - multi-component fields transform like

General assumption: theory is Poincare invariant - so a field (aka particle) must transform under a representation of the Poincare group

$$\phi'_i(x') = R_i^{\ j}(\Lambda)\phi_j(x) \tag{4.89}$$

Scalar field - (spin 0 representation)

Most trivial case with $R(\Lambda) = 1$

$$\phi'(\Lambda x + a) = \phi(x) \qquad \to \qquad \phi'(x) = \phi(\Lambda^{-1}x) \tag{4.90}$$
$$\to \qquad \phi'(x) \simeq \phi(x - \omega x - \epsilon) \tag{4.91}$$

then

$$\delta\phi(x) \equiv \phi'(x) - \phi(x) \tag{4.92}$$
$$\simeq \phi(x - [\omega x + \epsilon]) - \phi(x) \tag{4.93}$$
$$= \partial_\mu\phi(x)\cdot(-\delta x) \tag{4.94}$$
$$= -\omega^{\mu\nu}x_\nu\partial_\mu\phi(x) - \epsilon^\mu\partial_\mu\phi(x) \tag{4.95}$$
$$= -\frac{1}{2}\omega_{\mu\nu}(x^\nu\partial^\mu - x^\mu\partial^\nu)\phi(x) - \epsilon^\mu\partial_\mu\phi(x) \tag{4.96}$$
$$= -\frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}\phi(x) - \epsilon^\mu\partial_\mu\phi(x) \tag{4.97}$$

with $L^{\mu\nu} = -i(x^\nu\partial^\mu - x^\mu\partial^\nu)$

Vector field - (spin 1 representation)

The second most trivial case $R(\Lambda) = \Lambda^\mu_{\ \nu} = \left(e^{-\frac{i}{2}\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta}}\right)^\mu_{\ \nu} \simeq \delta^\mu_\nu - \frac{i}{2}(\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta})^\mu_{\ \nu} = \delta^\mu_\nu + \frac{1}{2}(\omega^\mu_{\ \beta}\delta^\beta_\nu - \omega_\alpha^{\ \mu}\delta^\alpha_\nu) = \delta^\mu_\nu + \omega^\mu_{\ \nu}$ with the representation of the Lie algebra $(\mathcal{J}^{\alpha\beta})^\mu_{\ \nu} = i(g^{\alpha\mu}\delta^\beta_\nu - \delta^\alpha_\nu g^{\beta\mu})$

$$A'^\mu(\Lambda x + a) = \Lambda^\mu_{\ \nu}A^\nu(x) \qquad \to \qquad A'^\mu(x) = \Lambda^\mu_{\ \nu}A_\nu(\Lambda^{-1}x) \tag{4.98}$$
$$\to \qquad A'^\mu(x) \simeq (\delta^\mu_\nu - \frac{i}{2}(\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta})^\mu_{\ \nu})A^\nu(x - \omega x - \epsilon) \tag{4.99}$$

then

$$\delta A^\mu(x) \equiv A'^\mu(x) - A^\mu(x) \tag{4.100}$$
$$= \delta^\mu_{\ \nu} A^\nu(x-\omega x-\epsilon) - \frac{i}{2}(\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta})^\mu_{\ \nu} A^\nu(x-\omega x-\epsilon) - A^\mu(x) \tag{4.101}$$
$$= \left(A^\mu(x) + \partial_\alpha A^\mu(x)[-\omega^{\alpha\beta}x_\beta - \epsilon^\alpha] + ...\right) - \frac{i}{2}(\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta})^\mu_{\ \nu} A^\nu(x) + \mathcal{O}(\omega^2) - A^\mu(x) \tag{4.102}$$
$$= -\omega^{\alpha\beta}x_\beta\partial_\alpha A^\mu(x) - \epsilon^\alpha\partial_\alpha A^\mu(x) - \frac{i}{2}(\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta})^\mu_{\ \nu} A^\nu(x) \tag{4.103}$$
$$= -\frac{i}{2}\omega_{\alpha\beta}L^{\alpha\beta}A^\mu - \epsilon^\alpha\partial_\alpha A^\mu - \frac{i}{2}(\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta})^\mu_{\ \nu} A^\nu(x) \tag{4.104}$$
$$= -\frac{i}{2}\omega_{\alpha\beta}\left(L^{\alpha\beta}A^\mu + (\mathcal{J}^{\alpha\beta})^\mu_{\ \ \nu} A^\nu\right) - \epsilon^\alpha\partial_\alpha A^\mu \tag{4.105}$$

Spinor field - (spin 1/2 representation)

Now the group rep. is $M_{\alpha\beta} = e^{-\frac{1}{2}\omega_{\mu\nu}S^{\mu\nu}}$ with $S^{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]$ (so $S^{\mu\nu}$ is a representation of the Lorentz algebra - which is automatically the case if $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \times 1_{n\times n}$)

$$\Psi'_\alpha(\Lambda x + a) = M_{\alpha\beta}(\Lambda)\Psi_\beta(x) \qquad \to \qquad \Psi'_\alpha(x) \simeq \left(\delta_{\alpha\beta} - \frac{i}{2}(\omega_{\mu\nu}S^{\mu\nu})_{\alpha\beta}\right)\Psi_\beta(x-\omega x-\epsilon) \tag{4.106}$$

then

$$\delta\Psi_\alpha(x) \equiv \Psi'_\alpha(x) - \Psi_\alpha(x) \tag{4.107}$$
$$\simeq \partial_\mu\Psi_\alpha(-\delta x) - \frac{i}{2}(\omega_{\mu\nu}S^{\mu\nu})_{\alpha\beta}\Psi_\beta(x) \tag{4.108}$$
$$\simeq (-\omega^{\mu\nu}x_\nu - \epsilon^\mu)\partial_\mu\Psi_\alpha - \frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})_{\alpha\beta}\Psi_\beta(x) \tag{4.109}$$
$$= -\frac{i}{2}\omega_{\rho\sigma}\left(L^{\rho\sigma}\Psi_\alpha(x) + (S^{\rho\sigma})_{\alpha\beta}\Psi_\alpha(x)\right) - \epsilon^\mu\partial_\mu\Psi_\alpha(x) \tag{4.110}$$

Arbitrary representation

$$\varphi'_\alpha(\Lambda x + a) = D(\Lambda)_{\alpha\beta}\varphi_\beta(x) \quad \to \quad \varphi'_\alpha(x) \simeq (\delta_{\alpha\beta} + \omega_{\mu\nu}\Sigma^{\mu\nu}_{\alpha\beta} + \epsilon ...)\varphi_\beta(x-\omega x-\epsilon) \tag{4.111}$$

then

$$\delta\varphi_\alpha(x) \equiv \varphi'_\alpha(x) - \varphi_\alpha(x) \tag{4.112}$$
$$= -\omega^{\rho\sigma}(\partial_\rho\varphi_\alpha)x_\sigma - \epsilon^\mu\partial_\mu\varphi_\alpha(x) + \omega_{\mu\nu}\Sigma^{\mu\nu}_{\alpha\beta}\varphi_\beta + ... \tag{4.113}$$

with

$$\text{scalar:} \qquad \Sigma^{\mu\nu}_{\alpha\beta} = 0 \tag{4.114}$$
$$\text{vector:} \qquad \Sigma^{\mu\nu}_{\alpha\beta} = \frac{1}{2}(g^{\alpha\mu}g^\nu_\beta - g^{\alpha\nu}g^\mu_\beta) \tag{4.115}$$
$$\text{spinor:} \qquad \Sigma^{\mu\nu}_{\alpha\beta} = ... \tag{4.116}$$

4.3.4 Noether Theorem

Noether Master equation

$$\partial_\mu\left\{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a + \mathcal{L}\delta x^\mu\right\} = 0 \tag{4.117}$$
$$\partial_\mu\left\{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a + g^{\mu\nu}\mathcal{L}\delta x_\nu\right\} = 0 \tag{4.118}$$

with inputs $\delta\phi_a$ and δx^μ.

The most interesting cases

1. Spacetime symmetries

   (a) Spacial translation $\delta x_\beta = \epsilon_\beta$ with associated $\delta\phi_a = -\epsilon^\mu \partial_\mu \phi_a = -\partial^\mu \phi_a \delta x_\mu$

$$\rightarrow \partial_\mu \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta\phi_a + g^{\mu\nu} \mathcal{L} \delta x_\nu \right\} = 0 \tag{4.119}$$

$$\rightarrow \partial_\mu \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial^\mu \phi_a - g^{\mu\nu} \mathcal{L} \right\} \delta x_\nu = 0 \tag{4.120}$$

$$\rightarrow T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial^\mu \phi_a - g^{\mu\nu} \mathcal{L} \tag{4.121}$$

   $T^{\mu\nu}$ is the canonical energy momentum tensor - four conserved currents (one for each ν) $\partial_\mu T^{\mu\nu} = \partial_0 T^{0\nu} + \partial_k T^{k\nu} = 0$ and four conserved quantities

$$\frac{d}{dt} P^\nu \equiv \frac{d}{dt} \int d^3x T^{0\nu} = \int d^3x \nabla_k T^{k\nu} = 0 \tag{4.122}$$

   (b) Lorentz transformation $\delta x_\nu = \omega_{\nu\rho} x^\rho$ with associated $\delta\phi_\alpha = -\omega_{\nu\rho} \Big((\partial^\nu \varphi_\alpha) x^\rho - \Sigma^{\nu\rho}_{\alpha\beta} \varphi_\beta \Big)$

$$\rightarrow \partial_\mu \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_\alpha)} \delta\phi_\alpha + g^{\mu\nu} \mathcal{L} \delta x_\nu \right\} = 0 \tag{4.123}$$

$$\rightarrow -\omega_{\nu\rho} \partial_\mu \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_\alpha)} \Big((\partial^\nu \varphi_\alpha) x^\rho - \Sigma^{\nu\rho}_{\alpha\beta} \varphi_\beta \Big) - g^{\mu\nu} \mathcal{L} x^\rho \right\} = 0 \tag{4.124}$$

$$\rightarrow -\omega_{\nu\rho} \partial_\mu \left\{ \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_\alpha)} \partial^\nu \varphi_\alpha - g^{\mu\nu} \mathcal{L} \right) x^\rho - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_\alpha)} \Sigma^{\nu\rho}_{\alpha\beta} \varphi_\beta \right\} = 0 \tag{4.125}$$

$$\rightarrow \omega_{\nu\rho} \partial_\mu \left\{ -T^{\mu\nu} x^\rho + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_\alpha)} \Sigma^{\nu\rho}_{\alpha\beta} \varphi_\beta \right\} = 0 \tag{4.126}$$

$$\rightarrow M^{\mu\nu\rho} = -T^{\mu\nu} x^\rho + T^{\mu\rho} x^\nu + 2 \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_\alpha)} \Sigma^{\nu\rho}_{\alpha\beta} \varphi_\beta \tag{4.127}$$

   because of $\omega_{\mu\nu} = -\omega_{\nu\mu}$ and $\Sigma^{\nu\rho}_{\alpha\beta} = -\Sigma^{\rho\nu}_{\alpha\beta}$
   ??Spin part - why??

$$S^{\nu\rho} = \int d^3x \; 2\pi_\alpha \Sigma^{\nu\rho}_{\alpha\beta} \phi_\beta \tag{4.128}$$

2. Inner symmetries

   (a) $\delta x^\mu = 0$ and $\phi'_a = R_a^b \phi_b$ meaning $\delta\phi_\alpha = ...$

4.3.5 Field representations for Poincare symmetry

[Kugo, p.15] Under an LT ($x' = \Lambda x$) a classical field transforms as

$$\phi_i(x) \rightarrow \phi'_i(x') = D(\Lambda)_i^{\;j} \phi_j(x) \tag{4.129}$$

while for a quantum state transforms we need a unitary matrix

$$|\Phi\rangle \rightarrow |\Phi'\rangle = U(\Lambda)|\Phi\rangle \tag{4.130}$$

$$|p\rangle \rightarrow |p'\rangle = U(\Lambda)|p\rangle \equiv |\Lambda p\rangle \tag{4.131}$$

Sidenote: from this we can deduce the following properties of $U(\Lambda)$

$$U(\Lambda)U(\Lambda)^\dagger = 1 \tag{4.132}$$

$$U(1) = 1 \tag{4.133}$$

$$U(\Lambda_1)U(\Lambda_2)^\dagger = U(\Lambda_1\Lambda_2) \tag{4.134}$$

$$U(\Lambda)^\dagger \hat{P} U(\Lambda)^\dagger = \Lambda\hat{P} \tag{4.135}$$

In quantum mechanics the classical field is associated with a matrix element of the field operator

$$\phi_i(x) \to \langle\Phi_\alpha|\hat{\phi}_i(x)|\Phi_\beta\rangle \tag{4.136}$$

the Lorentz transformed field $\phi_i'(x')$ is associated with the transformed matrix element of the field operator $\hat{\phi}$

$$\phi_i'(x') \to \langle\Phi_\alpha'|\hat{\phi}_i(x')|\Phi_\beta'\rangle \tag{4.137}$$

Using the definitions above we see

$$D(\Lambda)_i^{\ j}\langle\Phi_\alpha|\hat{\phi}_j(x)|\Phi_\beta\rangle = \langle\Phi_\alpha'|\hat{\phi}_j(x')|\Phi_\beta'\rangle = \langle\Phi_\alpha|U^{-1}(\Lambda)\hat{\phi}_i(x')U(\Lambda)|\Phi_\beta\rangle \tag{4.138}$$

As this must hold for any states we obtain

$$U^{-1}(\Lambda)\hat{\phi}_i(x')U(\Lambda) = D(\Lambda)_i^{\ j}\hat{\phi}_j(x) \tag{4.139}$$

or equivalently

$$U(\Lambda)\hat{\phi}_i(x)U^{-1}(\Lambda) = D^{-1}(\Lambda)_i^{\ j}\hat{\phi}_j(x') \tag{4.140}$$

It is reasonable to assume that $U(\Lambda)$ should contain an (hermitian) operator $\hat{P}^\mu$, $\hat{J}^{\rho\sigma}$ for each of the $4+6$ generators

$$U(\Lambda, a) = e^{i(a_\mu\hat{P}^\mu - \frac{1}{2}\omega_{\rho\sigma}\hat{J}^{\rho\sigma})} \tag{4.141}$$

$$\simeq 1 + ia_\mu\hat{P}^\mu - \frac{i}{2}\omega_{\rho\sigma}\hat{J}^{\rho\sigma} \tag{4.142}$$

Depending on the transformation properties of the fields the operators $\hat{P}^\mu$, $\hat{J}^{\rho\sigma}$ maybe look different.

- Translations and $\hat{P}^\mu$: Under linear translations each component of any field (independent of the field transformation properties) should behave like a scalar

$$\hat{P}_\mu = i\partial_\mu \qquad \text{(momentum operator)} \tag{4.143}$$

- Boosts/rotations and $\hat{J}^{\rho\sigma}$:

$$\text{scalar} \qquad \hat{J}^{\rho\sigma} = i(x_\rho\partial_\sigma - x_\sigma\partial_\rho) \qquad \text{(angular momentum operator)} \tag{4.144}$$

4.3.6 Field representations for internal symmetry

$$\phi_i(x) \to \phi_j'(x) = D(\Lambda)_i^{\ j}\phi_j(x) \tag{4.145}$$

$$U(g)\hat{\phi}_i(x)U^{-1}(g) = D(\Lambda)_i^j\hat{\phi}_i(x') \tag{4.146}$$

With operators $\hat{Q}^a$ and matrices T^a

$$U(g) = e^{-i\theta_a Q^a} \simeq 1 - i\theta_a\hat{Q}^a \tag{4.147}$$

$$D(g)_i^j = e^{-i\theta_a T^a} \simeq \delta_i^j - i\theta_a(T^a)_i^j \tag{4.148}$$

$$[\hat{Q}^a, \hat{\phi}_i(x)] = (T^a)_i^j\phi_j(x) \tag{4.149}$$

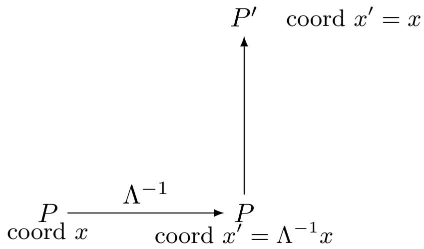
4.3.7 Examples

$$\text{scalar field} \qquad U^\dagger(\Lambda,a)\hat{\phi}(\Lambda x + a)U(\Lambda,a) = \phi(x) \tag{4.150}$$

$$\text{spinor field} \qquad U^\dagger(\Lambda,a)\hat{\Psi}(\Lambda x + a)U(\Lambda,a) = S(\Lambda)\hat{\Psi}(x) \tag{4.151}$$

$$\text{vector field} \qquad U^\dagger(\Lambda,a)\hat{A}^\nu(\Lambda x + a)U(\Lambda,a) = \Lambda^\nu{}_\mu\hat{A}^\mu(x) \tag{4.152}$$

4.4 Classical field theory



Infinitesimal Lorentz transformation with generator $(J_V^{\mu\nu})^\rho{}_\sigma = i(\eta^{\mu\rho}\delta^\nu_\sigma - \eta^{\nu\rho}\delta^\mu_\sigma)$. Coordinates of point P are x^ρ - same point P has coordinates x'^ρ after the Lorentz trafo

$$
\begin{aligned}
x'^\rho &= x^\rho + \delta x^\rho && (4.153)\\
&= x^\rho - \frac{i}{2}\omega_{\mu\nu}(J_V^{\mu\nu})^\rho{}_\sigma x^\sigma && (4.154)
\end{aligned}
$$

Local variation at fixed point P (but different coordinate $x' = \Lambda x$ after traft)

$$
\delta\phi \equiv \phi'(x') - \phi(x) \qquad (4.155)
$$

variation at fixed coordinate x

$$
\begin{aligned}
\delta_0\phi &\equiv \phi'(x) - \phi(x) && (4.156)\\
&= \phi'(x' - \delta x) - \phi(x) && (4.157)\\
&= \cancel{\phi'(x')} - \partial_\mu\phi'|_{x'-\delta x}\delta x - \cancel{\phi(x)} && (4.158)\\
&= -\delta x^\rho\,\partial_\rho\phi'(x) && (4.159)\\
&= -\delta x^\rho\,\partial_\rho\phi(x) \qquad \text{why?} && (4.160)\\
&= \frac{i}{2}\omega_{\mu\nu}(J_V^{\mu\nu})^\rho{}_\sigma x^\sigma\,\partial_\rho\phi(x) && (4.161)\\
&= \frac{i}{2}\omega_{\mu\nu}(\eta^{\mu\rho}\delta^\nu_\sigma - \eta^{\mu\sigma}\delta^\nu_\rho)x^\sigma\,\partial_\rho\phi(x) && (4.162)\\
&= \frac{i}{2}\omega_{\mu\nu}(x^\nu\,\partial^\mu - x^\mu\,\partial^\nu)\phi(x) && (4.163)\\
&= \frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}\phi(x) && (4.164)
\end{aligned}
$$

where we used $\phi'(x' - \delta x) = \phi(x' - \delta x)$.

Total variation

$$
\begin{aligned}
\tilde{\delta}\phi_r(x) &\equiv \phi'_r(x) - \phi_r(x) && (4.165)\\
&= \phi'_r(x) - \phi'_r(x') + \phi'_r(x') - \phi_r(x) && (4.166)\\
&= \frac{\partial\phi'_r(x)}{\partial x^\mu}\delta x^\mu + \delta\phi_r(x) && (4.167)\\
&\simeq \frac{\partial\phi_r(x)}{\partial x^\mu}\delta x^\mu + \delta\phi_r(x) && (4.168)
\end{aligned}
$$

4.5 Quantization - real scalar spin 0 field (Klein Gordon field)

4.5.1 Classical

a) Lagrangian for scalar field $\phi = \phi(\mathbf{x}, t)$

$$\mathcal{L}(\phi, \partial\phi) = \frac{1}{2} g^{\mu\nu} \partial_\nu \phi \partial_\mu \phi - \frac{m^2}{2}\phi^2 \tag{4.169}$$

b) Euler Lagrange equation

$$0 = \partial_\rho \left(\frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} \qquad \to \partial_\rho \partial^\rho \phi + m^2 \phi = 0 \tag{4.170}$$

c) Conjugated momentum

$$\pi(\mathbf{x}, t) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \qquad \to \pi = \dot{\phi} \tag{4.171}$$

d) Hamilton density

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} \qquad \to \mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2 \tag{4.172}$$

e) Hamiltonian

$$H = \int d^3x \; \frac{1}{2}\pi(x)^2 - \phi(x) \triangle \phi(x) + \frac{m^2}{2}\phi(x)^2 \tag{4.173}$$

f) Poisson brackets I (field and momentum)

$$\{\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)\} = \int d^3z \left(\frac{\partial \phi(\mathbf{x}, t)}{\partial \phi(\mathbf{z}, t)} \frac{\partial \pi(\mathbf{y}, t)}{\partial \pi(\mathbf{z}, t)} - \frac{\partial \phi(\mathbf{x}, t)}{\partial \pi(\mathbf{z}, t)} \frac{\partial \pi(\mathbf{y}, t)}{\partial \phi(\mathbf{z}, t)} \right) \tag{4.174}$$

$$= \delta^{(3)}(\mathbf{x} - \mathbf{y}) \tag{4.175}$$

$$\{\phi(\mathbf{x}, t), \phi(\mathbf{y}, t)\} = 0 \tag{4.176}$$

$$\{\pi(\mathbf{x}, t), \pi(\mathbf{y}, t)\} = 0 \tag{4.177}$$

g) Poisson brackets (Hamiltonian and field) II

$$\{H, \phi(\mathbf{y}, t)\} = -\pi(\mathbf{y}, t) \tag{4.178}$$

$$\{H, \pi(\mathbf{y}, t)\} = m^2\phi(\mathbf{y}, t) - \triangle\phi(\mathbf{y}) \tag{4.179}$$

h) Equations of motion

$$\dot{\phi}(\mathbf{y}, t) = -\{H, \phi\} \qquad \to \qquad \dot{\phi}(\mathbf{y}, t) = \pi(\mathbf{y}, t) \tag{4.180}$$

$$\dot{\pi}(\mathbf{y}, t) = -\{H, \phi\} \qquad \to \qquad \dot{\pi}(\mathbf{y}, t) = -m^2\phi(\mathbf{y}, t) + \triangle\phi(\mathbf{y}) \tag{4.181}$$

$$\to \ddot{\phi}(\mathbf{y}, t) + \triangle\phi(\mathbf{y}) - m^2\phi(\mathbf{y}, t) = 0 \tag{4.182}$$

$$\to \Box\phi(\mathbf{y}) + m^2\phi(\mathbf{y}, t) = 0 \tag{4.183}$$

i) Noether theorem

$$\phi_i(x) \to \phi_i'(x) = \phi_i(x) + \delta\phi_i(x) \tag{4.184}$$

$$j^\rho = \frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi_i)}\delta\phi_i - X^\rho \quad \to \quad \partial_\rho j^\rho = 0 \tag{4.185}$$

$$= \frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi_i)}\delta\phi_i + \mathcal{L}\delta x^\rho \tag{4.186}$$

Using Poincare invariance

$$x' = \Lambda x + a \simeq x + \omega x + \epsilon \tag{4.187}$$

$$\delta x^\alpha \equiv x'^\alpha - x^\alpha \tag{4.188}$$

$$= \omega^\alpha_{\ \beta} x^\beta + \epsilon^\alpha \tag{4.189}$$

Implied scalar field change

$$\phi'(\Lambda x + a) = \phi(x) \qquad \to \qquad \phi'(x) \simeq \phi(x - \omega x - \epsilon) \tag{4.190}$$

$$\delta\phi \equiv \phi'(x) - \phi(x) \tag{4.191}$$

$$= \partial_\mu \phi(x) \cdot (-\delta x^\mu) \tag{4.192}$$

$$= -\omega^{\mu\nu} x_\nu \partial_\mu \phi(x) - \epsilon^\mu \partial_\mu \phi(x) \tag{4.193}$$

Implied Langrangian (scalar) change

$$\mathcal{L}'(\Lambda x + a) = \mathcal{L}(x) \qquad \to \qquad \mathcal{L}'(x) = \mathcal{L}(x - \omega x - \epsilon) \tag{4.194}$$

$$\delta\mathcal{L}(x) \equiv \mathcal{L}'(x) - \mathcal{L}(x) \tag{4.195}$$

$$= (-\delta x^\mu)\, \partial_\mu \mathcal{L} \tag{4.196}$$

$$= -\partial_\mu(\delta x^\mu\, \mathcal{L}) + \mathcal{L}\partial_\mu(\delta x^\mu) \tag{4.197}$$

$$= \partial_\mu(-\omega^\mu_{\ \nu} x^\nu \mathcal{L} - \epsilon^\mu \mathcal{L}) \tag{4.198}$$

$$\to X^\mu = -\omega^\mu_{\ \nu} x^\nu \mathcal{L} - \epsilon^\mu \mathcal{L} = -\mathcal{L}\delta x^\mu \tag{4.199}$$

Now we can calculate the Noether current

$$j^\rho = (\partial^\rho \phi)\,[-\omega^{\mu\nu} x_\nu \partial_\mu \phi - \epsilon^\mu \partial_\mu \phi] + \omega^{\mu\nu} x_\nu \mathcal{L} + \epsilon^\rho \mathcal{L} \tag{4.200}$$

Translational invariance ($\sim -\epsilon^\mu$ coeff)

$$T^\rho_{\ \mu} = (\partial^\rho \phi)(\partial_\mu \phi) - g^\rho_{\ \mu} \mathcal{L} \tag{4.201}$$

$$T^{\rho\mu} = (\partial^\rho \phi)(\partial^\mu \phi) - g^{\rho\mu} \mathcal{L} \tag{4.202}$$

Lorenz invariance ($\sim \frac{1}{2}\omega^{\mu\nu}$ coeff)

$$\mathcal{M}^\rho_{\ \mu\nu} = (\partial^\rho \phi)(x_\mu \partial_\nu - x_\nu \partial_\mu)\phi + (g^\rho_{\ \mu} x_\nu - g^\rho_{\ \nu} x_\mu)\mathcal{L} \tag{4.203}$$

$$\mathcal{M}^{\rho\mu\nu} = (\partial^\rho \phi)(x^\mu \partial^\nu - x^\nu \partial^\mu)\phi + (g^{\rho\mu} x^\nu - g^{\rho\nu} x^\mu)\mathcal{L} \tag{4.204}$$

$$= T^{\rho\nu} x^\mu - T^{\rho\mu} x^\nu \tag{4.205}$$

Conserved quantities (with $\partial^\mu \phi = g^{\mu\nu}\partial_\nu \phi$ we see $\partial^0 \phi = g^{0\nu}\partial_\nu \phi = g^{00}\partial_0 \phi = \partial_0 \phi \equiv \dot{\phi}$

$$P^\mu = \int d^3x\; T^{0\mu} \tag{4.206}$$

$$= \int d^3x\; (\dot{\phi}\partial^\mu \phi) - \frac{1}{2} g^{0\mu}(\partial_\alpha \phi \partial^\alpha \phi - m^2 \phi^2) \tag{4.207}$$

$$\text{energy} \to P^0 = \frac{1}{2}\int d^3x\; \dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2 \tag{4.208}$$

$$\text{momentum} \to P^k = \int d^3x\; \dot{\phi}(\partial^k \phi) \tag{4.209}$$

$$J^{\mu\nu} = \int d^3x \; M^{0\mu\nu} \tag{4.210}$$
$$= \int d^3x \; (T^{0\nu}x^{\mu} - T^{0\mu}x^{\nu}) \tag{4.211}$$
$$\text{rotation} \rightarrow J^{ik} = \int d^3x \; (T^{0k}x^{i} - T^{0i}x^{k}) \tag{4.212}$$
$$= \int d^3x \; \dot{\phi}(x^i\partial^k - x^k\partial^i)\phi \tag{4.213}$$
$$\text{angular momentum} \rightarrow J_j \equiv \frac{1}{2}\epsilon_{jik}J^{ik} \tag{4.214}$$
$$= \frac{1}{2}\int d^3x \; (\epsilon_{jik}T^{0k}x^i - \epsilon_{jik}T^{0i}x^k) \tag{4.215}$$
$$= \int d^3x \; (\epsilon_{jik}x^iT^{0k}) \tag{4.216}$$
$$= \int d^3x \; (\mathbf{x} \times \mathcal{P}) \tag{4.217}$$
$$\text{boost} \rightarrow J^{0k} = \int d^3x \; \dot{\phi}(x^0\partial^k - x^k\partial^0)\phi + x^k\mathcal{L} \tag{4.218}$$

4.5.2 Quantized

a) Quantization (obtained from Poisson brackets I)

$$[\hat{\phi}(\mathbf{x},t), \hat{\phi}(\mathbf{y},t)] = 0 \tag{4.219}$$
$$[\hat{\phi}(\mathbf{x},t), \hat{\pi}(\mathbf{y},t)] = i\delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{4.220}$$
$$[\hat{\pi}(\mathbf{x},t), \hat{\pi}(\mathbf{y},t)] = 0 \tag{4.221}$$
$$\mathcal{H} = \mathcal{H}(\hat{\phi}, \hat{\pi}) \tag{4.222}$$

b) Time evolution in the Heisenberg picture (calculated from $\hat{\mathcal{H}}, \hat{\phi}$ and $\hat{\pi}$)

$$\dot{\hat{\phi}}(x) = i[\hat{H}, \hat{\phi}(x)] = \hat{\pi}(x) \tag{4.223}$$
$$\dot{\hat{\pi}}(x) = i[\hat{H}, \hat{\pi}(x)] = \triangle\hat{\phi}(x) - m^2\hat{\phi}(x) \tag{4.224}$$

c) Equations of motion (operator identity)

$$(\Box + m^2)\hat{\phi}(x) = 0 \tag{4.225}$$

d) (free) field operators (Heisenberg picture) are derived as ansatz to solve the equations of motion

$$\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}(\hat{a}_{\mathbf{p}}e^{-ipx} + \hat{a}^{\dagger}_{\mathbf{p}}e^{ipx}) \tag{4.226}$$
$$\hat{\phi}(\mathbf{x},t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}(\hat{a}_{\mathbf{p}}e^{-i(E_pt-\mathbf{p}\cdot\mathbf{x})} + \hat{a}^{\dagger}_{\mathbf{p}}e^{i(E_pt-\mathbf{p}\cdot\mathbf{x})}) \tag{4.227}$$

e) Commutators of the ladder operators

$$[a_{\mathbf{p}}, a_{\mathbf{q}}] = 0 \tag{4.228}$$
$$[a_{\mathbf{p}}, a^{\dagger}_{\mathbf{q}}] = (2\pi)^3 2E_{\mathbf{p}}\delta^{(3)}(\mathbf{p}-\mathbf{q}) \tag{4.229}$$
$$[a^{\dagger}_{\mathbf{p}}, a^{\dagger}_{\mathbf{q}}] = 0 \tag{4.230}$$

Hamiltonian

$$\hat{H} = \frac{1}{2}\int d^3x\; \hat{\pi}(x)^2 + (\nabla\hat{\phi}(x))^2 + m^2\hat{\phi}(x)^2 \tag{4.231}$$

$$= \int d^3\tilde{p}\; E_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} + \frac{1}{2}\int d^3p\delta^{(3)}(0) \tag{4.232}$$

The calculation of the commutator is now simple

$$[\hat{H}, \hat{a}^\dagger_{\mathbf{p}}] = E_{\mathbf{q}}\hat{a}^\dagger_{\mathbf{q}} \tag{4.233}$$

f) Conserved quantities

$$\hat{P}^0 = \frac{1}{2}\int d^3x\; \hat{\pi}^2 + (\nabla\hat{\phi})^2 + m^2\hat{\phi}^2 \tag{4.234}$$

$$= \int d^3\tilde{p}\; E_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} + \frac{1}{2}\int d^3p\,\delta^{(3)}(0) \tag{4.235}$$

$$\hat{P}^k = \int d^3x\; \hat{\pi}(\partial^k\hat{\phi}) \tag{4.236}$$

$$= \int d^3x\, \mathbf{p}\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} \tag{4.237}$$

$$\hat{J}^{ik} = \int d^3x\; \dot{\hat{\phi}}(x^i\partial^k - x^k\partial^i)\hat{\phi} \tag{4.238}$$

$$= \int d^3x\; \hat{\pi}(x^i\partial^k - x^k\partial^i)\hat{\phi} \tag{4.239}$$

$$\hat{J}^{0k} = \int d^3x\; \dot{\hat{\phi}}(x^0\partial^k - x^k\partial^0)\hat{\phi} + \frac{1}{2}x^k(\partial^\mu\hat{\phi}\partial_\mu\hat{\phi} - m^2\hat{\phi}^2) \tag{4.240}$$

$$= \int d^3x\; (x^0\dot{\hat{\phi}}\partial^k\hat{\phi} - x^k\dot{\hat{\phi}}^2) + \frac{1}{2}x^k(\dot{\hat{\phi}}^2 - (\partial_k\hat{\phi})^2 - m^2\hat{\phi}^2) \tag{4.241}$$

$$= \int d^3x\; x^0\dot{\hat{\phi}}\partial^k\hat{\phi} + \frac{1}{2}x^k(-\dot{\hat{\phi}}^2 - (\partial_k\hat{\phi})^2 - m^2\hat{\phi}^2) \tag{4.242}$$

$$= \int d^3x\; x^0\hat{\pi}\partial^k\hat{\phi} + \frac{1}{2}x^k(-\hat{\pi}^2 - (\partial_k\hat{\phi})^2 - m^2\hat{\phi}^2) \tag{4.243}$$

g) Commutation relations

$$[P^\mu, P^\nu] = \tag{4.244}$$

$$[P^\mu, J^{\rho\sigma}] = \tag{4.245}$$

h) Commutation relations

$$[\hat{P}^k, \hat{\phi}(y)] = \int d^3x\; [\hat{\pi}(x)(\partial^k\hat{\phi}(x)), \hat{\phi}(y)] \tag{4.246}$$

$$= \int d^3x\; \hat{\pi}(x)(\partial^k\hat{\phi}(x))\hat{\phi}(y) - \hat{\phi}(y)\hat{\pi}(x)(\partial^k\hat{\phi}(x)) \tag{4.247}$$

$$= \int d^3x\; \hat{\pi}(x)(\partial^k\hat{\phi}(x))\hat{\phi}(y) - \hat{\pi}(x)\hat{\phi}(y)(\partial^k\hat{\phi}(x)) + i\int d^3x\; \delta^{(3)}(\mathbf{x}-\mathbf{y})\partial^k\hat{\phi}(x) \tag{4.248}$$

$$= i\partial^k\hat{\phi}(y) \tag{4.249}$$

$$[\hat{P}^0, \hat{\phi}(y)] = \tag{4.250}$$

$$\tag{4.251}$$

$$[\hat{J}^{ik}, \hat{\phi}(y)] = \int d^3x \, [\hat{\pi}(x)(x^i\partial^k - x^k\partial^i)\hat{\phi}(x), \hat{\phi}(y)] \tag{4.252}$$

$$= \int d^3x \, x^i[\hat{\pi}(x)(\partial^k\hat{\phi}(x)), \hat{\phi}(y)] - x^k[\hat{\pi}(x)(\partial^i\hat{\phi}(x)), \hat{\phi}(y)] \tag{4.253}$$

$$= \int d^3x \, \left(-x^i\delta^{(3)}(\mathbf{x}-\mathbf{y})(\partial^k\hat{\phi}(x)) + x^k\delta^{(3)}(\mathbf{x}-\mathbf{y})(\partial^i\hat{\phi}(x))\right) \tag{4.254}$$

$$= -i(y^i\partial^k - y^k\partial^i)\hat{\phi}(y) \tag{4.255}$$

$$[\hat{J}^{0k}, \hat{\phi}(y)] = \tag{4.256}$$

$$[P^\mu, a^\dagger_{\mathbf{p}}] = p^\mu a^\dagger_{\mathbf{p}} \tag{4.257}$$

$$[J^{\rho\sigma}, a^\dagger_{\mathbf{p}}] = \tag{4.258}$$

$$\tag{4.259}$$

Then

$$\left(1 - \frac{i}{2}\omega_{\mu\nu}\hat{J}^{\mu\nu}\right)\hat{\phi}(x)\left(1 + \frac{i}{2}\omega_{\mu\nu}\hat{J}^{\mu\nu}\right) = \tag{4.260}$$

i) Hilbert space

- as commutator algebra for $\hat{a}$ and $\hat{a}^\dagger$ is as for harmonic oscillator we utilize the same Hilbert space (with $E_{\mathbf{p}} = \sqrt{m^2 + \mathbf{p}^2}$ and therefore $p = (E_{\mathbf{p}}, \mathbf{p}) = (\sqrt{m^2 + \mathbf{p}^2}, \mathbf{p})$ - so $\mathbf{p}$ defines p)

$$a_{\mathbf{p}}|0\rangle \equiv 0 \tag{4.261}$$

$$|p\rangle \equiv \sqrt{2E_{\mathbf{p}}}a^\dagger_{\mathbf{p}}|0\rangle \tag{4.262}$$

$$\langle p|p'\rangle = (2\pi)^3 2E_{\mathbf{p}}\delta^3(\mathbf{p} - \mathbf{p}') \tag{4.263}$$

$$\hat{P}^\mu|p\rangle = \sqrt{2E_{\mathbf{p}}}\hat{P}^\mu a^\dagger_{\mathbf{p}}|0\rangle = \sqrt{2E_{\mathbf{p}}}(a^\dagger_{\mathbf{p}}\hat{P}^\mu + p^\mu a^\dagger_{\mathbf{p}})|0\rangle = p^\mu|p\rangle \tag{4.264}$$

$$\hat{P}^\mu|p_1, ..., p_n\rangle = (p^\mu_1 + ... + p^\mu_n)|p_1, ..., p_n\rangle \tag{4.265}$$

$$U(\Lambda)|p\rangle = |\Lambda p\rangle = \sqrt{2E_{\Lambda\mathbf{p}}}a^\dagger_{\Lambda\mathbf{p}}|0\rangle \tag{4.266}$$

$$U(\Lambda)|p_1, p_2, ...\rangle = |\Lambda p_1, \Lambda p_2, ...\rangle \tag{4.267}$$

$$U(\Lambda)a^\dagger_{\mathbf{p}}U^{-1}(\Lambda) = \sqrt{\frac{E_{\Lambda\mathbf{p}}}{E_{\mathbf{p}}}}a^\dagger_{\Lambda\mathbf{p}} \tag{4.268}$$

$$U(\Lambda)a_{\mathbf{p}}U^{-1}(\Lambda) = \sqrt{\frac{E_{\Lambda\mathbf{p}}}{E_{\mathbf{p}}}}a_{\Lambda\mathbf{p}} \tag{4.269}$$

$$\tag{4.270}$$

4.6 Quantization - complex scalar spin 0 field

4.6.1 Classical

a) Lagrangian

For complex scalar field $\varphi = \varphi(\mathbf{x}, t) = \varphi_1(\mathbf{x}, t) + i\varphi_2(\mathbf{x}, t)$. The two real (fundamental) fields can be chosen as φ_1 and φ_2 or as φ and φ^*

$$\mathcal{L}(\varphi, \varphi^*, \partial\varphi, \partial\varphi^*) = (\partial_\mu \varphi^*)(\partial^\mu \varphi) - m^2 \varphi^* \varphi \tag{4.271}$$

b) Euler Lagrange equation

$$\partial_\mu \partial^\mu \varphi^* + m^2 \varphi^* = 0 \tag{4.272}$$

$$\partial_\mu \partial^\mu \varphi + m^2 \varphi = 0 \tag{4.273}$$

c) Conjugated momentum

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \varphi)} = \partial^0(\varphi^*) \qquad \rightarrow \pi = \partial^0 \varphi^* \tag{4.274}$$

$$\pi^* = \frac{\partial \mathcal{L}}{\partial(\partial_0 \varphi^*)} = \partial^0(\varphi) \qquad \rightarrow \pi^* = \partial^0 \varphi = (\partial^0 \varphi^*)^* \equiv (\pi)^* \tag{4.275}$$

d) Hamiltonian density

e) Hamiltonian

$$H = \int d^3x \left(\pi^* \pi + (\nabla \varphi^*) \cdot (\nabla \varphi) + m^2 \varphi^* \varphi \right) \tag{4.276}$$

f) Poisson brackets I (field and momentum)

g) Poisson brackets II (Hamiltonian and field)

h) Equations of motion

i) Noether theorem

$$\phi_i(x) \rightarrow \phi'_i(x) = \phi_i(x) + \delta\phi_i(x) \tag{4.277}$$

$$j^\rho = \frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi_i)} \delta\phi_i - X^\rho \quad \rightarrow \quad \partial_\rho j^\rho = 0 \tag{4.278}$$

$$= \frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi_i)} \delta\phi_i + \mathcal{L} \delta x^\rho \tag{4.279}$$

- Poincare invariance leads to energy, momentum and angular momentum conservation
- One more internal symmetry (meaning $\delta x = 0$)

$$\varphi'(x) = e^{-i\alpha} \varphi(x) \simeq (1 - i\alpha)\varphi(x) \tag{4.280}$$

$$\delta\varphi(x) \equiv \varphi'(x) - \varphi(x) \tag{4.281}$$

$$\simeq -i\varphi(x)\delta\alpha \tag{4.282}$$

$$\delta\varphi^*(x) \simeq +i\varphi^*(x)\delta\alpha \tag{4.283}$$

then

$$j = (\partial^\mu \varphi^*)\delta\varphi + (\partial^\mu \varphi)\delta\varphi^* \tag{4.284}$$

$$= i\alpha[-(\partial^\mu \varphi^*)\varphi + (\partial^\mu \varphi)\varphi^*] \tag{4.285}$$

4.6.2 Quantized

4.7 Quantization - spin-½ field (Dirac field)

4.7.1 Prelim

Pauli matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.286}$$

$$\rightarrow [\sigma_j, \sigma_k] = 2i\epsilon_{jkl}\sigma_l \tag{4.287}$$

$$\rightarrow \{\sigma_j, \sigma_k\} = 2\delta_{jk}\sigma_0 \tag{4.288}$$

and we also define

$$\sigma^\mu = (\sigma_0, \sigma_k) \tag{4.289}$$

$$\tilde{\sigma}^\mu = (\sigma_0, -\sigma_k) \tag{4.290}$$

General Dirac matrices

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}1_{n\times n} \qquad \text{Dirac algebra} \tag{4.291}$$

$$\rightarrow (\gamma^0)^2 = +1_{n\times n} \tag{4.292}$$

$$\rightarrow (\gamma^k)^2 = -1_{n\times n} \tag{4.293}$$

$$S^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu] \qquad n\text{-dimensional rep. of Lorentz algebra because ...} \tag{4.294}$$

$$\rightarrow [S^{\mu\nu}, S^{\rho\sigma}] = i(g^{\nu\rho}S^{\mu\sigma} - g^{\mu\rho}S^{\nu\sigma} - g^{\nu\sigma}S^{\mu\rho} + g^{\mu\sigma}S^{\nu\rho}) \tag{4.295}$$

Theorem: There is **exactly one irreducible representation** of the Dirac matrices and the irrep is 4-dimensional.

Representations of the γ-matrices

1. **Weyl/chiral basis/High-energy representation** - for 4d-Minkowski space SO(1,3) - needs 4 γ matrices which are coincidentally 4×4 matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1_{2\times2} \\ 1_{2\times2} & 0 \end{pmatrix} \qquad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} \tag{4.296}$$

$$\rightarrow \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix} \tag{4.297}$$

$$\rightarrow S^{0k} \equiv \frac{i}{4}[\gamma^0, \gamma^k] = \frac{i}{2}\gamma^0\gamma^k = -\frac{i}{2}\begin{pmatrix} \sigma_k & 0 \\ 0 & -\sigma_k \end{pmatrix} \tag{4.298}$$

$$\rightarrow S^{jk} \equiv \frac{i}{4}[\gamma^j, \gamma^k] = \frac{1}{2}\epsilon^{jkl}\begin{pmatrix} \sigma_l & 0 \\ 0 & \sigma_l \end{pmatrix} \tag{4.299}$$

we see that in the Weyl basis $(\gamma^0)^\dagger = \gamma^0, (\gamma^k)^\dagger = -\gamma^k$ and $(\gamma^0)^2 = 1_{4\times4}$ and $(\gamma^k)^2 = -1_{4\times4}$.

2. **Dirac basis representation**

$$\gamma^0 = \begin{pmatrix} 1_{2\times2} & 0 \\ 0 & -1_{2\times2} \end{pmatrix} \qquad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} \tag{4.300}$$

we see that in the Dirac basis $(\gamma^0)^\dagger = \gamma^0, (\gamma^k)^\dagger = -\gamma^k$ and $(\gamma^0)^2 = 1_{4\times4}$ and $(\gamma^k)^2 = -1_{4\times4}$.

Notation

In the Weyl representation we define

$$\tilde{V} \equiv \sigma^\mu V_\mu = \sigma^\mu g_{\mu\nu} V^\nu = \begin{pmatrix} V_0 + V_3 & V_1 - iV_2 \\ V_1 + iV_2 & V_0 - V_3 \end{pmatrix} \tag{4.301}$$

$$\rightarrow \det(\sigma^\mu V_\mu) = V_0^2 - V_3^2 - V_1^2 - V_2^2 = g^{\mu\nu} V_\mu V_\nu \tag{4.302}$$

$$\not{p} \equiv \gamma^\mu p_\mu = \begin{pmatrix} 0 & \sigma^\mu p_\mu \\ \tilde{\sigma}^\mu p_\mu & 0 \end{pmatrix} \tag{4.303}$$

$$\rightarrow \not{p}\not{p} = \gamma^\mu \gamma^\nu p_\mu p_\nu = \frac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) p_\mu p_\nu = g^{\mu\nu} p_\mu p_\nu = p^2 1_{4\times 4} \tag{4.304}$$

Dirac equation

With the Dirac spinor

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} \tag{4.305}$$

we can write the Dirac equation (which then implies Klein-Gordon equation)

$$(\not{p} - mc)\begin{pmatrix} \phi \\ \chi \end{pmatrix} = 0 \tag{4.306}$$

$$\rightarrow (\not{p} + mc)(\not{p} - mc)\begin{pmatrix} \phi \\ \chi \end{pmatrix} = 0 \tag{4.307}$$

$$\rightarrow (p^2 - m^2c^2)\begin{pmatrix} \phi \\ \chi \end{pmatrix} = 0 \tag{4.308}$$

In components this means

$$(\sigma^\mu p_\mu)\chi = mc\phi \tag{4.309}$$

$$(\tilde{\sigma}^\mu p_\mu)\phi = mc\chi \tag{4.310}$$

Lorentz Transformation

- Observe: For real ξ^μ the 2×2 matrices $\tilde{\xi} = \sigma^\mu \xi_\mu$ (see definition above) as well as $A(\sigma^\mu \xi_\mu)A^\dagger$ are self-adjoint and we can write (assuming $\eta_\nu \equiv \Lambda(A)\xi_\mu$ depends linearly on ξ - where $\Lambda(A)$ is a 4×4 matrix)

$$A(\sigma^\mu \xi_\mu)A^\dagger = \sigma^\mu \eta_\mu = \sigma^\mu(\Lambda(A)\xi_\mu) \tag{4.311}$$

- $\Lambda(A)$ is a representation of A because of

$$\rightarrow \sigma^\mu(\Lambda(A_1A_2)\xi)_\mu = A_1A_2(\sigma^\mu \xi_, u)(A_1A_2)^\dagger = \ldots = \sigma^\mu(\Lambda(A_1)\Lambda(A_2)\xi)_\mu \tag{4.312}$$

- If $A \in \mathrm{SL}(2,\mathbb{C})$ - meaning $\det A = 1$ and

$$\xi^\mu \xi_\mu = \det(\sigma^\mu \xi_\mu) = \det(A\sigma^\mu \xi_\mu A^\dagger) = \det(\sigma^\mu(\Lambda(A)\xi)_\mu) = (\Lambda(A)\xi)^\mu(\Lambda(A)\xi)_\mu \tag{4.313}$$

  then $\Lambda(A)$ is Lorentz Transformation for each $A \in \mathrm{SL}(2,\mathbb{C})$ - and also a 4-dimensional representation of $\mathrm{SL}(2,\mathbb{C})$

- For a LT invariant theory the LT transformed spinors must also fulfill the Dirac equation: Now consider LT

$$x' = \Lambda(A)x + a, \qquad p' = \Lambda(A)p \tag{4.314}$$

then we use the Dirac equation (in the LT transformed system) and the identity above

$$(\sigma^\mu p'_\mu)\chi' = (\sigma^\mu \Lambda(A) p_\mu)\chi' = A(\sigma^\mu p_\mu)A^\dagger \chi' \stackrel{!}{=} mc\phi' \tag{4.315}$$

$$(\tilde{\sigma}^\mu p'_\mu)\phi' = (\tilde{\sigma}^\mu \Lambda(A) p_\mu)\phi' = A(\tilde{\sigma}^\mu p_\mu)A^\dagger \phi' \stackrel{!}{=} mc\chi' \tag{4.316}$$

substituting $\chi'(x') = A^{\dagger^{-1}}\chi(x)$ and $\phi'(x') = A\phi(x)$

$$A(\sigma^\mu p_\mu)A^\dagger \chi' \stackrel{!}{=} mc\phi' \quad \to \quad A(\sigma^\mu p_\mu)\chi(x) = Amc\phi \tag{4.317}$$

$$A(\tilde{\sigma}^\mu p_\mu)A^\dagger \phi' \stackrel{!}{=} mc\chi' \quad \to \quad A(\tilde{\sigma}^\mu p_\mu)\phi(x) = Amc\chi \tag{4.318}$$

we recover the Dirac equation. So we conclude that Dirac spinors transform like

$$\psi'(x') = \begin{pmatrix} \phi'(x') \\ \chi'(x') \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A^{\dagger^{-1}} \end{pmatrix} \begin{pmatrix} \phi(x) \\ \chi(x) \end{pmatrix} \equiv S(A)\psi(x) \tag{4.319}$$

where $((A^\dagger)^{-1} = (A^{-1})^\dagger)$

$$S(A) = \begin{pmatrix} A & 0 \\ 0 & A^{\dagger^{-1}} \end{pmatrix}, \qquad S(A)^{-1} = \begin{pmatrix} A^{-1} & 0 \\ 0 & A^\dagger \end{pmatrix}, \qquad S(A)^\dagger = \begin{pmatrix} A^\dagger & 0 \\ 0 & A^{-1} \end{pmatrix} \tag{4.320}$$

- Then

$$S(A)(\gamma^\mu p_\mu)S(A)^{-1} = \begin{pmatrix} A & 0 \\ 0 & A^{\dagger^{-1}} \end{pmatrix} \left[p_0 \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix} + p_k \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} + \right] \begin{pmatrix} A^{-1} & 0 \\ 0 & A^\dagger \end{pmatrix} \tag{4.321}$$

$$= \begin{pmatrix} 0 & A(\sigma^\mu p_\mu)A^\dagger \\ A^{\dagger^{-1}}(\tilde{\sigma}^\mu p_\mu)A^{-1} & 0 \end{pmatrix} \tag{4.322}$$

$$= \ldots \tag{4.323}$$

$$= \Lambda^{-1}(A)^\mu_\nu \gamma^\nu p_\mu \tag{4.324}$$

or equivalently:

$$S(A)^{-1}\gamma^\mu S(A) = \Lambda^\mu_\nu \gamma^\nu \tag{4.325}$$

- And

$$\gamma^0 S^\dagger \gamma^0 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix} \begin{pmatrix} A^\dagger & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix} = \begin{pmatrix} A^{-1} & 0 \\ 0 & A^\dagger \end{pmatrix} = S^{-1} \tag{4.326}$$

- Consider parity transformation P

$$x' = Px = (x^0, -\vec{x}), \qquad p' = Pp = (p^0, -\vec{p}) \tag{4.327}$$

$$\to \sigma^\mu p'_\mu = \tilde{\sigma}^\mu p_\mu \tag{4.328}$$

$$\to \tilde{\sigma}^\mu p'_\mu = \sigma^\mu p_\mu \tag{4.329}$$

then

$$\psi'(x') = \begin{pmatrix} \phi'(x') \\ \chi'(x') \end{pmatrix} = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix} \begin{pmatrix} \phi(x) \\ \chi(x) \end{pmatrix} \equiv S(P)\psi(x) \tag{4.330}$$

with

$$S(P) = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix} = \gamma^0 \tag{4.331}$$

then

$$
\begin{aligned}
S(P)^{-1}\gamma^0 S(P) &= \gamma^0\gamma^0\gamma^0 = \gamma^0 && (4.332)\\
S(P)^{-1}\gamma^k S(P) &= \gamma^0\gamma^k\gamma^0 && (4.333)\\
&= (2g^{0k}1_{n\times n} - \gamma^k\gamma^0)\gamma^0 && (4.334)\\
&= -\gamma^k \underbrace{\gamma^0\gamma^0}_{=1} && (4.335)\\
&= -\gamma^k && (4.336)
\end{aligned}
$$

- Conjugate spinor

$$
\begin{aligned}
\bar{\psi}(x) &\equiv \psi^\dagger(x)\gamma^0 && (4.337)\\
&= (\phi^*(x)\ \chi^*(x))\begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix} = (\chi^*(x)\ \phi^*(x)) && (4.338)\\
\bar{\psi}'(x') &= (S(A)\psi(x))^\dagger\gamma^0 && (4.339)\\
&= \psi^\dagger(x)S(A)^\dagger\gamma^0 && (4.340)\\
&= \psi^\dagger(x)\gamma^0 S(A)^{-1} && (4.341)\\
&= \bar{\psi}(x)S(A)^{-1} && (4.342)
\end{aligned}
$$

- $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

$$
\begin{aligned}
P_+ &= \frac{1+\gamma_5}{2}, \qquad P_- = \frac{1-\gamma_5}{2} && (4.343)\\
&\to P_+ + P_- = 1 && (4.344)\\
&\to P_\pm^2 = 1 && (4.345)\\
&\to P_\pm^\dagger = P_\pm && (4.346)\\
&\to P_+\begin{pmatrix}\phi\\ \chi\end{pmatrix} = \begin{pmatrix}0\\ \chi\end{pmatrix}, \qquad P_-\begin{pmatrix}\phi\\ \chi\end{pmatrix} = \begin{pmatrix}\phi\\ 0\end{pmatrix} && (4.347)
\end{aligned}
$$

For the **Weyl** and the **Dirac** representation we can can show (using the hermiticity relations)

$$\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger \tag{4.348}$$

In general

$$
\begin{aligned}
[\gamma^\mu, S^{\rho\sigma}] &= (J^{\rho\sigma})^\mu{}_\nu\gamma^\nu && (4.349)\\
\gamma^0 S^{\rho\sigma}\gamma^0 &= (S^{\rho\sigma})^\dagger && (4.350)
\end{aligned}
$$

And

$$
\begin{aligned}
\bar{\Psi} &\equiv \Psi^\dagger\gamma^0 && (4.351)\\
\Psi &\to S(\Lambda)\Psi \qquad \text{Dirac spinor} && (4.352)\\
\bar{\Psi} &\to \bar{\Psi}S^{-1}(\Lambda) && (4.353)\\
x' &= \Lambda x + a \simeq (1+\omega)x + \epsilon && (4.354)\\
\Psi'(\Lambda x + a) &= S(\Lambda)\Psi(x) \qquad \text{Dirac spinor field} && (4.355)
\end{aligned}
$$

$$\to \Psi'(x) = \Psi(x) - \frac{i}{2}\omega_{\rho\sigma}\left(S^{\rho\sigma} + L^{\rho\sigma}\right)\Psi(x) - \epsilon^\mu\partial_\mu\Psi(x) \qquad \text{with} \qquad L^{\rho\sigma} = i(x^\rho\partial^\sigma - x^\sigma\partial^\rho) \tag{4.356}$$

4.7.2 Classical

a) Lagrangian with $\bar{\Psi} = \Psi^\dagger\gamma^0$

$$\mathcal{L}(\Psi, \partial\Psi, \bar{\Psi}, \partial\bar{\Psi}) = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi \tag{4.357}$$

b) Euler-Lagrange equation

$$\bar{\Psi} : \quad \rightarrow \quad (i\gamma^\mu\partial_\mu - m)\Psi(x) = 0 \tag{4.358}$$
$$\Psi : \quad \rightarrow \quad (m + i\gamma^\mu\partial_\mu)\bar{\Psi}(x) = 0 \tag{4.359}$$

c) Conjugated momenta

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\Psi}} = i\gamma^0\bar{\Psi}(x) = i\Psi^\dagger(x) \tag{4.360}$$
$$\bar{\pi}(x) = 0 \tag{4.361}$$

d) Hamiltonian

$$\mathcal{H} = \pi\Psi + \bar{\pi}\bar{\Psi} - \mathcal{L} \tag{4.362}$$
$$= \pi(x)\gamma^0\gamma^k\partial_k\Psi(x) + m\bar{\Psi}\Psi \tag{4.363}$$

e) Poisson brackets

$$\{\Psi(\mathbf{x}, t), \pi(\mathbf{y}, t)\} = \delta^{(3)}(\mathbf{x} - \mathbf{y}) \tag{4.364}$$
$$\{\Psi(\mathbf{x}, t), \Psi(\mathbf{y}, t)\} = 0 \tag{4.365}$$
$$\{\pi(\mathbf{x}, t), \pi(\mathbf{y}, t)\} = 0 \tag{4.366}$$

f) Noether theorem

$$\phi_i(x) \rightarrow \phi'_i(x) = \phi_i(x) + \delta\phi_i(x) \tag{4.367}$$
$$j^\rho = \frac{\partial\mathcal{L}}{\partial(\partial_\rho\phi_i)}\delta\phi_i - X^\rho \quad \rightarrow \quad \partial_\rho j^\rho = 0 \tag{4.368}$$

with Poincare invariance

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\Psi \tag{4.369}$$
$$x' = \Lambda x + a \simeq x + \omega x + \epsilon \tag{4.370}$$

Implied spinor field change

$$\Psi'(\Lambda x + a) = S(\Lambda)\Psi(x) \qquad \rightarrow \qquad \Psi'(x) \simeq S(\Lambda)\Psi(x - \omega x - \epsilon) \tag{4.371}$$
$$\delta\Psi \equiv \Psi'(x) - \Psi(x) \tag{4.372}$$
$$= -\frac{i}{2}\omega_{\rho\sigma}\left(S^{\rho\sigma} + L^{\rho\sigma}\right)\Psi(x) - \epsilon^\mu\partial_\mu\Psi(x) \qquad \text{with } L^{\rho\sigma} = i(x^\rho\partial^\sigma - x^\sigma\partial^\rho) \tag{4.373}$$

Implied Langrangian (scalar) change

$$\mathcal{L}'(\Lambda x + a) = \mathcal{L}(x) \qquad \rightarrow \qquad \mathcal{L}'(x) = \mathcal{L}(x - \omega x - \epsilon) \tag{4.374}$$
$$\delta\mathcal{L}(x) \equiv \mathcal{L}'(x) - \mathcal{L}(x) \tag{4.375}$$
$$= \partial_\mu(-\omega^\mu{}_\nu x^\nu\mathcal{L} - \epsilon^\mu\mathcal{L}) \tag{4.376}$$
$$\rightarrow X^\mu = -\omega^\mu{}_\nu x^\nu\mathcal{L} - \epsilon^\mu\mathcal{L} \tag{4.377}$$

Now we can calculate the Noether current

$$j^\mu = \bar{\Psi} i\gamma^\rho \left(\epsilon^\mu \partial_\mu \Psi - \frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu} + L^{\mu\nu})\Psi \right) + \epsilon^\rho \mathcal{L} + \omega^{\rho\sigma} x_\sigma \mathcal{L} \tag{4.378}$$

Translational invariance ($\sim \epsilon^\mu$ coeff)

$$T^\rho_{\ \mu} = \bar{\Psi} i\gamma^\rho \partial_\mu \Psi - g^\rho_{\ \mu}\mathcal{L} \tag{4.379}$$

Lorenz invariance ($\sim \omega^{\mu\nu}/2$ coeff)

$$\mathcal{M}^\rho_{\ \mu\nu} = \bar{\Psi}\gamma^\rho (S^{\mu\nu} + L^{\mu\nu})\Psi + (g^\rho_{\ \mu} x_\nu - g^\rho_{\ \nu} x_\mu)\mathcal{L} \tag{4.380}$$

4.7.3 Quantized

a) Quantization (obtained from Poisson brackets I)

$$[\Psi(\mathbf{x},t), \pi(\mathbf{y},t)]_+ = i\delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{4.381}$$
$$[\Psi(\mathbf{x},t), \Psi(\mathbf{y},t)]_+ = 0 \tag{4.382}$$
$$[\pi(\mathbf{x},t), \pi(\mathbf{y},t)]_+ = 0 \tag{4.383}$$

b) Time evolution in the Heisenberg picture (calculated from $\hat{\mathcal{H}}, \hat{\phi}$ and $\hat{\pi}$)

c) Equations of motion

d) Field operators

$$\hat{\Psi}(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}} \sum_{s=\pm 1/2} \left(u(p,s)\hat{a}(p,s)e^{-ipx} + v(p,s)\hat{b}^\dagger(p,s)e^{ipx} \right) \tag{4.384}$$

e) Commutators of the ladder operators

$$[\hat{a}(p,s), \hat{a}(p',s')] = 0 \tag{4.385}$$

f) Hilbert space

g)

$$[\hat{a}(p,s), J_z] = s\,\hat{a}(p,s) \tag{4.386}$$
$$[J_z, \hat{a}^\dagger(p,s)] = +s\,\hat{a}^\dagger(p,s) \tag{4.387}$$
$$[J_z, \hat{b}^\dagger(p,s)] = +s\,\hat{b}^\dagger(p,s) \tag{4.388}$$

h) With helicity $s = \pm\frac{1}{2}$

$$a^\dagger|0\rangle = |\mathrm{Fermion}, p, s\rangle \qquad p^0 = \sqrt{\mathbf{p}^2 + m^2} \tag{4.389}$$
$$b^\dagger|0\rangle = |\mathrm{AntiFermion}, p, s\rangle \qquad p^0 = \sqrt{\mathbf{p}^2 + m^2} \tag{4.390}$$
$$\rightarrow \langle 0|\hat{\Psi}(x)|\mathrm{F}, p, s\rangle = e^{-ipx}u(p,s) \tag{4.391}$$
$$\rightarrow \langle 0|\bar{\hat{\Psi}}(x)|\mathrm{AF}, p, s\rangle = e^{-ipx}\bar{v}(p,s) \tag{4.392}$$

4.8 Quantization - massive spin-1 field (Proca field)

4.8.1 Prelim Facts

$$\bar{\psi}_1\gamma^\mu \tag{4.393}$$
$$psi_2 \tag{4.394}$$

4.8.2 Classical

a) Lagrangian with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$\mathcal{L}(A_\mu, \partial_\nu A_\mu) = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{m^2}{2}A^\mu A_\mu \tag{4.395}$$

b) Euler-Lagrange equations

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial(\partial_\alpha A_\beta)} &= -\frac{1}{4}\left(g^{\mu\rho}g^{\nu\sigma}(\delta^\alpha_\rho\delta^\beta_\sigma - \delta^\alpha_\sigma\delta^\beta_\rho)F_{\mu\nu} + F^{\mu\nu}(\delta^\alpha_\mu\delta^\beta_\nu - \delta^\alpha_\nu\delta^\beta_\mu)\right) && (4.396)\\
&= -\frac{1}{4}\left((g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha})F_{\mu\nu} + (F^{\alpha\beta} - F^{\beta\alpha})\right) && (4.397)\\
&= -\frac{1}{4}\left((F^{\alpha\beta} - F^{\beta\alpha}) + (F^{\alpha\beta} - F^{\beta\alpha})\right) && (4.398)\\
&= -F^{\alpha\beta} && (4.399)\\
\frac{\partial\mathcal{L}}{\partial A_\beta} &= m^2 g^{\mu\nu}A_\mu\delta^\beta_\nu && (4.400)\\
&= m^2 A^\beta && (4.401)\\
&\to \partial_\alpha F^{\alpha\beta} + m^2 A^\beta = 0 \qquad \text{(Proca equation)} && (4.402)
\end{aligned}$$

And by ∂_β using symmetries

$$\begin{aligned}
0 &= \partial_\beta\partial_\alpha F^{\alpha\beta} + m^2\partial_\beta A^\beta && (4.403)\\
&= \frac{1}{2}(\partial_\beta\partial_\alpha F^{\alpha\beta} + \partial_\alpha\partial_\beta F^{\alpha\beta}) + m^2\partial_\beta A^\beta && (4.404)\\
&= \frac{1}{2}(\partial_\beta\partial_\alpha F^{\alpha\beta} + \partial_\beta\partial_\alpha F^{\beta\alpha}) + m^2\partial_\beta A^\beta && (4.405)\\
&= \frac{1}{2}\partial_\beta\partial_\alpha(F^{\alpha\beta} + F^{\beta\alpha}) + m^2\partial_\beta A^\beta && (4.406)\\
&\to \partial_\beta A^\beta = 0 && (4.407)
\end{aligned}$$

And

$$\begin{aligned}
0 &= \partial_\alpha(\partial^\alpha A^\beta - \partial^\beta A^\alpha) + m^2 A^\beta && (4.408)\\
&= \Box A^\beta - \partial^\beta\partial_\alpha A^\alpha + m^2 A^\beta && (4.409)\\
&\to (\Box + m^2)A^\beta = 0 && (4.410)
\end{aligned}$$

c) Conjugated momentum

$$\pi^\beta = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\beta)} \tag{4.411}$$
$$= -F^{0\beta} = F^{\beta 0} \tag{4.412}$$
$$= -\dot{A}^\beta + \partial^\beta A^0 \tag{4.413}$$
$$\pi^0 \equiv 0 \qquad \text{(2nd class constraint)} \tag{4.414}$$
$$\to \partial_\alpha F^{\alpha 0} + m^2 A^0 = 0 \tag{4.415}$$
$$\to \partial_\alpha \pi^\alpha + m^2 A^0 = 0 \tag{4.416}$$
$$\to A^0 = -\frac{1}{m^2}\partial_\alpha \pi^\alpha = -\frac{1}{m^2}\partial_k \pi^k \qquad \text{(auxiliary field)} \tag{4.417}$$

d) Hamiltonian

$$\mathcal{H} = \pi_k \dot{A}^k - \mathcal{L}|_{\pi_k = F_{k0},\, A_0 = -\frac{1}{m^2}\partial_k \pi^k} \tag{4.418}$$
$$= \pi_k(\partial^k A^0 - \pi^k) + \frac{1}{4}F^{ik}F_{ik} - \frac{m^2}{2}A^k A_k + \frac{1}{4}F^{00}F_{00} + \frac{1}{4}F^{0k}F_{0k} + \frac{1}{4}F^{k0}F_{k0} - \frac{m^2}{2}A^0 A_0 \tag{4.419}$$
$$= -\pi_k \pi^k - \frac{1}{m^2}\pi_k \Box \pi^k + \frac{1}{4}F^{ik}F_{ik} - \frac{m^2}{2}A^k A_k + 0 + \frac{1}{2}\pi^k \pi_k - \frac{1}{2m^2}(\partial_k \pi^k)(\partial_k \pi^k) \tag{4.420}$$
$$= -\frac{1}{2}\pi_k \pi^k + \frac{1}{2m^2}(\partial_k \pi^k)(\partial_k \pi^k) + \frac{1}{4}F^{ik}F_{ik} - \frac{m}{2}A^k A_k \tag{4.421}$$

e) Poisson brackets I

$$\{A^i(\mathbf{x},t), A^j(\mathbf{y},t)\} = 0 \tag{4.422}$$
$$\{A^i(\mathbf{x},t), \pi^j(\mathbf{y},t)\} = i g^{ij}\delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{4.423}$$
$$\{\pi^i(\mathbf{x},t), \pi^j(\mathbf{y},t)\} = 0 \tag{4.424}$$

f) Noether theorem

$$A^\mu(x) \to A'^\mu(x) = A^\mu(x) + \delta A^\mu(x) \tag{4.425}$$
$$j^\rho = \frac{\partial \mathcal{L}}{\partial(\partial_\rho A^\mu)}\delta A^\mu - X^\rho \quad \to \quad \partial_\rho j^\rho = 0 \tag{4.426}$$

with Poincare invariance

$$x' = \Lambda x + a \simeq x + \omega x + \epsilon \tag{4.427}$$
$$\delta x^\alpha \equiv x'^\alpha - x^\alpha \tag{4.428}$$
$$= \omega^\alpha{}_\beta x^\beta + \epsilon^\alpha \tag{4.429}$$

Implied vector field change

$$A'^\mu(\Lambda x + a) = \Lambda^\mu{}_\nu A^\nu(x) \qquad \to \qquad A'^\mu(x) \simeq (1 + \omega^\mu{}_\nu)A^\nu(x - \omega x - \epsilon) \tag{4.430}$$
$$\delta A^\mu \equiv A'^\mu(x) - A^\mu(x) \tag{4.431}$$
$$= -\omega^{\alpha\beta}x_\beta \partial_\alpha A^\mu(x) - \epsilon^\alpha \partial_\alpha A^\mu(x) + \omega^\mu{}_\nu A^\nu(x) \tag{4.432}$$

Implied Langrangian (scalar) change

$$\mathcal{L}'(\Lambda x + a) = \mathcal{L}(x) \qquad \rightarrow \qquad \mathcal{L}'(x) = \mathcal{L}(x - \omega x - \epsilon) \tag{4.433}$$
$$\delta\mathcal{L}(x) \equiv \mathcal{L}'(x) - \mathcal{L}(x) \tag{4.434}$$
$$= (-\delta x^\mu)\, \partial_\mu \mathcal{L} \tag{4.435}$$
$$= -\partial_\mu(\delta x^\mu\, \mathcal{L}) + \mathcal{L}\partial_\mu(\delta x^\mu) \tag{4.436}$$
$$= \partial_\mu(-\omega^\mu_{\ \nu} x^\nu \mathcal{L} - \epsilon^\mu \mathcal{L}) \tag{4.437}$$
$$\rightarrow X^\mu = -\delta x^\mu\, \mathcal{L} = -\omega^\mu_{\ \nu} x^\nu \mathcal{L} - \epsilon^\mu \mathcal{L} \tag{4.438}$$

Now we can calculate the Noether current

$$j^\rho = -F^\rho_{\ \mu}\left[-\omega^{\alpha\beta} x_\beta \partial_\alpha A^\mu - \epsilon^\alpha \partial_\alpha A^\mu + \omega^\rho_{\ \beta} A^\beta\right] + \omega^\rho_{\ \beta} x^\beta \mathcal{L} + \epsilon^\rho \mathcal{L} \tag{4.439}$$

Translational invariance ($\sim -\epsilon^\alpha$ coeff)

$$T^\rho_{\ \alpha} = -F^\rho_{\ \mu}(\partial_\alpha A^\mu) - g^\rho_{\ \alpha}\mathcal{L} \tag{4.440}$$
$$T^{\rho\alpha} = -F^{\rho\mu}(\partial^\alpha A_\mu) - g^{\rho\alpha}\mathcal{L} \tag{4.441}$$

Lorenz invariance ($\sim \frac{1}{2}\omega^{\alpha\beta}$ coeff)

$$\mathcal{M}^\rho_{\ \alpha\beta} = 2(-F^\rho_{\ \mu}[-x_\beta \partial_\alpha A^\mu + A_\beta] + g^\rho_{\ \alpha} x_\beta \mathcal{L}) \tag{4.442}$$
$$= 2(-x_\beta[-F^\rho_{\ \mu}(\partial_\alpha A^\mu) - g^\rho_{\ \alpha}\mathcal{L}] - F^\rho_{\ \mu} A_\beta) \tag{4.443}$$
$$= 2(-x_\beta T^\rho_{\ \alpha} - F^\rho_{\ \mu} A_\beta) \tag{4.444}$$
$$= \ldots \tag{4.445}$$
$$\mathcal{M}^{\rho\alpha\beta} = x^\alpha T^{\rho\beta} - x^\beta T^{\rho\alpha} + (-F^{\rho\mu} g_\mu^{\ \alpha} A^\beta + F^{\rho\mu} g_\mu^{\ \beta} A^\alpha) \tag{4.446}$$

Conserved quantities

$$P^\mu = \int d^3x\; T^{0\mu} \tag{4.447}$$
$$= \int d^3x\; [-F^{0\nu}(\partial^\mu A_\nu) - g^{0\mu}\mathcal{L}] \tag{4.448}$$
$$= \int d^3x\; [\pi^\nu(\partial^\mu A_\nu) - g^{0\mu}\mathcal{L}] \tag{4.449}$$
$$= \int d^3x\; [\pi^k(\partial^\mu A_k) - g^{0\mu}\mathcal{L}] \tag{4.450}$$
$$\rightarrow P^0 = \int d^3x\; [\pi^k \dot{A}_k - \mathcal{L}] = \text{Legendre trafo of } \mathcal{L} \tag{4.451}$$
$$= H \tag{4.452}$$
$$\rightarrow P^i = \int d^3x\; \pi^k(\partial^i A_k) \tag{4.453}$$

$$J^{\alpha\beta} = \int d^3x\; M^{0\alpha\beta} \tag{4.454}$$
$$= \int d^3x\; [x^\alpha T^{0\beta} - x^\beta T^{0\alpha} + (-F^{0\mu} g_\mu^{\ \alpha} A^\beta + F^{0\mu} g_\mu^{\ \beta} A^\alpha)] \tag{4.455}$$
$$= \int d^3x\; [x^\alpha T^{0\beta} - x^\beta T^{0\alpha} + (\pi^k g_k^{\ \alpha} A^\beta - \pi^k g_k^{\ \beta} A^\alpha)] \tag{4.456}$$
$$= \int d^3x\; [x^\alpha T^{0\beta} - x^\beta T^{0\alpha} + (\pi^\alpha A^\beta - \pi^\beta A^\alpha)] \qquad (\text{with } \pi^0 = 0) \tag{4.457}$$

4.8.3 Quantized

a) Quantization - operators $\hat{A}^i, \hat{\pi}^j$ and $\hat{A}^0 \equiv -\frac{1}{m}\partial^i \hat{\pi}_i$

$$[\hat{A}^i, \hat{A}^j] = 0 \tag{4.458}$$
$$[\hat{A}^i, \hat{\pi}^j] = ig^{ij}\delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{4.459}$$
$$[\hat{\pi}^i, \hat{\pi}^j] = 0 \tag{4.460}$$

b) Time evolution in the Heisenberg picture

$$\dot{\hat{A}}^i = i[\hat{H}, \hat{A}^i] \tag{4.461}$$
$$= -\hat{pi}^i - \frac{1}{m^2}\partial^i\partial^j\hat{\pi}_j \tag{4.462}$$
$$= -\hat{\pi}^i + \partial^i \hat{A}^0 \tag{4.463}$$
$$\dot{\hat{\pi}}^i = i[\hat{H}, \hat{\pi}^i] \tag{4.464}$$
$$= \partial_j \hat{F}^{ji} + m^2 \hat{A}^i \tag{4.465}$$

c) Equations of motion

d) Field operators (solution of $(\Box + m^2)\hat{A}^\nu = 0$ with $\hat{A}^\nu = (\hat{A}^\nu)^\dagger$

$$\hat{A}^\mu(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\sum_{\lambda=1,2,3}\left(\epsilon^\mu(p,\lambda)\hat{a}(p,\lambda)e^{-ipx} + \epsilon^\mu(p,\lambda)\hat{a}^\dagger(p,\lambda)e^{ipx}\right) \tag{4.466}$$

- Additional equation $\partial_\mu \hat{A}^\mu = 0$ requires $\epsilon^\mu(p,\lambda)p_\mu = 0$ for all λ
- And

$$\sum_{\lambda=1,2,3}\epsilon^\mu(p,\lambda)\epsilon^\nu(p,\lambda) = -g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2} \tag{4.467}$$

e) Commutators for ladder operators

$$[a(p,\lambda), a(q,\lambda')] = 0 \tag{4.468}$$
$$[a(p,\lambda), a^\dagger(q,\lambda')] = \delta_{\lambda\lambda'}(2\pi)^3 2E_{\mathbf{p}}\delta^{(3)}(\mathbf{p}-\mathbf{q}) \tag{4.469}$$
$$[a^\dagger(p,\lambda), a^\dagger(q,\lambda')] = 0 \tag{4.470}$$

$$[\hat{H}, \hat{A}^k(x)] = -i\dot{\hat{A}}^k(x) \tag{4.471}$$
$$\to [\hat{H}, \hat{a}^\dagger(p,\lambda)] = p^0\hat{a}^\dagger(p,\lambda) \tag{4.472}$$
$$\to [\hat{H}, \hat{a}(p,\lambda)] = p^0\hat{a}(p,\lambda) \tag{4.473}$$

f) Hilbert space

- as commutator algebra for $\hat{a}$ and $\hat{a}^\dagger$ is as for harmonic oscillator we utilize the same Hilbert space
- Vacuum $|0\rangle$ with

$$a(p,\lambda)|0\rangle = 0 \tag{4.474}$$

- The single particle states ($\lambda = 1, 2, 3$) for each $\mathbf{p}$ (with $p^0 = \sqrt{\mathbf{p}^2 + m^2}$) - so three internal degrees of freedom (spin)

$$|p, \lambda\rangle \equiv a(p, \lambda)|0\rangle \tag{4.475}$$

  States have positive norm and energy (in analogy to harmonic oscillator)

g) Bosonic multi-particle states (because of commutation relations)

$$|p', \lambda', p, \lambda\rangle \equiv a(p', \lambda')a(p, \lambda)|0\rangle \tag{4.476}$$

4.9 Quantization - massless spin-1 field (Maxwell field)

4.9.1 Classical

4.10 Quantization - spin 3/2 field (Rarita–Schwinger field)