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1 Advanced Topics in Gravity – Exercise sheet 4 - 2025-06-08

1.1 Exercise 1 - Penrose diagrams

Depict the Penrose diagram of the Schwarzschild spacetime when it has negative mass. We recommend to follow these steps:

1. Write the negative-mass Schwarzschild metric and analyze it.
2. Write the coordinate transformation and give the conformally compactified metric.
3. Identify infinities and singularities.
4. Draw the diagram.

Hint: Recall that this spacetime possesses a naked singularity. This kind of singularity is not covered by any horizon and it is going to be timelike.

1. With $\mu = -M > 0$

$$ds^2 = \left(1 + \frac{2G\mu}{r}\right) dt^2 - \left(1 + \frac{2G\mu}{r}\right)^{-1} dr^2 - r^2[d\vartheta^2 + \sin^2\vartheta d\phi^2] \tag{1}$$

We see that the metric coefficient $g_{tt} > 0$ everywhere - so there is no horizon. The center $r = 0$ is still a singularity

$$R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = \frac{48G^2\mu^2}{r^6} \tag{2}$$

but now it is naked (not covered by a horizon). The metric remains asymptotically flat for $r \to \infty$.

2. We define the tortoise coordinate again

$$\frac{dr^*}{dr} = \frac{1}{1 + \frac{2G\mu}{r}} \tag{3}$$

$$\to r^* = r - 2G\mu \log\left[1 + \frac{r}{2G\mu}\right] \tag{4}$$

And now we define advanced and retarded null-coordinates

$$v = t + r^* \tag{5}$$

$$u = t - r^* \tag{6}$$

then

$$\left(1+\frac{2G\mu}{r}\right)dudv = \left(1+\frac{2G\mu}{r}\right)(dt-dr^*)(dt+dr^*) \tag{7}$$

$$= \left(1+\frac{2G\mu}{r}\right)(dt^2-(dr^*)^2) \tag{8}$$

$$= \left(1+\frac{2G\mu}{r}\right)dt^2 - \frac{1}{1+\frac{2G\mu}{r}}dr^2 \tag{9}$$

resulting in

$$ds^2 = \left(1+\frac{2G\mu}{r}\right)du\,dv - r^2[d\vartheta^2+\sin^2\vartheta d\phi^2] \tag{10}$$

Now we compactify ($\Omega = \cos U \cos V$)

$$u = \tan U \tag{11}$$

$$v = \tan V \tag{12}$$

$$\to du\,dv = \frac{1}{\cos^2 U}\frac{1}{\cos^2 V}dU\,dV \tag{13}$$

$$\to ds^2 = \frac{1}{\cos^2 U\ \cos^2 V}\left(1+\frac{2G\mu}{r}\right)dU\,dV - r^2[d\vartheta^2+\sin^2\vartheta d\phi^2] \tag{14}$$

$$= \Omega^{-2}\left[\left(1+\frac{2G\mu}{r}\right)dU\,dV - r^2\Omega^2(d\vartheta^2+\sin^2\vartheta d\phi^2)\right] \tag{15}$$

The boundary $\Omega = 0$ is where $U = \pm\pi/2$ or $V = \pm\pi/2$ (or both), i.e. $u, v = \pm\infty$. Since $r\Omega \neq 0$ (I checked with Mathematica) at $\Omega = 0$, the boundary consists of (pieces of) a light cone. Now we do just a rotation to bring it into the normal shape of a Penrose diagram

$$T = \frac{1}{2}(V+U) \tag{16}$$

$$= \frac{1}{2}(\arctan v + \arctan u) \tag{17}$$

$$= \frac{1}{2}(\arctan[t+r^*] + \arctan[t-r^*]) \tag{18}$$

$$R = \frac{1}{2}(V-U) \tag{19}$$

$$= \frac{1}{2}(\arctan v - \arctan u) \tag{20}$$

$$= \frac{1}{2}(\arctan[t+r^*] - \arctan[t-r^*]) \tag{21}$$

Then

$$dU = dT - dR, \qquad dV = dT + dR \tag{22}$$

$$\to dUdV = dT^2 - dR^2 \tag{23}$$

$$ds^2 = \Omega^{-2}\left[\left(1+\frac{2G\mu}{r}\right)(dT^2-dR^2) - r^2\Omega^2(d\vartheta^2+\sin^2\vartheta d\phi^2)\right] \tag{24}$$

So the spacetime is similar to a Minkowski space BUT with a singularity at $r = 0$

3. See Mathematica code
   - $r = 0$ is the naked singularity
   - the null surface $V = \frac{1}{2}\pi$ called $\mathcal{I}^+$ (future lightlike infinity)

- the null surface $U = -\frac{1}{2}\pi$ called $\mathcal{I}^-$ (past lightlike infinity)
- the point $U = V = \pi/2$ is called i^+ (future timelike infinity)
- the point $V = \pi/2, U = -\pi/2$ is called i^0 (spacelike infinity)

4. See Mathematica code ($r, t =$const. plotted for $\mu = 1$)

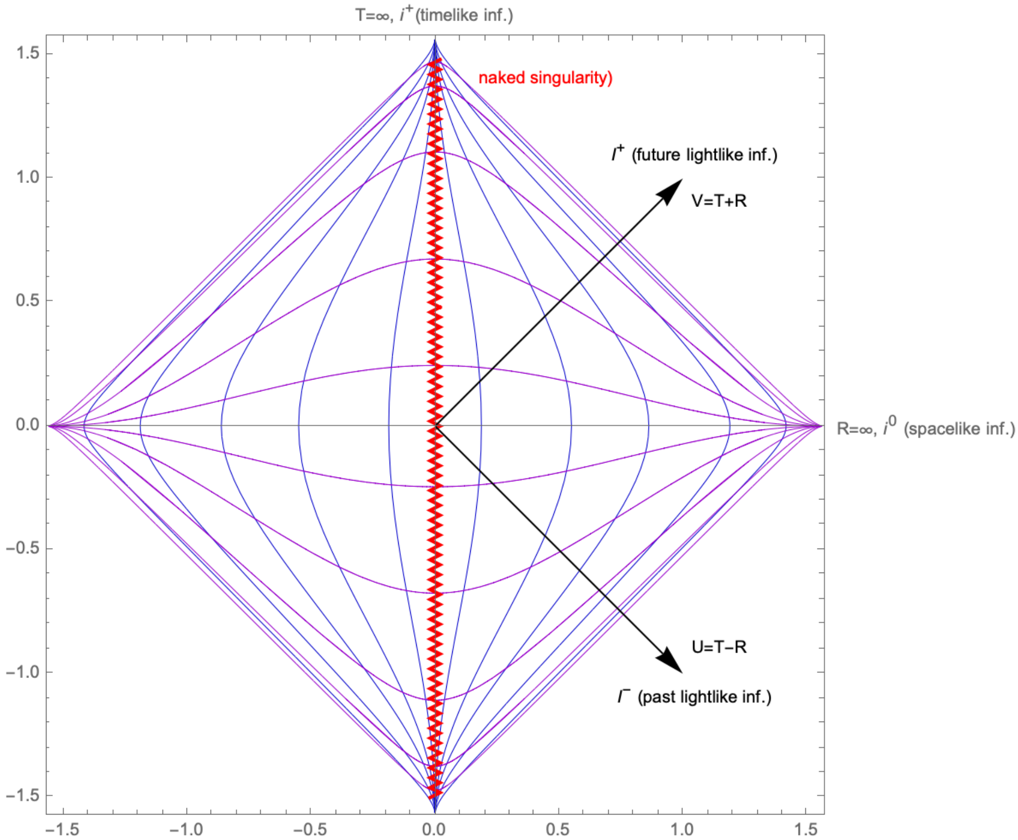

Figure 1: Mathematica calculations