0.1 GRAU - Elektrodynamik Aufgabensammlung

0.1.1 Exercise 8.1 - Hertzscher Dipol - NOT DONE YET

With $\vec{p}_0 = q\vec{d}$ and $d \ll r$

$$\rho(\vec{r}, t) = q\delta(\vec{r} - \frac{\vec{d}}{2}\cos(\omega t)) + (-q)\delta(\vec{r} + \frac{\vec{d}}{2}\cos(\omega t)) \tag{1}$$

$$\rho(\vec{r}, t_{\text{ret}}) = q\delta(\vec{r} - \frac{\vec{d}}{2}\cos(\omega(t - \frac{|\vec{r} - \vec{r'}|}{c}))) + (-q)\delta(\vec{r} + \frac{\vec{d}}{2}\cos(\omega(t - \frac{|\vec{r} - \vec{r'}|}{c}))) \tag{2}$$

$$\simeq q\delta(\vec{r} - \frac{\vec{d}}{2}\cos(\omega(t - \frac{r}{c}))) + (-q)\delta(\vec{r} + \frac{\vec{d}}{2}\cos(\omega(t - \frac{r}{c}))) \tag{3}$$

$$\tag{4}$$

$$\phi(\vec{r}, t) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r'}, t_{\text{ret}})}{|\vec{r} - \vec{r'}|} d^3\vec{r'} \tag{5}$$

$$= \frac{q}{4\pi\varepsilon_0} \int \frac{\delta(\vec{r'} - \frac{\vec{d}}{2}\cos(\omega[t - \frac{r}{c}])) - \delta(\vec{r'} + \frac{\vec{d}}{2}\cos(\omega[t - \frac{r}{c}]))}{\sqrt{r^2 + r'^2 - 2\vec{r} \cdot \vec{r'}}} d^3\vec{r'} \tag{6}$$

$$= \frac{q}{4\pi\varepsilon_0} \int \frac{\delta(\vec{r'} - \frac{\vec{d}}{2}\cos(\omega[t - \frac{r}{c}])) - \delta(\vec{r'} + \frac{\vec{d}}{2}\cos(\omega[t - \frac{r}{c}]))}{r\sqrt{1 + \frac{r'^2}{r^2} - 2\frac{\vec{r} \cdot \vec{r'}}{r^2}}} d^3\vec{r'} \tag{7}$$

$$\simeq \frac{q}{4\pi\varepsilon_0} \int \frac{\delta(\vec{r'} - \frac{\vec{d}}{2}\cos(\omega[t - \frac{r}{c}])) - \delta(\vec{r'} + \frac{\vec{d}}{2}\cos(\omega[t - \frac{r}{c}]))}{r} \left(1 + \frac{\vec{r} \cdot \vec{r'}}{r^2} + \frac{r'^2}{r^2}\right) d^3\vec{r'} \tag{8}$$

$$= \frac{q}{4\pi\varepsilon_0} \frac{1}{r} \left[\left(1 + \frac{\vec{r} \cdot \left[\frac{\vec{d}}{2}\cos(\omega[t - \frac{r}{c}])\right]}{r^2}\right) - \left(1 - \frac{\vec{r} \cdot \left[\frac{\vec{d}}{2}\cos(\omega[t - \frac{r}{c}])\right]}{r^2}\right) \right] \tag{9}$$

$$= \frac{q}{4\pi\varepsilon_0} \frac{1}{r} \frac{\vec{r} \cdot \vec{d}\cos(\omega[t - \frac{r}{c}])}{r^2} \tag{10}$$

$$\tag{11}$$

0.2 ZANGWILL - Classical Electrodynamics

0.2.1 Exercise 10.1 In-Plane Field of a Current Strip

We start with the Biot-Savart law (10.15)

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int d^3x' \frac{\mathbf{j}(\mathbf{x'}) \times (\mathbf{x} - \mathbf{x'})}{|\mathbf{x} - \mathbf{x'}|^3} \tag{12}$$

with

$$\mathbf{j} = (0, 0, K)\delta(x')\Theta(y')\Theta(y' - b) \tag{13}$$

$$\mathbf{x} - \mathbf{x'} = (0, a + y', z')^T \tag{14}$$

$$\mathbf{j}(\mathbf{x'}) \times (\mathbf{x} - \mathbf{x'}) = (a + y')K\delta(x')\Theta(y')\Theta(y' - b) \tag{15}$$

then

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 K}{4\pi} \int_{-\infty}^{\infty} dx' \int_0^b dy' \int_{-\infty}^{\infty} dz' \frac{(a+y')}{\sqrt{(a+y)^2+z'^2}^3} \delta(x') \tag{16}$$

$$= \frac{\mu_0 K}{4\pi} \int_0^b dy' \int_{-\infty}^{\infty} dz' \frac{(a+y')}{\sqrt{(a+y')^2+z'^2}^3} \tag{17}$$

$$= \frac{\mu_0 K}{4\pi} \int_0^b dy' \frac{2}{a+y} \tag{18}$$

$$= \frac{\mu_0 K}{2\pi} \log \frac{a+b}{a} \tag{19}$$

$$= \frac{\mu_0 I}{2\pi b} \log \frac{a+b}{a} \tag{20}$$

$$\tag{21}$$

0.3   Stratton - Electrodynamgnetic Theory

0.3.1   Problem III.1 Coodinate transform

a. Starting

$$\xi + i\eta = f(x+iy) = f(\alpha(x,y)) \tag{22}$$

$$\rightarrow d\xi + id\eta = \frac{\partial f(\alpha)}{\partial \alpha} d\alpha \tag{23}$$

$$= \frac{\partial f(\alpha)}{\partial \alpha} \left(\frac{\partial \alpha}{\partial x} dx + \frac{\partial \alpha}{\partial y} dy \right) \tag{24}$$

$$= \frac{\partial f(\alpha)}{\partial \alpha} (dx + idy) \tag{25}$$

$$= f' \cdot (dx + idy) \tag{26}$$

then calculating the absolute square

$$|f'|^2(dx^2+dy^2) = \frac{1}{h^2}(dx^2+dy^2) \tag{27}$$

$$= |d\xi + id\eta|^2 \tag{28}$$

$$= (d\xi + id\eta)(d\xi - id\eta) \tag{29}$$

$$= d\xi^2 + d\eta^2 \tag{30}$$

then with $dz = d\zeta$

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{31}$$

$$= h^2(d\xi^2 + d\eta^2) + d\zeta^2 \tag{32}$$

b. The metric is diagonal $g_{ij} = \text{diag}(h^2, h^2, 1)$ then

$$\mathbf{d\eta} \cdot \mathbf{d\xi} = (0 \, d\eta \, 0) \begin{pmatrix} h^2 & 0 & 0 \\ 0 & h^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} d\xi \\ 0 \\ 0 \end{pmatrix} = 0 \tag{33}$$

c.   (a) Let's look at the inverse transformation

$$dx = \frac{1}{2}\left(\frac{-id\eta + d\xi}{f'^*} + \frac{id\eta + d\xi}{f'} \right) \tag{34}$$

$$dy = \frac{1}{2}\left(\frac{id\eta + d\xi}{f'^*} + \frac{id\eta - d\xi}{f'} \right) \tag{35}$$

With the two vectors in cartesian coords

$$\mathbf{v}_1 = \alpha_1 \mathbf{dx} + \beta_1 \mathbf{dy} \qquad \mathbf{v}_2 = \alpha_2 \mathbf{dx} + \beta_2 \mathbf{dy} \tag{36}$$

and in the ξ, η coords

$$\mathbf{v}_1 = \frac{1}{2}\left(\frac{\alpha_1 + i\beta_1}{f'^*} + \frac{a_1 - i\beta_1}{f'}\right)\mathbf{d\xi} + \frac{1}{2}\left(\frac{-i\alpha_1 + \beta_1}{f'^*} + \frac{ia_1 + \beta_1}{f'}\right)\mathbf{d\eta} \tag{37}$$

$$\mathbf{v}_2 = \frac{1}{2}\left(\frac{\alpha_2 + i\beta_2}{f'^*} + \frac{a_2 - i\beta_2}{f'}\right)\mathbf{d\xi} + \frac{1}{2}\left(\frac{-i\alpha_2 + \beta_2}{f'^*} + \frac{ia_2 + \beta_2}{f'}\right)\mathbf{d\eta} \tag{38}$$

The angle between to vectors is in both cases given by

$$\frac{\langle \mathbf{v}_1, \mathbf{v}_2 \rangle}{|\mathbf{v}_1||\mathbf{v}_2|} = \frac{g_{ij}v_1^i v_2^j}{\sqrt{g_{ij}v_1^i v_1^j}\sqrt{g_{ij}v_2^i v_2^j}} = \frac{\alpha_1\alpha_2 + \beta_1\beta_2}{\sqrt{\alpha_1^2 + \beta_1^2}\sqrt{\alpha_2^2 + \beta_2^2}} \tag{39}$$

So the transform conserves angles and therefore does NOT change shapes.

(b) Let's calculate the Laplace-Beltrami operator with $|g| = h^2$ and $g^{-1} = \mathrm{diag}(h^{-2}, h^{-2}, 1)$

$$\Delta = \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j\right) \tag{40}$$

$$= \frac{1}{h^2}\left(\partial_\xi(h^2\frac{1}{h^2}\partial_\xi) + \partial_\eta(h^2\frac{1}{h^2}\partial_\eta) + \partial_\zeta(h^2\partial_\zeta)\right) \tag{41}$$

$$= \frac{1}{h^2}(\partial_{\xi\xi} + \partial_{\eta\eta}) + \partial_{\zeta\zeta} \tag{42}$$

0.4 JACKSON - Classical Electrodynamics

0.4.1 Exercise 1.3 Charge densities and the Dirac delta function

$$\rho_a = \frac{Q}{4\pi R^2}\delta(r - R) \quad \rightarrow \quad \int \rho_a d^3r = 4\pi\frac{Q}{4\pi R^2}\int_0^\infty \delta(r - R)r^2\,dr \tag{43}$$

$$= Q \tag{44}$$

$$\rho_b = \frac{\lambda}{2\pi b}\delta(r - b) \quad \rightarrow \quad \int \rho_b d^3r = \frac{\lambda}{2\pi b}2\pi\int_0^L dz\int_0^\infty \delta(r - b)r\,dr \tag{45}$$

$$= \lambda L \tag{46}$$

$$\rho_c = \frac{Q}{\pi R^2}\theta(R - r)\delta(z) \quad \rightarrow \quad \int \rho_c d^3r = \frac{Q}{\pi R^2}2\pi\int dz\int_0^\infty \theta(r - R)r\,dr \tag{47}$$

$$= \frac{Q}{\pi R^2}2\pi\int dz\int_0^R r\,dr \tag{48}$$

$$= \frac{Q}{\pi R^2}2\pi\frac{R^2}{2} = Q \tag{49}$$

Now we got curvilinear coordinates so we need an additional $1/r$ scaling

$$\rho_d = \frac{Q}{\pi R^2 r}\theta(R - r)\delta(\vartheta - \pi/2) \quad \rightarrow \quad \int \rho_d d^3r = \frac{Q}{\pi R^2}2\pi\int_0^\infty \frac{r^2}{r}\theta(R - r)\int_0^\pi \delta(\vartheta - \pi/2)\sin\vartheta\,d\vartheta \tag{50}$$

$$= \frac{Q}{\pi R^2}2\pi\int_0^R r\int_0^\pi \delta(\vartheta - \pi/2)\sin\vartheta\,d\vartheta \tag{51}$$

$$= \frac{Q}{\pi R^2}2\pi\frac{R^2}{2}\sin\pi/2 = Q \tag{52}$$

0.4.2   Exercise 1.4 Charged spheres

We can utilize the Gauss theorem

$$\oint_S \vec{E} \cdot \vec{n} dA = \frac{1}{\epsilon_0} \int_V \rho(x) d^3 x \tag{53}$$

$$4\pi r^2 E_r = \frac{q_r}{\epsilon_0} \tag{54}$$

$$E_r = \frac{q_r}{4\pi\epsilon_0 r^2} \tag{55}$$

assuming a radial electrical field.

- Conducting sphere

$$\rho_{\text{cond}} = Q\delta(r - a) \tag{56}$$

$$E_r = \frac{1}{4\pi\epsilon_0} \cdot \begin{cases} 0 & r < a \\ Q/r^2 & r > a \end{cases} \tag{57}$$

- Uniform sphere

$$\rho_{\text{hom}} = Q\theta(a - r) \tag{58}$$

$$E_r = \frac{1}{4\pi\epsilon_0} \cdot \begin{cases} Q/a^3 \cdot r & r < a \\ Q/r^2 & r > a \end{cases} \tag{59}$$

- Nonuniform sphere

$$\rho_{\text{inhom}} = Q\frac{n + 3}{a^{n+3}} r^n \quad (r < a) \tag{60}$$

$$E_r = \frac{1}{4\pi\epsilon_0} \cdot \begin{cases} Qa^{n+3}r^{n+1} & r < a \\ Q/r^2 & r > a \end{cases} \tag{61}$$
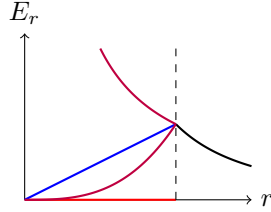

Figure 1: Jackson problem (1.4)

0.4.3   Exercise 1.5 Charge density of hydrogen atom

With the potential

$$\Phi = \frac{q}{4\pi\epsilon_0} \frac{e^{-\alpha r}}{r} \left(1 + \frac{\alpha r}{2}\right) \tag{62}$$

we calculate for $r > 0$

$$\rho_1 = -\epsilon_0 \triangle \Phi \tag{63}$$

$$= -\epsilon_0 \frac{1}{r^2} \partial_r (r^2 \partial_r \Phi) \tag{64}$$

$$= -\frac{q}{4\pi} e^{-\alpha r} \frac{\alpha^3}{2} \tag{65}$$

$$= -\frac{q}{\pi a_0^3} e^{-2r/a_0} \tag{66}$$

For $r = 0$ we have

$$\Phi(r \to 0) = \frac{q}{4\pi\epsilon_0 r} \tag{67}$$

$$\to \quad \rho_0 = q\delta(r) \tag{68}$$

Therefore

$$\rho = \rho_0 + \rho_1 \tag{69}$$

$$= q\left(\delta^{(3)}(r) - \frac{1}{\pi a_0^3}e^{-2r/a_0}\right) \tag{70}$$

Calculating the total charge

$$Q_0 = q\int d^3r\delta(r) = q \tag{71}$$

$$Q_1 = 4\pi\int_0^\infty r^2\rho_1 dr \tag{72}$$

$$= -\frac{4\pi q}{\pi a_0^3}\int_0^\infty r^2 e^{-2r/a_0}dr \tag{73}$$

$$= -\frac{4\pi q}{\pi a_0^3}\frac{a_0^3}{8}\int_0^\infty z^2 e^{-z}dz \tag{74}$$

$$= -\frac{4\pi q}{\pi a_0^3}\frac{a_0^3}{8}\Gamma(3) \tag{75}$$

$$= -q \tag{76}$$

0.4.4 Exercise 1.6 Simple capacitors

(a) Assuming only front and back surfaces contribute

$$2E_x A = \frac{Q}{\epsilon_0} \tag{77}$$

$$\to \quad E_x = \frac{Q}{2\epsilon_0 A} \tag{78}$$

$$\to \quad \phi = -\frac{Q}{2\epsilon_0 A}x \tag{79}$$

$$\to \quad \phi_{\text{tot}}(x) = -\frac{Q}{2\epsilon_0 A}x - \frac{-Q}{2\epsilon_0 A}(d-x) \tag{80}$$

$$= -\frac{Q}{2\epsilon_0 A}(x - (d-x)) \tag{81}$$

$$= -\frac{Q}{2\epsilon_0 A}(2x - d) \tag{82}$$

$$\to \quad C = \frac{Q}{\Delta\phi} = \frac{Q}{-\frac{Q}{2\epsilon_0 A}(-d-d)} \tag{83}$$

$$= \epsilon_0\frac{A}{d} \tag{84}$$

(b) The outer sphere does not contribute to the total potential as it is field free

$$4\pi r^2 E_r = \frac{Q}{\epsilon_0} \tag{85}$$

$$\rightarrow \quad E_r = \frac{Q}{4\pi\epsilon_0 r^2} \tag{86}$$

$$\rightarrow \quad \phi = \frac{Q}{4\pi\epsilon_0 r} \tag{87}$$

$$\rightarrow \quad \phi_{\text{tot}} = \frac{Q}{4\pi\epsilon_0 r} \quad (a < r < b) \tag{88}$$

$$\rightarrow \quad C = \frac{Q}{\Delta\phi} = \frac{Q}{\frac{Q}{4\pi\epsilon_0 b} - \frac{Q}{4\pi\epsilon_0 a}} \tag{89}$$

$$= \epsilon_0 \frac{4\pi ab}{b-a} \tag{90}$$

(c)

$$2\pi r L E_r = \frac{Q}{\epsilon_0} \tag{91}$$

$$\rightarrow \quad E_r = \frac{Q}{2\pi r L \epsilon_0} \tag{92}$$

$$\rightarrow \quad \phi = -\frac{Q}{2\pi L \epsilon_0} \log r \tag{93}$$

$$\rightarrow \quad \phi_{\text{tot}} = -\frac{Q}{2\pi L \epsilon_0} \log r \quad (a < r < b) \tag{94}$$

$$\rightarrow \quad C = \frac{Q}{\Delta\phi} = \frac{Q}{-\frac{Q}{2\pi L \epsilon_0} \log b + \frac{Q}{2\pi L \epsilon_0} \log a} \tag{95}$$

$$= \frac{2\pi L \epsilon_0}{\log a/b} \tag{96}$$

(d) ...

0.4.5   Exercise 1.7 Capacity of two parallel cylinders

Gauss law for one cylinder

$$\oint_S \vec{E} \cdot \vec{n} dA = \frac{1}{\epsilon_0} \int_V \rho(x) d^3 x \tag{97}$$

$$2\pi r L E_r = \frac{\rho_1 L}{\epsilon_0} \tag{98}$$

$$E_r = \frac{\rho}{2\pi\epsilon_0 r} \tag{99}$$

$$\phi = -\frac{\rho}{2\pi\epsilon_0} \ln r \tag{100}$$

For $d \gg a_{1,2}$ the potential of one cylinder on the surface of the second cylinder is constant - which means that the potential can be approximated by the sum of the potential of both cylinders (no

need to make it complicated)

$$\phi(\vec{r}) = \phi_1 + \phi_2 \tag{101}$$

$$= -\frac{\rho_1}{2\pi\epsilon_0}\ln|\vec{r}| - \frac{\rho_2}{2\pi\epsilon_0}\ln|\vec{r} - \vec{d}| \tag{102}$$

$$= -\frac{\rho}{2\pi\epsilon_0}\ln|\vec{r}| + \frac{\rho}{2\pi\epsilon_0}\ln|\vec{r} - \vec{d}| \tag{103}$$

$$= -\frac{\rho}{2\pi\epsilon_0}\left(\ln|\vec{r}| - \ln|\vec{r} - \vec{d}|\right) \tag{104}$$

$$= -\frac{\rho}{2\pi\epsilon_0}\ln\frac{|\vec{r}|}{|\vec{r} - \vec{d}|} \tag{105}$$

$$= -\frac{\rho}{\pi\epsilon_0}\ln\sqrt{\frac{|\vec{r}|}{|\vec{r} - \vec{d}|}} \tag{106}$$

Then the potential difference between to surfaces is given by (with $\vec{n} = \vec{d}/d$ and $\rho = \rho_1 = -\rho_2$)

$$\Delta\phi = \phi(a_1\vec{n}) - \phi((d - a_2)\vec{n}) \tag{107}$$

$$= -\frac{\rho}{\pi\epsilon_0}\left(\ln\sqrt{\frac{a_1}{d - a_1}} - \ln\sqrt{\frac{d - a_2}{a_2}}\right) \tag{108}$$

$$= \frac{\rho}{\pi\epsilon_0}\left(\ln\sqrt{\frac{d - a_1}{a_1}} + \ln\sqrt{\frac{d - a_2}{a_2}}\right) \tag{109}$$

$$\simeq \frac{\rho}{\pi\epsilon_0}\left(\ln\sqrt{\frac{d}{a_1}} + \ln\sqrt{\frac{d}{a_2}}\right) \tag{110}$$

$$\simeq \frac{\rho}{\pi\epsilon_0}\ln\frac{d}{\sqrt{a_1 a_2}} \tag{111}$$

With $C = Q/U$ we have

$$C = \frac{\rho L}{\Delta\phi} = \frac{\pi\epsilon_0 L}{\ln\frac{d}{\sqrt{a_1 a_2}}} \tag{112}$$

which is the desired result. The numbers are 0.49mm, 1.47mm and 4.92mm.

0.4.6 Exercise 1.8 Energy of capacitors

$$W = \frac{1}{2}\int \rho(x)\phi(x)d^3x = -\frac{\epsilon_0}{2}\int \phi\triangle\phi d^3x = \frac{\epsilon_0}{2}\int (\nabla\phi)^2 d^3x = \frac{\epsilon_0}{2}\int |\vec{E}|^2 d^3x \tag{113}$$

(a) With $\vec{E}_{\text{tot}} = -\nabla\phi_{\text{tot}}$ and $Q = C \cdot U$

$$W_{\text{plate}} = \frac{\epsilon_0}{2}\cdot\left(\frac{Q}{\epsilon_0 A}\right)^2\cdot(Ad) = \frac{Q^2 d}{2\epsilon_0 A} \tag{114}$$

$$= \frac{U^2 d}{2\epsilon_0 A}\left(\frac{\epsilon_0 A}{d}\right)^2 = \frac{\epsilon_0 A U^2}{2d} \tag{115}$$

$$W_{\text{sphere}} = \frac{\epsilon_0}{2}4\pi\int_a^b r^2\frac{Q^2}{16\pi^2\epsilon_0^2 r^4}dr = \frac{Q^2}{8\pi\epsilon_0}\left(\frac{1}{b} - \frac{1}{a}\right) \tag{116}$$

$$= \frac{U^2}{8\pi\epsilon_0}\left(\frac{a - b}{ab}\right)\cdot\left(\epsilon_0\frac{4\pi ab}{b - a}\right)^2 = 2\pi\epsilon_0 U^2\frac{ab}{b - a} \tag{117}$$

$$W_{\text{cylinder}} = \frac{\epsilon_0}{2} 2\pi L \int_a^b \left(\frac{Q}{2\pi\epsilon_0 L r} \right)^2 r \, dr = \frac{Q^2}{4\pi\epsilon_0 L} \log \frac{b}{a} \tag{118}$$

$$= \frac{U^2}{4\pi\epsilon_0 L} \log \frac{b}{a} \left(\frac{2\pi\epsilon_0 L}{\log b/a} \right)^2 = \frac{\pi\epsilon_0 L U^2}{\log b/a} \tag{119}$$

(b)

$$w_{\text{plate}} = \text{const} \tag{120}$$
$$w_{\text{sphere}} \sim r^{-4} \tag{121}$$
$$w_{\text{cylinder}} \sim r^{-2} \tag{122}$$

0.4.7   Exercise 5.1 Biot–Savart law NOT DONE YET

With

$$\nabla_{x'} \frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \tag{123}$$

we consider a loop of radius a in the $x - y$ plane

$$d\mathbf{B} = \frac{\mu_0 I}{4\pi} d\mathbf{l}' \times \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \tag{124}$$

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 I}{4\pi} \oint_C d\mathbf{l}' \times \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \tag{125}$$

$$= \frac{\mu_0 I}{4\pi} \oint_C d\mathbf{l}' \times \left(\nabla_{x'} \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \tag{126}$$

with P in the $x - z$ plane

$$(\mathbf{x} - \mathbf{x}')^2 = (r\cos\theta)^2 + ((r\sin\theta)^2 + a^2 - 2ar\sin\theta\cos\phi') \tag{127}$$

$$= r^2 + a^2 - 2ar\sin\theta\cos\phi' \tag{128}$$

0.4.8   Exercise 9.1 Rotating charge and current densities - NOT DONE YET

With $r = |\mathbf{x}|$ and $r' = |\mathbf{x}'|$

$$\mathbf{A}(\mathbf{x}, t) = \frac{\mu_0}{4\pi} \int dt' \int d^3\mathbf{x}' \frac{\mathbf{J}(\mathbf{x}', t')}{|\mathbf{x} - \mathbf{x}'|} \delta \left(t' + \frac{|\mathbf{x} - \mathbf{x}'|}{c} - t \right) \tag{129}$$

$$= \frac{\mu_0}{4\pi} \int dt' \int d^3\mathbf{x}' \frac{\mathbf{J}(\mathbf{x}', t - |\mathbf{x} - \mathbf{x}'|/c)}{|\mathbf{x} - \mathbf{x}'|} \tag{130}$$

$$= \frac{\mu_0}{4\pi} \sum_{l,m} \frac{4\pi}{2l+1} \frac{q_{lm}}{r^{l+1}} Y_{lm}(\vartheta, \varphi) \tag{131}$$

$$q_{lm}(t) = \int d^3\mathbf{x}' \, r'^l \, Y_{lm}^*(\vartheta', \varphi') \, \mathbf{J}(\mathbf{x}', t - |\mathbf{x} - \mathbf{x}'|/c) \tag{132}$$

$$\mathbf{J}(\mathbf{x}', t) = \rho(\mathbf{x}', t)\mathbf{v} = (\mathbf{\Omega} \times \mathbf{x}')\rho(\mathbf{x}', t) \tag{133}$$

0.4.9   Exercise 9.2 Rotating quadrupole - NOT DONE YET

Lets look at a single rotating point charge first

$$\rho(\mathbf{x}', t') = \frac{1}{r'^2 \sin\theta'} q\delta(r' - R)\delta(\phi' - \omega t')\delta(\theta' - \pi/2) \tag{134}$$

$$\mathbf{J}(\mathbf{x}', t') = \rho\mathbf{v} \tag{135}$$

$$= \frac{1}{r'^2 \sin\theta'} q\delta(r' - R)\delta(\phi' - \omega t')\delta(\theta' - \pi/2)R\omega\mathbf{e}_\phi \tag{136}$$

$$Y_{00} = \frac{1}{\sqrt{4\pi}} \qquad Y_{10} = \sqrt{\tfrac{3}{4\pi}}\cos\theta \qquad Y_{20} = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1) \quad (137)$$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi} \quad Y_{1,-1} = \sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\phi} \quad Y_{21} = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi} \quad (138)$$

$$Y_{2,-1} = \sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{-i\phi} \quad Y_{22} = \sqrt{\tfrac{15}{32\pi}}\sin^2\theta\, e^{2i\phi} \quad Y_{2,-2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{-2i\phi} \quad (139)$$

$$\rho(\mathbf{x}, t) = q\delta\left(x - \frac{a}{\sqrt{2}}\cos\omega t\right)\delta\left(y - \frac{a}{\sqrt{2}}\sin\omega t\right)\delta(z) + q\delta\left(x + \frac{a}{\sqrt{2}}\cos\omega t\right)\delta\left(y + \frac{a}{\sqrt{2}}\sin\omega t\right)\delta(z) \tag{140}$$

$$- q\delta\left(x + \frac{a}{\sqrt{2}}\sin\omega t\right)\delta\left(y - \frac{a}{\sqrt{2}}\cos\omega t\right)\delta(z) - q\delta\left(x - \frac{a}{\sqrt{2}}\sin\omega t\right)\delta\left(y + \frac{a}{\sqrt{2}}\cos\omega t\right)\delta(z) \tag{141}$$

$$\tag{142}$$

0.4.10 Exercise 12.1 Lagrangian of point charge

1. With $U^\alpha = \frac{dx_\alpha}{ds}$

$$L = -\frac{mU_\alpha U^\alpha}{2} - \frac{q}{c}U_\alpha A^\alpha \tag{143}$$

$$\frac{\partial L}{\partial x_\beta} = -\frac{q}{c}U_\alpha \frac{\partial A^\alpha}{\partial x_\beta} \tag{144}$$

$$\frac{\partial L}{\partial U_\beta} = -mU^\beta - \frac{q}{c}A^\beta \tag{145}$$

$$-m\frac{d}{ds}\left(\frac{dU^\beta}{ds}\right) - \frac{q}{c}\frac{dA^\beta}{ds} + \frac{q}{c}U_\alpha \frac{\partial A^\alpha}{\partial x_\beta} = 0 \tag{146}$$

$$m\frac{d^2 x^\beta}{ds^2} + \frac{q}{c}\frac{dA^\beta}{ds} - \frac{q}{c}\frac{dx_\alpha}{ds}\frac{\partial A^\alpha}{\partial x_\beta} = 0 \tag{147}$$

$$m\frac{d^2 x^\beta}{ds^2} + \frac{q}{c}\left(\frac{\partial A^\beta}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial s}\right) - \frac{q}{c}\frac{dx_\alpha}{ds}\frac{\partial A^\alpha}{\partial x_\beta} = 0 \tag{148}$$

$$m\frac{d^2 x^\beta}{ds^2} + \frac{q}{c}\frac{\partial x^\alpha}{\partial s}\left(\frac{\partial A^\beta}{\partial x^\alpha} - \frac{\partial A^\alpha}{\partial x_\beta}\right) = 0 \tag{149}$$

$$m\frac{d^2 x^\beta}{ds^2} + \frac{q}{c}\frac{\partial x^\alpha}{\partial s}F^{\alpha\beta} = 0 \tag{150}$$

2. Bit of a odd sign convention for the canonical momentum

$$P^\beta = -\frac{\partial L}{\partial U_\beta} = mU^\beta + \frac{q}{c}A^\beta \quad \rightarrow \quad U^\beta = \frac{1}{m}\left(P^\beta - \frac{q}{c}A^\beta\right) \tag{151}$$

$$H = P^\alpha U_\alpha + L \tag{152}$$

$$= P^\alpha \frac{1}{m}\left(P_\alpha - \frac{q}{c}A_\alpha\right) - \frac{m}{2}\frac{1}{m}\left(P_\alpha - \frac{q}{c}A_\alpha\right)\frac{1}{m}\left(P_\alpha - \frac{q}{c}A_\alpha\right) - \frac{q}{c}\frac{1}{m}\left(P_\alpha - \frac{q}{c}A_\alpha\right)A^\alpha \tag{153}$$

$$= \frac{1}{2m}\left(P^\alpha - \frac{q}{c}A^\alpha\right)\left(P_\alpha - \frac{q}{c}A_\alpha\right) \tag{154}$$

In space-time coordinates we can write

$$H = \frac{1}{2m}\left((p_0)^2 - \vec{p}^2 + \frac{q^2}{c^2}[\phi^2 - \vec{A}^2] + \frac{2q}{c}[\vec{p}\cdot\vec{A} - p^0\phi]\right) \tag{155}$$

$$= \frac{1}{2m}\left((\gamma mc)^2 - (\gamma m\vec{v})^2 + \frac{q^2}{c^2}[\phi^2 - \vec{A}^2] + \frac{2q}{c}[\gamma m\vec{v}\cdot\vec{A} - \gamma mc\phi]\right) \tag{156}$$

$$= \frac{\gamma^2 mc^2}{2}\left(1 - \frac{\vec{v}^2}{c^2}\right) + \frac{q^2}{2mc^2}[\phi^2 - \vec{A}^2] + q\gamma[\frac{1}{c}\vec{v}\cdot\vec{A} - \phi] \tag{157}$$

$$= \frac{mc^2}{2} + \frac{q^2}{2mc^2}[\phi^2 - \vec{A}^2] + q\gamma[\frac{1}{c}\vec{v}\cdot\vec{A} - \phi] \tag{158}$$

0.5 Schwinger - Classical Electrodynamics

0.5.1 Exercise 9.1 Lagrangian of a particle in an electromagnetic field

$$L = \mathbf{p}\cdot\left(\frac{d\mathbf{r}}{dt} - \mathbf{v}\right) + \frac{1}{2}mv^2 - e\phi + \frac{e}{c}\mathbf{v}\cdot\mathbf{A} \tag{159}$$

0.5.2 Exercise 31.1 Potentials of moving point charge

$$w = z - vt \rightarrow \frac{\partial}{\partial z} = \frac{\partial w}{\partial z}\frac{\partial}{\partial w} \tag{160}$$

$$\rightarrow \frac{\partial^2}{\partial z^2} = \frac{\partial^2 w}{\partial z^2}\frac{\partial}{\partial w} + \left(\frac{\partial w}{\partial z}\right)^2\frac{\partial^2}{\partial w^2} = \frac{\partial^2}{\partial w^2} \tag{161}$$

$$\rightarrow \frac{\partial^2}{\partial t^2} = \frac{\partial^2 w}{\partial t^2}\frac{\partial}{\partial w} + \left(\frac{\partial w}{\partial t}\right)^2\frac{\partial^2}{\partial w^2} = v^2\frac{\partial^2}{\partial w^2} \tag{162}$$

then

$$\triangle - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial w^2} - \frac{v^2}{c^2}\frac{\partial^2}{\partial w^2} \tag{163}$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(1 - \frac{v^2}{c^2}\right)\frac{\partial^2}{\partial w^2} \tag{164}$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial u^2} \tag{165}$$

with with $u = w/\sqrt{1 - v^2/c^2}$. The wave equation can then be rewritten

$$-\Box\phi = 4\pi\rho \tag{166}$$

$$= 4\pi e\delta(x)\delta(y)\delta(z - vt) \tag{167}$$

$$-\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial u^2}\right)\phi = 4\pi e\delta(x)\delta(y)\delta\left(\sqrt{1 - \frac{v^2}{c^2}}u\right) \tag{168}$$

$$= \frac{4\pi}{\sqrt{1 - \frac{v^2}{c^2}}}e\delta(x)\delta(y)\delta(u) \tag{169}$$

Using the Green function of the Coulomb equation (13.3) we obtain

$$\phi = \frac{e}{\sqrt{1 - \frac{v^2}{c^2}}\sqrt{u^2 + x^2 + y^2}} \tag{170}$$

$$= \frac{e}{\sqrt{w^2 + (1 - \frac{v^2}{c^2})(x^2 + y^2)}} \tag{171}$$

$$= \frac{e}{\sqrt{(z - vt)^2 + (1 - \frac{v^2}{c^2})(x^2 + y^2)}} \tag{172}$$

For the vector potential we can calculate similarly

$$-\Box\vec{A} = 4\pi\frac{\vec{j}}{c} \tag{173}$$

$$-\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial u^2}\right)\vec{A} = 4\pi e\frac{\vec{v}}{c}\delta(x)\delta(y)\delta\left(\sqrt{1 - \frac{v^2}{c^2}}u\right) \tag{174}$$

$$= \frac{4\pi}{\sqrt{1 - \frac{v^2}{c^2}}}e\frac{\vec{v}}{c}\delta(x)\delta(y)\delta(u) \tag{175}$$

which gives $\vec{A} = \vec{v}/c\phi$.

0.5.3   Exercise 31.2 Fields of moving point charge

$$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} \tag{176}$$

$$= \frac{e}{2}\left((z - vt)^2 + (1 - \frac{v^2}{c^2})(x^2 + y^2)\right)^{-3/2}[(1 - \frac{v^2}{c^2})2x, (1 - \frac{v^2}{c^2})2y, 2(z - vt)(1 - \frac{v^2}{c^2})] \tag{177}$$

$$= e(1 - \frac{v^2}{c^2})\left((z - vt)^2 + (1 - \frac{v^2}{c^2})(x^2 + y^2)\right)^{-3/2}[x, y, (z - vt)] \tag{178}$$

$$\vec{B} = \nabla \times \vec{A} \tag{179}$$

$$= -e\frac{v}{c}(1 - \frac{v^2}{c^2})\left((z - vt)^2 + (1 - \frac{v^2}{c^2})(x^2 + y^2)\right)^{-3/2}[y, x, 0] \tag{180}$$

0.5.4   Exercise 31.4 Wave equation for fields

With

$$\nabla \times \vec{B} = \frac{1}{c}\frac{\partial}{\partial t}\vec{E} + \frac{4\pi}{c}\vec{j_e} \tag{181}$$

$$\nabla \cdot \vec{E} = 4\pi\rho_e \tag{182}$$

$$-\nabla \times \vec{E} = \frac{1}{c}\frac{\partial}{\partial t}\vec{B} + \frac{4\pi}{c}\vec{j_m} \tag{183}$$

$$\nabla \cdot \vec{B} = 4\pi\rho_m \tag{184}$$

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we obtain

$$\nabla \times \nabla \times \vec{B} = \nabla(\nabla \cdot \vec{B}) - \triangle \vec{B} \tag{185}$$

$$= 4\pi \nabla \rho_m - \triangle \vec{B} \tag{186}$$

$$= \frac{1}{c}\frac{\partial}{\partial t}\nabla \times \vec{E} + \frac{4\pi}{c}\nabla \times \vec{j}_e \tag{187}$$

$$= -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{B} - \frac{4\pi}{c^2}\frac{\partial}{\partial t}\vec{j}_m + \frac{4\pi}{c}\nabla \times \vec{j}_e \tag{188}$$

$$\rightarrow -\Box\vec{B} = -4\pi\nabla\rho_m + \frac{4\pi}{c}(\nabla \times \vec{j}_e - \frac{1}{c}\frac{\partial}{\partial t}\vec{j}_m) \tag{189}$$

$$\nabla \times \nabla \times \vec{E} = \nabla(\nabla \cdot \vec{E}) - \triangle \vec{E} \tag{190}$$

$$= 4\pi \nabla \rho_e - \triangle \vec{E} \tag{191}$$

$$= -\frac{1}{c}\frac{\partial}{\partial t}\nabla \times \vec{B} - \frac{4\pi}{c}\nabla \times \vec{j}_m \tag{192}$$

$$= -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} - \frac{4\pi}{c^2}\frac{\partial}{\partial t}\vec{j}_e - \frac{4\pi}{c}\nabla \times \vec{j}_m \tag{193}$$

$$\rightarrow -\Box\vec{E} = -4\pi\nabla\rho_e + \frac{4\pi}{c}(\nabla \times \vec{j}_m - \frac{1}{c}\frac{\partial}{\partial t}\vec{j}_e) \tag{194}$$

0.5.5   Exercise 31.5 Lienard-Wiechert potentials

We start with the scalar potential

$$\phi(\mathbf{r}, t) = \int d\vec{r}' dt' \frac{\delta(\frac{1}{c}|\mathbf{r} - \mathbf{r}'| - (t - t'))}{|\mathbf{r} - \mathbf{r}'|}\rho(\mathbf{r}', t') \tag{195}$$

$$= \int d\mathbf{r}' dt' \frac{\delta(t' - t + \frac{1}{c}|\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|}e\delta(\mathbf{r}' - \mathbf{r}_B(t')) \tag{196}$$

$$= \int dt' \frac{e}{|\mathbf{r} - \mathbf{r}_B(t')|}\delta(t' - t + \frac{1}{c}|\mathbf{r} - \mathbf{r}_B(t')|) \tag{197}$$

with

$$\delta(t' - t + \frac{1}{c}|\mathbf{r} - \mathbf{r}_B(t')|) = \delta(f(t')) \tag{198}$$

$$= \sum_{t_{ret}} \frac{\delta(t' - t_{ret})}{|f'(t_{ret})|} \tag{199}$$

where

$$f(t') = t' - t + \frac{1}{c}|\mathbf{r} - \mathbf{r}'_B(t')| \tag{200}$$

$$f(t_{ret}) = t_{ret} - t + \frac{1}{c}|\mathbf{r} - \mathbf{r}_B(t_{ret})| = 0 \tag{201}$$

$$\rightarrow t_{ret} = t - \frac{1}{c}|\mathbf{r} - \mathbf{r}_B(t_{ret})| \tag{202}$$

$$f'(t') = 1 + \frac{1}{c}\partial_{t'}|\mathbf{r} - \mathbf{r}_B(t')| \tag{203}$$

$$= 1 + \frac{1}{c}\frac{2\mathbf{r}_B \cdot \mathbf{v}_B(t') - 2\mathbf{r} \cdot \mathbf{v}_B(t')}{2|\mathbf{r} - \mathbf{r}'_B(t')|} \tag{204}$$

$$\text{using} \quad |\mathbf{r} - \mathbf{r}_B(t')| = \sqrt{r^2 + r_B^2 - 2\mathbf{r} \cdot \mathbf{r}_B} \tag{205}$$

$$= 1 + \frac{1}{c}\frac{(\mathbf{r}_B(t') - \mathbf{r}) \cdot \mathbf{v}_B(t')}{|\mathbf{r} - \mathbf{r}_B(t')|} \tag{206}$$

then

$$\delta(t' - t + \frac{1}{c}|\mathbf{r} - \mathbf{r}_B(t')|) = \frac{\delta(t' - t_{ret})}{|f'(t_{ret})|} \tag{207}$$

$$= \frac{\delta(t' - t_{ret})}{1 + \frac{1}{c}\frac{(\mathbf{r}_B(t_{ret}) - \mathbf{r}) \cdot \mathbf{v}_B(t_{ret})}{|\mathbf{r} - \mathbf{r}_B(t_{ret})|}} \tag{208}$$

and therefore

$$\phi(\mathbf{r}, t) = \int dt' \frac{e}{|\mathbf{r} - \mathbf{r}_B(t')|} \delta(t' - t + \frac{1}{c}|\mathbf{r} - \mathbf{r}_B(t')|) \tag{209}$$

$$= \int dt' \frac{e}{|\mathbf{r} - \mathbf{r}_B(t')|} \frac{\delta(t' - t_{ret})}{1 + \frac{1}{c}\frac{(\mathbf{r}_B(t_{ret}) - \mathbf{r}) \cdot \mathbf{v}_B(t_{ret})}{|\mathbf{r} - \mathbf{r}_B(t_{ret})|}} \tag{210}$$

$$= \frac{e}{|\mathbf{r} - \mathbf{r}_B(t_{ret})| + \frac{1}{c}(\mathbf{r}_B(t_{ret}) - \mathbf{r}) \cdot \mathbf{v}_B(t_{ret})} \tag{211}$$

$$= \frac{e}{|\mathbf{r} - \mathbf{r}_B(t_{ret})| - [\mathbf{r} - \mathbf{r}_B(t_{ret})] \cdot \frac{\mathbf{v}_B(t_{ret})}{c}} \tag{212}$$

Now let's look at the vector potential

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{c} \int d\vec{r}' dt' \frac{\delta(\frac{1}{c}|\mathbf{r} - \mathbf{r}'| - (t - t'))}{|\mathbf{r} - \mathbf{r}'|} \mathbf{j}(\mathbf{r}', t') \tag{213}$$

$$= \frac{e}{c} \int d\vec{r}' dt' \frac{\delta(\frac{1}{c}|\mathbf{r} - \mathbf{r}'| - (t - t'))}{|\mathbf{r} - \mathbf{r}'|} \mathbf{v}(t') \delta(\mathbf{r}' - \mathbf{r}_B(t')) \tag{214}$$

$$= \frac{e}{c} \int dt' \frac{\delta(\frac{1}{c}|\mathbf{r} - \mathbf{r}_B(t')| - (t - t'))}{|\mathbf{r} - \mathbf{r}_B(t')|} \mathbf{v}(t') \tag{215}$$

$$= ... \tag{216}$$

$$= \frac{\mathbf{v}_B(t')}{c} \phi(\mathbf{r}, t) \tag{217}$$

0.5.6 Exercise 38.1 Total radiated power

We observe

$$\frac{\lambda}{(1 + \lambda\beta)^4} = -\frac{1}{\beta(1 + \lambda\beta)^4} + \frac{1}{\beta(1 + \lambda\beta)^3}. \tag{218}$$

Then

$$f(\lambda) = \frac{2}{(1 + \lambda\beta)^3} \left(-\frac{\beta^2}{2} + \frac{\lambda\beta}{8} \frac{\beta^2 - 1}{1 + \lambda\beta} \right) \tag{219}$$

$$= -\beta^2 \frac{1}{(1 + \lambda\beta)^3} + \frac{\beta(\beta^2 - 1)}{4} \frac{\lambda}{(1 + \lambda\beta)^4} \tag{220}$$

$$= \left(-\beta^2 + \frac{\beta^2 - 1}{4} \right) \frac{1}{(1 + \lambda\beta)^3} - \frac{(\beta^2 - 1)}{4} \frac{1}{(1 + \lambda\beta)^4} \tag{221}$$

$$\int_{-1}^{1} f(\lambda) d\lambda = -\frac{1 + 3\lambda^2}{4} \tag{222}$$

0.6 WALD - Advanced Classical Electrodynamics

0.6.1 Problem 2.3 The proton and the hydrogen atom

(a) Using Gauss law with spherical symmetry inside the nucleus

$$4\pi r^2 E_r(r) = \frac{1}{\varepsilon_0}\frac{4}{3}\pi r^3 \rho \tag{223}$$

$$4\pi R^2 E_r(r) = \frac{1}{\varepsilon_0}\frac{4}{3}\pi R^3 \rho = \frac{Q}{\varepsilon_0} \quad \rightarrow \quad \rho = \frac{Q}{\frac{4}{3}\pi R^3} \tag{224}$$

$$E_r(r) = \frac{1}{\varepsilon_0}\frac{Qr}{4\pi R^3} \tag{225}$$

and outside

$$E_r(r) = \frac{1}{\varepsilon_0}\frac{Q}{4\pi r^2} \tag{226}$$

Then the field energy is

$$\mathcal{E} = \frac{\varepsilon_0}{2}4\pi\left[\left(\frac{Q}{4\pi\varepsilon_0 R^3}\right)^2\int_0^R r^2 r^2 \, dr + \left(\frac{Q}{4\pi\varepsilon_0}\right)^2\int_0^R r^2\frac{1}{r^4}\, dr\right] \tag{227}$$

$$= \frac{3Q^2}{20\pi\varepsilon_0 R} \tag{228}$$

$$= 1.4\cdot 10^{-13}\text{J} \tag{229}$$

$$= 0.87\text{MeV} \tag{230}$$

while $mc^2 = 939\text{MeV}$.

(b) Interaction energy - we assume the proton to be a point charge

$$\mathcal{E} = \varepsilon_0\int d^3x\, \mathbf{E}_{\text{proton}}\cdot\mathbf{E}_{1\text{s}} \tag{231}$$

$$= \varepsilon_0\int d^3x\, E_{\text{proton,r}}\cdot E_{1\text{s,r}} \tag{232}$$

$$\simeq 4\pi\varepsilon_0\int_0^\infty dr r^2\frac{1}{\varepsilon_0}\frac{Q}{4\pi r^2}\cdot\frac{-Qe^{-2r/a}}{\pi a^3} \tag{233}$$

$$\simeq -\frac{Q^2}{\pi a^3}\int_0^\infty dr e^{-2r/a} \tag{234}$$

$$\simeq \frac{Q^2}{\pi a^3}\frac{a}{2}\left[e^{-2r/a}\right]_0^\infty \tag{235}$$

$$\simeq -\frac{Q^2}{\pi a^3}\frac{a}{2} = -\frac{Q^2}{2\pi a^2} \tag{236}$$

$$\simeq -9.1\text{eV} \tag{237}$$

with $\langle T\rangle = \frac{1}{2}\langle V\rangle$ we get to 13.6eV.

0.6.2 Problem 2.4 Potential from oddly shaped charge distribution

We need to calculate

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(r')}{|\vec{r}-\vec{r}'|}d^3\vec{r}' \tag{238}$$

trying to avoid the brute force calculation we see that we can expand the charge distribution in a finite series of Legendre polynomials

$$(1 - \cos\theta)^2 = \frac{4}{3}P_0(\cos\theta) - 2P_1(\cos\theta) + \frac{2}{3}P_2(\cos\theta) \tag{239}$$

where we used the orthogonality of the Legendre polynomials to project out the coefficients

$$\int_{-1}^{+1} P_n(x)P_m(x)dx = \frac{2}{2n+1}\delta_{mn} \tag{240}$$

$$\int_0^\pi P_n(\cos\theta)P_m(\cos\theta)\sin\theta\,d\theta = \frac{2}{2n+1}\delta_{mn} \tag{241}$$

with the multipole expansion (**for the outside**)

$$\frac{1}{|\vec{r} - \vec{r'}|} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}} \tag{242}$$

$$= \frac{1}{r\sqrt{1 + \frac{r'^2}{r^2} - 2\frac{r'}{r}\cos\theta}} \tag{243}$$

$$= \frac{1}{r}\sum_l P_l(\cos\theta)\frac{r'^l}{r^l} \tag{244}$$

we can insert all into the Coulomb integral and utilize the orthogonality of the Legendre polynomials again

$$\phi(\vec{r}) = \frac{2\pi\alpha}{4\pi\varepsilon_0 r}\int_0^R dr' r'^2 (R - r')\sum_l \frac{r'^l}{r^l}\int d\theta P_l(\cos\theta)\left(\frac{4}{3}P_0(\cos\theta) - 2P_1(\cos\theta) + \frac{2}{3}P_2(\cos\theta)\right)\sin\theta \tag{245}$$

$$= \frac{\alpha}{2\varepsilon_0 r}\int_0^R dr' r'^2 \frac{r'^l}{r^l}(R - r')\left[\frac{4}{3}\frac{2}{2\cdot 0 + 1} - 2\frac{2}{2\cdot 1 + 1}\frac{r'}{r} + \frac{2}{3}\frac{2}{2\cdot 2 + 1}\frac{r'^2}{r^2}\right] \tag{246}$$

$$= ... \tag{247}$$

$$= \frac{\alpha R^4}{9r\varepsilon_0}\left(1 - \frac{3R}{10r} + \frac{R^2}{25r^2}\right) \tag{248}$$

0.6.3 Exercise 5.1

Vacuum equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \mathbf{B} = \mu_0\mathbf{J} + \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} \tag{249}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \tag{250}$$

then

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t}\nabla \times \mathbf{B} \tag{251}$$

$$\nabla(\nabla \cdot \mathbf{E}) - \triangle^2\mathbf{E} = -\mu_0\frac{\partial\mathbf{J}}{\partial t} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} \tag{252}$$

$$\rightarrow \Box\mathbf{E} = \mu_0\frac{\partial\mathbf{J}}{\partial t} + \frac{1}{\epsilon_0}\nabla\rho \tag{253}$$

$$\nabla \times (\nabla \times \mathbf{B}) = \mu_0 \nabla \times \mathbf{J} + \frac{1}{c^2} \frac{\partial}{\partial t} \nabla \times \mathbf{E} \tag{254}$$

$$\nabla(\nabla \cdot \mathbf{B}) - \triangle^2 \mathbf{B} = \mu_0 \nabla \times \mathbf{J} - \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} \tag{255}$$

$$\rightarrow \Box \mathbf{B} = -\mu_0 \nabla \times \mathbf{J} \tag{256}$$

Construction a solution: Now observe that the charge continuity equation $\dot{\rho} + \nabla \cdot \mathbf{J} = 0$ can not be recovered from the two equations. So lets assume $\mathbf{J} = 0$ and $\rho(t) = q(t)\delta(\mathbf{x})$ then we set

$$\mathbf{B} = 0 \tag{257}$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q(t)}{r^2} \tag{258}$$

which violates $\nabla \times \mathbf{B}$

0.7 SMYTHE - Static and Dynamic Electricity

0.7.1 Exercise 1.1 Two coaxial rings and a point charge

Total charge of an axial ringlike charge distribution

$$Q = \int \rho_0(\varphi')\delta(z' - 0)\delta(r' - a)d\varphi' dz' dr' \tag{259}$$

$$= 2\pi a \rho_0 \tag{260}$$

which means that the 1-dimensional charge density is $\rho_0 = Q/2\pi a$. The axial potential of a single ring is then

$$\phi(z) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho_0 \delta(z' - 0)\delta(r' - a)}{\sqrt{a^2 + z^2}} r d\varphi' dz' dr' \tag{261}$$

$$= \frac{1}{4\pi\varepsilon_0} 2\pi a \rho_0 \frac{1}{\sqrt{a^2 + z^2}} \tag{262}$$

$$= \frac{Q}{4\pi\varepsilon_0} \frac{1}{\sqrt{a^2 + z^2}} \tag{263}$$

therefore we get for the energies

$$W_1 = \frac{qQ_1}{4\pi\varepsilon_0} \frac{1}{a} + \frac{qQ_2}{4\pi\varepsilon_0} \frac{1}{\sqrt{a^2 + b^2}} \tag{264}$$

$$W_2 = \frac{qQ_1}{4\pi\varepsilon_0} \frac{1}{\sqrt{a^2 + b^2}} + \frac{qQ_2}{4\pi\varepsilon_0} \frac{1}{a} \tag{265}$$

solving the linear system for the charges $Q_{1,2}$ we obtain

$$Q_1 = \frac{4\pi\varepsilon_0}{qb^2} \sqrt{a^2 + b^2} \left(\sqrt{a^2 + b^2} W_1 - aW_2 \right) \tag{266}$$

$$Q_2 = \frac{4\pi\varepsilon_0}{qb^2} \sqrt{a^2 + b^2} \left(-aW_1 + \sqrt{a^2 + b^2} W_2 \right). \tag{267}$$

0.7.2 Exercise 1.3 Flux of two point charges through circle

For the flux we have

$$N \equiv \int \vec{E} \cdot d\vec{A} \tag{268}$$

$$= \int E \cos(\vec{E}, \vec{n}) \, dA \tag{269}$$

$$= 2\pi \int \frac{q}{4\pi\varepsilon_0(a^2 + r^2)} \frac{a}{\sqrt{a^2 + r^2}} r dr - 2\pi \int \frac{Q}{4\pi\varepsilon_0(a^2 + r^2)} \frac{a}{\sqrt{a^2 + r^2}} r dr \tag{270}$$

$$= \frac{2\pi a}{4\pi\varepsilon_0}(q - Q) \int_0^a \frac{1}{(a^2 + r^2)^{3/2}} r dr \tag{271}$$

$$= \frac{1}{4\varepsilon_0}(q - Q)\left(2 - \sqrt{2}\right) \tag{272}$$

therefore

$$Q = q - \frac{4N\varepsilon_0}{2 - \sqrt{2}}. \tag{273}$$

0.7.3 Exercise 1.4 Concentric charged rings

The axial potential of a single ring is with radius a and charge $Q = 2\pi a \rho_0$ is

$$\phi(x) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho_0 \delta(z' - 0)\delta(r' - a)}{\sqrt{a^2 + x^2}} r d\varphi' dz' dr' \tag{274}$$

$$= \frac{1}{4\pi\varepsilon_0} 2\pi a \rho_0 \frac{1}{\sqrt{a^2 + x^2}} \tag{275}$$

$$= \frac{Q}{4\pi\varepsilon_0} \frac{1}{\sqrt{a^2 + x^2}} \tag{276}$$

The total potential and the resulting electrical field is therefore

$$\phi(x) = -\frac{Q}{4\pi\varepsilon_0} \frac{1}{\sqrt{a_1^2 + x^2}} + \frac{\sqrt{27}Q}{4\pi\varepsilon_0} \frac{1}{\sqrt{a_2^2 + x^2}} \tag{277}$$

$$E_x = -\frac{\partial\phi}{\partial x} \tag{278}$$

$$= \frac{Qx}{4\pi\varepsilon_0}\left(-\frac{1}{(a_2^2 + x^2)^{3/2}} + \frac{\sqrt{27}}{(a_2^2 + x^2)^{3/2}}\right) \tag{279}$$

which only vanishes for

$$x = 0, \pm\sqrt{\frac{-3a_1^2 + a2^2}{2}}. \tag{280}$$

Due to the radial symmetry the other field components at this points vanish too.

0.7.4   Exercise 1.19C Charged disc

$$\phi(z) = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma}{\sqrt{\rho'^2 + z^2}} \rho' \, d\rho' \, d\varphi' \tag{281}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{\sigma}{z} \int \frac{1}{\sqrt{1 + \rho'^2/z^2}} \rho' \, d\rho' \, d\varphi' \tag{282}$$

$$= \frac{1}{2\epsilon_0} \frac{\sigma}{z} \int_0^R \frac{1}{\sqrt{1 + \rho'^2/z^2}} \rho' \, d\rho' \tag{283}$$

$$= \frac{\sigma}{2\epsilon_0} z \left(\sqrt{1 + R^2/z^2} - 1 \right) \tag{284}$$

then we calculate the field

$$E(z) = -\frac{\partial \phi}{\partial z} \tag{285}$$

$$= \frac{\sigma}{2\epsilon_0} \left(1 - \frac{1}{\sqrt{1 + R^2/z^2}} \right) \tag{286}$$

and obtain

$$E = \frac{\sigma}{2\epsilon_0} \left\{ 1 - \frac{1}{\sqrt{26}}, 1 - \frac{3}{\sqrt{34}}, 1 - \frac{1}{\sqrt{2}}, 1 - \frac{7}{\sqrt{74}} \right\} \tag{287}$$

0.7.5   Exercise 12.1 Linear quadrupole

$$\beta = \omega\sqrt{\mu\epsilon} \tag{288}$$

$$q_{zz}^{(2)} = a^2 q \sin \omega t \tag{289}$$

$$8\pi\epsilon\vec{Z}_{zz} = a^2 q \sin \omega t \left(\frac{\beta}{r} - \frac{j}{r^2} \right) (\vec{r}_1 \cos \theta - \vec{\theta} \sin \theta) \cos \theta e^{-j\beta r} \tag{290}$$

$$\tag{291}$$