

0.1 ANDREWS - Number theory

0.1.1 Problem 1.1

Lets cut the chase

$$\frac{n(n+1)(2n+1)}{6} + (n+1)^2 = (n+1) \frac{n(2n+1) + 6(n+1)}{6} \quad (1)$$

$$= \frac{(n+1)}{6} (2n^2 + 7n + 6) \quad (2)$$

$$= \frac{(n+1)}{6} (n+2)(2n+3) \quad (3)$$

$$= \frac{(n+1)}{6} (n+2)(2(n+1)+1) \quad (4)$$

$$= \frac{(n+1)(n+2)(2(n+1)+1)}{6} \quad (5)$$

$$(6)$$

0.2 MORRIS - Georgi - Lie Algebras in Particle Physics 2nd ed.

0.2.1 Problem 1.A

We call the elements a, b, e - as we know a unique neutral element must exist

\circ	e	a	b
e	e	a	b
a	a	a^2	$b \circ a$
b	b	$a \circ b$	b^2

(7)

We have 4 fields to fill

- a needs an inverse - only element left is b meaning $b = a^{-1}$ and therefore $a \circ b = b \circ a = e$
- a^2 can't be e (because $e^2 = e$), a^2 can't be a (because $a \circ e = a$) therefore $a^2 = b$

\circ	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

(8)

0.3 MORRIS - Topology without tears

0.3.1 Problem 1.1.7

(a)

$$\tau_{X1} = \{X, \emptyset\} \quad (9)$$

$$\tau_{X2} = \{X, \emptyset, \{a\}\} \quad (10)$$

$$\tau_{X3} = \{X, \emptyset, \{b\}\} \quad (11)$$

$$\tau_{X4} = \{X, \emptyset, \{a\}, \{b\}\} \quad (12)$$

(b)

$$\tau_{Y1} = \{Y, \emptyset\}, \quad (13)$$

$$\tau_{Y2} = \{Y, \emptyset, \{a\}\}, \tau_{Y3} = \{Y, \emptyset, \{b\}\}, \tau_{Y4} = \{Y, \emptyset, \{c\}\}, \quad (14)$$

$$\tau_{Y5} = \{Y, \emptyset, \{a, b\}\}, \tau_{Y6} = \{Y, \emptyset, \{b, c\}\}, \tau_{Y7} = \{Y, \emptyset, \{a, c\}\}, \quad (15)$$

$$\tau_{Y8} = \{Y, \emptyset, \{a\}, \{b\}, \{a, b\}\}, \tau_{Y9} = \{Y, \emptyset, \{a\}, \{c\}, \{a, c\}\}, \tau_{Y10} = \{Y, \emptyset, \{b\}, \{c\}, \{b, c\}\}, \quad (16)$$

$$\tau_{Y11} = \{Y, \emptyset, \{a\}, \{b, c\}\}, \tau_{Y12} = \{Y, \emptyset, \{b\}, \{a, c\}\}, \tau_{Y13} = \{Y, \emptyset, \{c\}, \{a, b\}\}, \quad (17)$$

$$\tau_{Y14} = \{Y, \emptyset, \{a\}, \{a, b\}\}, \tau_{Y15} = \{Y, \emptyset, \{b\}, \{a, b\}\}, \tau_{Y16} = \{Y, \emptyset, \{a\}, \{a, c\}\}, \quad (18)$$

$$\tau_{Y17} = \{Y, \emptyset, \{c\}, \{a, c\}\}, \tau_{Y18} = \{Y, \emptyset, \{b\}, \{b, c\}\}, \tau_{Y19} = \{Y, \emptyset, \{c\}, \{b, c\}\}, \quad (19)$$

$$\tau_{Y20} = \{Y, \emptyset, \{a\}, \{a, c\}, \{a, b\}\}, \tau_{Y21} = \{Y, \emptyset, \{b\}, \{a, b\}, \{b, c\}\}, \tau_{Y22} = \{Y, \emptyset, \{c\}, \{b, c\}, \{a, c\}\}, \quad (20)$$

$$\tau_{Y23} = \{Y, \emptyset, \{a\}, \{b\}, \{a, c\}, \{a, b\}\}, \tau_{Y24} = \{Y, \emptyset, \{a\}, \{c\}, \{a, c\}, \{a, b\}\}, \quad (21)$$

$$\tau_{Y25} = \{Y, \emptyset, \{a\}, \{b\}, \{b, c\}, \{a, b\}\}, \tau_{Y26} = \{Y, \emptyset, \{b\}, \{c\}, \{b, c\}, \{a, b\}\} \quad (22)$$

$$\tau_{Y27} = \{Y, \emptyset, \{a\}, \{c\}, \{a, c\}, \{a, b\}\}, \tau_{Y28} = \{Y, \emptyset, \{a\}, \{b\}, \{a, c\}, \{a, b\}\}, \quad (23)$$

$$\tau_{Y29} = \{Y, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\} \quad (24)$$

0.4 BARTON - Elements of Greens functions and propagation

0.4.1 Problem 1.1 - Delta function

$$(i) \int_{-1}^2 dx \delta(x) \cos(2x) = \cos(0) = 1$$

$$(ii) \int_{-1}^2 dx \delta(2x) \cos(x) = \int_{-2}^4 (dy/2) \delta(y) \cos(y/2) = \cos(0)/2 = 1/2$$

$$(iii) \int_{-\infty}^{\infty} dx \delta'(x) \exp(ix) = 0 - i \int_{-\infty}^{\infty} \delta(x) \exp(ix) = -i$$

$$(iv) \int_0^{\infty} dx \delta'(\sqrt{2}x - 1) \tan^{-1}(x) = \int_0^{\infty} \frac{\delta(x-1/\sqrt{2})}{|\sqrt{2}|} \tan^{-1}(x) = \frac{\tan^{-1}(1/\sqrt{2})}{\sqrt{2}}$$

0.4.2 Problem 1.3 - Delta function

$$\int_0^{\infty} dx \delta(\cos(x)) e^{-x} = \sum_{x_n \in \{\pi/2 + n\pi\}} \int_0^{\infty} dx \frac{\delta(x - x_n)}{|\sin x_n|} e^{-x} \quad (25)$$

$$= e^{-\pi/2} (e^{-0\pi} + e^{-1\pi} + e^{-2\pi} + \dots) \quad (26)$$

$$= \frac{e^{-\pi/2}}{1 - e^{-\pi}} \quad (27)$$

$$= \frac{1}{e^{\pi/2} - e^{-\pi/2}} \quad (28)$$

$$(29)$$

0.5 WYLD - Mathematical methods for physics

0.5.1 Problem 10.2 - Bernoulli numbers

a) Rewriting

$$\frac{z}{e^z - 1} = \frac{z}{z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots} \quad (30)$$

$$= \frac{1}{1 + \frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4!} + \dots} \quad (31)$$

$$= B_0 + \frac{B_1 z}{1!} + \frac{B_2 z^2}{2!} + \frac{B_3 z^3}{3!} + \frac{B_4 z^4}{4!} \dots \quad (32)$$

then

$$1 = \left(B_0 + \frac{B_1 z}{1!} + \frac{B_2 z^2}{2!} + \frac{B_3 z^3}{3!} + \frac{B_4 z^4}{4!} + \dots \right) \left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4!} + \dots \right) \quad (33)$$

and we compare the polynomial coefficients in LHS and RHS for each order

$$z^0 : \quad 1 = B_0 \cdot 1 \quad \rightarrow \quad B_0 = 1 \quad (34)$$

$$z^1 : \quad 0 = B_0 \frac{1}{2!} + B_1 \quad \rightarrow \quad B_1 = -\frac{1}{2} \quad (35)$$

$$z^2 : \quad 0 = B_0 \frac{1}{3!} + B_1 \frac{1}{2!} + \frac{1}{2!} B_2 \quad \rightarrow \quad B_2 = 2 \left(-\frac{1}{3!} + \frac{1}{4} \right) = \frac{1}{6} \quad (36)$$

$$z^3 : \quad 0 = B_0 \frac{1}{4!} + B_1 \frac{1}{3!} + \frac{1}{2!2!} B_2 + \frac{1}{4!} B_3 \quad \rightarrow \quad B_3 = 0 \quad (37)$$

$$z^4 : \quad 0 = B_0 \frac{1}{5!} + B_1 \frac{1}{4!} + \frac{1}{3!2!} B_2 + \frac{1}{2!3!} B_3 + \frac{1}{4!} B_4 \quad \rightarrow \quad B_4 = \frac{1}{30} \quad (38)$$

$$(39)$$

b) Rewriting

$$\frac{z}{e^z - 1} + \frac{z}{2} = z \frac{2 + (e^z - 1)}{2(e^z - 1)} \quad (40)$$

$$= z \frac{2 + e^{z/2}(e^{z/2} - e^{-z/2})}{2e^{z/2}(e^{z/2} - e^{-z/2})} \quad (41)$$

$$= z \frac{2e^{-z/2} + (e^{z/2} - e^{-z/2})}{2(e^{z/2} - e^{-z/2})} \quad (42)$$

$$= \frac{z}{2} \frac{e^{z/2} + e^{-z/2}}{e^{z/2} - e^{-z/2}} \quad (43)$$

and now it is obvious. For c) we can rewrite this as via $z \rightarrow 2iz$

$$\frac{2iz}{e^{2iz} - 1} + \frac{2iz}{2} = iz \frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} \quad (44)$$

c)

$$\cot z = \frac{\cos z}{\sin z} = \frac{e^{iz} + e^{-iz}}{2} \frac{2i}{e^{iz} - e^{-iz}} = i \frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} \quad (45)$$

$$z \cot z = (iz) \frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} \quad (46)$$

$$= \frac{2iz}{e^{2iz} - 1} + iz \quad (47)$$

$$= iz + \left(1 + B_1(2iz) + \frac{B_2}{2!}(2iz)^2 + \frac{B_4}{4!}(2iz)^4 + \dots \right) \quad (48)$$

$$= 1 - \frac{1}{3}z^2 + \underbrace{\frac{16i^4}{24 \cdot (-30)}}_{=-2/90} z^4 - \dots \quad (49)$$

0.5.2 Problem 11.1 - Integral $\int_0^\infty dx \frac{x^2}{(x^2+a^2)^2}$

The zero of $\frac{x^2}{(x^2+a^2)^2}$ are $\pm ia$ (lets assume a is positive) so we can decompose into the common partial fractions

$$\frac{x^2}{(x^2+a^2)^2} = \left(\frac{x}{(x-ia)(x+ia)} \right)^2 = \frac{1}{4} \left(\frac{1}{x-ia} + \frac{1}{x+ia} \right)^2. \quad (50)$$

Using the residual theorem (closing the loop above as $f(x) \sim z^{-2}$) gives

$$\int_0^\infty \frac{x^2}{(x^2+a^2)^2} = \frac{1}{2} \int_{-\infty}^\infty \frac{x^2}{(x^2+a^2)^2} = \frac{1}{2} \left[2\pi i \text{Res}(f(ia)) - \int_{\text{C above}} f(x) dx \right] \quad (51)$$

$$= i\pi \text{Res}(f(ia)) \quad (52)$$

There are two methods to calculate the residuum

1. Direct: As ia is a second order pole we have

$$\text{Res}(f(ia)) = \frac{1}{(2-1)!} \frac{d^{2-1}}{dx^{2-1}} (x-ia)^2 \frac{x^2}{(x+ia)^2(x-ia)^2} \Big|_{x=ia} = \frac{2x(x+ia)^2 - x^2 2(x+ia)}{(x+ia)^4} = \frac{1}{4ia} \quad (53)$$

2. Laurent series at ai :

$$\frac{x^2}{(x^2+a^2)^2} = \left(\frac{x}{(x-ia)(x+ia)} \right)^2 = \frac{1}{4} \left(\frac{1}{x-ia} + \frac{1}{x+ia} \right)^2 \quad (54)$$

$$= \frac{1}{4} \left(\frac{1}{x-ia} + \frac{1}{2ia} \frac{1}{1 - \frac{x-ia}{-2ia}} \right)^2 \quad (\text{using geometric series trick}) \quad (55)$$

$$= \frac{1}{4} \left(\frac{1}{x-ia} + \frac{1}{2ia} \left[1 + \frac{x-ia}{-2ia} + \left(\frac{x-ia}{-2ia} \right)^2 + \left(\frac{x-ia}{-2ia} \right)^3 + \dots \right] \right)^2 \quad (56)$$

$$= -\frac{1}{16a^2} \left(\frac{2ia}{x-ia} + 1 + \frac{x-ia}{-2ia} + \left(\frac{x-ia}{-2ia} \right)^2 + \left(\frac{x-ia}{-2ia} \right)^3 + \dots \right)^2 \quad (57)$$

$$= -\frac{1}{16a^2} \left(\frac{(2ia)^2}{(x-ia)^2} + 2 \frac{2ia}{x-ia} + \left(1 + 2 \frac{x-ia}{-2ia} \frac{2ia}{x-ia} + \dots \right) \right) \quad (58)$$

then

$$\text{Res}(f(ia)) = -\frac{1}{16a^2} 4ia = \frac{1}{4ai} \quad (59)$$

and finally

$$\int_0^\infty \frac{x^2}{(x^2 + a^2)^2} = \frac{\pi}{4a} \quad (60)$$

0.5.3 Problem 11.2 - Integral $\int_0^\infty dx \frac{1}{x^4 + 5x^2 + 6}$

Rewriting and utilizing the residual theorem

$$\int_0^\infty dx \frac{1}{x^4 + 5x^2 + 6} = \frac{1}{2} \int_{-\infty}^\infty dx \frac{1}{x^4 + 5x^2 + 6} \quad (61)$$

$$= \frac{1}{2} \int_{-\infty}^\infty dx \frac{1}{(x - i\sqrt{3})(x + i\sqrt{3})(x - i\sqrt{2})(x + i\sqrt{2})} \quad (62)$$

$$= \frac{1}{2} \left(2\pi i \sum_{a_i = \{i\sqrt{2}, i\sqrt{3}\}} \text{Res}(f(a_i)) - \int_{\text{C above}} \right) \quad (63)$$

then

$$\text{Res}(f(i\sqrt{2})) = \lim_{x \rightarrow i\sqrt{2}} (x - i\sqrt{2}) \frac{1}{(x - i\sqrt{3})(x + i\sqrt{3})(x - i\sqrt{2})(x + i\sqrt{2})} \quad (64)$$

$$= \lim_{x \rightarrow i\sqrt{2}} \frac{1}{(x - i\sqrt{3})(x + i\sqrt{3})(x + i\sqrt{2})} \quad (65)$$

$$= -\frac{i}{2\sqrt{2}} \quad (66)$$

$$\text{Res}(f(i\sqrt{3})) = \frac{i}{2\sqrt{3}} \quad (67)$$

and

$$\int_0^\infty dx \frac{1}{x^4 + 5x^2 + 6} = \frac{\pi}{12} (3\sqrt{2} - 2\sqrt{3}) \quad (68)$$

0.5.4 Problem 11.3 - Integral $\int_0^\infty dx \frac{1}{x^4 + 1}$

Same idea as above -residue theorem, closing the half circle above (integral vanishes because $f(x) x^{-4}$),

$$\int_0^\infty dx \frac{1}{x^4 + 1} = \frac{1}{2} \int_{-\infty}^\infty dx \frac{1}{(x - e^{i\pi/4})(x - e^{i3\pi/4})(x - e^{-i\pi/4})(x - e^{-i3\pi/4})} \quad (69)$$

$$= \frac{1}{2} 2\pi i \left[\text{Res}(f(e^{i\pi/4})) + \text{Res}(f(e^{i3\pi/4})) \right] \quad (70)$$

$$= \frac{1}{2} 2\pi i \left[\frac{1}{4} e^{-3i\pi/4} + \frac{1}{4} e^{-i\pi/4} \right] \quad (71)$$

$$= \frac{\pi}{2\sqrt{2}} \quad (72)$$

0.5.5 Problem 11.4 - Integral $\int d^3k \frac{e^{i\vec{k}\cdot\vec{r}}}{k^2+a^2}$

$$\int d^3k \frac{e^{i\vec{k}\cdot\vec{r}}}{k^2+a^2} = 2\pi \int k^2 dk \frac{e^{ikr \cos \theta}}{k^2+a^2} \sin \theta d\theta \quad (73)$$

$$= -\frac{2\pi}{ir} \int \frac{k^2}{k} dk \frac{e^{-ikr} - e^{ikr}}{k^2+a^2} \quad (74)$$

$$= -\frac{2\pi}{2ir} \int_0^\infty dk \left(\frac{1}{k-ia} + \frac{1}{k+ia} \right) (e^{-ikr} - e^{ikr}) \quad (75)$$

$$= -\frac{\pi}{2ir} \int_{-\infty}^\infty dk \left(\frac{1}{k-ia} + \frac{1}{k+ia} \right) (e^{-ikr} - e^{ikr}) \quad (76)$$

Using the residue theorem - for e^{ikr} we close the loop above and for e^{-ikr} below (the way the integral along the loops vanish)

$$\int d^3k \frac{e^{i\vec{k}\cdot\vec{r}}}{k^2+a^2} = -\frac{\pi}{2ir} \int_{-\infty}^\infty dk \left(\frac{e^{-ikr}}{k-ia} + \frac{-e^{ikr}}{k-ia} + \frac{e^{-ikr}}{k+ia} + \frac{-e^{ikr}}{k+ia} \right) \quad (77)$$

$$= -\frac{\pi}{2ir} \int_{-\infty}^\infty dk \left(0 + \frac{-e^{ikr}}{k-ia} + \frac{e^{-ikr}}{k+ia} + 0 \right) \quad (78)$$

$$= -\frac{\pi}{2ir} 2\pi i \left(-e^{ik(ia)} + (-1)e^{-ik(-ia)} \right) \quad (\text{Curve below and negative winding}) \quad (79)$$

$$= \frac{2\pi^2}{r} e^{-ka} \quad (80)$$

0.5.6 Problem 11.5 - Integral $\int d^3k \frac{e^{i\vec{k}\cdot\vec{r}}}{k^2-a^2-i\varepsilon}$

Same as in 11.4

$$\int d^3k \frac{e^{i\vec{k}\cdot\vec{r}}}{k^2-a^2-i\varepsilon} = -\frac{2\pi}{ir} \int_0^\infty \frac{k^2}{k} dk \frac{e^{-ikr} - e^{ikr}}{k^2-a^2-i\varepsilon} \quad (81)$$

$$= -\frac{2\pi}{4ir} \int_{-\infty}^\infty dk \left(\frac{1}{k+(a+i\frac{\varepsilon}{2a})} + \frac{1}{k-(a+i\frac{\varepsilon}{2a})} \right) (e^{-ikr} - e^{ikr}) \quad (82)$$

$$= -\frac{2\pi}{4ir} \int dk \left(\frac{e^{-ikr}}{k+(a+i\frac{\varepsilon}{2a})} + \frac{-e^{ikr}}{k-(a+i\frac{\varepsilon}{2a})} \right) \quad (83)$$

$$= -\frac{2\pi}{4ir} 2\pi i \left(-e^{-i(-1)(a+i\frac{\varepsilon}{2a})r} - e^{i(a+i\frac{\varepsilon}{2a})r} \right) \quad (84)$$

$$= \frac{2\pi^2}{2r} \left(e^{i(a+i\frac{\varepsilon}{2a})r} + e^{i(a+i\frac{\varepsilon}{2a})r} \right) \quad (85)$$

$$= \frac{2\pi^2}{r} e^{i(a+i\frac{\varepsilon}{2a})r} \quad (86)$$

$$= \frac{2\pi^2}{r} e^{iar} \quad (87)$$

0.6 STONE, GOLDBART - Mathematics for physics: A guided tour for graduate students (2009)

0.6.1 Problem 1.1

$$\frac{\partial L}{\partial \dot{x}} = \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \quad (88)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{\ddot{x} \sqrt{\dot{x}^2 + \dot{y}^2} - \dot{x} \frac{\dot{x}\ddot{x} + \dot{y}\ddot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}}{\dot{x}^2 + \dot{y}^2} = 0 \quad (89)$$

$$\rightarrow \ddot{x}(\dot{x}^2 + \dot{y}^2) - \dot{x}(\dot{x}\ddot{x} + \dot{y}\ddot{y}) = 0 \quad (90)$$

$$\rightarrow \ddot{x}\dot{y}^2 - \dot{x}\dot{y}\ddot{y} = 0 \quad (91)$$

$$\rightarrow \dot{y}(\ddot{x}\dot{y} - \dot{x}\ddot{y}) = 0 \quad (92)$$

$$(93)$$

0.7 BENDER, ORSZAG - Advanced Mathematical Methods for Scientists and Engineers

0.7.1 Problem 1.1

$$1. \quad y' = e^{x+y}$$

$$\int \frac{dy}{e^y} = \int e^x dx \quad (94)$$

$$-e^{-y} = e^x + c \quad (95)$$

$$y = -\log(-e^x + c) \quad (96)$$

$$2. \quad y' = xy + x + y + 1$$

$$\frac{dy}{y+1} = x+1 \quad (97)$$

$$\log y + 1 = \frac{x^2}{2} + x + c \quad (98)$$

$$y = c' e^{x/2(x+2)} - 1 \quad (99)$$

0.7.2 Problem 1.2

$$y'' = yy'/x$$

1. Equidimensional-in-s equation

$$x = e^t \quad (100)$$

$$\frac{d}{dx} = \frac{dt}{dx} \frac{d}{dt} \quad (101)$$

$$= \frac{1}{x} \frac{d}{dt} \quad (102)$$

$$\frac{d^2}{dx^2} = \frac{dt}{dx} \frac{d}{dt} \left(\frac{1}{x} \frac{d}{dt} \right) \quad (103)$$

$$= \frac{1}{x} \left(-\frac{1}{x^2} x \frac{d}{dt} + \frac{1}{x} \frac{d^2}{dt^2} \right) \quad (104)$$

$$= \frac{1}{x^2} \left(-\frac{d}{dt} + \frac{d^2}{dt^2} \right) \quad (105)$$

now with $y = y(t)$

$$-y' + y'' = yy' \quad (106)$$

2. Autonomous equation

$$y' \equiv u(y) \quad (107)$$

$$y'' = \frac{du}{dy} \frac{dy}{dt} = \dot{u}y' \quad (108)$$

now with $u = u(y)$

$$-u + \dot{u}u = yu \quad (109)$$

$$\dot{u} = y + 1 \quad (110)$$

3. integration

$$u = \frac{y^2}{2} + y + c_0 \quad (111)$$

4. resubstitution I (with $\tan z = i \frac{e^{-iz} - e^{iz}}{e^{-iz} + e^{iz}}$)

$$y' = \frac{y^2}{2} + y + c_0 \quad (112)$$

$$t + c_3 = \int \frac{dy}{y^2/2 + y + c_0} \quad (113)$$

$$= 2 \frac{1}{2\sqrt{1-2c_0}} \int dy \left(-\frac{1}{y+1+\sqrt{1-2c_0}} + \frac{1}{y+1-\sqrt{1-2c_0}} \right) \quad (114)$$

$$= \frac{1}{\sqrt{1-2c_0}} \left(-\log[y+1+\sqrt{1-2c_0}] + \log[y+1-\sqrt{1-2c_0}] \right) \quad (115)$$

$$= \frac{1}{\sqrt{1-2c_0}} \log \frac{y+1-\sqrt{1-2c_0}}{y+1+\sqrt{1-2c_0}} \quad (116)$$

$$= \frac{1}{\sqrt{1-2c_0}} \log \frac{-i\sqrt{1-2c_0} \left(-i + \frac{i(y+1)}{\sqrt{1-2c_0}} \right)}{i\sqrt{1-2c_0} \left(-i - \frac{i(y+1)}{\sqrt{1-2c_0}} \right)} \quad (117)$$

$$= \frac{1}{\sqrt{1-2c_0}} \log \frac{-\left(-i + \frac{i(y+1)}{\sqrt{1-2c_0}} \right)}{\left(-i - \frac{i(y+1)}{\sqrt{1-2c_0}} \right)} \quad (118)$$

$$= \frac{2}{\sqrt{1-2c_0}} \log \sqrt{-\frac{-i + \frac{i(y+1)}{\sqrt{1-2c_0}}}{-i - \frac{i(y+1)}{\sqrt{1-2c_0}}}} \quad (119)$$

$$= \frac{2}{i\sqrt{1-2c_0}} \arctan \left(-\frac{i(y+1)}{\sqrt{1-2c_0}} \right) \quad (120)$$

5. resubstitution II

$$\log x + c_3 = \frac{2}{i\sqrt{1-2c_0}} \arctan \frac{y+1}{i\sqrt{1-2c_0}} \quad (121)$$

$$\tan \left[\frac{\sqrt{2c_0-1}}{2} (\log x + c_3) \right] = \frac{y+1}{\sqrt{2c_0-1}} \quad (122)$$

$$y = \sqrt{2c_0-1} \tan \left[\frac{\sqrt{2c_0-1}}{2} (\log x + c_3) \right] - 1 \quad (123)$$

$$y = 2c_1 \tan [c_1 \log x + c_2] - 1 \quad (124)$$

This solution has poles at

$$\log x_P = \frac{\pi/2 + k\pi - c_2}{c_1} \quad (125)$$

while the special solution $-2/(c_4 + \log x) - 1$ has a pole at

$$\log x_P = -c_4 \quad (126)$$

???

0.7.3 Problem 1.10

With $y = e^{rx}$ the equation $y''' - 3y'' + 3y' - y = 0$ becomes

$$r^3 - 3r^2 + 3r - 1 = 0 \quad (127)$$

$$(r - 1)^3 = 0 \quad (128)$$

then $y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x$.

0.7.4 Problem 1.11

We guess $y_1 = e^{-x}$ and have another guess $y_2 = e^{-x}u(x)$ we see

$$r^{-x}(u'' + xu') = 0 \quad (129)$$

$$v' + xv = 0 \quad (130)$$

$$v = c_0 e^{-x^2/2} \quad (131)$$

$$u = c_1 \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) + c_2 \quad (132)$$

and therefore $y = c_3 e^{-x} + c_4 e^{-x} \left[\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) + c_5 \right]$

0.7.5 Problem 1.23

Calculating the gradient

$$\nabla z = e^{-(x^4+4y^2)}(-4x^3, -8y) \quad (133)$$

$$= -4e^{-(x^4+4y^2)}(x^3, 2y) \quad (134)$$

Equation of motions $\ddot{\vec{x}} = -\nabla V$ are

$$\ddot{x} = 4e^{-(x^4+4y^2)}x^3 \quad (135)$$

$$\ddot{y} = 4e^{-(x^4+4y^2)}2y \quad (136)$$

with the initial conditions $x_0 = 0 = y_0$. To make this simpler to solve we rescale ($\tilde{t} = \alpha t$) the time variable

$$\frac{\partial}{\partial t} = \frac{\partial \tilde{t}}{\partial t} \frac{\partial}{\partial \tilde{t}} \quad (137)$$

$$= \alpha \frac{\partial}{\partial \tilde{t}} \quad (138)$$

$$\frac{\partial^2}{\partial t^2} = \frac{\partial^2 \tilde{t}}{\partial t^2} \frac{\partial}{\partial \tilde{t}} + \left(\frac{\partial \tilde{t}}{\partial t} \right)^2 \frac{\partial^2}{\partial \tilde{t}^2} \quad (139)$$

$$= \alpha^2 \frac{\partial^2}{\partial \tilde{t}^2}. \quad (140)$$

0.7.6 Problem 1.31

(a) Multiply by y and observe $yy' \sim (y^2)'$ and substitute $z = y^2$

$$y' = \frac{y}{x} + \frac{1}{y} \quad (141)$$

$$yy' - \frac{1}{x}y^2 - 1 = 0 \quad (142)$$

$$\frac{1}{2}(y^2)' - \frac{1}{x}y^2 - 1 = 0 \quad (143)$$

$$\frac{1}{2}z' - \frac{1}{x}z - 1 = 0 \quad (z = y^2) \quad (144)$$

$$z' - \frac{2}{x}z - 2 = 0 \quad (145)$$

General solution of the homogeneous equation

$$\frac{z'}{z} = \frac{2}{x} \rightarrow z_H = cx^2 \quad (146)$$

For special solution of the inhomogeneous equation - varying constants

$$z_I = C(x)x^2 \quad (147)$$

$$\rightarrow C'x^2 + 2xC - \frac{2}{x}Cx^2 - 2 = 0 \quad (148)$$

$$\rightarrow C' = \frac{2}{x^2} \quad (149)$$

$$\rightarrow C = -\frac{2}{x} \quad (150)$$

therefore

$$z = z_H + z_I \quad (151)$$

$$= x(cx - 2) \quad (152)$$

$$y = \pm \sqrt{x(cx - 2)} \quad (153)$$

(b) Nothing obvious pops into the eye so we make a desperate try $z = y/x$

$$z' = \frac{y'x - y}{x^2} \rightarrow y' = z'x + z \quad (154)$$

then

$$y' = \frac{xy}{x^2 + y^2} \quad (155)$$

$$z'x + z = \frac{zx^2}{x^2 + z^2x^2} \quad (156)$$

$$= \frac{z'}{1 + z^2} \quad (157)$$

$$z'x = \frac{z - z(1 + z^2)}{1 + z^2} \quad (158)$$

$$= \frac{-z^3}{1 + z^2} \quad (159)$$

Now we can separate and integrate on both sides

$$\frac{1+z^2}{z^3}dz = -\frac{dx}{x} \quad (160)$$

$$\int \left(\frac{1}{z^2} + \frac{1}{z} \right) dz = \int \frac{dx}{x} \quad (161)$$

$$-\frac{1}{z} + \log z = \log x + c \quad (162)$$

$$-\frac{x}{y} + \log \frac{y}{x} = \log x + c \quad (163)$$

$$-\frac{x}{y} + \log y = 2 \log x + c \quad (164)$$

(c) Try the obvious $z = x + y$

$$y' = x^2 + 2xy + y^2 \quad (165)$$

$$y' = (x + y)^2 \quad (166)$$

$$\rightarrow z' - 1 = z^2 \quad (167)$$

Now separate and integrate (subs $z = \tan t$)

$$\frac{dz}{z^2 + 1} = dx \quad (168)$$

$$\arctan z = x + c \quad (169)$$

$$y = \tan(x + c) - x \quad (170)$$

(d) Rewriting the ODE we see similarities to the quotient rule

$$\frac{yy''}{(y')^2} = 2 \quad (171)$$

Let's guess

$$\left(\frac{y}{y'} \right)' = \frac{y'y' - yy''}{(y')^2} = 1 - \frac{yy''}{(y')^2} \quad (172)$$

so we can rewrite the ODE

$$\frac{yy''}{(y')^2} = 1 - \left(\frac{y}{y'} \right)' = 2 \quad (173)$$

then we can solve

$$\left(\frac{y}{y'} \right)' = -1 \quad (174)$$

$$\frac{y}{y'} = -x + c_1 \quad (175)$$

$$\frac{y'}{y} = \frac{1}{-x + c_1} \quad (176)$$

$$\log y = -\log(-x + c_1) + c_2 \quad (177)$$

$$y = \frac{c_3}{c_1 - x} \quad (178)$$

(e)

$$\frac{y'}{y^2} = \frac{1}{x^2} + \frac{1}{x} \quad (179)$$

$$-\frac{1}{y} = -\frac{1}{x} + \log x + c \quad (180)$$

$$y = \frac{1}{\frac{1}{x} - \log x + c} = \frac{1}{1 - x \log x + xc} \quad (181)$$

(f) With $f = xy$

$$x^2 y' + xy + y^2 = 0 \quad (182)$$

$$xy' + y + \frac{y^2}{x} = 0 \quad (183)$$

$$f' + \frac{f^2}{x^3} = 0 \quad (184)$$

$$\frac{f'}{f^2} + \frac{1}{x^3} = 0 \rightarrow -\frac{1}{f} - \frac{1}{2}x^{-2} + c = 0 \quad (185)$$

$$xy = f = -\frac{1}{\frac{1}{2x^2} + c} \quad (186)$$

$$y = -\frac{1}{\frac{1}{2x} + xc} = -\frac{2x}{1 + 2x^2c} \quad (187)$$

(g)

$$xy' = y(1 - \log x + \log y) \quad (188)$$

$$\frac{xy'}{y} = (1 - \log \frac{y}{x}) \quad (189)$$

(n) Observe $(\frac{y}{x})' = \frac{xy' - y}{x^2}$ then

$$xy' - y = xe^{y/x} \quad (190)$$

$$x^2 \left(\frac{y}{x}\right)' = xe^{y/x} \quad (191)$$

$$\frac{f'}{e^{-f}} = \frac{1}{x} \rightarrow -e^{-f} = \log x + c \quad (192)$$

$$-f = \log(-\log x - c) \quad (193)$$

$$y = -x \log(-\log x - c) \quad (194)$$

$$(195)$$

(o) Lets try $(x^m y^n)' = mx^{m-1}y^n + nx^m y^{n-1}$ and rewrite

$$y' = \frac{x^4 - 3x^2 y^2 - y^3}{2x^3 y + 3y^2 x} \quad (196)$$

$$2x^3 y y' + 3y^2 x y' = x^4 - 3x^2 y^2 - y^3 \quad (197)$$

then with

$$(x^3 y^2)' = 2x^3 y y' + 3x^2 y^2 \rightarrow 2x^3 y y' = (x^3 y^2)' - 3x^2 y^2 \quad (198)$$

$$(x y^3)' = 3x y^2 y' + y^3 \rightarrow 3x y^2 y' = (x y^3)' - y^3 \quad (199)$$

we can rewrite the LHS

$$2x^3 y y' + 3y^2 x y' = (x^3 y^2)' + (x y^3)' - 3x^2 y^2 - y^3 \quad (200)$$

$$= (x^3 y^2 + x y^3)' - 3x^2 y^2 - y^3 \quad (201)$$

putting it back into the ODE

$$(x^3y^2 + xy^3)' - 3x^2y^2 - y^3 = x^4 - 3x^2y^2 - y^3 \quad (202)$$

$$(x^3y^2 + xy^3)' = x^4 \quad (203)$$

$$\rightarrow x^3y^2 + xy^3 = \frac{1}{5}x^5 + c \quad (204)$$

$$\rightarrow xy^2(x^2 + y) = \frac{1}{5}x^5 + c \quad (205)$$

(t) Observe $-\left(\frac{x}{y}\right)' = \frac{xy' - y}{y^2}$ then

$$xy' = y + \sqrt{xy} \quad (206)$$

$$\frac{xy'}{y^2} = \frac{y}{y^2} + \frac{\sqrt{xy}}{y^2} \quad (207)$$

$$-\left(\frac{x}{y}\right)' = \frac{\sqrt{xy}}{y^2} = \frac{x}{x} \frac{\sqrt{xy}}{y^2} = \frac{1}{x} \sqrt{\frac{x^3}{y^3}} \quad (208)$$

$$-f' = \frac{1}{x} f^{3/2} \rightarrow -2f^{-1/2} = \log x + c \quad (209)$$

$$y = \frac{x}{4}(\log x + c) \quad (210)$$

(x) First the homog. equations

$$\frac{y'}{y} + \frac{1}{(x-1)(x-2)} = 0 \quad (211)$$

$$\frac{y'}{y} - \left(\frac{1}{x-1} - \frac{1}{x-2}\right) = 0 \quad (212)$$

$$\log y_h - \log \frac{x-1}{x-2} = C \quad (213)$$

$$y_h = C \frac{x-1}{x-2} \quad (214)$$

Now variations of constants and resubstitute

$$y = C(x) \frac{x-1}{x-2} \quad (215)$$

$$\rightarrow (x-1)^2 C'(x) = 2 \quad (216)$$

$$\rightarrow C(x) = -\frac{2}{x-1} + c \quad (217)$$

$$\rightarrow y = \left(-\frac{2}{x-1} + c\right) \frac{x-1}{x-2} \quad (218)$$

$$\rightarrow y = \frac{-2}{x-2} \quad (219)$$

(y) Playing around a bit we see $(xe^{-y})' = e^{-y} - xy'e^{-y}$ and then

$$y' = \frac{1}{x + e^y} \quad (220)$$

$$xy' + y'e^y = 1 \quad (221)$$

$$xy'e^{-y} - e^{-y} + y' = 0 \quad (222)$$

$$-(xe^{-y})' + y' = 0 \quad (223)$$

$$-xe^{-y} + y = c \quad (224)$$

$$ye^y = ce^y + x \quad (225)$$

and we recognize the productlog (Lambert W function).

(z) With substitution $f = xy$

$$xy' + y = y^2 x^4 \quad (226)$$

$$(xy)' = (xy)^2 x^2 \quad (227)$$

$$f' = f^2 x^2 \quad (228)$$

$$\rightarrow \frac{f'}{f^2} = x^2 \quad \rightarrow \quad -f^{-1} = \frac{x^3}{3} + c \quad (229)$$

$$\rightarrow y = -\frac{1}{x} \frac{3}{x^3 + \bar{c}} \quad (230)$$

0.7.7 Problem 7.1

Inserting the series expansion into the equation and sorting by powers of ϵ

(a)

$$a_0 + a_0^2 = 0 \quad (231)$$

$$6 + a_1(1 + 2a_0) = 0 \quad (232)$$

$$a_1^2 + a_2(1 + 2a_0) = 0 \quad (233)$$

then coefficients upto second order (for both zeros) are

$$a_0 = -1 \quad \rightarrow \quad a_1 = 6 \quad \rightarrow \quad a_2 = 36 \quad (234)$$

$$\rightarrow x_- = -1 + 6\epsilon + 36\epsilon^2 \quad (235)$$

$$a_0 = 0 \quad \rightarrow \quad a_1 = -6 \quad \rightarrow \quad a_2 = -36 \quad (236)$$

$$\rightarrow x_+ = -6\epsilon - 36\epsilon^2 \quad (237)$$

which is consistent with the series expansion of the analytical roots

$$x_{\pm} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - 6\epsilon} \quad (238)$$

$$= -\frac{1}{2} \pm \frac{1}{2} \sqrt{1 - 24\epsilon} \quad (239)$$

(b)

$$1 + a_0^3 = 0 \quad (240)$$

$$-a_0 + 3a_0^2 a_1 = 0 \quad (241)$$

$$-a_1 + 3a_0 a_1^2 + 3a_0^2 a_2 = 0 \quad (242)$$

then

$$a_0 = 1 \quad \rightarrow \quad a_1 = 1/3 \quad \rightarrow \quad a_2 = 0 \quad (243)$$

$$\rightarrow x_0 = 1 + \frac{1}{3}\epsilon + 0\epsilon^2 \quad (244)$$

$$a_0 = e^{-2\pi i/3} \quad \rightarrow \quad a_1 = \frac{1}{3}e^{2\pi i/3} \quad \rightarrow \quad a_2 = \frac{i}{3\sqrt{3}} \quad (245)$$

$$\rightarrow x_1 = e^{-2\pi i/3} + \frac{1}{3}e^{2\pi i/3}\epsilon + \frac{i}{3\sqrt{3}}\epsilon^2 \quad (246)$$

$$a_0 = e^{-2\pi i/3} \quad \rightarrow \quad a_1 = \frac{1}{3}e^{2\pi i/3} \quad \rightarrow \quad a_2 = -\frac{i}{3\sqrt{3}} \quad (247)$$

$$\rightarrow x_2 = e^{2\pi i/3} + \frac{1}{3}e^{-2\pi i/3}\epsilon - \frac{i}{3\sqrt{3}}\epsilon^2 \quad (248)$$

(c)

0.7.8 Problem 7.3

With

$$x = a_0 + a_1\epsilon + a_2\epsilon^2 + \dots \quad (249)$$

$$x^k = a_0^k + k a_0^{k-1} a_1 \epsilon + \left[\binom{k}{2} a_0^{k-2} a_1^2 + k a_0^{k-1} a_2 \right] \epsilon^2 + \dots \quad (250)$$

$$(x+1)^n = \sum_k \binom{n}{k} x^k \quad (251)$$

$$= 1 + nx + \frac{n(n-1)}{2} x^2 + \dots \quad (252)$$

we obtain for each power of ϵ

$$\sum_{k=0}^n \binom{n}{k} a_0^k = 0 \quad (253)$$

$$\sum_{k=1}^n \binom{n}{k} k a_0^{k-1} a_1 = a_0 \quad (254)$$

$$\sum_{k=2}^n \binom{n}{k} \left[\binom{k}{2} a_0^{k-2} a_1^2 + k a_0^{k-1} a_2 \right] = a_1 \quad (255)$$

which we can solve

$$0 = \sum_{k=0}^n \binom{n}{k} a_0^k = (a_0 + 1)^n \rightarrow a_0 = -1$$

then

$$a_1 = \frac{a_0}{\sum_{k=1}^n \binom{n}{k} k a_0^{k-1}} = \frac{a_0}{n(1+a_0)^{n-1}} \rightarrow a_1 = -\infty \quad (256)$$

0.8 STEIN, SHAKARCHI - Princeton Lectures in Analysis vol 1 - Fourier Analysis An Introduction

0.8.1 1. Exercise 1

(a) Circle

(b) $x^2 + y^2 = 0 \rightarrow x = 0 = y \rightarrow z = 0$

(c) $|\lambda z| = |(\lambda x) + i(\lambda y)| = \sqrt{(\lambda x)^2 + i(\lambda y)^2} = \sqrt{(\lambda^2 x^2 + \lambda^2 y^2)} = |\lambda| \sqrt{x^2 + y^2} = |\lambda| |z|$

(d) $|z_1 z_2| = |r_1 e^{i\phi_1} \cdot r_2 e^{i\phi_2}| = |(r_1 r_2) e^{i(\phi_1 + \phi_2)}| = |\lambda z| = |\lambda| |z| = |r_1 r_2| = r_1 r_2$

0.8.2 1. Problem 1

0.9 ARFKEN, WEBER - Mathematical Methods for physicists 7th ed

0.9.1 6.5.19

(a) Lets generalize the problem a bit ($k, m \rightarrow k_1, k_2, k_3, m_1, m_2$)

$$L = T - V \quad (257)$$

$$= \frac{m_1}{2} \dot{x}_1^2 + \frac{m_1}{2} \dot{x}_2^2 - \frac{k_1}{2} (x_1 - 0 - l_1)^2 - \frac{k_2}{2} (x_2 - x_1 - l_2)^2 - \frac{k_3}{2} (L - x_2 - l_3)^2 \quad (258)$$

Using the Euler-Lagrange equations for x_1 and x_2

$$-k_1(x_1 - l_1) + k_2(x_2 - x_1 - l_2) - m_1\ddot{x}_1 = 0 \quad (259)$$

$$-k_2(x_2 - x_1 - l_2) + k_3(L - x_2 - l_3) - m_2\ddot{x}_2 = 0 \quad (260)$$

and simplifying

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 - k_1l_1 + k_2l_2 = 0 \quad (261)$$

$$m_2\ddot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 - l_2l_2 - k_3L + k_3l_3 = 0 \quad (262)$$

(b) Finding the eigenvalues of the Hessian

$$\begin{pmatrix} (k_1 + k_2)/m_1 & -k_2/m_1 \\ -k_2/m_2 & (k_2 + k_3)/m_2 \end{pmatrix} \quad (263)$$

we get

$$\omega_A^2 = \frac{k_2 + k_3}{m_2} + \frac{k_1 + k_2}{m_1} - \frac{1}{2} \sqrt{\frac{(k_2 + k_3)^3}{m_2^2} - 2 \frac{(k_2(k_3 - k_2) + k_1(k_2 + k_2))}{m_1 m_2} + \frac{(k_1 + k_2)^2}{m_1^2}} \quad (264)$$

$$\rightarrow \frac{k}{m} \quad (265)$$

$$\omega_B^2 = \frac{k_2 + k_3}{m_2} + \frac{k_1 + k_2}{m_1} + \frac{1}{2} \sqrt{\frac{(k_2 + k_3)^3}{m_2^2} - 2 \frac{(k_2(k_3 - k_2) + k_1(k_2 + k_2))}{m_1 m_2} + \frac{(k_1 + k_2)^2}{m_1^2}} \quad (266)$$

$$\rightarrow 3 \frac{k}{m} \quad (267)$$

(c) The associated eigenvectors are

$$X_A = (1, 1) \quad (268)$$

$$X_B = (-1, 1) \quad (269)$$

0.10 ARNOL'D - Ordinary differential equations

0.10.1 Sample Examination Problem 2

$$\ddot{x} = 1 + 2 \sin x \quad \rightarrow \quad \begin{aligned} \dot{x} &= y \\ \dot{y} &= 1 + 2 \sin x \end{aligned} \quad (270)$$

0.11 ARNOL'D - A mathematical trivium

0.11.1 Problem 4

Calculate the 100th derivative of the function $\frac{x^2+1}{x^3-x}$.

Rewrite the function as

$$\frac{x^2 + 1}{x^3 - x} = \frac{x^2 + 1}{x(x+1)(x-1)} \quad (271)$$

$$= -\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x-1} \quad (272)$$

$$\frac{d}{dx}(x+a)^{-1} = -(x+a)^{-2} \quad (273)$$

$$\frac{d^{100}}{dx^{100}}(x+a)^{-1} = 100!(x+a)^{-101} \quad (274)$$

$$(275)$$

Then

$$\frac{d^{100}}{dx^{100}} \left(\frac{x^2+1}{x^3-x} \right) = 100! \left(-\frac{1}{x^{101}} + \frac{1}{(x+1)^{101}} + \frac{1}{(x-1)^{101}} \right) \quad (276)$$

0.11.2 Problem 13

Calculate with 5% relative error $\int_1^{10} x^x dx$.

Analytic integration seems not possible

$$\int_1^{10} x^x dx < \int_1^{10} 10^x dx = \int_1^{10} e^{x \log 10} dx = \frac{1}{\log 10} e^{x \log 10} \Big|_1^{10} = \frac{1}{\log 10} 10^x \Big|_1^{10} \approx 4.35 \cdot 10^9 \quad (277)$$

0.11.3 Problem 20

$$\ddot{x} = x + A\dot{x}^2 \quad x(0) = 1, \dot{x}(0) = 0 \quad (278)$$

Using the standard perturbation theory approach we assume $x(t) = x_0(t) + Ax_1(t) + A^2x_2(t) + \dots$. Inserting into the ODE gives

$$\ddot{x}_0 + A\ddot{x}_1 + A^2\ddot{x}_2 + \dots = x_0 + Ax_1 + A^2x_2 + \dots + A(\dot{x}_0 + A\dot{x}_1 + A^2\dot{x}_2 + \dots)^2. \quad (279)$$

Sorting by powers of A we obtain a set of ODEs

$$A^0 : \quad \ddot{x}_0 = x_0 \quad (280)$$

$$A^1 : \quad \ddot{x}_1 = x_1 + \dot{x}_0^2 \quad (281)$$

$$A^2 : \quad \ddot{x}_2 = x_2 + 2\dot{x}_0\dot{x}_1. \quad (282)$$

The first ODE can be solved directly

$$x_0 = c_1 e^t + c_2 e^{-t}. \quad (283)$$

The second ODE then transforms into

$$\ddot{x}_1 = x_1 + c_1^2 e^{2t} + c_2^2 e^{-2t} - 2c_1 c_2 \quad (284)$$

with the homogeneous solution

$$x_{1H} = c_3 e^t + c_4 e^{-t}. \quad (285)$$

For the particular solution we try the ansatz (inspired by the inhomogeneity)

$$x_{1S} = \alpha + \beta e^{2t} + \gamma e^{-2t} \quad (286)$$

$$= 2c_1 c_2 + \frac{c_1^2}{3} e^{2t} + \frac{c_2^2}{3} e^{-2t} \quad (287)$$

then

$$x_1 = x_{1H} + x_{1S} \quad (288)$$

$$= c_3 e^t + c_4 e^{-t} + 2c_1 c_2 + \frac{c_1^2}{3} e^{2t} + \frac{c_2^2}{3} e^{-2t} \quad (289)$$

Imposing initial conditions on x_0 gives

$$c_1 = c_2 = \frac{1}{2} \quad \rightarrow \quad x_0 = \cosh t \quad (290)$$

$$c_3 = c_4 = -\frac{1}{3} \quad \rightarrow \quad x_1 = -\frac{2}{3} \cosh t + \frac{1}{2} + \frac{1}{6} \cosh 2t \quad (291)$$

and therefore

$$\left. \frac{dx(t)}{dA} \right|_{A=0} = \frac{1}{2} - \frac{2}{3} \cosh t + \frac{1}{6} \cosh 2t \quad (292)$$

0.11.4 Problem 23

Solve the quasi-homogeneous equation $y' = x + \frac{x^3}{y}$.

Sharp look

$$\left(\frac{y}{x} \right)' = \frac{y'x - y}{x^2} \quad (293)$$

$$= \frac{y'}{x} - \frac{y}{x^2} \quad (294)$$

then

$$y' = x + \frac{x^3}{y} \quad (295)$$

$$\frac{y'}{x} = 1 + \frac{x^2}{y} \quad (296)$$

0.11.5 Problem 50

Assume real and $k > 0$. Using the residual theorem we obtain

$$\int_{-\infty}^{\infty} \frac{e^{ikx}}{1+x^2} = \int_{-\infty}^{\infty} \frac{e^{ikx}}{(x+i)(x-i)} \quad (297)$$

$$= \frac{1}{-2i} \int_{-\infty}^{\infty} \left(\frac{1}{x+i} - \frac{1}{x-i} \right) e^{ikx} \quad (298)$$

$$= -\frac{1}{-2i} \int_{-\infty}^{\infty} \frac{e^{ikx}}{x-i} \quad (299)$$

$$= -\frac{1}{-2i} (2\pi i) e^{iki} \quad (300)$$

$$= \pi e^{-k} \quad (301)$$

0.11.6 Problem 85

In three dimensions we have

$$x^2 + y^2 + z^2 + xy + yz + zx = 1 \quad (302)$$

which can be written as

$$\vec{x}^T A \vec{x} = 1 \quad (303)$$

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1 \quad (304)$$

With an orthorgonal matrix S ($S^{-1} = S^T$) we can rotate the ellipsoid to line it up with the coordinate axes (choose S such that $D_A = S^{-1}AS$ is diagonal)

$$1 = \vec{x}^T A \vec{x} \quad (305)$$

$$= \vec{x}^T (SS^{-1})A(SS^{-1}\vec{x}) \quad (306)$$

$$= (\vec{x}^T S)S^{-1}AS(S^{-1}\vec{x}) \quad (307)$$

$$= (\vec{x}^T S)S^{-1}AS(S^T \vec{x}) \quad (308)$$

$$= (S^T \vec{x})^T S^{-1}AS(S^T \vec{x}) \quad (309)$$

$$= (S^T \vec{x})^T D_A (S^T \vec{x}) \quad (310)$$

For this we need to find the eigensystem $\{\vec{v}_i, \lambda_i\}$ of A . The characteristic polynomial is given by

$$\lambda^3 - 3\lambda^2 + \frac{9}{4}\lambda - \frac{1}{2} = 0. \quad (311)$$

Then

$$S = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad (312)$$

$$D_A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \quad (313)$$

the length of the principal axes are therefore 4, 1 and 1.

0.12 NEEDHAM - Visual Complex Analysis

0.12.1 Exercise 1.7

$$|z - a| = |z - b| \quad (314)$$

$$(z - a)(z - a)^* = (z - b)(z - b)^* \quad (315)$$

$$zz^* - az^* - za^* - aa^* = zz^* - bz^* - zb^* - bb^* \quad (316)$$

$$(-a + b)z^* + (-a^* + b^*)z = aa^* - bb^* \quad (317)$$

$$(-a + b - a^* + b^*)x + (a - b - a^* + b^*)iy = aa^* - bb^* \quad (318)$$

$$2\Re(b - a)x + 2\Im(b - a)yi = aa^* - bb^* \quad (319)$$

Looking at it from a vector space perspective - set of all points which have same distance from a and b . So its the perpendicular bisector (Mittelsenkrechte).

0.12.2 Exercise 1.33

1. It is a polynomial of ninth order $(z - 1)^{10} = z^{10} \rightarrow -10z^9 + 45z^8 + \dots + 1 = 0$. We can rewrite it as $(z - 1)^{10} = (z - 0)^{10}$

2.

$$w^{10} = 1 \rightarrow w_k = e^{2\pi ik/10} \quad k \in \{0, \dots, 9\} \quad (320)$$

$$\rightarrow z_k = \frac{1}{1 - z_k} = \frac{1}{1 - e^{2\pi ik/10}} \quad (321)$$

3.

$$z_k = \frac{1}{1 - e^{\pi ik/5}} = \frac{1}{e^{\pi ik/10}(e^{-\pi ik/10} - e^{\pi ik/10})} = \frac{e^{-\pi ik/10}}{-2i \sin(\pi k/10)} \quad (322)$$

$$= \frac{\cos(-\pi k/10) + i \sin(-\pi k/10)}{-2i \sin(\pi k/10)} = \frac{i}{2} \tan[\pi k/10] + \frac{1}{2} \quad (323)$$

0.12.3 Exercise 9.1

Recognizing the Law of Cosine - we can rewrite

$$\int_0^{2\pi} \frac{dt}{1 + a^2 - 2a \cos t} = \int_0^{2\pi} \frac{dt}{|1 - ae^{it}|^2} \quad (324)$$

$$= \int_0^{2\pi} \frac{dt}{(1 - ae^{it})(1 - ae^{-it})} \quad \frac{dz}{dt} = ie^{it} = iz \quad (325)$$

$$= \oint_C \frac{dz}{iz(1 - az)(1 - a\bar{z})} \quad (326)$$

$$= \oint_C \frac{-idz}{(1 - az)(z - a\bar{z})} \quad (327)$$

$$= \oint_C \frac{-idz}{(1 - az)(z - a)} \quad (328)$$

$$= \oint_C \frac{idz}{(az - 1)(z - a)} \quad (329)$$

then using the residuum theorem we get

$$\oint_C \frac{idz}{(az-1)(z-a)} = \frac{i}{1-a^2} \oint_C \frac{1}{z-1/a} - \frac{1}{z-a} dz \quad (330)$$

$$= \frac{i}{1-a^2} \left(\oint_C \frac{dz}{z-1/a} - \oint_C \frac{dz}{z-a} \right) \quad (331)$$

$$= \frac{i}{1-a^2} (0 - 2\pi i) \quad (332)$$

$$= \frac{2\pi}{1-a^2} \quad (333)$$

0.13 TALL, STEWARD - Complex Analysis 2018

0.13.1 Problem 11.1 - Laurent expansion

(i) Using the common geometric series trick ($|z/3| < 1$)

$$\frac{1}{z-3} = -\frac{1}{3} \frac{1}{1-z/3} \stackrel{\text{GS}}{=} -\frac{1}{3} \left(1 + \frac{z}{3} + \frac{z^2}{9} + \frac{z^3}{27} \dots \right) = -\frac{1}{3} \sum_{k=0}^{\infty} \frac{z^k}{3^k} = -\sum_{k=0}^{\infty} \frac{z^k}{3^{k+1}} \quad (334)$$

(ii)

$$\frac{1}{(z-a)^k} = \frac{1}{(-a)^k} \frac{1}{(1-z/a)^k} = \frac{1}{(-a)^k} \left(1 + \frac{z}{a} + \frac{z^2}{a^2} + \frac{z^3}{a^3} + \dots \right)^k \quad (335)$$

$$= \frac{1}{(-a)^k} \left(1 + k \frac{z}{a} + \left[\binom{k}{2} + k \right] \frac{z^2}{a^2} + \left[\binom{k}{3} + \binom{k}{2}(k-2) + k \right] \frac{z^3}{a^3} \right. \quad (336)$$

$$\left. + \left[\binom{k}{4} + \binom{k}{3}(k-2) + k(k-1) + k \right] \frac{z^4}{a^4} + \dots \right) \quad (337)$$

0.14 AHLFORS - Complex Calculus

0.14.1 Chap 1.1

1. (a)

$$(1+2i)^3 = 1 + 3(2i)^2 + 3 \cdot 2i + (2i)^3 \quad (338)$$

$$= 1 - 12 + 6i - 8i \quad (339)$$

$$= -11 - 2i \quad (340)$$

(b)

$$\frac{5}{-3+4i} = \frac{5(-3-4i)}{(-3+4i)(-3-4i)} \quad (341)$$

$$= \frac{-15-20i}{25} \quad (342)$$

$$= -\frac{3}{5} - \frac{4}{5}i \quad (343)$$

(c)

$$\left(\frac{2+i}{3-2i}\right)^2 = \frac{3+4i}{5-12i} \quad (344)$$

$$= \frac{(3+4i)(5+12i)}{169} \quad (345)$$

$$= \frac{15-48+20i+36i}{169} \quad (346)$$

$$= -\frac{33}{169} + \frac{56}{169}i \quad (347)$$

(d)

$$(1+i)^n + (1-i)^n = \sqrt{2}^n \left(e^{i\pi n/4} + e^{-i\pi n/4} \right) \quad (348)$$

$$= 2^{(n+1)/2} \cos \frac{n\pi}{4} \quad (349)$$

2. (a)

$$z^4 = (x+iy)^4 \quad (350)$$

$$= x^4 + 4x^3(iy) + 6x^2(iy)^2 + 4x(iy)^3 + (iy)^4 \quad (351)$$

$$= x^4 - 6x^2y^2 + y^4 + (4x^3y - 4xy^3)i \quad (352)$$

(b)

$$1/z = \frac{x-iy}{x^2+y^2} \quad (353)$$

(c)

$$\frac{z-1}{z+1} = \frac{(x-1)+iy}{(x+1)+iy} \quad (354)$$

$$= \frac{x^2+y^2-1+2xyi}{(x+1)^2+y^2} \quad (355)$$

(d)

$$1/z^2 = \frac{1}{x^2-y^2+2xyi} \quad (356)$$

$$= \frac{x^2-y^2-2xyi}{(x^2-y^2+2xyi)(x^2-y^2-2xyi)} \quad (357)$$

$$= \frac{x^2-y^2-2xyi}{(x^2+y^2)^2} \quad (358)$$

3. (a) With $\alpha = \pm 1$

$$(-1+i\alpha\sqrt{3})^2 = 1-3\alpha^2-i2\sqrt{3}\alpha \quad (359)$$

$$(-1+i\alpha\sqrt{3})^3 = -1+9\alpha^2+3\sqrt{3}\alpha(1-\alpha^2)i \quad (360)$$

then we see $(-1+i\alpha\sqrt{3})^3 = 9$ for $\alpha = \pm 1$.

(b) With $\alpha, \beta = \pm 1$

$$(-\beta + i\alpha\sqrt{3})^6 = (-\beta + i\alpha\sqrt{3})^{3 \cdot 2} \quad (361)$$

$$= \beta^{3 \cdot 2} \left(\underbrace{\left(-1 + i \frac{\alpha}{\beta} \sqrt{3} \right)^3}_{=1 \text{ see (a)}} \right)^2 \quad (362)$$

$$= \beta^6 \cdot 1^6 \quad (363)$$

$$= 1 \quad (364)$$

0.14.2 Chap 1.2

1. (a)

$$i = (x + iy)^2 \quad (365)$$

$$= x^2 - y^2 + 2xyi \quad (366)$$

then

$$x^2 - y^2 = 0 \quad 2xy = 1 \quad \rightarrow \quad \frac{1}{4y^2} - y^2 = 0 \quad (367)$$

$$z_1 = \frac{1+i}{\sqrt{2}} \quad (368)$$

$$z_2 = \frac{-1-i}{\sqrt{2}} \quad (369)$$

(b)

$$-i = (x + iy)^2 \quad (370)$$

$$= x^2 - y^2 + 2xyi \quad (371)$$

then

$$x^2 - y^2 = 0 \quad 2xy = -1 \quad \rightarrow \quad \frac{1}{4y^2} - y^2 = 0 \quad (372)$$

$$z_1 = \frac{-1+i}{\sqrt{2}} \quad (373)$$

$$z_2 = -\frac{1-i}{\sqrt{2}} \quad (374)$$

(c)

$$1 + i = (x + iy)^2 \quad (375)$$

$$= x^2 - y^2 + 2xyi \quad (376)$$

then

$$x^2 - y^2 = 1 \quad 2xy = 1 \quad \rightarrow \quad \frac{1}{4y^2} - y^2 = 1 \quad (377)$$

$$z_1 = \frac{1}{2\sqrt{\frac{1}{\sqrt{2}} - \frac{1}{2}}} + \sqrt{\frac{1}{\sqrt{2}} - \frac{1}{2}}i \quad (378)$$

$$z_2 = -\frac{1}{2\sqrt{\frac{1}{\sqrt{2}} - \frac{1}{2}}} - \sqrt{\frac{1}{\sqrt{2}} - \frac{1}{2}}i \quad (379)$$

(d)

$$\sqrt{\frac{1-i\sqrt{3}}{2}} = (x+iy)^2 \quad (380)$$

$$= x^2 - y^2 + 2xyi \quad (381)$$

then

$$x^2 - y^2 = \frac{1}{2} \quad 2xy = -\frac{\sqrt{3}}{2} \quad \rightarrow \quad \frac{1}{4y^2} - y^2 = \frac{1}{2} \quad (382)$$

$$z_1 \dots \quad (383)$$

$$z_2 \dots \quad (384)$$

0.15 STEIN, SHAKARCHI - Princeton Lectures in Analysis - Vol 1 Fourier-Analysis

0.15.1 Problem 1.1

$$u(x, y) = v(x)w(y) \quad (385)$$

$$\Delta u = v_{xx}w + vw_{yy} = 0 \quad (386)$$

Using a separation constant c^2 gives

$$v_{xx} + c^2v = 0 \quad \rightarrow \quad v_c(x) = C \sin cx + D \cos cx \quad (387)$$

$$w_{yy} - c^2w = 0 \quad \rightarrow \quad w_c(x) = E \sinh cy + F \cosh cy \quad (388)$$

then the general solution is (setting $F = 1$)

$$u_c(x, y) = (C \sin cx + D \cos cx)(E \sinh cy + \cosh cy) \quad (389)$$

Now we can look at the boundary conditions

$$u_c(x, 0) = (C \sin cx + D \cos cx) \stackrel{!}{=} A_k \sin kx \quad (390)$$

$$u_c(x, 1) = (C \sin cx + D \cos cx)(E \sinh c + \cosh c) \stackrel{!}{=} B_k \sin kx \quad (391)$$

$$u_c(0, y) = D(E \sinh cy + \cosh cy) \stackrel{!}{=} 0 \quad (392)$$

$$u_c(\pi, y) = (C \sin c\pi + D \cos c\pi)(E \sinh cy + \cosh cy) \stackrel{!}{=} 0 \quad (393)$$

then we see (using \sin, \cos being a complete orth. system)

$$\rightarrow c = k, D = 0, C = A_k \quad (394)$$

$$\rightarrow B_k = C(E \sinh c + \cosh c) \quad (395)$$

$$\rightarrow E = \left(\frac{B_k}{A_k} - \cosh k \right) \frac{1}{\sinh k} \quad (396)$$

And therefore

$$u(x, y) = \sum_c u_c \quad (397)$$

$$= \sum_k u_k \quad (398)$$

$$= \sum_k A_k \sin kx \left[\left(\frac{B_k}{A_k} - \cosh k \right) \frac{1}{\sinh k} \sinh ky + \cosh ky \right] \quad (399)$$

$$= \sum_k \sin kx \left[(B_k - A_k \cosh k) \frac{1}{\sinh k} \sinh ky + A_k \cosh ky \right] \quad (400)$$

$$= \sum_k \sin kx \left[(B_k \sinh ky - A_k \cosh k \sinh ky) \frac{1}{\sinh k} + A_k \cosh ky \sinh k \frac{1}{\sinh k} \right] \quad (401)$$

$$= \sum_k \sin kx \left[B_k \frac{\sinh ky}{\sinh k} + A_k \frac{-\cosh k \sinh ky + \cosh ky \sinh k}{\sinh k} \right] \quad (402)$$

$$= \sum_k \sin kx \left[B_k \frac{\sinh ky}{\sinh k} + A_k \frac{\cosh k \sinh(-ky) + \cosh(-ky) \sinh k}{\sinh k} \right] \quad (403)$$

$$= \sum_k \sin kx \left[B_k \frac{\sinh ky}{\sinh k} + A_k \frac{\sinh k(1-y)}{\sinh k} \right] \quad (404)$$

0.16 SPIVAK - Calculus on Manifolds

0.17 O'NEILL - Elementary Differential Geometry

0.17.1 Problem 1.1 - 1

(a) $x^2 y^3 \sin[z]^2$

(b) $x^2 y \sin[z] + 2xy^2 \sin[z]$

(c) $2x^2 y \cos[z]$

(d) $x^2 \cos[x^2 y]$

0.17.2 Problem 1.1 - 2

(a) 0

(b) $-19/2$

(c) $a^2 + a - 1$

(d) $t^4 - t^7$

0.17.3 Problem 1.1 - 3

(a) $xy \cos[xy] + \sin[xy] - yz \sin[xz]$

(b) $xe^{x^2+y^2+z^2} \cos(e^{x^2+y^2+z^2})$

0.17.4 Problem 1.1 - 4

(a) $-y^2 + 2(x + y)$

(b) $-2e^{2x+y}$

(c) $4x$

0.18 BOOTHBY - An Introduction to Differential Manifolds and Riemannian Geometry**0.19 BURKE - Applied Differential Geometry****0.20 O'NEILL - Semi-Riemannian Geometry - With Applications to Relativity****0.21 HUBBERT - Vector Calculus, Linear Algebra, and Differential Forms****0.22 FLANDERS - Differential Forms with Applications to the Physical Sciences****0.23 MORSE, FESHBACH - Methods of mathematical physics****0.23.1 Problem 1.1**

With

$$\cot^2 \psi = \frac{\cos^2 \psi}{\sin^2 \psi} = \frac{\cos^2 \psi}{1 - \cos^2 \psi} \quad (405)$$

we can obtain a quadratic equation

$$(x^2 + y^2) \cos^2 \psi (1 - \cos^2 \psi) + z^2 \cos^2 \psi = a^2 (1 - \cos^2 \psi) \quad (406)$$

$$\cos^4 \psi - \frac{x^2 + y^2 + z^2 + a^2}{x^2 + y^2} \cos^2 \psi + \frac{a^2}{x^2 + y^2} = 0 \quad (407)$$

with the solution

$$\cos^2 \psi = \frac{x^2 + y^2 + z^2 + a^2}{2(x^2 + y^2)} \pm \sqrt{\frac{(x^2 + y^2 + z^2 + a^2)^2}{4(x^2 + y^2)^2} - \frac{4a^2(x^2 + y^2)}{4(x^2 + y^2)^2}} \quad (408)$$

$$= \frac{x^2 + y^2 + z^2 + a^2 \pm \sqrt{(x^2 + y^2 + z^2 + a^2)^2 - 4a^2(x^2 + y^2)}}{2(x^2 + y^2)} \quad (409)$$

To obtain the gradient we differentiate the surface equation implicitly with respect to x, y and z

$$2x \cos^2 \psi - 2(x^2 + y^2) \cos \psi \sin \psi \frac{\partial \psi}{\partial x} - 2z^2 \cot \psi \csc^2 \psi \frac{\partial \psi}{\partial x} = 0 \quad (410)$$

$$\rightarrow \frac{\partial \psi}{\partial x} = \psi_x = \frac{x \cos^2 \psi}{z^2 \cot \psi \csc^2 \psi + (x^2 + y^2) \sin \psi \cos \psi} \quad (411)$$

$$2y \cos^2 \psi - 2(x^2 + y^2) \cos \psi \sin \psi \frac{\partial \psi}{\partial x} - 2z^2 \cot \psi \csc^2 \psi \frac{\partial \psi}{\partial x} = 0 \quad (412)$$

$$\rightarrow \frac{\partial \psi}{\partial y} = \psi_y = \frac{y \cos^2 \psi}{z^2 \cot \psi \csc^2 \psi + (x^2 + y^2) \sin \psi \cos \psi} \quad (413)$$

$$-2(x^2 + y^2) \cos \psi \sin \psi \frac{\partial \psi}{\partial z} + 2z \cot^2 \psi - 2z^2 \cot \psi \csc^2 \psi \frac{\partial \psi}{\partial z} = 0 \quad (414)$$

$$\rightarrow \frac{\partial \psi}{\partial z} = \psi_z = \frac{z \cot^2 \psi}{z^2 \cot \psi \csc^2 \psi + (x^2 + y^2) \cos \psi \sin \psi} \quad (415)$$

The direction cosines are then given by

$$\cos \alpha = \frac{\psi_x}{\sqrt{\psi_x^2 + \psi_y^2 + \psi_z^2}} = \frac{2\sqrt{2}x \sin^2 \psi}{\sqrt{8z^2 + (x^2 + y^2)(3 - 4 \cos 2\psi + \cos 4\psi)}} \quad (416)$$

$$\cos \beta = \frac{\psi_y}{\sqrt{\psi_x^2 + \psi_y^2 + \psi_z^2}} = \frac{2\sqrt{2}y \sin^2 \psi}{\sqrt{8z^2 + (x^2 + y^2)(3 - 4 \cos 2\psi + \cos 4\psi)}} \quad (417)$$

$$\cos \gamma = \frac{\psi_z}{\sqrt{\psi_x^2 + \psi_y^2 + \psi_z^2}} = \frac{2\sqrt{2}z}{\sqrt{8z^2 + (x^2 + y^2)(3 - 4 \cos 2\psi + \cos 4\psi)}}. \quad (418)$$

The second derivatives (for the Laplacian) can again be calculated via (lengthy) implicit differentiation and substituting the first derivatives from above. Adding them up gives zero which implies $\Delta \psi = 0$.

The surface equations $\psi = \text{const}$ can be written in form of an ellipsoid

$$\frac{x^2}{a^2 \sec^2 \psi} + \frac{y^2}{a^2 \sec^2 \psi} + \frac{z^2}{a^2 \tan^2 \psi} = 1 \quad (419)$$

which degenerates to a flat pancake for $\psi = 0, \pi$.

0.23.2 Problem 4.1 - NOT DoNE yet

Standard trick

$$x = \tan \vartheta/2 \rightarrow d\theta = \frac{2dx}{1+x^2}, \sin \vartheta = \frac{2x}{1+x^2}, \cos \vartheta = \frac{1-x^2}{1+x^2} \quad (420)$$

$$\int_0^{2\pi} \frac{\sin^2 \vartheta d\vartheta}{a + b \cos \vartheta} = \int_?^? \frac{8x^3 \cdot dx}{(1+x^2)^3(a + b \frac{1-x^2}{1+x^2})} \quad (421)$$

0.23.3 Problem 6.3 - NOT DoNE yet

Fourier series of initial condition on the interval $[0, \pi]$

$$\psi(t, 0) = \psi_0(x) = \frac{b_0}{2} + \sum_{k=1}^{\infty} (a_k \sin 2kx + b_k \cos 2kx) \quad (422)$$

$$a_k = \frac{2}{\pi} \int_0^{\pi} \psi_0(y) \sin 2ky dy \quad (423)$$

$$b_k = \frac{2}{\pi} \int_0^{\pi} \psi_0(y) \cos 2ky dy \quad (424)$$

0.24 WOIT - Quantum Theory, Groups and Representations

0.24.1 Problem B.1-3

Rotations of the 2D-plane

$$D_\phi^2 = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \quad (425)$$

$$D_\phi^2 D_\theta^2 = \begin{pmatrix} \cos \theta \cos \phi - \sin \theta \sin \phi & -\cos \phi \sin \theta - \cos \theta \sin \phi \\ \cos \phi \sin \theta + \cos \theta \sin \phi & \cos \theta \cos \phi - \sin \theta \sin \phi \end{pmatrix} \quad (426)$$

$$= \begin{pmatrix} \cos(\phi + \theta) & -\sin(\phi + \theta) \\ \sin(\phi + \theta) & \cos(\phi + \theta) \end{pmatrix} \quad (427)$$

$$= D_{\phi+\theta}^2 \quad (428)$$

can also be represented by

$$D_\phi^1 = e^{i\phi} \quad (429)$$

$$D_\phi^1 D_\theta^1 = e^{i\phi} e^{i\theta} = e^{i(\phi+\theta)} \quad (430)$$

$$= D_{\phi+\theta}^1. \quad (431)$$

Furthermore there is also the trivial representation

$$D_\phi^{1'} = 1 \quad (432)$$

$$D_\phi^{1'} D_\theta^1 = 1 \cdot 1 = 1 \quad (433)$$

$$= D_{\phi+\theta}^{1'} \quad (434)$$

0.24.2 Problem B.1-4

The time evolution is given by

$$|\Psi(t)\rangle = e^{-iHt} |\Psi(0)\rangle \quad (435)$$

$$= \left(\sum_{k=0}^{\infty} \frac{(-iHt)^k}{k!} \right) |\Psi(0)\rangle \quad (436)$$

We see

$$H = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad H^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \quad H^3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix} \quad (437)$$

and calculate

$$\sum_{k=0}^{\infty} \frac{(-it)^{2k}}{(2k)!} = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{(2k)!} = \cos(t) \quad (438)$$

$$\sum_{k=0}^{\infty} \frac{(-it)^{2k+1}}{(2k+1)!} = (-i) \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k+1}}{(2k+1)!} = -i \sin(t) \quad (439)$$

$$\sum_{k=0}^{\infty} \frac{(-i2t)^k}{k!} = \cos(2t) - i \sin(2t) = e^{-i2t} \quad (440)$$

which gives

$$e^{-iHt} = \begin{pmatrix} \cos(t) & -i \sin(t) & 0 \\ -i \sin(t) & \cos(t) & 0 \\ 0 & 0 & e^{-i2t} \end{pmatrix} \quad (441)$$

and therefore

$$|\Psi(t)\rangle = \begin{pmatrix} \psi_1 \cos(t) - \psi_2 i \sin(t) \\ -\psi_1 i \sin(t) + \psi_2 \cos(t) \\ \psi_3 e^{-2it} \end{pmatrix} \quad (442)$$

. To check the result one can calculate both sides of $i\partial_t|\Psi(t)\rangle = H|\Psi(t)\rangle$.

0.24.3 Problem B.2-1

1. With $M = PDP^{-1}$ we have $M^2 = PDP^{-1}PDP^{-1} = PDDP^{-1}$ and see

$$e^{tM} = \sum_{k=0}^{\infty} \frac{(tM)^k}{k!} = \sum_{k=0}^{\infty} \frac{(tPDP^{-1})^k}{k!} = \sum_{k=0}^{\infty} \frac{P(tD)^k P^{-1}}{k!} \quad (443)$$

$$= P \left(\sum_{k=0}^{\infty} \frac{(tD)^k}{k!} \right) P^{-1} = Pe^{tD}P^{-1}. \quad (444)$$

The eigenvalues of M are given by

$$-\lambda^3 - (-\lambda)(-\pi^2) = 0 \rightarrow \lambda_1 = i\pi, \lambda_2 = -i\pi, \lambda_3 = 0 \quad (445)$$

with the eigenvectors

$$\vec{v}_1 = (-i, 1, 0) \quad (446)$$

$$\vec{v}_2 = (i, 1, 0) \quad (447)$$

$$\vec{v}_3 = (0, 0, 1) \quad (448)$$

we obtain

$$M = PDP^{-1} \quad (449)$$

$$= \begin{pmatrix} -i & i & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} i\pi & 0 & 0 \\ 0 & -i\pi & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} i/2 & 1/2 & 0 \\ -i/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (450)$$

With

$$\sum_{k=0}^{\infty} \frac{(i\pi)^k}{k!} = e^{i\pi} \quad (451)$$

$$\sum_{k=0}^{\infty} \frac{(-i\pi)^k}{k!} = e^{-i\pi} \quad (452)$$

we see

$$tD^k = \begin{pmatrix} (i\pi t)^k & 0 & 0 \\ 0 & (-i\pi t)^k & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (453)$$

$$e^{tD} = \sum_{k=0}^{\infty} \frac{(tD)^k}{k!} = \begin{pmatrix} e^{i\pi t} & 0 & 0 \\ 0 & e^{-i\pi t} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (454)$$

and therefore

$$e^{tM} = Pe^{tD}P^{-1} \quad (455)$$

$$= \begin{pmatrix} \frac{1}{2}(e^{-i\pi t} + e^{i\pi t}) & -\frac{1}{2}i(e^{i\pi t} - e^{-i\pi t}) & 0 \\ -\frac{1}{2}i(e^{-i\pi t} - e^{i\pi t}) & \frac{1}{2}(e^{-i\pi t} + e^{i\pi t}) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (456)$$

$$= \begin{pmatrix} \cos(\pi t) & \sin(\pi t) & 0 \\ -\sin(\pi t) & \cos(\pi t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (457)$$

2. Brute force calculation of the matrix powers reveals

$$(tM)^2 = \begin{pmatrix} -(t\pi)^2 & 0 & 0 \\ 0 & -(t\pi)^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (tM)^3 = \begin{pmatrix} 0 & -(t\pi)^3 & 0 \\ (t\pi)^3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (458)$$

$$(tM)^4 = \begin{pmatrix} (t\pi)^4 & 0 & 0 \\ 0 & (t\pi)^4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (tM)^5 = \begin{pmatrix} 0 & (t\pi)^5 & 0 \\ -(t\pi)^5 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (459)$$

With

$$1 - \frac{1}{2!}(\pi t)^2 + \frac{1}{4!}(\pi t)^4 + \dots = \cos(\pi t) \quad (460)$$

$$\pi t - \frac{1}{3!}(\pi t)^3 + \frac{1}{5!}(\pi t)^5 + \dots = \sin(\pi t) \quad (461)$$

$$-\pi t + \frac{1}{3!}(\pi t)^3 - \frac{1}{5!}(\pi t)^5 + \dots = (-\pi t) + \frac{1}{3!}(-\pi t)^3 - \frac{1}{5!}(-\pi t)^5 + \dots \quad (462)$$

$$= \sin(-\pi t) \quad (463)$$

$$= -\sin(\pi t) \quad (464)$$

we obtain

$$e^{tM} = \begin{pmatrix} \cos(\pi t) & \sin(\pi t) & 0 \\ -\sin(\pi t) & \cos(\pi t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (465)$$

Problem B.2-2

For the Hamiltonian

$$H = -B_x \sigma_1 = \begin{pmatrix} 0 & -B_x \\ -B_x & 0 \end{pmatrix} \quad (466)$$

we find the eigensystem

$$E_1 = -B_x \quad |\psi_1\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (467)$$

$$E_2 = +B_x \quad |\psi_2\rangle = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \quad (468)$$

The Hamiltonian (with full units) is given by

$$H = -g \frac{q\hbar}{2m} \frac{\sigma_1}{2} B_x \quad (469)$$

which translates into energies of

$$E_1 = -g \frac{q\hbar}{4m} B_x \quad (470)$$

$$E_2 = g \frac{q\hbar}{4m} B_x. \quad (471)$$

The time evolution is then given by

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}Ht}|\psi(0)\rangle \quad (472)$$

$$= e^{-i\frac{gq}{4m}\sigma_1 t}|\psi(0)\rangle \quad (473)$$

$$= \left[\cos\left(\frac{gq}{4m}\sigma_1 t\right) - i \sin\left(\frac{gq}{4m}\sigma_1 t\right) \right] |\psi(0)\rangle \quad (474)$$

$$= \left[\cos\left(\frac{gq}{4m}t\right) \mathbb{I}_2 - i \sin\left(\frac{gq}{4m}t\right) \sigma_1 \right] |\psi(0)\rangle \quad (475)$$

$$= \begin{pmatrix} \cos\left(\frac{gqt}{4m}\right) & -i \sin\left(\frac{gqt}{4m}\right) \\ -i \sin\left(\frac{gqt}{4m}\right) & \cos\left(\frac{gqt}{4m}\right) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (476)$$

$$= \begin{pmatrix} \cos\left(\frac{gqt}{4m}\right) \\ -i \sin\left(\frac{gqt}{4m}\right) \end{pmatrix} \quad (477)$$

where we used $\sigma_1^{2n} = \mathbb{I}^n = \mathbb{I}$.

0.25 BAEZ, MUNIAIN - Gauge Fields, Knots and Gravity

0.25.1 Problem I.1 - Plane waves in vacuum

With

$$\vec{\mathcal{E}} = \vec{E} e^{-i(\omega t - \vec{k}\vec{x})} \quad (478)$$

we calculate in cartesian coordinates

$$1. \nabla \cdot \vec{\mathcal{E}} = 0$$

$$\nabla \cdot \vec{\mathcal{E}} = \partial_a \mathcal{E}_a \quad (479)$$

$$= \partial_a (e^{-i(\omega t - \vec{k}\vec{x})} E_a \vec{e}^a) \quad (480)$$

$$= \delta_{ab} i k_b E_a e^{-i(\omega t - \vec{k}\vec{x})} \vec{e}^a \quad (481)$$

$$= i k_b E_b e^{-i(\omega t - \vec{k}\vec{x})} \vec{e}^a \quad (482)$$

$$= 0 \quad (483)$$

where we assumed $E_a = \text{const}$ and used

$$0 = \vec{k} \cdot \vec{E} \quad (484)$$

$$= k_a \vec{e}^a E_a \vec{e}^a \quad (485)$$

$$= k_a E_a \quad (486)$$

$$2. \nabla \times \vec{\mathcal{E}} = i \frac{\partial \vec{\mathcal{E}}}{\partial t}$$

$$\nabla \times \vec{\mathcal{E}} = \epsilon_{abc} \partial_b \mathcal{E}_c \vec{e}_a \quad (487)$$

$$= \epsilon_{abc} E_c \vec{e}_a \partial_b (e^{-i(\omega t - \vec{k}\vec{x})}) \quad (488)$$

$$= \epsilon_{abc} E_c \vec{e}_a \delta_{bd} i k_d e^{-i(\omega t - \vec{k}\vec{x})} \quad (489)$$

$$= i(\epsilon_{abc} k_b E_c \vec{e}_a) e^{-i(\omega t - \vec{k}\vec{x})} \quad (490)$$

$$= i(-i\omega E_a \vec{e}^a) e^{-i(\omega t - \vec{k}\vec{x})} \quad (491)$$

$$= i(E_a \vec{e}^a)(-i\omega) e^{-i(\omega t - \vec{k}\vec{x})} \quad (492)$$

$$= i \vec{E} \frac{\partial}{\partial t} e^{-i(\omega t - \vec{k}\vec{x})} \quad (493)$$

$$= i \frac{\partial \vec{\mathcal{E}}}{\partial t} \quad (494)$$

where we used (typo in the book!)

$$-i\omega\vec{E} = \vec{k} \times \vec{E} \quad (495)$$

$$= \epsilon_{abc} k_b E_c \vec{e}_a \quad (496)$$

0.25.2 Problem I.7 - Adding and multiplying vector fields

1. With $(v + w)f \equiv (f) + w(f)$

$$(a) \quad (v + w)(f + g) = v(f + g) + w(f + g) = vf + vg + wf + wg = (v + w)f + (v + w)g$$

$$(b) \quad (v + w)(\alpha f) = v(\alpha f) + w(\alpha f) = \alpha vf + \alpha wf = \alpha(v + w)f$$

$$(c) \quad (v + w)(fg) = v(fg) + w(fg) = v(f)g + f v(g) + w(f)g + f w(g) = [(v + w)f]g + f[(v + w)g]$$

2. With $(gv)(f) \equiv gv(f)$

$$(a) \quad (gv)(f + h) = gv(f + h) = gv(f) + gv(h) = g(v(f) + v(h)) = gv(f) + gv(h)$$

$$(b) \quad gv(\alpha f) = gv(\alpha f) = g\alpha v(f) = \alpha gv(f)$$

$$(c) \quad (gv)(fh) = gv(fh) = g(v(f)h + f v(h)) = (gv)(f)h + f(gv)(h)$$

0.26 KREYSZIG - Introduction to functional analysis

0.26.1 Problem 1.1 Problem 1

Real line: $x \in \mathbb{R}$ with $d(x, y) = |x - y|$

M1 d is real finite, nonnegative: obvious

M2 $d(x, y) = 0$ iff $x = y$: obvious

M3 $d(x, y) = d(y, x)$: obvious

M4 $x < z < y$: $d(x, y) = d(x, z) + d(z, y)$

0.26.2 Problem 11.3 Problem 3

Physicist: Ground state of the harmonic osci. - time-independent Schroedinger equation for harmonic oscillator

$$\psi'_0 = -s e^{-s^2/2} \quad (497)$$

$$= -s \psi_0 \quad (498)$$

$$\psi''_0 = -e^{-s^2/2} + s^2 e^{-s^2/2} \quad (499)$$

$$= -\psi_0 + s^2 \psi_0 \quad (500)$$

$$= -(1 - s^2) \psi_0 \quad (501)$$

$$\rightarrow \psi''_0 + (1 - s^2) \psi_0 = 0 \quad (502)$$

0.27 GARRITY et al. - Algebraic Geometry: A Problem Solving Approach

0.28 GARRITY, NEUMANN-CHUN - Electricity and magnetism for mathematicians - A guided path from Maxwell's equations to Yang-Mills

0.29 GUIDRY - Symmetry, Broken Symmetry, and Topology in Modern Physics

0.29.1 Problem 15.1 - Poincaré transformation I

$$g(b, \Lambda) \rightarrow x' = \Lambda x + b \quad (503)$$

$$g(b', \Lambda') \circ g(b, \Lambda) \rightarrow x'' = \Lambda' x' + b' \quad (504)$$

$$= \Lambda'(\Lambda x + b) + b' \quad (505)$$

$$= \Lambda' \Lambda x + \Lambda' b + b' \rightarrow g(\Lambda' b + b', \Lambda' \Lambda) \quad (506)$$

0.29.2 Problem 15.2 - Poincaré transformation II

$$g(b', I') \circ g(0, \Lambda) = g(b', \Lambda) \quad (507)$$

$$\Lambda x + b = T(b) \circ \Lambda x \quad (508)$$

0.29.3 Problem 15.3 - Poincaré transformation III**0.30 BOLTYANSKII, EFREMOVICH - Intuitive Combinatorial Topology****0.31 NAKAHARA - Geometry, Topology and Physics****0.32 FRANKEL - The Geometry of Physics****0.33 SEXL, URBANTKE - Relativity, Groups, Particles****0.34 SCHERER - Symmetrien und Gruppen in der Teilchenphysik****0.34.1 Problem 3.11 - Taylor series**

$$e^{tC} e^{tD} e^{-tC} e^{-tD} \quad (509)$$

$$\simeq \left(1 + tC + \frac{t^2}{2}C^2 + \dots\right) \left(1 + tD + \frac{t^2}{2}D^2 + \dots\right) \left(1 - tC + \frac{t^2}{2}C^2 + \dots\right) \left(1 - tD + \frac{t^2}{2}D^2 + \dots\right) \quad (510)$$

$$= 1 + t(C + D - C - D) + \frac{t^2}{2}(C^2 + D^2 + C^2 + D^2) + t^2(CD - C^2 - CD) \quad (511)$$

$$+ t^2(-DC - D^2) + t^2(CD) + \mathcal{O}(t^3) \quad (512)$$

$$= 1 + t \cdot 0 + t^2(C^2 + D^2) + t^2(-C^2 - DC - D^2 + CD) + \mathcal{O}(t^3) \quad (513)$$

$$= 1 + t^2[C, D] + \mathcal{O}(t^3) \quad (514)$$

$$(515)$$