0.1 Feynman, Hibbs - Quantum mechanics and path integrals 2ed

0.1.1 2.1

With $\dot{x} = 0$ and $\dot{x}$ =const we see

$$\begin{aligned} S &= \int_{t_a}^{t_b} L dt && (1)\\ &= \frac{m}{2}\int_{t_a}^{t_b} \dot{x}^2 dt && (2)\\ &= \frac{m}{2}\left[\dot{x}x|_{t_a}^{t_b} - \int_{t_a}^{t_b} x\ddot{x}dt\right] && (3)\\ &= \frac{m}{2}\frac{x_b - x_a}{t_b - t_b}(x_b - x_a) && (4)\\ &= \frac{m}{2}\frac{(x_b - x_a)^2}{t_b - t_b} && (5) \end{aligned}$$

0.1.2 2.2

With the solution of the equation of motion

$$\begin{aligned} \ddot{x} + \omega^2 x = 0 \quad &\rightarrow \quad x = x_0 \sin(\omega t + \varphi_0) = (x_0 \cos\varphi_0)\sin\omega t + (x_0 \sin\varphi_0)\cos\omega t && (6)\\ &\rightarrow \quad \dot{x} = (x_0\omega\cos\varphi_0)\cos\omega t - (x_0\omega\sin\varphi_0)\sin\omega t && (7) \end{aligned}$$

then with (x_a, x_b, t_a, t_b) we can solve for x_0 and φ_0

$$\begin{aligned} x_0\cos\varphi_0 &= \frac{x_a\cos\omega t_b - x_b\cos\omega t_a}{\cos\omega t_b \sin\omega t_a - \cos\omega t_a \sin\omega t_b} && (8)\\ &= \frac{x_a\cos\omega t_b - x_b\cos\omega t_a}{\sin\omega(t_a - t_b)} && (9)\\ x_0\sin\varphi_0 &= -\frac{x_a\frac{\sin\omega t_b}{\sin\omega t_a} - x_b\tan\omega t_a}{-\sin\omega t_b + \cos\omega t_b\tan\omega t_a} && (10)\\ &= \frac{x_b\sin\omega t_a - x_a\sin\omega t_b}{\sin\omega(t_a - t_b)} && (11) \end{aligned}$$

and therefore

$$\begin{aligned} v_a &= \frac{x_a\cos\omega t_b - x_b\cos\omega t_a}{\sin\omega(t_a - t_b)}\sin\omega t_a + \frac{x_b\sin\omega t_a - x_a\sin\omega t_b}{\sin\omega(t_a - t_b)}\sin\omega t_a && (12)\\ &= -\frac{1}{\sin\omega T}\left[(x_a\cos\omega t_b - x_b\cos\omega t_a)\sin\omega t_a + (x_b\sin\omega t_a - x_a\sin\omega t_b)\sin\omega t_a\right] && (13)\\ &= -\frac{1}{\sin\omega T}\left[x_a(\cos\omega t_b\sin\omega t_a - \sin\omega t_a\sin\omega t_b) + x_b(\sin^2\omega t_a - \cos\omega t_a\sin\omega t_a)\right] && (14)\\ v_b &= \frac{x_a\cos\omega t_b - x_b\cos\omega t_a}{\sin\omega(t_a - t_b)}\sin\omega t_b + \frac{x_b\sin\omega t_a - x_a\sin\omega t_b}{\sin\omega(t_a - t_b)}\sin\omega t_b && (15)\\ &= -\frac{1}{\sin\omega T}\left[x_a(\cos\omega t_b\sin\omega t_b - \sin^2\omega t_b) + x_b(\sin\omega t_a\sin\omega t_b - \cos\omega t_a\sin\omega t_b)\right] && (16) \end{aligned}$$

Now we can write

$$
\begin{aligned}
S &= \int_{t_a}^{t_b} L dt & (17)\\
&= \frac{m}{2}\int_{t_a}^{t_b} (\dot{x}^2 - \omega^2 x^2) dt & (18)\\
&= \frac{m}{2} x_0^2\omega^2 \int_{t_a}^{t_b} dt \left(\cos^2(\omega t+\varphi) - \sin^2(\omega t+\varphi)\right) & (19)\\
&= \frac{m}{2} x_0^2\omega^2 \int_{t_a}^{t_b} dt \cos(2[\omega t+\varphi]) & (20)\\
&= \frac{m}{4} x_0^2\omega \sin(2[\omega t+\varphi])|_{t_a}^{t_b} & (21)\\
&= \frac{m}{2} x_0^2\omega \sin(\omega t+\varphi)\cos(\omega t+\varphi)|_{t_a}^{t_b} & (22)\\
&= \frac{m}{2} x\dot{x}|_{t_a}^{t_b} & (23)\\
&= \frac{m}{2}(x_b v_b - x_a v_a) & (24)\\
&= \frac{m\omega}{2\sin\omega T}\left[(x_a^2+x_b^2)\cos\omega T - 2x_a x_b\right] & (25)
\end{aligned}
$$

0.1.3 2.3

$$
m\ddot{x} + f = 0 \qquad \to \qquad x(t) = -\frac{f}{2m}t^2 + v_a t + x_a \tag{26}
$$

then

$$
\begin{aligned}
S &= \int_{t_a}^{t_b} \frac{m}{2}\left(-\frac{f}{m}t\right)^2 - \frac{f^2}{2m}t^2 - f v_a t + f x_a\, dt & (27)\\
&= \int_{t_a}^{t_b} -\frac{f^2}{m}t^2 - f v_a t + f x_a\, dt & (28)\\
&= -\frac{f^2}{3m}(t_b^3 - t_a^3) - \frac{f v_a}{2}(t_b^2 - t_a^2) + f x_a(t_b - t_a) & (29)\\
&= -\frac{f^2}{3m}(t_b^3 - t_a^3) - v_a m(x_b - v_a t_b - x_a - x_a + v_a t_a + x_a) + f x_a(t_b - t_a) & (30)\\
&= -\frac{f^2}{3m}(t_b^3 - t_a^3) - v_a m(x_b - x_a) + v_a^2 m(t_b - t_a) + f x_a(t_b - t_a) & (31)\\
& & (32)
\end{aligned}
$$

0.2 Straumann - Quantenmechanik 2ed

0.2.1 2.1 - Spectral oscillator density

The vanishing electrical field in the surface requires for each standing wave

$$
k_i = \frac{\pi}{L} n_i. \tag{33}
$$

and

$$
\begin{aligned}
k^2 &= k_x^2 + k_y^2 + k_z^2 & (34)\\
\Delta V &= \frac{\pi^3}{L^3}. & (35)
\end{aligned}
$$

With $k = 2\pi/\lambda = \omega/c$ we have $dk = \frac{d\omega}{c}$ and the volume of a sphere in k-space is given by

$$V(k) = \frac{4}{3}\pi k^3 \tag{36}$$

$$dV = 4\pi k^2 dk = 4\pi \frac{\omega^2}{c^2}\frac{d\omega}{c} = 4\pi(2\pi)^3\frac{\nu^2}{c^3}d\nu \tag{37}$$

The number of oscillator are then given by the number of points in the positive quadrant (all k_i positive) time two (polarization)

$$dN(\nu) = 2\frac{V(\nu)/8}{\Delta V} = L^3\frac{8\pi}{c^3}\nu^2 d\nu \tag{38}$$

0.2.2 2.2 - Energy variance of the harmonic oscillator

First we obtain an expression for T

$$E = \frac{h\nu}{e^{h\nu/kT}-1} \quad \rightarrow \quad \frac{h\nu}{kT} = \ln\left(\frac{h\nu}{E}+1\right) \tag{39}$$

which we can use in

$$\frac{dS}{dE} = \frac{1}{T} = \frac{k}{h\nu}\ln\left(\frac{h\nu}{E}+1\right) \tag{40}$$

and take one more derivative

$$\frac{d^2S}{dE^2} = -\frac{k}{h\nu}\frac{\frac{h\nu}{E^2}}{\frac{h\nu}{E}+1} \tag{41}$$

$$= -k\frac{1}{h\nu E + E^2}. \tag{42}$$

Now we see

$$\langle(\Delta E)^2\rangle = E^2 + Eh\nu. \tag{43}$$

0.2.3 3.6 - 1D molecular potential

With the given coordinate transformation we get for the single terms

$$e^{-\alpha x} = \frac{\alpha\hbar\xi}{2\sqrt{2mA}} \tag{44}$$

$$e^{-2\alpha x} = \frac{(\alpha\hbar\xi)^2}{8mA} \tag{45}$$

$$\frac{\partial}{\partial x} = \frac{\partial\xi}{\partial x}\frac{\partial}{\partial\xi} \tag{46}$$

$$= -\alpha\xi\frac{\partial}{\partial\xi} \tag{47}$$

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2\xi}{\partial x^2}\frac{\partial}{\partial\xi} + \left(\frac{\partial\xi}{\partial x}\right)^2\frac{\partial^2}{\partial\xi^2} \tag{48}$$

$$= \alpha^2\xi\frac{\partial}{\partial\xi} + (\alpha\xi)^2\frac{\partial^2}{\partial\xi^2} \tag{49}$$

and combined

$$-\frac{\hbar^2}{2m}\partial_{xx}\psi + A(e^{-2\alpha x} - 2e^{-\alpha x})\psi = E\psi \tag{50}$$

$$-\frac{\hbar^2}{2m}\left(\alpha^2\xi\frac{\partial}{\partial\xi} + (\alpha\xi)^2\frac{\partial^2}{\partial\xi^2}\right)\psi + A\left(\frac{(\alpha\hbar\xi)^2}{8mA} - 2\frac{\alpha\hbar\xi}{2\sqrt{2mA}}\right)\psi = E\psi \tag{51}$$

$$\left(\alpha^2\xi\frac{\partial}{\partial\xi} + (\alpha\xi)^2\frac{\partial^2}{\partial\xi^2}\right)\psi - \frac{2mA}{\hbar^2}\left(\frac{(\alpha\hbar\xi)^2}{8mA} - 2\frac{\alpha\hbar\xi}{2\sqrt{2mA}}\right)\psi = -\frac{2mE}{\hbar^2}\psi \tag{52}$$

$$\left(\frac{1}{\xi}\frac{\partial}{\partial\xi} + \frac{\partial^2}{\partial\xi^2}\right)\psi - \frac{2mA}{\alpha^2\xi^2\hbar^2}\left(\frac{(\alpha\hbar\xi)^2}{8mA} - 2\frac{\alpha\hbar\xi}{2\sqrt{2mA}}\right)\psi = -\frac{2mE}{\hbar^2\alpha^2\xi^2}\psi \tag{53}$$

$$\left(\frac{1}{\xi}\frac{\partial}{\partial\xi} + \frac{\partial^2}{\partial\xi^2}\right)\psi + \left(-\frac{1}{4} + \frac{\sqrt{2mA}}{\alpha\hbar\xi}\right)\psi = -\frac{2mE}{\hbar^2\alpha^2\xi^2}\psi \tag{54}$$

$$\left(\frac{1}{\xi}\frac{\partial}{\partial\xi} + \frac{\partial^2}{\partial\xi^2}\right)\psi + \left(-\frac{1}{4} + \frac{n+s+\frac{1}{2}}{\xi}\right)\psi = \frac{s^2}{\xi^2}\psi \tag{55}$$

$$\left(\frac{\partial^2}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial}{\partial\xi}\right)\psi + \left(-\frac{1}{4} + \frac{n+s+\frac{1}{2}}{\xi} - \frac{s^2}{\xi^2}\right)\psi = 0. \tag{56}$$

The units of ξ is $\sqrt{\mathrm{kg}\cdot\mathrm{J}}/\mathrm{m}^{-1}\mathrm{Js} = 1$ so ξ in dimensionless.

1. Case $\xi \gg 1$ ($x \to -\infty$) Dropping all $1/\xi$ terms

$$\psi'' - \frac{1}{4}\psi = 0 \quad \to \quad \psi = c_1 e^{\xi/2} + c_2 e^{-\xi/2} \tag{57}$$

2. Case $0 < \xi \ll 1$ ($x \to +\infty$) Ansatz $\psi \sim \xi^m$

$$m(m-1)\xi^{m-2} + m\xi^{m-2} - \frac{1}{4}\xi^m + \left(n+s+\frac{1}{2}\right)\xi^{m-1} - s^2\xi^{m-2} = 0 \tag{58}$$

$$\left[(m^2 - s^2) - \frac{1}{4}\xi^2 + \left(n+s+\frac{1}{2}\right)\xi\right]\xi^{m-2} = 0 \tag{59}$$

which for small ξ becomes

$$(m^2 - s^2)\xi^{m-2} = 0 \quad \to \quad \psi = \xi^{\pm s} \tag{60}$$

With the two asymptotics we can make a physically sensible ansatz for a full solutions $\psi = \xi^s e^{-\xi/2} u(\xi)$ which leads to

$$\xi u'' + (2s + 1 - \xi)u' + nu = 0 \tag{61}$$

To solve this equation we use the Sommerfeld polynomial method

$$u = \sum_k a_k\xi^k \quad \to \quad \sum_k k(k-1)a_k\xi^{k-1} + (2s+1)ka_k\xi^{k-1} - ka_k\xi^k + na_k\xi^k = 0 \tag{62}$$

$$\sum_k (k+1)ka_{k+1}\xi^k + (2s+1)(k+1)a_{k+1}\xi^k - ka_k\xi^k + na_k\xi^k = 0 \tag{63}$$

$$a_{k+1} = \frac{k-n}{(k+1)(2s+1+k)}a_k. \tag{64}$$

The requirement for the series to cut off (making u a finite order polynomial) is $n_k = k$. The energies of the bound states are therefore

$$E_k = -\frac{\alpha^2\hbar^2}{2m}s_k^2 \tag{65}$$

$$= -\frac{\alpha^2\hbar^2}{2m}\left[\frac{\sqrt{2mA}}{\alpha\hbar} - (k+1/2)\right]^2 \tag{66}$$

$$= -A\left[1 - \frac{\alpha\hbar}{\sqrt{2mA}}(k+1/2)\right]^2 \tag{67}$$

where the only valid k are the ones where E_k is in $[-A, 0]$.

0.3 Schwinger - Quantum Mechanics Symbolism of Atomic Measurements

0.3.1 2.1

Observe

$$\int_{-\infty}^{\infty}\left(\theta(x+a)+\theta(a-x)\right)e^{ikx}dx = \int_{-a}^{a}e^{ikx}dx \tag{68}$$

$$= \frac{1}{ik}\left(e^{ika} - e^{-ika}\right) \tag{69}$$

$$= 2a\frac{\sin ka}{ka} \tag{70}$$

$$\lim_{P\to\infty}\int_{-\infty}^{\infty}\frac{d\chi}{\pi}\frac{\sin\chi}{\chi}e^{ik\left(q'+\frac{\chi}{P}\right)} = \frac{1}{\pi}e^{ikq'}\lim_{P\to\infty}\int_{-\infty}^{\infty}d\chi\frac{\sin\chi}{\chi}e^{i\frac{k}{P}\chi} \tag{71}$$

0.4 Weinberg - Quantum Mechanics 2nd edition

0.4.1 1.1

- The solution of for a free particle in the interval $-a < x < a$ is given by

$$\left[-\frac{\hbar^2}{2M}\frac{d^2}{dx^2} - E\right]\phi = 0 \tag{72}$$

$$\left[\frac{d^2}{dx^2} + \frac{2ME}{\hbar^2}\right]\phi = 0 \tag{73}$$

$$\to \phi = A\sin\left(\frac{\sqrt{2ME}}{\hbar}x\right) + B\cos\left(\frac{\sqrt{2ME}}{\hbar}x\right) \tag{74}$$

  with the two boundary conditions

$$A\sin\left(\frac{\sqrt{2ME}}{\hbar}(-a)\right) + B\cos\left(\frac{\sqrt{2ME}}{\hbar}(-a)\right) = 0 \tag{75}$$

$$A\sin\left(\frac{\sqrt{2ME}}{\hbar}a\right) + B\cos\left(\frac{\sqrt{2ME}}{\hbar}a\right) = 0. \tag{76}$$

The possible energy eigenvalues are therefore

$$A=0, \quad \frac{\sqrt{2ME_{2n+1}}}{\hbar}a=(2n+1)\frac{\pi}{2} \quad \rightarrow \quad E_{2n+1}=\frac{\pi^2\hbar^2}{8Ma^2}(2n+1)^2 \tag{77}$$
$$\rightarrow \quad \phi=\frac{1}{\sqrt{a}}\cos\left(x\frac{\pi}{2a}(2n+1)\right) \tag{78}$$

$$B=0, \quad \frac{\sqrt{2ME_{2n}}}{\hbar}a=2n\frac{\pi}{2} \quad \rightarrow \quad E_{2n}=\frac{\pi^2\hbar^2}{8Ma^2}(2n)^2 \tag{79}$$
$$\rightarrow \quad \phi=\frac{1}{\sqrt{a}}\sin\left(x\frac{\pi}{2a}(2n)\right) \tag{80}$$

where we calculated the normalization via

$$\int_{-a}^{a}\sin^2(kx)dx=\int_{-a}^{a}(1-\cos^2(kx))dx \tag{81}$$
$$=2a-\int_{-a}^{a}\cos^2(kx)dx \quad \rightarrow \int_{-a}^{a}\sin^2(kx)dx=a. \tag{82}$$

- Lets first calculate the normalization

$$\int_{-a}^{a}(a^2-x^2)^2dx = a^4x-2a^2\frac{x^3}{3}+\frac{x^5}{5}\Big|_{-a}^{a} \tag{83}$$
$$=a^4(2a)-\frac{2}{3}a^2(16a^3)+\frac{1}{5}(64a^5) \tag{84}$$
$$=\left(2-\frac{4}{3}+\frac{2}{5}\right)a^5=\frac{16}{15}a^5 \tag{85}$$

and then obtain

$$\int_{-a}^{a}\frac{1}{\sqrt{\frac{16a^5}{15}}}\left(a^2-x^2\right)\frac{1}{\sqrt{a}}\cos\left(\frac{\pi x}{2a}\right)dx=\frac{8\sqrt{15}}{\pi^3} \tag{86}$$

0.4.2 1.2

- We can write the Hamiltonian as

$$H=\frac{\vec{P}^2}{2M}+\frac{M\omega_0^2}{2}\vec{X}^2 \tag{87}$$
$$=\sum_{k=1}^{3}\frac{p_k^2}{2M}+\frac{M\omega_0^2}{2}x_k^2 \tag{88}$$

the energy is therefore given by

$$E_{n_1,n_2,n_3}=\hbar\omega_0\left(n_1+n_2+n_3+\frac{3}{2}\right) \tag{89}$$
$$N_{n=n_1+n_2+n_3}=\sum_{k=0}^{n}(k+1) \tag{90}$$
$$=\frac{n(n+1)}{2}+n+1 \tag{91}$$
$$=\frac{(n+1)(n+2)}{2} \tag{92}$$

- With (1.4.5), (1.4.15) and $\omega_{01} = \omega_0$ we have

$$\vec{x}]_{01} = e^{i\omega_0 t}\sqrt{\frac{\hbar}{2M\omega_0}} \tag{93}$$

$$A_{n=1}^{n=0} = \frac{4e^2\omega_0^3}{3c^3\hbar}\,|[\vec{x}]_{01}|^2 \tag{94}$$

$$= \frac{2e^2\omega_0^2}{3c^3 M} \tag{95}$$

where with (1.4.15).

0.5 Hannabuss - An Introduction to Quantum Theory

0.5.1 Problem 12.2 - Harmonic oscillator with x^4 perturbation

With

$$[a, a^\dagger] = aa^\dagger - a^\dagger a = 1 \tag{96}$$

$$[a, x] = \sqrt{\frac{\hbar}{2m\omega}}[a, a + a^\dagger] = \sqrt{\frac{\hbar}{2m\omega}}\left([a, a] + [a, a^\dagger]\right) = \sqrt{\frac{\hbar}{2m\omega}}[a, a^\dagger] = \sqrt{\frac{\hbar}{2m\omega}} \tag{97}$$

$$[a^n, x] = ... = \sqrt{\frac{\hbar}{2m\omega}}[a^n, a^\dagger] = \sqrt{\frac{\hbar}{2m\omega}}(a^n a^\dagger - a^\dagger a^n) = \sqrt{\frac{\hbar}{2m\omega}}(a^n a^\dagger - (a^\dagger a)a^{n-1}) \tag{98}$$

$$= \sqrt{\frac{\hbar}{2m\omega}}(a^n a^\dagger - (aa^\dagger - 1)a^{n-1}) \tag{99}$$

$$= \sqrt{\frac{\hbar}{2m\omega}}(a^n a^\dagger + a^{n-1} - aa^\dagger a^{n-1}) \tag{100}$$

$$= \sqrt{\frac{\hbar}{2m\omega}}(a^n a^\dagger + a^{n-1} - a(aa^\dagger - 1)a^{n-2}) \tag{101}$$

$$= ... = \sqrt{\frac{\hbar}{2m\omega}}na^{n-1} \tag{102}$$

the first order energy perturbation can be written as

$$\Delta E_n^{(1)} = \langle\psi_n^{(0)}|H_1|\psi_n^{(0)}\rangle \tag{103}$$

$$= \frac{1}{n!}\langle 0|a^n x^4 (a^\dagger)^n|0\rangle \tag{104}$$

$$= \frac{1}{n!}\langle 0|\left(xa^n + \sqrt{\frac{\hbar}{2m\omega}}na^{n-1}\right)x^3(a^\dagger)^n|0\rangle \tag{105}$$

$$= \frac{1}{n!}\langle 0|xa^n x^3(a^\dagger)^n|0\rangle + \frac{n}{n!}\sqrt{\frac{\hbar}{2m\omega}}\langle 0|a^{n-1}x^3(a^\dagger)^n|0\rangle \tag{106}$$

$$= \frac{1}{n!}\langle 0|x\left(xa^n + \sqrt{\frac{\hbar}{2m\omega}}na^{n-1}\right)x^2(a^\dagger)^n|0\rangle + \frac{n}{n!}\sqrt{\frac{\hbar}{2m\omega}}\langle 0|\left(xa^{n-1} + \sqrt{\frac{\hbar}{2m\omega}}(n-1)a^{n-2}\right)x^2(a^\dagger)^n|0\rangle \tag{107}$$

$$= \frac{1}{n!}\langle 0|x^2 a^n x^2(a^\dagger)^n|0\rangle + 2\frac{n}{n!}\sqrt{\frac{\hbar}{2m\omega}}\langle 0|xa^{n-1}x^2(a^\dagger)^n|0\rangle + \frac{n(n-1)}{n!}\sqrt{\frac{\hbar}{2m\omega}}^2\langle 0|a^{n-2}x^2(a^\dagger)^n|0\rangle \tag{108}$$

$$\tag{109}$$

$$= \frac{1}{n!}\langle 0|x^3 a^n x(a^\dagger)^n|0\rangle + 3\frac{n}{n!}\sqrt{\frac{\hbar}{2m\omega}}\langle 0|x^2 a^{n-1} x(a^\dagger)^n|0\rangle + 3\frac{n(n-1)}{n!}\sqrt{\frac{\hbar}{2m\omega}}^2 \langle 0|x a^{n-2} x(a^\dagger)^n|0\rangle \tag{110}$$

$$+ \frac{n(n-1)(n-2)}{n!}\sqrt{\frac{\hbar}{2m\omega}}^3 \langle 0|a^{n-3} x(a^\dagger)^n|0\rangle \tag{111}$$

$$= \frac{1}{n!}\langle 0|x^4 a^n (a^\dagger)^n|0\rangle + 4\frac{n}{n!}\sqrt{\frac{\hbar}{2m\omega}}\langle 0|x^3 a^{n-1}(a^\dagger)^n|0\rangle + 6\frac{n(n-1)}{n!}\sqrt{\frac{\hbar}{2m\omega}}^2 \langle 0|x^2 a^{n-2}(a^\dagger)^n|0\rangle \tag{112}$$

$$+ 4\frac{n(n-1)(n-2)}{n!}\sqrt{\frac{\hbar}{2m\omega}}^3 \langle 0|x a^{n-3}(a^\dagger)^n|0\rangle + \frac{n(n-1)(n-2)(n-3)}{n!}\sqrt{\frac{\hbar}{2m\omega}}^4 \langle 0|a^{n-4}(a^\dagger)^n|0\rangle \tag{113}$$

$$= \langle 0|x^4|0\rangle + \frac{4n}{\sqrt{1!}}\sqrt{\frac{\hbar}{2m\omega}}\langle 0|x^3|1\rangle + \frac{6n(n-1)}{\sqrt{2!}}\sqrt{\frac{\hbar}{2m\omega}}^2 \langle 0|x^2|2\rangle \tag{114}$$

$$+ \frac{4n(n-1)(n-2)}{\sqrt{3!}}\sqrt{\frac{\hbar}{2m\omega}}^3 \langle 0|x|3\rangle + \frac{n(n-1)(n-2)(n-3)}{\sqrt{4!}}\sqrt{\frac{\hbar}{2m\omega}}^4 \langle 0|4\rangle \tag{115}$$

where we used $\frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle = |n\rangle$ and $\frac{\sqrt{k!}}{\sqrt{n!}}a^{n-k}|n\rangle = |k\rangle$. Using additionally information about the unperturbed solution

$$H_0(y) = 1 \tag{116}$$

$$H_1(y) = 2y \tag{117}$$

$$H_2(y) = 4y^2 - 2 \tag{118}$$

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) e^{-m\omega x^2/2\hbar} \tag{119}$$

we can rewrite

$$x^2|0\rangle \simeq \sqrt{\frac{\hbar}{m\omega}}^2 \left(\sqrt{\frac{m\omega}{\hbar}}^2 x^2\right) \frac{1}{\sqrt{2^0 0!}} H_0(\sqrt{\frac{m\omega}{\hbar}}x) \tag{120}$$

$$= \sqrt{\frac{\hbar}{m\omega}}^2 \left(\frac{1}{4}H_2(\sqrt{\frac{m\omega}{\hbar}}x) + \frac{1}{2}H_0(\sqrt{\frac{m\omega}{\hbar}}x)\right) \underbrace{\frac{1}{\sqrt{2^0 0!}} H_0(\sqrt{\frac{m\omega}{\hbar}}x)}_{=1} \tag{121}$$

$$= \sqrt{\frac{\hbar}{m\omega}}^2 \left[\frac{\sqrt{2^2 2!}}{4}\frac{1}{\sqrt{2^2 2!}}H_2(\sqrt{\frac{m\omega}{\hbar}}x) + \frac{1}{2}\frac{1}{\sqrt{2^0 0!}}H_0(\sqrt{\frac{m\omega}{\hbar}}x)\right] \tag{122}$$

$$= \sqrt{\frac{\hbar}{m\omega}}^2 \left[\frac{\sqrt{2}}{2}|2\rangle + \frac{1}{2}|0\rangle\right] \tag{123}$$

and

$$x|1\rangle \simeq \sqrt{\frac{\hbar}{m\omega}} \left(\sqrt{\frac{m\omega}{\hbar}}x\right) \frac{1}{\sqrt{2^1 1!}} H_1(\sqrt{\frac{m\omega}{\hbar}}x) \tag{124}$$

$$= \sqrt{\frac{\hbar}{m\omega}} \frac{1}{\sqrt{2^1 1!}} \left(\frac{1}{2}H_2(\sqrt{\frac{m\omega}{\hbar}}x) + H_0(\sqrt{\frac{m\omega}{\hbar}}x)\right) \tag{125}$$

$$= \sqrt{\frac{\hbar}{m\omega}} \left(\frac{1}{2\sqrt{2}}\sqrt{2^2 2!}\frac{1}{\sqrt{2^2 2!}}H_2(\sqrt{\frac{m\omega}{\hbar}}x) + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2^0 0!}}H_0(\sqrt{\frac{m\omega}{\hbar}}x)\right) \tag{126}$$

$$= \sqrt{\frac{\hbar}{m\omega}} \left(|2\rangle + \frac{1}{\sqrt{2}}|0\rangle\right) \tag{127}$$

then with $\langle m|n\rangle = \delta_{mn}$

$$\langle 0|x^4|0\rangle = \langle 0|x^2 \cdot x^2|0\rangle = \sqrt{\frac{\hbar}{m\omega}}^4 \left(\frac{2}{4} + \frac{1}{4}\right) = \frac{3}{4}\frac{\hbar^2}{m^2\omega^2} \tag{128}$$

$$\langle 0|x^3|1\rangle = \langle 0|x^2 \cdot x|1\rangle = \sqrt{\frac{\hbar}{m\omega}}^3 \left(\frac{\sqrt{2}}{2} + \frac{1}{2\sqrt{2}}\right) = \frac{3}{2\sqrt{2}}\frac{\hbar}{m\omega}\sqrt{\frac{\hbar}{m\omega}} \tag{129}$$

$$\langle 0|x^2|2\rangle = \langle 0|x^2 \cdot 1|2\rangle = \sqrt{\frac{\hbar}{m\omega}}^2 \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}\frac{\hbar}{m\omega} \tag{130}$$

$$\langle 0|x|3\rangle = 0 \tag{131}$$

$$\langle 0|4\rangle = 0 \tag{132}$$

we obtain

$$\Delta E_n^{(1)} = \frac{3}{4}\frac{\hbar^2}{m^2\omega^2} + 4n\sqrt{\frac{\hbar}{2m\omega}}\frac{3}{2\sqrt{2}}\frac{\hbar}{m\omega}\sqrt{\frac{\hbar}{m\omega}} + \frac{6n(n-1)}{\sqrt{2!}}\sqrt{\frac{\hbar}{2m\omega}}^2\frac{\sqrt{2}}{2}\frac{\hbar}{m\omega} + 0 + 0 \tag{133}$$

$$= \frac{3\hbar^2}{4m^2\omega^2}\left(1 + 2n + 2n^2\right) \tag{134}$$

0.5.2 Problem 12.3 - Harmonic oscillator with other perturbations

(i) Calculating the first order energy correction using $x = \sqrt{\hbar/2m\omega}(a + a^\dagger)$

$$\Delta E_n^{(1)} = \langle \psi_n^{(0)}|x|\psi_n^{(0)}\rangle \tag{135}$$

$$= \sqrt{\hbar/2m\omega}\langle \psi_n^{(0)}|a + a^\dagger|\psi_n^{(0)}\rangle \tag{136}$$

$$= \sqrt{\hbar/2m\omega}\langle n|a + a^\dagger|n\rangle \tag{137}$$

$$= \sqrt{\hbar/2m\omega}\left(\sqrt{n}\langle n-1|n\rangle + \sqrt{n+1}\langle n|n+1\rangle\right) \tag{138}$$

$$= 0 \tag{139}$$

Calculating the second order energy correction

$$\Delta E_n^{(2)} = \sum_{k\neq n}\frac{|\langle k^{(0)}|x|n^{(0)}\rangle|^2}{E_n^{(0)} - E_k^{(0)}} \tag{140}$$

$$= \sqrt{\hbar/2m\omega}\sum_{k\neq n}\frac{|\langle k^{(0)}|a + a^\dagger|n^{(0)}\rangle|^2}{(n-k)\hbar\omega} \tag{141}$$

$$= \sqrt{\hbar/2m\omega}\sum_{k\neq n}\frac{|\sqrt{k}\langle (k-1)^{(0)}|n^{(0)}\rangle + \sqrt{n+1}\langle k^{(0)}|(n+1)^{(0)}\rangle|^2}{(n-k)\hbar\omega} \tag{142}$$

$$= \sqrt{\hbar/2m\omega}\sum_{k\neq n}\frac{|\sqrt{k}\delta_{k-1,n} + \sqrt{n+1}\delta_{k,n+1}|^2}{(n-k)\hbar\omega} \tag{143}$$

$$= \sqrt{\hbar/2m\omega}\sum_{k\neq n}\frac{k\delta_{k-1,n} + 2\sqrt{k(n+1)}\delta_{k-1,n}\delta_{k,n+1} + (n+1)\delta_{k,n+1}}{(n-k)\hbar\omega} \tag{144}$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\frac{n+1}{[n-(n+1)]\hbar\omega} + \frac{2\sqrt{(n+1)(n+1)}}{[n-(n+1)]\hbar\omega} + \frac{n+1}{[n-(n+1)]\hbar\omega}\right) \tag{145}$$

$$= \sqrt{\frac{1}{2m\hbar\omega^3}}\left(-(n+1) - 2(n+1) - (n+1)\right) \tag{146}$$

$$= -4(n+1)\sqrt{\frac{1}{2m\hbar\omega^3}} \tag{147}$$

(ii)

0.6 Schwabl - Quantum Mechanics 4th ed

0.6.1 Problem 17.1 - 3d Harmonic oscillator

(a) Represent the 3d oscillator by three 1d oscillators

$$H = \frac{\mathbf{p}^2}{2m} + \frac{m\omega^2}{2}\mathbf{x}^2 \tag{148}$$
$$= \frac{p_x^2 + p_y^2 + p_z^2}{2m} + \frac{m\omega^2}{2}(x^2 + y^2 + z^2) \tag{149}$$
$$= \sum_k^3 \frac{p_k^2}{2m} + \frac{m\omega^2}{2}x_k^2 \tag{150}$$
$$= \hbar\omega \sum_k^3 \left(a_k^\dagger a_k + \frac{1}{2}\right) \tag{151}$$
$$= \hbar\omega \sum_k^3 \left(n_k + \frac{1}{2}\right) \tag{152}$$
$$\rightarrow E = \hbar\omega \left(n_x + n_y + n_y + \frac{3}{2}\right) \tag{153}$$

level	1	2	3	4	...	N
energy	3/2	5/2	7/2	9/2	...	$3/2 + N$
multi	1	3	6	10	...	$N(N+1)/2$

The eigenfunctions are then

$$\psi(\mathbf{x}) = \psi_{n_x}(x)\psi_{n_y}(y)\psi_{n_z}(z) \tag{154}$$

(b)

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} - \frac{2m[V(r) - E]}{\hbar^2}\right) R(r) = 0 \tag{155}$$
$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} + \frac{2mE}{\hbar^2} - \frac{m^2\omega^2}{\hbar^2}r^2\right) R(r) = 0 \tag{156}$$

For the asymptotics $r \rightarrow 0$ we set $R(r) = u(r)/r$ and obtain

$$\left(\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2}\right) u(r) = 0 \tag{157}$$

assuming $E - V(r)$ is small compared to the $1/r^2$. This gives

$$u(r) = Ar^{l+1} + Br^{-l} \tag{158}$$
$$\rightarrow u(r) = Ar^{l+1} \tag{159}$$

We therefore guess the solution as $R(r) \sim r^l e^{-\alpha r^2}(a_0 + a_1 r + a_2 r^2 + ...) = r^l e^{-\alpha r^2} f(r)$ and substitute into the ODE obtaining a system of algebraic equations for the a_i and E. For the lowed energy levels we obtain

$$l=0 \quad R(r)=e^{-\frac{m\omega}{2\hbar}r^2} \quad \rightarrow \quad E=\frac{3}{2}\hbar\omega \tag{160}$$

$$R(r)=e^{-\frac{m\omega}{2\hbar}r^2}\left(1-\frac{2m\omega r^2}{3\hbar}\right) \quad \rightarrow \quad E=\frac{7}{2}\hbar\omega \tag{161}$$

$$l=1 \quad R(r)=e^{-\frac{m\omega}{2\hbar}r^2}r \quad \rightarrow \quad E=\frac{5}{2}\hbar\omega \tag{162}$$

$$l=2 \quad R(r)=e^{-\frac{m\omega}{2\hbar}r^2}r^2 \quad \rightarrow \quad E=\frac{7}{2}\hbar\omega \tag{163}$$

Making the calculation more robust we insert a full series expansion $f(r)=\sum_k a_k r^k$ into the radial equation

$$\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{l(l+1)}{r^2}+\frac{2mE}{\hbar^2}-\frac{m^2\omega^2}{\hbar^2}r^2\right)R(r)=0$$

$$rf''+2(1+l-2\alpha r^2)f'-r\left(-\frac{2mE}{\hbar^2}+\alpha(3+2l-2\alpha r^2)+\frac{m^2\omega^2}{\hbar^2}\right)f=0$$

$$f''+2\frac{1+l-2\alpha r^2}{r}f'-\left(-\frac{2mE}{\hbar^2}+\alpha(3+2l-2\alpha r^2)+\frac{m^2\omega^2}{\hbar^2}\right)f=0$$

$$\sum_k\left[k(k-1)a_k+2(1+l-2\alpha r^2)ka_k-\left(-\frac{2mE}{\hbar^2}+\alpha(3+2l-2\alpha r^2)+\frac{m^2\omega^2}{\hbar^2}\right)a_k r^2\right]r^{k-2}=0$$

$$\sum_k\left[k(k-1)a_k+2(1+l)ka_k-2\alpha(k-2)a_{k-2}-\frac{m(m\omega^2-2E)}{\hbar^2}a_{k-2}+\alpha(3+2l)a_{k-2}-2\alpha^2r^2a_kr^2\right]r^{k-2}=0$$

0.6.2 Problem 17.2 - Delta-shell potential

With

$$y=r/a \tag{164}$$

$$\frac{d}{dr}=\frac{\partial y}{\partial r}\frac{d}{dy}=\frac{1}{a}\frac{d}{dy} \tag{165}$$

$$\frac{d^2}{dr^2}=\frac{d}{dr}\left(\frac{1}{a}\frac{d}{dy}\right)=\frac{1}{a^2}\frac{d}{dy} \tag{166}$$

we can rewrite

$$\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{l(l+1)}{r^2}-\frac{2m[V(r)-E]}{\hbar^2}\right)R(r)=0 \tag{167}$$

$$\left(\frac{1}{a^2}\frac{d^2}{dy^2}+\frac{2}{ya}\frac{1}{a}\frac{d}{dy}-\frac{l(l+1)}{y^2a^2}-\frac{2m}{\hbar^2}\left[-\lambda\frac{\hbar^2}{2m}\delta(r-a)\right]+\frac{2mE}{\hbar^2}\right)R(r)=0 \tag{168}$$

$$\left(\frac{1}{a^2}\frac{d^2}{dy^2}+\frac{2}{ya}\frac{1}{a}\frac{d}{dy}-\frac{l(l+1)}{y^2a^2}+\lambda\delta(r-a)+\frac{2mE}{\hbar^2}\right)R(r)=0 \tag{169}$$

$$\left(\frac{d^2}{dy^2}+\frac{2}{y}\frac{d}{dy}-\frac{l(l+1)}{y^2}+ga\delta(r-a)+\frac{2ma^2E}{\hbar^2}\right)R(r)=0 \tag{170}$$

and see

$$y\neq 1 \quad \left(\frac{d^2}{dy^2}+\frac{2}{y}\frac{d}{dy}-\frac{l(l+1)}{y^2}+ak^2\right)R(y)=0 \tag{171}$$

$$k^2=g+\frac{2maE}{\hbar^2} \tag{172}$$

Independent solutions

$$R(y) = Aj_l(y\sqrt{ka}) + By_l(y\sqrt{ka}) \tag{173}$$

Here the requirements for the wavefunction

- regular at the origin with $R(r) \sim r^l$
- continuous (not differentiable) at $r = a$ (or $y = 1$)
- jump of the first derivative of ga
- exponential decay outside to ensure normalizability

and here a quick overview of the two functions and a special linear combination

$$\begin{aligned}
&j_l(x) = (-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\sin x}{x} && y_l(x) = -(-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\cos x}{x} && h_0^{(1)}(x) = j_l(ix) + iy_l(ix) \\
&j_0(x) = \frac{\sin x}{x} && y_0(x) = -\frac{\cos x}{x} && h_0^{(1)}(x) = -\frac{e^{-x}}{x} \\
&j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} && y_1(x) = -\frac{\cos x}{x} - \frac{\sin x}{x} && h_1^{(1)}(x) = i(1+x)\frac{e^{-x}}{x^2} \\
&J_2(x) = ... && y_l(x) = ... && h_2^{(1)}(x) = (x^2 + 3x + 3)\frac{e^{-x}}{x^3}
\end{aligned}$$

We see that j_l is suitable for the inside and $h_l^{(1)}$ for the outside.

$$R(\rho) = \begin{cases} Aj_l(\rho) & r < a \\ Ch_l^{(1)}(\rho) & r > a \end{cases} \tag{174}$$

0.7 Shankar - Modern Quantum Mechanics 3rd ed

0.7.1 13.3.1 Pion rest energy

Remebering Yukawa potential and fixing units in the exponential

$$V(r) \sim \frac{e^{-mr}}{r} = \frac{e^{-\frac{mcr}{\hbar}}}{r} \tag{175}$$

Range is given by

$$\frac{m_\pi c d_\pi}{\hbar} \sim 1 \tag{176}$$

$$\rightarrow m_\pi = \frac{\hbar}{c d_\pi} = 200\text{MeV} \tag{177}$$

0.7.2 13.3.2 de Broglie wavelength

With $E = \frac{p^2}{2m} = qU$ we have

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mqU}} = 0.86\text{Å} \tag{178}$$

0.7.3 13.3.3 Balmer and Lyman lines in sun spectrum

$$E_2 - E_1 = \frac{1}{2}mc^2\alpha^2\left(\frac{1}{1^2} - \frac{1}{2^2}\right) \tag{179}$$

$$= \frac{3}{8}mc^2\alpha^2 \tag{180}$$

$$= 10.2\text{eV} \tag{181}$$

$$\frac{E_2 - E_1}{kT_{6,000K}} = \frac{10.2}{20\frac{1}{40}} = 20.4 \qquad \to \frac{P(n=2)}{P(n=1)} = 5.5\cdot 10^{-9} \tag{182}$$

$$\frac{E_2 - E_1}{kT_{100,000K}} = \frac{10.2}{333\frac{1}{40}} = 1.2 \qquad \to \frac{P(n=2)}{P(n=1)} = 1.2 \tag{183}$$

0.7.4 13.3.4 Energy levels of multi-electron atoms - NOT DONE YET

We always remember

$$E_n = \frac{1}{2}mc^2\frac{(\alpha Z)^2}{n^2} \tag{184}$$

Justification - Virial theorem $E_{kin} \sim E_{pot}$

$$E_n = \langle n|H|n\rangle \sim \langle n|V_C|n\rangle \sim \langle n|\frac{Ze^2}{r}|n\rangle \tag{185}$$

0.8 Zettili - Quantum Mechanics - Concepts and Applications 2nd ed

0.9 Banks - Quantum Mechanics

0.9.1 Exercise 13.1 - Cubic and Quartic perturbed harmonic oscillator

We split the Hamiltonian and see

$$H = H_0 + a\left(X^3 + \frac{b}{a}X^4\right) \tag{186}$$

$$E_n^{(0)} = \hbar\omega\left(n + \frac{1}{2}\right) \tag{187}$$

$$|n^{(0)}\rangle = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) \tag{188}$$

then

$$E_n = E_n^{(0)} + a\langle n^{(0)}|X^3 + \frac{b}{a}X^4|n^{(0)}\rangle + a^2\sum_{k\neq n}\frac{|\langle k^{(0)}|X^3 + \frac{b}{a}X^4|n^{(0)}\rangle|^2}{E_n^{(0)} - E_k^{(0)}} \tag{189}$$

we can use the identities for the Hermite polynomials

$$xH_n(x) = nH_{n-1}(x) + \frac{1}{2}H_{n+1}(x) \tag{190}$$

$$x^2H_n(x) = n(n-1)H_{n-2}(x) + \frac{2n+1}{2}H_n(x) + \frac{1}{4}H_{n+2}(x) \tag{191}$$

$$x^3H_n(x) = n(n-1)(n-2)H_{n-3}(x) + \left(\frac{n(n-1)}{2} + \frac{(2n+1)n}{2}\right)H_{n-1} + \left(\frac{2n+1}{4} + \frac{n+2}{4}\right)H_{n+1}(x) + \frac{1}{8}H_{n+3}(x) \tag{192}$$

$$= n(n-1)(n-2)H_{n-3}(x) + \frac{3n^2}{2}H_{n-1}(x) + 3\frac{n+1}{4}H_{n+1}(x) + \frac{1}{8}H_{n+3}(x) \tag{193}$$

$$x^4H_n(x) = n(n-1)(n-2)(n-3)H_{n-4}(x) + (2n^2-3n+1)nH_{n-2}(x) + \frac{3}{4}(2n^2+2n+1)H_n(x) \tag{194}$$

$$+\frac{1}{4}(2n+3)H_{n+2}(x) + \frac{1}{16}H_{n+4}(x) \tag{195}$$

and see

$$x^3|n^{(0)}\rangle = x^3\frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-\frac{m\omega x^2}{2\hbar}}H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) \tag{196}$$

$$= \left(\sqrt{\frac{\hbar}{m\omega}}x\right)^3\frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-\frac{m\omega x^2}{2\hbar}}\left(\sqrt{\frac{m\omega}{\hbar}}x\right)^3H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) \tag{197}$$

$$= \left(\sqrt{\frac{\hbar}{m\omega}}\right)^3\frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-\frac{m\omega x^2}{2\hbar}}\Bigg[n(n-1)(n-2)H_{n-3}(\sqrt{\frac{m\omega}{\hbar}}x) \tag{198}$$

$$+\frac{3n^2}{2}H_{n-1}(\sqrt{\frac{m\omega}{\hbar}}x) + 3\frac{n+1}{4}H_{n+1}(\sqrt{\frac{m\omega}{\hbar}}x) + \frac{1}{8}H_{n+3}(\sqrt{\frac{m\omega}{\hbar}}x)\Bigg] \tag{199}$$

$$= \left(\sqrt{\frac{\hbar}{m\omega}}\right)^3\Bigg[n(n-1)(n-2)\frac{\sqrt{2^{n-3}(n-3)!}}{\sqrt{2^n n!}}|(n-3)^{(0)}\rangle + \frac{3n^2}{2}\frac{\sqrt{2^{n-1}(n-1)!}}{\sqrt{2^n n!}}|(n-1)^{(0)}\rangle \tag{200}$$

$$+\ \frac{3n+1}{4}\frac{\sqrt{2^{n+1}(n+1)!}}{\sqrt{2^n n!}}|(n-1)^{(0)}\rangle + \frac{1}{8}\frac{\sqrt{2^{n+1}(n+1)!}}{\sqrt{2^n n!}}|(n+1)^{(0)}\rangle\Bigg] \tag{201}$$

$$x^4|n^{(0)}\rangle = \frac{\hbar^2}{m^2\omega^2}\Bigg[n(n-1)(n-2)(n-3)\frac{\sqrt{2^{n-4}(n-4)!}}{\sqrt{2^n n!}}|(n-4)^{(0)}\rangle + (2n^2-3n+1)n\frac{\sqrt{2^{n-2}(n-2)!}}{\sqrt{2^n n!}}|(n-2)^{(0)}\rangle \tag{202}$$

$$+\ \frac{3}{4}(2n^2+2n+1)\||n^{(0)}\rangle + \frac{1}{4}(2n+3)\frac{\sqrt{2^{n+2}(n+2)!}}{\sqrt{2^n n!}}|(n+2)^{(0)}\rangle + \frac{1}{16}\frac{\sqrt{2^{n+4}(n+4)!}}{\sqrt{2^n n!}}|(n+4)^{(0)}\rangle\Bigg] \tag{203}$$

Then

$$\langle n^{(0)}|X^3|n^{(0)}\rangle = 0 \tag{204}$$

$$\langle n^{(0)}|X^4|n^{(0)}\rangle = \frac{3}{4}[2n(n+1)+1]\frac{\hbar^2}{m^2\omega^2} \tag{205}$$

and the first order corrections are given by

$$a\langle n^{(0)}|X^3 + \frac{b}{a}X^4|n^{(0)}\rangle = a\langle n^{(0)}|X^3|n^{(0)}\rangle + a\langle n^{(0)}|\frac{b}{a}X^4|n^{(0)}\rangle \tag{206}$$

$$= \frac{3}{4}[2n(n+1)+1]\frac{\hbar^2}{m^2\omega^2}b \tag{207}$$

Also

$$\langle n^{(0)}|X^3|(n-3)^{(0)}\rangle = n(n-1)(n-2)\frac{\sqrt{2^{n-3}(n-3)!}}{\sqrt{2^n n!}}\left(\frac{\hbar}{m\omega}\right)^{3/2} \tag{208}$$

$$= \frac{\sqrt{n(n-1)(n-2)}}{\sqrt{8}}\left(\frac{\hbar}{m\omega}\right)^{3/2} \tag{209}$$

$$\langle n^{(0)}|X^3|(n-1)^{(0)}\rangle = \frac{3n^2}{2}\frac{\sqrt{2^{n-1}(n-1)!}}{\sqrt{2^n n!}}\left(\frac{\hbar}{m\omega}\right)^{3/2} \tag{210}$$

$$= \frac{3n^{3/2}}{\sqrt{8}}\left(\frac{\hbar}{m\omega}\right)^{3/2} \tag{211}$$

$$\langle n^{(0)}|X^3|(n+1)^{(0)}\rangle = \frac{3}{4}(n+1)\frac{\sqrt{2^{n+1}(n+1)!}}{\sqrt{2^n n!}}\left(\frac{\hbar}{m\omega}\right)^{3/2} \tag{212}$$

$$= \frac{3}{\sqrt{8}}(n+1)^{3/2}\left(\frac{\hbar}{m\omega}\right)^{3/2} \tag{213}$$

$$\langle n^{(0)}|X^3|(n+3)^{(0)}\rangle = \frac{1}{8}\frac{\sqrt{2^{n+3}(n+3)!}}{\sqrt{2^n n!}}\left(\frac{\hbar}{m\omega}\right)^{3/2} \tag{214}$$

$$= \frac{1}{\sqrt{8}}\sqrt{(n+1)(n+2)(n+3)}\left(\frac{\hbar}{m\omega}\right)^{3/2} \tag{215}$$

$$\langle n^{(0)}|X^4|(n-4)^{(0)}\rangle = n(n-1)(n-2)(n-3)\frac{\sqrt{2^{n-4}(n-4)!}}{\sqrt{2^n n!}}\left(\frac{\hbar}{m\omega}\right)^{2} \tag{216}$$

$$= \frac{1}{4}\sqrt{n(n-1)(n-2)(n-3)}\left(\frac{\hbar}{m\omega}\right)^{2} \tag{217}$$

$$\langle n^{(0)}|X^4|(n-2)^{(0)}\rangle = (2n^2-3n+1)n\frac{\sqrt{2^{n-2}(n-2)!}}{\sqrt{2^n n!}}\left(\frac{\hbar}{m\omega}\right)^{2} \tag{218}$$

$$= \frac{1}{2}(2n-1)\sqrt{n(n-1)}\left(\frac{\hbar}{m\omega}\right)^{2} \tag{219}$$

$$\langle n^{(0)}|X^4|(n)^{(0)}\rangle = \frac{3}{4}(2n^2+2n+1)\left(\frac{\hbar}{m\omega}\right)^{2} \tag{220}$$

$$\langle n^{(0)}|X^4|(n+2)^{(0)}\rangle = \frac{1}{4}(2n+3)\frac{\sqrt{2^{n+2}(n+2)!}}{\sqrt{2^n n!}}\left(\frac{\hbar}{m\omega}\right)^{2} \tag{221}$$

$$= \frac{1}{2}(2n+3)\sqrt{(n+1)(n+2)}\left(\frac{\hbar}{m\omega}\right)^{2} \tag{222}$$

$$\langle n^{(0)}|X^4|(n+4)^{(0)}\rangle = \frac{1}{16}\frac{\sqrt{2^{n+4}(n+4)!}}{\sqrt{2^n n!}}\left(\frac{\hbar}{m\omega}\right)^{2} \tag{223}$$

$$= \frac{1}{4}\sqrt{(n+1)(n+2)(n+3)(n+4)}\left(\frac{\hbar}{m\omega}\right)^{2} \tag{224}$$

and the second order corrections are given by

$$a^2\sum_{k\neq n}\frac{|\langle k^{(0)}|X^3+\frac{b}{a}X^4|n^{(0)}\rangle|^2}{E_n^{(0)}-E_k^{(0)}} = a^2\sum_{k\neq n}\frac{|\langle n^{(0)}|X^3+\frac{b}{a}X^4|k^{(0)}\rangle|^2}{(n-k)\hbar\omega} \tag{225}$$

$$= a^2\sum_{k\neq n}\frac{|\langle n^{(0)}|X^3|k^{(0)}\rangle+\frac{b}{a}\langle n^{(0)}|X^4|k^{(0)}\rangle|^2}{(n-k)\hbar\omega} \tag{226}$$

because X^3 and X^4 terms do not mix AND terms like $n-4$ vanish for $n = 1, 2, 3$ we can write

$$E_n^{(2)} = \sum_{k\in\{n-4,\ldots,n+4\}} \frac{a^2|\langle n^{(0)}|X^3|k^{(0)}\rangle|^2 + b^2|\langle n^{(0)}|X^4|k^{(0)}\rangle|^2}{(n-k)\hbar\omega} \tag{227}$$

$$= -a^2\frac{1}{8}\frac{\hbar^2}{m^3\omega^4}(30n^2+30n+11) - b^2\frac{1}{16}\frac{\hbar^3}{m^4\omega^5}(68n^3+102n^2+118n+42) \tag{228}$$

0.9.2 Exercise 13.2 - Quartic perturbed harmonic oscillator

Substituting all into the Schroedinger equation

$$-\frac{\hbar^2}{2m}\psi'' - \frac{m\omega^2}{2}x^2\psi + bx^4\psi = E\psi \tag{229}$$

$$\sum_{k=0} b^k\left(-\frac{\hbar^2}{2m}\left[P_k''(x) - \frac{2m\omega}{\hbar}xP_k'(x) + \frac{m^2\omega^2}{\hbar^2}x^2P_k(x) - \frac{m\omega}{\hbar}P_k(x)\right] + \frac{m\omega^2}{2}x^2P_k(x) + bx^4P_k(x)\right)e^{-\frac{m\omega}{2\hbar}x^2} \tag{230}$$

$$= \sum_{k=0} b^kE_k \cdot \sum_{l=0} b^lP_l(x)e^{-\frac{m\omega}{2\hbar}x^2} \tag{231}$$

with $E_0 = \frac{1}{2}\hbar\omega$ (the book value of $\hbar\omega$ seems wrong). Now we can sort by powers of b

Zeroth order - using $E_0 = \hbar\omega/2$

$$b^0: \quad -\frac{\hbar^2}{2m}\left[P_0''(x) - \frac{2m\omega}{\hbar}xP_0'(x) + \frac{m^2\omega^2}{\hbar^2}x^2P_0(x) - \frac{m\omega}{\hbar}P_0(x)\right] + \frac{m\omega^2}{2}x^2P_0(x) = E_0P_0(x) \tag{232}$$

$$P_0''(x) - \frac{2m\omega}{\hbar}xP_0'(x) - \frac{m}{\hbar}\left(\omega - \frac{2E_0}{\hbar}\right)P_0(x) = 0 \tag{233}$$

$$\rightarrow P_0(x) = 1 \tag{234}$$

First order - using $E_0 = \hbar\omega/2$ and $P_0(x) = 1$

$$b^1: \quad -\frac{\hbar^2}{2m}\left[P_1''(x) - \frac{2m\omega}{\hbar}xP_1'(x) + \frac{m^2\omega^2}{\hbar^2}x^2P_1(x) - \frac{m\omega}{\hbar}P_1(x)\right] + \frac{m\omega^2}{2}x^2P_1(x) + x^4P_0(x) \tag{235}$$

$$= E_0P_1(x) + E_1P_0(x) \tag{236}$$

$$P_1''(x) - \frac{2m\omega}{\hbar}xP_1'(x) - \frac{2m}{\hbar^2}x^4 + \frac{m\omega}{\hbar}P_1(x) + \frac{2mE_1}{\hbar^2} = 0 \tag{237}$$

$$\rightarrow P_1(x) = -\frac{1}{4\hbar\omega}x^4 - \frac{3}{4m\omega^2}x^2 + c_1 \tag{238}$$

$$\rightarrow E_1(x) = \frac{3\hbar^2}{4m^2\omega^2} \tag{239}$$

Second order - using $E_0 = \hbar\omega/2$, $E_1(x) = \frac{3\hbar^2}{4m^2\omega^2}$ and $P_0(x) = 1$, $P_1(x) = -\frac{1}{4\hbar\omega}x^4 - \frac{3}{4m\omega^2}x^2 + c_1$

$$b^2 : \quad -\frac{\hbar^2}{2m}\left[P_2''(x) - \frac{2m\omega}{\hbar}xP_2'(x) + \frac{m^2\omega^2}{\hbar^2}x^2P_2(x) - \frac{m\omega}{\hbar}P_2(x)\right] + \frac{m\omega^2}{2}x^2P_2(x) + x^4P_1(x) \tag{240}$$

$$= E_0P_2(x) + E_1P_1(x) + E_2P_0(x) \tag{241}$$

$$P_2''(x) - \frac{2m\omega}{\hbar}xP_2'(x) - \frac{2m}{\hbar^2}x^4P_1(x) + \frac{m\omega}{\hbar}P_1(x) + \frac{2mE_1}{\hbar^2} = 0 \tag{242}$$

$$\to P_2(x) = \frac{1}{32\hbar^2\omega^2}x^8 + \frac{13}{48m\omega^3\hbar}x^6 + \frac{31\hbar - 8m^2\omega^3c_0}{32m^2\omega^4\hbar}x^4 + \frac{3(7\hbar - 2m^2\omega^3c_0)}{8m^3\omega^5}x^2 + c_2 \tag{243}$$

$$\to E_2(x) = -\frac{21\hbar^3}{8m^4\omega^5} \tag{244}$$

Then

$$E = \frac{1}{2}\hbar\omega + \frac{3\hbar^2}{4m^2\omega^2}b - \frac{21\hbar^3}{8m^4\omega^5}b^2 + ... \tag{245}$$

0.9.3 Exercise 13.3 - Normal matrix

A normal matrix A has the property $A^\dagger A = AA^\dagger$

0.10 Sakurai, Napolitano - Modern Quantum Mechanics 3rd ed

0.10.1 5.1 - Harmonic oscillator with linear perturbation

The Hamiltonians are given by

$$\hat{H}_0 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega_o^2x^2 \tag{246}$$

$$\hat{H}_1 = bx \tag{247}$$

We remember

$$\phi_0(x) = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4} e^{-m\omega_0x^2/2\hbar} \tag{248}$$

$$E_0 = \frac{1}{2}\hbar\omega_0 \tag{249}$$

$$\phi_n(x) = \frac{1}{\sqrt{2^nn!}}\left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4} e^{-m\omega_0x^2/2\hbar}H_n\left(\sqrt{\frac{m\omega_0}{\hbar}}x\right) \tag{250}$$

$$E_n = \hbar\omega_0\left(n + \frac{1}{2}\right) \tag{251}$$

1. Time independent perturbation theory gives

$$\Delta E_n^{(1)} = \langle n^{(0)}|\hat{H}_1|n^{(0)}\rangle \tag{252}$$

$$\Delta E_0^{(1)} = \langle 0^{(0)}|\hat{H}_1|0^{(0)}\rangle = 0 \tag{253}$$

The first order energy shift vanishes because of the wave function is even and H_1 is odd. For the first order perturbation of the wave function we observe

$$H_1(x) = 2xH_0(x) \quad \to \quad \hat{H}_1|0^{(0)}\rangle = \frac{b}{2}\sqrt{2}\sqrt{\frac{\hbar}{m\omega_0}}|1^{(0)}\rangle \tag{254}$$

$$\langle m^{(0)}|n^{(0)}\rangle = \delta_{nm} \tag{255}$$

Now we can calculate

$$|n^{(1)}\rangle = \sum_{k\neq n} \frac{\langle k^{(0)}|\hat{H}_1|n^{(0)}\rangle}{E_n^{(0)} - E_k^{(0)}}|k^{(0)}\rangle \tag{256}$$

$$|0^{(1)}\rangle = \frac{\langle 0^{(0)}|\hat{H}_1|1^{(0)}\rangle}{E_0^{(0)} - E_1^{(0)}}|1^{(0)}\rangle \tag{257}$$

$$= -\frac{1}{\hbar\omega_0} b\sqrt{\frac{\hbar}{2m\omega_0}}|1^{(0)}\rangle \tag{258}$$

$$= -b\sqrt{\frac{1}{2m\hbar\omega_0^3}}|1^{(0)}\rangle \tag{259}$$

Second order enegy perturbation

$$\Delta E_n^{(2)} = \langle n^{(0)}|\hat{H}_1|n^{(1)}\rangle = \sum_{k\neq n} \frac{|\langle k^{(0)}|\hat{H}_1|n^{(0)}\rangle|^2}{E_n^{(0)} - E_k^{(0)}} \tag{260}$$

$$\Delta E_0^{(2)} = \langle 0^{(0)}|\hat{H}_1|0^{(1)}\rangle \tag{261}$$

$$= b\sqrt{\frac{\hbar}{2m\omega_0}}\langle 1^{(0)}|0^{(1)}\rangle \tag{262}$$

$$= b\sqrt{\frac{\hbar}{2m\omega_0}}\langle 1^{(0)}|\left(-b\sqrt{\frac{1}{2m\hbar\omega_0^3}}\right)|1^{(0)}\rangle \tag{263}$$

$$= -b^2\frac{1}{2m\omega_0^2} \tag{264}$$

2. The linear perturbation does not change the shape of the potential - only shifts the minimum

$$V(x) = \frac{m\omega_0^2}{2}x^2 + bx = \frac{m\omega_0^2}{2}\left(x + \frac{b}{m\omega_0^2}\right)^2 - \frac{b^2}{2m\omega_0^2} \tag{265}$$

$$\Delta E^{(\infty)} = -\frac{b^2}{2m\omega_0^2} \tag{266}$$

So the second order gives the exact result - interesting to see if higher orders would all vanish or give oscillating contributions.

0.10.2 5.2 - Potential well with linear slope

We will treat the slope as a perturbation with

$$\hat{H}_1 = \frac{V}{L}x \tag{267}$$

Therefore the unperturbed wave functions are given by

$$\phi_n = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L} \qquad E_n = \frac{\pi^2\hbar^2}{2mL^2}n^2 \tag{268}$$

Then

$$\Delta E_n^{(1)} = \langle n^{(0)}|\hat{H}_1|n^{(0)}\rangle \tag{269}$$
$$= \frac{V}{L}\frac{2}{L}\int_0^L x\sin^2\frac{n\pi x}{L}dx \tag{270}$$
$$= \frac{2V}{L^2}\int_0^L x\sin^2\frac{n\pi x}{L}dx \tag{271}$$
$$= \frac{2V}{L^2}\int_0^L x\left(1-\cos^2\frac{n\pi x}{L}\right)dx \tag{272}$$
$$= \frac{2V}{L^2}\frac{L^2}{2} - \Delta E_n^{(1)} \tag{273}$$

meaning $\Delta E_n^{(1)} = V/2$.

0.10.3 5.3 - Relativistic perturbation

We can approximate the kinetic energy by

$$E = \sqrt{m^2c^4 + p^2c^2} \tag{274}$$
$$\approx mc^2 + \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + \frac{p^6}{16m^5c^4} + \cdots \tag{275}$$
$$= mc^2 + \frac{mc^2}{2}\frac{p^2}{m^2c^2} - \frac{mc^2}{8}\frac{p^4}{m^4c^4} + \cdots \tag{276}$$
$$= mc^2\left(1 + \frac{1}{2}\beta^2 - \frac{1}{8}\beta^4 + \cdots\right) \tag{277}$$

so

$$\hat{H}_0 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2 \tag{278}$$
$$\hat{H}_1 = -\frac{1}{8m^3c^2}p^4 = -\frac{\hbar^4}{8m^3c^2}\frac{d^4}{dx^4} \tag{279}$$

and we remember

$$\phi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar} \tag{280}$$
$$E_0 = \frac{1}{2}\hbar\omega_0 \tag{281}$$

then

$$\Delta E_0^{(1)} = \langle 0^{(0)}|\hat{H}_1|0^{(0)}\rangle \tag{282}$$
$$= -\frac{\hbar^4}{8m^3c^2}\int_{-\infty}^{\infty}\phi_0(x)^*\frac{d^4}{d^4x}\phi_0(x)dx \tag{283}$$
$$= -\frac{3\hbar^2\omega^2}{32mc^2} \tag{284}$$

0.10.4 5.4 - Diatomic atomic rotor - NOT DONE YET

Hamiltonian of the problem is given by

$$H = \frac{L^2}{2I} \quad\to\quad \hat{H} = -\frac{\hbar^2}{2I}\frac{d^2}{d\varphi^2} \tag{285}$$

with the unperturbed solutions

$$\phi_n^{(0)} = Ce^{in\phi} \qquad E_n^{(0)} = \frac{\hbar^2 n^2}{2I} \tag{286}$$

where only E_0 is non-degenerate (all other are double degenerated). For the perturbation we use the Hamiltonian

$$\hat{H}_1 = Ed\cos\varphi \tag{287}$$

Hmmm....

0.10.5 5.6 - Two dimensional potential well

As the problem separates

$$\left(\hat{H}_x + \hat{H}_y\right)\phi_x\phi_y = (E_x + E_y)\phi_x\phi_y \tag{288}$$

$$\phi_y\hat{H}_x\phi_x + \phi_x\hat{H}_y\phi_y = (E_x + E_y)\phi_x\phi_y \tag{289}$$

$$\frac{\hat{H}_x\phi_x}{\phi_x} + \frac{\hat{H}_y\phi_y}{\phi_y} = (E_x + E_y) \tag{290}$$

the wave function can be written as a product of the 1-dimensional wave functions

$$\phi_{n_x,n_y} = \sqrt{\frac{2}{L}}\sqrt{\frac{2}{L}}\sin\left(\frac{n_x\pi}{L}x\right)\sin\left(\frac{n_y\pi}{L}y\right) \tag{291}$$

$$E_{n_x,n_y} = \frac{\pi^2\hbar^2}{2mL^2}(n_x^2 + n_y^2) \tag{292}$$

So

$$\phi_{1,1} \quad \rightarrow \quad E_{1,1} = 2\frac{\pi^2\hbar^2}{2mL^2} \tag{293}$$

$$\phi_{2,1},\phi_{1,2} \quad \rightarrow \quad E_{2,1} = 5\frac{\pi^2\hbar^2}{2mL^2} \tag{294}$$

$$\phi_{2,2} \quad \rightarrow \quad E_{1,1} = 8\frac{\pi^2\hbar^2}{2mL^2} \tag{295}$$

for the non-degenerated levels $E_{1,1}$ and $E_{2,2}$ we get

$$\Delta E_{1,1}^{(1)} = \langle 1,1^{(0)}|\hat{H}_1|1,1^{(0)}\rangle \tag{296}$$

$$= \frac{1}{4}\lambda L^2 \tag{297}$$

$$\Delta E_{2,2}^{(1)} = \langle 2,2^{(0)}|\hat{H}_1|2,2^{(0)}\rangle \tag{298}$$

$$= \frac{1}{4}\lambda L^2 \tag{299}$$

and for the degenerated levels $E_{1,2}/E_{2,1}$ we get

$$H = \begin{pmatrix} \langle 1,2^{(0)}|\hat{H}_1|1,2^{(0)}\rangle & \langle 1,2^{(0)}|\hat{H}_1|2,1^{(0)}\rangle \\ \langle 2,1^{(0)}|\hat{H}_1|1,2^{(0)}\rangle & \langle 2,1^{(0)}|\hat{H}_1|2,1^{(0)}\rangle \end{pmatrix} \tag{300}$$

with

$$H_{aa} = \langle 1,2^{(0)}|\hat{H}_1|1,2^{(0)}\rangle = \frac{\lambda L^2}{4} \tag{301}$$

$$H_{ab} = \langle 1,2^{(0)}|\hat{H}_1|2,1^{(0)}\rangle = \frac{256\lambda L^2}{81\pi^4} \tag{302}$$

$$H_{bb} = \langle 2,1^{(0)}|\hat{H}_1|2,1^{(0)}\rangle = \frac{\lambda L^2}{4} \tag{303}$$

and $\hat{H}_1 = \lambda xy$ Diagonalising the matrix H gives the perturbation

$$\Delta E^{(1)}_{12,21} = \frac{\lambda L^2}{4} - \frac{256\lambda L^2}{81\pi^4} \tag{304}$$
$$\Delta E^{(1)}_{12,21} = \frac{\lambda L^2}{4} + \frac{256\lambda L^2}{81\pi^4} \tag{305}$$
$$\tag{306}$$

0.10.6 5.8 - Quadratically perturbed harmonic oscillator

$$\hat{H}_1 = \epsilon\frac{1}{2}m\omega^2 x^2 \tag{307}$$

$$H_0(x) = 1 \tag{308}$$
$$H_2(x) = 4x^2 - 2 \quad \rightarrow \quad x^2 = \frac{H_2}{4} + \frac{1}{2} \tag{309}$$

0.10.7 5.13 - Two-dimensional infinite square well - NOT DONE YET

a. Separation ansatz

$$\left[-\frac{\hbar^2}{2m}(\partial_{xx} + \partial_{yy}) - E_{kl}\right]\psi_k(x)\psi_l(y) = 0 \tag{310}$$
$$\frac{1}{\psi_k(x)}\left(-\frac{\hbar^2}{2m}\partial_{xx}\right)\psi_k(x) = E_{kl} = \frac{1}{\psi_l(y)}\left(-\frac{\hbar^2}{2m}\partial_{yy}\right)\psi_l(y) \tag{311}$$

giving with boundary condition $\psi = 0$

$$\psi_{kl}(x,y) = \sqrt{\frac{2}{a}}\sqrt{\frac{2}{a}}\sin(\frac{k\pi}{a}x)\sin(\frac{k\pi}{a}y) \tag{312}$$
$$E_{kl} = \frac{\pi^2\hbar^2}{2ma^2}(k^2 + l^2) \tag{313}$$

Then the three lowest energy eigenstates are

$$E_{11} = 2\cdot\frac{\pi^2\hbar^2}{2ma^2} \tag{314}$$
$$E_{21} = 5\cdot\frac{\pi^2\hbar^2}{2ma^2} \tag{315}$$
$$E_{12} = 5\cdot\frac{\pi^2\hbar^2}{2ma^2} \tag{316}$$

b.

$$E^{(1)}_{11} = \lambda\langle 1,1|xy|1,1\rangle = \lambda\frac{a^2}{4} \tag{317}$$
$$E^{(2)}_{11} = \lambda^2 \tag{318}$$

0.10.8 5.42 - Triton beta decay - NOT DONE YET

a. With the generic $1s$ wave function

$$\psi_{10} = \frac{1}{\sqrt{4\pi}}\sqrt{\frac{1}{2}\left(\frac{2Z}{a_\mu}\right)^3} e^{-Zr/a_\mu} \tag{319}$$

$$a_\mu = \frac{1}{\mu}a_0 \tag{320}$$

$$\mu = \frac{mM}{m+M} \tag{321}$$

we get with the initial state ($Z = 1$, $M = 3m1837$) and the final state ($Z = 2$, $M = 3m1837$) then

$$(i|f) = 4\pi\int_0^\infty r^2\psi_i\psi_f = \frac{16\sqrt{2}}{27} \tag{322}$$

so the probability is 512/729.

b.

0.10.9 8.1 - Natural units

1. Proton Mass

$$E_p = m_p c^2/e = 0.937\text{GeV} \tag{323}$$

2. With $\Delta p \cdot \Delta x \geq \hbar/2$ and $E = \sqrt{m^2c^4 + p^2c^2} \approx pc$

$$E = \Delta pc/e = 98.6\text{MeV} \tag{324}$$

Alternatively we have $E = \frac{\hbar c}{e \cdot dx}$ meaning $1\text{fm} = \frac{1}{197.3\text{MeV}}$ and therefore

$$E = \frac{\hbar}{2\cdot\Delta x}c = 197.3/2\text{MeV} \tag{325}$$

3. Solving for α, β, γ

$$M_P = G^\alpha c^\beta \hbar^\gamma \tag{326}$$

$$= \left(\frac{\text{Nm}^2}{\text{kg}^2}\right)^\alpha \left(\frac{\text{m}}{\text{s}}\right)^\beta (\text{Js})^\gamma \tag{327}$$

$$= \sqrt{\frac{\hbar c}{G}} \tag{328}$$

$$E_P = \sqrt{\frac{\hbar c}{G}}c^2\frac{1}{e} = 1.22\cdot 10^{19}\text{GeV} \tag{329}$$

0.10.10 8.2 - Minkowski Metric

The definition implies that $\eta_{\lambda\nu}$ is the inverse of $\eta^{\lambda\nu}$ - simple calculation shows that they are identical. Now we can calculate

$$\eta^{\mu\lambda}\eta^{\nu\sigma}\eta_{\lambda\sigma} = \eta^{\nu\sigma}\delta^\mu_\sigma \tag{330}$$

$$= \eta^{\nu\mu} \tag{331}$$

and

$$a^\mu b_\mu = a_\alpha\eta^{\alpha\mu}b^\beta\eta_{\beta\mu} = a_\alpha b^\beta\delta^\alpha_\beta = a_\alpha b^\alpha \tag{332}$$

0.11 Bethe, Jackiw - Intermediate Quantum Mechanics

0.11.1 1.1 - Atomic units

Set $\hbar = e = m_e = 1$ and $a_B = \frac{4\pi\varepsilon_0\hbar^2}{m_e e^2} = 1$ then $4\pi\varepsilon_0 = 1$ and therefore $\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} = 1/c$

1. energy: $E_{1s} = \frac{1}{2}m_e c^2\alpha^2$ therefore 1 a.u. $= 2 \times 13.6$eV

2. momentum: $p = m_e c$ therefore 1 a.u. $= 2 \cdot 10^{-31}\text{kg} \times 3 \cdot 10^8 \text{m/s}^2 = 2.73 \cdot 10^{-22}$J

3. angular momentum: $L = \hbar$ therefore 1 a.u. $= 1.04 \cdot 10^{-34}$Js

0.11.2 1.7 - Hydrogen atom with finite nucleus

The field of a uniform sphere of charge Q can be found by Gauss law

$$E_r = \frac{1}{4\pi\epsilon_0} \cdot \begin{cases} Q/a^3 \cdot r & r < R \\ Q/r^2 & r > R \end{cases} \tag{333}$$

The potential is then given by

$$\phi = \frac{1}{4\pi\epsilon_0} \cdot \begin{cases} Q/2R\left(3 - \frac{r^2}{R^2}\right) & r < R \\ Q/r & r > R \end{cases} \tag{334}$$

Treating this as a perturbation problem the energy shift can be calculated via the perturbation Hamiltonian (switching the electrostatic energy within the finite nucleus)

$$\begin{aligned} H_1 &= (q\phi_{\text{finite}} - q\phi_{\text{point}})\theta(R-r) && (335)\\ &= -e\left(\phi_{\text{finite}} - \phi_{\text{point}}\right)\theta(R-r) && (336)\\ &= -\frac{e}{4\pi\epsilon_0}\left(\frac{Ze}{2R}\left[3 - \frac{r^2}{R^2}\right] - \frac{Ze}{r}\right)\theta(R-r) && (337)\\ &= -\frac{Ze^2}{4\pi\epsilon_0}\left(\frac{1}{2R}\left[3 - \frac{r^2}{R^2}\right] - \frac{1}{r}\right)\theta(R-r) && (338) \end{aligned}$$

with $R = r_0 A^{1/3}$. With the radial wavefunction (in Mathematica notation)

$$R_{nl}(r) = \frac{2}{n^2}\sqrt{\frac{(n-l-1)!Z^3}{(n+l)!a_B^3}}\left(\frac{2Zr}{na_B}\right)^l e^{-Zr/na_B} L_{n-l-1}^{2l+1}(\frac{2Zr}{na_B}) \tag{339}$$

we can do a series expansion at $r = 0$ and use the first term (as nucleus is small)

$$\begin{aligned} &R_{10}^2 \simeq 4Z^3 && (340)\\ &R_{20}^2 \simeq \frac{1}{2}Z^3 \qquad R_{21}^2 \simeq \frac{1}{24}Z^5r^2 && (341)\\ &R_{30}^2 \simeq \frac{4}{27}Z^3 \qquad R_{31}^2 \simeq \frac{32}{2187}Z^5r^2 \qquad R_{32}^2 \simeq \frac{8}{98415}Z^7r^4 && (342)\\ &R_{40}^2 \simeq \frac{1}{16}Z^3 \qquad R_{41}^2 \simeq \frac{5}{768}Z^5r^2 \qquad R_{42}^2 \simeq \frac{1}{20400}Z^7r^4 \qquad R_{43}^2 \simeq \frac{1}{20643840}Z^9r^6 && (343) \end{aligned}$$

then

$$\Delta E_{nl} = \int_0^R r^2 R_{nl}(r)^2 H_1(r) \tag{344}$$

$$\Delta E_{10} = -\frac{2}{5} r_0^2 A^{2/3} Z^4 \tag{345}$$

$$\Delta E_{20} = -\frac{1}{20} r_0^2 A^{2/3} Z^4 \qquad \Delta E_{21} = -\frac{1}{1120} r_0^4 A^{4/3} Z^6 \tag{346}$$

$$\Delta E_{30} = -\frac{2}{135} r_0^2 A^{2/3} Z^4 \qquad \Delta E_{31} = -\frac{8}{25515} r_0^4 A^{4/3} Z^6 \qquad \Delta E_{32} = -\frac{4}{6200145} r_0^6 A^2 Z^8 \tag{347}$$

$$\Delta E_{40} = -\frac{1}{160} r_0^2 A^{2/3} Z^4 \qquad \Delta E_{41} = -\frac{1}{7168} r_0^4 A^{4/3} Z^6 \qquad \Delta E_{42} = -\frac{1}{2580480} r_0^6 A^2 Z^8 \qquad \Delta E_{43} = -\frac{1}{5449973760} r_0^8 A^8 \tag{348}$$

and

$$\Delta E_{2p\to 1s} = \left(-\frac{1}{1120} r_0^4 A^{4/3} Z^6\right) - \left(-\frac{2}{5} r_0^2 A^{2/3} Z^4\right) \tag{349}$$

$$\Delta E_{H:\,2p\to 1s} = 2.05593 \cdot 10^{-10} = 0.000045\mathrm{cm}^{-1} \tag{350}$$

$$\Delta E_{Pb:\,2p\to 1s} = 0.003981 = 873.8\mathrm{cm}^{-1} \tag{351}$$

0.11.3 1.9 - Exponential potential

The Schroedinger equation is given by

$$-\frac{1}{2}\triangle_r \psi + V\psi = E\psi \tag{352}$$

$$-\frac{1}{2r^2}\partial_r(r^2\partial_r\psi) - \frac{a^2}{8}e^{-r/2r_0}\psi = E\psi \tag{353}$$

$$-\frac{1}{2}\left(\frac{2}{r}\psi' + \psi''\right) - \frac{a^2}{8}e^{-r/2r_0}\psi = E\psi \tag{354}$$

Ansatz $\psi(r) = u(r)/r$

$$-\frac{1}{2}\left(\frac{2}{r}\frac{u'r-u}{r^2} + \frac{(u''r+u'-u')r^2 - 2r(u'r-u)}{r^4}\right) - \frac{a^2}{8}e^{-r/2r_0}\frac{u}{r} = E\frac{u}{r} \tag{355}$$

$$-\frac{u'}{r^2} + \frac{u}{r^3} - \frac{u''}{2r} + \frac{2u'}{r^2} - \frac{u}{r^3} - \frac{a^2}{8}e^{-r/2r_0}\frac{u}{r} = E\frac{u}{r} \tag{356}$$

$$-\frac{u''}{2r} + \frac{u'}{r^2} - \frac{a^2}{8}e^{-r/2r_0}\frac{u}{r} = E\frac{u}{r} \tag{357}$$

$$\tag{358}$$

Stepwise calculation for the verification of the solution

$$r^2\partial_r\psi = u'r - u \tag{359}$$

$$= \frac{1}{2}\left[J_{n-1}(.) - J_{n+1}(.)\right] a r_0 e^{-\frac{r}{2r_0}} \frac{-1}{2r_0} r - J_n(.) \tag{360}$$

$$= -\frac{1}{4}\left[J_{n-1}(.) - J_{n+1}(.)\right] a r e^{-\frac{r}{2r_0}} - J_n(.) \tag{361}$$

$$= -\frac{1}{4}\left[J_{n-1}(.) - \left(\frac{2n}{a r_0 e^{-r/2r_0}} J_n(.) - J_{n-1}(.)\right)\right] a r e^{-\frac{r}{2r_0}} - J_n(.) \tag{362}$$

$$= -\frac{1}{4}\left[2J_{n-1}(.) - \frac{2n}{a r_0 e^{-r/2r_0}} J_n(.)\right] a r e^{-\frac{r}{2r_0}} - J_n(.) \tag{363}$$

$$= -\frac{1}{2}J_{n-1}(.) a r e^{-\frac{r}{2r_0}} + \left(\frac{nr}{2r_0} - 1\right) J_n(.) \tag{364}$$

$$\frac{1}{r^2}\partial_r(r^2\partial_r\psi) = -\frac{1}{2}\left(J_{n-1} - J_{n+1}\right)a^2\frac{r_0}{r} - \frac{1}{2}J_{n-1}(.)\frac{a}{r^2}e^{-\frac{r}{2r_0}} - \frac{1}{2}J_{n-1}(.)\frac{-a}{2rr_0}e^{-\frac{r}{2r_0}} \tag{365}$$

$$+\frac{n}{2r_0r^2}J_n(.) + \left(\frac{nr}{2r_0} - 1\right)\frac{1}{2r^2}(J_{n-1}(.) - J_{n+1}(.))ar_0e^{-\frac{r}{2r_0}}\frac{-1}{2r_0} \tag{366}$$

$$= (J_{n-1} - J_{n+1})\left[-\frac{a^2r_0}{2r} - \left(\frac{nr}{2r_0} - 1\right)\frac{ar_0}{4r^2r_0}\right]e^{-\frac{r}{2r_0}} \tag{367}$$

0.12 Gottfried, Tung - Quantum Mechanics: Fundamentals, 2nd ed

0.13 Merzbacher - Quantum Mechanics, 3rd ed

0.14 Jackson - A Course in Quantum Mechanics

0.14.1 1.1 Lorentzian wave package

(a)

$$\psi(x,0) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dp\, e^{ixp/\hbar}\sqrt{\frac{2}{\pi\hbar}}\frac{\alpha^{3/2}}{(p-p_0)^2+\alpha^2} \tag{368}$$

$$= \frac{\alpha}{\pi\hbar}\int_{-\infty}^{\infty} dp\, e^{ixp/\hbar}\frac{\alpha^{1/2}}{(p-p_0)^2+\alpha^2} \tag{369}$$

$$= \frac{\alpha^{3/2}}{\pi\hbar}e^{ixp_0/\hbar}\int_{-\infty}^{\infty} d\hat{p}\, e^{ix\hat{p}/\hbar}\frac{1}{\hat{p}^2+\alpha^2} \tag{370}$$

$$= \frac{\alpha^{3/2}}{\pi\hbar}\frac{i}{2\alpha}e^{ixp_0/\hbar}\int_{-\infty}^{\infty} d\hat{p}\, e^{ix\hat{p}/\hbar}\left(\frac{1}{\hat{p}+i\alpha} - \frac{1}{\hat{p}-i\alpha}\right) \tag{371}$$

Close loop above for $x > 0$ (half loop integral vanishes) and below for $x < 0$

$$\psi(x>0,0) = \frac{i\sqrt{\alpha}}{2\pi\hbar}e^{ixp_0/\hbar}\cdot 2\pi i\,\mathrm{Res}(i\alpha, I_2) \tag{372}$$

$$= -\frac{\sqrt{\alpha}}{\hbar}e^{ixp_0/\hbar}\cdot\left(-e^{ix(i\alpha)/\hbar}\right) \tag{373}$$

$$= \frac{\sqrt{\alpha}}{\hbar}e^{-x\alpha/\hbar}\cdot e^{ixp_0/\hbar} \tag{374}$$

$$\rightarrow \psi(x,0) = \frac{\sqrt{\alpha}}{\hbar}e^{-|x|\alpha/\hbar}\cdot e^{ixp_0/\hbar} \tag{375}$$

α is the width of the package in momentum space. $1/\alpha$ is the package width in space.

(b)

$$\int \phi^*\phi\, dp = \frac{1}{\hbar} \tag{376}$$

$$\int \phi^* p\phi\, dp = \frac{p_0}{\hbar} \quad\rightarrow\quad \langle p\rangle = p_0 \tag{377}$$

$$\int \phi^* p^2\phi\, dp = \frac{p_0^2+\alpha^2}{\hbar} \quad\rightarrow\quad \langle p^2\rangle = p_0^2+\alpha^2 \tag{378}$$

$$\int \psi^* \psi \, dp = \frac{1}{\hbar} \tag{379}$$
$$\int \psi^* x \psi \, dp = 0 \quad \rightarrow \quad \langle x \rangle = 0 \tag{380}$$
$$\int \psi^* x^2 \psi \, dp = \frac{\hbar}{2\alpha^2} \quad \rightarrow \quad \langle x^2 \rangle = \frac{\hbar^2}{2\alpha^2} \tag{381}$$

(c)

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \frac{\hbar}{\sqrt{2}\alpha} \tag{382}$$
$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \alpha \tag{383}$$
$$\Delta x \cdot \Delta p = \frac{\hbar}{\sqrt{2}} > \hbar/2 \tag{384}$$

0.14.2 1.2 1D Box with vanishing walls

With

$$\psi(x,t) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right) e^{-i\frac{\pi^2 \hbar}{2mL^2}t} \tag{385}$$
$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \sqrt{\frac{2}{L}} \int_0^L dx \, \psi(x,0) \, e^{ipx/\hbar} \tag{386}$$
$$= \frac{1}{\sqrt{\pi\hbar L}} \int_0^L dx \, \sin\left(\frac{\pi x}{L}\right) e^{ipx/\hbar} \tag{387}$$

Two times integration by parted gives

$$\phi(p) = \frac{1}{\sqrt{\pi\hbar L}} \frac{\pi L \hbar^2 \left(1 + e^{ipL/\hbar}\right)}{\hbar^2 \pi^2 - p^2 L^2} \tag{388}$$

Now we can use the Schroedinger equation with $\phi(p,t) = \phi(p) f(t)$

$$i\hbar \partial_t \phi(p,t) = \frac{p^2}{2m} \phi(p,t) \tag{389}$$
$$\rightarrow i\hbar \partial_t f(t) = \frac{p^2}{2m} f(t) \tag{390}$$
$$\rightarrow f(t) = c \, e^{-i\frac{p^2}{2\hbar m}t} \tag{391}$$
$$\rightarrow \phi(p,t) = \frac{1}{\sqrt{\pi\hbar L}} \frac{\pi L \hbar^2 \left(1 + e^{ipL/\hbar}\right)}{\hbar^2 \pi^2 - p^2 L^2} e^{-i\frac{p^2}{2\hbar m}t} \tag{392}$$
$$\rightarrow \rho(p,t) = \phi^* \phi = \frac{2\hbar^4 L^2 \pi^2}{(p^2 L^2 + \hbar^2 \pi^2)^2} \left(1 + \cos\frac{pL}{\hbar}\right) \tag{393}$$

0.14.3 1.3 Protonium

$$\mu_x = \frac{m_p m_x}{m_p + m_x} \tag{394}$$

$$E_n^{(x)} = -\frac{\mu_x c^2}{2}\alpha^2 \frac{1}{n^2} \tag{395}$$

$$r_B^{(x)} = \frac{\hbar c}{\alpha\, \mu_x c^2} \tag{396}$$

$$\frac{dP}{dt} = \Gamma_{f\to i} = \frac{2\pi}{\hbar}|\langle f|H'|i\rangle|^2 \rho(E_f) \tag{397}$$

$$\sim \frac{1}{\tau} \tag{398}$$

(a) We have now with $m_p = 938.272\text{MeV}$, $\alpha = 1/137$ and $c = 1$

x	m_x	μ_x	r_B	E_{1s}	E_{2p}	$\Delta E_{2p/1s}$
e^-	511.0keV	510.7keV	$5.3 \cdot 10^{-11}$m	13.6eV	3.4eV	10.2eV
μ^-	105.7MeV	95.0MeV	$2.8 \cdot 10^{-13}$m	2,530eV	632eV	1,898eV
π^-	139.6MeV	121.5MeV	$2.2 \cdot 10^{-13}$m	3,237eV	809eV	2,428eV
K^-	493.6MeV	299.3MeV	$9.0 \cdot 10^{-14}$m	8,616eV	2,154eV	6,462eV
$\bar{\mathrm{p}}$	938.3MeV	469.1MeV	$5.8 \cdot 10^{-14}$m	12,498eV	3,123eV	9,373eV
Σ^-	1197.4MeV	526.1MeV	$5.1 \cdot 10^{-14}$m	14,019eV	3,505eV	10,510eV
Ξ^-	1321.7MeV	548.7MeV	$4.9 \cdot 10^{-14}$m	14,618eV	3,654eV	10,963eV

(b) Different values for hydrogen can be found $\tau_H = 1.76 \cdot 10^{-9}\text{s}$ and $\Gamma(2p \to 1s) = 6.2 \cdot 10^8 \text{s}^{-1}$. Full valuation of Fermis golden rule gives

$$\tau = \left(\frac{3}{2}\right)^8 \frac{r_B}{c\alpha^4} \tag{399}$$

$$\tau \sim r_B \sim \frac{1}{\mu} \tag{400}$$

$$\tau_{p\bar{p}} = \frac{\mu_H}{\mu_{p\bar{p}}}\tau_H = 1.73 \cdot 10^{-12}\text{s} \tag{401}$$

Rational: dipole matrix element scales with r_B (smaller object means smaller dipole)

$$\langle f|H'|i\rangle \sim \langle f|\mathbf{x}|i\rangle \sim r_B \tag{402}$$

$$|\langle f|H'|i\rangle|^2 \sim r_B^2 \tag{403}$$

$$\rho \sim \frac{1}{\Delta E_{s/p}} \sim \frac{1}{\mu} \sim r_B \tag{404}$$

$$\Gamma \sim r_B^3 \tag{405}$$

Hmmmmm ....

0.14.4 2.5 Unitary operators

Unitary: $U^\dagger = U^{-1}$ meaning $U^\dagger U = 1$. We see $(U^n)^\dagger = (U...U)^\dagger = (U^\dagger ... U^\dagger) = (U^\dagger)^n$

- $U_1 = e^{iK}$: Lets start with

$$U_1^\dagger = \left(e^{iK}\right)^\dagger = \left(\sum_n \frac{1}{n!}(iK)^n\right)^\dagger = \sum_n \frac{1}{n!}\left((iK)^n\right)^\dagger = \sum_n \frac{(-i)^n}{n!}(K^\dagger)^n = \sum_n \frac{1}{n!}(-iK^\dagger)^n \tag{406}$$

$$= e^{-iK^\dagger} \qquad \text{with } K = K^\dagger \tag{407}$$

$$= e^{-iK} \tag{408}$$

Now with $[K, K] = 0$ (meaning we can Taylor-expand each exponential and flip term by term so $e^X e^Y = e^{X+Y}$ if $[X, Y] = 0$)

$$U_1^\dagger U_1 = e^{-iK} e^{iK} = e^{-iK+iK} = e^0 = 1 \tag{409}$$

- $U_2 = (1 + iK)(1 - iK)^{-1}$

$$\begin{aligned}
U_2^\dagger &= \left((1+iK)(1-iK)^{-1}\right)^\dagger && (410)\\
&= \left((1-iK)^{-1}\right)^\dagger (1+iK)^\dagger \qquad \text{with } K = K^\dagger && (411)\\
&= \left((1-iK)^{-1}\right)^\dagger (1-iK) && (412)
\end{aligned}$$

then

$$\begin{aligned}
U_2^\dagger U_2 &= \left((1-iK)^{-1}\right)^\dagger (1-iK)(1+iK)(1-iK)^{-1} && (413)\\
&= \left((1-iK)^{-1}\right)^\dagger (1-iK+iK+K^2)(1-iK)^{-1} && (414)\\
&= \left((1-iK)^{-1}\right)^\dagger (1+iK)(1-iK)(1-iK)^{-1} && (415)\\
&= \left((1-iK)^{-1}\right)^\dagger (1+iK) \qquad \text{with } B^\dagger A^\dagger = (AB)^\dagger && (416)\\
&= \left((1+iK)^\dagger (1-iK)^{-1}\right)^\dagger \qquad \text{with } K = K^\dagger && (417)\\
&= \left((1-iK)(1-iK)^{-1}\right)^\dagger && (418)\\
&= 1^\dagger = 1 && (419)
\end{aligned}$$

- $U_2' = (1 - iK)^{-1}(1 + iK)$. Assume $U_2' = U_2$ then

$$\begin{aligned}
1 &= (U_2')^{-1} U_2' && (420)\\
&= U_2^{-1} U_2' && (421)\\
&= U_2^\dagger U_2' && (422)\\
&= \underbrace{\left((1-iK)^{-1}\right)^\dagger (1-iK)}_{U_2^\dagger} \underbrace{(1-iK)^{-1}(1+iK)}_{U_2'} && (423)\\
&= \left((1-iK)^{-1}\right)^\dagger (1+iK) && (424)\\
&= \left((1+iK)^\dagger (1-iK)^{-1}\right)^\dagger && (425)\\
&= \left((1-iK)(1-iK)^{-1}\right)^\dagger && (426)\\
&= 1^\dagger = 1 && (427)
\end{aligned}$$

0.15 Basdevant - The Quantum machanics solver 3rd ed.

0.15.1 Exercise 8.1 - Neutrino Oscillations in Vacuum - NOT DONE YET

1.

$$\Delta = E_2 - E_1 \tag{428}$$
$$= \sqrt{p^2c^2 + m_2^2c^4} - \sqrt{p^2c^2 + m_1^2c^4} \tag{429}$$
$$= pc\sqrt{1 + \frac{m_2^2c^4}{p^2c^2}} - pc\sqrt{1 + \frac{m_1^2c^4}{p^2c^2}} \tag{430}$$
$$\simeq pc\left(1 + \frac{m_2^2c^4}{2p^2c^2} - 1 - \frac{m_1^2c^4}{2p^2c^2}\right) \tag{431}$$
$$= \frac{c^3}{2p}(m_2^2 - m_1^2) \tag{432}$$

2.

$$\Delta(2 \times 10^5 \mathrm{eV}/c) = 2 \times 10^{-10} \mathrm{eV} \tag{433}$$
$$\Delta(8 \times 10^6 \mathrm{eV}/c) = 5 \times 10^{-12} \mathrm{eV} \tag{434}$$

3. (a)

$$|\nu_e(t)\rangle = \cos\theta\, e^{-iE_1t/\hbar}|\nu_1\rangle + \sin\theta\, e^{-iE_2t/\hbar}|\nu_2\rangle \tag{435}$$
$$= \cos\theta\, e^{-iE_1t/\hbar}|\nu_1\rangle + \sin\theta\, e^{-i(E_1+\Delta)t/\hbar}|\nu_2\rangle \tag{436}$$
$$= e^{-iE_1t/\hbar}\left(\cos\theta|\nu_1\rangle + \sin\theta\, e^{-i\Delta t/\hbar}|\nu_2\rangle\right) \tag{437}$$
$$\tag{438}$$

(b)

$$\langle\nu_e|\nu_e(t)\rangle = (\langle\nu_1|\cos\theta + \langle\nu_2|\sin\theta)e^{-iE_1t/\hbar}\left(\cos\theta|\nu_1\rangle + \sin\theta\, e^{-i\Delta t/\hbar}|\nu_2\rangle\right) \tag{439}$$
$$= e^{-iE_1t/\hbar}(\cos^2\theta + \sin^2\theta\, e^{-i\Delta t/\hbar}) \tag{440}$$
$$|\langle\nu_e|\nu_e(t)\rangle|^2 = |\cos^2\theta + \sin^2\theta\, e^{-i\Delta t/\hbar}|^2 \tag{441}$$
$$= (\cos^2\theta + \sin^2\theta\, e^{-i\Delta t/\hbar})(\cos^2\theta + \sin^2\theta\, e^{+i\Delta t/\hbar}) \tag{442}$$
$$= \cos^4\theta + \sin^4\theta + 2\sin^2\theta\cos^2\theta\cos\Delta t/\hbar \tag{443}$$

4. (a) We check

$$\langle\nu_e|\nu_e(0)\rangle = \cos^4\theta + \sin^4\theta + 2\sin^2\theta\cos^2\theta = (\cos^2\theta + \sin^2\theta)^2 = 1 \tag{444}$$

then

$$\frac{\Delta \cdot t}{\hbar} = \Delta\frac{d_{ES}}{\hbar c} = 1.52 \cdot 10^8 \tag{445}$$
$$N = \frac{\Delta \cdot t}{2\pi\hbar} = 2.42 \cdot 10^7 \tag{446}$$

(b)