$$\left(\int_{-\infty}^{\infty} dx e^{-x^2}\right)^2 = \int_{-\infty}^{\infty} dx e^{-x^2} \cdot \int_{-\infty}^{\infty} dy e^{-y^2} \tag{1}$$

$$= \int_{\mathbb{R}^2} e^{-(x^2+y^2)} dx\, dy \tag{2}$$

$$= \int_0^{2\pi} \int_0^{2\pi} e^{-r^2} r dr \tag{3}$$

$$= -2\pi \left. \frac{e^{-r^2}}{2} \right|_0^{\infty} = \pi \tag{4}$$

0.1 Common integrals

$$\int_{-\infty}^{\infty} dx e^{-ax^2} = \sqrt{\frac{\pi}{a}} \qquad a > 0, a \in \mathbb{R} \tag{5}$$

$$\int_{-\infty}^{\infty} dx e^{-ax^2+bx+c} = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}+c} \qquad a > 0, a, b, c \in \mathbb{R} \tag{6}$$

$$\int_{-\infty}^{\infty} dx e^{iax^2} = \sqrt{\frac{\pi}{a}} e^{\frac{i\pi}{4}} \qquad a > 0, a \in \mathbb{R} \tag{7}$$

$$\text{modified Bessel} \quad K_0(a\beta) = \int_0^{\infty} dx \frac{\cos(ax)}{\sqrt{\beta^2 + x^2}} \qquad \text{Gradshteyn, Ryzhik 7ed (3.754)} \tag{8}$$

$$\text{modified Bessel} \quad K_1(a\beta) = \frac{1}{\beta} \int_0^{\infty} dx \frac{x \sin(ax)}{\sqrt{\beta^2 + x^2}} \tag{9}$$

0.2 Common Fourier integrals

$$\int_{-\infty}^{\infty} dy e^{-ay^2} e^{-iby} = \sqrt{\frac{\pi}{a}} e^{-\frac{b^2}{4a}} \qquad a > 0, a, b \in \mathbb{R} \tag{10}$$

$$\int_{-\infty}^{\infty} dy e^{iay^2} e^{-iby} = \sqrt{\frac{\pi}{a}} e^{\frac{i}{4}\left(\pi - \frac{b^2}{a}\right)} \qquad a > 0, a, b \in \mathbb{R} \tag{11}$$

$$\int_{-\infty}^{\infty} dy e^{-(a+ic)y^2} e^{-iby} = \sqrt{\frac{\pi}{a+ic}} e^{-\frac{b^2}{4(a+ic)}} \qquad a > 0, a, b, c \in \mathbb{R} \tag{12}$$

$$= \sqrt{\frac{\pi}{a^2+c^2}} \sqrt{a-ic}\, e^{-\frac{b^2}{4(a^2+c^2)}(a-ic)} \tag{13}$$

0.3 Residue theorem

$$\int_{\Gamma} f = 2\pi i \sum_{a \in D_{\mathrm{Singu}}} \mathrm{ind}_{\Gamma}(a)\, \mathrm{Res}_a f \tag{14}$$

Winding number $\mathrm{ind}_{\Gamma}(a)$, Residue $\mathrm{Res}_a f = c_{-1}$ from Laurent series at singularity a

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n \tag{15}$$

0.4 Common contour integrals

$$G(t-t') = -\frac{1}{2\pi}\int_{-\infty}^{+\infty} dE \frac{e^{-iE(t-t')}}{E^2-\omega^2+i\epsilon} = \frac{i}{2\omega}e^{i\omega|t|} \qquad \text{Sredniki (7.12)} \tag{16}$$

$$D(x-y) = \frac{4\pi}{(2\pi)^3}\int_0^{\infty} dp \frac{p^2 e^{i\sqrt{p^2+m^2}t}}{2\sqrt{p^2+m^2}} = \frac{1}{4\pi^2}\int_m^{\infty} dE\sqrt{E^2-m^2}e^{iEt} \qquad \text{PS (2.51)} \tag{17}$$

$$D(x-y) = \frac{-i}{2(2\pi)^2 r}\int_{-\infty}^{+\infty} dp \frac{pe^{ipr}}{\sqrt{p^2+m^2}} = \frac{1}{4\pi^2 r}\int_m^{\infty} d\rho \frac{\rho e^{-\rho r}}{\sqrt{\rho^2-m^2}} \qquad \text{PS (2.52)} \tag{18}$$

$$V(r) = \frac{1}{(2\pi)^2 ir}\int_{-\infty}^{\infty} dp \frac{pe^{ipr}}{p^2+m^2} = \frac{1}{4\pi r}e^{-mr} \qquad \text{PS (4.126)} \tag{19}$$

0.5 Feynman integral tricks

0.5.1 First example

$$\int_0^{\infty} \frac{e^{-t^2(x^2+1)}}{x^2+1}dx \tag{20}$$

0.5.2 Second example

We are trying to evaluate the integral without using contour integrals

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\log(x^4+1)}{x^2+1} &= 2\int_0^{\infty} \frac{\log(x^4+1)}{x^2+1}dx && (21)\\
&= 2\int_0^{\infty} \frac{\log[(x^2-i)(x^2+i)]}{x^2+1}dx && (22)\\
&= 2\int_0^{\infty} \frac{\log(x^2-i)}{x^2+1} + \frac{\log(x^2+i)}{x^2+1}dx && (23)\\
&= 2\left(I(-i)+I(i)\right) && (24)
\end{aligned}$$

Now the trick - come up with a parameter t inside the integral

$$\begin{aligned}
I(t) &= \int_0^{\infty} \frac{\log(x^2+t)}{x^2+1}dx && (25)\\
I(0) &= \int_0^{\infty} \frac{2\log(x)}{x^2+1}dx && (26)\\
&\overset{x=1/u}{=} \int_{\infty}^{0} \frac{2\log(1/u)}{1/u^2+1}\frac{-1}{u^2}du && (27)\\
&= (-1)^2\int_0^{\infty} \frac{-2\log(u)}{1+u^2}du && (28)\\
&= -I(0) = 0 && (29)
\end{aligned}$$

and differentiate with respect to t (without checking if are allowed to switch the integral and the differentiation)

$$\frac{dI}{dt} = \int_0^\infty \frac{1}{x^2+1}\frac{1}{x^2+t}dx \tag{30}$$
$$= \int_0^\infty \frac{1/(t-1)}{x^2+1} + \frac{1/(1-t)}{x^2+t}dx \tag{31}$$
$$= \frac{\arctan x}{t-1}\bigg|_0^\infty - \frac{\arctan \frac{x}{\sqrt{t}}}{(t-1)\sqrt{t}}\bigg|_0^\infty \tag{32}$$
$$= \frac{\pi}{2}\frac{1}{t-1}\frac{\sqrt{t}-1}{\sqrt{t}} \tag{33}$$
$$= \frac{\pi}{2}\frac{1}{\sqrt{t}(1+\sqrt{t})} \tag{34}$$

and now we can integrate

$$I(t) = \frac{\pi}{2}\int \frac{1}{\sqrt{t}(1+\sqrt{t})}dt \tag{35}$$
$$\overset{u=\sqrt{t}}{=} \frac{\pi}{2}\int \frac{1}{u(1+u)}2u\,du \qquad \frac{du}{dt} = \frac{1}{2\sqrt{t}}, \to dt = 2u\,du \tag{36}$$
$$= \pi\log(1+u) + c \qquad \text{with } I(0) = 0 \to c = 0 \tag{37}$$
$$= \pi\log(1+\sqrt{t}) \tag{38}$$

then with $i = e^{i\pi/2+2\pi k}$ and $-i = e^{-i\pi/2+2\pi n}$

$$\int_{-\infty}^\infty \frac{\log(x^4+1)}{x^2+1} = 2\left(I(-i) + I(i)\right) \tag{39}$$
$$= 2\pi\left(\log(1+\sqrt{-i}) + \log(1+\sqrt{i})\right) \tag{40}$$
$$= 2\pi\log[(1+\sqrt{-i})(1+\sqrt{i})] \tag{41}$$
$$= 2\pi\log[1+\sqrt{-i}+\sqrt{i}+\sqrt{-i^2}] \tag{42}$$
$$= 2\pi\log[2+\sqrt{-i}+\sqrt{i}] \tag{43}$$
$$= 2\pi\log[2+e^{-i\pi/4}e^{i\pi n}+e^{i\pi/4}e^{i\pi k}] \tag{44}$$
$$= 2\pi\log[2+\sqrt{2}] \qquad \text{just setting } n,k=0 \text{ to ensure a real solution} \tag{45}$$
$$= \pi\log[(2+\sqrt{2})^2] \tag{46}$$
$$= \log[(6+4\sqrt{2})^\pi] \tag{47}$$
$$\tag{48}$$

0.6 Fourier transformation

Starting from the Fourier integral theorem we have some freedom to distribute the 2π between back and forth transformation ($a, b \in \mathbb{R}$)

$$F(k) = \sqrt{\frac{|b|}{(2\pi)^{1-a}}}\int_{-\infty}^\infty f(x)e^{ibkx}dx \quad \leftrightarrow \quad f(x) = \sqrt{\frac{|b|}{(2\pi)^{1+a}}}\int_{-\infty}^\infty F(t)e^{-ibkx}dk \tag{49}$$

0.7 Laplace transformation

Origin: Power series

$$\sum_{n=0}^{\infty} a_n x^n \simeq A(x) \tag{50}$$

$$\sum_{n=0}^{\infty} a(n) x^n = A(x) \tag{51}$$

with $n \in \mathbb{N}$ and $a(n) \in \mathbb{R}$. Examples

$$a(n) = 1 \qquad \rightarrow \qquad A(x) = \frac{1}{1-x} \qquad |x| < 1 \tag{52}$$

$$a(n) = \frac{1}{n} \qquad \rightarrow \qquad A(x) = -\log(1-x) \tag{53}$$

$$a(n) = \frac{1}{n!} \qquad \rightarrow \qquad A(x) = e^x \tag{54}$$

Now extend $n \in \mathbb{N} \rightarrow t \in \mathbb{R}$

$$\int_0^{\infty} a(t) x^t dt = A(x) \tag{55}$$

$$\int_0^{\infty} a(t) e^{\log x \; t} dt = A(x) \tag{56}$$

$$\int_0^{\infty} a(t) e^{-s \; t} dt = A(e^s) \tag{57}$$

$$\int_0^{\infty} f(t) e^{-st} dt = F(s) \tag{58}$$

meaning the Laplace trafo is the continuous analog of the discrete power series.

Now

$$Y(s) = \int_0^{\infty} f(t) e^{-st} dt \tag{59}$$

then

$$\int_0^{\infty} f'(t) e^{-st} dt = f(t) e^{-st}\Big|_0^{\infty} - \int_0^{\infty} f(t)(-s) e^{-st} dt \tag{60}$$

$$= -f(0) + s \int_0^{\infty} f(t) e^{-st} dt \tag{61}$$

$$= sY(s) - f(0) \tag{62}$$

$$\int_0^{\infty} f''(t) e^{-st} dt = ... \tag{63}$$

$$= s^2 Y(s) - sf'(0) - f(0) \tag{64}$$

0.8 Delta distribution

$$x\delta(x) = 0\delta(ax) = \frac{1}{|a|}\delta(x) \tag{65}$$

$$\int \delta(x)e^{-ikx}dx = 1 \tag{66}$$
$$\int e^{ik(x-y)}dk = 2\pi\delta(x-y) \tag{67}$$

$$
\begin{aligned}
\int g(x)\delta(f(x))dx &= \sum_{x_i:f(x_i)=0} \int_{x_i-\epsilon}^{x_i+\epsilon} g(x)\delta(f(x))dx && (68)\\
&= \sum_{x_i} \int_{x_i-\epsilon}^{x_i+\epsilon} g(x)\delta\left(f(x_i)+f'(x_i)(x-x_i)+\frac{1}{2}f''(x_i)(x-x_i)^2+...\right)dx && (69)\\
&= \sum_{x_i} \int_{x_i-\epsilon}^{x_i+\epsilon} g(x)\delta\left(f'(x_i)(x-x_i)\right)dx && (70)\\
&= \sum_{x_i} \int_{(x_i-\epsilon)f'}^{(x_i+\epsilon)f'} g\left(\frac{u}{f'(x_i)}\right)\delta\left(u-f'(x_i)x_i)\right)\frac{1}{f'(x_i)}du && (71)\\
&= \sum_{x_i} \int_{(x_i-\epsilon)|f'|}^{(x_i+\epsilon)|f'|} g\left(\frac{u}{f'(x_i)}\right)\frac{1}{|f'(x_i)|}\delta\left(u-f'(x_i)x_i)\right)du && (72)\\
&= \sum_{x_i} g(x_i)\frac{1}{|f'(x_i)|} && (73)
\end{aligned}
$$

Important restriction: x_i are the **simple** zeros

0.9 Bessel functions

- Bessel ODE $x^2y'' + xy' + (x^2 - \nu^2)y = 0 \qquad \mathrm{Re}\,\nu \geq 0$

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 && (74)\\
&= \begin{cases} c_1J_\nu + c_2J_{-\nu} & \nu \notin \mathbb{Z} \\ c_1J_\nu + c_2Y_\nu & \nu = 0,1,2,... \end{cases} && (75)
\end{aligned}
$$

$$
\begin{aligned}
J_\nu(x) &= \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k} && \text{Bessel function} && (76)\\
Y_\nu(x) &= \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi} && \text{Neumann/Weber function} && (77)\\
Y_n(x) &= \lim_{\alpha\to n} Y_\alpha(x) && && (78)\\
H_\nu^{(1)}(x) &= J_\nu(x) + iY_\nu(x) && \text{Hankel function 1. kind} && (79)\\
H_\nu^{(2)}(x) &= J_\nu(x) - iY_\nu(x) && \text{Hankel function 2. kind} && (80)
\end{aligned}
$$

If $\nu = n$ then

$$
\begin{aligned}
J_n &= \frac{1}{\pi}\int_0^\pi \cos(x\sin\varphi - n\varphi)d\varphi && (81)\\
&= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{i(x\sin\varphi - n\varphi)}d\varphi && (82)
\end{aligned}
$$

- Modified Bessel ODE $x^2y'' + xy' - (x^2+\nu^2)y = 0$

$$I_\nu(x) = i^{-\nu} J_\nu(ix) \tag{83}$$
$$K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu\pi} \tag{84}$$
$$K_n(x) = \lim_{\alpha\to n} K_\alpha(x) \tag{85}$$
$$\tag{86}$$

If $\operatorname{Re} x > 0$ then

$$K_n = \int_0^\pi e^{-x\cosh t}\cosh \nu t\, dt \tag{87}$$

0.10 Γ, ζ function

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt \tag{88}$$
$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} \tag{89}$$

then with $t/n = x$ and $dx = dt/n$

$$\zeta(s)\Gamma(s) = \sum_{n=1}^\infty \int_0^\infty \frac{1}{n^s} t^{s-1}e^{-t}dt \tag{90}$$
$$= \sum_{n=1}^\infty \int_0^\infty \frac{1}{n^s} t^{s-1}e^{-t} n\, dx \tag{91}$$
$$= \sum_{n=1}^\infty \int_0^\infty \frac{t^{s-1}}{n^{s-1}} e^{-nx}dx \tag{92}$$
$$= \int_0^\infty x^{s-1}\sum_{n=1}^\infty e^{-nx}dx \tag{93}$$
$$= \int_0^\infty \frac{x^{s-1}}{e^x - 1}dx \tag{94}$$
$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x - 1}dx \tag{95}$$

0.11 n-dimensional unit spheres

$$\pi^{n/2} = \left(\int_{-\infty}^\infty dt e^{-t^2}\right)^n \tag{96}$$
$$= \int_{R^n} e^{-|x|^2}dx \tag{97}$$
$$= \int_0^\infty \int_{\omega_n} e^{-r^2} r^{n-1} dr\, ds \tag{98}$$
$$= \int_{\omega_n} ds \cdot \int_0^\infty e^{-r^2} r^{n-1} dr \tag{99}$$
$$= |\omega_n| \cdot \frac{1}{2}\int_0^\infty e^{-\rho}\rho^{\frac{n}{2}-1}d\rho \tag{100}$$
$$= |\omega_n| \cdot \frac{1}{2}\Gamma\left(\frac{n}{2}\right) \tag{101}$$

Therefore

$$|\omega_n| = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \tag{102}$$

$$V_n = |\omega_n| \int_0^1 r^{n-1} dr \tag{103}$$

$$= \frac{|\omega_n|}{n} \tag{104}$$

0.12 Vector Analysis

Identities

$$\nabla \times \nabla\phi \equiv 0 \tag{105}$$

$$\nabla \cdot \nabla \times \mathbf{A} \equiv \mathbf{0} \tag{106}$$

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \triangle\mathbf{A} \tag{107}$$

Gauss and Stokes Theorem

$$\oint_{\partial V} \mathbf{A} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{A}\, dV \tag{108}$$

$$\oint_{\partial S} \mathbf{A} \cdot d\mathbf{r} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} \tag{109}$$

Helmholtz-Hodge decomposition

$$\mathbf{E} = \mathbf{E}_{\parallel} + \mathbf{E}_{\perp} \tag{110}$$

$$\nabla \times \mathbf{E}_{\parallel} = 0 \tag{111}$$

$$\nabla \cdot \mathbf{E}_{\perp} = 0 \tag{112}$$

With

$$-\frac{1}{4\pi} \triangle \frac{1}{|\mathbf{x} - \mathbf{x}'|} = \delta(\mathbf{x} - \mathbf{x}') \tag{113}$$

$$\triangle\mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla \times \nabla \times \mathbf{E} \tag{114}$$

we can construct

$$\mathbf{E}(\mathbf{x}) = \int \mathbf{E}(\mathbf{x}')\delta(\mathbf{x} - \mathbf{x}')dx' \tag{115}$$

$$= -\frac{1}{4\pi} \int \mathbf{E}(\mathbf{x}') \triangle \frac{1}{|\mathbf{x} - \mathbf{x}'|} dx' \tag{116}$$

$$= -\frac{1}{4\pi} \triangle \int \mathbf{E}(\mathbf{x}') \frac{1}{|\mathbf{x} - \mathbf{x}'|} dx' \tag{117}$$

$$= -\frac{1}{4\pi} \nabla \int \mathbf{E}(\mathbf{x}') \nabla \cdot \frac{1}{|\mathbf{x} - \mathbf{x}'|} dx' + \frac{1}{4\pi} \nabla \times \int \mathbf{E}(\mathbf{x}') \nabla \times \frac{1}{|\mathbf{x} - \mathbf{x}'|} dx' \tag{118}$$

$$\tag{119}$$

0.13 Laplace operator

$$\nabla \cdot X = \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}X^i\right) \tag{120}$$
$$(\nabla f)^i = g^{ij}\partial_j f \tag{121}$$
$$\triangle f = \nabla \cdot \nabla f \tag{122}$$
$$= \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j f\right) \tag{123}$$
$$= \sum_i \frac{\partial^2}{\partial x_j^2} \tag{124}$$

$$\frac{\partial}{\partial x_i}\frac{\partial f}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{\partial y_j}{\partial x_i}\frac{\partial f}{\partial y_j}\right) \tag{125}$$
$$= \frac{\partial y_k}{\partial x_i}\frac{\partial}{\partial y_k}\left(\frac{\partial y_j}{\partial x_i}\frac{\partial f}{\partial y_j}\right) \tag{126}$$
$$= \frac{\partial^2 y_j}{\partial x_i^2}\frac{\partial f}{\partial y_j} + \frac{\partial y_j}{\partial x_i}\frac{\partial y_k}{\partial x_i}\frac{\partial^2 f}{\partial y_j \partial y_k} \tag{127}$$
$$\tag{128}$$

With $f = f(r)$ and $r = \sqrt{x_1^2 + ... + x_n^2}$ we have

$$\triangle f(r) = \sum_i \frac{r - x_i \frac{x_i}{r}}{r^2}\frac{\partial f}{\partial r} + \frac{x_i^2}{r^2}\frac{\partial^2 f}{\partial r^2} \tag{129}$$
$$= \frac{nr - r}{r^2}\frac{\partial f}{\partial r} + \frac{r^2}{r^2}\frac{\partial^2 f}{\partial r^2} \tag{130}$$
$$= \frac{(n-1)}{r}\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial r^2} \tag{131}$$

0.14 ODE solving strategies

0.14.1 Special ODEs

Bernoulli	$y' + p(x)y + q(x)y^n = 0$	
Ricatti	$y' + p(x)y + q(x)y^2 = r(x)$	
d'Alembert	$y = x \cdot g(y') + h(y')$	
Exact	$M(x,y) + N(x,y)y' = 0 \quad (\partial_y M = \partial_x N)$	
Airy	$y'' - xy = 0$	
Bessel	$x^2y'' + xy' + (x^2 - \nu^2)y = 0$	$\mathrm{Re}\,\nu \geq 0$
modified Bessel	$x^2y'' + xy' - (x^2 + \nu^2)y = 0$	
Hermite	$y'' - 2xy' + 2ny = 0$	$n = 0, 1, 2, 3, ...$
Laguerre	$xy'' + (1-x)y' + ny = 0$	$n = 0, 1, 2, 3, ...$
Legendre	$(1-x^2)y'' - 2xy' + n(n+1)y = 0$	$n = 0, 1, 2, 3, ...$
Weber Hermite	$y'' + \left(\nu + \frac{1}{2} - \frac{1}{4}x^2\right)y = 0$	

0.14.2 1st order ODE

1. Is separable $y' = g(x)h(y) \quad \rightarrow \quad \int \frac{dy}{h(y)} = \int g(x)dx$ - done.

2. Is linear homogen $y' + f(x)y = 0$ go to 1.

$$y(x) = Ce^{-\int f(x)dx} \tag{132}$$

3. Is linear inhomogen $y' + f(x)y = g(x)$ general solution

$$y(x) = y_{\text{hom}}(x) + y_{\text{spec}}(x) \tag{133}$$
$$y_{\text{hom}}(x) = Ce^{-\int f(x)dx} \tag{134}$$
$$y_{\text{spec}}(x) = C(x)e^{-\int f(x)dx} \tag{135}$$
$$\rightarrow C(x)'e^{-\int f(x)dx} - f(x)C(x)e^{-\int f(x)dx} + f(x)C(x)e^{-\int f(x)dx} = g(x) \tag{136}$$
$$\rightarrow C(x)' = g(x)e^{\int f(x)dx} \tag{137}$$
$$\rightarrow C(x) = \int g(x)e^{\int f(x)dx}dx + c_1 \tag{138}$$
$$y(x) = Ce^{-\int f(x)dx} + \left(\int g(x)e^{\int f(x)dx}dx + c_1\right)e^{-\int f(x)dx} \tag{139}$$

solve homogen (go to 2) then variation of constants.

4. Is linear Bernoulli $y' + f(x)y + g(x)y^n = 0$ divide by y^n and subs $z = \frac{1}{y^{n-1}}$ then go to 3.

5. Is linear Ricatti $y' + f(x)y + g(x)y^2 = r(x)$ substitude with $y = Q\frac{w'}{w}$ to linearize it

$$Q'\frac{w'}{w} + \frac{Qw''}{w} - \frac{Qw'^2}{w^2} + fQ\frac{w'}{w} + gQ^2\frac{w'^2}{w^2} = r \tag{140}$$
$$Q'\frac{w'}{w} + \frac{Qw''}{w} + (gQ-1)\,Q\frac{w'^2}{w^2} + fQ\frac{w'}{w} = r \qquad gQ - 1 = 0 \tag{141}$$
$$Q'w' + Qw'' + fQw' = rw \tag{142}$$
$$w'' + \frac{Q' + fQ}{Q}w' - \frac{r}{Q}w = 0 \quad Q = 1/g, Q' = -1/g^2 \tag{143}$$
$$w'' + \left(-\frac{1}{g} + f\right)w' - rgw = 0 \tag{144}$$

6. Is exact $M(x,y) + N(x,y)y' = 0$ with $M_y = N_y$ then solution $\Phi(x,y) = C$ because

$$0 = \frac{d\Phi(x,y)}{dx} = \frac{\partial\Phi}{\partial x} + \frac{\partial\Phi}{\partial y}\frac{dy}{dx} \tag{145}$$
$$d\Phi = M(x,y)dx + N(x,y)dy \tag{146}$$
$$\Phi(x,y) = \int_{y_0}^{y} N(x,v)dv + \int_{x_0}^{x} M(u,y)du \tag{147}$$

or easier

$$\frac{\partial\Phi}{\partial x} = M \quad \rightarrow \quad \Phi = \int Mdx + G(y) \quad \rightarrow \quad \frac{\partial\Phi}{\partial y} = N \tag{148}$$

7. If nothing works try if of form $y' = f\left(\frac{y}{x}\right)$ and subs $z = y/x$ and go to 1.

8. If still nothing works try $y = u(x) \cdot v(x)$

0.14.3 2nd order ODE

Linear homegeneous equation - we can simplify

$$y'' + a(x)y' + b(x)y = 0 \quad \to \quad y = f(x)u \tag{149}$$
$$f''u + 2f'u' + u'' + a(f'u + fu') + bfu = 0 \tag{150}$$
$$u'' + (2f' + af)u' + (f'' + af' + bf)u = 0 \quad \to \quad 2f' + af = 0 \tag{151}$$
$$u'' + q(x)u = 0 \tag{152}$$

Why is it hard to solve 2nd order ODE

$$y'' + a(x)y' + b(x)y = 0 \tag{153}$$
$$(D^2 + aD + b)y = 0 \tag{154}$$

Lets try to factorize the differential operator

$$(D + A)(D + B)y = 0 \tag{155}$$
$$(D^2 + (A + B)D + B' + AB)y = 0 \tag{156}$$

Once we know A and B we can solve

$$(D + B)y \equiv w \tag{157}$$
$$w' + Aw = 0 \quad \text{(very simple to solve for } w) \tag{158}$$
$$y' + by = w \quad \text{(simple to solve for } y) \tag{159}$$

But how to find A and B

$$A + B = a, \quad B' = -AB + b \tag{160}$$
$$\to B' = -aB + B^2 + b \tag{161}$$

which gives the Riccati equation - but linearizing it leads back to the same linear, homogeneous 2nd order equation (which we started with).

1. Is inhomogeneous equation with constant coefficients $ay'' + by' + cy = r(x)$

$$a[s^2Y - sy(0) - y'(0)] + b[sY - y(0)] + cY = \mathcal{L}(r(x)) \tag{162}$$
$$(as^2 + bs + c)Y - asy(0) - ay'(0) - by(0) = \mathcal{L}(r(x)) \tag{163}$$
$$Y = \frac{\mathcal{L}(r(x)) + (as + b)y(0) + ay'(0)}{as^2 + bs + c} \tag{164}$$

... write me ...

0.14.4 n-th order ODE

1. Linear homogen $c_n y^n + ... + c_2 y'' + c_1 y' + c_0 y = 0$ ansatz $y = e^{\alpha x}$ then solve polynom for α, for repeated root α_1 try $y = xe^{\alpha_1 x}, x^2 e^{\alpha_1 x}, ...$

0.15 Greenfunctions and ODEs

0.15.1 Harmonic Oscillator

$$\ddot{G}(t - t') + 2\gamma\dot{G}(t - t') + \omega_0^2 G(t - t') = \delta(t - t') \tag{165}$$

with $G(t-t') = (2\pi)^{-1/2}\int e^{i\omega(t-t')}y(\omega)d\omega$

$$\frac{1}{\sqrt{2\pi}}\int e^{i\omega(t-t')}\left((i\omega)^2 y + 2\gamma(i\omega)y + \omega_0 y\right)d\omega = \frac{1}{2\pi}\int e^{i\omega(t-t')}d\omega \tag{166}$$

then

$$y(\omega) = \frac{1}{\sqrt{2\pi}}\frac{1}{\omega_0^2-\omega^2+2i\gamma\omega} \tag{167}$$

$$G(t-t') = \frac{1}{2\pi}\int\frac{e^{-i\omega(t-t')}}{\omega_0^2-\omega^2+2i\gamma\omega}d\omega \tag{168}$$

and the general solution is given by

$$\ddot{x}(t) + 2\gamma\dot{x}(t) + \omega_0^2 x(t) = f(t) \quad \rightarrow \quad x(t) = \int G(t-t')f(t) \tag{169}$$

0.16 PDEs

0.16.1 Transport equation $u_t + cu_x = 0$

Imagine $x = x(t)$ so $u(x,t) = u(x(t),t)$, then formally we write

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} \tag{170}$$

$$= u_t + u_x\frac{\partial x}{\partial t} \tag{171}$$

$$u_t + cu_x = 0 \quad \Leftrightarrow \quad \frac{du}{dt} = 0, \quad \frac{\partial x}{\partial t} = c \tag{172}$$

Solving the two ODEs

$$\frac{\partial x}{\partial t} = c \quad \rightarrow \quad x = ct + x_0 \tag{173}$$

$$\frac{du}{dt} = 0 \quad \rightarrow \quad u(x = ct + x_0, t) = u(x = x_0, t) \tag{174}$$

$$\rightarrow \quad u(x,t) = u(x-ct,0) = u_0(x-ct) \tag{175}$$

0.16.2 Transport equation $u_t + cu_x = g(t)$

$$u(x,t) = u_0(x,t) + \frac{1}{c}\int_{x-ct}^{x} g\left(t + \frac{\xi - x}{c}\right)d\xi \tag{176}$$

0.16.3 Transport equation $u_t + cu_x = g(x,t)$

$$u(x,t) = u_0(x,t) + \frac{1}{c}\int_{x-ct}^{x} g\left(\xi, t + \frac{\xi - x}{c}\right)d\xi \tag{177}$$

0.17 Greenfunctions and PDEs

The Greensfunction $G(x,y)$ for a general PDE $D_x u(x) = f(x)$ is defined by

$$D_x G(x,y) = \delta(x-y). \tag{178}$$

This means that general solution of the PDE can be expressed as

$$u(x) = \int G(x,y)f(y)dy \tag{179}$$

because

$$D_x u(x) = D_x \int G(x,y)f(y)dy \tag{180}$$

$$= \int D_x G(x,y)f(y)dy \tag{181}$$

$$= \int \delta(x-y)f(y)dy \tag{182}$$

$$= f(x) \tag{183}$$

0.17.1 Poisson equation $\triangle u(x) = f(x)$

The n-dimensional Fourier transform of $\triangle_x G(x,y) = \delta(x-y)$ and integration by parts gives

$$\frac{1}{(2\pi)^{n/2}} \int d^n x \, \triangle_x G(x,y) e^{-ikx} = \frac{1}{(2\pi)^{n/2}} \underbrace{\int d^n x \, \delta(x-y) e^{-ikx}}_{=e^{-iky}} \tag{184}$$

$$\frac{1}{(2\pi)^{n/2}} \int d^n x \, G(x,y)(-ik)^2 e^{-ikx} = \frac{1}{(2\pi)^{n/2}} e^{-iky} \tag{185}$$

$$(-ik)^2 g(k) = \frac{1}{(2\pi)^{n/2}} e^{-iky} \tag{186}$$

$$\rightarrow g(k) = -\frac{1}{(2\pi)^{n/2}} \frac{1}{k^2} e^{-iky} \tag{187}$$

we can now use the Fourier transform of the Greensfunction and transform back.

- Case $n = 1$: The function has a pole at $k = 0$ and the Laurent series is given by

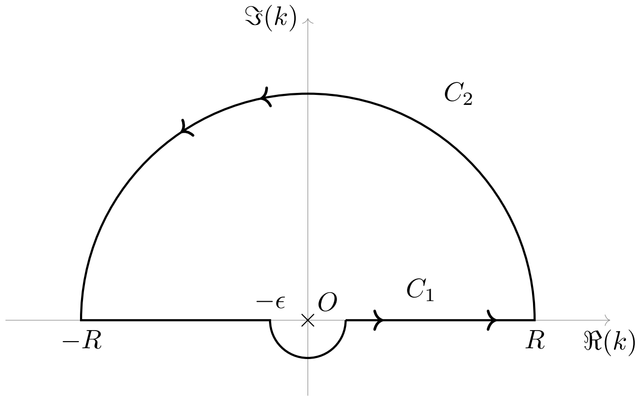
$$\frac{e^{ik(x-y)}}{k^2} = \frac{1}{k^2} + i(x-y)\frac{1}{k} - \frac{(x-y)^2}{2} - \frac{i(x-y)^3}{6} k + ... \tag{188}$$

with Res $= i(x-y)$. We can now use the residue theorem to evaluate the integral

$$G(x,y) = -\frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \, \frac{e^{ik(x-y)}}{k^2} = -\frac{1}{2\pi} \int_{C_1} dk \, \frac{e^{ik(x-y)}}{k^2} \tag{189}$$

$$= -\frac{1}{2\pi} \left(\underbrace{\int_C dk \, \frac{e^{ik(x-y)}}{k^2}}_{=2\pi i \text{ Res}} - \underbrace{\int_{C_2} dk \, \frac{e^{ik(x-y)}}{k^2}}_{=0} \right) \tag{190}$$

$$= (x-y) \tag{191}$$



- Case $n = 2$:

$$G(x,y) = -\frac{1}{2\pi} \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_1 dk_2 \, \frac{e^{i(k_1(x_1-y1)+k_2(x_2-y2))}}{k_1^2 + k_2^2} \tag{192}$$

$$= -\frac{1}{4\pi^2} \int_0^{\infty} \int_0^{2\pi} dk \, d\phi \, \frac{e^{ik|x-y|\cos\phi}}{k^2} k \tag{193}$$

$$= \frac{1}{2\pi} \int_0^{\infty} dk \frac{1}{k} \frac{1}{2\pi} \int_0^{2\pi} d\phi \, e^{ik|x-y|\cos\phi} \tag{194}$$

$$= \frac{1}{2\pi} \int_0^{\infty} dk \, \frac{J_0(k|x-y|)}{k} = -\frac{1}{2\pi} \int_0^{\infty|x-y|} dk' \, \frac{J_0(k')}{k'} \tag{195}$$

The last integral diverges but we try a nasty trick (?!?)

$$\frac{dG}{dx} = -\frac{1}{2\pi}\frac{d}{dx}\int_0^\infty dk\,\frac{J_0(k|x-y|)}{k} \tag{196}$$
$$= -\frac{1}{2\pi}\int_0^\infty dk\,J_1(k|x-y|) \tag{197}$$
$$= -\frac{1}{2\pi}\frac{1}{|x-y|} \tag{198}$$

Now simple integration yields

$$G(x,y) = -\frac{1}{2\pi}\log(|x-y|) \tag{199}$$

- Case $n = 3$:

$$G(x,y) = \frac{1}{(2\pi)^3}\int d^3k\,\frac{1}{k^2}e^{ik(x-y)} \tag{200}$$
$$= \frac{1}{(2\pi)^3}\int dk\,\underbrace{\int d\phi}_{=|\omega_2|}\int d\theta\,e^{ik|x-y|\cos\theta}\sin\theta \tag{201}$$
$$= -\frac{1}{(2\pi)^2}\int dk\int_{-1}^{+1} e^{ik|x-y|\cos\theta}d\cos\theta \tag{202}$$
$$= -\frac{1}{(2\pi)^2}\int dk\frac{e^{ik|x|}-e^{-ik|x-y|}}{ik|x-y|} \tag{203}$$
$$= -\frac{1}{2\pi^2}\int_0^\infty dk\frac{\sin k|x-y|}{k|x-y|} \tag{204}$$
$$= -\frac{1}{2\pi^2}\frac{1}{|x-y|}\int_0^\infty dk'\frac{\sin k'}{k'} \tag{205}$$
$$= -\frac{1}{4\pi}\frac{1}{|x-y|} \tag{206}$$

- Case $n > 3$: ...

Alternatively we can use the Gauss theorem with $\vec{F} = \nabla_x G(x,y)$

$$\int_V \nabla\cdot\vec{F}dx = \int_{\partial V}\vec{F}\cdot d\vec{S} \tag{207}$$
$$\int_{K_r(y)} \triangle_x G(x,y)dx = \int_{\partial_{K_r(y)}} \nabla G\cdot d\vec{S} \tag{208}$$
$$1 = \frac{\partial G(r,0)}{\partial r}|\omega_n|r^{n-1} \tag{209}$$
$$\frac{\partial G(r,0)}{\partial r} = \frac{r^{-n+1}}{|\omega_n|} \tag{210}$$
$$G(x,y) = \begin{cases} \frac{1}{|\omega_2|}\log|x-y| & n=2 \\ -\frac{1}{|\omega_n|(n-2)}\frac{1}{|x-y|^{n-2}} & n\geq 3 \end{cases} \tag{211}$$

0.17.2 Wave equation $\left(\frac{1}{c^2}\partial_{tt} - \triangle\right) u(x,t) = j(x,t)$

- The free fundamental solution (no source with $j(x,t) = 0$)

$$u(\vec{x},t) = e^{-i(k_0 t - \vec{k}\vec{x})} \tag{212}$$
$$\to \quad -\frac{k_0^2}{c^2} + \vec{k}^2 + \mu^2 = 0 \tag{213}$$
$$\to \quad k_0 = \pm c\sqrt{\vec{k}^2} \tag{214}$$

- The free solution (no source with $f(x,t) = 0$) with initial conditions

$$u(\vec{x},0) = u_0(\vec{x}), \qquad \left.\frac{\partial u}{\partial x}\right|_{t=0} = u_1(\vec{x}) \tag{215}$$

Then we find by applying the differential operator to the Fourier transformation

$$u(\vec{x},t) = \frac{1}{(2\pi)^4}\int d^3k \int d\omega\, \tilde{u}(\vec{k},\omega) e^{i(\vec{k}\vec{x}-\omega t)} \tag{216}$$
$$\left(\frac{1}{c^2}\partial_{tt} - \triangle\right) u(\vec{x},t) = \frac{1}{(2\pi)^4}\int d^3k \int d\omega\, \tilde{u}(\vec{k},\omega) \left(\frac{1}{c^2}\partial_{tt} - \triangle\right) e^{i(\vec{k}\vec{x}-\omega t)} = 0 \tag{217}$$
$$\left(\frac{\omega^2}{c^2} - \vec{k}^2\right)\tilde{u}(\vec{k},\omega) = 0 \qquad \to \omega = \pm ck \tag{218}$$

This leads to the ansatz which we can transform back

$$\tilde{u}(\vec{k},\omega) = a_+(\vec{k})\delta(\omega + ck) + a_-(\vec{k})\delta(\omega - ck) \tag{219}$$
$$u(\vec{x},t) = \frac{1}{(2\pi)^4}\int d^3k \int d\omega\, \Big(a_+(\vec{k})\delta(\omega + ck) + a_-(\vec{k})\delta(\omega - ck)\Big) e^{i(\vec{k}\vec{r}-\omega t)} \tag{220}$$
$$= \frac{1}{(2\pi)^4}\int d^3k\, \Big(a_+(\vec{k})e^{i(\vec{k}\vec{x}+ckt)} + a_-(\vec{k})e^{i(\vec{k}\vec{x}-ckt)}\Big) \tag{221}$$

Obeying the initial conditions

$$u_0(\vec{x}) = \frac{1}{(2\pi)^4}\int d^3k\, e^{i\vec{k}\vec{x}} \Big(a_+(\vec{k}) + a_-(\vec{k})\Big) \tag{222}$$
$$u_1(\vec{x}) = \frac{i}{(2\pi)^4}\int d^3k\, cke^{i\vec{k}\vec{x}} \Big(a_+(\vec{k}) - a_-(\vec{k})\Big) \tag{223}$$

then leads to expressions for $a_\pm$

$$\int d\vec{x} e^{-i\vec{q}\vec{x}} u_0(\vec{x}) = \frac{1}{(2\pi)^4}\int d^3k \int d\vec{x}\, e^{i(\vec{k}-\vec{q})\vec{x}} \Big(a_+(\vec{k}) + a_-(\vec{k})\Big) \tag{224}$$
$$= \frac{1}{2\pi}\int d^3k \delta(\vec{k}-\vec{q}) \Big(a_+(\vec{k}) + a_-(\vec{k})\Big) \tag{225}$$
$$= \frac{1}{2\pi}\left(a_+(\vec{q}) + a_-(\vec{q})\right) \tag{226}$$
$$\int d\vec{x} e^{-i\vec{q}\vec{x}} u_1(\vec{x}) = \frac{icq}{2\pi}\left(a_+(\vec{q}) - a_-(\vec{q})\right) \tag{227}$$
$$\to a_\pm(\vec{q}) = \pi\int d\vec{x} e^{-i\vec{q}\vec{x}} \left(u_0(\vec{x}) \mp \frac{i}{cq} u_1(\vec{x})\right) \tag{228}$$

Inserting $a_\pm$ into the original Fourier transform (and renaming the integration variable x by y)

$$u(\vec{x},t) = \frac{1}{2(2\pi)^3}\int d^3y \int d^3k e^{i\vec{k}(\vec{x}-\vec{y})}\left[\left(u_0(\vec{y}) - \frac{i}{ck}u_1(\vec{y})\right)e^{ickt} + \left(u_0(\vec{y}) + \frac{i}{cq}u_1(\vec{y})\right)e^{-ickt}\right] \tag{229}$$

$$= \frac{1}{2(2\pi)^3}\int d^3y \int d^3k e^{i\vec{k}(\vec{x}-\vec{y})}\left[\left(e^{ickt} + e^{-ickt}\right)u_0(\vec{y}) - \frac{i}{ck}\left(e^{ickt} - e^{-ickt}\right)u_1(\vec{y})\right] \tag{230}$$

$$= \int d^3y \left[\partial_t D(\vec{x}-\vec{y},t)u_0(\vec{y}) + D(\vec{x}-\vec{y},t)u_1(\vec{y})\right] \tag{231}$$

with

$$D(\vec{z},t) = -\frac{i}{2(2\pi)^3}\int d^3k \frac{e^{i\vec{k}\vec{z}}}{ck}(e^{ickt} - e^{-ickt}) \tag{232}$$

The above calculation is basically valid in any dimension so we will get explicit expressions for $n = 1,2,3$

1. $D(\vec{z},t)$ can be simplified (in one dimensions)

$$D(z,t) = -\frac{i}{2(2\pi)}\int dk \frac{e^{ikz}}{ck}(e^{ickt} - e^{-ickt}) \tag{233}$$

$$= -\frac{i}{4\pi c}\int_{-\infty}^{\infty} dk \frac{1}{k}\left(e^{-ik(-z-ct)} - e^{-ik(-z+ct)}\right) \tag{234}$$

$$= -\frac{i}{4\pi c}\left[-i\pi \mathrm{sgn}(-z-ct) + i\pi \mathrm{sgn}(-z+ct)\right] \tag{235}$$

$$= \frac{1}{4c}\left[\mathrm{sgn}(z+ct) + \mathrm{sgn}(-z+ct)\right] \tag{236}$$

$$= \frac{1}{4c}\left[\mathrm{sgn}(z+ct) - \mathrm{sgn}(z-ct)\right] \tag{237}$$

$$= \begin{cases} +\frac{1}{2c} & |z| < ct, \quad t > 0 \\ 0 & |z| > ct, \\ -\frac{1}{2c} & |z| < ct, \quad t < 0 \end{cases} \tag{238}$$

Which vanishes outside the light cone but NOT inside. The explicit solution (for $t > 0$) is then given as

$$\partial_t D(x-\xi,t) = \frac{1}{4c}\left[2\delta(x-\xi+ct)c + 2\delta(-(x-\xi)+ct)c\right] \tag{239}$$

$$u(x,t) = \int d\xi \frac{1}{4c}\left[2\delta(x-\xi+ct)c + 2\delta(-(x-\xi)+ct)c\right]u_0(\xi) \tag{240}$$

$$+ \frac{1}{4c}\int_{-\infty}^{+\infty}\left[\mathrm{sgn}((x-\xi)+ct) + \mathrm{sgn}(-(x-\xi)+ct)\right]u_1(\xi)d\xi \tag{241}$$

$$= \frac{1}{2}\left[u_0(x+ct) + u_0(x-ct)\right] + \frac{1}{2c}\int_{K(x)_{ct}} u_1(\xi)d\xi \tag{242}$$

where $K(x)_{ct}$ is a 1-dimensional sphere of radius ct around x - meaning the interval $[x-ct, x+ct]$.

2. $D(\vec{z},t)$ can be simplified (in two dimensions)

$$D(\vec{z},t) = -\frac{i}{2(2\pi)^2}\int d^2k \frac{e^{i\vec{k}\vec{z}}}{ck}(e^{ickt}-e^{-ickt}) \tag{243}$$
$$= -\frac{i}{2(2\pi)^2}\int_0^\infty dk\, k\frac{1}{ck}(e^{ickt}-e^{-ickt})\int d\phi\, e^{ikz\cos\phi} \tag{244}$$
$$= -\frac{i}{2(2\pi)^2 c}\int_0^\infty dk\, (e^{ickt}-e^{-ickt})\cdot 2\pi J_0(kz) \tag{245}$$
$$= -\frac{i(2\pi)2i}{2(2\pi)^2 c}\int_0^\infty dk\ \cdot J_0(kz)\sin(ctk) \tag{246}$$
$$= \frac{1}{2\pi c}\begin{cases} 0 & 0<ct<z \\ \frac{1}{\sqrt{c^2t^2-z^2}} & 0<z<ct \end{cases} \tag{247}$$

where we used 6.671-7 of Gradshteyn, Ryzhik - Table of integrals, series and products 7ed. The explicit solution is then given by

$$u(\vec{x},t) = \frac{1}{2\pi c}\partial_t\left(\int_{K(\vec{x})_{ct}} d^2\xi \frac{u_0(\vec{\xi})}{\sqrt{c^2t^2-|\vec{x}-\vec{\xi}|^2}}\right) + \frac{1}{2\pi c}\int_{K(\vec{x})_{ct}} d^2\xi \frac{u_1(\vec{\xi})}{\sqrt{c^2t^2-|\vec{x}-\vec{\xi}|^2}} \tag{248}$$

where $K(\vec{x})_{ct}$ is a disc of radius ct at $\vec{x}$.

3. $D(\vec{z},t)$ can be simplified (in three dimensions)

$$D(\vec{z},t) = -\frac{i}{2(2\pi)^3}\int d^3k \frac{e^{i\vec{k}\vec{z}}}{ck}(e^{ickt}-e^{-ickt}) \tag{249}$$
$$= -\frac{i}{2(2\pi)^3}\int_0^\infty dk\, k^2\int d\phi\int d\theta\sin\theta\frac{e^{ikz\cos\theta}}{ck}2i\sin(ckt) \tag{250}$$
$$= -\frac{(2\pi)(2i)i}{2(2\pi)^3 c}\int_0^\infty dk\, k\sin(ckt)\int d\theta\sin\theta e^{ikz\cos\theta} \tag{251}$$
$$= \frac{1}{(2\pi)^2 c}\int_0^\infty dk\, k\sin(ckt)\frac{i}{kz}e^{ikz\cos\theta}|_0^\pi \tag{252}$$
$$= \frac{i}{(2\pi)^2 cz}\int_0^\infty dk\sin(ckt)\left(e^{-ikz}-e^{ikz}\right) \tag{253}$$
$$= \frac{i}{(2\pi)^2 cz}\int_0^\infty dk\frac{i}{2}(e^{-ickt}-e^{ickt})\left(e^{-ikz}-e^{ikz}\right) \tag{254}$$
$$= \frac{-1}{2(2\pi)^2 cz}\int_0^\infty dk(e^{-ik(ct+z)}-e^{ik(-ct+z)}-e^{-ik(-ct+z)}+e^{ik(ct+z)}) \tag{255}$$
$$= \frac{-1}{2(2\pi)^2 cz}\int_{-\infty}^\infty dk(-e^{ik(-ct+z)}+e^{ik(ct+z)}) \tag{256}$$
$$= \frac{-1}{4\pi cz}\left[\delta(z+ct)-\delta(z-ct)\right] \tag{257}$$

as $z,c>0$ we have

$$D(\vec{z},t) = \frac{1}{4\pi zc}\begin{cases} -\delta(|\vec{z}|+ct) & (t<0) \\ 0 & (t=0) \\ +\delta(|\vec{z}|-ct) & (t>0) \end{cases} \tag{258}$$

Vanishes outside and inside the light cone but NOT on the light cone. The explicit solution is then given as

$$
\begin{aligned}
u(\vec{x},t) &= \int d^3\xi \left[\partial_t D(\vec{x}-\vec{\xi},t)u_0(\vec{\xi}) + D(\vec{x}-\vec{\xi},t)u_1(\vec{\xi})\right] && (259)\\
&= \frac{1}{4\pi c}\partial_t \int d^3\xi \, \frac{\delta(|\vec{x}-\vec{\xi}|-ct)}{|\vec{x}-\vec{\xi}|} u_0(\vec{\xi}) + \frac{1}{4\pi c}\int d^3\xi \, \frac{\delta(|\vec{x}-\vec{\xi}|-ct)}{|\vec{x}-\vec{\xi}|} u_1(\vec{\xi}) && (260)\\
&= \frac{1}{4\pi c}\partial_t \int d^3\xi' \, \frac{\delta(|\vec{\xi}'|-ct)}{|\vec{\xi}'|} u_0(\vec{x}-\vec{\xi}') + \frac{1}{4\pi c}\int d^3\xi' \, \frac{\delta(|\vec{\xi}'|-ct)}{|\vec{\xi}'|} u_1(\vec{x}-\vec{\xi}') && (261)\\
&= \frac{1}{4\pi c}\partial_t \int d\Omega_{\vec{\chi}} \int d\xi' \, \xi'^2 \frac{\delta(|\vec{\xi}'|-ct)}{|\vec{\xi}'|} u_0(\vec{x}-\vec{\xi}') + \frac{1}{4\pi c}\int d\Omega_{\vec{\chi}} \int d\xi' \, \xi'^2 \frac{\delta(|\vec{\xi}'|-ct)}{|\vec{\xi}'|} u_1(\vec{x}-\vec{\xi}') && (262)\\
&= \frac{1}{4\pi}\partial_t \left(t\int d\Omega_{\vec{\chi}} \, u_0(\vec{x}-ct\vec{\chi})\right) + \frac{t}{4\pi}\int d\Omega_{\vec{\chi}} \, u_1(\vec{x}-ct\vec{\chi}) && (263)\\
&= ... && (264)\\
&= \frac{t}{4\pi(ct)^2}\partial_t \left(t\int_{\partial K(\vec{x})_{ct}} u_0(\xi)dA_\xi\right) + \frac{t}{4\pi(ct)^2}\int_{\partial K(\vec{x})_{ct}} u_1(\xi)dA_\xi && (265)
\end{aligned}
$$

- Sourced solution

  Knowing the Greens function defined by

$$
\left(\frac{1}{c^2}\partial_{tt} - \triangle\right) G(\vec{x}-\vec{x}',t-t') = \delta(\vec{x}-\vec{x}')\delta(t-t') \qquad (266)
$$

  allows us to write the solutions of $\left(\frac{1}{c^2}\partial_{tt} - \triangle\right) u(x,t) = j(x,t)$ as

$$
u(\vec{x},t) = \int d^n x' \, dt' \, G(\vec{x}-\vec{x}',t-t') j(\vec{x}',t') \qquad (267)
$$

  because

$$
\begin{aligned}
\left(\frac{1}{c^2}\partial_{tt} - \triangle\right) u(\vec{x},t) &= \int d^n x' \, dt' \, \left(\frac{1}{c^2}\partial_{tt} - \triangle\right) G(\vec{x}-\vec{x}',t-t') j(\vec{x}',t') && (268)\\
&= \int d^n x' \, dt' \, \delta(\vec{x}-\vec{x}')\delta(t-t') j(\vec{x}',t') && (269)\\
&= j(\vec{x},t) && (270)
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{1}{c^2}\partial_{tt} - \triangle\right) G(\vec{x}-\vec{x}',t-t') &= \delta(\vec{x}-\vec{x}')\delta(t-t') && (271)\\
G(\vec{x}-\vec{x}',t-t') &= G(\vec{r},\tau) && (272)\\
&= \frac{1}{(2\pi)^{n+1}}\int d^n k d\omega \, \tilde{G}(\vec{k},\omega) e^{i(\vec{k}\vec{r}-\omega\tau)} && (273)
\end{aligned}
$$

  then

$$
\begin{aligned}
\left(-\frac{\omega^2}{c^2} + k^2\right)\tilde{G}(k,\omega) &= 1 && (274)\\
\tilde{G}(k,\omega) &= \frac{c^2}{-\omega^2 + c^2k^2} = \frac{c}{2k}\left(\frac{1}{\omega+ck} - \frac{1}{\omega-ck}\right) && (275)
\end{aligned}
$$

and therefore

$$G(\vec{r},\tau) = \frac{1}{(2\pi)^{n+1}} \int d^n k d\omega\, \tilde{G}(\vec{k},\omega) e^{i(\vec{k}\vec{r}-\omega\tau)} \tag{276}$$
$$= \frac{1}{(2\pi)^{n+1}} \int d^n k\, e^{i\vec{k}\vec{r}} \int d\omega\, \frac{c^2}{-\omega^2 + c^2 k^2} e^{-i\omega\tau} \tag{277}$$
$$= \frac{c}{2(2\pi)^{n+1}} \int d^n k\, \frac{1}{k} e^{i\vec{k}\vec{r}} \int d\omega \left(\frac{1}{\omega + ck} - \frac{1}{\omega - ck} \right) e^{i\omega\tau} \tag{278}$$

Now we need to transform back - but the result depends on the number of space dimensions

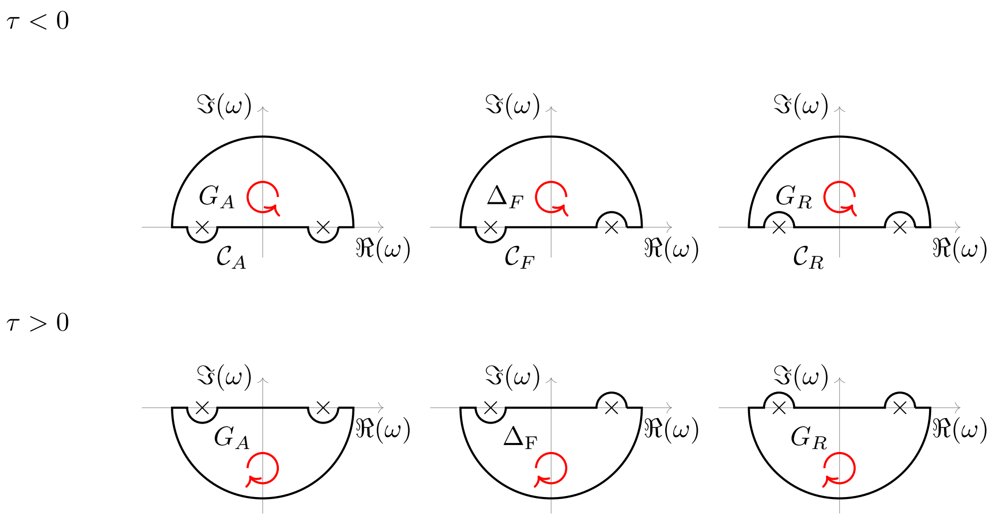


Figure 1: Possible contours for the Fourier back transformation of the Greens functions of the wave equation caused by the poles $\pm ck$

1. Case $n = 1$

$$G(r,\tau) = \frac{1}{(2\pi)^2} \int dk d\omega\, \frac{c^2}{-\omega^2 + c^2 k^2} e^{i(kr-\omega\tau)} \tag{279}$$
$$= \frac{1}{(2\pi)^2} \int dk\, e^{ikr} \int d\omega\, \frac{c^2}{-\omega^2 + c^2 k^2} e^{-i\omega\tau} \tag{280}$$
$$= \frac{c}{2(2\pi)^2} \int dk\, \frac{e^{ikr}}{k} \int d\omega \left(\frac{1}{\omega + ck} - \frac{1}{\omega - ck} \right) e^{-i\omega\tau} \tag{281}$$
$$\tag{282}$$

Now we can evaluate using the residue theorem (additional factor -1 if contour closes in mathematical negative direction)

$$
\begin{aligned}
G_R(r,\tau>0) &= \frac{c}{2(2\pi)^2}\int dk\,\frac{e^{ikr}}{k}(-1)2\pi i\left(e^{ick\tau}-e^{-ick\tau}\right) && (283)\\
&= -\frac{ic}{4\pi}\int_{-\infty}^{\infty} dk\,\frac{1}{k}\left(e^{ik(r+c\tau)}-e^{ik(r-c\tau)}\right) && (284)\\
&= -\frac{ic}{4\pi}\left(i\sqrt{\frac{\pi}{2}}\mathrm{sign}(r+c\tau)-i\sqrt{\frac{\pi}{2}}\mathrm{sign}(r-c\tau)\right) && (285)\\
&= \frac{c}{4\sqrt{2\pi}}\left(\mathrm{sgn}(r+c\tau)-\mathrm{sgn}(r-c\tau)\right) && (286)\\
&= \begin{cases} +\frac{c}{4\sqrt{2\pi}} & |z|<c\tau,\quad \tau>0\\ 0 & |z|>ct,\\ -\frac{c}{4\sqrt{2\pi}} & |z|<c\tau,\quad \tau<0\end{cases} && (287)\\
G_R(r,\tau<0) &= 0 && (288)\\
G_A(r,\tau>0) &= 0 && (289)\\
G_A(r,\tau<0) &= ... && (290)
\end{aligned}
$$

2. Case $n=2$???

3. Case $n=3$

$$
\begin{aligned}
G(\vec r,\tau) &= \frac{1}{(2\pi)^4}\int d^3k\; e^{i\vec k\vec r}\int d\omega\;\frac{c^2}{-\omega^2+c^2k^2}e^{-i\omega\tau} && (291)\\
&= \frac{2\pi}{(2\pi)^4}\int dk\,k^2\,d\theta\,\sin\theta\; e^{ikr\cos\theta}\int d\omega\;\frac{c^2}{-\omega^2+c^2k^2}e^{-i\omega\tau} && (292)\\
&= \frac{2\pi}{(2\pi)^4 ir}\int dk\,k\,(e^{ikr}-e^{-ikr})\int d\omega\;\frac{c^2}{-\omega^2+c^2k^2}e^{-i\omega\tau} && (293)
\end{aligned}
$$

The poles at $\omega=\pm ck$ make the value of the integral not unique. Using the residue theorem we can evaluate the integral but the value will depend on the chosen contour - which means the Greens function is NOT unique!

Applying the wave operator to the solution we obtain

$$
\begin{aligned}
G(\vec r,\tau) &= \frac{1}{(2\pi)^4}\int d^3k\;\int d\omega\;\frac{c^2}{-\omega^2+c^2k^2}e^{i(\vec k\vec r-\omega\tau)} && (294)\\
\Box G(\vec r,\tau) &= \frac{1}{(2\pi)^4}\int d^3k\;\int d\omega\; e^{i(\vec k\vec r-\omega\tau)} = \delta(\vec r)\delta(\tau) && (295)
\end{aligned}
$$

where we now can integrate along any ω-contour (even along the ω axis) as the poles are gone. This means the all contours give (potentially different) but valid Greens functions. The physical interpretation is that the different Greens function depend on the boundary conditions.

Now we can evaluate using the residue theorem (additional factor -1 if contour closes in mathematical negative direction)

$$G_R(\vec{r},\tau>0) = \frac{c^2}{(2\pi)^3 ir}\int_0^\infty dk\,k\,(e^{ikr}-e^{-ikr})\int d\omega\,\frac{1}{-\omega^2+c^2k^2}e^{-i\omega\tau} \quad (296)$$
$$= \frac{c}{2(2\pi)^3 ir}\int_0^\infty dk\,(e^{ikr}-e^{-ikr})\int d\omega\left(\frac{1}{\omega+ck}-\frac{1}{\omega-ck}\right)e^{-i\omega\tau} \quad (297)$$
$$= \frac{c}{2(2\pi)^3 ir}\int_0^\infty dk\,(e^{ikr}-e^{-ikr})(-1)2\pi i\left(e^{ick\tau}-e^{-ick\tau}\right) \quad (298)$$
$$= \frac{c}{2(2\pi)^2 r}\int_0^\infty dk\,(e^{ikr}-e^{-ikr})\left(e^{-ick\tau}-e^{ick\tau}\right) \quad (299)$$
$$= \frac{c}{2(2\pi)^2 r}\int_{-\infty}^\infty dk\,(e^{ik(r-c\tau)}-e^{-ik(r+c\tau)}) \quad (300)$$
$$= \frac{c}{4\pi r}\left(\delta(r-c\tau)-\delta(r+c\tau)\right) \quad (301)$$
$$= \frac{c}{4\pi r}\delta(r-c\tau) \qquad (r,\tau>0) \quad (302)$$
$$G_R(\vec{x}-\vec{x}',t-t'>0) = \frac{c}{4\pi}\frac{\delta(|\vec{x}-\vec{x}'|-c(t-t'))}{|\vec{x}-\vec{x}'|} \quad (303)$$
$$G_R(\vec{x}-\vec{x}',t-t'<0) = 0 \quad (304)$$
$$G_A(\vec{x}-\vec{x}',t-t'>0) = 0 \quad (305)$$
$$G_A(\vec{x}-\vec{x}',t-t'<0) = ... \quad (306)$$

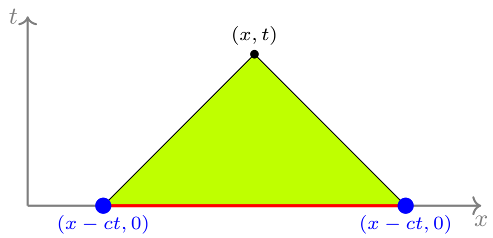
- Summary

$$u(x,t)_R^{1\mathrm{D}} = \frac{1}{2}\left[u_0(x+ct)+u_0(x-ct)\right]+\frac{1}{2c}\int_{K^{1\mathrm{D}}(x)_{ct}}u_1(\xi)d\xi+\frac{c}{2\sqrt{2\pi}}\int_{\mathrm{pLC}}j(\xi,\tau)\,d\xi d\tau \quad (307)$$

$$u(\vec{x},t)_R^{2\mathrm{D}} = \frac{1}{2\pi c}\partial_t\left(\int_{K^{2\mathrm{D}}(\vec{x})_{ct}}d^2\xi\frac{u_0(\vec{\xi})}{\sqrt{c^2t^2-|\vec{x}-\vec{\xi}|^2}}\right)+\frac{1}{2\pi c}\int_{K^{2\mathrm{D}}(\vec{x})_{ct}}d^2\xi\frac{u_1(\vec{\xi})}{\sqrt{c^2t^2-|\vec{x}-\vec{\xi}|^2}}+??? \quad (308)$$

$$u(\vec{x},t)_R^{3\mathrm{D}} = \frac{t}{4\pi(ct)^2}\partial_t\left(t\int_{\partial K^{3\mathrm{D}}(\vec{x})_{ct}}u_0(\xi)dA_\xi\right)+\frac{t}{4\pi(ct)^2}\int_{\partial K^{3\mathrm{D}}(\vec{x})_{ct}}u_1(\xi)dA_\xi+\frac{c}{4\pi}\int_{\partial\mathrm{pLC}}\frac{j(\vec{\xi},\tau)}{|\vec{r}-\vec{\xi}|}d^3\xi d\tau \quad (309)$$

$1d$



0.17.3 Klein-Gordon equation $\left(\frac{1}{c^2}\partial_{tt} - \triangle + \mu^2\right) u(x,t) = j(x,t)$

- The free fundamental solution (no source with $j(x,t) = 0$)

$$u(\vec{x},t) = e^{-i(k_0 t - \vec{k}\vec{x})} \tag{310}$$

$$\rightarrow \quad \frac{(-ik_0)^2}{c^2} - (i\vec{k})^2 + \mu^2 = 0 \tag{311}$$

$$\rightarrow \quad -\frac{k_0^2}{c^2} + \vec{k}^2 + \mu^2 = 0 \tag{312}$$

$$\rightarrow \quad k_0 = \omega = \pm c\sqrt{\vec{k}^2 + \mu^2} \tag{313}$$

- Free solutions of

$$\left(\frac{1}{c^2}\partial_{tt} - \triangle + \mu^2\right) u(\vec{x},t) = 0 \tag{314}$$

$$\mathit{\Delta}(x) = -\int_C \frac{d^4k}{(2\pi)^4}\frac{e^{-ikx}}{k^2 - \mu^2} = \frac{1}{2i}\int \frac{d^3k}{(2\pi)^3}\frac{e^{-ikx} - e^{ikx}}{\sqrt{\vec{k}^2 + \mu^2}} \tag{315}$$

$$\mathit{\Delta}^{\pm}(x) = -\int_{C^{\pm}} \frac{d^4k}{(2\pi)^4}\frac{e^{-ikx}}{k^2 - \mu^2} = \mp\frac{i}{2}\int \frac{d^3k}{(2\pi)^3}\frac{e^{\mp ikx}}{\sqrt{\vec{k}^2 + \mu^2}} \tag{316}$$

- Sourced solution
  Using

$$\delta(\vec{x}) = \frac{1}{2\pi}\int dk e^{i\vec{k}\vec{x}} \tag{317}$$

$$G(\vec{x},t) = \frac{1}{(2\pi)^d}\int d\vec{k}\, d\omega\, g(\vec{k},\omega) e^{i\vec{k}\vec{x}} e^{-i\omega t} \tag{318}$$

  note the sign change of the frequency/time transform.

  1. Case $n = 1$
     To find the Green function perform a 2d Fourier transform

$$\left(\frac{1}{c^2}\partial_{tt} - \triangle + \mu^2\right) G(x - x_0, t - t_0) = \delta(x - x_0)\delta(t - t_0) \tag{319}$$

$$\left(-\frac{\omega^2}{c^2} + k^2 + \mu^2\right)\tilde{G}(k,\omega) = 1 \tag{320}$$

$$\tilde{G}(k,\omega) = \frac{c^2}{-\omega^2 + c^2(k^2 + \mu^2)} \tag{321}$$

$$= \frac{c}{2\sqrt{k^2 + \mu^2}}\left(\frac{1}{\omega + c\sqrt{k^2 + \mu^2}} - \frac{1}{\omega - c\sqrt{k^2 + \mu^2}}\right) \tag{322}$$

     First Fourier back transformation of ω to t

$$w(k, t - t_0) = \frac{1}{2\pi}\int d\omega\, v(\vec{k},\omega) e^{-i\omega(t - t_0)} \tag{323}$$

$$= \frac{c}{(2\pi)2\sqrt{k^2 + \mu^2}}\int_{-\infty}^{+\infty} d\omega\, e^{-i\omega(t - t_0)}\left(\frac{1}{\omega + c\sqrt{k^2 + \mu^2}} - \frac{1}{\omega - c\sqrt{k^2 + \mu^2}}\right) \tag{324}$$

Name	Symbol	Contour
Feynman propagator	Δ_F	$\mathcal{C}_F$
Dyson propagator	Δ_D	$\mathcal{C}_D$
Retarded propagator	Δ_R	$\mathcal{C}_R$
Advanced propagator	Δ_A	$\mathcal{C}_A$
Principle-part propagator	$\bar{\Delta}$	$\bar{\mathcal{C}}$

Table 1: Greens functions

we recognize the two poles at $\pm c\sqrt{k^2+\mu^2}$ on the real axis. Using the residue theorem we can decide pick four (five) contours which subsequently result in different Green functions

$t - t_0 < 0$

$t - t_0 > 0$

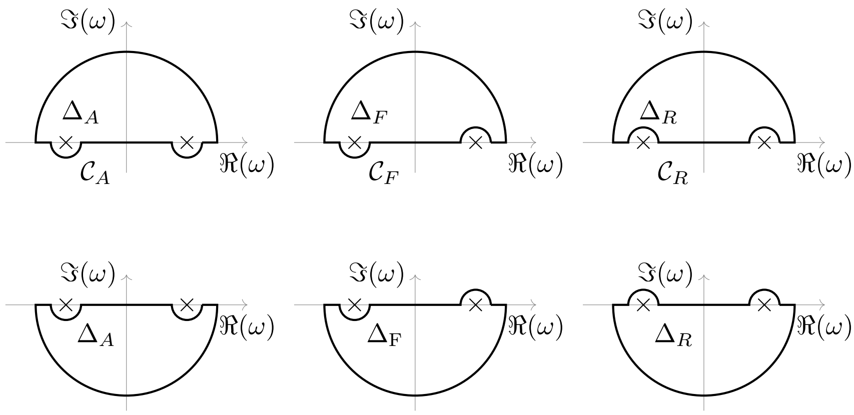


Figure 2: Possible contours for the Fourier back transformation of the one dimensional Klein-Gordon Greens functions caused by the poles $\pm c\sqrt{k^2+\mu^2}$

$$\int_{-\infty}^{\infty} f\,d\omega + \int_{\text{half circ}} f\,d\omega = 2\pi i \text{Res} f \tag{325}$$

$t - t_0 < 0:$

$$w_A(k, t-t_0) = \frac{2\pi i c}{4\pi\sqrt{k^2+\mu^2}}\left[e^{ic(t-t_0)\sqrt{k^2+\mu^2}} - e^{-ic(t-t_0)\sqrt{k^2+\mu^2}}\right] \tag{326}$$

$$= \frac{-c}{\sqrt{k^2+\mu^2}}\sin\left(c(t-t_0)\sqrt{k^2+\mu^2}\right) \tag{327}$$

$$w_F(k, t-t_0) = \frac{ic}{2\sqrt{k^2+\mu^2}}\left[-e^{-ic(t-t_0)\sqrt{k^2+\mu^2}}\right] \tag{328}$$

$$w_R(k, t-t_0) = 0 \tag{329}$$

$t - t_0 > 0:$

$$w_A(k, t-t_0) = 0 \tag{330}$$

$$w_F(k, t-t_0) = \frac{2\pi i c}{4\pi\sqrt{k^2+\mu^2}}\left[-e^{-ic(t-t_0)\sqrt{k^2+\mu^2}}\right] \tag{331}$$

$$w_R(k, t-t_0) = \frac{2\pi i c}{4\pi\sqrt{k^2+\mu^2}}\left[e^{ic(t-t_0)\sqrt{k^2+\mu^2}} - e^{-ic(t-t_0)\sqrt{k^2+\mu^2}}\right] \tag{332}$$

$$= \frac{-ic}{\sqrt{k^2+\mu^2}}\sin\left(c(t-t_0)\sqrt{k^2+\mu^2}\right) \tag{333}$$

Second Fourier back transformation

$$u(x,t) = \frac{1}{2\pi}\int dk\, e^{ikx} w(k,t) \tag{334}$$

$$= \frac{c}{4\pi^2}\int dk \frac{e^{ikx}}{\mu\sqrt{k^2/\mu^2+1}}\left[e^{-ict\mu\sqrt{k^2/\mu^2+1}} - e^{ict\mu\sqrt{k^2/\mu^2+1}}\right] \tag{335}$$

Now substitude $k/\mu = \sinh s$ and $1+\sinh^2 s = \cosh^2 s$

$$u(x,t) = \frac{c}{4\pi^2}\int \mu\cosh s\, ds \frac{e^{ix\mu\sinh s}}{\mu\cosh s}\left[e^{-ict\mu\cosh s} - e^{ict\mu\cosh s}\right] \tag{336}$$

$$= \frac{c}{4\pi^2}\int ds\, e^{ix\mu\sinh s}\left[e^{-ict\mu\cosh s} - e^{ict\mu\cosh s}\right] \tag{337}$$

as well as

$$x = \frac{1}{\mu}z\cosh y \tag{338}$$

$$ct = \frac{1}{\mu}z\sinh y \tag{339}$$

$$\to x^2 - c^2t^2 = \frac{1}{\mu^2}z^2 \tag{340}$$

which gives

$$u(x,t) = \frac{c}{4\pi^2}\int ds\, e^{iz\cosh y\sinh s}\left[e^{-iz\sinh y\cosh s} - e^{iz\sinh y\cosh s}\right] \tag{341}$$

$$= \frac{c}{4\pi^2}\int ds\, (e^{iz(\cosh y\sinh s - \sinh y\cosh s)} - e^{iz(\cosh y\sinh s + \sinh y\cosh s)}) \tag{342}$$

$$= \frac{c}{4\pi^2}\int ds\, (e^{iz\sinh(s-y)} - e^{iz\sinh(s+y)}) \tag{343}$$

$$= \frac{c}{4\pi^2}\int ds\, \left[\cos(z\sinh(s-y)) + i\sin(z\sinh(s-y)) - e^{iz\sinh(s+y)}\right] \tag{344}$$

$$z^2 = \mu^2(x^2 - c^2t^2) \tag{345}$$

$$\psi_0(x,t) = \frac{i}{\pi c}\partial_t\int_0^\infty dy\, \cos(z\sinh y) \quad \text{for} \quad \psi_0(x,0) = \delta(x) \tag{346}$$

$$\psi(x,t) = \int dy\, f(y)\psi_0(x-y,t) \quad \text{for} \quad \psi(x,0) = f(x) \tag{347}$$

2. Case $n = 3$

0.17.4 Helmholtz equation $(\triangle + k^2)u(x) = f(x)$

The Greens function is given by $(\triangle_x + k^2)G(x,y) = \delta(x-y)$

0.17.5 Feynman propagator $\left(\triangle - k^2\right) u(x) = f(x)$

0.17.6 Heat equation $\left(\partial_t - k\triangle\right) u(x) = f(x)$

Homogenous case $\left(\partial_t - k\triangle\right) G(x,t) = 0$

$$G(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}} \tag{348}$$

0.17.7 Relativistic Heat equation $\left(\partial_{tt} + 2\gamma\partial_t - c^2\triangle\right) u(x) = f(x)$

0.17.8 Sine-Gordon equation equation $\left(\frac{1}{c^2}\partial_{tt} - \triangle +\right) u(x,t) + \sin u(x,t) = 0$

0.17.9 Kortegweg-De Vries equation equation $\partial_t u + 6u \cdot \partial_x u + \partial_{xxx} u = 0$

0.18 Perturbation and divergent series

$$A_n = \sum_{k=0}^{n} a_k \tag{349}$$

$$A = \lim_{n\to\infty} A_n \tag{350}$$

0.18.1 Shanks transform (convergence accelerators)

For a slowly converging sum

$$S(A_n) = \frac{A_{n+1}A_{n-1} - A_n^2}{A_{n+1} - 2A_n + A_{n-1}} \tag{351}$$

the transformed partial sum $S(S(S(S(A_n))))$ might converge much faster than A_n.

0.18.2 Richardson extrapolations (convergence accelerators)

For a slowly converging sum the extrapolation

$$R_1 = (n+1)S_{n+1} - nS_n \tag{352}$$

$$R_2 = \frac{1}{2}\left[(n+2)^2 S_{n+2} - 2(n+1)^S_{n+1} + n^2 S_n\right] \tag{353}$$

$$R_n = ... \tag{354}$$

0.19 Probability

- Hypothesis H: Steve is a librarian
- Evidence E: Steve likes reading books

Question: Whats the probability of the hypothesis is true given the evidence is true $P(H|E)$

$$P(H|E) \equiv \frac{P(E \cap H)}{P(E)} \qquad P(E|H) \equiv \frac{P(E \cap H)}{P(H)} \tag{355}$$

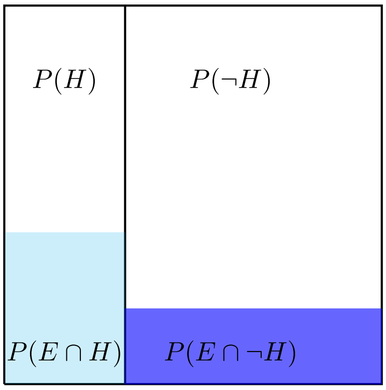
$$\rightarrow \qquad P(H|E) = \frac{P(E|H) \cdot P(H)}{P(E)} = \frac{P(H) \cdot P(E|H)}{P(H) \cdot P(E|H) + P(\neg H) \cdot P(E|\neg H)} \tag{356}$$

alternatively

$$P(H|E) = \frac{\#\text{allPeople} \cdot P(H) \cdot P(E|H)}{\#\text{allPeople} \cdot P(H) \cdot P(E|H) + \#\text{allPeople} \cdot P(\neg H) \cdot P(E|\neg H)} \tag{357}$$

$$= \frac{P(H) \cdot P(E|H)}{P(H) \cdot P(E|H) + P(\neg H) \cdot P(E|\neg H)} \tag{358}$$

$$= \frac{P(H) \cdot P(E|H)}{P(E)} \tag{359}$$



0.20 Matrices

1. inverse $A^{-1}A = \mathbb{I}$
   - therefore $\mathbb{I} = (AB)(B^{-1}A^{-1}) \quad \rightarrow \quad (AB)^{-1} = B^{-1}A^{-1}$
2. Hermitian transpose $A^\dagger = (\overline{A})^T = \overline{A^T}$
   - $(AB)^\dagger = B^\dagger A^\dagger$ therefore $\mathbb{I} = (AA^{-1})^\dagger = (A^{-1})^\dagger A^\dagger \quad \rightarrow \quad (A^\dagger)^{-1} = (A^{-1})^\dagger$

$$\langle x|Ay\rangle = \sum_k x_k^* (\vec{A}_{\text{row}k} \cdot \vec{y}) = \sum_{k,l} x_k^* A_{kl} y_l \tag{360}$$

$$\langle Bx|y\rangle = \sum_k (\vec{B}_{\text{row}k} \cdot \vec{x})^* y_k = \sum_{k,l} B_{kl}^* x_l^* y_k \tag{361}$$

3. Real symmetric $A^T = A$
   - only real eigenvalues
   - always diagonalizable

4. Hermitian $A^T = \overline{A}$ or better $A^\dagger = A$
    - only real eigenvalues
    - always diagonalizable
5. Orthorgonal $A^T = A^{-1}$
    - at most eigenvalues ± 1
    - always invertible
6. Unitary $A^\dagger = A^{-1}$
    - at most eigenvalues of from $e^{-i\alpha}$
    - always invertible

0.21 Matrix exponentials

$$
\begin{aligned}
e^X &= \sum_{n=0} \frac{1}{n!} X^n && (362)\\
\det e^X &= e^{\operatorname{tr} X} && (363)\\
\left(e^X\right)^{-1} &= e^{-X} && (364)\\
e^X e^Y &= e^{X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X,Y]]-\frac{1}{12}[Y,[X,Y]]+\ldots} && (365)
\end{aligned}
$$

0.22 Diagonalization

Any matrix A is called diagonalizable if there exists an invertible matrix S such that

$$D_A = S^{-1} A S \tag{366}$$

is a diagonal matrix. The diagonalizability of A is equivalent to the fact that the $\{\vec{v}_i\}$ are all linearly independent. Necessary condition

- n distinct eigenvalues
- if there is eigenvalue with multiplicity k then it must have k linearly independent eigenvectors

To find S and D_A one has to find the eigensystem $\{\lambda_i, \vec{v}_i\}$ with $A\vec{v}_i = \lambda_i \vec{v}_i$. Then $D_A S$ and S can be written as $S = (\vec{v}_1, ..., \vec{v}_n)$ and $D_A = \operatorname{diag}(\lambda_1, ..., \lambda_n)$ because $AS = (A\vec{v}_1, ..., A\vec{v}_n) = (\lambda_1 \vec{v}_1, ..., \lambda_n \vec{v}_n) = SD_A$.

0.23 Functional derivatives

Let $F[\phi]$ a functional, i.e. a mapping from a Banach space $\mathcal{M}$ to the field of real or complex numbers. The functional (Frechet) derivative $\delta F[\phi]/\delta\phi$ is defined by

$$
\begin{aligned}
\delta F &= \int dx \frac{\delta F[\phi]}{\delta \phi(x)} \cdot \delta\phi(x) && (367)\\
&= \int dx \frac{\delta F[\phi]}{\delta \phi(x)} \cdot \epsilon \delta(x-y) && (368)\\
&= \epsilon \frac{\delta F[\phi]}{\delta \phi(y)} && (369)\\
&= F[\phi + \epsilon\delta(x-y)] - F[\phi] && (370)
\end{aligned}
$$

which means

$$\frac{\delta F[\phi]}{\delta\phi[y]} = \lim_{\epsilon\to 0}\frac{F[\phi+\epsilon\delta(x-y)]-F[\phi]}{\epsilon} \tag{371}$$

$$F[\phi+\epsilon\delta(x-y)] = F[\phi]+\epsilon\frac{\delta F[\phi]}{\delta\phi(y)} \tag{372}$$

$$= F[\phi]+\epsilon\int dx\frac{\delta F[\phi]}{\delta\phi(x)}\cdot\delta(x-y) \tag{373}$$

- Product rule $F[\phi] = G[\phi]H[\phi]$

$$\frac{\delta F[\phi]}{\delta\phi(x)} = \frac{\delta(G[\phi]H[\phi])}{\delta\phi} \tag{374}$$

$$= \lim_{\epsilon\to 0}\frac{G[\phi+\epsilon\delta(x-y)]H[\phi+\epsilon\delta(x-y)]-G[\phi]H[\phi]}{\epsilon} \tag{375}$$

$$= \lim_{\epsilon\to 0}\frac{\left(G[\phi]+\epsilon\frac{\delta G}{\delta\phi}\right)\left(H[\phi]+\epsilon\frac{\delta H}{\delta\phi}\right)-G[\phi]H[\phi]}{\epsilon} \tag{376}$$

$$= \lim_{\epsilon\to 0}\frac{G[\phi]H[\phi]+\epsilon G[\phi]\frac{\delta H}{\delta\phi}+\frac{\delta G}{\delta\phi}H[\phi]+\epsilon^2\frac{\delta G}{\delta\phi}\frac{\delta H}{\delta\phi}-G[\phi]H[\phi]}{\epsilon} \tag{377}$$

$$= G[\phi]\frac{\delta H[\phi]}{\delta\phi(x)}+\frac{\delta G[\phi]}{\delta\phi(x)}H[\phi] \tag{378}$$

- Chain rule $F[G[\phi]]$

$$\delta F = \int dx\frac{\delta F[G[\phi]]}{\delta\phi(x)}\delta\phi(x) \tag{379}$$

$$\delta G = \int dy\frac{\delta G[\phi]}{\delta\phi(y)}\delta\phi(y) \tag{380}$$

$$\delta F = \int dz\frac{\delta F[G]}{\delta G(z)}\delta G(z) \tag{381}$$

$$= \int dz\frac{\delta F[G]}{\delta G(z)}\int dy\frac{\delta G[\phi]}{\delta\phi(y)}\delta\phi(y) \tag{382}$$

$$= \int dy\underbrace{\int dz\frac{\delta F[G]}{\delta G(z)}\frac{\delta G[\phi]}{\delta\phi(y)}}_{=\frac{\delta F[G[\phi]]}{\delta\phi(y)}}\delta\phi(y) \tag{383}$$

$$\frac{\delta F[G[\phi]]}{\delta\phi(y)} = \int dz\frac{\delta F[G]}{\delta G(z)}\frac{\delta G[\phi]}{\delta\phi(y)} \tag{384}$$

- Chain rule (special case) $F[g[\phi]]$

$$\frac{\delta F[g[\phi]]}{\delta\phi(y)} = ... \tag{385}$$

$$= \frac{\delta F}{\delta g(\phi(y))}\frac{dg(\phi)}{d\phi(y)} \tag{386}$$

Some examples

1. $F[\phi] = \int dx\phi(x)\delta(x)$

$$\frac{\delta F[\phi]}{\delta\phi(y)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int dx(\phi(x)+\epsilon\delta(x-y))\delta(x))-\int dx\,\phi(x)\delta(x)\right) \tag{387}$$

$$= \int dx\,\delta(x-y))\delta(x) \tag{388}$$

$$= \delta(y) \tag{389}$$

2. $F[\phi] = \int dx\phi(x)$

$$\frac{\delta F[\phi]}{\delta\phi(y)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int dx(\phi(x)+\epsilon\delta(x-y))) - \int dx\,\phi(x)\right) \tag{390}$$
$$= \int dx\,\delta(x-y) \tag{391}$$
$$= 1 \tag{392}$$

3. $F_x[\phi] = \phi(x)$

$$\frac{\delta\phi(x)}{\delta\phi(y)} = \frac{\delta F_x[\phi]}{\delta\phi(y)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left((\phi(x)+\epsilon\delta(x-y)) - \phi(x)\right) \tag{393}$$
$$= \delta(x-y) \tag{394}$$

4. $F[\phi] = \int dx\phi(x)^n$

$$\frac{\delta F[\phi]}{\delta\phi(y)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int dx(\phi(x)+\epsilon\delta(x-y)))^n - \int dx\,\phi(x)^n\right) \tag{395}$$
$$= \int dx\,n\phi(x)^{n-1}\delta(x-y) \tag{396}$$
$$= n\phi(y)^{n-1} \tag{397}$$

5. $F[\phi] = \int dx\left(\frac{\phi(x)}{dx}\right)^n$

$$\frac{\delta F_y[\phi]}{\delta\phi(y)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int dx(\frac{d}{dx}\phi(x)+\epsilon\frac{d}{dx}\delta(x-y))^n - \int dx\,(\frac{d}{dx}\phi(x))^n\right) \tag{398}$$
$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int dx(\frac{d}{dx}\phi(x))^n + n(\frac{d}{dx}\phi(x))^{n-1}\epsilon\frac{d}{dx}\delta(x-y) + O(\epsilon^2) - \int dx\,(\frac{d}{dx}\phi(x))^n\right) \tag{399}$$
$$= \int dx\,n(\frac{d}{dx}\phi(x))^{n-1}\frac{d}{dx}\delta(x-y) \tag{400}$$
$$= -n\frac{d}{dx}(\frac{d}{dx}\phi(x))^{n-1} \tag{401}$$

6. $F_y[\phi] = \int dzK(y,z)\phi(z)$

$$\frac{\delta F_y[\phi]}{\delta\phi(x)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int dz(K(y,z)(\phi(z)+\epsilon\delta(z-x)) - \int dzK(y,z)\phi(z)\right) \tag{402}$$
$$= \int dz\,K(y,z)\delta(z-x) \tag{403}$$
$$= K(y,x) \tag{404}$$

7. $F_x[\phi] = \nabla\phi(x)$

$$\frac{\delta F[\phi]}{\delta\phi(y)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\nabla_x(\phi(x)+\epsilon\delta(x-y)) - \nabla_x\phi(x)\right) \tag{405}$$
$$= \nabla_x\delta(x-y) \tag{406}$$

8. $F[\phi] = g\left(G[\phi(x)]\right)$

$$\frac{\delta F[\phi]}{\delta\phi(y)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}g(G[\phi(x)+\epsilon\delta(x-y)]) - g(G[\phi(x)]) \tag{407}$$
$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}g(G[\phi(x)] + \epsilon\frac{\delta G}{\delta\phi}) - g(G[\phi(x)]) \tag{408}$$
$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}g(G[\phi(x)]) + g'\epsilon\frac{\delta G}{\delta\phi} - g(G[\phi(x)]) \tag{409}$$
$$= \frac{\delta G}{\delta\phi}g'(G[\phi(x)]) \tag{410}$$

0.24 Complex Calculus

- Cauchy–Riemann equations $f(x+iy) = u(x,y) + iv(x,y)$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{411}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{412}$$

- Cauchy integral formula

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz \tag{413}$$
$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}dz \tag{414}$$

- Taylor series

$$f(a+h) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a-h}dz \tag{415}$$
$$= \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\frac{1}{1-\frac{h}{z-a}}dz \tag{416}$$
$$= \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\sum_k \frac{h^k}{(z-a)^k}dz \qquad \text{(geometric series)} \tag{417}$$
$$= \sum_k \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{k+1}}dz\cdot h^k \qquad \text{(quick and dirty - exchanging integral and sum)} \tag{418}$$
$$= \sum_k \frac{f^{(k)}(a)}{k!}\,h^k \tag{419}$$

0.25 Space hierarchy

1. K-Vector space $(K, \oplus, \odot)$
   - set V, field K with $(K, +, \cdot)$
   - vector addition $\oplus : V\times V \to V$
   - scalar multiplication $\odot : K\times V\to V$
2. Topological vector space

- K-vector space
- continuous (smooth) vector addition and scalar multiplication

3. Metric (vector) space (M, d)

   - set M, metric $d : M \times M \to \mathbb{R}$
   - $d(x, y) = 0 \Leftrightarrow x = y$
   - $d(x, y) = d(y, x)$
   - $d(x, y) + d(y, z) \geq d(x, z)$
   - from the requirements above follows $d(x, y) \geq 0$

4. Normed vector space $(V, \|\cdot\|)$

   - K-vector space V, norm $\|\cdot\| : V \to \mathbb{R}$
   - Typically $K \in (\mathbb{R}, \mathbb{C})$ to have a definition of $|\lambda|$
   - $\|x\| \geq 0$
   - $\|x\| = 0 \Leftrightarrow x = 0$
   - $\|\lambda x\| = |\lambda|\|x\|$ with $\lambda \in K$
   - $\|x\| + \|y\| \geq \|x + y\|$
   - with $d(x, y) := \|x - y\|$ every normed vector space has also a metric
   - a metric does NOT induce a always norm as the linearity/homogeneity of the norm is not guaranteed

5. Banach space (complete normed vector space)

   - normed K-vector space $(V, \|\cdot\|)$ with $K \in (\mathbb{R}, \mathbb{C})$
   - completeness: every Cauchy sequence converges (with the metric induced by the norm) to a well defined limit
   - if the space is just a metric space (without a norm) the space is called Cauchy space

6. Hilbert space (complete vector space with a scalar product)

   - K-vector space V with $K \in (\mathbb{R}, \mathbb{C})$
   - scalar product $\langle \cdot, \cdot \rangle : V \times V \to K$
   - $\langle \lambda x_1 + x_2, y \rangle = \langle \lambda x_1, y \rangle + \langle \lambda x_2, y \rangle$
   - $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$ for $\lambda \in K$
   - $\langle x, y \rangle = \overline{\langle y, x \rangle}$ which implies $\langle x, x \rangle \in \mathbb{R}$
   - $\langle x, x \rangle > 0$
   - $\langle x, x \rangle = 0 \Leftrightarrow x = 0$
   - completeness: every Cauchy sequence converges (with the metric induced by the norm which is itself induced by the scalar product) to a well defined limit
   - without completeness the space is called Pre-Hilbert space

0.26 Tensors

- For a vector $\mathbf{A}$ the expression $\mathbf{A}^2$ is the squared distance between tip and tail.
- The inner product of two vectors can then be defined by the parallelogram law

$$\mathbf{A}\cdot\mathbf{B} \equiv \frac{1}{4}\left[(\mathbf{A}+\mathbf{B})^2 - (\mathbf{A}-\mathbf{B})^2\right] \tag{420}$$

- A rank-n tensor $\mathbf{T} = \mathbf{T}(\_,\_,\_)$ is real-valued linear function of n vectors.

$$\mathbf{T}(\alpha\mathbf{A} + \mu\mathbf{B}, \mathbf{C}, \mathbf{D}) = \alpha\mathbf{T}(\mathbf{A}, \mathbf{C}, \mathbf{D}) + \beta\mathbf{T}(\mathbf{B}, \mathbf{C}, \mathbf{D}) \tag{421}$$

- Metric tensor

$$\mathbf{g}(\mathbf{A}, \mathbf{B}) \equiv \mathbf{A}\cdot\mathbf{B} \tag{422}$$

- A vector is a tensor of rank one

$$\mathbf{A}(\mathbf{C}) \equiv \mathbf{A}\cdot\mathbf{C} \tag{423}$$

- Tensor product

$$\mathbf{A}\otimes\mathbf{B}\otimes\mathbf{C}(\mathbf{E}, \mathbf{F}, \mathbf{G}) \equiv \mathbf{A}(\mathbf{E})\mathbf{B}(\mathbf{F})\mathbf{C}(\mathbf{G}) = (\mathbf{A}\cdot\mathbf{E})(\mathbf{B}\cdot\mathbf{F})(\mathbf{C}\cdot\mathbf{G}) \tag{424}$$

- Contraction

$$1\&3\ \text{contraction}(\mathbf{A}\otimes\mathbf{B}\otimes\mathbf{C}\otimes\mathbf{D}) \equiv (\mathbf{A}\cdot\mathbf{C})\mathbf{B}\otimes\mathbf{D} \tag{425}$$

- Orthogonal basis

$$\mathbf{e}_j\cdot\mathbf{e}_k = \delta_{jk} \tag{426}$$

- Component expansion

$$\mathbf{A} = A_j\mathbf{e}_j \quad\rightarrow\quad A_j = \mathbf{A}(\mathbf{e}_j) = \mathbf{A}\cdot\mathbf{e}_j \tag{427}$$

$$\mathbf{T} = T_{abc}\mathbf{e}_a\otimes\mathbf{e}_b\otimes\mathbf{e}_c \quad\rightarrow\quad T_{ijk} = \mathbf{T}(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k) \tag{428}$$

$$1\&3\ \text{contraction}(\mathbf{R}) \quad\rightarrow\quad R_{ijik} \tag{429}$$

$$\mathbf{g} \quad\rightarrow\quad g_{jk} = \mathbf{g}(\mathbf{e}_j, \mathbf{e}_k) = \mathbf{e}_j\cdot\mathbf{e}_k = \delta_{jk} \tag{430}$$

0.27 Tensors Index rules

A_{ij} - i-th row and j-th column

$$C = AB \tag{431}$$

$$C_{ij} = \sum_k A_{ik}C_{kj} \tag{432}$$

Matrix - Vector

$$\Lambda\mathbf{a} \rightarrow \Lambda^i{}_j a^j \tag{433}$$

Vector - Vector

$$\begin{aligned}
\mathbf{a}\cdot\mathbf{b} &\equiv G(\mathbf{a},\mathbf{b}) && (434)\\
&= G(\sum_i a^i\mathbf{e}_i, \sum_j b^j\mathbf{e}_j) && (435)\\
&= \sum_{ij} a^i b^j G(\mathbf{e}_i,\mathbf{e}_j) && (436)\\
&= \sum_{ij} a^i b^j g_{ij} && (437)\\
&= a^i g_{ij} b^j \quad = \mathbf{a}^T\ G\ \mathbf{b} && (438)\\
&= a_i b^i && (439)
\end{aligned}$$

Matrix - Matrix

$$\begin{aligned}
\eta_{\alpha\beta} dx^\alpha dx^\beta &= \eta_{\mu\nu}(\Lambda^\mu{}_\alpha dx^\alpha)(\Lambda^\nu{}_\beta dx^\beta) && (440)\\
\mathbf{dx}^T\eta\mathbf{dx} &= (\Lambda\mathbf{dx})^T\eta\Lambda\mathbf{dx} = \mathbf{dx}^T(\Lambda^T\eta\Lambda)\mathbf{dx} && (441)\\
\eta &= \Lambda^T\eta\Lambda && (442)\\
\eta_{\alpha\beta} &= \Lambda^\mu{}_\alpha\eta_{\mu\nu}\Lambda^\nu{}_\beta && (443)
\end{aligned}$$

$$\begin{aligned}
F^{ab} &= \Lambda^a_c\Lambda^b_d F^{cd} \qquad \to \Lambda F\Lambda^T && (444)\\
F_{ab} &= \Lambda^c_a\Lambda^d_b F_{cd} \qquad \to \Lambda^T F\Lambda && (445)
\end{aligned}$$

$$\begin{aligned}
F^a_b &= \eta^{ac}F_{cb} \qquad \to \eta F && (446)\\
F^{ad} &= \eta^{db}\eta^{ac}F_{cb} = \eta^{ac}F_{cb}\eta^{bd} \qquad \to \eta F\eta^T && (447)\\
F_{ab}F^{ab} &= -F_{ba}F^{ab} \qquad \to -\mathrm{tr}(FF) && (448)
\end{aligned}$$

0.28 Direct Sum and Tensor Products

Two vector spaces V, W with $\dim V = n$, $\dim W = m$ and with bases

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \in V \tag{449}$$

$$\vec{f}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \vec{f}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \vec{f}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \in W \tag{450}$$

0.28.1 Direct sum

Then

$$\vec{v}\oplus\vec{w} = \begin{pmatrix} \vec{v} \\ \vec{w} \end{pmatrix} \tag{451}$$

and therefore (counting basis vectors) $\dim V\oplus W = n+m$. Linear maps (via matrices) can be written as

$$(A\oplus B)(\vec{v}\oplus\vec{w}) = \begin{pmatrix} A & 0_{n\times m} \\ 0_{n\times m} & B \end{pmatrix}\begin{pmatrix} \vec{v} \\ \vec{w} \end{pmatrix} = (A\vec{v})\oplus(B\vec{w}) \tag{452}$$

and also

$$\det(A \oplus B) = (\det A)(\det B) \tag{453}$$
$$\mathrm{tr}(A \oplus B) = (\mathrm{tr}A) + (\mathrm{tr}B) \tag{454}$$

0.28.2 Tensor product

Then

$$\vec{v} \otimes \vec{w} = (\sum_i v^i \vec{e}_i) \otimes (\sum_j w^j \vec{f}_j) \tag{455}$$
$$= \sum_{i,j} v^i w^j (\vec{e}_i \otimes \vec{f}_j) \tag{456}$$

and therefore (counting basis vectors $\vec{e}_i \otimes \vec{f}_j$) $\dim V \otimes W = n \cdot m$. The basis vectors can be written explicitly (stacking n zero vectors $\vec{0}_m$ on top of each other)

$$\vec{b}_k = \vec{e}_i \otimes \vec{f}_j \tag{457}$$
$$= \left(\begin{array}{l} \vec{0}_m \\ \hline \dots \\ \hline 0 \\ \dots \\ 1 \quad (i^{\text{th}} \text{ slot } j^{\text{th}} \text{position}) \\ \dots \\ 0 \\ \hline \dots \\ \hline \vec{0}_m \end{array} \right) \quad \rightarrow \quad \vec{v} \otimes \vec{w} = \left(\begin{array}{c} v^1 w^1 \\ v^1 w^2 \\ v^1 w^3 \\ \hline v^2 w^1 \\ v^2 w^2 \\ v^2 w^3 \end{array} \right) \tag{458}$$

Alternatively visualise it as

$$\vec{v} \otimes \vec{w} = \vec{v}\vec{w}^\dagger = \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} (w^1 \; w^2 \; w^3) = \begin{pmatrix} v^1 w^1 & v^1 w^2 & v^1 w^3 \\ v^2 w^1 & v^2 w^2 & v^2 w^3 \end{pmatrix} \tag{459}$$

Linear maps (via matrices) can be written as

$$(A \otimes B)(\vec{v} \otimes \vec{w}) = \begin{pmatrix} a_{11}B & \dots & a_{1n}B \\ \dots & \dots & \dots \\ a_{n1}B & \dots & a_{nn}B \end{pmatrix} \begin{pmatrix} v_1 \vec{w} \\ \dots \\ v_n \vec{w} \end{pmatrix} = (A\vec{v}) \otimes (B\vec{w}) \tag{460}$$

Be careful with sloppy notation for addition of operators - $A + B$ is not defined on V or W but CAN be defined on $V \otimes W$ via

$$A + B \equiv A \otimes I_{m \times m} + I_{n \times n} \otimes B \tag{461}$$

and also

$$\det(A \otimes B) = (\det A)^m (\det B)^n \tag{462}$$
$$\mathrm{tr}(A \otimes B) = (\mathrm{tr}A) \cdot (\mathrm{tr}B) \tag{463}$$

0.29 Clebsch-Gordon decomposition in QM

Falling from the sky

$$\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} = \mathcal{H}_{|j_1 - j_2|} \oplus \dots \oplus \mathcal{H}_{j_1 + j_2} \tag{464}$$

Using the notation

$\mathcal{H}_{j_1}$	$\dim \mathcal{H}_{j_1}$	$\mathbf{2j_1+1}$	Multiplett
$\mathcal{H}_0$	0	$\mathbf{0}$	Singlet
$\mathcal{H}_{1/2}$	2	$\mathbf{2}$	Doublet
$\mathcal{H}_1$	3	$\mathbf{3}$	Triplett
$\mathcal{H}_{3/2}$	4	$\mathbf{4}$	Quartett

Examples

$$j_1 = \frac{1}{2}, j_2 = \frac{1}{2} \quad \rightarrow \quad \mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{3} \quad \rightarrow \quad \text{1 singlet, 1 triplett} \tag{465}$$

$$j_1 = 1, j_2 = 1 \quad \rightarrow \quad \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{5} \quad \rightarrow \quad \text{1 singlet, 1 triplett, 1 quintett} \tag{466}$$

Extension to three angular momentum

$$\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3} = (\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2}) \otimes \mathcal{H}_{j_3} = (\mathcal{H}_{|j_1-j_2|} \otimes \mathcal{H}_{j_3}) \oplus ... \oplus (\mathcal{H}_{j_1+j_2} \otimes \mathcal{H}_{j_3}) \tag{467}$$

then

$$\begin{aligned} j_1 = 1, j_2 = 1, j_3 = 1 \quad & \rightarrow (\mathcal{H}_1 \otimes \mathcal{H}_1) \otimes \mathcal{H}_1 = (\mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2) \otimes \mathcal{H}_1 && (468)\\ & = (\mathcal{H}_0 \otimes \mathcal{H}_1) \oplus (\mathcal{H}_1 \otimes \mathcal{H}_1) \oplus (\mathcal{H}_2 \otimes \mathcal{H}_1) && (469)\\ & = (\mathcal{H}_1) \oplus (\mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2) \oplus (\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3) && (470)\\ & \rightarrow \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{3} \oplus \mathbf{3} \oplus \mathbf{5} \oplus \mathbf{5} \oplus \mathbf{7} \quad \rightarrow \quad ... && (471) \end{aligned}$$

0.30 Tensorproduct in QM

Given two Hilbert spaces $\mathcal{V}_1, \mathcal{V}_2$ with complete orthonormal basis $\{|u_{i\in I}\rangle\}$ and $\{|v_{k\in K}\rangle\}$. The tensor product of two states $|\psi\rangle \in \mathcal{V}_1, |\phi\rangle \in \mathcal{V}_2$ is defined by

$$|\psi\rangle_1 \otimes |\phi\rangle_2 = |\phi\rangle_2 \otimes |\psi\rangle_1 \in \mathcal{V} \tag{472}$$

with linearity restrictions

$$a(|\psi\rangle_1 \otimes |\phi\rangle_2) = (a|\psi\rangle_1) \otimes |\phi\rangle_2 = |\psi\rangle_1 \otimes (a|\phi\rangle_2) \tag{473}$$

$$(|\psi_1\rangle_1 + |\psi_2\rangle_1) \otimes |\phi\rangle_2 = |\psi_1\rangle_1 \otimes |\phi\rangle_2 + |\psi_2\rangle_1 \otimes |\phi\rangle_2 \tag{474}$$

Combining two basis vectors from each of the two Hilbert spaces gives as basis of $\mathcal{V}$

$$\{|u_i\rangle_1 \otimes |v_j\rangle_2\} \tag{475}$$

meaning that $\dim \mathcal{V} = \dim \mathcal{V}_1 \cdot \dim \mathcal{V}_2$. Therefore each element of $\mathcal{V}$ can be represented by

$$|\chi\rangle = \sum_{ij} a_{ij}(|u_i\rangle_1 \otimes |v_j\rangle_2) \tag{476}$$

while the tensor product of two states is

$$|\psi\rangle_1 \otimes |\phi\rangle_2 = \left(\sum_i c_i|u_i\rangle\right) \otimes \left(\sum_i d_i|v_i\rangle\right) \tag{477}$$

$$= \sum_{ij} c_i d_j(|u_i\rangle_1 \otimes |v_j\rangle_2) \tag{478}$$

Question: Can every state in $\mathcal{V}$ be expressed as a tensor product of two states in $\mathcal{V}_1$ and $\mathcal{V}_2$? **Answer:** No! (simple counter example). States which can not be written as a product are called entangled states.

The scalar product of $\mathcal{V}$ space can be defined by

$$ {}_1\langle\psi| \otimes {}_2\langle\phi|)(|\psi'\rangle_1 \otimes |\phi'\rangle_2) = {}_1\langle\psi|\psi'\rangle_1\, {}_2\langle\phi|\phi'\rangle_2 \tag{479}$$

With orthonormal basis $\{u\} \in \mathcal{V}_1$ and $\{v\} \in \mathcal{V}_2$

$$\langle u_i|u_k\rangle = \delta_{ik} \tag{480}$$
$$\langle v_j|v_l\rangle = \delta_{jl} \tag{481}$$

then

$$ {}_1\langle u_i| \otimes {}_2\langle v_j|)(|u_k\rangle_1 \otimes |v_l\rangle_2) = {}_1\langle u_i|u_k\rangle_1\, {}_2\langle v_j|v_l\rangle_2 = \delta_{ik}\delta_{jl} \tag{482}$$

meaning the basis of $\mathcal{V}$ is also orthonormal.

Operators A and B defined on Hilbert spaces $\mathcal{V}_1$ and $\mathcal{V}_2$ can be promoted to operators on $\mathcal{V}$ by

$$A \to A \otimes 1_B \tag{483}$$
$$B \to 1_A \otimes B \tag{484}$$
$$A + B \equiv A \otimes 1_B + 1_A \otimes B \tag{485}$$

then

$$\begin{aligned}
(A \otimes B)|\chi\rangle &:= (A|\phi\rangle_1) \otimes (B|\phi\rangle_2) && (486)\\
&= \sum_{ij} a_{ij}(A \otimes B)(|u_j\rangle_1) \otimes |v_j\rangle_2) && (487)\\
&= \sum_{ij} a_{ij}(A|u_j\rangle_1) \otimes (B|v_j\rangle_2) && (488)
\end{aligned}$$

or

$$\begin{aligned}
(A \otimes B)(|\phi\rangle_1 \otimes |\phi\rangle_2) &:= (A|\phi\rangle_1) \otimes (B|\phi\rangle_2) && (489)\\
&= \sum_{ij} c_i d_j (A|u_j\rangle_1) \otimes (B|v_j\rangle_2) && (490)
\end{aligned}$$

with eigenvectors $A|\phi_a\rangle = a|\phi_a\rangle$ and $B|\phi_b\rangle = b|\phi_b\rangle$

$$(A + B)(|\phi_a\rangle \otimes |\phi_b\rangle) = ... = (a + b)|\phi_a\rangle \otimes |\phi_b\rangle \tag{491}$$

Examples

$$\mathbb{R}^3 \otimes \mathbb{R}^3 \simeq \mathbb{R}^9 \tag{492}$$
$$\mathbb{R} \otimes \mathbb{R} \otimes \mathbb{R} \simeq \mathbb{R} \tag{493}$$
$$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \simeq \mathbb{R}^3 \tag{494}$$

0.31 Pauli Matrices

Properties of the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \tag{495}$$
$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = i\epsilon_{ijk}\frac{\sigma_k}{2} \qquad \{\sigma_i, \sigma_j\} = 2\delta_{ij} \tag{496}$$
$$\mathrm{Tr}\sigma_i = 0 \qquad \mathrm{Tr}(\sigma_i\sigma_j) = 2\delta_{ij} \tag{497}$$
$$\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k \qquad \sigma_i^2 = 1 \tag{498}$$
$$\sum_i (\sigma_i)_{ab}(\sigma_i)_{cd} = 2(\delta_{bc}\delta_{ad} - \frac{1}{2}\delta_{ab}\delta_{cd}) \tag{499}$$

The fundamental representation of SU(2) is given by 2×2 matrices U (with $U^\dagger U = 1$ and $\det U = 1$) which operate on two-component column vectors (fundamental doublet or Pauli spinor) $\xi' = U\xi$. A general matrix U can be expressed as

$$U = e^{\frac{i}{2}\theta_i \sigma_i}. \tag{500}$$

0.32 Division Algebras

There are exactly four real division algebras with an identity element

- $\mathbb{R} = \{1\}$ (dimension 1)
- $\mathbb{C} = \{1, i\}$ (dimension 2) with $i^2 = -1$
- $\mathbb{H} = \{1, i_1, i_2, i_3\}$ (dimension 4) with $i_1^2 = i_2^2 = i_3^2 = -1$
- $\mathbb{O} = \{1, i_1, ..., i_7\}$ (dimension 8) with $i_k^2 = -1$

where the i_k obey

$$i_k \circ i_l + i_l \circ i_k = 2\delta_{kl} \tag{501}$$

This can be generalized to a Clifford algebra by

$$i_k \circ i_l + i_l \circ i_k = 2\sigma_k \delta_{kl} \tag{502}$$

where $\sigma_k = \pm 1$. If $\sigma_1 = ... = \sigma_p = -1$ and $\sigma_{p+1} = ... = \sigma_q = +1$ it is called $\mathrm{Cl}(p, q, \mathbb{R})$. Some examples are:

- $\mathrm{Cl}(1, 0, \mathbb{R}) \cong \mathbb{C}$
- $\mathrm{Cl}(2, 0, \mathbb{R}) \cong \mathbb{H}$
- $\mathrm{Cl}(3, 1, \mathbb{R}) \to$ spin 1/2

0.33 Clifford Algebras

$\mathrm{Cliff}(1, d-1)$ is defined as set of d matrices of shape $n \times n$

$$\{(\gamma^\mu)^A{}_B\}_{\mu \in \{0,1,\ldots,d-1\}} \qquad A, B = 1, ..., n \tag{503}$$

which obey

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbf{1}_{n\times n} \tag{504}$$

$$(\gamma^\mu)^A{}_B(\gamma^\nu)^B{}_C + (\gamma^\nu)^A{}_B(\gamma^\mu)^B{}_C = 2\eta^{\mu\nu}(\mathbf{1}_{n\times n})^A{}_C \tag{505}$$

with spinor indices A, B and space-time indices μ, ν.

Some properties of Clifford algebras

- With $\mathrm{diag}\,\eta^{\mu\nu} = (1, -1, -1, -1)$
$$(\gamma^0)^2 = \mathbf{1}_{n\times n} \qquad (\gamma^i)^2 = -\mathbf{1}_{n\times n} \tag{506}$$
- The irreducible representations of $\mathrm{Cliff}(1, d-1)$ have dimensions
  - d even: $n = 2^{d/2}$
  - d odd: $n = 2^{(d-1)/2}$

d	1	2	3	4	5, 6, 7, 8
algebra	Cliff$(1,0)$	Cliff$(1,1)$	Cliff$(1,2)$	Cliff$(1,3)$	Cliff$(1,d-1)$
n	1	2	2	4	4, 8, 8, 16

Table 2: •

- The d matrices $\{\gamma^\mu\}$ of the Clifford algebra Cliff$(1,d-1)$ induce $d(d-1)/2$ matrices $S^{\rho\sigma}$

$$(S^{\rho\sigma})^A_{\ B} = \frac{i}{4}[\gamma^\rho,\gamma^\sigma]^A_{\ B} \tag{507}$$

  which form a representation of the Lie algebra (of the Lorentz group) $\mathfrak{so}(1,d-1)$.

- For $d=4$ we have Cliff(1,3) which contains $1+(d-1)=4$ matrices of shape 4×4

$$\gamma^0,\gamma^1,\gamma^2,\gamma^3 \tag{508}$$

  which induces 6 matrices $S^{01},S^{12},S^{23},S^{02},S^{03},S^{13}$ which are the generators of $\mathfrak{so}(1,3)$.

- Chiral/Dirac representation of Cliff(1,3)

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1}_{2\times 2} \\ \mathbf{1}_{2\times 2} & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{509}$$

  The γ's act on a complex vector space - the space of Dirac spinors $(\gamma^\mu)^A_{\ B}\psi^B$. A Lorentz trafo look like

$$\psi^A(x) \to \left[e^{-i\omega_{\rho\sigma}S^{\rho\sigma}}\right]^A_{\ B}\psi^B(x) \tag{510}$$

$$= \left[e^{-i(\omega_{01}S^{01}+\ldots+\omega_{23}S^{23})}\right]^A_{\ B}\psi^B(x) \tag{511}$$

$$= \left[e^{\frac{1}{4}(\omega_{01}[\gamma^0,\gamma^1]+\ldots+\omega_{23}[\gamma^2,\gamma^3])}\right]^A_{\ B}\psi^B(x) \tag{512}$$

0.34 Spinors

3D vector $(x,y,z\in\mathbb{R})$ can be written as a Pauli vector (via the Pauli matrices) which can be written as a product of Pauli spinors $(\xi_1,\xi_2\in\mathbb{C})$

$$\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Longleftrightarrow x\sigma_x + y\sigma_y + z\sigma_z = \vec{v}\cdot\vec{\sigma} \tag{513}$$

$$= x\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + y\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + z\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \tag{514}$$

$$= \begin{pmatrix} z & x-yi \\ x+yi & -z \end{pmatrix} \tag{515}$$

$$= \begin{pmatrix} -\xi_1\xi_2 & \xi_1^2 \\ -\xi_2^2 & \xi_1\xi_2 \end{pmatrix} \tag{516}$$

$$= \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}\begin{pmatrix} -\xi_2 & \xi_1 \end{pmatrix} \tag{517}$$

Rotating a vector

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{518}$$

Rotating the associated Pauli vector

$$\begin{pmatrix} \cos\theta/2 & i\sin\theta/2 \\ i\sin\theta/2 & \cos\theta/2 \end{pmatrix} \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix} \begin{pmatrix} \cos\theta/2 & i\sin\theta/2 \\ i\sin\theta/2 & \cos\theta/2 \end{pmatrix}^\dagger \tag{519}$$

$$= \underbrace{\begin{pmatrix} \cos\theta/2 & i\sin\theta/2 \\ i\sin\theta/2 & \cos\theta/2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}}_{\dots} \underbrace{\begin{pmatrix} -\xi_2 & \xi_1 \end{pmatrix} \begin{pmatrix} \cos\theta/2 & i\sin\theta/2 \\ i\sin\theta/2 & \cos\theta/2 \end{pmatrix}^\dagger}_{\dots} \tag{520}$$

As you need two Pauli spinores to represent a 3-vector and their associated rotations contain only half the angles we can regard a spinor as a rank 1/2 tensor.

3-vectors are represented by two Pauli spinors while 4-vectors are represented by Weyl spinors.

Weyl representation of 4-vectors

$$\vec{X} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \Longleftrightarrow ct\mathbb{I} + x\sigma_x + y\sigma_y + z\sigma_z = ct\mathbb{I} + \vec{x}\cdot\vec{\sigma} \tag{521}$$

$$= X^\mu \sigma_\mu \tag{522}$$

$$= \begin{pmatrix} ct+z & x-yi \\ x+yi & ct-z \end{pmatrix} \tag{523}$$

For 4-vectors we replace the σ by the γ-matrices and obtain Weyl spinors

$$\vec{X} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \Longleftrightarrow ct\gamma^0 + x\gamma^1 + y\gamma^2 + z\gamma^3 \tag{524}$$

- Grassmann algebra (exterior algebra): contains a wedge product
- Clifford algebra (geometric algebra): contains a wedge product and a scalar product

algebra	signature	equation	object
$Cl_{4,2}$	$+,+,+,+,-,-$	Twistor	twistor
$Cl_{1,3}$	$+,-,-,-$	Dirac	rel spin-1/2
$Cl_{3,0}$	$+,+,+$	Pauli	spin-1/2
$Cl_{0,1}$	$-$	Schroedinger	spin-0