0.1 Grau - Elektrodynamik Aufgabensammlung

0.1.1 Exercise 8.1 - Hertzscher Dipol - NOT DONE YET

With $\vec{p}_0 = q\vec{d}$ and $d \ll r$

$$\rho(\vec{r},t) = q\delta(\vec{r} - \frac{\vec{d}}{2}\cos(\omega t)) + (-q)\delta(\vec{r} + \frac{\vec{d}}{2}\cos(\omega t)) \tag{1}$$
$$\rho(\vec{r},t_{\mathrm{ret}}) = q\delta(\vec{r} - \frac{\vec{d}}{2}\cos(\omega(t - \frac{|\vec{r}-\vec{r}'|}{c}))) + (-q)\delta(\vec{r} + \frac{\vec{d}}{2}\cos(\omega(t - \frac{|\vec{r}-\vec{r}'|}{c}))) \tag{2}$$
$$\simeq q\delta(\vec{r} - \frac{\vec{d}}{2}\cos(\omega(t - \frac{r}{c}))) + (-q)\delta(\vec{r} + \frac{\vec{d}}{2}\cos(\omega(t - \frac{r}{c}))) \tag{3}$$
$$\tag{4}$$

$$\phi(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}',t_{\mathrm{ret}})}{|\vec{r}-\vec{r}'|}d^3\vec{r}' \tag{5}$$
$$= \frac{q}{4\pi\varepsilon_0}\int \frac{\delta(\vec{r}' - \frac{\vec{d}}{2}\cos(\omega[t-\frac{r}{c}])) - \delta(\vec{r}' + \frac{\vec{d}}{2}\cos(\omega[t-\frac{r}{c}]))}{\sqrt{r^2 + r'^2 - 2\vec{r}\cdot\vec{r}'}}d^3\vec{r}' \tag{6}$$
$$= \frac{q}{4\pi\varepsilon_0}\int \frac{\delta(\vec{r}' - \frac{\vec{d}}{2}\cos(\omega[t-\frac{r}{c}])) - \delta(\vec{r}' + \frac{\vec{d}}{2}\cos(\omega[t-\frac{r}{c}]))}{r\sqrt{1 + \frac{r'^2}{r^2} - 2\frac{\vec{r}\cdot\vec{r}'}{r^2}}}d^3\vec{r}' \tag{7}$$
$$\simeq \frac{q}{4\pi\varepsilon_0}\int \frac{\delta(\vec{r}' - \frac{\vec{d}}{2}\cos(\omega[t-\frac{r}{c}])) - \delta(\vec{r}' + \frac{\vec{d}}{2}\cos(\omega[t-\frac{r}{c}]))}{r}\left(1 + \frac{\vec{r}\cdot\vec{r}'}{r^2} + \frac{r'^2}{r^2}\right)d^3\vec{r}' \tag{8}$$
$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{r}\left[\left(1 + \frac{\vec{r}\cdot\left[\frac{\vec{d}}{2}\cos(\omega[t-\frac{r}{c}])\right]}{r^2}\right) - \left(1 - \frac{\vec{r}\cdot\left[\frac{\vec{d}}{2}\cos(\omega[t-\frac{r}{c}])\right]}{r^2}\right)\right] \tag{9}$$
$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{r}\frac{\vec{r}\cdot\vec{d}\cos(\omega[t-\frac{r}{c}]))}{r^2} \tag{10}$$
$$\tag{11}$$

0.2 Zangwill - Classical Electrodynamics

0.2.1 Exercise 10.1 In-Plane Field of a Current Strip

We start with the Biot-Savart law (10.15)

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{\mathbf{j}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} \tag{12}$$

with

$$\mathbf{j} = (0,0,K)\delta(x')\Theta(y')\Theta(y'-b) \tag{13}$$
$$\mathbf{x}-\mathbf{x}' = (0,a+y',z')^T \tag{14}$$
$$\mathbf{j}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}') = (a+y')K\delta(x')\Theta(y')\Theta(y'-b) \tag{15}$$

then

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 K}{4\pi}\int_{-\infty}^{\infty} dx' \int_0^b dy' \int_{-\infty}^{\infty} dz' \frac{(a+y')}{\sqrt{(a+y)^2+z'^2}^3}\delta(x') \tag{16}$$
$$= \frac{\mu_0 K}{4\pi}\int_0^b dy' \int_{-\infty}^{\infty} dz' \frac{(a+y')}{\sqrt{(a+y')^2+z'^2}^3} \tag{17}$$
$$= \frac{\mu_0 K}{4\pi}\int_0^b dy' \frac{2}{a+y} \tag{18}$$
$$= \frac{\mu_0 K}{2\pi}\log\frac{a+b}{a} \tag{19}$$
$$= \frac{\mu_0 I}{2\pi b}\log\frac{a+b}{a} \tag{20}$$

0.3 Stratton - Electrodynamgnetic Theory

0.3.1 Problem III.1 Coordinate transform

a. Starting

$$\xi + i\eta = f(x+iy) = f(\alpha(x,y)) \tag{21}$$
$$\to d\xi + id\eta = \frac{\partial f(\alpha)}{\partial \alpha} d\alpha \tag{22}$$
$$= \frac{\partial f(\alpha)}{\partial \alpha}\left(\frac{\partial \alpha}{\partial x}dx + \frac{\partial \alpha}{\partial y}dy\right) \tag{23}$$
$$= \frac{\partial f(\alpha)}{\partial \alpha}(dx + idy) \tag{24}$$
$$= f' \cdot (dx + idy) \tag{25}$$

then calculating the absolute square

$$|f'|^2(dx^2+dy^2) = \frac{1}{h^2}(dx^2+dy^2) \tag{26}$$
$$= |d\xi + id\eta|^2 \tag{27}$$
$$= (d\xi + id\eta)(d\xi - id\eta) \tag{28}$$
$$= d\xi^2 + d\eta^2 \tag{29}$$

then with $dz = d\zeta$

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{30}$$
$$= h^2(d\xi^2 + d\eta^2) + d\zeta^2 \tag{31}$$

b. The metric is diagonal $g_{ij} = \mathrm{diag}(h^2, h^2, 1)$ then

$$\mathbf{d}\eta \cdot \mathbf{d}\xi = (0\, d\eta\, 0)\begin{pmatrix} h^2 & 0 & 0 \\ 0 & h^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} d\xi \\ 0 \\ 0 \end{pmatrix} = 0 \tag{32}$$

c. (a) Let's look at the inverse transformation

$$dx = \frac{1}{2}\left(\frac{-id\eta + d\xi}{f'^*} + \frac{id\eta + d\xi}{f'}\right) \tag{33}$$
$$dy = \frac{1}{2}\left(\frac{id\eta + d\xi}{f'^*} + \frac{id\eta - d\xi}{f'}\right) \tag{34}$$

With the two vectors in cartesian coords

$$\mathbf{v}_1 = \alpha_1 \mathbf{dx} + \beta_1 \mathbf{dy} \qquad \mathbf{v}_2 = \alpha_2 \mathbf{dx} + \beta_2 \mathbf{dy} \tag{35}$$

and in the ξ, η coords

$$\mathbf{v}_1 = \frac{1}{2}\left(\frac{\alpha_1 + i\beta_1}{f'^*} + \frac{a_1 - i\beta_1}{f'}\right)\mathbf{d}\xi + \frac{1}{2}\left(\frac{-i\alpha_1 + \beta_1}{f'^*} + \frac{ia_1 + \beta_1}{f'}\right)\mathbf{d}\eta \tag{36}$$

$$\mathbf{v}_2 = \frac{1}{2}\left(\frac{\alpha_2 + i\beta_2}{f'^*} + \frac{a_2 - i\beta_2}{f'}\right)\mathbf{d}\xi + \frac{1}{2}\left(\frac{-i\alpha_2 + \beta_2}{f'^*} + \frac{ia_2 + \beta_2}{f'}\right)\mathbf{d}\eta \tag{37}$$

The angle between to vectors is in both cases given by

$$\frac{\langle \mathbf{v}_1, \mathbf{v}_2 \rangle}{|\mathbf{v}_1||\mathbf{v}_2|} = \frac{g_{ij} v_1^i v_2^j}{\sqrt{g_{ij} v_1^i v_1^j}\sqrt{g_{ij} v_2^i v_2^j}} = \frac{\alpha_1\alpha_2 + \beta_1\beta_2}{\sqrt{\alpha_1^2 + \beta_1^2}\sqrt{\alpha_2^2 + \beta_2^2}} \tag{38}$$

So the transform conserves angles and therefore does NOT change shapes.

(b) Let's calculate the Laplace-Beltrami operator with $|g| = h^2$ and $g^{-1} = \mathrm{diag}(h^{-2}, h^{-2}, 1)$

$$\Delta = \frac{1}{\sqrt{|g|}}\partial_i \left(\sqrt{|g|} g^{ij} \partial_j\right) \tag{39}$$

$$= \frac{1}{h^2}\left(\partial_\xi (h^2 \frac{1}{h^2}\partial_\xi) + \partial_\eta (h^2 \frac{1}{h^2}\partial_\eta) + \partial_\zeta (h^2 \partial_\zeta)\right) \tag{40}$$

$$= \frac{1}{h^2}(\partial_{\xi\xi} + \partial_{\eta\eta}) + \partial_{\zeta\zeta} \tag{41}$$

0.4 Jackson - Classical Electrodynamics

0.4.1 Exercise 1.3 Charge densities and the Dirac delta function

$$\rho_a = \frac{Q}{4\pi R^2}\delta(r - R) \quad \to \quad \int \rho_a d^3 r = 4\pi \frac{Q}{4\pi R^2}\int_0^\infty \delta(r - R) r^2 \, dr \tag{42}$$

$$= Q \tag{43}$$

$$\rho_b = \frac{\lambda}{2\pi b}\delta(r - b) \quad \to \quad \int \rho_b d^3 r = \frac{\lambda}{2\pi b} 2\pi \int_0^L dz \int_0^\infty \delta(r - b) r \, dr \tag{44}$$

$$= \lambda L \tag{45}$$

$$\rho_c = \frac{Q}{\pi R^2}\theta(R - r)\delta(z) \quad \to \quad \int \rho_c d^3 r = \frac{Q}{\pi R^2} 2\pi \int dz \int_0^\infty \theta(r - R) r \, dr \tag{46}$$

$$= \frac{Q}{\pi R^2} 2\pi \int dz \int_0^R r \, dr \tag{47}$$

$$= \frac{Q}{\pi R^2} 2\pi \frac{R^2}{2} = Q \tag{48}$$

Now we got curvilinear coordinates so we need an additional $1/r$ scaling

$$\rho_d = \frac{Q}{\pi R^2 r}\theta(R - r)\delta(\vartheta - \pi/2) \quad \to \quad \int \rho_d d^3 r = \frac{Q}{\pi R^2} 2\pi \int_0^\infty \frac{r^2}{r}\theta(R - r) \int_0^\pi \delta(\vartheta - \pi/2) \sin\vartheta \, d\vartheta \tag{49}$$

$$= \frac{Q}{\pi R^2} 2\pi \int_0^R r \int_0^\pi \delta(\vartheta - \pi/2) \sin\vartheta \, d\vartheta \tag{50}$$

$$= \frac{Q}{\pi R^2} 2\pi \frac{R^2}{2} \sin \pi/2 = Q \tag{51}$$

0.4.2 Exercise 1.4 Charged spheres

We can utilize the Gauss theorem

$$\oint_S \vec{E}\cdot\vec{n}dA = \frac{1}{\epsilon_0}\int_V \rho(x)d^3x \tag{52}$$
$$4\pi r^2 E_r = \frac{q_r}{\epsilon_0} \tag{53}$$
$$E_r = \frac{q_r}{4\pi\epsilon_0 r^2} \tag{54}$$

assuming a radial electrical field.

- Conducting sphere

$$\rho_{\mathrm{cond}} = Q\delta(r-a) \tag{55}$$
$$E_r = \frac{1}{4\pi\epsilon_0}\cdot\begin{cases} 0 & r<a \\ Q/r^2 & r>a \end{cases} \tag{56}$$

- Uniform sphere

$$\rho_{\mathrm{hom}} = Q\theta(a-r) \tag{57}$$
$$E_r = \frac{1}{4\pi\epsilon_0}\cdot\begin{cases} Q/a^3\cdot r & r<a \\ Q/r^2 & r>a \end{cases} \tag{58}$$

- Nonuniform sphere

$$\rho_{\mathrm{inhom}} = Q\frac{n+3}{a^{n+3}}r^n \quad (r<a) \tag{59}$$
$$E_r = \frac{1}{4\pi\epsilon_0}\cdot\begin{cases} Qa^{n+3}r^{n+1} & r<a \\ Q/r^2 & r>a \end{cases} \tag{60}$$

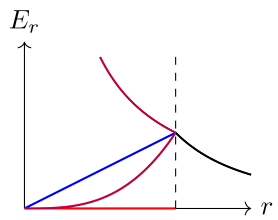

Figure 1: Jackson problem (1.4)

0.4.3 Exercise 1.5 Charge density of hydrogen atom

With the potential

$$\Phi = \frac{q}{4\pi\epsilon_0}\frac{e^{-\alpha r}}{r}\left(1+\frac{\alpha r}{2}\right) \tag{61}$$

we calculate for $r>0$

$$\rho_1 = -\epsilon_0\triangle\Phi \tag{62}$$
$$= -\epsilon_0\frac{1}{r^2}\partial_r(r^2\partial_r\Phi) \tag{63}$$
$$= -\frac{q}{4\pi}e^{-\alpha r}\frac{\alpha^3}{2} \tag{64}$$
$$= -\frac{q}{\pi a_0^3}e^{-2r/a_0} \tag{65}$$

For $r = 0$ we have

$$\Phi(r \to 0) = \frac{q}{4\pi\epsilon_0 r} \tag{66}$$
$$\to \quad \rho_0 = q\delta(r) \tag{67}$$

Therefore

$$\rho = \rho_0 + \rho_1 \tag{68}$$
$$= q\left(\delta^{(3)}(r) - \frac{1}{\pi a_0^3} e^{-2r/a_0}\right) \tag{69}$$

Calculating the total charge

$$Q_0 = q \int d^3 r \delta(r) = q \tag{70}$$
$$Q_1 = 4\pi \int_0^\infty r^2 \rho_1 dr \tag{71}$$
$$= -\frac{4\pi q}{\pi a_0^3} \int_0^\infty r^2 e^{-2r/a_0} dr \tag{72}$$
$$= -\frac{4\pi q}{\pi a_0^3} \frac{a_0^3}{8} \int_0^\infty z^2 e^{-z} dz \tag{73}$$
$$= -\frac{4\pi q}{\pi a_0^3} \frac{a_0^3}{8} \Gamma(3) \tag{74}$$
$$= -q \tag{75}$$

0.4.4 Exercise 1.6 Simple capacitors

(a) Assuming only front and back surfaces contribute

$$2E_x A = \frac{Q}{\epsilon_0} \tag{76}$$
$$\to \quad E_x = \frac{Q}{2\epsilon_0 A} \tag{77}$$
$$\to \quad \phi = -\frac{Q}{2\epsilon_0 A} x \tag{78}$$
$$\to \quad \phi_{\text{tot}}(x) = -\frac{Q}{2\epsilon_0 A} x - \frac{-Q}{2\epsilon_0 A}(d - x) \tag{79}$$
$$= -\frac{Q}{2\epsilon_0 A}(x - (d - x)) \tag{80}$$
$$= -\frac{Q}{2\epsilon_0 A}(2x - d) \tag{81}$$
$$\to \quad C = \frac{Q}{\Delta\phi} = \frac{Q}{-\frac{Q}{2\epsilon_0 A}(-d - d)} \tag{82}$$
$$= \epsilon_0 \frac{A}{d} \tag{83}$$

(b) The outer sphere does not contribute to the total potential as it is field free

$$4\pi r^2 E_r = \frac{Q}{\epsilon_0} \tag{84}$$
$$\rightarrow \quad E_r = \frac{Q}{4\pi\epsilon_0 r^2} \tag{85}$$
$$\rightarrow \quad \phi = \frac{Q}{4\pi\epsilon_0 r} \tag{86}$$
$$\rightarrow \quad \phi_{\text{tot}} = \frac{Q}{4\pi\epsilon_0 r} \quad (a < r < b) \tag{87}$$
$$\rightarrow \quad C = \frac{Q}{\Delta\phi} = \frac{Q}{\frac{Q}{4\pi\epsilon_0 b} - \frac{Q}{4\pi\epsilon_0 a}} \tag{88}$$
$$= \epsilon_0 \frac{4\pi ab}{b-a} \tag{89}$$

(c)

$$2\pi r L E_r = \frac{Q}{\epsilon_0} \tag{90}$$
$$\rightarrow \quad E_r = \frac{Q}{2\pi r L \epsilon_0} \tag{91}$$
$$\rightarrow \quad \phi = -\frac{Q}{2\pi L \epsilon_0} \log r \tag{92}$$
$$\rightarrow \quad \phi_{\text{tot}} = -\frac{Q}{2\pi L \epsilon_0} \log r \quad (a < r < b) \tag{93}$$
$$\rightarrow \quad C = \frac{Q}{\Delta\phi} = \frac{Q}{-\frac{Q}{2\pi L\epsilon_0}\log b + \frac{Q}{2\pi L\epsilon_0}\log a} \tag{94}$$
$$= \frac{2\pi L \epsilon_0}{\log a/b} \tag{95}$$

(d) ...

0.4.5 Exercise 1.7 Capacity of two parallel cylinders

Gauss law for one cylinder

$$\oint_S \vec{E} \cdot \vec{n} dA = \frac{1}{\epsilon_0} \int_V \rho(x) d^3x \tag{96}$$
$$2\pi r L E_r = \frac{\rho_1 L}{\epsilon_0} \tag{97}$$
$$E_r = \frac{\rho}{2\pi\epsilon_0 r} \tag{98}$$
$$\phi = -\frac{\rho}{2\pi\epsilon_0} \ln r \tag{99}$$

For $d \gg a_{1,2}$ the potential of one cylinder on the surface of the second cylinder is constant - which means that the potential can be approximated by the sum of the potential of both cylinders (no need to make it complicated)

$$\begin{aligned}
\phi(\vec{r}) &= \phi_1 + \phi_2 && (100)\\
&= -\frac{\rho_1}{2\pi\epsilon_0}\ln|\vec{r}| - \frac{\rho_2}{2\pi\epsilon_0}\ln|\vec{r}-\vec{d}| && (101)\\
&= -\frac{\rho}{2\pi\epsilon_0}\ln|\vec{r}| + \frac{\rho}{2\pi\epsilon_0}\ln|\vec{r}-\vec{d}| && (102)\\
&= -\frac{\rho}{2\pi\epsilon_0}\left(\ln|\vec{r}| - \ln|\vec{r}-\vec{d}|\right) && (103)\\
&= -\frac{\rho}{2\pi\epsilon_0}\ln\frac{|\vec{r}|}{|\vec{r}-\vec{d}|} && (104)\\
&= -\frac{\rho}{\pi\epsilon_0}\ln\sqrt{\frac{|\vec{r}|}{|\vec{r}-\vec{d}|}} && (105)
\end{aligned}$$

Then the potential difference between to surfaces is given by (with $\vec{n} = \vec{d}/d$ and $\rho = \rho_1 = -\rho_2$)

$$\begin{aligned}
\Delta\phi &= \phi(a_1\vec{n}) - \phi((d-a_2)\vec{n}) && (106)\\
&= -\frac{\rho}{\pi\epsilon_0}\left(\ln\sqrt{\frac{a_1}{d-a_1}} - \ln\sqrt{\frac{d-a_2}{a_2}}\right) && (107)\\
&= \frac{\rho}{\pi\epsilon_0}\left(\ln\sqrt{\frac{d-a_1}{a_1}} + \ln\sqrt{\frac{d-a_2}{a_2}}\right) && (108)\\
&\simeq \frac{\rho}{\pi\epsilon_0}\left(\ln\sqrt{\frac{d}{a_1}} + \ln\sqrt{\frac{d}{a_2}}\right) && (109)\\
&\simeq \frac{\rho}{\pi\epsilon_0}\ln\frac{d}{\sqrt{a_1a_2}} && (110)
\end{aligned}$$

With $C = Q/U$ we have

$$C = \frac{\rho L}{\Delta\phi} = \frac{\pi\epsilon_0 L}{\ln\frac{d}{\sqrt{a_1a_2}}} \tag{111}$$

which is the desired result. The numbers are 0.49mm, 1.47mm and 4.92mm.

0.4.6 Exercise 1.8 Energy of capacitors

$$W = \frac{1}{2}\int\rho(x)\phi(x)d^3x = -\frac{\epsilon_0}{2}\int\phi\Delta\phi d^3x = \frac{\epsilon_0}{2}\int(\nabla\phi)^2d^3x = \frac{\epsilon_0}{2}\int|\vec{E}|^2d^3x \tag{112}$$

(a) With $\vec{E}_{\text{tot}} = -\nabla\phi_{\text{tot}}$ and $Q = C\cdot U$

$$\begin{aligned}
W_{\text{plate}} &= \frac{\epsilon_0}{2}\cdot\left(\frac{Q}{\epsilon_0A}\right)^2\cdot(Ad) = \frac{Q^2d}{2\epsilon_0A} && (113)\\
&= \frac{U^2d}{2\epsilon_0A}\left(\frac{\epsilon_0A}{d}\right)^2 = \frac{\epsilon_0AU^2}{2d} && (114)
\end{aligned}$$

$$\begin{aligned}
W_{\text{sphere}} &= \frac{\epsilon_0}{2}4\pi\int_a^b r^2\frac{Q^2}{16\pi^2\epsilon_0^2r^4}dr = \frac{Q^2}{8\pi\epsilon_0}\left(\frac{1}{b}-\frac{1}{a}\right) && (115)\\
&= \frac{U^2}{8\pi\epsilon_0}\left(\frac{a-b}{ab}\right)\cdot\left(\epsilon_0\frac{4\pi ab}{b-a}\right)^2 = 2\pi\epsilon_0U^2\frac{ab}{b-a} && (116)
\end{aligned}$$

$$W_{\text{cylinder}} = \frac{\epsilon_0}{2} 2\pi L \int_a^b \left(\frac{Q}{2\pi\epsilon_0 L r}\right)^2 r\, dr = \frac{Q^2}{4\pi\epsilon_0 L} \log\frac{b}{a} \tag{117}$$

$$= \frac{U^2}{4\pi\epsilon_0 L} \log\frac{b}{a} \left(\frac{2\pi\epsilon_0 L}{\log b/a}\right)^2 = \frac{\pi\epsilon_0 L U^2}{\log b/a} \tag{118}$$

(b)

$$w_{\text{plate}} = \text{const} \tag{119}$$

$$w_{\text{sphere}} \sim r^{-4} \tag{120}$$

$$w_{\text{cylinder}} \sim r^{-2} \tag{121}$$

0.4.7 Exercise 5.1 Biot–Savart law NOT DONE YET

With

$$\nabla_{x'} \frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \tag{122}$$

we consider a loop of radius a in the $x-y$ plane

$$d\mathbf{B} = \frac{\mu_0 I}{4\pi} d\mathbf{l}' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \tag{123}$$

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 I}{4\pi} \oint_C d\mathbf{l}' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \tag{124}$$

$$= \frac{\mu_0 I}{4\pi} \oint_C d\mathbf{l}' \times \left(\nabla_{x'} \frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) \tag{125}$$

with P in the $x-z$ plane

$$(\mathbf{x}-\mathbf{x}')^2 = (r\cos\theta)^2 + ((r\sin\theta)^2 + a^2 - 2ar\sin\theta\cos\phi') \tag{126}$$

$$= r^2 + a^2 - 2ar\sin\theta\cos\phi' \tag{127}$$

0.4.8 Exercise 9.1 Rotating charge and current densities - NOT DONE YET

With $r = |\mathbf{x}|$ and $r' = |\mathbf{x}'|$

$$\mathbf{A}(\mathbf{x},t) = \frac{\mu_0}{4\pi} \int dt' \int d^3\mathbf{x}' \frac{\mathbf{J}(\mathbf{x}',t')}{|\mathbf{x}-\mathbf{x}'|} \delta\left(t' + \frac{|\mathbf{x}-\mathbf{x}'|}{c} - t\right) \tag{128}$$

$$= \frac{\mu_0}{4\pi} \int dt' \int d^3\mathbf{x}' \frac{\mathbf{J}(\mathbf{x}', t - |\mathbf{x}-\mathbf{x}'|/c)}{|\mathbf{x}-\mathbf{x}'|} \tag{129}$$

$$= \frac{\mu_0}{4\pi} \sum_{l,m} \frac{4\pi}{2l+1} \frac{q_{lm}}{r^{l+1}} Y_{lm}(\vartheta,\varphi) \tag{130}$$

$$q_{lm}(t) = \int d^3\mathbf{x}'\, r'^l\, Y^*_{lm}(\vartheta',\varphi')\, \mathbf{J}(\mathbf{x}', t - |\mathbf{x}-\mathbf{x}'|/c) \tag{131}$$

$$\mathbf{J}(\mathbf{x}',t) = \rho(\mathbf{x}',t)\mathbf{v} = (\mathbf{\Omega}\times\mathbf{x}')\rho(\mathbf{x}',t) \tag{132}$$

0.4.9 Exercise 9.2 Rotating quadrupole - NOT DONE YET

Lets look at a single rotating point charge first

$$\rho(\mathbf{x}',t') = \frac{1}{r'^2\sin\theta'} q\delta(r'-R)\delta(\phi'-\omega t')\delta(\theta'-\pi/2) \tag{133}$$

$$\mathbf{J}(\mathbf{x}',t') = \rho\mathbf{v} \tag{134}$$

$$= \frac{1}{r'^2\sin\theta'} q\delta(r'-R)\delta(\phi'-\omega t')\delta(\theta'-\pi/2) R\omega\mathbf{e}_\phi \tag{135}$$

$$Y_{00} = \frac{1}{\sqrt{4\pi}} \qquad Y_{10} = \sqrt{\tfrac{3}{4\pi}}\cos\theta \qquad Y_{20} = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1) \tag{136}$$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi} \quad Y_{1,-1} = \sqrt{\tfrac{3}{8\pi}}\sin\theta\, e^{-i\phi} \quad Y_{21} = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi} \tag{137}$$

$$Y_{2,-1} = \sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{-i\phi} \quad Y_{22} = \sqrt{\tfrac{15}{32\pi}}\sin^2\theta\, e^{2i\phi} \quad Y_{2,-2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{-2i\phi} \tag{138}$$

$$\rho(\mathbf{x},t) = q\delta\left(x - \frac{a}{\sqrt{2}}\cos\omega t\right)\delta\left(y - \frac{a}{\sqrt{2}}\sin\omega t\right)\delta(z) + q\delta\left(x + \frac{a}{\sqrt{2}}\cos\omega t\right)\delta\left(y + \frac{a}{\sqrt{2}}\sin\omega t\right)\delta(z) \tag{139}$$

$$- q\delta\left(x + \frac{a}{\sqrt{2}}\sin\omega t\right)\delta\left(y - \frac{a}{\sqrt{2}}\cos\omega t\right)\delta(z) - q\delta\left(x - \frac{a}{\sqrt{2}}\sin\omega t\right)\delta\left(y + \frac{a}{\sqrt{2}}\cos\omega t\right)\delta(z) \tag{140}$$

$$\tag{141}$$

0.4.10 Exercise 12.1 Lagrangian of point charge

1. With $U^\alpha = \frac{dx_\alpha}{ds}$

$$L = -\frac{mU_\alpha U^\alpha}{2} - \frac{q}{c}U_\alpha A^\alpha \tag{142}$$

$$\frac{\partial L}{\partial x_\beta} = -\frac{q}{c}U_\alpha \frac{\partial A^\alpha}{\partial x_\beta} \tag{143}$$

$$\frac{\partial L}{\partial U_\beta} = -mU^\beta - \frac{q}{c}A^\beta \tag{144}$$

$$-m\frac{d}{ds}\left(\frac{dU^\beta}{ds}\right) - \frac{q}{c}\frac{dA^\beta}{ds} + \frac{q}{c}U_\alpha\frac{\partial A^\alpha}{\partial x_\beta} = 0 \tag{145}$$

$$m\frac{d^2x^\beta}{ds^2} + \frac{q}{c}\frac{dA^\beta}{ds} - \frac{q}{c}\frac{dx_\alpha}{ds}\frac{\partial A^\alpha}{\partial x_\beta} = 0 \tag{146}$$

$$m\frac{d^2x^\beta}{ds^2} + \frac{q}{c}\left(\frac{\partial A^\beta}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial s}\right) - \frac{q}{c}\frac{dx_\alpha}{ds}\frac{\partial A^\alpha}{\partial x_\beta} = 0 \tag{147}$$

$$m\frac{d^2x^\beta}{ds^2} + \frac{q}{c}\frac{\partial x^\alpha}{\partial s}\left(\frac{\partial A^\beta}{\partial x^\alpha} - \frac{\partial A^\alpha}{\partial x_\beta}\right) = 0 \tag{148}$$

$$m\frac{d^2x^\beta}{ds^2} + \frac{q}{c}\frac{\partial x^\alpha}{\partial s}F^{\alpha\beta} = 0 \tag{149}$$

2. Bit of a odd sign convention for the canonical momentum

$$P^\beta = -\frac{\partial L}{\partial U_\beta} = mU^\beta + \frac{q}{c}A^\beta \quad \to \quad U^\beta = \frac{1}{m}\left(P^\beta - \frac{q}{c}A^\beta\right) \tag{150}$$

$$H = P^\alpha U_\alpha + L \tag{151}$$

$$= P^\alpha\frac{1}{m}\left(P_\alpha - \frac{q}{c}A_\alpha\right) - \frac{m}{2}\frac{1}{m}\left(P_\alpha - \frac{q}{c}A_\alpha\right)\frac{1}{m}\left(P_\alpha - \frac{q}{c}A_\alpha\right) - \frac{q}{c}\frac{1}{m}\left(P_\alpha - \frac{q}{c}A_\alpha\right)A^\alpha \tag{152}$$

$$= \frac{1}{2m}\left(P^\alpha - \frac{q}{c}A^\alpha\right)\left(P_\alpha - \frac{q}{c}A_\alpha\right) \tag{153}$$

In space-time coordinates we can write

$$
\begin{aligned}
H &= \frac{1}{2m}\left((p_0)^2 - \vec{p}^2 + \frac{q^2}{c^2}[\phi^2 - \vec{A}^2] + \frac{2q}{c}[\vec{p}\cdot\vec{A} - p^0\phi]\right) && (154)\\
&= \frac{1}{2m}\left((\gamma mc)^2 - (\gamma m\vec{v})^2 + \frac{q^2}{c^2}[\phi^2 - \vec{A}^2] + \frac{2q}{c}[\gamma m\vec{v}\cdot\vec{A} - \gamma mc\phi]\right) && (155)\\
&= \frac{\gamma^2 mc^2}{2}\left(1 - \frac{\vec{v}^2}{c^2}\right) + \frac{q^2}{2mc^2}[\phi^2 - \vec{A}^2] + q\gamma[\frac{1}{c}\vec{v}\cdot\vec{A} - \phi] && (156)\\
&= \frac{mc^2}{2} + \frac{q^2}{2mc^2}[\phi^2 - \vec{A}^2] + q\gamma[\frac{1}{c}\vec{v}\cdot\vec{A} - \phi] && (157)
\end{aligned}
$$

0.4.11 Exercise 14.1 Lienard-Wiechert fields for particle with constant velocity - NOT DONE YET

With

$$
\begin{aligned}
A^\alpha(x) &= \frac{4\pi}{c}\int d^4x'\, D_r(x-x')J^\alpha(x') && (158)\\
J^\alpha(x') &= ec\int d\tau\, V^\alpha(\tau)\delta^{(4)}(x' - r(\tau)) && (159)\\
D_r(x-x') &= \frac{\theta(t-t')}{4\pi R}\delta(t-t'-R) && (160)\\
&= \frac{1}{2\pi}\theta(t-t')\delta[(t-t'-R)(t-t'+R)] && (161)\\
&= \frac{1}{2\pi}\theta(t-t')\delta[(t-t')^2 - (\vec{x}-\vec{x}')^2] \qquad (a-b)(a+b) = a^2 - b^2 && (162)\\
&= \frac{1}{2\pi}\theta(t-t')\delta[(x-x')^2] && (163)\\
R &= |\vec{x}-\vec{x}'| && (164)\\
A^\alpha(x) &= 4\pi e\int d\tau \int d^4x'\, \frac{\theta(t-t')}{4\pi R}\delta(t-t'-R)V^\alpha(\tau)\delta^{(4)}(x'-r(\tau)) && (165)
\end{aligned}
$$

Constant velocity means

$$
\begin{aligned}
V^\alpha(\tau) &= \frac{dr}{d\tau} = \frac{dr}{dt}\frac{dt}{d\tau} = \gamma\cdot(c,\vec{v}) && (166)\\
r^\alpha(\tau) &= \gamma\cdot(c\tau, \vec{v}\tau) && (167)\\
\to J^\alpha(x') &= ec\gamma\cdot(c,\vec{v}) && (168)
\end{aligned}
$$

then

$$
A^\alpha(x) = \frac{4\pi}{c}\int d^4x'\, \frac{\theta(t-t')}{4\pi R}\delta(t-t'-R)J^\alpha(x') \qquad (169)
$$

0.5 Schwinger - Classical Electrodynamics

0.5.1 Exercise 9.1 Lagrangian of a particle in an electromagnetic field

$$
L = \mathbf{p}\cdot\left(\frac{d\mathbf{r}}{dt} - \mathbf{v}\right) + \frac{1}{2}mv^2 - e\phi + \frac{e}{c}\mathbf{v}\cdot\mathbf{A} \qquad (170)
$$

0.5.2 Exercise 31.1 Potentials of moving point charge

$$w = z - vt \to \frac{\partial}{\partial z} = \frac{\partial w}{\partial z}\frac{\partial}{\partial w} \tag{171}$$

$$\to \frac{\partial^2}{\partial z^2} = \frac{\partial^2 w}{\partial z^2}\frac{\partial}{\partial w} + \left(\frac{\partial w}{\partial z}\right)^2 \frac{\partial^2}{\partial w^2} = \frac{\partial^2}{\partial w^2} \tag{172}$$

$$\to \frac{\partial^2}{\partial t^2} = \frac{\partial^2 w}{\partial t^2}\frac{\partial}{\partial w} + \left(\frac{\partial w}{\partial t}\right)^2 \frac{\partial^2}{\partial w^2} = v^2\frac{\partial^2}{\partial w^2} \tag{173}$$

then

$$\triangle - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial w^2} - \frac{v^2}{c^2}\frac{\partial^2}{\partial w^2} \tag{174}$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(1 - \frac{v^2}{c^2}\right)\frac{\partial^2}{\partial w^2} \tag{175}$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial u^2} \tag{176}$$

with with $u = w/\sqrt{1 - v^2/c^2}$. The wave equation can then be rewritten

$$-\Box\phi = 4\pi\rho \tag{177}$$

$$= 4\pi e\delta(x)\delta(y)\delta(z - vt) \tag{178}$$

$$-\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial u^2}\right)\phi = 4\pi e\delta(x)\delta(y)\delta\left(\sqrt{1 - \frac{v^2}{c^2}}u\right) \tag{179}$$

$$= \frac{4\pi}{\sqrt{1 - \frac{v^2}{c^2}}}e\delta(x)\delta(y)\delta\left(u\right) \tag{180}$$

Using the Green function of the Coulomb equation (13.3) we obtain

$$\phi = \frac{e}{\sqrt{1 - \frac{v^2}{c^2}}\sqrt{u^2 + x^2 + y^2}} \tag{181}$$

$$= \frac{e}{\sqrt{w^2 + (1 - \frac{v^2}{c^2})(x^2 + y^2)}} \tag{182}$$

$$= \frac{e}{\sqrt{(z - vt)^2 + (1 - \frac{v^2}{c^2})(x^2 + y^2)}} \tag{183}$$

For the vector potential we can calculate similarly

$$-\Box\vec{A} = 4\pi\frac{\vec{j}}{c} \tag{184}$$

$$-\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial u^2}\right)\vec{A} = 4\pi e\frac{\vec{v}}{c}\delta(x)\delta(y)\delta\left(\sqrt{1 - \frac{v^2}{c^2}}u\right) \tag{185}$$

$$= \frac{4\pi}{\sqrt{1 - \frac{v^2}{c^2}}}e\frac{\vec{v}}{c}\delta(x)\delta(y)\delta\left(u\right) \tag{186}$$

which gives $\vec{A} = \vec{v}/c\phi$.

0.5.3 Exercise 31.2 Fields of moving point charge

$$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \tag{187}$$
$$= \frac{e}{2}\left((z-vt)^2 + (1-\frac{v^2}{c^2})(x^2+y^2)\right)^{-3/2}[(1-\frac{v^2}{c^2})2x, (1-\frac{v^2}{c^2})2y, 2(z-vt)(1-\frac{v^2}{c^2})] \tag{188}$$
$$= e(1-\frac{v^2}{c^2})\left((z-vt)^2 + (1-\frac{v^2}{c^2})(x^2+y^2)\right)^{-3/2}[x, y, (z-vt)] \tag{189}$$
$$\vec{B} = \nabla \times \vec{A} \tag{190}$$
$$= -e\frac{v}{c}(1-\frac{v^2}{c^2})\left((z-vt)^2 + (1-\frac{v^2}{c^2})(x^2+y^2)\right)^{-3/2}[y, x, 0] \tag{191}$$

0.5.4 Exercise 31.4 Wave equation for fields

With

$$\nabla \times \vec{B} = \frac{1}{c}\frac{\partial}{\partial t}\vec{E} + \frac{4\pi}{c}\vec{j}_e \tag{192}$$
$$\nabla \cdot \vec{E} = 4\pi\rho_e \tag{193}$$
$$-\nabla \times \vec{E} = \frac{1}{c}\frac{\partial}{\partial t}\vec{B} + \frac{4\pi}{c}\vec{j}_m \tag{194}$$
$$\nabla \cdot \vec{B} = 4\pi\rho_m \tag{195}$$

we obtain

$$\nabla \times \nabla \times \vec{B} = \nabla(\nabla \cdot \vec{B}) - \triangle\vec{B} \tag{196}$$
$$= 4\pi\nabla\rho_m - \triangle\vec{B} \tag{197}$$
$$= \frac{1}{c}\frac{\partial}{\partial t}\nabla \times \vec{E} + \frac{4\pi}{c}\nabla \times \vec{j}_e \tag{198}$$
$$= -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{B} - \frac{4\pi}{c^2}\frac{\partial}{\partial t}\vec{j}_m + \frac{4\pi}{c}\nabla \times \vec{j}_e \tag{199}$$
$$\to -\Box\vec{B} = -4\pi\nabla\rho_m + \frac{4\pi}{c}(\nabla \times \vec{j}_e - \frac{1}{c}\frac{\partial}{\partial t}\vec{j}_m) \tag{200}$$

$$\nabla \times \nabla \times \vec{E} = \nabla(\nabla \cdot \vec{E}) - \triangle\vec{E} \tag{201}$$
$$= 4\pi\nabla\rho_e - \triangle\vec{E} \tag{202}$$
$$= -\frac{1}{c}\frac{\partial}{\partial t}\nabla \times \vec{B} - \frac{4\pi}{c}\nabla \times \vec{j}_m \tag{203}$$
$$= -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} - \frac{4\pi}{c^2}\frac{\partial}{\partial t}\vec{j}_e - \frac{4\pi}{c}\nabla \times \vec{j}_m \tag{204}$$
$$\to -\Box\vec{E} = -4\pi\nabla\rho_e + \frac{4\pi}{c}(\nabla \times \vec{j}_m - \frac{1}{c}\frac{\partial}{\partial t}\vec{j}_e) \tag{205}$$

0.5.5 Exercise 31.5 Lienard-Wiechert potentials

We start with the scalar potential

$$\phi(\mathbf{r},t) = \int d\vec{r}' dt' \frac{\delta(\frac{1}{c}|\mathbf{r}-\mathbf{r}'| - (t-t'))}{|\mathbf{r}-\mathbf{r}'|}\rho(\mathbf{r}',t') \tag{206}$$

$$= \int d\mathbf{r}' dt' \frac{\delta(t'-t+\frac{1}{c}|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|} e\delta(\mathbf{r}'-\mathbf{r}_B(t')) \tag{207}$$

$$= \int dt' \frac{e}{|\mathbf{r}-\mathbf{r}_B(t')|}\delta(t'-t+\frac{1}{c}|\mathbf{r}-\mathbf{r}_B(t')|) \tag{208}$$

with

$$\delta(t'-t+\frac{1}{c}|\mathbf{r}-\mathbf{r}_B(t')|) = \delta(f(t')) \tag{209}$$

$$= \sum_{t_{ret}} \frac{\delta(t'-t_{ret})}{|f'(t_{ret})|} \tag{210}$$

where

$$f(t') = t'-t+\frac{1}{c}|\mathbf{r}-\mathbf{r}'_B(t')| \tag{211}$$

$$f(t_{ret}) = t_{ret}-t+\frac{1}{c}|\mathbf{r}-\mathbf{r}_B(t_{ret})| = 0 \tag{212}$$

$$\rightarrow t_{ret} = t-\frac{1}{c}|\mathbf{r}-\mathbf{r}_B(t_{ret})| \tag{213}$$

$$f'(t') = 1+\frac{1}{c}\partial_{t'}|\mathbf{r}-\mathbf{r}_B(t')| \tag{214}$$

$$= 1+\frac{1}{c}\frac{2\mathbf{r}_B\cdot\mathbf{v}_B(t')-2\mathbf{r}\cdot\mathbf{v}_B(t')}{2|\mathbf{r}-\mathbf{r}'_B(t')|} \tag{215}$$

$$\text{using} \quad |\mathbf{r}-\mathbf{r}_B(t')| = \sqrt{r^2+r_B^2-2\mathbf{r}\cdot\mathbf{r}_B} \tag{216}$$

$$= 1+\frac{1}{c}\frac{(\mathbf{r}_B(t')-\mathbf{r})\cdot\mathbf{v}_B(t')}{|\mathbf{r}-\mathbf{r}_B(t')|} \tag{217}$$

then

$$\delta(t'-t+\frac{1}{c}|\mathbf{r}-\mathbf{r}_B(t')|) = \frac{\delta(t'-t_{ret})}{|f'(t_{ret})|} \tag{218}$$

$$= \frac{\delta(t'-t_{ret})}{1+\frac{1}{c}\frac{(\mathbf{r}_B(t_{ret})-\mathbf{r})\cdot\mathbf{v}_B(t_{ret})}{|\mathbf{r}-\mathbf{r}_B(t_{ret})|}} \tag{219}$$

and therefore

$$\phi(\mathbf{r},t) = \int dt' \frac{e}{|\mathbf{r}-\mathbf{r}_B(t')|}\delta(t'-t+\frac{1}{c}|\mathbf{r}-\mathbf{r}_B(t')|) \tag{220}$$

$$= \int dt' \frac{e}{|\mathbf{r}-\mathbf{r}_B(t')|}\frac{\delta(t'-t_{ret})}{1+\frac{1}{c}\frac{(\mathbf{r}_B(t_{ret})-\mathbf{r})\cdot\mathbf{v}_B(t_{ret})}{|\mathbf{r}-\mathbf{r}_B(t_{ret})|}} \tag{221}$$

$$= \frac{e}{|\mathbf{r}-\mathbf{r}_B(t_{ret})|+\frac{1}{c}(\mathbf{r}_B(t_{ret})-\mathbf{r})\cdot\mathbf{v}_B(t_{ret})} \tag{222}$$

$$= \frac{e}{|\mathbf{r}-\mathbf{r}_B(t_{ret})|-[\mathbf{r}-\mathbf{r}_B(t_{ret})]\cdot\frac{\mathbf{v}_B(t_{ret})}{c}} \tag{223}$$

Now let's look at the vector potential

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{c}\int d\vec{r}'dt'\frac{\delta(\frac{1}{c}|\mathbf{r}-\mathbf{r}'|-(t-t'))}{|\mathbf{r}-\mathbf{r}'|}\mathbf{j}(\mathbf{r}',t') \tag{224}$$

$$= \frac{e}{c}\int d\vec{r}'dt'\frac{\delta(\frac{1}{c}|\mathbf{r}-\mathbf{r}'|-(t-t'))}{|\mathbf{r}-\mathbf{r}'|}\mathbf{v}(t')\delta(\mathbf{r}'-\mathbf{r}_B(t')) \tag{225}$$

$$= \frac{e}{c}\int dt'\frac{\delta(\frac{1}{c}|\mathbf{r}-\mathbf{r}_B(t')|-(t-t'))}{|\mathbf{r}-\mathbf{r}_B(t')|}\mathbf{v}(t') \tag{226}$$

$$= \dots \tag{227}$$

$$= \frac{\mathbf{v}_B(t')}{c}\phi(\mathbf{r},t) \tag{228}$$

0.5.6 Exercise 38.1 Total radiated power

We observe

$$\frac{\lambda}{(1+\lambda\beta)^4} = -\frac{1}{\beta(1+\lambda\beta)^4} + \frac{1}{\beta(1+\lambda\beta)^3}. \tag{229}$$

Then

$$f(\lambda) = \frac{2}{(1+\lambda\beta)^3}\left(-\frac{\beta^2}{2} + \frac{\lambda\beta}{8}\frac{\beta^2-1}{1+\lambda\beta}\right) \tag{230}$$

$$= -\beta^2\frac{1}{(1+\lambda\beta)^3} + \frac{\beta(\beta^2-1)}{4}\frac{\lambda}{(1+\lambda\beta)^4} \tag{231}$$

$$= \left(-\beta^2 + \frac{\beta^2-1}{4}\right)\frac{1}{(1+\lambda\beta)^3} - \frac{(\beta^2-1)}{4}\frac{1}{(1+\lambda\beta)^4} \tag{232}$$

$$\int_{-1}^{1} f(\lambda)d\lambda = -\frac{1+3\lambda^2}{4} \tag{233}$$

0.6 Wald - Advanced Classical Electrodynamics

0.6.1 Problem 2.3 The proton and the hydrogen atom

(a) Using Gauss law with spherical symmetry inside the nucleus

$$4\pi r^2 E_r(r) = \frac{1}{\varepsilon_0}\frac{4}{3}\pi r^3\rho \tag{234}$$

$$4\pi R^2 E_r(r) = \frac{1}{\varepsilon_0}\frac{4}{3}\pi R^3\rho = \frac{Q}{\varepsilon_0} \quad \to \quad \rho = \frac{Q}{\frac{4}{3}\pi R^3} \tag{235}$$

$$E_r(r) = \frac{1}{\varepsilon_0}\frac{Qr}{4\pi R^3} \tag{236}$$

and outside

$$E_r(r) = \frac{1}{\varepsilon_0}\frac{Q}{4\pi r^2} \tag{237}$$

Then the field energy is

$$\mathscr{E} = \frac{\varepsilon_0}{2}4\pi\left[\left(\frac{Q}{4\pi\varepsilon_0 R^3}\right)^2\int_0^R r^2 r^2\,dr + \left(\frac{Q}{4\pi\varepsilon_0}\right)^2\int_0^R r^2\frac{1}{r^4}\,dr\right] \tag{238}$$

$$= \frac{3Q^2}{20\pi\varepsilon_0 R} \tag{239}$$

$$= 1.4\cdot 10^{-13}\mathrm{J} \tag{240}$$

$$= 0.87\mathrm{MeV} \tag{241}$$

while $mc^2 = 939\text{MeV}$.

(b) Interaction energy - we assume the proton to be a point charge

$$\begin{aligned}
\mathscr{E} &= \varepsilon_0 \int d^3x\, \mathbf{E}_{\text{proton}} \cdot \mathbf{E}_{1\text{s}} && (242)\\
&= \varepsilon_0 \int d^3x\, E_{\text{proton,r}} \cdot E_{1\text{s,r}} && (243)\\
&\simeq 4\pi\varepsilon_0 \int_0^\infty drr^2 \frac{1}{\varepsilon_0}\frac{Q}{4\pi r^2} \cdot \frac{-Qe^{-2r/a}}{\pi a^3} && (244)\\
&\simeq -\frac{Q^2}{\pi a^3}\int_0^\infty dre^{-2r/a} && (245)\\
&\simeq \frac{Q^2}{\pi a^3}\frac{a}{2}\left[e^{-2r/a}\right]_0^\infty && (246)\\
&\simeq -\frac{Q^2}{\pi a^3}\frac{a}{2} = -\frac{Q^2}{2\pi a^2} && (247)\\
&\simeq -9.1\text{eV} && (248)
\end{aligned}$$

with $\langle T\rangle = \frac{1}{2}\langle V\rangle$ we get to 13.6eV.

0.6.2 Problem 2.4 Potential from oddly shaped charge distribution

We need to calculate

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r')}{|\vec{r}-\vec{r}'|}d^3\vec{r}' \tag{249}$$

trying to avoid the brute force calculation we see that we can expand the charge distribution in a finite series of Legendre polynomials

$$(1-\cos\theta)^2 = \frac{4}{3}P_0(\cos\theta) - 2P_1(\cos\theta) + \frac{2}{3}P_2(\cos\theta) \tag{250}$$

where we used the orthogonality of the Legendre polynomials to project out the coefficients

$$\int_{-1}^{+1} P_n(x)P_m(x)dx = \frac{2}{2n+1}\delta_{mn} \tag{251}$$

$$\int_0^\pi P_n(\cos\theta)P_m(\cos\theta)\sin\theta\, d\theta = \frac{2}{2n+1}\delta_{mn} \tag{252}$$

with the multipole expansion (**for the outside**)

$$\begin{aligned}
\frac{1}{|\vec{r}-\vec{r}'|} &= \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\theta}} && (253)\\
&= \frac{1}{r\sqrt{1+\frac{r'^2}{r^2}-2\frac{r'}{r}\cos\theta}} && (254)\\
&= \frac{1}{r}\sum_l P_l(\cos\theta)\frac{r'^l}{r^l} && (255)
\end{aligned}$$

we can insert all into the Coulomb integral and utilize the orthogonality of the Legendre polynomials again

$$\phi(\vec{r}) = \frac{2\pi\alpha}{4\pi\varepsilon_0 r}\int_0^R dr' r'^2 (R-r')\sum_l \frac{r'^l}{r^l}\int d\theta P_l(\cos\theta)\left(\frac{4}{3}P_0(\cos\theta) - 2P_1(\cos\theta) + \frac{2}{3}P_2(\cos\theta)\right)\sin\theta \tag{256}$$

$$= \frac{\alpha}{2\varepsilon_0 r}\int_0^R dr' r'^2 \frac{r'^l}{r^l}(R-r')\left[\frac{4}{3}\frac{2}{2\cdot 0+1} - 2\frac{2}{2\cdot 1+1}\frac{r'}{r} + \frac{2}{3}\frac{2}{2\cdot 2+1}\frac{r'^2}{r^2}\right] \tag{257}$$

$$= \ldots \tag{258}$$

$$= \frac{\alpha R^4}{9r\varepsilon_0}\left(1 - \frac{3R}{10r} + \frac{R^2}{25r^2}\right) \tag{259}$$

0.6.3 Exercise 5.1

Vacuum equations

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla\times\mathbf{B} = \mu_0\mathbf{J} + \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} \tag{260}$$

$$\nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \tag{261}$$

then

$$\nabla\times(\nabla\times\mathbf{E}) = -\frac{\partial}{\partial t}\nabla\times\mathbf{B} \tag{262}$$

$$\nabla(\nabla\cdot\mathbf{E}) - \triangle^2\mathbf{E} = -\mu_0\frac{\partial\mathbf{J}}{\partial t} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} \tag{263}$$

$$\rightarrow \Box\mathbf{E} = \mu_0\frac{\partial\mathbf{J}}{\partial t} + \frac{1}{\epsilon_0}\nabla\rho \tag{264}$$

$$\nabla\times(\nabla\times\mathbf{B}) = \mu_0\nabla\times\mathbf{J} + \frac{1}{c^2}\frac{\partial}{\partial t}\nabla\times\mathbf{E} \tag{265}$$

$$\nabla(\nabla\cdot\mathbf{B}) - \triangle^2\mathbf{B} = \mu_0\nabla\times\mathbf{J} - \frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2} \tag{266}$$

$$\rightarrow \Box\mathbf{B} = -\mu_0\nabla\times\mathbf{J} \tag{267}$$

Construction a solution: Now observe that the charge continuity equation $\dot{\rho} + \nabla\cdot\mathbf{J} = 0$ can not be recovered from the two equations. So lets assume $\mathbf{J} = 0$ and $\rho(t) = q(t)\delta(\mathbf{x})$ then we set

$$\mathbf{B} = 0 \tag{268}$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q(t)}{r^2} \tag{269}$$

which violates $\nabla\times\mathbf{B}$

0.7 Smythe - Static and Dynamic Electricity

0.7.1 Exercise 1.1 Two coaxial rings and a point charge

Total charge of an axial ringlike charge distribution

$$Q = \int \rho_0(\varphi')\delta(z'-0)\delta(r'-a)d\varphi' dz' dr' \tag{270}$$

$$= 2\pi a\rho_0 \tag{271}$$

which means that the 1-dimensional charge density is $\rho_0 = Q/2\pi a$. The axial potential of a single ring is then

$$\phi(z) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho_0\delta(z'-0)\delta(r'-a)}{\sqrt{a^2+z^2}} r d\varphi' dz' dr' \tag{272}$$
$$= \frac{1}{4\pi\varepsilon_0} 2\pi a\rho_0 \frac{1}{\sqrt{a^2+z^2}} \tag{273}$$
$$= \frac{Q}{4\pi\varepsilon_0}\frac{1}{\sqrt{a^2+z^2}} \tag{274}$$

therefore we get for the energies

$$W_1 = \frac{qQ_1}{4\pi\varepsilon_0}\frac{1}{a} + \frac{qQ_2}{4\pi\varepsilon_0}\frac{1}{\sqrt{a^2+b^2}} \tag{275}$$
$$W_2 = \frac{qQ_1}{4\pi\varepsilon_0}\frac{1}{\sqrt{a^2+b^2}} + \frac{qQ_2}{4\pi\varepsilon_0}\frac{1}{a} \tag{276}$$

solving the linear system for the charges $Q_{1,2}$ we obtain

$$Q_1 = \frac{4\pi\varepsilon_0}{qb^2}\sqrt{a^2+b^2}\left(\sqrt{a^2+b^2}W_1 - aW_2\right) \tag{277}$$
$$Q_2 = \frac{4\pi\varepsilon_0}{qb^2}\sqrt{a^2+b^2}\left(-aW_1 + \sqrt{a^2+b^2}W_2\right). \tag{278}$$

0.7.2 Exercise 1.3 Flux of two point charges through circle

For the flux we have

$$N \equiv \int \vec{E}\cdot d\vec{A} \tag{279}$$
$$= \int E\cos(\vec{E},\vec{n})\, dA \tag{280}$$
$$= 2\pi\int \frac{q}{4\pi\varepsilon_0(a^2+r^2)}\frac{a}{\sqrt{a^2+r^2}} r dr - 2\pi\int \frac{Q}{4\pi\varepsilon_0(a^2+r^2)}\frac{a}{\sqrt{a^2+r^2}} r dr \tag{281}$$
$$= \frac{2\pi a}{4\pi\varepsilon_0}(q-Q)\int_0^a \frac{1}{(a^2+r^2)^{3/2}} r dr \tag{282}$$
$$= \frac{1}{4\varepsilon_0}(q-Q)\left(2-\sqrt{2}\right) \tag{283}$$

therefore

$$Q = q - \frac{4N\varepsilon_0}{2-\sqrt{2}}. \tag{284}$$

0.7.3 Exercise 1.4 Concentric charged rings

The axial potential of a single ring is with radius a and charge $Q = 2\pi a\rho_0$ is

$$\phi(x) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho_0\delta(z'-0)\delta(r'-a)}{\sqrt{a^2+x^2}} r d\varphi' dz' dr' \tag{285}$$
$$= \frac{1}{4\pi\varepsilon_0} 2\pi a\rho_0 \frac{1}{\sqrt{a^2+x^2}} \tag{286}$$
$$= \frac{Q}{4\pi\varepsilon_0}\frac{1}{\sqrt{a^2+x^2}} \tag{287}$$

The total potential and the resulting electrical field is therefore

$$\phi(x) = -\frac{Q}{4\pi\varepsilon_0}\frac{1}{\sqrt{a_1^2+x^2}} + \frac{\sqrt{27}Q}{4\pi\varepsilon_0}\frac{1}{\sqrt{a_2^2+x^2}} \tag{288}$$

$$E_x = -\frac{\partial\phi}{\partial x} \tag{289}$$

$$= \frac{Qx}{4\pi\varepsilon_0}\left(-\frac{1}{(a_2^2+x^2)^{3/2}} + \frac{\sqrt{27}}{(a_2^2+x^2)^{3/2}}\right) \tag{290}$$

which only vanishes for

$$x = 0, \pm\sqrt{\frac{-3a_1^2 + a2^2}{2}}. \tag{291}$$

Due to the radial symmetry the other field components at this points vanish too.

0.7.4 Exercise 1.19C Charged disc

$$\phi(z) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma}{\sqrt{\rho'^2+z^2}}\rho'\,d\rho'\,d\varphi' \tag{292}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{\sigma}{z}\int \frac{1}{\sqrt{1+\rho'^2/z^2}}\rho'\,d\rho'\,d\varphi' \tag{293}$$

$$= \frac{1}{2\epsilon_0}\frac{\sigma}{z}\int_0^R \frac{1}{\sqrt{1+\rho'^2/z^2}}\rho'\,d\rho' \tag{294}$$

$$= \frac{\sigma}{2\epsilon_0}z\left(\sqrt{1+R^2/z^2}-1\right) \tag{295}$$

then we calculate the field

$$E(z) = -\frac{\partial\phi}{\partial z} \tag{296}$$

$$= \frac{\sigma}{2\epsilon_0}\left(1-\frac{1}{\sqrt{1+R^2/z^2}}\right) \tag{297}$$

and obtain

$$E = \frac{\sigma}{2\epsilon_0}\left\{1-\frac{1}{\sqrt{26}}, 1-\frac{3}{\sqrt{34}}, 1-\frac{1}{\sqrt{2}}, 1-\frac{7}{\sqrt{74}}\right\} \tag{298}$$

0.7.5 Exercise 12.1 Linear quadrupole

$$\beta = \omega\sqrt{\mu\epsilon} \tag{299}$$

$$q_{zz}^{(2)} = a^2 q\sin\omega t \tag{300}$$

$$8\pi\epsilon\vec{Z}_{zz} = a^2 q\sin\omega t\left(\frac{\beta}{r}-\frac{j}{r^2}\right)(\vec{r}_1\cos\theta - \vec{\theta}\sin\theta)\cos\theta e^{-j\beta r} \tag{301}$$

$$\tag{302}$$