Book of Solutions

C Thierfelder

May 2020

1 Introduction

There is a theory which states that if ever anyone discovers exactly what the Universe is for and why it is here, it will instantly disappear and be replaced by something even more bizarre and inexplicable. There is another theory which states that this has already happened.

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2 Useful formulas

$$\left(\int_{-\infty}^{\infty} dx e^{-x^2}\right)^2 = \int_{-\infty}^{\infty} dx e^{-x^2} \cdot \int_{-\infty}^{\infty} dy e^{-y^2} \tag{1}$$

$$= \int_{\mathbb{R}^2} e^{-(x^2+y^2)} dx\, dy \tag{2}$$

$$= \int_0^{2\pi} \int_0^{2\pi} e^{-r^2} r dr \tag{3}$$

$$= -2\pi \left. \frac{e^{-r^2}}{2} \right|_0^{\infty} = \pi \tag{4}$$

Common integrals

$$\int_{-\infty}^{\infty} dx e^{-ax^2} = \sqrt{\frac{\pi}{a}} \qquad a > 0, a \in \mathbb{R} \tag{5}$$

$$\int_{-\infty}^{\infty} dx e^{-ax^2+bx+c} = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}+c} \qquad a > 0, a, b, c \in \mathbb{R} \tag{6}$$

$$\int_{-\infty}^{\infty} dx e^{iax^2} = \sqrt{\frac{\pi}{a}} e^{\frac{i\pi}{4}} \qquad a > 0, a \in \mathbb{R} \tag{7}$$

Common Fourier integrals

$$\int_{-\infty}^{\infty} dy e^{-ay^2} e^{-iby} = \sqrt{\frac{\pi}{a}} e^{-\frac{b^2}{4a}} \qquad a > 0, a, b \in \mathbb{R} \tag{8}$$

$$\int_{-\infty}^{\infty} dy e^{iay^2} e^{-iby} = \sqrt{\frac{\pi}{a}} e^{\frac{i}{4}\left(\pi - \frac{b^2}{a}\right)} \qquad a > 0, a, b \in \mathbb{R} \tag{9}$$

$$\int_{-\infty}^{\infty} dy e^{-(a+ic)y^2} e^{-iby} = \sqrt{\frac{\pi}{a+ic}} e^{-\frac{b^2}{4(a+ic)}} \qquad a > 0, a, b, c \in \mathbb{R} \tag{10}$$

$$= \sqrt{\frac{\pi}{a^2+c^2}} \sqrt{a-ic}\, e^{-\frac{b^2}{4(a^2+c^2)}(a-ic)} \tag{11}$$

Fourier transformation

Starting from the Fourier integral theorem we have some freedom to distribute the 2π between back and forth transformation ($a, b \in \mathbb{R}$)

$$F(k) = \sqrt{\frac{|b|}{(2\pi)^{1-a}}} \int_{-\infty}^{\infty} f(x) e^{ibkx} dx \quad \leftrightarrow \quad f(x) = \sqrt{\frac{|b|}{(2\pi)^{1+a}}} \int_{-\infty}^{\infty} F(t) e^{-ibkx} dk \tag{12}$$

Delta distribution

$$\int \delta(x) e^{-ikx} dx = 1 \tag{13}$$

$$\int e^{ik(x-y)} dk = 2\pi \delta(x-y) \tag{14}$$

Matrices

1. inverse $A^{-1}A = \mathbb{I}$
   - therefore $\mathbb{I} = (AB)(B^{-1}A^{-1}) \quad \rightarrow \quad (AB)^{-1} = B^{-1}A^{-1}$
2. Hermitian transpose $A^\dagger = (\overline{A})^T = \overline{A^T}$
   - $(AB)^\dagger = B^\dagger A^\dagger$ therefore $\mathbb{I} = (AA^{-1})^\dagger = (A^{-1})^\dagger A^\dagger \quad \rightarrow \quad (A^\dagger)^{-1} = (A^{-1})^\dagger$
3. Orthorgonal $A^T = A^{-1}$
4. Unitary $A^\dagger = A^{-1}$
5. Hermitian $A^\dagger = A$

Diagonalization

Any matrix A is called diagonalizable if there exists an invertible matrix S such that

$$D_A = S^{-1}AS \tag{15}$$

is a diagonal matrix. The diagonalizability of A is equivalent to the fact that the $\{\vec{v}_i\}$ are all linearly independent.

To find S and D_A one has to find the eigensystem $\{\lambda_i, \vec{v}_i\}$ with $A\vec{v}_i = \lambda_i\vec{v}_i$. Then $D_A S$ and S can be written as $S = (\vec{v}_1, ..., \vec{v}_n)$ and $D_A = \text{diag}(\lambda_1, ..., \lambda_n)$ because $AS = (A\vec{v}_1, ..., A\vec{v}_n) = (\lambda_1\vec{v}_1, ..., \lambda_n\vec{v}_n) = SD_A$.

Functional derivatives

Let $F[\phi]$ a functional, i.e. a mapping from a Banach space $\mathcal{M}$ to the field of real or complex numbers. The functional (Frechet) derivative $\delta F[\phi]/\delta\phi$ is defined by

$$\begin{aligned}
\delta F &= \int dx \frac{\delta F[\phi]}{\delta\phi(x)} \cdot \delta\phi(x) && (16)\\
&= \int dx \frac{\delta F[\phi]}{\delta\phi(x)} \cdot \epsilon\delta(x-y) && (17)\\
&= \epsilon\frac{\delta F[\phi]}{\delta\phi(y)} && (18)\\
&= F[\phi + \epsilon\delta(x-y)] - F[\phi] && (19)
\end{aligned}$$

which means

$$\begin{aligned}
\frac{\delta F[\phi]}{\delta\phi[y]} &= \lim_{\epsilon\to 0}\frac{F[\phi+\epsilon\delta(x-y)] - F[\phi]}{\epsilon} && (20)\\
F[\phi + \epsilon\delta(x-y)] &= F[\phi] + \epsilon\frac{\delta F[\phi]}{\delta\phi(y)} && (21)\\
&= F[\phi] + \epsilon\int dx \frac{\delta F[\phi]}{\delta\phi(x)} \cdot \delta(x-y) && (22)
\end{aligned}$$

- Product rule $F[\phi] = G[\phi]H[\phi]$

$$\frac{\delta F[\phi]}{\delta\phi(x)} = \frac{\delta(G[\phi]H[\phi])}{\delta\phi} \tag{23}$$

$$= \lim_{\epsilon\to 0} \frac{G[\phi+\epsilon\delta(x-y)]H[\phi+\epsilon\delta(x-y)] - G[\phi]H[\phi]}{\epsilon} \tag{24}$$

$$= \lim_{\epsilon\to 0} \frac{\left(G[\phi]+\epsilon\frac{\delta G}{\delta\phi}\right)\left(H[\phi]+\epsilon\frac{\delta H}{\delta\phi}\right) - G[\phi]H[\phi]}{\epsilon} \tag{25}$$

$$= \lim_{\epsilon\to 0} \frac{G[\phi]H[\phi] + \epsilon G[\phi]\frac{\delta H}{\delta\phi} + \frac{\delta G}{\delta\phi}H[\phi] + \epsilon^2\frac{\delta G}{\delta\phi}\frac{\delta H}{\delta\phi} - G[\phi]H[\phi]}{\epsilon} \tag{26}$$

$$= G[\phi]\frac{\delta H[\phi]}{\delta\phi(x)} + \frac{\delta G[\phi]}{\delta\phi(x)}H[\phi] \tag{27}$$

- Chain rule $F[G[\phi]]$

$$\delta F = \int dx \frac{\delta F[G[\phi]]}{\delta\phi(x)}\delta\phi(x) \tag{28}$$

$$\delta G = \int dy \frac{\delta G[\phi]}{\delta\phi(y)}\delta\phi(y) \tag{29}$$

$$\delta F = \int dz \frac{\delta F[G]}{\delta G(z)}\delta G(z) \tag{30}$$

$$= \int dz \frac{\delta F[G]}{\delta G(z)} \int dy \frac{\delta G[\phi]}{\delta\phi(y)}\delta\phi(y) \tag{31}$$

$$= \int dy \underbrace{\int dz \frac{\delta F[G]}{\delta G(z)}\frac{\delta G[\phi]}{\delta\phi(y)}}_{=\frac{\delta F[G[\phi]]}{\delta\phi(y)}} \delta\phi(y) \tag{32}$$

$$\frac{\delta F[G[\phi]]}{\delta\phi(y)} = \int dz \frac{\delta F[G]}{\delta G(z)}\frac{\delta G[\phi]}{\delta\phi(y)} \tag{33}$$

- Chain rule (special case) $F[g[\phi]]$

$$\frac{\delta F[g[\phi]]}{\delta\phi(y)} = ... \tag{34}$$

$$= \frac{\delta F}{\delta g(\phi(y))}\frac{dg(\phi)}{d\phi(y)} \tag{35}$$

Some examples

1. $F[\phi] = \int dx\phi(x)\delta(x)$

$$\frac{\delta F[\phi]}{\delta\phi(y)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int dx(\phi(x)+\epsilon\delta(x-y))\delta(x)) - \int dx\,\phi(x)\delta(x)\right) \tag{36}$$

$$= \int dx\,\delta(x-y))\delta(x) \tag{37}$$

$$= \delta(y) \tag{38}$$

2. $F[\phi] = \int dx\phi(x)$

$$\frac{\delta F[\phi]}{\delta\phi(y)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int dx(\phi(x)+\epsilon\delta(x-y))) - \int dx\,\phi(x)\right) \tag{39}$$

$$= \int dx\,\delta(x-y) \tag{40}$$

$$= 1 \tag{41}$$

3. $F_x[\phi] = \phi(x)$

$$\frac{\delta\phi(x)}{\delta\phi(y)} = \frac{\delta F_x[\phi]}{\delta\phi(y)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left((\phi(x)+\epsilon\delta(x-y))-\phi(x)\right) \tag{42}$$
$$= \delta(x-y) \tag{43}$$

4. $F[\phi] = \int dx\phi(x)^n$

$$\frac{\delta F[\phi]}{\delta\phi(y)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int dx(\phi(x)+\epsilon\delta(x-y)))^n - \int dx\,\phi(x)^n\right) \tag{44}$$
$$= \int dx\, n\phi(x)^{n-1}\delta(x-y) \tag{45}$$
$$= n\phi(y)^{n-1} \tag{46}$$

5. $F[\phi] = \int dx \left(\frac{\phi(x)}{dx}\right)^n$

6. $F_y[\phi] = \int dz K(y,z)\phi(z)$

$$\frac{\delta F_y[\phi]}{\delta\phi(x)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int dz(K(y,z)(\phi(z)+\epsilon\delta(z-x)) - \int dz K(y,z)\phi(z)\right) \tag{47}$$
$$= \int dz\, K(y,z)\delta(z-x) \tag{48}$$
$$= K(y,x) \tag{49}$$

7. $F_x[\phi] = \nabla\phi(x)$

$$\frac{\delta F[\phi]}{\delta\phi(y)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\nabla_x(\phi(x)+\epsilon\delta(x-y)) - \nabla_x\phi(x)\right) \tag{50}$$
$$= \nabla_x\delta(x-y) \tag{51}$$

8. $F[\phi] = g\left(G[\phi(x)]\right)$

$$\frac{\delta F[\phi]}{\delta\phi(y)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}g(G[\phi(x)+\epsilon\delta(x-y)]) - g(G[\phi(x)]) \tag{52}$$
$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}g(G[\phi(x)] + \epsilon\frac{\delta G}{\delta\phi}) - g(G[\phi(x)]) \tag{53}$$
$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}g(G[\phi(x)]) + g'\epsilon\frac{\delta G}{\delta\phi} - g(G[\phi(x)]) \tag{54}$$
$$= \frac{\delta G}{\delta\phi}g'(G[\phi(x)]) \tag{55}$$

Space hierarchy

1. K-Vector space $(K,\oplus,\odot)$
   - set V, field K with $(K,+,\cdot)$
   - vector addition $\oplus : V\times V\to V$
   - scalar multiplication $\odot : K\times V\to V$
2. Topological vector space
   - K-vector space

- continuous (smooth) vector addition and scalar multiplication

3. Metric (vector) space (M, d)
   - set M, metric $d : M \times M \to \mathbb{R}$
   - $d(x, y) = 0 \Leftrightarrow x = y$
   - $d(x, y) = d(y, x)$
   - $d(x, y) + d(y, z) \geq d(x, z)$
   - from the requirements above follows $d(x, y) \geq 0$

4. Normed vector space $(V, \|\cdot\|)$
   - K-vector space V, norm $\|\cdot\| : V \to \mathbb{R}$
   - Typically $K \in (\mathbb{R}, \mathbb{C})$ to have a definition of $|\lambda|$
   - $\|x\| \geq 0$
   - $\|x\| = 0 \Leftrightarrow x = 0$
   - $\|\lambda x\| = |\lambda|\|x\|$ with $\lambda \in K$
   - $\|x\| + \|y\| \geq \|x + y\|$
   - with $d(x, y) := \|x - y\|$ every normed vector space has also a metric
   - a metric does NOT induce a always norm as the linearity/homogeneity of the norm is not guaranteed

5. Banach space (complete normed vector space)
   - normed K-vector space $(V, \|\cdot\|)$ with $K \in (\mathbb{R}, \mathbb{C})$
   - completeness: every Cauchy sequence converges (with the metric induced by the norm) to a well defined limit
   - if the space is just a metric space (without a norm) the space is called Cauchy space

6. Hilbert space (complete vector space with a scalar product)
   - K-vector space V with $K \in (\mathbb{R}, \mathbb{C})$
   - scalar product $\langle \cdot, \cdot \rangle : V \times V \to K$
   - $\langle \lambda x_1 + x_2, y \rangle = \langle \lambda x_1, y \rangle + \langle \lambda x_2, y \rangle$
   - $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$ for $\lambda \in K$
   - $\langle x, y \rangle = \overline{\langle y, x \rangle}$ which implies $\langle x, x \rangle \in \mathbb{R}$
   - $\langle x, x \rangle > 0$
   - $\langle x, x \rangle = 0 \Leftrightarrow x = 0$
   - completeness: every Cauchy sequence converges (with the metric induced by the norm which is itself induced by the scalar product) to a well defined limit
   - without completeness the space is called Pre-Hilbert space

Tensors

- For a vector $\mathbf{A}$ the expression $\mathbf{A}^2$ is the squared distance between tip and tail.
- The inner product of two vectors can then be defined by the parallelogram law

$$\mathbf{A} \cdot \mathbf{B} \equiv \frac{1}{4} \left[(\mathbf{A} + \mathbf{B})^2 - (\mathbf{A} - \mathbf{B})^2 \right] \tag{56}$$

- A rank-n tensor $\mathbf{T} = \mathbf{T}(\_,\_,\_)$ is real-valued linear function of n vectors.

$$\mathbf{T}(\alpha\mathbf{A} + \mu\mathbf{B}, \mathbf{C}, \mathbf{D}) = \alpha\mathbf{T}(\mathbf{A}, \mathbf{C}, \mathbf{D}) + \beta\mathbf{T}(\mathbf{B}, \mathbf{C}, \mathbf{D}) \tag{57}$$

- Metric tensor

$$\mathbf{g}(\mathbf{A}, \mathbf{B}) \equiv \mathbf{A} \cdot \mathbf{B} \tag{58}$$

- A vector is a tensor of rank one

$$\mathbf{A}(\mathbf{C}) \equiv \mathbf{A} \cdot \mathbf{C} \tag{59}$$

- Tensor product

$$\mathbf{A} \otimes \mathbf{B} \otimes \mathbf{C}(\mathbf{E}, \mathbf{F}, \mathbf{G}) \equiv \mathbf{A}(\mathbf{E})\mathbf{B}(\mathbf{F})\mathbf{C}(\mathbf{G}) = (\mathbf{A} \cdot \mathbf{E})(\mathbf{B} \cdot \mathbf{F})(\mathbf{C} \cdot \mathbf{G}) \tag{60}$$

- Contraction

$$1\&3 \text{ contraction}(\mathbf{A} \otimes \mathbf{B} \otimes \mathbf{C} \otimes \mathbf{D}) \equiv (\mathbf{A} \cdot \mathbf{C})\mathbf{B} \otimes \mathbf{D} \tag{61}$$

- Orthogonal basis

$$\mathbf{e}_j \cdot \mathbf{e}_k = \delta_{jk} \tag{62}$$

- Component expansion

$$\mathbf{A} = A_j\mathbf{e}_j \quad \to \quad A_j = \mathbf{A}(\mathbf{e}_j) = \mathbf{A} \cdot \mathbf{e}_j \tag{63}$$

$$\mathbf{T} = T_{abc}\mathbf{e}_a \otimes \mathbf{e}_b \otimes \mathbf{e}_c \quad \to \quad T_{ijk} = \mathbf{T}(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k) \tag{64}$$

$$1\&3 \text{ contraction}(\mathbf{R}) \quad \to \quad R_{ijik} \tag{65}$$

$$\mathbf{g} \quad \to \quad g_{jk} = \mathbf{g}(\mathbf{e}_j, \mathbf{e}_k) = \mathbf{e}_j \cdot \mathbf{e}_k = \delta_{jk} \tag{66}$$

3 Primer special relativity

Definition of line element

$$\begin{aligned} ds^2 &= dx^\mu dx_\nu = \eta_{\mu\nu} dx^\mu dx^\nu && (67)\\ &= dx^T \eta dx && (68)\end{aligned}$$

Definition of Lorentz transformation

$$dx^\mu = \Lambda^\mu{}_\nu dx^\nu \tag{69}$$

By postulate the line element ds is invariant under Lorentz transformation

$$\begin{aligned} ds^2 &= \eta_{\mu\nu} dx^\mu dx^\nu && (70)\\ &\stackrel{!}{=} \eta_{\alpha\beta} \Lambda^\alpha{}_\mu dx^\mu \Lambda^\beta{}_\nu dx^\nu \quad \to \quad \eta_{\mu\nu} = \eta_{\alpha\beta}\Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu && (71)\end{aligned}$$

or analog

$$\begin{aligned} ds^2 &= dx^T \eta dx && (72)\\ &\stackrel{!}{=} (\Lambda dx)^T \eta (\Lambda dx) && (73)\\ &= dx^T \Lambda^T \eta \Lambda dx \quad \to \quad \eta = \Lambda^T \eta \Lambda && (74)\end{aligned}$$

Observation with the eigentime $d\tau = ds/c$ and 3-velocity $dx^i = v^i dt$

$$\frac{ds^2}{d\tau^2} = c^2 = c^2 \frac{dt^2}{d\tau^2} - \frac{dx^i}{dt}\frac{dx_i}{dt}\left(\frac{dt}{d\tau}\right)^2 \tag{75}$$

$$1 = \frac{dt^2}{d\tau^2}\left(1 - \frac{v^i v_i}{c^2}\right) \quad \to \quad \frac{dt}{d\tau} \equiv \gamma = \left(\sqrt{1-\frac{v^2}{c^2}}\right)^{-1} \tag{76}$$

Definition of 4-velocity with 3-velocity $d\vec{x} = \vec{v} dt$

$$\begin{aligned} u^\mu &\equiv \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt}\frac{dt}{d\tau} = \quad \to \quad u^\mu u_\mu = \eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = \frac{ds^2}{d\tau^2} = c^2 && (77)\\ &= (c, \vec{v})\gamma && (78)\end{aligned}$$

Object moving in x direction with v meaning $dx = v \cdot dt$ compared to rest frame $dx' = 0$

$$\begin{aligned} c^2 dt'^2 = ds^2 &= c^2 dt^2 - v^2 dt^2 && (79)\\ &= c^2 dt^2 \left(1 - \frac{v^2}{c^2}\right) && (80)\end{aligned}$$

$$dt' = \frac{ds}{c} \equiv d\tau = dt\sqrt{1-\frac{v^2}{c^2}} = \frac{dt}{\gamma} \tag{81}$$

Definition 4-momentum (using the 3-momentum $\vec{p} = \gamma m \vec{v}$)

$$\begin{aligned} p^\mu \equiv m u^\mu = (\gamma m c, \gamma m \vec{v}) = \left(\frac{E_p}{c}, \vec{p}\right) \quad &\to \quad p^\mu p_\mu = m^2 u^\mu u_\mu = m^2 c^2 && (82)\\ &\to \quad (p^0)^2 - p^i p_i = m^2 c^2 && (83)\\ &\to \quad p^0 = \sqrt{m^2 c^2 + \vec{p}^2} && (84)\\ &\to \quad E_p = \sqrt{m^2 c^4 + \vec{p}^2 c^2} && (85)\\ &\qquad\quad = \frac{mc^2}{\sqrt{1-\frac{\vec{v}^2}{c^2}}} && (86)\end{aligned}$$

First we observe

$$\eta_{\mu\nu} = \eta_{\alpha\beta}\Lambda^{\alpha}_{\ \mu}\Lambda^{\beta}_{\ \nu} \tag{87}$$
$$\det(\eta) = \det(\Lambda)^2\det(\eta) \tag{88}$$
$$1 = \det(\Lambda)^2. \tag{89}$$

Now we see

$$\Lambda_{\gamma}^{\ \nu}\Lambda^{\gamma}_{\ \mu} = \eta_{\alpha\gamma}\eta^{\nu\beta}\Lambda^{\alpha}_{\ \beta}\Lambda^{\gamma}_{\ \mu} \tag{90}$$
$$= \eta^{\nu\beta}(\eta_{\alpha\gamma}\Lambda^{\alpha}_{\ \beta}\Lambda^{\gamma}_{\ \mu}) \tag{91}$$
$$= \eta^{\nu\beta}\eta_{\beta\mu} \tag{92}$$
$$= \delta^{\nu}_{\ \mu} \tag{93}$$

which means in matrix notation $\Lambda_{\gamma}^{\ \nu} = (\Lambda^{-1})^{\nu}_{\ \gamma}$. General transformation laws for tensors of first order

$$V'^{\alpha} = \Lambda^{\alpha}_{\ \beta}V^{\beta} \tag{94}$$
$$\eta_{\alpha\mu}V'^{\alpha} = \eta_{\alpha\mu}\Lambda^{\alpha}_{\ \beta}V^{\beta} = \eta_{\alpha\mu}\Lambda^{\alpha}_{\ \beta}(\eta^{\nu\beta}V_{\nu}) \tag{95}$$
$$V'_{\mu} = \Lambda_{\mu}^{\ \nu}V_{\nu} \tag{96}$$
$$\rightarrow \quad \Lambda_{\mu}^{\ \nu} = \eta_{\alpha\mu}\eta^{\nu\beta}\Lambda^{\alpha}_{\ \beta} \tag{97}$$

and second order

$$T'^{\alpha\beta} = \Lambda^{\alpha}_{\ \mu}\Lambda^{\beta}_{\ \nu}T^{\mu\nu} \tag{98}$$
$$\eta_{\alpha\delta}\eta_{\beta\gamma}T'^{\alpha\beta} = \eta_{\alpha\delta}\eta_{\beta\gamma}\Lambda^{\alpha}_{\ \mu}\Lambda^{\beta}_{\ \nu}T^{\mu\nu} = \eta_{\alpha\delta}\eta_{\beta\gamma}\Lambda^{\alpha}_{\ \mu}\Lambda^{\beta}_{\ \nu}(\eta^{\mu\rho}\eta^{\nu\sigma}T_{\rho\sigma}) \tag{99}$$
$$T'_{\delta\gamma} = \Lambda_{\delta}^{\ \rho}\Lambda_{\gamma}^{\ \sigma}T_{\rho\sigma}. \tag{100}$$

The general transformation is therefore given by

$$T'_{\mu_1\mu_2\ldots}{}^{\nu_1\nu_2\ldots} = \Lambda_{\mu_1}{}^{\rho_1}\Lambda_{\mu_2}{}^{\rho_2}\ldots\Lambda^{\nu_1}{}_{\sigma_1}\Lambda^{\nu_2}{}_{\sigma_2}\ldots T'_{\rho_1\rho_2\ldots}{}^{\sigma_1\sigma_2\ldots} \tag{101}$$

There exist two invariant tensors

$$\eta'_{\mu\nu} = \eta_{\alpha\beta}\Lambda^{\alpha}_{\ \mu}\Lambda^{\beta}_{\ \nu} = \Lambda_{\beta\mu}\Lambda^{\beta}_{\ \nu} = \eta_{\mu\sigma}\Lambda_{\beta}^{\ \sigma}\Lambda^{\beta}_{\ \nu} = \eta_{\mu\sigma}\delta^{\sigma}_{\ \nu} = \eta_{\mu\nu} \tag{102}$$
$$\epsilon'^{\mu\nu\rho\sigma} = \Lambda^{\mu}_{\ \alpha}\Lambda^{\nu}_{\ \beta}\Lambda^{\rho}_{\ \gamma}\Lambda^{\sigma}_{\ \delta}\epsilon'^{\alpha\beta\gamma\delta} \equiv \epsilon^{\mu\nu\rho\sigma}\det(\Lambda) = \pm\epsilon^{\mu\nu\rho\sigma} \tag{103}$$

Due to the possibility of the minus sign the Levi-Civita symbol ϵ is sometimes called pseudo-tensor.

4 Groups

4.1 Overview

$\mathbb{F}$	GL$(n,\mathbb{F})$	SL$(n,\mathbb{F})$	U(n)	SU(n)	O(n)	SO(n)
$\mathbb{R}$	n^2	n^2-1	-	-	$n(n-1)/2$	$n(n-1)/2$
$\mathbb{C}$	$2n^2$	$2(n^2-1)$	n^2	n^2-1	$n(n-1)$	$n(n-1)$

Table 1: Dimensions of common Lie groups (number of independent real parameters)

Observation: $\dim(\mathrm{SO}(n,\mathbb{F})) = \dim(\mathrm{O}(n,\mathbb{F}))$ - sign that SO(n) is not connected

4.2 SO(2)

There are infinitely many (non-equivalent) 1-dimensional standard irreps

$$D^k(\alpha) = e^{-ik\alpha},\ k = 0, \pm1, \pm2, ... \tag{104}$$

4.3 SO(3)

4.4 SU(2)

Finite dimensional irreps of the Lorentz group are labeled by l with

$$l \in \left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2, ...\right\}. \tag{105}$$

and have dimension $2l+1$. For two irreps with $l \geq m$ the tensor product representations decomposes as (Clebsch-Gordan decomposition)

$$\begin{aligned} V_l \otimes V_m &\cong \bigoplus_{j=l-m}^{l+m} V_j && (106)\\ &= V_{l+m} \oplus V_{l+m-1} \oplus ... \oplus V_{l-m+1} \oplus V_{l-m} && (107)\\ \dim(V_l \otimes V_m) &= (2l+1)(2m+1) && (108)\\ \dim(V_{l+m} \oplus ... \oplus V_{l-m}) &= \sum_{k=0}^{2m} 2[(l-m)+k]+1 && (109)\\ &= (2m+1)[2(l-m)+1] + 2\frac{2m(2m+1)}{2} && (110)\\ &= (2m+1)(2l+1) && (111) \end{aligned}$$

4.5 SU(3)

4.6 Lorentz group O(1,3)

There are the obvious tensor representations for tensors of first and second order

$$\begin{aligned} [D(\Lambda)]^{\alpha}_{\ \beta} = \Lambda^{\alpha}_{\ \beta} \quad &\rightarrow \quad V^{\alpha} = [D(\Lambda)]^{\alpha}_{\ \beta} V^{\beta} = \Lambda^{\alpha}_{\ \beta} V^{\beta} && (112)\\ [D(\Lambda)]_{\alpha\beta}^{\ \ \gamma\delta} = \Lambda_{\alpha}^{\ \gamma}\Lambda_{\beta}^{\ \delta} \quad &\rightarrow \quad T_{\alpha\beta} = [D(\Lambda)]_{\alpha\beta}^{\ \ \gamma\delta} T_{\gamma\delta} = \Lambda_{\alpha}^{\ \gamma}\Lambda_{\beta}^{\ \delta} T_{\gamma\delta} && (113) \end{aligned}$$

which are 4 and 16 dimensional.

Infinitesimal Lorentz transformations can be written as

$$\Lambda^{\alpha}_{\ \beta} = \delta^{\alpha}_{\ \beta} + \omega^{\alpha}_{\ \beta} \qquad (|\omega^{\alpha}_{\ \beta}| \ll 1). \tag{114}$$

The first order approximation gives an additional restriction

$$\begin{aligned} \eta_{\mu\nu} &= \eta_{\alpha\beta}\Lambda^{\alpha}_{\ \mu}\Lambda^{\beta}_{\ \nu} = \eta_{\alpha\beta}(\delta^{\alpha}_{\ \mu} + \omega^{\alpha}_{\ \mu})(\delta^{\beta}_{\ \nu} + \omega^{\beta}_{\ \nu}) = \eta_{\mu\nu} + \eta_{\mu\beta}\omega^{\beta}_{\ \nu} + \eta_{\alpha\nu}\omega^{\alpha}_{\ \mu} && (115)\\ &\rightarrow \omega_{\mu\nu} = -\omega_{\nu\mu} && (116) \end{aligned}$$

which implies six independent components. As the four dimensional representation of the infinitesimal transformation is close to unity it can then be written as

$$D(\Lambda) = D(1+\omega) = 1 + \frac{1}{2}\omega^{\alpha\beta}\sigma_{\alpha_\beta} \tag{117}$$

where the six ω components correspond to the six matrices $\sigma_{01}, \sigma_{02}, \sigma_{03}, \sigma_{12}, \sigma_{13}, \sigma_{23}$ which are the generators of the group.

Finite dimensional irreps of the Lorentz group are labeled by two parameters (μ,ν) with

$$\mu,\nu \in \left\{0,\frac{1}{2},1,\frac{3}{2},2,...\right\}. \tag{118}$$

and have dimension $(2\mu+1)(2\nu+1)$

$$\begin{aligned} M^2 &= \mu(\mu+1) \\ N^2 &= \nu(\nu+1) \\ j &\in |\mu-\nu|,...,(\mu+\nu) \end{aligned}$$

irrep	dim	j	example
$(0,0)$	1	0	Scalar
$(\frac{1}{2},0)$	2	$\frac{1}{2}$	Left-handed Weyl spinor
$(0,\frac{1}{2})$	2	$\frac{1}{2}$	Right-handed Weyl spinor
$(\frac{1}{2},\frac{1}{2})$	4	0,1	4-Vector A^μ
$(1,0)$	3	1	Self-dual 2-form
$(0,1)$	3	1	Anti-self-dual 2-form
$(1,1)$	9	0,1,2	Traceless symmetric 2^{nd} rank tensor

rep	dim	j	example
$(\frac{1}{2},0)\oplus(0,\frac{1}{2})$	-	-	Dirac bispinor $\psi^\alpha \quad \alpha\in\{1,2,3,4\}$
$(\frac{1}{2},\frac{1}{2})\otimes\left[(\frac{1}{2},0)\oplus(0,\frac{1}{2})\right]$	-	-	Rarita-Schwinger field $\psi^\alpha \quad \alpha\in\{1,2,3,4\}$
$(0,1)\oplus(0,1)$	-	-	Parity invariant field of 2-forms

5 Mathematical

5.1 Arnol'd - A mathematical trivium

Problem 4

$$\frac{x^2+1}{x^3-x} = \frac{x^2+1}{x(x+1)(x-1)} \tag{119}$$
$$= -\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x-1} \tag{120}$$

$$\frac{d}{dx}(x+a)^{-1} = -(x+a)^{-2} \tag{121}$$
$$\frac{d^{100}}{dx^{100}}(x+a)^{-1} = 100!(x+a)^{-101} \tag{122}$$
$$\tag{123}$$

$$\frac{d^{100}}{dx^{100}}\left(\frac{x^2+1}{x^3-x}\right) = 100!\left(-\frac{1}{x^{101}} + \frac{1}{(x+1)^{101}} + \frac{1}{(x-1)^{101}}\right) \tag{124}$$

Problem 13

$$\int_1^{10} x^x dx = \int_1^{10} e^{x\log x} dx \tag{125}$$

Problem 20

$$\ddot{x} = x + A\dot{x}^2 \qquad x(0) = 1, \dot{x}(0) = 0 \tag{126}$$

Using the standard perturbation theory approach we assume $x(t) = x_0(t) + Ax_1(t) + A^2x_2(t) + ...$ Inserting into the ODE gives

$$\ddot{x}_0 + A\ddot{x}_1 + A^2\ddot{x}_2 + ... = x_0 + Ax_1 + A^2x_2 + ... + A\left(\dot{x}_0 + A\dot{x}_1 + A^2\dot{x}_2 + ...\right)^2. \tag{127}$$

Sorting by powers of A we obtain a set of ODEs

$$A^0: \qquad \ddot{x}_0 = x_0 \tag{128}$$
$$A^1: \qquad \ddot{x}_1 = x_1 + \dot{x}_0^2 \tag{129}$$
$$A^2: \qquad \ddot{x}_2 = x_2 + 2\dot{x}_0\dot{x}_1. \tag{130}$$

The first ODE can be solved directly

$$x_0 = c_1e^t + c_2e^{-t}. \tag{131}$$

The second ODE then transforms into

$$\ddot{x}_1 = x_1 + c_1^2e^{2t} + c2^2e^{-2t} - 2c_1c_2 \tag{132}$$

with the homogeneous solution

$$x_{1H} = c_3e^t + c_4e^{-t}. \tag{133}$$

For the particular solution we try the ansatz (inspired by the inhomogeneity)

$$x_{1S} = \alpha + \beta e^{2t} + \gamma e^{-2t} \tag{134}$$
$$= 2c_1c_2 + \frac{c_1^2}{3}e^{2t} + \frac{c_2^2}{3}e^{-2t} \tag{135}$$

then

$$x_1 = x_{1H} + x_{1S} \tag{136}$$
$$= c_3e^t + c_4e^{-t} + 2c_1c_2 + \frac{c_1^2}{3}e^{2t} + \frac{c_2^2}{3}e^{-2t} \tag{137}$$

Imposing initial conditions on x_0 gives

$$c_1 = c_2 = \frac{1}{2} \quad \rightarrow \quad x_0 = \cosh t \tag{138}$$
$$c_3 = c_4 = -\frac{1}{3} \quad \rightarrow \quad x_1 = -\frac{2}{3}\cosh t + \frac{1}{2} + \frac{1}{6}\cosh 2t \tag{139}$$

and therefore

$$\left.\frac{dx(t)}{dA}\right|_{A=0} = \frac{1}{2} - \frac{2}{3}\cosh t + \frac{1}{6}\cosh 2t \tag{140}$$

Problem 23

$$y' = x + \frac{x^3}{y} \tag{141}$$

Problem 85

In three dimensions we have

$$x^2 + y^2 + z^2 + xy + yz + zx = 1 \tag{142}$$

which can be written as

$$\vec{x}^T A\vec{x} = 1 \tag{143}$$
$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1 \tag{144}$$

With an orthorgonal matrix S ($S^{-1} = S^T$) we can rotate the ellipsoid to line it up with the coordinate axes (choose S such that $D_A = S^{-1}AS$ is diagonal)

$$1 = \vec{x}^T A\vec{x} \tag{145}$$
$$= \vec{x}^T(SS^{-1})A(SS^{-1}\vec{x} \tag{146}$$
$$= (\vec{x}^T S)S^{-1}AS(S^{-1}\vec{x}) \tag{147}$$
$$= (\vec{x}^T S)S^{-1}AS(S^T\vec{x}) \tag{148}$$
$$= (S^T\vec{x})^T S^{-1}AS(S^T\vec{x}) \tag{149}$$
$$= (S^T\vec{x})^T D_A(S^T\vec{x}) \tag{150}$$

For this we need to find the eigensystem $\{\vec{v}_i, \lambda_i\}$ of A. The characteristic polynomial is given by

$$\lambda^3 - 3\lambda^2 + \frac{9}{4}\lambda - \frac{1}{2} = 0. \tag{151}$$

Then

$$S = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{152}$$

$$D_A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \tag{153}$$

the length of the principal axes are therefore 4, 1 and 1.

5.2 Spivak - Calculus on Manifolds

5.3 Flanders - Differential Forms with Applications to the Physical Sciences

5.4 Morse, Feshbach - Methods of mathematical physics

Problem 1.1

With

$$\cot^2\psi = \frac{\cos^2\psi}{\sin^2\psi} = \frac{\cos^2\psi}{1-\cos^2\psi} \tag{154}$$

we can obtain a quadratic equation

$$(x^2+y^2)\cos^2\psi(1-\cos^2\psi) + z^2\cos^2\psi = a^2(1-\cos^2\psi) \tag{155}$$

$$\cos^4\psi - \frac{x^2+y^2+z^2+a^2}{x^2+y^2}\cos^2\psi + \frac{a^2}{x^2+y^2} = 0 \tag{156}$$

with the solution

$$\cos^2\psi = \frac{x^2+y^2+z^2+a^2}{2(x^2+y^2)} \pm \sqrt{\frac{(x^2+y^2+z^2+a^2)^2}{4(x^2+y^2)^2} - \frac{4a^2(x^2+y^2)}{4(x^2+y^2)^2}} \tag{157}$$

$$= \frac{x^2+y^2+z^2+a^2 \pm \sqrt{(x^2+y^2+z^2+a^2)^2 - 4a^2(x^2+y^2)}}{2(x^2+y^2)} \tag{158}$$

To obtain the gradient we differentiate the surface equation implicitly with respect to x, y and z

$$2x\cos^2\psi - 2(x^2+y^2)\cos\psi\sin\psi\frac{\partial\psi}{\partial x} - 2z^2\cot\psi\csc^2\psi\frac{\partial\psi}{\partial x} = 0 \tag{159}$$

$$\rightarrow \frac{\partial\psi}{\partial x} = \psi_x = \frac{x\cos^2\psi}{z^2\cot\psi\csc^2\psi + (x^2+y^2)\sin\psi\cos\psi} \tag{160}$$

$$2y\cos^2\psi - 2(x^2+y^2)\cos\psi\sin\psi\frac{\partial\psi}{\partial x} - 2z^2\cot\psi\csc^2\psi\frac{\partial\psi}{\partial x} = 0 \tag{161}$$

$$\rightarrow \frac{\partial\psi}{\partial y} = \psi_y = \frac{y\cos^2\psi}{z^2\cot\psi\csc^2\psi + (x^2+y^2)\sin\psi\cos\psi} \tag{162}$$

$$-2(x^2+y^2)\cos\psi\sin\psi\frac{\partial\psi}{\partial z} + 2z\cot^2\psi - 2z^2\cot\psi\csc^2\psi\frac{\partial\psi}{\partial z} = 0 \tag{163}$$

$$\rightarrow \frac{\partial\psi}{\partial z} = \psi_z = \frac{z\cot^2\psi}{z^2\cot\psi\csc^2\psi + (x^2+y^2)\cos\psi\sin\psi} \tag{164}$$

The direction cosines are then given by

$$\cos\alpha = \frac{\psi_x}{\sqrt{\psi_x^2+\psi_y^2+\psi_z^2}} = \frac{2\sqrt{2}x\sin^2\psi}{\sqrt{8z^2+(x^2+y^2)(3-4\cos 2\psi+\cos 4\psi)}} \tag{165}$$

$$\cos\beta = \frac{\psi_y}{\sqrt{\psi_x^2+\psi_y^2+\psi_z^2}} = \frac{2\sqrt{2}y\sin^2\psi}{\sqrt{8z^2+(x^2+y^2)(3-4\cos 2\psi+\cos 4\psi)}} \tag{166}$$

$$\cos\gamma = \frac{\psi_z}{\sqrt{\psi_x^2+\psi_y^2+\psi_z^2}} = \frac{2\sqrt{2}z}{\sqrt{8z^2+(x^2+y^2)(3-4\cos 2\psi+\cos 4\psi)}}. \tag{167}$$

The second derivatives (for the Laplacian) can again be calculated via (lengthy) implicit differentiation and substituting the first derivatives from above. Adding them up gives zero which implies $\triangle\psi = 0$.

The surface equations $\psi = \text{const}$ can be written in form of an ellipsoid

$$\frac{x^2}{a^2\sec^2\psi} + \frac{y^2}{a^2\sec^2\psi} + \frac{z^2}{a^2\tan^2\psi} = 1 \tag{168}$$

which degenerates to a flat pancake for $\psi = 0, \pi$.

Problem 4.1

Standard trick

$$x = \tan\vartheta/2 \to d\theta = \frac{2dx}{1+x^2},\ \sin\vartheta = \frac{2x}{1+x^2},\ \cos\vartheta = \frac{1-x^2}{1+x^2} \tag{169}$$

$$\int_0^{2\pi} \frac{\sin^2\vartheta d\vartheta}{a+b\cos\vartheta} = \int_?^? \frac{8x^3\cdot dx}{(1+x^2)^3(a+b\frac{1-x^2}{1+x^2})} \tag{170}$$

5.5 Woit - Quantum Theory, Groups and Representations

Problem B.1-3

Rotations of the 2D-plane

$$D^2_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \tag{171}$$

$$D^2_\phi D^2_\theta = \begin{pmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -\cos\phi\sin\theta - \cos\theta\sin\phi \\ \cos\phi\sin\theta + \cos\theta\sin\phi & \cos\theta\cos\phi - \sin\theta\sin\phi \end{pmatrix} \tag{172}$$

$$= \begin{pmatrix} \cos(\phi+\theta) & -\sin(\phi+\theta) \\ \sin(\phi+\theta) & \cos(\phi+\theta) \end{pmatrix} \tag{173}$$

$$= D^2_{\phi+\theta} \tag{174}$$

can also be represented by

$$D^1_\phi = e^{i\phi} \tag{175}$$

$$D^1_\phi D^1_\theta = e^{i\phi}e^{i\theta} = e^{i(\phi+\theta)} \tag{176}$$

$$= D^1_{\phi+\theta}. \tag{177}$$

Furthermore there is also the trivial representation

$$D^{1'}_\phi = 1 \tag{178}$$

$$D^{1'}_\phi D^1_\theta = 1\cdot 1 = 1 \tag{179}$$

$$= D^{1'}_{\phi+\theta} \tag{180}$$

Problem B.1-4

The time evolution is given by

$$
\begin{aligned}
|\Psi(t)\rangle &= e^{-iHt}|\Psi(0)\rangle && (181)\\
&= \left(\sum_{k=0}^{\infty}\frac{(-iHt)^k}{k!}\right)|\Psi(0)\rangle && (182)
\end{aligned}
$$

We see

$$
H = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad H^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \qquad H^3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix} \tag{183}
$$

and calculate

$$
\begin{aligned}
\sum_{k=0}^{\infty}\frac{(-it)^{2k}}{(2k)!} &= \sum_{k=0}^{\infty}(-1)^k\frac{t^{2k}}{(2k)!} = \cos(t) && (184)\\
\sum_{k=0}^{\infty}\frac{(-it)^{2k+1}}{(2k+1)!} &= (-i)\sum_{k=0}^{\infty}(-1)^k\frac{t^{2k+1}}{(2k+1)!} = -i\sin(t) && (185)\\
\sum_{k=0}^{\infty}\frac{(-i2t)^k}{k!} &= \cos(2t) - i\sin(2t) = e^{-i2t} && (186)
\end{aligned}
$$

which gives

$$
e^{-iHt} = \begin{pmatrix} \cos(t) & -i\sin(t) & 0 \\ -i\sin(t) & \cos(t) & 0 \\ 0 & 0 & e^{-2it} \end{pmatrix} \tag{187}
$$

and therefore

$$
|\Psi(t)\rangle = \begin{pmatrix} \psi_1\cos(t) - \psi_2 i\sin(t) \\ -\psi_1 i\sin(t) + \psi_2\cos(t) \\ \psi_3 e^{-2it} \end{pmatrix} \tag{188}
$$

. To check the result one can calculate both sides of $i\partial_t|\Psi(t)\rangle = H|\Psi(t)\rangle$.

Problem B.2-1

1. With $M = PDP^{-1}$ we have $M^2 = PDP^{-1}PDP^{-1} = PDDP^{-1}$ and see

$$
\begin{aligned}
e^{tM} &= \sum_{k=0}^{\infty}\frac{(tM)^k}{k!} = \sum_{k=0}^{\infty}\frac{(tPDP^{-1})^k}{k!} = \sum_{k=0}^{\infty}\frac{P(tD)^kP^{-1}}{k!} && (189)\\
&= P\left(\sum_{k=0}^{\infty}\frac{(tD)^k}{k!}\right)P^{-1} = Pe^{tD}P^{-1}. && (190)
\end{aligned}
$$

The eigenvalues of M are given by

$$
-\lambda^3 - (-\lambda)(-\pi^2) = 0 \quad \rightarrow \quad \lambda_1 = i\pi,\ \lambda_2 = -i\pi,\ \lambda_3 = 0 \tag{191}
$$

with the eigenvectors

$$
\begin{aligned}
\vec{v}_1 &= (-i, 1, 0) && (192)\\
\vec{v}_2 &= (i, 1, 0) && (193)\\
\vec{v}_3 &= (0, 0, 1) && (194)
\end{aligned}
$$

we obtain

$$\begin{aligned}
M &= PDP^{-1} &(195)\\
&= \begin{pmatrix} -i & i & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} i\pi & 0 & 0 \\ 0 & -i\pi & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} i/2 & 1/2 & 0 \\ -i/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} &(196)
\end{aligned}$$

With

$$\sum_{k=0}^{\infty} \frac{(i\pi)^k}{k!} = e^{i\pi} \tag{197}$$

$$\sum_{k=0}^{\infty} \frac{(-i\pi)^k}{k!} = e^{-i\pi} \tag{198}$$

we see

$$tD^k = \begin{pmatrix} (i\pi t)^k & 0 & 0 \\ 0 & (-i\pi t)^k & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{199}$$

$$e^{tD} = \sum_{k=0}^{\infty} \frac{(tD)^k}{k!} = \begin{pmatrix} e^{i\pi t} & 0 & 0 \\ 0 & e^{-i\pi t} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{200}$$

and therefore

$$\begin{aligned}
e^{tM} &= Pe^{tD}P^{-1} &(201)\\
&= \begin{pmatrix} \frac{1}{2}(e^{-i\pi t} + e^{i\pi t}) & -\frac{1}{2}i(e^{i\pi t} - e^{-i\pi t}) & 0 \\ -\frac{1}{2}i(e^{-i\pi t} - e^{i\pi t}) & \frac{1}{2}(e^{-i\pi t} + e^{i\pi t}) & 0 \\ 0 & 0 & 1 \end{pmatrix} &(202)\\
&= \begin{pmatrix} \cos(\pi t) & \sin(\pi t) & 0 \\ -\sin(\pi t) & \cos(\pi t) & 0 \\ 0 & 0 & 1 \end{pmatrix} &(203)
\end{aligned}$$

2. Brute force calculation of the matrix powers reveals

$$(tM)^2 = \begin{pmatrix} -(t\pi)^2 & 0 & 0 \\ 0 & -(t\pi)^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (tM)^3 = \begin{pmatrix} 0 & -(t\pi)^3 & 0 \\ (t\pi)^3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{204}$$

$$(tM)^4 = \begin{pmatrix} (t\pi)^4 & 0 & 0 \\ 0 & (t\pi)^4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (tM)^5 \begin{pmatrix} 0 & (t\pi)^5 & 0 \\ -(t\pi)^5 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{205}$$

With

$$\begin{aligned}
1 - \frac{1}{2!}(\pi t)^2 + \frac{1}{4!}(\pi t)^4 + ... &= \cos(\pi t) &(206)\\
\pi t - \frac{1}{3!}(\pi t)^3 + \frac{1}{5!}(\pi t)^5 + ... &= \sin(\pi t) &(207)\\
-\pi t + \frac{1}{3!}(\pi t)^3 - \frac{1}{5!}(\pi t)^5 + ... &= (-\pi t) + \frac{1}{3!}(-\pi t)^3 - \frac{1}{5!}(-\pi t)^5 + ... &(208)\\
&= \sin(-\pi t) &(209)\\
&= -\sin(\pi t) &(210)
\end{aligned}$$

we obtain

$$e^{tM} = \begin{pmatrix} \cos(\pi t) & \sin(\pi t) & 0 \\ -\sin(\pi t) & \cos(\pi t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{211}$$

Problem B.2-2

For the Hamiltonian

$$H = -B_x \sigma_1 = \begin{pmatrix} 0 & -B_x \\ -B_x & 0 \end{pmatrix} \tag{212}$$

we find the eigensystem

$$E_1 = -B_x \quad |\psi_1\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{213}$$

$$E_2 = +B_x \quad |\psi_2\rangle = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{214}$$

The Hamiltonian (with full units) is given by

$$H = -g\frac{q\hbar}{2m}\frac{\sigma_1}{2}B_x \tag{215}$$

which translates into energies of

$$E_1 = -g\frac{q\hbar}{4m}B_x \tag{216}$$

$$E_2 = g\frac{q\hbar}{4m}B_x. \tag{217}$$

The time evolution is them given by

$$\begin{aligned}
|\psi(t)\rangle &= e^{-\frac{i}{\hbar}Ht}|\psi(0)\rangle && (218)\\
&= e^{-i\frac{gq}{4m}\sigma_1 t}|\psi(0)\rangle && (219)\\
&= \left[\cos\left(\frac{gq}{4m}\sigma_1 t\right) - i\sin\left(\frac{gq}{4m}\sigma_1 t\right)\right]|\psi(0)\rangle && (220)\\
&= \left[\cos\left(\frac{gq}{4m}t\right)\mathbb{I}_2 - i\sin\left(\frac{gq}{4m}t\right)\sigma_1\right]|\psi(0)\rangle && (221)\\
&= \begin{pmatrix} \cos\left(\frac{gqt}{4m}\right) & -i\sin\left(\frac{gqt}{4m}\right) \\ -i\sin\left(\frac{gqt}{4m}\right) & \cos\left(\frac{gqt}{4m}\right) \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} && (222)\\
&= \begin{pmatrix} \cos\left(\frac{gqt}{4m}\right) \\ -i\sin\left(\frac{gqt}{4m}\right) \end{pmatrix} && (223)
\end{aligned}$$

where we used $\sigma_1^{2n} = \mathbb{I}^n = \mathbb{I}$.

5.6 Baez, Muniain - Gauge Fields, Knots and Gravity

Problem I.1 - Plane waves in vacuum

With

$$\vec{\mathcal{E}} = \vec{E}e^{-i(\omega t - \vec{k}\vec{x})} \tag{224}$$

we calculate in cartesian coordinates

1. $\nabla \cdot \vec{\mathcal{E}} = 0$

$$\begin{aligned}
\nabla \cdot \vec{\mathcal{E}} &= \partial_a \mathcal{E}_a && (225)\\
&= \partial_a(e^{-i(\omega t - \vec{k}\vec{x})})E_a \vec{e}^a && (226)\\
&= \delta_{ab} i k_b E_a e^{-i(\omega t - \vec{k}\vec{x})}\vec{e}^a && (227)\\
&= i k_b E_b e^{-i(\omega t - \vec{k}\vec{x})}\vec{e}^a && (228)\\
&= 0 && (229)
\end{aligned}$$

where we assumed $E_a = \text{const}$ and used

$$0 = \vec{k} \cdot \vec{E} \tag{230}$$
$$= k_a \vec{e}^a E_a \vec{e}^a \tag{231}$$
$$= k_a E_a \tag{232}$$

2. $\nabla \times \vec{\mathcal{E}} = i \frac{\partial \vec{\mathcal{E}}}{\partial t}$

$$\nabla \times \vec{\mathcal{E}} = \epsilon_{abc} \partial_b \mathcal{E}_c \vec{e}_a \tag{233}$$
$$= \epsilon_{abc} E_c \vec{e}_a \partial_b (e^{-i(\omega t - \vec{k}\vec{x})}) \tag{234}$$
$$= \epsilon_{abc} E_c \vec{e}_a \delta_{bd} i k_d e^{-i(\omega t - \vec{k}\vec{x})} \tag{235}$$
$$= i(\epsilon_{abc} k_b E_c \vec{e}_a) e^{-i(\omega t - \vec{k}\vec{x})} \tag{236}$$
$$= i(-i\omega E_a \vec{e}^a) e^{-i(\omega t - \vec{k}\vec{x})} \tag{237}$$
$$= i(E_a \vec{e}^a)(-i\omega) e^{-i(\omega t - \vec{k}\vec{x})} \tag{238}$$
$$= i\vec{E} \frac{\partial}{\partial t} e^{-i(\omega t - \vec{k}\vec{x})} \tag{239}$$
$$= i \frac{\partial \vec{\mathcal{E}}}{\partial t} \tag{240}$$

where we used (typo in the book!)

$$-i\omega \vec{E} = \vec{k} \times \vec{E} \tag{241}$$
$$= \epsilon_{abc} k_b E_c \vec{e}_a \tag{242}$$

6 Many-body physics

6.1 Coleman - Introduction to Many-Body Physics

Problem 2.1 - Specific heat capacity of a solid

Using the Boltzmann statistics and $E_n = \hbar\omega(n + \frac{1}{2})$ the energy E of a system of N_{AV} harmonic oscillators (in 3d!!) is given by

$$E = 3N_{AV} \frac{\sum_n E_n e^{-\frac{E_n}{k_B T}}}{\sum_n e^{-\frac{E_n}{k_B T}}} \tag{243}$$
$$= 3N_{AV} \frac{\sum_n \left(\hbar\omega \left[n + \frac{1}{2}\right]\right) e^{-\frac{n\hbar\omega}{k_B T}} e^{-\frac{\hbar\omega}{2k_B T}}}{\sum_n e^{-\frac{n\hbar\omega}{k_B T}} e^{-\frac{\hbar\omega}{2k_B T}}} \tag{244}$$
$$= 3N_{AV} \hbar\omega \left(\frac{1}{2} + \frac{\sum_n n e^{-\frac{n\hbar\omega}{k_B T}}}{\sum_n e^{-\frac{n\hbar\omega}{k_B T}}} \right) \tag{245}$$
$$= 3N_{AV} \hbar\omega \left(\frac{1}{2} + \frac{1}{e^{\frac{\hbar\omega}{k_B T}} - 1} \right) \tag{246}$$

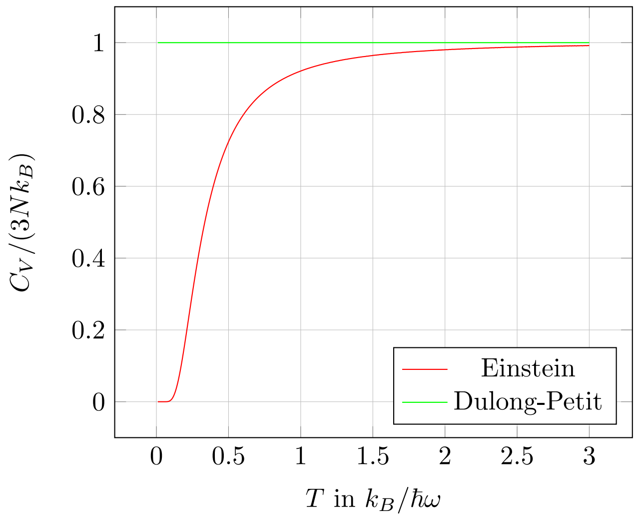
where we used the sum formulas

$$s_1 = \sum_{n=0} q^n = \frac{1}{1-q} \tag{247}$$
$$s_2 = \sum_{n=0} n q^n = q \frac{ds_1}{dq} = \frac{q}{(1-q)^2} \tag{248}$$

the specific heat can the be calculated as

$$C_V = \frac{dE}{dT} \tag{249}$$
$$= 3N_{AV}\hbar\omega \frac{\exp\left[\frac{\hbar\omega}{kT}\right]\frac{\hbar\omega}{kT^2}}{\left[\exp\left[\frac{\hbar\omega}{kT}\right]-1\right]^2} \tag{250}$$
$$= 3N_{AV}k \frac{x^2\exp(x)}{[\exp(x)-1]^2} \tag{251}$$
$$= 3N_{AV}k \frac{x^2}{[\exp(x/2)-\exp(-x/2)]^2} \tag{252}$$
$$= 3N_{AV}k \frac{x^2}{[\exp(x/2)-\exp(-x/2)]^2} \tag{253}$$
$$= 3N_{AV}k \left(\frac{x/2}{\sinh(x/2)}\right)^2 \tag{254}$$
$$\tag{255}$$

The Dulong-Petit rule says $k/2$ per harmonic degree of freedom which means in 3d that $C_V/N = 3k$ (for each harmonic degree there is also a kinetic one - so $f = 6$)



6.2 Lancaster, Blundell - Quantum Field Theory for the gifted amateur

Problem 1.1 - Snell's law via Fermat's principle

The light travels from point A in medium 1 to point B in medium 2. We assume a vertical medium boundary at x_0 and that the light travels within a medium in the straight line. This makes y_0 the free parameter and the the travel time is given by

$$t = \frac{s_{A0}}{c/n_1} + \frac{s_{0B}}{c/n_2} \tag{256}$$
$$= \sqrt{\frac{(x_A - x_0)^2 + (y_A - y_0)^2}{c/n_1}} + \sqrt{\frac{(x_0 - x_B)^2 + (y_0 - y_B)^2}{c/n_2}} \tag{257}$$

The local extrema of the travel time is given by

$$
\begin{aligned}
0 &= \frac{dt}{dy_0} && (258)\\
&= \frac{y_A - y_0}{s_{A0}c/n_1} + \frac{y_0 - y_B}{s_{0B}c/n_2} && (259)\\
&= \frac{\sin\alpha}{c/n_1} - \frac{\sin\beta}{c/n_2} && (260)
\end{aligned}
$$

and therefore

$$
n_1 \sin\alpha = n_2 \sin\beta. \tag{261}
$$

6.3 Srednicki - Quantum Field Theory

Problem 6.1 - Path integral in quantum mechanics

(a) The transition amplitude $\langle q''|e^{-iH(t''-t')}|q'\rangle$ (particle to start at q', t' and ends at position q'' at time t'') can be written in the Heisenberg picture as

$$
\begin{aligned}
\langle q''|e^{-iH(t''-t')}|q'\rangle &= \langle q''|e^{-iHt''}e^{iHt''}e^{-iH(t''-t')}e^{-iHt'}e^{iHt'}|q'\rangle && (262)\\
&= \langle q'', t''|e^{iHt''}e^{iH(t''-t')}e^{-iHt'}|q', t'\rangle && (263)\\
&= \langle q'', t''|q', t'\rangle. && (264)
\end{aligned}
$$

Now we can do the standard path integral derivation

$$
\begin{aligned}
\langle q'', t''|q', t'\rangle &= \int \left(\prod_{j=1}^{N} dq_j\right) \langle q''|e^{-iH\delta t}|q_N\rangle\langle q_N|e^{-iH\delta t}|q_{N-1}\rangle \dots \langle q_1|e^{-iH\delta t}|q'\rangle && (265)\\
&= \int \left(\prod_{j=1}^{N} dq_j\right) \int \frac{dp_N}{2\pi} e^{-iH(p_N,q_N)\delta t} e^{ip_N(q'-q_N)} \dots \int \frac{dp'}{2\pi} e^{-iH(p',q')\delta t} e^{ip'(q_1-q')} && (266)\\
&= \int \left(\prod_{j=1}^{N} dq_j\right) \left(\prod_{k=0}^{N} \frac{dp_k}{2\pi} e^{ip_k(q_{k+1}-q_k)} e^{-iH(p_k,q_k)\delta t}\right) \quad (q_0 = q', q_{N+1} = q'') && (267)
\end{aligned}
$$

which under Weyl ordering (see Greiner, Reinhard - field quantization) has to be replaced by

$$
\begin{aligned}
\langle q'', t''|q', t'\rangle &= \int \left(\prod_{j=1}^{N} dq_j\right) \left(\prod_{k=0}^{N} \frac{dp_k}{2\pi} e^{ip_k(q_{k+1}-q_k)} e^{-iH(p_k,\bar{q}_k)\delta t}\right) \quad \bar{q}_k = (q_{k+1}+q_k)/2 && (268)\\
&= \int \left(\prod_{j=1}^{N} dq_j\right) \left(\prod_{k=0}^{N} \frac{dp_k}{2\pi} e^{i[p_k\dot{q}_k - H(p_k,\bar{q}_k)]\delta t}\right) \quad \dot{q}_k = (q_{k+1}-q_k)/\delta t && (269)\\
&= \int \left(\prod_{j=1}^{N} dq_j\right) \left(\prod_{k=0}^{N} \frac{dp_k}{2\pi}\right) \left(e^{i\sum_{n=0}^{N}[p_n\dot{q}_n - H(p_n,\bar{q}_n)]\delta t}\right) && (270)\\
&= \int \mathcal{D}q\mathcal{D}p \exp\left[i\int_{t'}^{t''} dt\,(p(t)\dot{q}(t) - H(p(t),q(t)))\right] && (271)
\end{aligned}
$$

Let's now assume $H(p,q)$ has only a quadratic term in p which is independent of q meaning

$$
H(p,q) = \frac{p^2}{2m} + V(q) \tag{272}
$$

then

$$\langle q''|e^{-iH(t''-t')}|q'\rangle = \int \left(\prod_{j=1}^{N} dq_j\right)\left(\prod_{k=0}^{N}\frac{dp_k}{2\pi}\right)\left(e^{i\sum_{n=0}^{N}[p_n\dot{q}_n-\frac{1}{2m}p_n^2-V(\bar{q}_n)]\delta t}\right) \tag{273}$$

We can evaluate a single integral using

$$\int_{-\infty}^{\infty} dx e^{-ax^2+bx+c} = \sqrt{\frac{\pi}{a}}e^{\frac{b^2}{4a}+c} \tag{274}$$

and obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dp_k \left(e^{i[p_k\dot{q}_k-\frac{1}{2m}p_k^2-V(\bar{q}_k)]\delta t}\right) = \frac{1}{2\pi}e^{-iV(\bar{q}_k)\delta t}\int dp_k\left(e^{i[p_k\dot{q}_k-\frac{1}{2m}p_k^2]\delta t}\right) \tag{275}$$

$$= \frac{1}{2\pi}e^{-iV(\bar{q}_k)\delta t}\sqrt{\frac{\pi}{i\frac{\delta t}{2m}}}e^{\frac{-\dot{q}_k^2\delta t^2}{4\frac{i\delta t}{2m}}} \tag{276}$$

$$= \frac{1}{2\pi}\sqrt{\frac{2\pi m}{i\delta t}}e^{i\left(\frac{m\dot{q}_k^2}{2}-V(\bar{q}_k)\right)\delta t} \tag{277}$$

$$= \sqrt{\frac{m}{2\pi i\delta t}}e^{iL(\bar{q}_k,\dot{q}_k)\delta t}. \tag{278}$$

As there are $N+1$ p-integrals we have

$$\mathcal{D}q = \left(\frac{m}{2\pi i\delta t}\right)^{(N+1)/2}\prod_{j=1}^{N} dq_j \tag{279}$$

(b) We now assume $V(q) = 0$

$$\langle q'',t''|q',t'\rangle = \lim_{N\to\infty}\left(\frac{m}{2\pi i\delta t}\right)^{\frac{N+1}{2}}\left(\prod_{j=1}^{N}\int_{-\infty}^{\infty} dq_j\, e^{i\frac{m\dot{q}_j^2}{2}\delta t}\right)e^{i\frac{m\dot{q}_0^2}{2}\delta t} \tag{280}$$

$$= \lim_{N\to\infty}\left(\frac{m}{2\pi i\delta t}\right)^{\frac{N+1}{2}}\int_{-\infty}^{\infty}\left(\prod_{j=1}^{N} dq_j\, e^{i\frac{m(q_{j+1}-q_j)^2}{2\delta t}}\right)e^{i\frac{m(q_1-q_0)^2}{2\delta t}} \tag{281}$$

$$= \lim_{N\to\infty}\left(\frac{m}{2\pi i\delta t}\right)^{\frac{N+1}{2}}\int_{-\infty}^{\infty}\left(\prod_{j=1}^{N} dq_j\right)\, e^{\frac{im}{2\delta t}\sum_{k=0}^{N}(q_{k+1}-q_k)^2} \tag{282}$$

Integrating one by one with

$$(q_{j+1}-q_j)^2+\frac{1}{n}(q_j-q_0)^2 = \frac{n+1}{n}q_j^2-2\left(q_{j+1}+\frac{1}{n}q_0\right)q_j+q_{j+1}^2+\frac{1}{n}q_0^2 \tag{283}$$

and

$$\int_{-\infty}^{\infty} dy e^{iay^2-iby} = \sqrt{\frac{\pi}{a}}e^{\frac{i}{4}\left(\pi-\frac{b^2}{a}\right)} \qquad a>0, a,b\in\mathbb{R} \tag{284}$$

$$\int_{-\infty}^{\infty} dy e^{ic\frac{n+1}{n}y^2-i2c(q_{j+1}+\frac{1}{n}q_0)y}\cdot e^{ic(q_{j+1}^2+q_0^2/n)} = \sqrt{\frac{\pi n}{c(n+1)}}e^{\frac{i}{4}\left(\pi-\frac{4nc(q_{j+1}+\frac{1}{n}q_0)^2}{n+1}\right)}\cdot e^{ic(q_{j+1}^2+q_0^2/n)} \tag{285}$$

$$= \sqrt{\frac{\pi n}{c(n+1)}}e^{\frac{\pi i}{4}}e^{-\frac{inc(q_{j+1}+\frac{1}{n}q_0)^2}{n+1}}\cdot e^{ic(q_{j+1}^2+q_0^2/n)} \tag{286}$$

$$= \sqrt{\frac{\pi n}{c(n+1)}}e^{\frac{\pi i}{4}}e^{\frac{ic(q_{j+1}+q_0)^2}{n+1}} \tag{287}$$

we obtain

$$\int_{-\infty}^{\infty}\left(\prod_{j=1}^{N} dq_j\right)\ e^{\frac{im}{2\delta t}\sum_{k=0}^{N}(q_{k+1}-q_k)^2} \tag{288}$$

$$= \int_{-\infty}^{\infty}\left(\prod_{j=2}^{N} dq_j\right)\ e^{\frac{im}{2\delta t}\sum_{k=2}^{N}(q_{k+1}-q_k)^2} \times \int_{-\infty}^{\infty} dq_1\, e^{\frac{im}{2\delta t}\left[(q_2-q_1)^2+(q_1-q_0)^2\right]} \tag{289}$$

$$= \int_{-\infty}^{\infty}\left(\prod_{j=2}^{N} dq_j\right)\ e^{\frac{im}{2\delta t}\sum_{k=2}^{N}(q_{k+1}-q_k)^2} e^{\frac{im}{2\delta t}(q_2^2+q_0^2)} \int_{-\infty}^{\infty} dq_1\, e^{\frac{im}{2\delta t}[2q_1^2-2q_1(q_2+q_0)]} \tag{290}$$

$$= \int_{-\infty}^{\infty}\left(\prod_{j=2}^{N} dq_j\right)\ e^{\frac{im}{2\delta t}\sum_{k=2}^{N}(q_{k+1}-q_k)^2} e^{\frac{im}{2\delta t}(q_2^2+q_0^2)} \sqrt{\frac{\pi\delta t}{m}} e^{\frac{i}{4}\left(\pi-\frac{m(q_2+q_0)^2}{\delta t}\right)} \tag{291}$$

$$= \int_{-\infty}^{\infty}\left(\prod_{j=2}^{N} dq_j\right)\ e^{\frac{im}{2\delta t}\sum_{k=2}^{N}(q_{k+1}-q_k)^2} e^{\frac{im}{2\delta t}(q_2^2+q_0^2)} \sqrt{\frac{\pi\delta t}{m}} e^{\frac{i\pi}{4}} e^{-\frac{im(q_2+q_0)^2}{4\delta t}} \tag{292}$$

$$= \sqrt{\frac{\pi\delta t}{m}} e^{\frac{i\pi}{4}} \int_{-\infty}^{\infty}\left(\prod_{j=2}^{N} dq_j\right)\ e^{\frac{im}{2\delta t}\sum_{k=2}^{N}(q_{k+1}-q_k)^2} e^{\frac{im}{4\delta t}(q_2-q_0)^2} \tag{293}$$

$$= \sqrt{\frac{\pi\delta t}{m}} e^{\frac{i\pi}{4}} \int_{-\infty}^{\infty}\left(\prod_{j=3}^{N} dq_j\right)\ e^{\frac{im}{2\delta t}\sum_{k=3}^{N}(q_{k+1}-q_k)^2} \times \int_{-\infty}^{\infty} dq_2\, e^{\frac{im}{2\delta t}(q_3-q_2)^2} e^{\frac{im}{4\delta t}(q_2-q_0)^2} \tag{294}$$

$$= \sqrt{\frac{\pi\delta t}{m}} e^{\frac{i\pi}{4}} \int_{-\infty}^{\infty}\left(\prod_{j=3}^{N} dq_j\right)\ e^{\frac{im}{2\delta t}\sum_{k=3}^{N}(q_{k+1}-q_k)^2} e^{\frac{im}{2\delta t}(q_3^2+\frac{1}{4}q_0^2)} \int_{-\infty}^{\infty} dq_2\, e^{\frac{im}{2\delta t}[\frac{3}{2}q_2^2-2(q_3+\frac{1}{2}q_0)q_2]} \tag{295}$$

$$= \sqrt{\frac{\pi\delta t}{m}} e^{\frac{i\pi}{4}} \int_{-\infty}^{\infty}\left(\prod_{j=3}^{N} dq_j\right)\ e^{\frac{im}{2\delta t}\sum_{k=3}^{N}(q_{k+1}-q_k)^2} e^{\frac{im}{2\delta t}(q_3^2+\frac{1}{4}q_0^2)} \sqrt{\frac{\pi 4\delta t}{3m}} e^{\frac{i}{4}\left(\pi-\frac{4m(q_3+q_0/2)^2}{3\delta t}\right)} \tag{296}$$

$$\tag{297}$$

6.4 Kachelriess - Quantum Fields - From the Hubble to the Planck scale

Problem 1.1 - Units

1. The fundamental constants are given by

$$k = 1.381\cdot 10^{-23}\mathrm{m}^2\mathrm{s}^{-2}\mathrm{kg}^1\mathrm{K}^{-1} \tag{298}$$

$$G = 6.674\cdot 10^{-11}\mathrm{m}^3\mathrm{s}^{-2}\mathrm{kg}^{-1} \tag{299}$$

$$\hbar = 1.054\cdot 10^{-34}\mathrm{m}^2\mathrm{s}^{-1}\mathrm{kg}^1 \tag{300}$$

$$c = 2.998\cdot 10^{-8}\mathrm{m}^1\mathrm{s}^{-1} \tag{301}$$

A newly constructed Planck constant has the general form

$$X_P = c^{\alpha_c}\cdot G^{\alpha_G}\cdot \hbar^{\alpha_\hbar}\cdot k^{\alpha_k} \tag{302}$$

and the dimension of X_P is given by $\mathrm{m}^{\beta_m}\mathrm{s}^{\beta_s}\mathrm{kg}^{\beta_{kg}}\mathrm{K}^{\beta_K}$ are determined by

$$\begin{aligned} \text{Meter} \quad & \beta_m = 2\alpha_k + 3\alpha_G + 2\alpha_h + \alpha_c && (303)\\ \text{Second} \quad & \beta_s = -2\alpha_k - 2\alpha_G - \alpha_c - \alpha_h && (304)\\ \text{Kilogram} \quad & \beta_{kg} = \alpha_k - \alpha_G + \alpha_h && (305)\\ \text{Kelvin} \quad & \beta_K = -\alpha_k && (306) \end{aligned}$$

Solving the linear system gives

$$\begin{aligned} l_P &= \sqrt{\frac{\hbar G}{c^3}} = 1.616 \cdot 10^{-35}\mathrm{m} && (307)\\ m_P &= \sqrt{\frac{\hbar c}{G}} = 2.176 \cdot 10^{-8}\mathrm{kg} && (308)\\ t_P &= \sqrt{\frac{\hbar G}{c^5}} = 5.391 \cdot 10^{-44}\mathrm{s} && (309)\\ T_P &= \sqrt{\frac{\hbar c^5}{Gk^2}} = 1.417 \cdot 10^{-32}\mathrm{K} && (310)\\ & && (311) \end{aligned}$$

As the constants are made up from QM, SR and GR constants they indicate magnitudes at which a quantum theory of gravity is needed to make a sensible predictions.

2. We use the definition $1\mathrm{barn} = 10^{-28}\mathrm{m}^2$

$$\begin{aligned} 1\mathrm{cm}^2 &= 10^{-4}\mathrm{m}^2 && (312)\\ 1\mathrm{mbarn} &= 10^{-31}\mathrm{m}^2 && (313)\\ &= 10^{-27}\mathrm{cm}^2 && (314) \end{aligned}$$

We also have $1\mathrm{eV} = 1.602 \cdot 10^{-19}\mathrm{As} \cdot 1\mathrm{V} = 1.602 \cdot 10^{-19}\mathrm{J}$

$$\begin{aligned} E = mc^2 \quad & \to \quad 1\mathrm{kg} \cdot c^2 = 8.987 \cdot 10^{16}\mathrm{J} = 5.609 \cdot 10^{35}\mathrm{eV} && (315)\\ & \to \quad 1\mathrm{GeV} = 1.782 \cdot 10^{-27}\mathrm{kg} && (316)\\ E = \hbar\omega \quad & \to \quad \frac{1}{1\mathrm{s}} \cdot \hbar = 1.054 \cdot 10^{-34}\mathrm{J} = 6.582 \cdot 10^{-16}\mathrm{eV} && (317)\\ & \to \quad 1\mathrm{GeV}^{-1} = 6.582 \cdot 10^{-25}\mathrm{s} && (318)\\ E = \frac{\hbar c}{\lambda} \quad & \to \quad \frac{1}{1\mathrm{m}} \cdot \hbar c = 3.161 \cdot 10^{-26}\mathrm{J} = 1.973 \cdot 10^{-7}\mathrm{eV} && (319)\\ & \to \quad 1\mathrm{GeV}^{-1} = 1.973 \cdot 10^{-16}\mathrm{m} && (320)\\ E \sim pc \quad & \to \quad 1\mathrm{kgms}^{-1} \cdot c = 2.998 \cdot 10^{8}\mathrm{J} = 1.871 \cdot 10^{27}\mathrm{eV} && (321)\\ & \to \quad 1\mathrm{GeV} = 5.344 \cdot 10^{-19}\mathrm{kgms}^{-1} && (322) \end{aligned}$$

therefore

$$\begin{aligned} 1\mathrm{GeV}^{-2} &= (1.973 \cdot 10^{-16}\mathrm{m})^2 && (323)\\ &= 3.893 \cdot 10^{-32}\mathrm{m}^2 && (324)\\ &= 0.389\mathrm{mbarn} && (325) \end{aligned}$$

Problem 3.2 - Maxwell Lagrangian

1. First we observe that

$$\begin{aligned}
F_{\mu\nu}F^{\mu\nu} &= (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) && (326)\\
&= (\partial_\mu A_\nu)(\partial^\mu A^\nu) - (\partial_\mu A_\nu)(\partial^\nu A^\mu) - \underbrace{(\partial_\nu A_\mu)(\partial^\mu A^\nu)}_{=(\partial_\mu A_\nu)(\partial^\nu A^\mu)} + \underbrace{(\partial_\nu A_\mu)(\partial^\nu A^\mu)}_{=(\partial_\mu A_\nu)(\partial^\mu A^\nu)} && (327)\\
&= 2\left((\partial_\mu A_\nu)(\partial^\mu A^\nu) - (\partial_\mu A_\nu)(\partial^\nu A^\mu)\right) && (328)\\
&= 2(\partial_\mu A_\nu)F^{\mu\nu} && (329)
\end{aligned}$$

we also find

$$\begin{aligned}
\delta\left(F_{\mu\nu}F^{\mu\nu}\right) &= 2\delta\left((\partial_\mu A_\nu)F^{\mu\nu}\right) && (330)\\
&= 2\left[\delta\left(\partial_\mu A_\nu\right)F^{\mu\nu} + (\partial_\mu A_\nu)\delta F^{\mu\nu}\right] && (331)\\
& && (332)
\end{aligned}$$

We start with the source free Maxwell equations $\partial_\mu F^{\mu\nu} = 0$

$$\begin{aligned}
0 &= C\int_\Omega d^4x\ (\delta A_\nu)\partial_\mu F^{\mu\nu} && (333)\\
&= C(\delta A_\nu)F^{\mu\nu}|_{\partial\Omega} - C\int_\Omega d^4x\ \partial_\mu(\delta A_\nu)F^{\mu\nu} && (334)\\
&= -C\int_\Omega d^4x\ \delta(\partial_\mu A_\nu)F^{\mu\nu} && (335)\\
&= -C\int_\Omega d^4x\ \left[\delta(\partial_\mu A_\nu F^{\mu\nu}) - (\partial_\mu A_\nu)\delta F^{\mu\nu}\right] && (336)\\
& && (337)
\end{aligned}$$

2.

3.

Problem 4.1 - $Z[J]$ at order λ in ϕ^4 theory

Lets start at (4.6a) with $\mathscr{L}_I = -\lambda/4!\phi^4$

$$\begin{aligned}
Z[J] &= \exp\left[\mathrm{i}\int d^4x\mathscr{L}_I\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}\right)\right]\int\mathcal{D}\phi\exp\left[\mathrm{i}\int d^4x(\mathscr{L}_0 + J\phi)\right] && (338)\\
&= \exp\left[\mathrm{i}\int d^4x\mathscr{L}_I\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}\right)\right]Z_0[J] && (339)\\
&= \exp\left[-\frac{\mathrm{i}\lambda}{4!}\int d^4x\left(\frac{\delta^4}{\delta J(x)^4}\right)\right]Z_0[J] && (340)\\
&= Z_0[J] - \frac{\mathrm{i}\lambda}{4!}\int d^4x\left(\frac{\delta^4 Z_0[J]}{\delta J(x)^4}\right) + \dots. && (341)
\end{aligned}$$

Using (4.7)

$$Z_0[J] = Z_0[0]\exp\left[-\frac{\mathrm{i}}{2}\int d^4yd^4zJ(y)\Delta_F(y-z)J(z)\right] = Z_0[0]e^{iW_0[J]} \qquad (342)$$

$$W_0[J] = -\frac{1}{2}\int d^4yd^4zJ(y)\Delta_F(y-z)J(z) \qquad (343)$$

we derive (4.10) in various steps

1. Calculating $\frac{\delta W_0[J]}{\delta J(x)}$

$$\frac{\delta W_0[J]}{\delta J(x)} = -\frac{1}{2}\lim_{\epsilon\to 0}\int d^4y d^4z \frac{\left(J(y)+\epsilon\delta^{(4)}(y-x)\right)\Delta_F(y-z)\left(J(z)+\epsilon\delta^{(4)}(z-x)\right) - W_0[J]}{\epsilon} \tag{344}$$

$$= -\frac{1}{2}\int d^4y d^4z \left[\delta^{(4)}(y-x)\Delta_F(y-z)J(z) + J(y)\Delta_F(y-z)\delta^{(4)}(z-x)\right] \tag{345}$$

$$= -\frac{1}{2}\int d^4z \Delta_F(x-z)J(z) - \frac{1}{2}\int d^4y J(y)\Delta_F(y-x) \tag{346}$$

$$= -\int d^4y \Delta_F(y-x)J(y) \tag{347}$$

where we used $\Delta_F(x) = \Delta_F(-x)$.

2. Calculating $\frac{\delta^2 W_0[J]}{\delta J(x)^2}$

$$\frac{\delta^2 W_0[J]}{\delta J(x)^2} = -\int d^4y \Delta_F(y-x)\frac{\delta J(y)}{\delta J(x)} \tag{348}$$

$$= -\int d^4y \Delta_F(y-x)\delta(y-x) \tag{349}$$

$$= -\Delta_F(0) \tag{350}$$

3. Calculating $\delta F[J]/\delta J(x)$ for $F[J] = f\left(W_0[J]\right)$

$$\frac{\delta F[J]}{\delta J(x)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}f(W_0[\phi(x)+\epsilon\delta(x-y)]) - f(W_0[\phi(x)]) \tag{351}$$

$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}f(W_0[\phi(x)] + \epsilon\frac{\delta W_0}{\delta\phi}) - f(W_0[\phi(x)]) \tag{352}$$

$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}f(W_0[\phi(x)]) + g'\epsilon\frac{\delta W_0}{\delta\phi} - f(W_0[\phi(x)]) \tag{353}$$

$$= f'(W_0[J])\frac{\delta W_0}{\delta J} \tag{354}$$

4. Calculating first derivative

$$\frac{\delta}{\mathrm{i}\delta J(x)}\exp\left(\mathrm{i}W_0[J]\right) = \frac{\delta W_0[J]}{\delta J(x)}\exp\left(\mathrm{i}W_0[J]\right) \tag{355}$$

5. Calculating second derivative (using the functional derivative product rule)

$$\left(\frac{\delta}{\mathrm{i}\delta J(x)}\right)^2\exp\left(\mathrm{i}W_0[J]\right) = \left(\left(\frac{\delta W_0[J]}{\delta J(x)}\right)^2 + \frac{1}{i}\frac{\delta^2 W_0[J]}{\delta J(x)^2}\right)\exp\left(\mathrm{i}W_0[J]\right) \tag{356}$$

6. Calculating third derivative

$$\left(\frac{\delta}{\mathrm{i}\delta J(x)}\right)^3\exp\left(\mathrm{i}W_0[J]\right) = \left(\left(\frac{\delta W_0[J]}{\delta J(x)}\right)^3 + \frac{3}{i}\frac{\delta^2 W_0[J]}{\delta J(x)^2}\frac{\delta W_0[J]}{\delta J(x)} + \frac{1}{i^2}\frac{\delta^3 W_0[J]}{\delta J(x)^3}\right)\exp\left(\mathrm{i}W_0[J]\right) \tag{357}$$

7. Calculating fourth derivative

$$\left(\frac{\delta}{\mathrm{i}\delta J(x)}\right)^4 \exp\left(\mathrm{i}W_0[J]\right) = \left(\left(\frac{\delta W_0[J]}{\delta J(x)}\right)^4 + \frac{6}{i}\frac{\delta^2 W_0[J]}{\delta J(x)^2}\left(\frac{\delta W_0[J]}{\delta J(x)}\right)^2 + \frac{3}{i^2}\left(\frac{\delta^2 W_0[J]}{\delta J(x)^2}\right)^2 + \right.$$
$$\left. + \frac{4}{i^2}\frac{\delta W_0[J]}{\delta J(x)}\frac{\delta^3 W_0[J]}{\delta J(x)^3} + \frac{1}{i^3}\frac{\delta^4 W_0[J]}{\delta J(x)^4}\right)\exp\left(\mathrm{i}W_0[J]\right)$$
$$= \left(\left(\frac{\delta W_0[J]}{\delta J(x)}\right)^4 + \frac{6}{i}\frac{\delta^2 W_0[J]}{\delta J(x)^2}\left(\frac{\delta W_0[J]}{\delta J(x)}\right)^2 + \frac{3}{i^2}\left(\frac{\delta^2 W_0[J]}{\delta J(x)^2}\right)^2\right)\exp\left(\mathrm{i}W_0[J]\right)$$

8. Substituting the functional derivatives

$$\left(\frac{\delta}{\mathrm{i}\delta J(x)}\right)^4 \exp\left(\mathrm{i}W_0[J]\right) = \left[\left(\int d^4y\Delta_F(y-x)J(y)\right)^4 + 6i\Delta_F(0)\left(\int d^4y\Delta_F(y-x)J(y)\right)^2\right.$$
$$\left. +3\left(i\Delta_F(0)\right)^2\right]\exp\left(\mathrm{i}W_0[J]\right)$$

Problem 19.1 - Dynamical stress tensor

Preliminaries

- The Laplace expansion of the determinate by row or column is given by

$$|g| = \sum_\kappa g_{\kappa\mu}G_{\kappa\mu} \quad \text{(no sum over } \mu!) \tag{358}$$

  with the cofactor matrix $G_{\kappa\mu}$ (matrix of determinants of minors of g).
- The inverse matrix is given by

$$g^{\alpha\beta} = \frac{1}{|g|}G_{\alpha\beta} \tag{359}$$

- Therefore we have

$$\frac{\partial|g|}{\delta g_{\alpha\beta}} = \frac{\partial\left(\sum_\kappa g_{\kappa\beta}G_{\kappa\beta}\right)}{\delta g_{\alpha\beta}} \tag{360}$$
$$= \delta_{\kappa\alpha}G_{\kappa\beta} \tag{361}$$
$$= G_{\alpha\beta} \tag{362}$$
$$= |g|g^{\alpha\beta} \tag{363}$$

Now we can calculate

$$\delta\sqrt{|g|} = \frac{\partial\sqrt{|g|}}{\delta g_{\mu\nu}}\delta g_{\mu\nu} = \frac{1}{2\sqrt{|g|}}\frac{\partial|g|}{\delta g_{\mu\nu}}\delta g_{\mu\nu} = \frac{1}{2}\sqrt{|g|}g^{\mu\nu}\delta g_{\mu\nu} \tag{364}$$
$$\frac{\delta\sqrt{|g(x)|}}{\delta g_{\mu\nu}(y)} = \frac{1}{2}\sqrt{|g|}\delta(x-y) \tag{365}$$

We now use the action and definition (7.49)

$$S_\mathrm{m} = \int d^4x\sqrt{|g|}\mathscr{L}_\mathrm{m} \tag{366}$$
$$T^{\mu\nu} = \frac{2}{\sqrt{|g|}}\frac{\delta S_\mathrm{m}}{\delta g^{\mu\nu}} \tag{367}$$
$$= \frac{2}{\sqrt{|g|}}\int d^4x\left[\frac{1}{2}\sqrt{|g|}g^{\mu\nu}\mathscr{L}_\mathrm{m} + \sqrt{|g|}\frac{\delta\mathscr{L}_\mathrm{m}}{\delta g_{\mu\nu}}\right] \tag{368}$$

Problem 19.6 - Dirac-Schwarzschild

1. (19.13) - adding the bi-spinor index might be helpful for some readers, see (B.27)
2. (19.13) vs (B.27) naming of generators $J^{\mu\nu}$ vs $\sigma_{\mu\nu}/2$

The Dirac equation in curved space is obtained (from the covariance principle) by replacing all derivatives ∂_k with covariant tetrad derivatives $\mathscr{D}_k$

$$(i\hbar\gamma^k \mathscr{D}_k + mc)\psi = 0 \tag{369}$$

Lets start with the Schwarzschild line element

$$\begin{aligned} ds^2 &= \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2(d\vartheta^2 + \sin^2\vartheta\, d\phi^2) && (370)\\ &= \eta_{mn} d\xi^m d\xi^n && (371)\end{aligned}$$

with

$$d\xi^0 = \left(1 - \frac{2M}{r}\right)^{1/2} dt, \quad d\xi^1 = \left(1 - \frac{2M}{r}\right)^{-1/2} dr, \quad d\xi^2 = r d\vartheta, \quad d\xi^3 = r\sin\vartheta d\phi. \tag{372}$$

and the tetrad fields e^m_μ can then be derived via $d\xi^m = e^m_\mu(x) dx^\mu$.

Problem 23.6 - Reflection formula

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx \tag{373}$$

Problem 23.7 - Unruh temperature

Problem 24.14 - Jeans length and the speed of sound

We start with the Euler equations

$$\begin{aligned} \frac{D\rho}{Dt} &= -\rho\nabla\cdot\vec{u} \quad &\rightarrow \quad &\frac{\partial\rho}{\partial t} + \vec{u}\cdot(\nabla\rho) + \rho(\nabla\cdot\vec{u}) = 0 && (374)\\ \frac{D\vec{u}}{Dt} &= -\nabla\left(\frac{P}{\rho}\right) + \vec{g} \quad &\rightarrow \quad &\frac{\partial\vec{u}}{\partial t} + \vec{u}\cdot(\nabla\vec{u}) + \frac{\nabla P}{\rho} = \vec{g}. && (375)\end{aligned}$$

With the perturbation ansatz (small perturbation in a resting fluid)

$$\begin{aligned} \rho &= \rho_0 + \varepsilon\rho_1(x,t) && (376)\\ P &= P_0 + \varepsilon P_1(x,t) && (377)\\ \vec{u} &= \varepsilon\vec{u}_1(x,t) && (378)\end{aligned}$$

and the Newton equation

$$\triangle\phi = 4\pi G\rho \quad \rightarrow \quad \nabla\cdot\vec{g}_1 = -4\pi G\rho_1 \tag{379}$$

we obtain (with the EoS $P = w\rho$) in order ε

$$\begin{aligned} \frac{\partial\rho_1}{\partial t} + \rho_0(\nabla\cdot\vec{u}_1) &= 0 && (380)\\ \frac{\partial\vec{u}_1}{\partial t} + \underbrace{\frac{1}{\rho_0}\nabla P_1}_{=\frac{w}{\rho_0}\nabla\rho_1} &= \vec{g}_1. && (381)\end{aligned}$$

Differentiating both (with respect to space and time) we obtain a wave equation

$$\frac{\partial^2 \rho_1}{\partial t^2} - w\triangle\rho_1 = 4\pi G\rho_0\rho_1 \tag{382}$$

with the speed of sound $c_s^2 = w$. Inserting the wave ansatz $\rho_1 \sim \exp[i(\vec{k}\cdot\vec{x} - \omega t)]$ yields the dispersion relation

$$\omega^2 = c_s^2 k^2 - 4\pi G\rho_0. \tag{383}$$

For wave numbers $k_J < \sqrt{4\pi G/c_s^2}$ the ω becomes complex which gives rise to exponentially growing modes. Therefore the Jeans length is given by

$$\lambda_J = \frac{2\pi}{k_J} = c_s\sqrt{\frac{\pi}{G\rho_0}} = \sqrt{\frac{\pi w}{G\rho_0}}. \tag{384}$$

Problem 26.4 - Fixed points of (26.18)

We start with

$$\text{(F1)} \qquad H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V + \rho\right) \tag{385}$$

$$\text{(F2)} \qquad \dot{H} = -4\pi G\left[\dot{\phi}^2 + (1+w_m)\rho\right] \tag{386}$$

$$\text{(KG)} \qquad \ddot{\phi} = -3H\dot{\phi} - V_{,\phi}. \tag{387}$$

Using $H = \dot{a}/a$, $N = \ln(a)$ and $\lambda = -V_{,\phi}/(\sqrt{8\pi G}V)$ we obtain for the time derivatives of x and y

$$\dot{V} = \frac{dV}{d\phi}\frac{d\phi}{dt} = V_{,\phi}\dot{\phi} \tag{388}$$

$$x = \sqrt{\frac{4}{3}\pi G}\frac{\dot{\phi}}{H} \quad \to \quad \frac{dx}{dt} = \frac{dx}{dN}\frac{d\ln(a)}{dt} = \frac{dx}{dN}H = \sqrt{\frac{4}{3}\pi G}\frac{\ddot{\phi}H - \dot{\phi}\dot{H}}{H^2} \tag{389}$$

$$y = \sqrt{\frac{8}{3}\pi G}\frac{\sqrt{V}}{H} \quad \to \quad \frac{dy}{dt} = \frac{dy}{dN}\frac{d\ln(a)}{dt} = \frac{dy}{dN}H = \sqrt{\frac{8}{3}\pi G}\frac{\frac{V_{,\phi}\dot{\phi}}{2\sqrt{V}} - \sqrt{V}\dot{H}}{H^2}. \tag{390}$$

With the substitutions

$$\dot{H} = -4\pi G\left[\dot{\phi}^2 + (1+w_m)\rho\right] \tag{391}$$

$$\ddot{\phi} = -3H\dot{\phi} - V_{,\phi} \tag{392}$$

$$V_{,\phi} = -\sqrt{8\pi G}\lambda V \tag{393}$$

$$\rho = \frac{3H^2}{8\pi G} - \frac{1}{2}\dot{\phi}^2 - V \tag{394}$$

$$\dot{\phi} = xH/\sqrt{\frac{4}{3}\pi G} \tag{395}$$

$$\sqrt{V} = yH/\sqrt{\frac{8}{3}\pi G} \tag{396}$$

we obtain

$$\frac{dx}{dN} = -3x + \frac{\sqrt{6}}{2}\lambda y^2 + \frac{3}{2}x[(1-w_m)x^2 + (1+w_m)(1-y^2)] \tag{397}$$

$$\frac{dy}{dN} = -\frac{\sqrt{6}}{2}\lambda xy + \frac{3}{2}y[(1-w_m)x^2 + (1+w_m)(1-y^2)]. \tag{398}$$

To find the fix points of (26.17) we need to solve

$$-3x + \frac{\sqrt{6}}{2}\lambda y^2 + \frac{3}{2}x[(1-w_m)x^2 + (1+w_m)(1-y^2)] = 0 \tag{399}$$

$$-\frac{\sqrt{6}}{2}\lambda xy + \frac{3}{2}y[(1-w_m)x^2 + (1+w_m)(1-y^2)] = 0. \tag{400}$$

- An obvious solution is

$$x_0 = 0, y_0 = 0. \tag{401}$$

- Two semi-obvious solutions can be found for $y = 0$ which solves the second equation and transforms the first to the quadratic equation $x^2 - 1 = 0$ which gives

$$x_1 = +1, y_1 = 0 \tag{402}$$

$$x_2 = -1, y_2 = 0. \tag{403}$$

- Substituting the square bracket of the second equation into the first and simplifying the second gives

$$-3x + \frac{\sqrt{6}}{2}\lambda(x^2 + y^2) = 0 \tag{404}$$

$$-\frac{\sqrt{6}}{2}\lambda x + \frac{3}{2}[1 + 2x^2 - (x^2 + y^2) - w_m((x^2 + x^2) - 1)] = 0. \tag{405}$$

  Now we can eliminate $x^2 + y^2$ and obtain a single quadratic equation in x

$$-\frac{\sqrt{6}}{2}\lambda x + \frac{3}{2}\left[1 + 2x^2 - \frac{\sqrt{6}}{\lambda}x - w_m\left(\frac{\sqrt{6}}{\lambda}x - 1\right)\right] = 0 \tag{406}$$

  which can be simplified to

$$x^2 - \frac{3(1+w_m) + \lambda^2}{\sqrt{6}\lambda}x + \frac{1+w_m}{2} = 0. \tag{407}$$

  This gives us two more solutions

$$x_3 = \frac{\lambda}{\sqrt{6}}, y_3 = \sqrt{1 - \frac{\lambda^2}{6}} \qquad (\lambda^2 < 6) \tag{408}$$

$$x_4 = \sqrt{\frac{3}{2}}\frac{1+w_m}{\lambda}, y_4 = \sqrt{\frac{3}{2}}\frac{\sqrt{1-w_m^2}}{\lambda} \qquad (w_m^2 < 1). \tag{409}$$

- Let's quickly check the stability of the fix points. The characteristic equation for the fix points of a 2d system is given by

$$\alpha^2 + a_1(x_i, y_i)\alpha + a_2(x_i, y_i) = 0 \tag{410}$$

$$a_1(x_i, y_i) = -\left(\frac{df_x}{dx} + \frac{df_y}{dy}\right)_{x=x_i, y=y_i} \tag{411}$$

$$a_2(x_i, y_i) = \left.\frac{df_x}{dx}\frac{df_y}{dy} - \frac{df_x}{dy}\frac{df_y}{dx}\right|_{x=x_i, y=y_i} \tag{412}$$

  with the stability classification (assuming for EoS parameter $w_m^2 < 1$)

type	condition	fix point 0	fix point 1	fix point 2
saddle node	$a_2 < 0$	$-1 < w_m < 1$	$\lambda > \sqrt{6}$	$\lambda < -\sqrt{6}$
unstable node	$0 < a_2 < a_1^2/4$	-	$\lambda < \sqrt{6}$	$\lambda > -\sqrt{6}$
unstable spiral	$a_1^2/4 < a_2, a_1 < 0$	-	-	-
center	$0 < a_2, a_1 = 0$	-	-	-
stable spiral	$a_1^2/4 < a_2, a_1 > 0$	-	-	-
stable node	$0 < a_2 < a_1^2/4$	-	-	-

type	fix point 3	fix point 4
saddle node	$3(1+w_m) < \lambda^2 < 6$	-
unstable node	-	-
unstable spiral	-	-
center	-	-
stable spiral	-	$\lambda^2 > \frac{24(1+w_m)^2}{7+9w_m}$
stable node	$\lambda^2 < 3(1+w_m)$	$\lambda^2 < \frac{24(1+w_m)^2}{7+9w_m}$

Problem 26.5 - Tracker solution

Inserting the ansatz

$$\phi(t) = C(\alpha, n) M^{1+\nu} t^{\nu} \tag{413}$$

into the ODE

$$\ddot{\phi} + \frac{3\alpha}{t}\dot{\phi} - \frac{M^{4+n}}{\phi^{n+1}} = 0 \tag{414}$$

gives

$$CM^{1+\nu}\nu(\nu-1)t^{\nu-2} + CM^{1+\nu}\frac{3\alpha}{t}t^{\nu-1} - \frac{M^{4+n}}{C^{n+1}M^{(n+1)(1+\nu)}t^{\nu(n+1)}} = 0 \tag{415}$$

$$CM^{1+\nu}\left[\nu(\nu-1)+3\alpha\right]t^{\nu-2} - \frac{M^{3-\nu(n+1)}}{C^{n+1}}t^{-\nu(n+1)} = 0 \tag{416}$$

From equating coefficients and powers (in t) we obtain

$$\nu = \frac{2}{2+n} \tag{417}$$

$$C(\alpha, n) = \left(\frac{(2+n)^2}{6\alpha(2+n)-2n}\right)^{\frac{1}{2+n}}. \tag{418}$$

7 Quantum Gravity

7.1 Ammon, Erdmenger - Gauge/Gravity Duality - Foundations and Applications

The authors use $d-1$ spacial dimension and the sign convention

$$\eta_{\mu\nu} = diag(-1, 1, ..., 1) \tag{419}$$

which implies

$$\Box = \partial^\mu \partial_\mu = -\partial_t^2 + \triangle \tag{420}$$
$$kx = -k^0 x^0 + \vec{k}\vec{x} \tag{421}$$

and results in a minus sign in the KG equation.

Problem 1.1.1 - Fourier representation of free scalar field

Ansatz (because KG equation looks quite similar to wave equation) $\phi(x) = a \cdot e^{ikx}$ with $x^\mu = (t, \vec{x})$, $k^\mu = (\omega, \vec{k})$ and $a \in \mathbb{C}$ meaning

$$e^{ikx} \equiv e^{ik^\mu x_\mu} = e^{i\eta_{\mu\nu}k^\mu x^\nu} = e^{i(-k^0x^0+\vec{k}\vec{x})} \tag{422}$$

Inserting into the equation of motion

$$(\Box - m^2)\phi(x) = (\partial^t\partial_t + \triangle - m^2)\phi(x) \tag{423}$$
$$= a(-\partial_t^2 + \triangle - m^2)e^{i(-\omega t+\vec{k}\vec{x})} \tag{424}$$
$$= a\left(\omega^2 + i^2\vec{k}^2 - m^2\right)e^{i(-\omega t+\vec{k}\vec{x})} = 0 \tag{425}$$

This implies $\omega^2 - \vec{k}^2 - m^2 = 0$ and therefore $\omega_k \equiv \omega = \sqrt{\vec{k}^2 + m^2}$. One particular solution is therefore $\phi(x) = a \cdot e^{ikx}|_{k^0=\omega_k}$. The general solution is then given by a superposition

$$\phi(x) = \int d^{d-1}\vec{k}\left[a(\vec{k})e^{ikx}\right] \tag{426}$$

to ensure a real valued ϕx we add the conjugate complex solution

$$\phi(x) = \int d^{d-1}\vec{k}\left[a(\vec{k})e^{ikx} + a^*(\vec{k})e^{-ikx}\right]. \tag{427}$$

The factor $(2\pi)^{1-d}/2\omega_k$ can be absorbed into $a(k)$.

Problem 1.1.2 - Lagrangian of self-interacting scalar field

The Lagrangian is then

$$\mathcal{L} = \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}} \tag{428}$$
$$= -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi(x)\partial_\nu\phi(x) - \frac{1}{2}m^2\phi(x)^2 - \frac{g}{4!}\phi(x)^4. \tag{429}$$

with the Euler-Lagrange equations

$$\partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0. \tag{430}$$

Therefore

$$\partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)}\right) = \partial_\alpha\left(-\frac{1}{2}\eta^{\mu\nu}[\delta_{\mu\alpha}\partial_\nu\phi + \partial_\mu\phi\delta_{\nu\alpha}]\right) \tag{431}$$
$$= \partial_\alpha\left(-\frac{1}{2}\eta^{\alpha\nu}\partial_\nu\phi - \frac{1}{2}\eta^{\mu\alpha}\partial_\mu\phi\right) \tag{432}$$
$$= -\partial_\alpha\left(\eta^{\alpha\beta}\partial_\beta\phi\right) \tag{433}$$
$$= -\partial^\beta\partial_\beta\phi \tag{434}$$
$$= -\Box\phi \tag{435}$$

and

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi - \frac{g}{3!}\phi^3. \tag{436}$$

The relevant term in the Euler-Lagrange equations is $\partial\mathcal{L}_{\text{int}}/\partial\phi = -g\phi^3/3!$. The modified equation of motion is therefore

$$(\Box - m^2)\phi(x) - \frac{g}{3!}\phi(x)^3 = 0 \tag{437}$$

Problem 1.1.3 - Complex scalar field

$$\mathcal{L}_{\text{free}} = -\partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi \tag{438}$$
$$= -\eta^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi - m^2\phi^*\phi \tag{439}$$
$$= -\frac{1}{2}\eta^{\mu\nu}\partial_\mu(\phi_1 - i\phi_2)\partial_\nu(\phi_1 + i\phi_2) - \frac{1}{2}m^2(\phi_1^2 + \phi_2^2) \tag{440}$$
$$= -\frac{1}{2}\eta^{\mu\nu}\left(\partial_\mu\phi_1\partial_\nu\phi_1 + i\partial_\mu\phi_1\partial_\nu\phi_2 - i\partial_\mu\phi_2\partial_\nu\phi_1 + \partial_\mu\phi_2\partial_\nu\phi_2\right) - \frac{1}{2}m^2(\phi_1^2 + \phi_2^2) \tag{441}$$
$$= -\frac{1}{2}\eta^{\mu\nu}\left(\partial_\mu\phi_1\partial_\nu\phi_1 + \partial_\mu\phi_2\partial_\nu\phi_2\right) - \frac{1}{2}m^2(\phi_1^2 + \phi_2^2) \tag{442}$$
$$= -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi_1\partial_\nu\phi_1 - \frac{1}{2}m^2\phi_1^2 - \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi_2\partial_\nu\phi_2 - \frac{1}{2}m^2\phi_2^2 \tag{443}$$
$$= \mathcal{L}_{\text{free1}} + \mathcal{L}_{\text{free2}} \tag{444}$$

Equations of motion for ϕ and ϕ^* are given by

$$\partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi^*)}\right) - \frac{\partial\mathcal{L}}{\partial\phi^*} = 0 \tag{445}$$
$$-\partial_\mu\partial^\mu\phi + m^2\phi = 0 \tag{446}$$
$$(\Box - m^2)\phi = 0 \tag{447}$$

and

$$\partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi^*)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0 \tag{448}$$
$$-\partial_\mu\partial^\mu\phi + m^2\phi^* = 0 \tag{449}$$
$$(\Box - m^2)\phi^* = 0 \tag{450}$$

Problem 1.2.1 - Time-independence of Noether charge

The conserved current is

$$\partial_\mu\mathcal{J}^\mu \equiv -\partial_0\mathcal{J}^0 + \partial_i\mathcal{J}^i = 0. \tag{451}$$

Spacial integration using Gauss law on the right hand side gives

$$\int_{\mathbb{R}^{d-1}} d^{d-1}\vec{x}\ \partial_0 \mathcal{J}^0 = \int_{\mathbb{R}^{d-1}} d^{d-1}\vec{x}\ \partial_i \mathcal{J}^i \tag{452}$$

$$\partial_0 \int_{\mathbb{R}^{d-1}} d^{d-1}\vec{x}\ \mathcal{J}^0 = \int_{\partial\mathbb{R}^{d-1}} dS\ \mathcal{J}^i \tag{453}$$

$$\partial_0 \mathcal{Q} = 0 \tag{454}$$

where we used that $\mathcal{J}^i$ is vanishing at infinity.

Problem 1.2.2 - Hamiltonian of scalar field

The Lagrangian of the real free scalar field is given by

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi(x)\partial_\nu\phi(x) - \frac{1}{2}m^2\phi(x)^2. \tag{455}$$

The canonical momentum is therefore

$$\begin{aligned} \Pi &= \frac{\partial\mathcal{L}}{\partial(\partial_t\phi)} && (456)\\ &= -\frac{1}{2}2\eta^{ti}\partial_i\phi - \frac{1}{2}2\eta^{tt}\partial_t\phi && (457)\\ &= \partial_t\phi. && (458) \end{aligned}$$

Using $\eta_{\mu\nu} = diag(-1, 1, ..., 1)$ the Hamiltonian $\mathcal{H} = \Theta^{tt} = \eta^{t\nu}\Theta^t{}_\nu = -\Theta^t{}_t$ is

$$\begin{aligned} \Theta^t{}_t &= -\frac{\partial\mathcal{L}}{\partial(\partial_t\phi)}\partial_t\phi + \mathcal{L} && (459)\\ &= -\Pi\cdot\partial_t\phi + \mathcal{L} && (460) \end{aligned}$$

and therefore

$$\begin{aligned} \mathcal{H} &= \Pi\partial_t\phi - \mathcal{L} && (461)\\ &= \Pi^2 - \left(-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi(x)\partial_\nu\phi(x) - \frac{1}{2}m^2\phi(x)^2\right) && (462)\\ &= \Pi^2 - \left(\frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi(x)^2\right) && (463)\\ &= \frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi(x)^2 && (464) \end{aligned}$$

Problem 1.2.3 - Symmetric energy-momentum tensor

The Lorentz transformation

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu \tag{465}$$

implies the field transformation

$$\begin{aligned} \phi(x^\mu) \to \tilde{\phi}(x^\mu) &= \phi(x^\mu - \omega^\mu{}_\rho x^\rho) && (466)\\ &= \phi(x^\mu) - \omega^\mu{}_\rho x^\rho\partial_\mu\phi && (467) \end{aligned}$$

under which the Lagrangian transforms as

$$
\begin{aligned}
\mathcal{L} \to \tilde{\mathcal{L}} &= \mathcal{L} + \frac{\partial \mathcal{L}}{\partial x^\mu} dx^\mu &(468)\\
&= \mathcal{L} - \omega^\nu{}_\rho x^\rho \partial_\mu (\delta^\mu_\nu \mathcal{L}) &(469)\\
&= \mathcal{L} + \partial_\mu (\omega^\nu{}_\rho x^\rho) \cdot (\delta^\mu_\nu \mathcal{L}) - \partial_\mu (\omega^\nu{}_\rho x^\rho \delta^\mu_\nu \mathcal{L}) &(470)\\
&= \mathcal{L} + \omega^\nu{}_\rho \delta^\rho_\mu \cdot (\delta^\mu_\nu \mathcal{L}) - \partial_\mu (\omega^\nu{}_\rho x^\rho \delta^\mu_\nu \mathcal{L}) &(471)\\
&= \mathcal{L} + \omega^\rho{}_\rho \mathcal{L} - \partial_\mu (\omega^\nu{}_\rho x^\rho \delta^\mu_\nu \mathcal{L}) &(472)\\
&= \mathcal{L} - \partial_\mu (\omega^\nu{}_\rho x^\rho \delta^\mu_\nu \mathcal{L}) &(473)
\end{aligned}
$$

where we used $\omega_{\mu\nu} = -\omega_{\nu\mu}$ meaning

$$
\begin{aligned}
\omega^\rho{}_\rho &= \eta^{\alpha\rho} \omega_{\alpha\rho} &(474)\\
&= \sum_\rho \eta^{0\rho} \omega_{0\rho} + \eta^{1\rho} \omega_{1\rho} + \eta^{2\rho} \omega_{2\rho} + \eta^{3\rho} \omega_{3\rho} &(475)\\
&= 0 &(476)
\end{aligned}
$$

in the last step (as η has only diagonal elements and the diagonal elements of ω are zero). With $\delta\phi = -\omega^\mu{}_\rho x^\rho \partial_\mu \phi$ and $X^\mu = -\omega^\nu{}_\rho x^\rho \delta^\mu_\nu \mathcal{L}$ we obtain for the conserved current

$$
\begin{aligned}
\mathcal{J}^\mu &= -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi + X^\mu &(477)\\
&= -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (-\omega^\nu{}_\rho x^\rho \partial_\nu \phi) + (-\omega^\nu{}_\rho x^\rho \delta^\mu_\nu \mathcal{L}) &(478)\\
&= (-\omega^\nu{}_\rho x^\rho) \left(-\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi + (\delta^\mu_\nu \mathcal{L}) \right) &(479)\\
&= (-\omega^\nu{}_\rho x^\rho) \Theta^\mu_\nu &(480)\\
&= (-\eta^{\nu\alpha} \omega_{\alpha\rho} x^\rho) \Theta^\mu_\nu &(481)\\
&= -\omega_{\alpha\rho} x^\rho \Theta^{\mu\alpha} &(482)\\
&= -\frac{1}{2} \omega_{\alpha\rho} (x^\rho \Theta^{\mu\alpha} - x^\alpha \Theta^{\mu\rho}) &(483)\\
&= -\frac{1}{2} \omega_{\alpha\rho} N^{\mu\rho\alpha} &(484)
\end{aligned}
$$

With $\partial_\mu \Theta^\mu_\nu = 0$ and $\partial_\mu N^{\mu\nu\rho} = 0$ we see

$$
\begin{aligned}
0 &= \partial_\mu N^{\mu\nu\rho} &(485)\\
&= \partial_\mu \left(x^\nu \Theta^{\mu\rho} - x^\rho \Theta^{\mu\nu} \right) &(486)\\
&= (\partial_\mu x^\nu) \Theta^{\mu\rho} + x^\nu (\partial_\mu \Theta^{\mu\rho}) - (\partial_\mu x^\rho) \Theta^{\mu\nu} - x^\rho (\partial_\mu \Theta^{\mu\nu}) &(487)\\
&= \delta^\nu_\mu \Theta^{\mu\rho} + x^\nu (\partial_\mu \Theta^{\mu\rho}) - \delta^\rho_\mu \Theta^{\mu\nu} - x^\rho (\partial_\mu \Theta^{\mu\nu}) &(488)\\
&= \Theta^{\nu\rho} - \Theta^{\rho\nu}. &(489)
\end{aligned}
$$

which means that the (canonical) energy-momentum tensor for Poincare invariant field theories is symmetric $\Theta^{\nu\rho} = \Theta^{\rho\nu}$.

Problem 1.2.4 - Callan-Coleman-Jackiw energy-momentum tensor

For the scalar field we have with $\mathcal{L} = -\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}m^2\phi^2$

$$\Theta^\mu_{\ \nu} = -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi + (\delta^\mu_{\ \nu}\mathcal{L}) \tag{490}$$

$$= -\left(-\frac{1}{2}\eta^{\alpha\beta}\delta^\mu_\alpha\partial_\beta\phi - \frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\phi\delta^\mu_\beta\right)\partial_\nu\phi + \delta^\mu_{\ \nu}\left(-\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}m^2\phi^2\right) \tag{491}$$

$$= \partial^\mu\phi\partial_\nu\phi - \frac{1}{2}\delta^\mu_{\ \nu}(\partial^\beta\phi\partial_\beta\phi + m^2\phi^2) \tag{492}$$

which gives in the massless case

$$\Theta^\mu_{\ \nu,\text{ massless}} = \partial^\mu\phi\partial_\nu\phi - \frac{1}{2}\delta^\mu_{\ \nu}\partial^\beta\phi\partial_\beta\phi \tag{493}$$

$$\Theta_{\mu\nu,\text{ massless}} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\partial^\beta\phi\partial_\beta\phi \tag{494}$$

The new improved or Callan–Coleman–Jackiw energy-momentum tensor for a single, real, massless scalar field in d-dimensional Minkowski space is obtained by adding a term proportional to $(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box)\phi^2$ where the proportionality constant is chosen to make the tensor traceless

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\partial_\rho\phi\partial^\rho\phi - \frac{d-2}{4(d-1)}\left(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box\right)\phi^2 \tag{495}$$

Let us now check the properties

1. symmetric: obvious
2. conserved: we use the equation of motion $\partial^\mu\partial_\mu\phi = \Box\phi = 0$

$$\partial_\mu T^{\mu\nu} = (\partial_\mu\partial^\mu\phi)\partial^\nu\phi + \partial^\mu\phi(\partial_\mu\partial^\nu\phi) \tag{496}$$

$$- \frac{1}{2}\eta^{\mu\nu}\left[(\partial_\mu\partial_\rho\phi)\partial^\rho\phi + \partial_\rho\phi(\partial_\mu\partial^\rho\phi)\right] \tag{497}$$

$$- \frac{d-2}{4(d-1)}\Box\partial^\nu\phi^2 + \frac{d-2}{4(d-1)}\eta^{\mu\nu}\partial_\mu\Box\phi^2 \tag{498}$$

$$= \partial^\mu\phi(\partial_\mu\partial^\nu\phi) - \frac{1}{2}\left[(\partial^\nu\partial_\rho\phi)\partial^\rho\phi + \partial_\rho\phi(\partial^\nu\partial^\rho\phi)\right] \tag{499}$$

$$= 0 \tag{500}$$

3. traceless:

$$T^\mu_\mu = \partial^\mu\phi\partial_\mu\phi - \frac{1}{2}\eta^\mu_{\ \mu}\partial_\rho\phi\partial^\rho\phi - \frac{d-2}{4(d-1)}\left(\partial^\mu\partial_\mu - \eta^\mu_{\ \mu}\Box\right)\phi^2 \tag{501}$$

$$= \partial^\mu\phi\partial_\mu\phi - \frac{d}{2}\partial_\rho\phi\partial^\rho\phi - \frac{d-2}{4(d-1)}\left(\partial^\mu\partial_\mu - d\cdot\partial^\mu\partial_\mu\right)\phi^2 \tag{502}$$

$$= \frac{2-d}{2}\partial_\rho\phi\partial^\rho\phi - \frac{d-2}{4(d-1)}(1-d)\partial^\mu\partial_\mu\phi^2 \tag{503}$$

$$= \frac{2-d}{2}\partial_\rho\phi\partial^\rho\phi + \frac{d-2}{4}\partial^\mu\partial_\mu\phi^2 \tag{504}$$

$$= \frac{2-d}{2}\partial_\rho\phi\partial^\rho\phi + \frac{d-2}{4}\partial^\mu(2\phi\partial_\mu\phi) \tag{505}$$

$$= \frac{2-d}{2}\left[\partial_\rho\phi\partial^\rho\phi - \partial^\mu\phi\partial_\mu\phi\right] + \frac{d-2}{2}\phi\cdot\Box\phi \tag{506}$$

$$= 0. \tag{507}$$

Problem 1.2.5 - Noether currents of complex scalar field

$$\mathcal{L}_{\text{free}} = -\partial^\mu \phi^* \partial_\mu \phi - m^2 \phi^* \phi \tag{508}$$
$$= -\eta^{\mu\nu} \partial_\nu \phi^* \partial_\nu \phi - m^2 \phi^* \phi \tag{509}$$

with the field transformations

$$\phi \to \phi' = e^{i\alpha}\phi = \phi + i\alpha\phi \tag{510}$$
$$\phi^* \to \phi^{*'} = e^{-i\alpha}\phi^* = \phi^* - i\alpha\phi^* \tag{511}$$
$$\mathcal{L} \to \mathcal{L}' = \mathcal{L} \tag{512}$$

we have $\delta\phi = i\alpha\phi$ and $\delta\phi^* = -i\alpha\phi^*$ and $X^\mu = 0$. With

$$\mathcal{J}^\sigma = -\frac{\partial \mathcal{L}}{\partial(\partial_\sigma \phi)} \delta\phi + X^\sigma \tag{513}$$

we obtain the the two fields

$$\mathcal{J}^\sigma = -\frac{\partial \mathcal{L}}{\partial(\partial_\sigma \phi)} \delta\phi - \frac{\partial \mathcal{L}}{\partial(\partial_\sigma \phi^*)} \delta\phi^* \tag{514}$$
$$= -(\eta^{\sigma\nu} \partial_\nu \phi^*) i\alpha\phi + (\eta^{\sigma\nu} \partial_\nu \phi) i\alpha\phi^* \tag{515}$$
$$= i\alpha \left[\phi^* (\partial^\sigma \phi) - \phi (\partial^\sigma \phi^*)\right] \tag{516}$$

Problem 1.2.6 - $O(n)$ invariance of action of n free scalar fields

For the n real scalar fields with equal mass m we have

$$\mathcal{L} = -\frac{1}{2} \sum_{j=1}^{n} \left[\eta^{\alpha\beta} (\partial_\alpha \phi_j)(\partial_\beta \phi^j) + m^2 (\phi^j)^2\right] \tag{517}$$

the action functional is then

$$S = \int d^d x \mathcal{L} \tag{518}$$
$$= -\frac{1}{2} \sum_{j=1}^{n} \int d^d x \left[\eta^{\alpha\beta} (\partial_\alpha \phi_j)(\partial_\beta \phi^j) + m^2 (\phi_j \phi^j)\right] \tag{519}$$

With $\phi'^j = R^j{}_k \phi^k$ and the definition of an orthogonal matrix R (inner product is invariant under rotation)

$$x^i x_i = x^i \delta_{ij} x^j \tag{520}$$
$$\stackrel{!}{=} R^i{}_a x^a \delta_{ij} R^j{}_b x^b \tag{521}$$
$$= \delta_{ij} R^j{}_b R^i{}_a x^a x^b \tag{522}$$
$$= R_{ib} R^i{}_a x^a x^b \tag{523}$$

we require $R_{ib}R^i{}_a = \delta_{ba}$. Then we can recalculate the action

$$S' = -\frac{1}{2}\sum_{j=1}^{n}\int d^dx\left[\eta^{\alpha\beta}(\partial_\alpha R_{ja}\phi^a)(\partial_\beta R^j{}_b\phi^b) + m^2(R_{ja}\phi^a\cdot R^j{}_b\phi^b)\right] \tag{524}$$

$$= -\frac{1}{2}\sum_{j=1}^{n}\int d^dx\left[\eta^{\alpha\beta}R_{ja}R^j{}_b(\partial_\alpha\phi^a)(\partial_\beta\phi^b) + m^2 R_{ja}R^j{}_b(\phi^a\cdot\phi^b)\right] \tag{525}$$

$$= -\frac{1}{2}\sum_{b=1}^{n}\int d^dx\left[\eta^{\alpha\beta}\delta_{ab}(\partial_\alpha\phi^a)(\partial_\beta\phi^b) + m^2\delta_{ab}(\phi^a\cdot\phi^b)\right] \tag{526}$$

$$= -\frac{1}{2}\sum_{b=1}^{n}\int d^dx\left[\eta^{\alpha\beta}(\partial_\alpha\phi_b)(\partial_\beta\phi^b) + m^2(\phi_b\cdot\phi^b)\right] \tag{527}$$

Analog for the complex case.

Problem 1.3.1 - Field commutators of scalar field

From the field

$$\hat\phi(x) = \frac{1}{(2\pi)^{d-1}}\int\frac{d^{d-1}\vec k}{2\omega_k}\left[\hat a(\vec k)e^{ikx} + \hat a^\dagger(\vec k)e^{-ikx}\right]_{k^0=\omega_k} \tag{528}$$

we can derive the conjugated momentum

$$\hat\Pi(x) = \partial_t\hat\phi \tag{529}$$

$$= \frac{1}{(2\pi)^{d-1}}\int\frac{d^{d-1}\vec k}{2\omega_k}\partial_t\left[\hat a(\vec k)e^{-i\omega_k t}e^{i\vec k\vec x} + \hat a^\dagger(\vec k)e^{i\omega_k t}e^{-i\vec k\vec x}\right] \tag{530}$$

$$= \frac{1}{(2\pi)^{d-1}}\int\frac{d^{d-1}\vec k}{2\omega_k}\left[\hat a(\vec k)(-i\omega_k)e^{ikx} + \hat a^\dagger(\vec k)(i\omega_k)e^{-ikx}\right]_{k^0=\omega_k} \tag{531}$$

$$= \frac{i}{2(2\pi)^{d-1}}\int d^{d-1}\vec k\left[-\hat a(\vec k)e^{ikx} + \hat a^\dagger(\vec k)e^{-ikx}\right]_{k^0=\omega_k}. \tag{532}$$

Now calculating the three commutation relations

- $[\hat\phi(t,\vec x), \hat\phi(t,\vec y)]$

$$= \frac{1}{(2\pi)^{2(d-1)}}\int\frac{d^{d-1}\vec k d^{d-1}\vec q}{4\omega_k\omega_q}\Big((\hat a(\vec k)e^{ikx} + \hat a^\dagger(\vec k)e^{-ikx})(\hat a(\vec q)e^{iqy} + \hat a^\dagger(\vec q)e^{-iqy}) - \tag{533}$$

$$(\hat a(\vec q)e^{iqy} + \hat a^\dagger(\vec q)e^{-iqy})(\hat a(\vec k)e^{ikx} + \hat a^\dagger(\vec k)e^{-ikx})\Big) \tag{534}$$

  the bracket can then be simplified

$$(\hat a(\vec k)e^{ikx} + \hat a^\dagger(\vec k)e^{-ikx})(\hat a(\vec q)e^{iqy} + \hat a^\dagger(\vec q)e^{-iqy}) - (\hat a(\vec q)e^{iqy} + \hat a^\dagger(\vec q)e^{-iqy})(\hat a(\vec k)e^{ikx} + \hat a^\dagger(\vec k)e^{-ikx}) \tag{535}$$

$$= [\hat a(\vec k), \hat a(\vec q)]e^{i(kx+qy)} + [\hat a(\vec k), \hat a^\dagger(\vec q)]e^{i(kx-qy)} + [\hat a^\dagger(\vec k), \hat a(\vec q)]e^{i(-kx+qy)} + [\hat a^\dagger(\vec k), \hat a^\dagger(\vec q)]e^{i(-kx-qy)} \tag{536}$$

$$= [\hat a(\vec k), \hat a^\dagger(\vec q)]e^{i(kx-qy)} - [\hat a(\vec q), \hat a^\dagger(\vec k)]e^{i(-kx+qy)} \tag{537}$$

$$= 2\omega_k(2\pi)^{d-1}\left(\delta^{d-1}(\vec k-\vec q)e^{i(kx-qy)} - \delta^{d-1}(\vec q-\vec k)e^{i(-kx+qy)}\right) \tag{538}$$

where we used the given commutation relations for $\hat{a}(\vec{k})$.

$$[\hat{\phi}(t,\vec{x}),\hat{\phi}(t,\vec{y})] = \frac{1}{(2\pi)^{2(d-1)}}\int \frac{d^{d-1}\vec{k}d^{d-1}\vec{q}}{4\omega_k\omega_q}2\omega_k(2\pi)^{d-1}\left(\delta^{d-1}(\vec{k}-\vec{q})e^{i(kx-qy)}-\delta^{d-1}(\vec{q}-\vec{k})e^{i(-kx+qy)}\right) \tag{539}$$

$$= \frac{1}{(2\pi)^{d-1}}\int \frac{d^{d-1}\vec{k}d^{d-1}\vec{q}}{2\omega_q}\left(\delta^{d-1}(\vec{k}-\vec{q})e^{i(kx-qy)}-\delta^{d-1}(\vec{q}-\vec{k})e^{i(-kx+qy)}\right) \tag{540}$$

$$= \frac{1}{(2\pi)^{d-1}}\int \frac{d^{d-1}\vec{k}d^{d-1}\vec{q}}{2\omega_q}\left(\delta^{d-1}(\vec{k}-\vec{q})e^{i(-\omega_k t+\vec{k}\vec{x}-[-\omega_q t+\vec{q}\vec{y}])}\right. \tag{541}$$

$$\left.-\delta^{d-1}(\vec{q}-\vec{k})e^{-i(-\omega_k t+\vec{k}\vec{x}-[-\omega_q t+\vec{q}\vec{y}])}\right) \tag{542}$$

$$= \frac{1}{(2\pi)^{d-1}}\int \frac{d^{d-1}\vec{k}d^{d-1}\vec{q}}{2\omega_q}\left(\delta^{d-1}(\vec{k}-\vec{q})e^{i(-[\omega_k-\omega_q]t+\vec{k}\vec{x}-\vec{q}\vec{y})}\right. \tag{543}$$

$$\left.-\delta^{d-1}(\vec{q}-\vec{k})e^{-i(-[\omega_k-\omega_q]t+\vec{k}\vec{x}-\vec{q}\vec{y})}\right) \tag{544}$$

$$= \frac{1}{(2\pi)^{d-1}}\int \frac{d^{d-1}\vec{k}}{2\omega_k}\left(e^{i\vec{k}(\vec{x}-\vec{y})}-e^{-i\vec{k}(\vec{x}-\vec{y})}\right) \tag{545}$$

$$= \frac{1}{2\omega_k}\left(\delta^{d-1}(\vec{y}-\vec{x})-\delta^{d-1}(\vec{x}-\vec{y})\right) \tag{546}$$

$$= 0 \tag{547}$$

where we used $\delta(x) = \int dk e^{-2\pi ikx}$ or $\delta^d(x) = \int \frac{d^dk}{(2\pi)^d}e^{-ikx}$.

- $[\hat{\Pi}(t,\vec{x}),\hat{\Pi}(t,\vec{y})]$ Not done yet
- $[\hat{\phi}(t,\vec{x}),\hat{\Pi}(t,\vec{y})]$ Not done yet

Problem 1.3.2 - Lorentz invariant integration measure

We use the property of the δ-function $\delta(f(x)) = \sum_i \frac{\delta(x-a_i)}{]f'(a_i)|}$ where a_i are the zeros of $f(x)$ and $\omega_k = \sqrt{\vec{k}^2+m^2}$. With $\int d^dk$ being manifestly Lorentz invariant

$$dk'^\mu = \Lambda^\mu_\nu dk^\nu \quad\rightarrow\quad \frac{dk'^\mu}{dk^\nu} = \Lambda^\mu_\nu \quad\rightarrow\quad \int d^dk' = |\det(\Lambda^\mu_\nu)|\int d^dk = \int d^dk \tag{548}$$

$\delta^d[k^2+m^2]$ being invariant and with $k^0 = \sqrt{\vec{k}^2+m^2}$ we see that k is inside the forward light cone and remains there under orthochrone transformation ($\Theta(k^0)$ is invariant for relevant k) we are convinced that the starting expression is Lorentz invariant (integration over the upper mass shell)

$$
\begin{aligned}
\int d^d\vec{k}\delta^d[k^2+m^2]\Theta(k^0) &= \int d^{d-1}\vec{k}\int dk^0\delta^d[k^2+m^2]\Theta(k^0) && (549)\\
&= \int d^{d-1}\vec{k}\int dk^0\delta^d[-(k^0)^2+\vec{k}^2+m^2]\Theta(k^0) && (550)\\
&= \int d^{d-1}\vec{k}\int dk^0\delta^d[\omega_k^2-(k^0)^2]\Theta(k^0) && (551)\\
&= \int d^{d-1}\vec{k}\int dk^0\left(\frac{\delta(k^0-\omega_k)}{2\omega_k}+\frac{\delta(k^0+\omega_k)}{2\omega_k}\right)\Theta(k^0) && (552)\\
&= \int \frac{d^{d-1}\vec{k}}{2\omega_k}\int dk^0\delta(k^0-\omega_k) && (553)\\
&= \int \frac{d^{d-1}\vec{k}}{2\omega_k}. && (554)
\end{aligned}
$$

As we started with a Lorentz invariant expression the derived measure is also invariant.

Problem 1.3.3 - Retarded Green function

$$
\begin{aligned}
\Delta_{\mathrm{F}} &= \int \frac{d^dk}{(2\pi)^d}\frac{e^{ik(x-y)}}{k^2+m^2-i\epsilon} && (555)\\
G_{\mathrm{R}} &= \int \frac{d^dk}{(2\pi)^d}\frac{e^{ik(x-y)}}{-(k^0+i\epsilon)^2+\vec{k}^2+m^2} && (556)
\end{aligned}
$$

For the poles of G_{R} we have

$$
\begin{aligned}
-(k^0+i\epsilon)^2+\vec{k}^2+m^2 &= 0 && (557)\\
k^0 &= -i\epsilon\pm\sqrt{\vec{k}^2+m^2} && (558)\\
&= -i\epsilon\pm\omega_k && (559)
\end{aligned}
$$

while we the poles of Δ_{F} are given by

$$
\begin{aligned}
-(k^0)^2+\vec{k}^2+m^2-i\epsilon &= 0 && (560)\\
k^0 &= \pm\sqrt{\vec{k}^2+m^2-i\epsilon} && (561)\\
&= \pm\sqrt{\omega_k^2-i\epsilon} && (562)
\end{aligned}
$$

Figure 1: Poles of G_{R} (circle) and Δ_{F} (triangle)

With $|\vec{k}\rangle = a^\dagger(\vec{k})|0\rangle$ and

$$
\hat{\phi}(x) = \frac{1}{(2\pi)^{d-1}}\int\frac{d^{d-1}\vec{k}}{2\omega_k}\left[\hat{a}(\vec{k})e^{ikx}+\hat{a}^\dagger(\vec{k})e^{-ikx}\right]_{k^0=\omega_k} \tag{563}
$$

we obtain

$$\hat{\phi}(x)\hat{\phi}(y) \sim \left(\hat{a}(\vec{k})e^{ikx} + \hat{a}^\dagger(\vec{k})e^{-ikx}\right)\left(\hat{a}(\vec{q})e^{iqy} + \hat{a}^\dagger(\vec{q})e^{-iqy}\right) \tag{564}$$

$$= \hat{a}(\vec{k})\hat{a}(\vec{q})e^{i(kx+qy)} + \hat{a}(\vec{k})\hat{a}^\dagger(\vec{q})e^{-i(-kx+qy)} + \hat{a}^\dagger(\vec{k})\hat{a}(\vec{q})e^{i(-kx+qy)} + \hat{a}^\dagger(\vec{k})\hat{a}^\dagger(\vec{q})e^{-i(kx+qy)} \tag{565}$$

$$= \hat{a}(\vec{k})\hat{a}(\vec{q})e^{i(kx+qy)} + \hat{a}(\vec{k})\hat{a}^\dagger(\vec{q})e^{-i(-kx+qy)} + \hat{a}^\dagger(\vec{k})\hat{a}^\dagger(\vec{q})e^{-i(kx+qy)} \tag{566}$$

$$+ \left(\hat{a}(\vec{q})\hat{a}^\dagger(\vec{k}) - 2\omega_k(2\pi)^{d-1}\delta^{d-1}(\vec{q}-\vec{k})\right)e^{i(-kx+qy)} \tag{567}$$

and therefore

$$\langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle = \frac{1}{(2\pi)^{2(d-1)}}\int \frac{d^{d-1}\vec{k}}{2\omega_k}\frac{d^{d-1}\vec{q}}{2\omega_q}\langle 0|\hat{a}(\vec{k})\hat{a}(\vec{q})|0\rangle e^{i(kx+qy)} + \langle 0|\hat{a}(\vec{k})\hat{a}^\dagger(\vec{q})|0\rangle e^{-i(-kx+qy)} \tag{568}$$

$$+ \langle 0|\hat{a}^\dagger(\vec{k})\hat{a}^\dagger(\vec{q})|0\rangle e^{-i(kx+qy)} + \left(\langle 0|\hat{a}(\vec{q})\hat{a}^\dagger(\vec{k})|0\rangle - 2\omega_k(2\pi)^{d-1}\delta^{d-1}(\vec{q}-\vec{k})\right)e^{i(-kx+qy)} \tag{569}$$

$$= \frac{1}{(2\pi)^{2(d-1)}}\int \frac{d^{d-1}\vec{k}}{2\omega_k}\frac{d^{d-1}\vec{q}}{2\omega_q}\langle\vec{k}|\vec{q}\rangle e^{-i(-kx+qy)} + \left(\langle\vec{q}|\vec{k}\rangle - 2\omega_k(2\pi)^{d-1}\delta^{d-1}(\vec{q}-\vec{k})\right)e^{i(-kx+qy)} \tag{570}$$

$$\tag{571}$$

Not done yet

Problem 1.3.4 - Feynman rules of ϕ^4 theory

Not done yet

Problem 1.3.5 - Convergence of perturbative expansion

Not done yet

Problem 1.3.6

Not done yet

Problem 1.3.7

Not done yet

Problem 1.3.8

Not done yet

8 String Theory

8.1 Zwiebach - A First Course in String Theory

8.2 Becker, Becker, Schwarz - String Theory and M-Theroy

8.3 Polchinski - String Theory Volumes 1 and 2

Problem 1.1 - Non-relativistic action limits

(a) We start with (1.2.2) and use $dt = \gamma d\tau$ and $u^\mu = \gamma(c, \vec{v})$ as well as $v \ll c$

$$S_{\text{pp}} = -mc \int d\tau \sqrt{-\dot{X}^\mu \dot{X}_\mu} \tag{572}$$

$$= -mc \int d\tau \sqrt{(c^2 - v^2)\gamma^2} \tag{573}$$

$$= -\int mc^2 \cdot dt \sqrt{1 - \frac{v^2}{c^2}} \tag{574}$$

$$\approx -\int dt \cdot mc^2 \left(1 - \frac{1}{2}\frac{v^2}{c^2}\right) \tag{575}$$

$$= -\int dt \left(mc^2 - \frac{1}{2}mv^2\right) \tag{576}$$

(b)

Not done yet

9 Astrophysics

9.1 Carroll, Ostlie - An Introduction to Modern Astrophysics

9.2 Weinberg - Lecture on Astrophysics

Problem 1 - Hydrostatics of spherical star

Gravitational force on a mass element must be balanced by the top and bottom pressure (buoyancy)

$$F_p^{\text{top}} - F_p^{\text{bottom}} = F_g \tag{577}$$

$$dA \cdot p\left(r + \frac{dr}{2}\right) - dA \cdot p\left(r - \frac{dr}{2}\right) = -g(r)\rho(r) \cdot dA \cdot dr \tag{578}$$

$$\frac{dp}{dr} = -g(r)\rho(r) \tag{579}$$

$$= -G\frac{\mathcal{M}(r)}{r^2}\rho(r) \tag{580}$$

and therefore

$$\rho(r)\mathcal{M}(r) = -\frac{dp}{dr}\frac{r^2}{G} \tag{581}$$

where

$$g(r) = G\frac{\mathcal{M}(r)}{r^2} = \frac{G}{r^2}\int_0^r 4\pi\rho(r')r'^2 dr'. \tag{582}$$

The gravitational binding energy Ω is given by

$$d\Omega = -G\frac{m_{\text{shell}}\mathcal{M}}{r} \tag{583}$$

$$\Omega = -G\int_0^R \frac{4\pi\rho(r)\mathcal{M}(r)}{r} r^2 dr \tag{584}$$

$$= -4\pi G\int_0^R r\rho(r)\mathcal{M}(r)dr \tag{585}$$

$$= 4\pi\int_0^R \frac{dp}{dr} r^3 dr \tag{586}$$

$$= 4\pi p r^3|_0^R - 3\cdot 4\pi\int_0^R p(r)r^2 dr \tag{587}$$

$$= 4\pi p_0 R^3 - 3\left(4\pi\int_0^R p(r)r^2 dr\right) \tag{588}$$

$$= 4\pi p_0 R^3 - 3\int_{K_R} p(\vec{r})d^3r. \tag{589}$$

Problem 2 - CNO cycle

$$\Gamma(ii) = \Gamma(iii) = \Gamma(iv) = \Gamma(v) = \Gamma(i) \tag{590}$$

$$\Gamma(vi) = P\cdot\Gamma(i) \tag{591}$$

$$\Gamma(vii) = \Gamma(viii) = \Gamma(ix) = \Gamma(x) = (1-P)\cdot\Gamma(i) \tag{592}$$

Check result!

Problem 3

Not done yet

Problem 4

Not done yet

Problem 5 - Radial density expansion for a polytrope

For the polytrope equation

$$p = K\rho^\Gamma \tag{593}$$

we obtain

$$\frac{dp}{d\rho} = K\Gamma\rho^{\Gamma-1} \tag{594}$$

$$= \Gamma\frac{p}{\rho} \tag{595}$$

With equations (1.1.4/5)

$$\frac{dp}{dr} = -\frac{G\mathcal{M}(r)\rho(r)}{r^2} \quad \rightarrow \quad \mathcal{M}(r) = -\frac{p' r^2}{G\rho} \tag{596}$$

$$\frac{d\mathcal{M}(r)}{dr} = 4\pi r^2\rho(r) \tag{597}$$

we can obtain a second order ODE by differentiating the first one and substituting $\mathcal{M}'$

$$\mathcal{M}' = -\frac{1}{G}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{d}{dr}p\right) \tag{598}$$

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{d}{dr}p\right) + G\mathcal{M}' = 0 \tag{599}$$

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{d}{dr}p\right) + 4\pi G r^2 \rho = 0 \tag{600}$$

now we can substitute the $p = K\rho^\Gamma$ and obtain

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{d}{dr}\rho^\Gamma\right) + \frac{4\pi G}{K} r^2 \rho = 0. \tag{601}$$

The Taylor expansion

$$\rho(r) = \rho(0)\left[1 + ar^2 + br^4 + ...\right] \tag{602}$$

$$\rho(r)^\Gamma = \rho(0)^\Gamma\left[1 + ar^2 + br^4 + ...\right]^\Gamma \tag{603}$$

$$= \rho(0)^\Gamma\left[1 + a\Gamma r^2 + \left(b\Gamma + \frac{1}{2}a^2\Gamma(\Gamma-1)\right) r^4 + ...\right] \tag{604}$$

$$\frac{1}{\rho} = \frac{1}{\rho(0)}\left[1 - ar^2 + (a^2 - b)r^4 + ...\right] \tag{605}$$

can be substituted into the ODE

$$\rho(0)^{\Gamma-1}\frac{d}{dr}\left(r^2\left[1 - ar^2 + (a^2-b)r^4 + ...\right]\left[a\Gamma 2r + \left(b\Gamma + \frac{1}{2}a^2\Gamma(\Gamma-1)\right)4r^3 + ...\right]\right) \tag{606}$$

$$+\frac{4\pi G}{K}\rho(0)\left[r^2 + ar^4 + br^6 + ...\right] = 0. \tag{607}$$

and sort by powers of r

$$\rho(0)^{\Gamma-1}\frac{d}{dr}\left(2\Gamma a r^3 + \left[-2\Gamma a^2 + 4\left(b\Gamma + \frac{1}{2}a^2\Gamma(\Gamma-1)\right)\right]r^5 + ...\right) + \frac{4\pi G}{K}\rho(0)\left[r^2 + ar^4 + br^6 + ...\right] = 0. \tag{608}$$

In second order of r we obtain

$$\rho(0)^{\Gamma-1} 2\Gamma a 3 + \frac{4\pi G}{K}\rho(0) = 0 \tag{609}$$

which results in

$$a = -\frac{2\pi G}{3\Gamma K \rho(0)^{\Gamma-2}} \tag{610}$$

Problem 6

Not done yet

Problem 7

Not done yet

Problem 8

Not done yet

Problem 9

Not done yet

Problem 10

Not done yet

Problem 11 - Modified Newtonian gravity

The modified Poisson equation is given by

$$\left(\triangle + \mathcal{R}^{-2}\right)\phi = 4\pi G\rho \tag{611}$$

with the Greens function

$$\left(\triangle + \mathcal{R}^{-2}\right)G(\vec{r}) = -\delta^3(\vec{r}). \tag{612}$$

The Fourier transform of the Greens function

$$G(\vec{k}) = \int d^3\vec{r}\, G(\vec{r})e^{-i\vec{k}\vec{r}} \tag{613}$$

and the field equations are given by

$$\begin{aligned}
\left[k^2 + \mathcal{R}^{-2}\right]G(\vec{k}) &= -1 && (614)\\
G(\vec{k}) &= \frac{1}{k^2 + \mathcal{R}^{-2}} && (615)\\
G(\vec{x}) &= \frac{1}{(2\pi)^3}\int d^3\vec{k}\frac{e^{i\vec{k}\vec{r}}}{k^2+\mathcal{R}^{-2}} && (616)\\
&= \frac{1}{(2\pi)^3}2\pi\int_0^\infty\int_0^\pi \frac{e^{ik_r\cdot r\cos\theta}}{k_r^2+\mathcal{R}^{-2}}k_r^2\sin\theta\, d\theta dk_r && (617)\\
&= \frac{1}{(2\pi)^3}2\pi\int_0^\infty\left[-\frac{e^{ik_r r\cos\theta}}{ik_r r}\right]_0^\pi \frac{1}{k_r^2+\mathcal{R}^{-2}}k_r^2 dk_r && (618)\\
&= \frac{1}{2\pi^2 r}\int_0^\infty \frac{k_r\sin(k_r r)}{k_r^2+\mathcal{R}^{-2}}dk_r && (619)\\
& && (620)
\end{aligned}$$

The integral can be can be calculated using the residual theorem

$$\begin{aligned}
\int_0^\infty \frac{k_r\sin(k_r r)}{k_r^2+\mathcal{R}^{-2}}dk_r &= \frac{1}{2}\int_{-\infty}^\infty \frac{k_r\sin(k_r r)}{k_r^2+\mathcal{R}^{-2}}dk_r && (621)\\
&= \frac{1}{2}\int_{-\infty}^\infty \frac{k_r\sin(k_r r)}{(k_r+i\mathcal{R}^{-1})(k_r-i\mathcal{R}^{-1})}dk_r && (622)\\
&= \frac{1}{2}\int_{-\infty}^\infty \frac{k_r\sin(k_r r)}{2k_r}\left(\frac{1}{k_r+i\mathcal{R}^{-1}}+\frac{1}{k_r-i\mathcal{R}^{-1}}\right)dk_r && (623)\\
&= \frac{1}{4}\int_{-\infty}^\infty \frac{\sin(k_r r)}{k_r+i\mathcal{R}^{-1}}dk_r + \frac{1}{4}\int_{-\infty}^\infty \frac{\sin(k_r r)}{k_r-i\mathcal{R}^{-1}}dk_r && (624)
\end{aligned}$$

Not done yet

Problem 12

Not done yet

10 General Physics

10.1 Feynman - Feynman Lectures on Physics

Section G1-1 - 1961 Sep 28 (1.16)

Section G1-2 - 1961 Sep 28 (1.15)

(a) We use the Penman equation to estimate the specific evaporation rate

$$\frac{dm}{dAdt} = \frac{mR_n + \rho_{\text{air}} c_p (\delta e) g_a}{\lambda_v (m + \gamma)} \tag{625}$$

$$= \frac{mR_n + \rho_{\text{air}} c_p (\delta e) g_a}{\lambda_v (m + \frac{c_p p}{\lambda_v MW_{\text{ratio}}})} \tag{626}$$

$$\approx \frac{mR_n}{\lambda_v (m + \frac{c_p p}{\lambda_v MW_{\text{ratio}}})}. \tag{627}$$

The total time is then given by

$$t = \frac{M}{\frac{dm}{dAdt} A} \tag{628}$$

$$= \frac{M}{\frac{dm}{dAdt} \pi r^2} \tag{629}$$

$$= \frac{M\lambda_v (m + \frac{c_p p}{\lambda_v MW_{\text{ratio}}})}{\pi r^2 m R_n} \tag{630}$$

with vapor the water vapor pressure

$$p_{\text{vap}} = \frac{101325\text{Pa}}{760} \exp\left[20.386 - \frac{5132K}{T}\right] \tag{631}$$

the slope of the saturation vapor pressure

$$m = \frac{\partial p_{\text{vap}}}{\partial T} = ... \tag{632}$$

the air heat capacity $c_p = 1.012\text{Jkg}^{-1}\text{K}^{-1}$, the latent heat of vaporization $\lambda_v = 2.26 \cdot 10^6\text{Jkg}^{-1}$, the net irradiance $R_n = 150\text{Wm}^{-2}$ (average day/night partly shade), the ratio molecular weight of water vapor/dry air $MW_{\text{ratio}} = 0.622$, the pressure $p = 10^5\text{Pa}$, the temperature $T = 298\text{K}$, the water weight $M = 0.5\text{kg}$ and the radius of the glass $r = 0.04\text{m}$. This results in $t = 26$ days.

(b) With the molar mass of water $m_{H2O} = 18\,\text{g} \cdot \text{mol}^{-1}$

$$N = \frac{dm}{dAdt} \frac{N_A}{m_{H20}} \tag{633}$$

$$= \frac{mR_n}{\lambda_v (m + \frac{c_p p}{\lambda_v MW_{\text{ratio}}})} \frac{N_A}{m_{H20}} \tag{634}$$

$$= 1.47 \cdot 10^{17}\text{cm}^{-1}\text{s}^{-1} \tag{635}$$

(c) The total mass of water vaporizing on earth in one year is

$$M_{\text{1y prec}} = \varepsilon_{\text{ocean}} 4\pi R_E^2 \frac{dm}{dAdt} t_{1y}. \tag{636}$$

with $\varepsilon_{\text{ocean}} = 0.7$. In equilibrium this must be equal to the total amount of precipitation. So the average rainfall height is

$$h = \frac{M_{1\text{y prec}}}{4\pi R_E^2 \rho_{\text{H2O}}} \tag{637}$$
$$= \frac{\varepsilon_{\text{ocean}} t_{1y}}{\rho_{\text{H2O}}} \frac{dm}{dAdt} \tag{638}$$
$$= 947\text{mm}. \tag{639}$$

which seems reasonable (given that the solar constant is $1,361\text{Wm}^{-2}$ the estimate of $R_n = 150\text{Wm}^{-2}$ seems ok).

Section G-1 - 1961 Oct 5 (?.??)

(a) $\sqrt{s/g}$

(b) mL/T^2

(c) $\rho g h$

(d) $\sqrt{p/\rho}$

(e) gT (need to use the period T as c is not a material constant due to strong dispersion)

(f) $\rho g H^2$

(g) $\sqrt{R/g}$ here we assume the hemisphere rests on the table upside down - so it acts like a pendulum

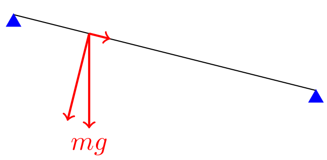
(h) $\sqrt{FL/m}$

Section G-2 - 1961 Oct 5 (?.??)

1. Equilibrium is given by condition

$$m_1 g = m_2 g \sin\alpha \tag{640}$$
$$= m_2 g \frac{x}{\sqrt{x^2+a^2}} \tag{641}$$
$$\rightarrow \quad m_1^2(x^2+a^2) = m_2^2 x^2 \tag{642}$$
$$\rightarrow \quad x = \frac{m_1 a}{\sqrt{m_2^2 - m_1^2}} \tag{643}$$
$$\tag{644}$$

2. General consideration



3.

4.

5.

Problem Set 3-1 - 1961 Nov 03 (3.16)

Direct measurement can be done for the

- radius of the earth $R_e = 6371\text{km}$
- orbital period of the moon $T_M = 28\text{d}$
- angular diameter of the moon $\delta = 30' = 0.5°$
- earths gravitational acceleration $g = 9.81\text{ms}^2$
- also Sputnik I orbital data can be looked up $a_{\text{satellite}} = R_E + 584\text{km}$ and $T_{\text{satellite}} = 96.2\text{min}$
- height difference between low and high tide $\Delta h = 1\text{m}$

1. We use Keplers 3rd law

$$\frac{a_M^3}{T_M^2} = \frac{a_{\text{satellite}}^3}{T_{\text{satellite}}^2} \tag{645}$$

$$a_M = a_{\text{sat}} \left(\frac{T_M}{T_{\text{sat}}}\right)^{2/3} \tag{646}$$

then the radius of the moon is given by

$$R_M = \frac{a_M}{2} \tan\delta = \frac{a_{\text{sat}}}{2} \left(\frac{T_M}{T_{\text{sat}}}\right)^{2/3} \tan\delta \tag{647}$$

and the mass by

$$m_M = \rho_M V_M = \frac{4}{3}\pi\rho_M R_M^3 \tag{648}$$

$$= \frac{4}{3}\pi\rho_M \left(\frac{a_{\text{sat}}}{2} \left(\frac{T_M}{T_{\text{sat}}}\right)^{2/3} \tan\delta\right)^3 \tag{649}$$

$$= \frac{1}{6}\pi\rho_M a_{\text{sat}}^3 \left(\frac{T_M}{T_{\text{sat}}}\right)^2 \tan^3\delta \tag{650}$$

$$\approx \frac{1}{6}\pi\rho_E a_{\text{sat}}^3 \left(\frac{T_M}{T_{\text{sat}}}\right)^2 \tan^3\delta \tag{651}$$

where we approximated the moon by the earth mass density. From the gravitational law we can obtain the earth density by

$$g = \frac{F_g}{m} = \frac{Gm_E}{R_E^2} \quad \rightarrow \quad m_E = \frac{gR_E^2}{G} \tag{652}$$

$$\rho_E = \frac{m_E}{V_E} = \frac{m_E}{\frac{4}{3}\pi R_E^3} = \frac{3g}{4\pi G R_E}. \tag{653}$$

Therefore the mass of the moon is given by

$$m_M \approx \frac{g}{8GR_E} a_{\text{sat}}^3 \left(\frac{T_M}{T_{\text{sat}}}\right)^2 \tan^3\delta \tag{654}$$

$$= 1.16 \cdot 10^{23}\text{kg}. \tag{655}$$

2. We use Keplers 3rd law (for the earth-moon system) and the gravitational law for the earth

$$\frac{a_M^3}{T_M^2} = \frac{G(m_E + m_M)}{4\pi^2} \approx \frac{Gm_E}{4\pi^2} = \frac{a_{\text{satellite}}^3}{T_{\text{satellite}}^2} \tag{656}$$

$$g = \frac{F_g}{m} = \frac{Gm_E}{R_E^2} \tag{657}$$

and obtain

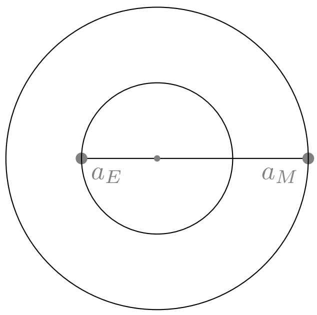
$$\frac{a_{\text{satellite}}^3}{T_{\text{satellite}}^2} = \frac{gR_E^2 + Gm_M}{4\pi^2} \tag{658}$$

$$m_M = \frac{4\pi^2}{G}\left(\frac{a_{\text{satellite}}^3}{T_{\text{satellite}}^2} - \frac{gR_E^2}{4\pi^2}\right) \tag{659}$$

$$= 7.07 \cdot 10^{21}\text{kg}. \tag{660}$$

This result is quite sensitive to the satellite orbital data.

3. We will use the earth tidal data. Lets assume circular orbits with $a_E + a_M = D$ which we can justify by observation (as the moon appears to have constant angular diameter). As reference system we use the center of mass of the system



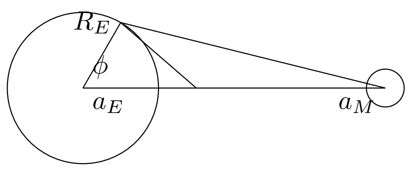
$$m_E\omega^2 a_E = \frac{Gm_E m_M}{D^2} = m_M\omega^2 a_M \tag{661}$$

$$\rightarrow a_E = \frac{m_M D}{m_E + m_M} \tag{662}$$

$$\rightarrow \omega^2 = \frac{G(m_E + m_M)}{D^3} \tag{663}$$

$$\rightarrow \omega^2 a_E^2 = \frac{Gm_M^2}{D(m_E + m_M)} \tag{664}$$

$$\rightarrow \frac{R_E}{a_E} = \frac{m_E + m_M}{m_M}\frac{R_E}{D} \tag{665}$$



The potential is then given by (gravity of moon and earth as well as the centripetal potential around the center of gravity)

$$
\begin{aligned}
V &= V_{\mathrm{G,moon}} + V_{\mathrm{G,earth}} + V_{\mathrm{cent}} && (666)\\
&= -\frac{Gm_M}{\sqrt{R_E^2 + D^2 - 2DR_E\cos\phi}} - \frac{Gm_E}{R_E} - \frac{1}{2}\omega^2(R_E^2 + a_E^2 - 2a_E R_E\cos\phi) && (667)\\
&= -\frac{Gm_M}{D\sqrt{\left(\frac{R_E}{D}\right)^2 + 1 - 2\frac{R_E}{D}\cos\phi}} - \frac{Gm_E}{R_E} - \frac{1}{2}\omega^2 a_E^2\left[\left(\frac{R_E}{a_E}\right)^2 + 1 - 2\frac{R_E}{a_E}\cos\phi\right] && (668)\\
&\approx -\frac{Gm_M}{D}\left(1 + \frac{R_E}{D}\cos\phi + \frac{3}{2}\left(\frac{R_E}{D}\right)^2\cos^2\phi\right) - \frac{Gm_E}{R_E} && (669)\\
&\qquad - \frac{1}{2}\omega^2 a_E^2\left[\left(\frac{R_E}{a_E}\right)^2 + 1 - 2\frac{R_E}{a_E}\cos\phi\right] && (670)\\
&\approx -\frac{Gm_M}{D}\left(1 + \frac{R_E}{D}\cos\phi + \frac{3}{2}\left(\frac{R_E}{D}\right)^2\cos^2\phi\right) - \frac{Gm_E}{R_E} && (671)\\
&\qquad - \frac{1}{2}\frac{Gm_M^2}{D(m_E + m_M)}\left[\left(\frac{m_E + m_M}{m_M}\frac{R_E}{D}\right)^2 + 1 - 2\frac{m_E + m_M}{m_M}\frac{R_E}{D}\cos\phi\right] && (672)\\
&\approx -\frac{Gm_M}{D} - \frac{3Gm_M}{2D}\frac{R_E^2}{D^2}\cos^2\phi - \frac{Gm_E}{R_E} && (673)\\
&\qquad - \frac{1}{2}\frac{Gm_M^2}{D(m_E + m_M)}\left[\left(\frac{m_E + m_M}{m_M}\frac{R_E}{D}\right)^2 + \frac{m_M^2 D^2}{m_M^2 D^2}\right] && (674)\\
&\approx -\frac{Gm_M}{D} - \frac{3Gm_M}{2D}\frac{R_E^2}{D^2}\cos^2\phi - \frac{Gm_E}{R_E} - \frac{G}{2}\left[(m_E + m_M)\frac{R_E^2}{D^3} + \frac{m_M^2}{m_E + m_M}\frac{1}{D}\right]. && (675)
\end{aligned}
$$

with the angular dependent tidal part

$$
V_{\mathrm{tidal}} = -\frac{3GR_E^2 m_M}{2D^3}\cos^2\phi. \tag{676}
$$

The tidal water surface would be formed by the the surface $r_{\mathrm{surf}}(\phi) = R_E + h$ of constant potential. The height difference between low and high tide can then be estimated by

$$
\begin{aligned}
-\frac{3GR_E^2 m_M}{2D^3} &= Gm_E\left(\frac{1}{R_E + h} - \frac{1}{R_E}\right) && (677)\\
&\approx Gm_E\left(\frac{1}{R_E\left(1 + \frac{h}{R_E}\right)} - \frac{1}{R_E}\right) && (678)\\
&\approx \frac{Gm_E}{R_E}\left(\left(1 - \frac{h}{R_E}\right) - 1\right) && (679)
\end{aligned}
$$

which gives

$$
h = \frac{3R_E^4}{2D^3}\frac{m_M}{m_E}. \tag{680}
$$

Using the results from above

$$
\begin{aligned}
m_E &= \frac{gR_E^2}{G} && (681)\\
\omega^2 &= \frac{G(m_E + m_M)}{D^3} && (682)\\
\rightarrow \quad D^3 &= \frac{G(m_E + m_M)}{\omega^2} = G(m_E + m_M)\frac{T_M^2}{4\pi^2} && (683)
\end{aligned}
$$

we obtain

$$h = \frac{6\pi^2 R_E^4 T_M^2}{G(m_E + m_M)T_M^2} \frac{m_M}{m_E}. \tag{684}$$

and can subsequently solve for m_M

$$\begin{aligned} m_M &= \frac{Ghm_E^2 T^2}{6\pi^2 R_E^4 - Ghm_E T^2} && (685)\\ &= \frac{m_E}{\frac{6\pi^2 R_E^4}{Ghm_E T^2} - 1} && (686)\\ &= \frac{g^2 h T_M^2 R_E^2}{G(6\pi^2 R_E^2 - ghT_M^2)} && (687)\\ &= \frac{gR_E^2}{G\left(\frac{6\pi^2 R_E^2}{ghT_M^2} - 1\right)} && (688)\\ &= 1.38 \cdot 10^{23}\text{kg} && (689) \end{aligned}$$

Problem Set 3-3 - 1961 Nov 03 (3.10)

(a) We use Keplers 3rd law for the earth

$$\frac{a_E^3}{T_E^2} = \frac{G(m_S + m_E)}{4\pi^2} \approx \frac{Gm_S}{4\pi^2} \tag{690}$$

$$\tag{691}$$

and the stars a and b

$$\begin{aligned} \frac{a^3}{T^2} &= \frac{G(m_A + m_B)}{4\pi^2} && (692)\\ \frac{(Ra_E)^3}{(TT_E)^2} &= \frac{R^3}{T^2}\frac{a_E^3}{T_E^2} = \frac{R^3}{T^2}\frac{Gm_S}{4\pi^2} = \frac{G(m_A + m_B)}{4\pi^2} && (693)\\ \rightarrow m_A + m_B &= \frac{R^3}{T^2} m_S = \frac{729}{25} m_S && (694) \end{aligned}$$

(b) For a the circular orbits we have the stability condition

$$\begin{aligned} m_A \omega^2 r_A = F_{AB} &= m_B \omega^2 r_B && (695)\\ \rightarrow m_A \omega v_A &= m_B \omega v_B && (696)\\ \rightarrow \frac{m_A}{m_B} &= \frac{v_B}{v_A} = \frac{1}{5} && (697) \end{aligned}$$

with $m_B = 5m_A$ we have

$$\begin{aligned} m_A &= \frac{243}{50} m_S && (698)\\ m_B &= \frac{243}{10} m_S. && (699) \end{aligned}$$

Problem Set 3-4 - 1961 Nov 03 (?.??)

$$g_M = \frac{GM_M}{R_M^2} = \frac{4}{3} G\rho_M R_M = \frac{4}{3} G(0.537\rho_E)(0.716R_E) = 0.384 \cdot g_E \tag{700}$$

10.2 Thorne, Blandford - Modern Classical Physics

Exercise 1.1 Practice: Energy Change for Charged Particle

With $E = p^2/2m$ and (1.7c) we obtain

$$\frac{dE}{dt} = \frac{d}{dt}\frac{p^2}{2m} = \frac{2\vec{p}\cdot d\vec{p}/dt}{2m} \tag{701}$$
$$= \frac{q}{m}\vec{p}\cdot(\vec{E}+\vec{v}\times\vec{B}) \tag{702}$$
$$= q\vec{v}\cdot(\vec{E}+\vec{v}\times\vec{B}) \tag{703}$$
$$= q\vec{v}\cdot\vec{E}. \tag{704}$$

As $\vec{v}\times\vec{B}$ is orthogonal to $\vec{v}$ (and $\vec{B}$) the scalar product $\vec{v}\cdot(\vec{v}\times\vec{B})$ vanishes.

Exercise 1.2 Practice: Particle Moving in a Circular Orbit

(a) With

$$\frac{d\vec{n}}{ds} = \frac{\vec{n}'-\vec{n}}{R\cdot d\phi} = \frac{\vec{v}'-\vec{v}}{vR\cdot d\phi} \tag{705}$$

we can calculate the norm

$$\left|\frac{d\vec{n}}{ds}\right| = \frac{\sqrt{v^2+v^2-2v^2\cos(d\phi)}}{vR\cdot d\phi} = \frac{v\sqrt{1-\cos(d\phi)}}{vR\cdot d\phi} = \frac{v\sqrt{2[1-\cos(d\phi)]}}{vR\cdot d\phi} \tag{706}$$
$$\approx \frac{vd\phi}{vR\cdot d\phi} = \frac{1}{R} \tag{707}$$

and the scalar product

$$\frac{d\vec{n}}{ds}\cdot\vec{n} = \frac{\vec{n}'\cdot\vec{n}-\vec{n}\cdot\vec{n}}{R\cdot d\phi} = \frac{n^2\cos(d\phi)-n^2}{vR\cdot d\phi} \tag{708}$$
$$\approx \frac{(1-d\phi^2/2)-1}{vR\cdot d\phi} = \frac{d\phi}{2vR} \tag{709}$$

which vanished for $d\phi\to 0$ and therefore implies that $d\vec{n}$ is orthogonal to $\vec{n}$ (and therefore points to the center).

(b) From (a) we know

$$\vec{R} = R^2\frac{d\vec{n}}{ds} = R^2\frac{d\vec{v}}{v\cdot ds} = R^2\frac{d\vec{v}}{v\cdot ds} = \frac{R^2}{v}\frac{d\vec{v}}{dt}\frac{dt}{ds} = \left(\frac{R}{v}\right)^2\vec{a} \tag{710}$$

Taking the absolute value we have

$$R = \frac{R^2}{v^2}a \quad\rightarrow\quad R = \frac{v^2}{a} \tag{711}$$

and therefore

$$\vec{R} = \frac{R^2}{v^2}\vec{a} = \frac{v^4}{v^2a^2}\vec{a} = \left(\frac{v}{a}\right)^2\vec{a}. \tag{712}$$

Exercise 1.3 Derivation: Component Manipulation Rules

1. (1.9g I) - using (1.9b), (1.9a) and (1.9c)

$$\mathbf{A} \cdot \mathbf{B} = (A_j \mathbf{e}_j) \cdot (B_k \mathbf{e}_k) = A_j B_k \mathbf{e}_j \cdot \mathbf{e}_k = A_j B_k \delta_{jk} = A_j B_j \tag{713}$$

2. (1.9g II) - using (1.9d) and (1.5a)

$$\mathbf{T} = T_{ijk} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \tag{714}$$
$$\mathbf{T}(\mathbf{A}, \mathbf{B}, \mathbf{C}) = T_{ijk} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k (\mathbf{A}, \mathbf{B}, \mathbf{C}) \tag{715}$$
$$= T_{ijk} (\mathbf{A} \cdot \mathbf{e}_i)(\mathbf{B} \cdot \mathbf{e}_j)(\mathbf{C} \cdot \mathbf{e}_k) \tag{716}$$
$$= T_{ijk} A_i B_j C_k \tag{717}$$

3. (1.9h) - using (1.9d), (1.6b), (1.9a) and (1.5a)

$$\mathbf{R} = R_{abcd} \mathbf{e}_a \otimes \mathbf{e}_b \otimes \mathbf{e}_c \otimes \mathbf{e}_d \tag{718}$$
$$\text{1\&3 contraction}(\mathbf{R}) = R_{abcd} (\mathbf{e}_a \cdot \mathbf{e}_c) \mathbf{e}_b \otimes \mathbf{e}_d \tag{719}$$
$$= R_{abcd} \delta_{ac} \mathbf{e}_b \otimes \mathbf{e}_d \tag{720}$$
$$= R_{abad} \mathbf{e}_b \otimes \mathbf{e}_d \tag{721}$$
$$\text{components of [1\&3 contraction}(\mathbf{R})] = R_{abad} \mathbf{e}_b \otimes \mathbf{e}_d (\mathbf{e}_j, \mathbf{e}_k) \tag{722}$$
$$= R_{abad} (\mathbf{e}_b \cdot \mathbf{e}_j)(\mathbf{e}_d \cdot \mathbf{e}_k) \tag{723}$$
$$= R_{abad} \delta_{bj} \delta_{dk} \tag{724}$$
$$= R_{ajak} \tag{725}$$

Exercise 1.4 Example and Practice: Numerics of Component Manipulations

$$\mathbf{C} = \mathbf{S}(\mathbf{A}, \mathbf{B}, \_) \tag{726}$$
$$= S_{ijk} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k (\mathbf{A}, \mathbf{B}, \_) \tag{727}$$
$$= S_{ijk} (\mathbf{A} \cdot \mathbf{e}_i)(\mathbf{B} \cdot \mathbf{e}_j) \mathbf{e}_k \tag{728}$$
$$= S_{ijk} A_i B_j \mathbf{e}_k \tag{729}$$

$$C_k = S_{11k} A_1 B_1 + S_{12k} A_1 B_2 \tag{730}$$
$$C_1 = 0, \quad C_2 = 0, \quad C_3 = S_{123} A_1 B_2 = 15 \tag{731}$$

$$\mathbf{D} = \mathbf{S}(\mathbf{A}, \_, \mathbf{B}) \tag{732}$$
$$= S_{ijk} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k (\mathbf{A}, \_, \mathbf{B}) \tag{733}$$
$$= S_{ijk} (\mathbf{A} \cdot \mathbf{e}_i)(\mathbf{B} \cdot \mathbf{e}_k) \mathbf{e}_j \tag{734}$$
$$= S_{ijk} A_i B_k \mathbf{e}_j \tag{735}$$

$$D_j = S_{1j1} A_1 B_1 + S_{1j2} A_1 B_2 = 0 \tag{736}$$

$$\mathbf{W} = \mathbf{A} \otimes \mathbf{B} \tag{737}$$
$$= (A_i \mathbf{e}_i) \otimes (B_j \mathbf{e}_j) \tag{738}$$
$$= A_i B_j \mathbf{e}_i \otimes \mathbf{e}_j \tag{739}$$

$$W_{11} = 12, \quad W_{12} = 15, \tag{740}$$

Exercise 1.5 Practice: Meaning of Slot-Naming Index Notation

(a) Somewhat guessing

$$
\begin{aligned}
A_i B_{jk} &\rightarrow A_i B_{jk} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k && (741)\\
&= (A_i \mathbf{e}_i) \otimes (B_{jk} \mathbf{e}_j \otimes \mathbf{e}_k) && (742)\\
&= A(\_) \otimes B(\_,\_) && (743)
\end{aligned}
$$

$$
\begin{aligned}
A_i B_{ji} &\rightarrow A_i B_{ji} \mathbf{e}_j && (744)\\
&= (\mathbf{A} \cdot \mathbf{e}_i) B_{ji} \mathbf{e}_j && (745)\\
&= B_{ji} \mathbf{e}_j \otimes \mathbf{e}_i (\_, \mathbf{A}) && (746)\\
&= \mathbf{B}(\_, \mathbf{A}) && (747)
\end{aligned}
$$

$$
S_{ijk} = S_{kji} \rightarrow ... \qquad (748)
$$

$$
A_i B_i = AiB_j g_{ij} \rightarrow \mathbf{A} \cdot \mathbf{B} = \mathbf{g}(\mathbf{A}, \mathbf{B}) \qquad (749)
$$

(b) Applying the standard machinery

$$
\begin{aligned}
\mathbf{T}(\_,\_,\mathbf{A}) &= T_{ijk} \mathbf{e}_i \otimes \mathbf{e}_j (\mathbf{A} \cdot \mathbf{e}_k) && (750)\\
&= T_{ijk} A_k \mathbf{e}_i \otimes \mathbf{e}_j && (751)\\
&\rightarrow T_{ijk} A_k && (752)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{S}(\mathbf{B},\_) &= S_{ab} (\mathbf{B} \cdot \mathbf{e}_a) \mathbf{e}_b && (753)\\
&= S_{ab} B_a \mathbf{e}_b && (754)\\
\mathbf{T}(\_, \mathbf{S}(\mathbf{B},\_),\_) &= T_{ijk} \mathbf{e}_i \otimes \mathbf{e}_k (S_{ab} B_a \mathbf{e}_b \cdot \mathbf{e}_j) && (755)\\
&= T_{ijk} \mathbf{e}_i \otimes \mathbf{e}_k (S_{ab} B_a \delta_{bj}) && (756)\\
&= T_{ijk} S_{aj} B_a \mathbf{e}_i \otimes \mathbf{e}_k && (757)\\
&\rightarrow T_{ijk} S_{aj} && (758)
\end{aligned}
$$

Exercise 1.6 Example and Practice: Rotation in x-y Plane

(a)

(b)

(c)

(d)

Exercise 1.7 Derivation: Properties of the Levi-Civita Tensor

Exercise 1.8 Example and Practice: Vectorial Identities for the Cross Product and Curl

(a)

(b)

(c)

Exercise 1.9 Example and Practice: Levi-Civita Tensor in 2-Dimensional Euclidean Space

(a)

(b)

Exercise 1.10 Derivation and Practice: Volume Elements in Cartesian Coordinates

Exercise 1.11 Example and Practice: Integral of a Vector Field over a Sphere

(a)

(b)

(c)

(d)

Exercise 1.12 Example: Faraday's Law of Induction

Exercise 1.13 Example: Equations of Motion for a Perfect Fluid

(a)

(b)

(c)

(d)

(e)

Exercise 1.14 Problem: Electromagnetic Stress Tensor

(a)

(b)

Exercise 1.15 Practice: Geometrized Units

(a) $t_P = \sqrt{G\hbar} \quad \rightarrow \quad \sqrt{\frac{G\hbar}{c^5}} = 5.39 \cdot 10^{-44}\text{s} \equiv 1.61 \cdot 10^{-35}\text{m}$

(b) $E = 2mc^2$

(c)

(d)

(e) $1\text{m} \equiv 3.33 \cdot 10^{-9}\text{s}$ and $1\text{yr} \equiv 9.45 \cdot 10^{15}\text{m}$

Exercise 3.3 Practice and Example: Regimes of Particulate and Wave - Like Behavior

(a) The Schwarzschild radius of the BH is

$$R_S = \frac{2GM}{c^2} = 44,466\text{m} \tag{759}$$

which gives a disk radius of $R = 7R_S = 311$km. With

$$F_{\text{Earth}} = \frac{dP}{dA} = \frac{dW}{dA\,dt} = \frac{dN \cdot E_{ph}c}{dA \cdot dl} = \left(\frac{dN}{d\mathcal{V}_x}\right)_{\text{Earth}} \cdot E_{\text{ph}}c \tag{760}$$

$$\left(\frac{dN}{d\mathcal{V}_x}\right)_{\text{Earth}} = \frac{F_{\text{Earth}}}{cE_{\text{ph}}} = 0.00104\text{m}^{-3} \tag{761}$$

$$F_{\text{CX1}} = \frac{r^2}{R^2}F_{\text{Earth}} \tag{762}$$

$$\left(\frac{dN}{d\mathcal{V}_x}\right)_{\text{CX1}} = \frac{F_{\text{CX1}}}{cE_{\text{ph}}} = \frac{r^2}{R^2}\frac{F_{\text{Earth}}}{cE_{\text{ph}}} = 3.72 \cdot 10^{25}\text{m}^{-3} \tag{763}$$

The momentum of the photons is $p = E/c$.

The mean occupation number is then

$$\eta = \frac{h^3}{g_s}\mathcal{N} = \frac{h^3}{g_s}\frac{dN}{d\mathcal{V}_x d\mathcal{V}_p} = \tag{764}$$

Exercise 7.1 Practice: Group and Phase Velocities

With the definition of phase and group velocities

$$\vec{v}_{ph} = \frac{\omega}{k}\frac{\vec{k}}{k} \tag{765}$$

$$\vec{v}_g = \nabla_k \omega \tag{766}$$

$$\omega_1(\vec{k}) = C|\vec{k}| \tag{767}$$
$$\rightarrow \vec{v}_{ph} = \frac{C|\vec{k}|}{k}\frac{\vec{k}}{k} = C\frac{\vec{k}}{k} \tag{768}$$
$$\rightarrow \vec{v}_g = C\frac{2\vec{k}}{2\sqrt{k^2}} = C\frac{\vec{k}}{k} \tag{769}$$
$$\omega_2(\vec{k}) = \sqrt{g|\vec{k}|} \tag{770}$$
$$\rightarrow \vec{v}_{ph} = \frac{\sqrt{g|\vec{k}|}}{k}\frac{\vec{k}}{k} = \sqrt{\frac{g}{k}}\frac{\vec{k}}{k} \tag{771}$$
$$\rightarrow \vec{v}_g = \sqrt{g}\frac{1}{2\sqrt{|\vec{k}|}}\frac{\vec{k}}{k} = \frac{1}{2}\sqrt{\frac{g}{k}}\frac{\vec{k}}{k} \tag{772}$$
$$\omega_3(\vec{k}) = \sqrt{\frac{D}{\Lambda}}\vec{k}^2 \tag{773}$$
$$\rightarrow \vec{v}_{ph} = \sqrt{\frac{D}{\Lambda}}\frac{\vec{k}^2}{k}\frac{\vec{k}}{k} = \sqrt{\frac{D}{\Lambda}}k\frac{\vec{k}}{k} \tag{774}$$
$$\rightarrow \vec{v}_g = \sqrt{\frac{D}{\Lambda}}2\vec{k} = 2\sqrt{\frac{D}{\Lambda}}k\frac{\vec{k}}{k} \tag{775}$$
$$\omega_4(\vec{k}) = \vec{a}\cdot\vec{k} \tag{776}$$
$$\rightarrow \vec{v}_{ph} = \frac{\vec{a}\cdot\vec{k}}{k}\frac{\vec{k}}{k} = \left(\vec{a}\cdot\frac{\vec{k}}{k}\right)\frac{\vec{k}}{k} \tag{777}$$
$$\rightarrow \vec{v}_g = \vec{a} \tag{778}$$

Exercise 7.2 Example: Gaussian Wave Packet and Its Dispersion

(a) Taylor expansion of the dispersion relation gives

$$\omega = \Omega(k) = \omega(k_0) + \left.\frac{\partial\omega(k)}{\partial k}\right|_{k=k_0}(k-k_0) + \frac{1}{2}\left.\frac{\partial^2\omega(k)}{\partial k^2}\right|_{k=k_0}(k-k_0)^2 \tag{779}$$
$$= \omega(k_0) + V_g|_{k=k_0}(k-k_0) + \frac{1}{2}\left.\frac{\partial V_g(k)}{\partial k}\right|_{k=k_0}(k-k_0)^2. \tag{780}$$

(b) The wave packet can then be written as

$$\psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk A(k)e^{i\alpha(k)}e^{i(kx-\omega t)} \tag{781}$$
$$= \frac{C}{2\pi}\int_{-\infty}^{\infty} dk e^{-\frac{(k-k_0)^2}{2\Delta k^2}}e^{i[\alpha_0 - x_0(k-k0)]}e^{i(kx-[\omega_0+V_g(k-k_0)+\frac{1}{2}V_g'(k-k_0)^2]t)} \tag{782}$$
$$= \frac{C}{2\pi}\int_{-\infty}^{\infty} dk e^{-\frac{(k-k_0)^2}{2\Delta k^2}}e^{i(\alpha_0+k_0x-\omega_0 t-(V_g t-x+x_0)(k-k_0)-\frac{1}{2}V_g' t(k-k_0)^2)} \tag{783}$$
$$= \frac{C}{2\pi}e^{i(\alpha_0+k_0x-\omega_0 t)}\int_{-\infty}^{\infty} dk e^{-i(V_g t-x+x_0)(k-k_0)}e^{-\frac{1}{2}(k-k_0)^2\left(\frac{1}{\Delta k^2}+iV_g' t\right)} \tag{784}$$
$$= \frac{C}{2\pi}e^{i(\alpha_0+k_0x-\omega_0 t)}\int_{-\infty}^{\infty} d\kappa e^{i(x-x_0-V_g t)\kappa}e^{-\frac{1}{2}\kappa^2\left(\frac{1}{\Delta k^2}+iV_g' t\right)} \tag{785}$$
$$\tag{786}$$

(c) With

$$\int_{-\infty}^{\infty} dy e^{-(a+ic)y^2} e^{-iby} = \sqrt{\frac{\pi}{a^2+c^2}} \sqrt{a-ic}\, e^{-\frac{b^2}{4(a^2+c^2)}(a-ic)} \qquad a>0, a,b,c \in \mathbb{R} \tag{787}$$

and the substitutions $a = \frac{1}{2\,\Delta k^2}, c = \frac{V_g' t}{2}$ and

$$a^2 + c^2 = \frac{1}{4\,\Delta k^2} \frac{1}{\Delta k^2} \left(1 + [V_g'(\Delta k)^2 t]^2\right) \tag{788}$$

$$= \frac{1}{4\,\Delta k^2} L^2 \tag{789}$$

$$= \frac{a}{2} L^2 \tag{790}$$

we obtain

$$\psi(x,t) = \frac{C}{2\pi} e^{i(\alpha_0 + k_0 x - \omega_0 t)} \sqrt{\frac{\pi}{aL^2}} \sqrt{a-ic}\, e^{-\frac{ab^2}{4(a^2+c^2)}} e^{-\frac{(-ic)b^2}{4(a^2+c^2)}} \tag{791}$$

$$= \frac{C}{2\pi} e^{i(\alpha_0 + k_0 x - \omega_0 t)} e^{\frac{2icb^2}{4aL^2}} \sqrt{\frac{\pi}{aL^2}} \sqrt{a-ic}\, e^{-\frac{(x-x_0-V_g t)^2}{2L^2}} \tag{792}$$

and therefore (with $|\sqrt{a-ic}| = \sqrt{|a-ic|} = \sqrt{\sqrt{aL^2}} = a^{1/4}\sqrt{L}$)

$$|\psi(x,t)| = \frac{C}{2\pi} \sqrt{\frac{\pi}{aL^2}} a^{1/4} \sqrt{L}\, e^{-\frac{(x-x_0-V_g t)^2}{2L^2}} \tag{793}$$

$$= \frac{C}{2\pi} \sqrt{\frac{\pi}{\sqrt{a}L}}\, e^{-\frac{(x-x_0-V_g t)^2}{2L^2}} \tag{794}$$

$$= \frac{C}{2} \sqrt{\frac{1}{\pi\sqrt{a}} \frac{1}{\sqrt{L}}}\, e^{-\frac{(x-x_0-V_g t)^2}{2L^2}}. \tag{795}$$

(d) At $t=0$ the packets width in position space is $L = 1/\Delta k$ while the width in momentum space is Δk which means the product is $\Delta x \cdot \Delta k = 1$.

(e) With the group velocity

$$V_g = \frac{1}{2}\sqrt{\frac{g}{k_0}} \tag{796}$$

$$V_g' = \frac{\partial V_g}{\partial k}|_{k=k_0} = -\frac{1}{4}\sqrt{\frac{g}{k_0^3}} \tag{797}$$

the width of the package is proportional to

$$L = \frac{1}{\Delta k}\sqrt{1 + \left(V_g'(\Delta k)^2 t\right)^2} \tag{798}$$

$$\rightarrow T_D = \frac{\sqrt{3}}{V_g'(\Delta k)^2} \tag{799}$$

$$\rightarrow T_D = \frac{4}{\Delta k^2}\sqrt{\frac{3k_0^3}{g}}. \tag{800}$$

The condition for the spread limitation is

$$S_{\text{HI-CA}} \leq V_g \cdot T_D \tag{801}$$

$$= \frac{1}{2}\sqrt{\frac{g}{k_0}} \frac{4}{\Delta k^2} \sqrt{\frac{3k_0^3}{g}} \tag{802}$$

$$= 2\sqrt{3}\frac{k_0}{\Delta k^2} \tag{803}$$

Exercise 7.3 Derivation and Example: Amplitude Propagation for Dispersionless Waves Expressed as Constancy of Something along a Ray

(a)

(b)

(c)

(d)

Exercise 7.4 Example: Energy Density and Flux, and Adiabatic Invariant, or a Dispersionless Wave

(a) For a generic Lagrangian density $\mathcal{L}$ we find

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial x_i}\right)}\delta\left(\frac{\partial\psi}{\partial x_i}\right) \tag{804}$$

$$= \frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial x_i}\right)}\frac{\partial}{\partial x_i}\left(\delta\psi\right) \tag{805}$$

$$\to \delta\int\mathcal{L}d^4x = \int\delta\mathcal{L}d^4x \tag{806}$$

$$= \int\left[\frac{\partial\mathcal{L}}{\partial\psi} - \frac{\partial}{\partial x_i}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial x_i}\right)}\right)\right]\delta\psi \tag{807}$$

$$\to 0 = \frac{\partial\mathcal{L}}{\partial\psi} - \frac{\partial}{\partial x_i}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial x_i}\right)}\right) \tag{808}$$

the general Euler-Lagrange equation. For the given density we can calculate the derivatives

$$\mathcal{L} = W\left[\frac{1}{2}\left(\frac{\partial\psi}{\partial t}\right)^2 - \frac{1}{2}C^2\left(\nabla\psi\right)^2\right] \tag{809}$$

$$\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial t}\right)} = W\frac{\partial\psi}{\partial t} \tag{810}$$

$$\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial x_i}\right)} = -WC^2\frac{\partial\psi}{\partial x_i} \tag{811}$$

and obtain

$$\frac{\partial}{\partial t}\left(W\frac{\partial\psi}{\partial t}\right) - \frac{\partial}{\partial x_i}\left(WC^2\frac{\partial\psi}{\partial x_i}\right) = 0. \tag{812}$$

(b) Using the definitions we obtain

$$\frac{\partial U}{\partial t} = \frac{\partial^2\psi}{\partial t^2}\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial t)} + \frac{\partial\psi}{\partial t}\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial t)}\right) - \frac{\partial\mathcal{L}}{\partial t} \tag{813}$$

$$\frac{\partial F_j}{\partial x_j} = \frac{\partial^2\psi}{\partial t\partial x_j}\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_j)} + \frac{\partial\psi}{\partial t}\frac{\partial}{\partial x_j}\left(\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_j)}\right) \tag{814}$$

and therefore

$$\frac{\partial U}{\partial t}+\frac{\partial F_j}{\partial x_j}=\frac{\partial^2\psi}{\partial t^2}\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial t)}+\frac{\partial\psi}{\partial t}\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial t)}\right)-\frac{\partial\mathcal{L}}{\partial t}+\frac{\partial^2\psi}{\partial t\partial x_j}\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_j)}+\frac{\partial\psi}{\partial t}\frac{\partial}{\partial x_j}\left(\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_j)}\right) \tag{815}$$

$$=\frac{\partial^2\psi}{\partial t^2}\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial t)}+\frac{\partial\psi}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\psi}\right)-\frac{\partial\mathcal{L}}{\partial t}+\frac{\partial^2\psi}{\partial t\partial x_j}\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_j)} \tag{816}$$

$$=\frac{\partial\psi}{\partial t}\left(-\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial t)}\right)+\frac{\partial\psi}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\psi}\right)-\frac{\partial\mathcal{L}}{\partial t}+\frac{\partial\psi}{\partial t}\left(-\frac{\partial}{\partial x_i}\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_j)}\right) \tag{817}$$

$$=\frac{\partial\psi}{\partial t}\left(-\frac{\partial\mathcal{L}}{\partial\psi}\right)+\frac{\partial\psi}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\psi}\right)-\frac{\partial\mathcal{L}}{\partial t} \tag{818}$$

$$=-\frac{\partial\mathcal{L}}{\partial t} \tag{819}$$

(c) Substituting $\mathcal{L}$ into the definitions yields

$$U=\frac{\partial\psi}{\partial t}\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial t)}-\mathcal{L} \tag{820}$$

$$=W\left(\frac{\partial\psi}{\partial t}\right)^2-\mathcal{L} \tag{821}$$

$$=W\left[\frac{1}{2}\left(\frac{\partial\psi}{\partial t}\right)^2+\frac{1}{2}C^2\left(\nabla\psi\right)^2\right] \tag{822}$$

$$F_j=\frac{\partial\psi}{\partial t}\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_j)} \tag{823}$$

$$=-\frac{\partial\psi}{\partial t}WC^2\frac{\partial\psi}{\partial x_j}. \tag{824}$$

(d) The momentum density is given by

$$\pi=\frac{\partial\mathcal{L}}{\partial\frac{\partial\phi}{\partial t}}=W\frac{\partial\phi}{\partial t} \tag{825}$$

$$\Pi=\int\pi d^3x=\int W\frac{\partial\phi}{\partial t}d^3x \tag{826}$$

$$J=\int_0^{\omega/2\pi}Ldt=\int_0^{\omega/2\pi}\int\mathcal{L}d^3xdt \tag{827}$$

$$= \tag{828}$$

Exercise 8.1 Practice: Convolutions and Fourier Transforms

(a) With $f_1(x) = e^{-\frac{x^2}{2\sigma^2}}$ and $f_2(x) = e^{-\frac{x}{h}}\theta(x)$ we obtain

$$\begin{aligned}
F_1(k) &= \int_{-\infty}^{\infty} f_1(x) e^{-ikx} dx && (829)\\
&= \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} e^{-ikx} dx && (830)\\
&= e^{-\frac{k^2\sigma^2}{2}} \int_{-\infty}^{\infty} e^{-\left(\frac{x}{\sqrt{2}\sigma}+\frac{ik\sigma}{\sqrt{2}}\right)^2} dx && (831)\\
&= e^{-\frac{k^2\sigma^2}{2}} \sqrt{2\sigma^2} \int_{-\infty}^{\infty} e^{-y^2} dy && (832)\\
&= \sqrt{2\pi\sigma^2} e^{-\frac{\sigma^2 k^2}{2}} && (833)\\
F_2(k) &= \int_{-\infty}^{\infty} f_2(x) e^{-ikx} dx && (834)\\
&= \int_{0}^{\infty} e^{-\frac{x}{h}} e^{-ikx} dx && (835)\\
&= -\frac{1}{h} e^{-\frac{x}{h}} e^{-ikx}\Big|_0^{\infty} - \int_0^{\infty} \left(-\frac{1}{h}\right) e^{-\frac{x}{h}} \frac{1}{(-ik)} e^{-ikx} dx && (836)\\
&= \frac{1}{h} - \frac{1}{ihk} \int_0^{\infty} e^{-\frac{x}{h}} e^{-ikx} dx && (837)\\
&= \ldots && (838)\\
&= \frac{1}{\frac{1}{h} + ik} && (839)
\end{aligned}$$

(b)

(c)

$$\begin{aligned}
f_1 \otimes f_2 &= \int_{-\infty}^{\infty} f_2(y-x) f_1(x) dx && (840)\\
&= \int_{-\infty}^{\infty} e^{-\frac{y-x}{h}} \theta(y-x) e^{-\frac{x^2}{2\sigma^2}} dx && (841)\\
&= \int_{-\infty}^{y} e^{-\frac{y-x}{h}} e^{-\frac{x^2}{2\sigma^2}} dx && (842)\\
&= \ldots && (843)
\end{aligned}$$

Not done yet

Exercise 13.1 Example: Earth's Atmosphere

(a) With $PV = Nk_BT$, $\rho = \frac{\mu m_p N}{V}$ and assuming $g = \text{const}$ we obtain

$$\begin{aligned}
\nabla P &= \rho \mathbf{g} && (844)\\
\frac{dP}{dz} &= -\rho g && (845)\\
&= -\frac{\mu m_p N}{V} g && (846)\\
&= -\mu m_p g \frac{P}{k_B T} && (847)\\
& && (848)
\end{aligned}$$

which can be solved by

$$\frac{dP}{P} = -\frac{\mu m_p g}{k_B T} \tag{849}$$

$$P(z) = P_0 \exp\left(-\frac{\mu m_p g}{k_B T} z\right). \tag{850}$$

With $\mu = 0.2 \cdot 2 \cdot 16 + 0.8 \cdot 2 \cdot 14 +$ ($20\%O_2/80\%N_2$) and $T = 220\text{K}$ we have

$$H = 6,400\text{m} \tag{851}$$

$$P(16\text{km}) = 0.083\text{bar} \tag{852}$$

$$\frac{P(35\text{km})}{P(16\text{km})} = 0.052 \tag{853}$$

(b) The isentropic condition $P \sim \rho^\gamma$ acts as an additional condition on top of the equations of state. It can be rewritten as

$$P\rho^{-\gamma} = \text{const} \tag{854}$$

$$PV^\gamma = \text{const} \tag{855}$$

$$P\left(\frac{T}{P}\right)^\gamma = \text{const} \tag{856}$$

$$TP^{\frac{1-\gamma}{\gamma}} = \text{const}. \tag{857}$$

Differentiating the last equation gives

$$\frac{dT}{dz} P^{\frac{1-\gamma}{\gamma}} + \left(\frac{1-\gamma}{\gamma}\right) P^{\frac{1-2\gamma}{\gamma}} \frac{dP}{dz} T = 0 \tag{858}$$

$$\rightarrow \frac{dT}{dz} = -\left(\frac{1-\gamma}{\gamma}\right) \frac{T}{P} \frac{dP}{dz} \tag{859}$$

Inserting the

$$\frac{dP}{dz} = -\mu m_p g \frac{P}{k_B T} \tag{860}$$

which we calculated in (a) we obtain

$$\frac{dT}{dz} = \left(\frac{1-\gamma}{\gamma}\right) \frac{\mu m_p g}{k_B}. \tag{861}$$

With this we calculate a lapse rate of 9.76K km^{-1}.

Exercise 13.2 Practise: Weight in Vacuum

$$F_b = \rho_{\text{air}} g V_{\text{body}} \tag{862}$$

$$= \rho_{\text{air}} g \frac{m_{\text{body}}}{\rho_{\text{body}}} \tag{863}$$

$$= 1\text{N} \tag{864}$$

where we used a mass of 100kg and $\rho_{\text{air}}/\rho_{\text{body}} = 0.001$.

Exercise 13.4 Example: Polytropes — The Power of Dimensionless Variables

(a) From

$$\frac{dP}{dr} = -\rho\frac{Gm}{r^2} \quad \rightarrow \quad m = -\frac{r^2}{G\rho}\frac{dP}{dr} \tag{865}$$

$$\frac{dm}{dr} = 4\pi\rho r^2 \tag{866}$$

we obtain by differentiation

$$\begin{aligned}
\frac{d^2P}{dr^2} &= -G\frac{(\frac{d\rho}{dr}m + \rho\frac{dm}{dr})r^2 - 2r\rho m}{r^4} &(867)\\
&= -\frac{G}{r^4}\left(\left[\frac{d\rho}{dr}m + \rho\frac{dm}{dr}\right]r^2 - 2r\rho m\right) &(868)\\
&= -\frac{G}{r^4}\left(\left[-\frac{r^2}{G\rho}\frac{dP}{dr}\frac{d\rho}{dr} + \rho 4\pi\rho r^2\right]r^2 + \frac{r^2}{G\rho}\frac{dP}{dr}2r\rho\right) &(869)\\
&= \left(\frac{1}{\rho}\frac{d\rho}{dr} - \frac{2}{r}\right)\frac{dP}{dr} - 4\pi G\rho^2 &(870)
\end{aligned}$$

(b) With the polytropic equation of state $P = K\rho^{1+1/n}$ we find for the derivatives of P

$$\frac{dP}{dr} = K\left(1+\frac{1}{n}\right)\rho^{1/n}\frac{d\rho}{dr} \tag{871}$$

$$\frac{d^2P}{dr^2} = K\left(1+\frac{1}{n}\right)\rho^{1/n}\left[\frac{1}{n}\rho^{-1}\left(\frac{d\rho}{dr}\right)^2 + \frac{d^2\rho}{dr^2}\right] \tag{872}$$

and therefore

$$\begin{aligned}
\frac{1}{n}\rho^{-1}\left(\frac{d\rho}{dr}\right)^2 + \frac{d^2\rho}{dr^2} &= \left(\frac{1}{\rho}\frac{d\rho}{dr} - \frac{2}{r}\right)\frac{d\rho}{dr} - \frac{n}{1+n}\frac{4\pi G\rho^{2-1/n}}{K} &(873)\\
\frac{d^2\rho}{dr^2} &= \left(\frac{n-1}{n}\frac{1}{\rho}\frac{d\rho}{dr} - \frac{2}{r}\right)\frac{d\rho}{dr} - \frac{n}{1+n}\frac{4\pi G}{K}\rho^{2-1/n} &(874)\\
\frac{d^2\rho}{dr^2} &= \frac{n-1}{n}\frac{1}{\rho}\left(\frac{d\rho}{dr}\right)^2 - \frac{2}{r}\frac{d\rho}{dr} - \frac{n}{1+n}\frac{4\pi G}{K}\rho^{2-1/n}. &(875)
\end{aligned}$$

(c) With

$$\begin{aligned}
\rho(r) &= \rho_c\theta^\alpha(r) &(876)\\
\frac{d\rho}{dr} &= \rho_c\alpha\theta^{\alpha-1}\frac{d\theta}{dr} &(877)\\
\left(\frac{d\rho}{dr}\right)^2 &= \rho_c^2\alpha^2\theta^{2(\alpha-1)}\left(\frac{d\theta}{dr}\right)^2 &(878)\\
\frac{d^2\rho}{dr^2} &= \rho_c\alpha(\alpha-1)\theta^{\alpha-2}\left(\frac{d\theta}{dr}\right)^2 + \rho_c\alpha\theta^{\alpha-1}\frac{d^2\theta}{dr^2} &(879)
\end{aligned}$$

we can rewrite the differential equation as

$$\rho_c\alpha(\alpha-1)\theta^{\alpha-2}\left(\frac{d\theta}{dr}\right)^2 + \rho_c\alpha\theta^{\alpha-1}\frac{d^2\theta}{dr^2} \tag{880}$$

$$= \frac{n-1}{n}\frac{1}{\rho_c\theta^\alpha}\rho_c^2\alpha^2\theta^{2(\alpha-1)}\left(\frac{d\theta}{dr}\right)^2 - \frac{2}{r}\rho_c\alpha\theta^{\alpha-1}\frac{d\theta}{dr} - \frac{n}{1+n}\frac{4\pi G}{K}\rho_c^{2-1/n}\theta^{\alpha(2-1/n)} \tag{881}$$

and see that for $n = \alpha$ the $(d\theta/dr)^2$ terms and the left and right side cancel out.

(d) With $n = \alpha$ the simplified equation is given by

$$\frac{d^2\theta}{dr^2} + \frac{2}{r}\frac{d\theta}{dr} + \frac{4\pi G\rho_c^{1-1/n}}{(n+1)K}\theta^n = 0 \tag{882}$$

$$\frac{d^2\theta}{dr^2} + \frac{2}{r}\frac{d\theta}{dr} + \frac{4\pi G}{(n+1)K\rho_c^{1/n-1}}\theta^n = 0 \tag{883}$$

(e) With

$$r = a\xi \tag{884}$$

$$\frac{d\theta}{dr} = \frac{d\theta}{d\xi}\frac{d\xi}{dr} = \frac{1}{a}\frac{d\theta}{d\xi} \tag{885}$$

$$\frac{d^2\theta}{dr^2} = \frac{1}{a^2}\frac{d^2\theta}{d\xi^2} \tag{886}$$

we obtain

$$\frac{d^2\theta}{d\xi^2} + \frac{2}{\xi}\frac{d\theta}{d\xi} + a^2\frac{4\pi G}{(n+1)K\rho_c^{1/n-1}}\theta^n = 0 \tag{887}$$

which for $a^{-2} = \frac{4\pi G}{(n+1)K\rho_c^{1/n-1}}$ gives the Lane-Emden equation in standard form

$$\frac{d^2\theta}{d\xi^2} + \frac{2}{\xi}\frac{d\theta}{d\xi} + \theta^n = 0 \tag{888}$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n. \tag{889}$$

(f)
- $\theta(\xi = 0) = 1$

$$\rightarrow \quad \rho(r = 0) = \rho_c \tag{890}$$

- $\theta'(\xi = 0) = 0$

$$\rightarrow \quad \frac{dP}{dr} = K\left(\frac{n+1}{n}\right)\rho^{1/n}\frac{d\rho}{dr} \tag{891}$$

$$= K\left(\frac{n+1}{n}\right)(\rho_c^{1/n}\theta)\frac{d(\rho_c\theta^n)}{d\xi}\frac{d\xi}{dr} \tag{892}$$

$$= K\left(\frac{n+1}{n}\right)\rho_c^{1+1/n}\theta n\theta^{n-1}\frac{d\theta}{d\xi}\frac{d\xi}{dr} \tag{893}$$

$$= K(n+1)\rho_c^{1+1/n}\theta^n\frac{d\theta}{d\xi}\frac{1}{a} \tag{894}$$

$$\rightarrow \left.\frac{dP}{dr}\right|_{r=0} = 0 \tag{895}$$

(g) The mass integral can be rewritten by using the Lane-Emden equation

$$M = 4\pi\int_0^R \rho(r)r^2dr \tag{896}$$

$$= 4\pi\rho_c a^3\int_0^{\xi_1}\theta^n\xi^2d\xi \tag{897}$$

$$= -4\pi\rho_c a^3\int_0^{\xi_1}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)d\xi \tag{898}$$

$$= -4\pi\rho_c a^3\left[\xi^2\frac{d\theta}{d\xi}\right]_0^{\xi_1} \tag{899}$$

$$= -4\pi\rho_c a^3\xi_1^2\theta'(\xi_1). \tag{900}$$

For the radius R we find

$$\begin{aligned}
R &= a\xi_1 && (901)\\
&= \left[\frac{(n+1)K\rho_c^{1/n-1}}{4\pi G}\right]^{\frac{1}{2}}\xi_1 && (902)\\
&= \left[\frac{(n+1)K}{4\pi G}\right]^{\frac{1}{2}}\rho_c^{(1-n)/2n}\xi_1 && (903)\\
& && (904)
\end{aligned}$$

$$\xi_1 = \left[\frac{(n+1)K}{4\pi G}\right]^{-\frac{1}{2}}\rho_c^{(n-1)/2n}R. \quad (905)$$

Furthermore we can write (somewhat arbitrarily)

$$\begin{aligned}
\xi_1^2 &= \xi_1^2 1^{\frac{3-n}{1-n}} = \xi_1^2\left(\frac{R}{a\xi_1}\right)^{\frac{3-n}{1-n}} && (906)\\
&= \xi_1^{2-\frac{3-n}{1-n}}\left(\frac{1}{a}\right)^{\frac{3-n}{1-n}}R^{\frac{3-n}{1-n}} && (907)\\
&= \xi_1^{\frac{n+1}{n-1}}a^{\frac{3-n}{n-1}}R^{\frac{3-n}{1-n}} && (908)
\end{aligned}$$

which results in

$$\begin{aligned}
M &= 4\pi\rho_c a^3\cdot\xi_1^{\frac{n+1}{n-1}}a^{\frac{3-n}{n-1}}R^{\frac{3-n}{1-n}}\theta'(\xi_1) && (909)\\
&= 4\pi\rho_c a^{2n/(n-1)}\cdot\xi_1^{\frac{n+1}{n-1}}R^{\frac{3-n}{1-n}}\theta'(\xi_1) && (910)\\
&= 4\pi\rho_c\left[\frac{(n+1)K\rho_c^{1/n-1}}{4\pi G}\right]^{n/(n-1)}\cdot\xi_1^{\frac{n+1}{n-1}}R^{\frac{3-n}{1-n}}\theta'(\xi_1) && (911)\\
&= 4\pi R^{\frac{3-n}{1-n}}\left[\frac{(n+1)K}{4\pi G}\right]^{n/(n-1)}\cdot\xi_1^{\frac{n+1}{n-1}}\theta'(\xi_1) && (912)
\end{aligned}$$

which is an expression without ρ_c.

(h) For $n=1$ we have

$$R = \left[\frac{K}{2\pi G}\right]^{1/2}\xi_1 \quad (913)$$

which means R is independent of mass and central pressure and therefore constant for all objects. So we conclude $R_S = R_J$.

For $n=1$ have have $\theta(\xi) = \sin\xi/\xi$ and find

$$\begin{aligned}
\theta' &= \frac{\xi\cos\xi - \sin\xi}{\xi^2} && (914)\\
\xi_1 &= \pi && (915)\\
\theta'(\xi_1) &= -1/\pi. && (916)
\end{aligned}$$

Therefore

$$R = \pi \left[\frac{K}{2\pi G}\right]^{1/2} \tag{917}$$

$$M = 4\pi^2 \left[\frac{K}{2\pi G}\right]^{3/2} \rho_c = 4\pi^2 \left[\frac{R}{\pi}\right]^3 \rho_c \tag{918}$$

$$= \frac{4R^3}{\pi}\rho_c \tag{919}$$

$$\to \rho_c = \frac{\pi M}{4R^3} = \frac{\pi}{3}\frac{\pi M}{4\frac{\pi}{3}R^3} = \frac{\pi^2}{3}\rho_{\text{avg}} \tag{920}$$

which gives $\rho_{c,J} = 4.6 \cdot 10^{12}\text{kg/m}^3$ and $\rho_{c,S} = 1.3 \cdot 10^{12}\text{kg/m}^3$.

Exercise 13.5 Example: Shape of a Constant-Density, Spinning Planet

(a) The gravitational potential is given by the integral of the mass distribution $\rho(\vec{r})$

$$\Phi(\vec{r}) = -G \int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3\vec{r}' \tag{921}$$

$$= -2\pi G \int \frac{\rho(\vec{r}')}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}} r'^2 \sin\theta d\theta dr' \tag{922}$$

$$= -\frac{2\pi G\rho}{r} \int_0^R \frac{1}{\sqrt{1 + (r'/r)^2 - 2r'/r\cos\theta}} d(\cos\theta) r'^2 dr' \tag{923}$$

$$= -\frac{2\pi G\rho}{r} \int_0^R r' \left(r + r' - \sqrt{(r - r')^2}\right) dr'. \tag{924}$$

For $\vec{r}$ inside the mass distribution the integral needs to be split

$$\Phi(\vec{r}) = -\frac{2\pi G\rho}{r} \left[\int_0^r r'\,(2r')\,dr' + \int_r^R r'\,(2r)\,dr'\right] \tag{925}$$

$$= -\frac{4\pi G\rho}{r} \left[\int_0^r r'^2 dr' + r \int_r^R r' dr'\right] \tag{926}$$

$$= -\frac{4\pi G\rho}{r} \left[\frac{r^3}{3} + r\frac{1}{2}(R^2 - r^2)\right] \tag{927}$$

$$= -4\pi G\rho \left[\frac{r^2}{3} + \frac{1}{2}(R^2 - r^2)\right] \tag{928}$$

$$= -4\pi G\rho \left[-\frac{r^2}{6} + \frac{R^2}{2}\right] \tag{929}$$

$$= \frac{2\pi G\rho}{3}\left[r^2 - 3R^2\right] \tag{930}$$

(b) For the centrifugal force and potential we find

$$F_{\text{cen}} = m\frac{v^2}{\varpi} = m\Omega^2\varpi \tag{931}$$

$$\Phi_{\text{cen}} = -\frac{1}{2}(\Omega\varpi)^2 \tag{932}$$

$$= -\frac{1}{2}(\Omega r\cos\theta)^2 \tag{933}$$

$$= -\frac{1}{2}(\Omega r\sin\theta')^2 \tag{934}$$

$$= -\frac{1}{2}(\vec{\Omega}\times\vec{r})^2 \tag{935}$$

we results in

$$\Phi = \frac{2\pi G\rho r^2}{3} - 2\pi G\rho R^2 - \frac{1}{2}(\Omega r\cos\theta)^2 \tag{936}$$

$$= \frac{2\pi G\rho r^2}{3} - 2\pi G\rho R^2 - \frac{1}{2}\Omega^2 r^2\cos^2\theta \tag{937}$$

$$= \frac{2\pi G\rho r^2}{3} - 2\pi G\rho R^2 - \frac{\Omega^2}{3}r^2\frac{1}{2}3\cos^2\theta \tag{938}$$

$$= \frac{2\pi G\rho r^2}{3} - 2\pi G\rho R^2 - \frac{\Omega^2 r^2}{3} - \frac{\Omega^2}{3}r^2\frac{1}{2}(3\cos^2\theta - 1) \tag{939}$$

$$= \frac{2\pi G\rho r^2}{3} - 2\pi G\rho R^2 - \frac{\Omega^2 r^2}{3} - \frac{\Omega^2}{3}r^2 P_2(\cos\theta) \tag{940}$$

$$\tag{941}$$

(c)

(d)

(e)

Exercise 13.7 Problem: A Hole in My Bucket

Applying the Bernoulli equation

$$\frac{1}{2}\rho v^2 + \rho g h = \text{const} \tag{942}$$

to the hole and the water surface we get

$$\frac{1}{2}\rho v_{\text{hole}}^2 = \rho g h + \frac{1}{2}\rho v_{\text{surf}}^2. \tag{943}$$

The change in volume is given by (neglecting limitations from the Hagen-Poiseulle equation but having a hole significantly smaller than the bucket surface - with hole has the same size the assumption of a static pressure does not make sense)

$$\frac{dV}{dt} = A_{\text{bucket}}v_{\text{surf}} = A_{\text{hole}}v_{\text{hole}} = A_{\text{bucket}}\frac{dh}{dt} \tag{944}$$

$$\rightarrow \quad v_{\text{hole}} = \frac{A_{\text{bucket}}}{A_{\text{hole}}}\frac{dh}{dt} \tag{945}$$

$$\rightarrow \quad v_{\text{surf}} = \frac{dh}{dt}. \tag{946}$$

With this the Bernoulli equation turns into

$$\frac{1}{2}\left(\frac{A_{\text{bucket}}}{A_{\text{hole}}}\right)^2\left(\frac{dh}{dt}\right)^2 = gh + \frac{1}{2}\left(\frac{dh}{dt}\right)^2 \tag{947}$$

$$\left(\frac{dh}{dt}\right)^2 = \frac{2g}{\left(\frac{A_{\text{bucket}}}{A_{\text{hole}}}\right)^2 - 1}h \tag{948}$$

$$= \frac{2A_{\text{hole}}^2 g}{A_{\text{bucket}}^2 - A_{\text{hole}}^2}h \tag{949}$$

$$\tag{950}$$

which can be solved by

$$\frac{dh}{\sqrt{h}} = -\sqrt{\frac{2A_{\text{hole}}^2 g}{A_{\text{bucket}}^2 - A_{\text{hole}}^2}}dt \tag{951}$$

$$2\sqrt{h} = -\sqrt{\frac{2A_{\text{hole}}^2 g}{A_{\text{bucket}}^2 - A_{\text{hole}}^2}} \cdot t + 2\sqrt{H_0} \tag{952}$$

$$h(t) = \left(\sqrt{H_0} - \sqrt{\frac{A_{\text{hole}}^2 g}{2(A_{\text{bucket}}^2 - A_{\text{hole}}^2)}} \cdot t\right)^2 \tag{953}$$

$$\approx \left(\sqrt{H_0} - \frac{A_{\text{hole}}}{A_{\text{surface}}}\sqrt{\frac{g}{2}} \cdot t\right)^2 \tag{954}$$

For the time T to empty the bucket we solve for $h(T) = 0$ and obtain

$$T = \sqrt{\frac{2H_0}{g}}\frac{A_{\text{bucket}}}{A_{\text{hole}}}. \tag{955}$$

in the case of small holes or buckets with thick walls D the Hagen-Poiseulle should be taken into account.

Exercise 14.1 Practice: Constant-Angular-Momentum Flow - Relative Motion of Fluid Elements

Taylor expansion of the components of the velocity field gives

$$v_j(\mathbf{x} + \boldsymbol{\xi}) = v_j(\mathbf{x}) + \left(\left.\frac{\partial v_j(\mathbf{y})}{\partial y_i}\right|_{y=x} \xi_i\right). \tag{956}$$

For the vector we can then write

$$\mathbf{v}(\mathbf{x} + \boldsymbol{\xi}) = \mathbf{v}(\mathbf{x}) + \left(\left.\frac{\partial v_j(\mathbf{y})}{\partial y_i}\right|_{y=x} \xi_i\right)\mathbf{e}_j \tag{957}$$

$$\nabla_{\boldsymbol{\xi}}\mathbf{v} \equiv \mathbf{v}(\mathbf{x} + \boldsymbol{\xi}) - \mathbf{v}(\mathbf{x}) \tag{958}$$

$$= \boldsymbol{\xi} \cdot \nabla\mathbf{v} \tag{959}$$

For the constant-angular-momentum flow we have

$$\mathbf{v} = \frac{1}{\varpi^2}\mathbf{j} \times \mathbf{x} \tag{960}$$

$$= \frac{1}{\varpi^2}j\varpi\mathbf{e}_\phi = \frac{j}{\varpi}\mathbf{e}_\phi. \tag{961}$$

The only non-vanishing component of $\nabla \mathbf{v}$ is

$$\frac{\partial v_\phi}{\partial \varpi} = -\frac{j}{\varpi^2}. \tag{962}$$

- tangential: $\boldsymbol{\xi} = \varpi\, d\phi \mathbf{e}_\phi = d\epsilon\, \mathbf{e}_\phi$ Not done yet
- radial: $\boldsymbol{\xi} = d\varpi\, \mathbf{e}_\varpi = d\epsilon\, \mathbf{e}_\varpi$ Not done yet

Exercise 14.2 Practice: Vorticity and Incompressibility

Vorticity: $\boldsymbol{\omega} = \nabla \times \mathbf{v}$. Compressibility: $\rho = \mathrm{const}$

$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0 \quad \rightarrow \quad \nabla \cdot \mathbf{v} = 0 \tag{963}$$

(a)

$$\nabla \times \mathbf{v} = \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \mathbf{e}_z = 0 \cdot \mathbf{e}_z \tag{964}$$

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 2y \tag{965}$$

(b)

$$\nabla \times \mathbf{v} = -2y \cdot \mathbf{e}_z \tag{966}$$

$$\nabla \cdot \mathbf{v} = 0 \tag{967}$$

(c)

$$\nabla \times \mathbf{v} = \frac{1}{\varpi} \left(\frac{\partial (\varpi v_\phi)}{\partial \varpi} - \frac{\partial v_\varpi}{\partial \phi} \right) \mathbf{e}_z = 2 \cdot \mathbf{e}_z \tag{968}$$

$$\nabla \cdot \mathbf{v} = \frac{1}{\varpi} \left(\frac{\partial (\varpi v_\varpi)}{\partial \varpi} + \frac{\partial v_\phi}{\partial \phi} \right) = 0 \tag{969}$$

(d)

$$\nabla \times \mathbf{v} = 0 \cdot \mathbf{e}_z \tag{970}$$

$$\nabla \cdot \mathbf{v} = 0 \tag{971}$$

Exercise 16.9 Example: Breaking of a Dam

The PDEs

$$h_t + h v_x + v h_x = 0 \tag{972}$$

$$v_t + v v_x + g h_x = 0 \tag{973}$$

can be written as

$$A u_t + B u_x = 0 \tag{974}$$

$$u = \begin{pmatrix} h \\ v \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} v & h \\ g & v \end{pmatrix} \tag{975}$$

Now

$$vh_t + vhv_x + v^2h_x = 0 \tag{976}$$
$$hv_t + hvv_x + ghh_x = 0 \tag{977}$$
$$\rightarrow (hv)_t = hv_t + vh_t \tag{978}$$
$$= -2vhv_x - v^2h_x - ghh_x \tag{979}$$
$$= -\partial_x \left(h \left[v^2 + \frac{1}{2} gh \right] \right) \tag{980}$$

Not done yet

10.3 Walter - Astronautics

Problem 1.1 - Balloon Propulsion

For the mass flow rate we have

$$\dot{m} = \rho \dot{V} \approx \rho A_t v_t \overset{!}{=} \frac{\rho V}{T} \quad \rightarrow \quad v_t = \frac{V}{A_t T} = 20\mathrm{m/s} \tag{981}$$

and the speed of sound in a diatomic gas ($f = 5$, $\rho_0 = 1.225\mathrm{kg/m}^3$, $P_0 = 101.3 \cdot 10^3 Pa$) is

$$c = \sqrt{\kappa \frac{p}{\rho}} = \sqrt{\frac{f+2}{f} \frac{P}{\rho}} = 340\mathrm{m/s} \tag{982}$$

which justifies $v_t \ll c$. Newtons second law gives for the momentum thrust

$$F_e = \frac{dp}{dt} = \dot{m} v_t = \frac{\rho V}{T} \frac{V}{A_t T} = \frac{\rho}{A_t} \left(\frac{V}{T} \right)^2 = 0.0258\mathrm{N} \tag{983}$$

From the Bernoulli equation we can obtain the pressure difference

$$P = P_0 + \frac{\rho}{2} v_t^2 \quad \rightarrow \quad P - P_0 = \frac{\rho}{2} v_t^2 \tag{984}$$

and can then calculate the pressure thrust

$$F_p = A_t (P - P_0) = \frac{A_t \rho}{2} v_t^2 = \frac{\rho V^2}{2 A_t T^2} = 0.0129\mathrm{N} \tag{985}$$

and see $F_e = 2F_p$.

Problem 1.2 - Nozzle Exit Area of an SSME

For the total thrust we have in vacuum and at sea level we have

$$F_{\mathrm{SL}} = A_t (P - P_0) + \dot{m} v_t \tag{986}$$
$$F_{\mathrm{V}} = A_t (P - 0) + \dot{m} v_t \tag{987}$$

which implies with $P_0 = 101.3\mathrm{Pa}$

$$A_t = \frac{F_{\mathrm{V}} - F_{\mathrm{SL}}}{P_0} = 4.55\mathrm{m}^2 \tag{988}$$

Problem 1.3 - Proof of $\eta_{\mathbf{VDF}} \leq 1$

$$\langle \nu_e \rangle_\mu = \frac{\int_0^{\pi/2} \nu_e(\theta) \cdot \mu(\theta) \sin\theta \, d\theta}{\int_0^{\pi/2} \mu(\theta) \sin\theta \, d\theta} \tag{989}$$

$$\langle \nu_e \rangle_\mu^2 \leq \langle \nu_e^2 \rangle_\mu \tag{990}$$

Not done yet

Problem 4.1 - Gas Velocity-Pressure Relation in a Nozzle

Using the ideal gas equation $pV = NkT$ we have for a adiabatic process

$$pV^\kappa = p\left(\frac{NkT}{p}\right)^\kappa \tag{991}$$

$$= p^{1-\kappa}T^\kappa \tag{992}$$

$$= \text{const} \tag{993}$$

$$\rightarrow \quad p^{\frac{1-\kappa}{\kappa}}T = p_0^{\frac{1-\kappa}{\kappa}}T_0 \tag{994}$$

and with $pV = nRT$

$$\rho = \frac{m}{V} = \frac{nM_p}{V} = \frac{M_p p}{RT} \quad \rightarrow \quad p = \frac{R}{M_p}\rho T \tag{995}$$

$$(\rho T)^{\frac{1-\kappa}{\kappa}}T = \text{const} \tag{996}$$

$$\rho^{1-\kappa}T = \text{const} \tag{997}$$

we obtain with $\kappa = \frac{2+n}{n}$ for the energy conversion efficiency

$$\eta = 1 - \frac{T}{T_0} = 1 - \left(\frac{p}{p_0}\right)^{\frac{\kappa-1}{\kappa}} = 1 - \left(\frac{\rho}{\rho_0}\right)^{\kappa-1} \tag{998}$$

$$= 1 - \left(\frac{p}{p_0}\right)^{\frac{2}{n+2}} = 1 - \left(\frac{\rho}{\rho_0}\right)^{\frac{2}{n}} \tag{999}$$

$$\tag{1000}$$

11 Doodling

Fundamental ingredients for a quantum theory are a set of states $\{|\psi\rangle\}$ and operators $\{\mathcal{O}\}$. The time development is governed by a Hamilton operator

$$i\hbar\partial_t|\psi\rangle = H|\psi\rangle \tag{1001}$$

Lets assume that momentum eigenstates are simultaneously eigenstates of H then a simple relativistic theory looks like

$$H|\vec{p}\rangle = E_{\vec{p}}|\vec{p}\rangle \tag{1002}$$

$$E_{\vec{p}} = +\sqrt{\vec{p}^2c^2 + m^2c^4} \tag{1003}$$

The time evolution of the wave function is given by

$$\psi(\vec{p},t) = e^{-iE_{\vec{p}}t}\psi(\vec{p},0) \tag{1004}$$

$$\psi(\vec{x},t) = \int d^3\vec{p}\, e^{i\vec{p}\vec{x}}\psi(\vec{p},t) \tag{1005}$$

$$= \int d^3\vec{p}\, e^{-i(E_{\vec{p}}t-\vec{p}\vec{x})}\psi(\vec{p},0) \tag{1006}$$

$$= \frac{1}{(2\pi)^3}\int d^3\vec{p}\, e^{-i(E_{\vec{p}}t-\vec{p}\vec{x})}\int d^3\vec{y}e^{-i\vec{p}\vec{y}}\psi(\vec{y},0) \tag{1007}$$

$$= \int d^3\vec{y}\left[\frac{1}{(2\pi)^3}\int d^3\vec{p}\, e^{-i(E_{\vec{p}}t-\vec{p}(\vec{x}-\vec{y}))}\right]\psi(\vec{y},0) \tag{1008}$$

$$\psi(\vec{x},t) = \int d^3\vec{y}\, G(\vec{x}-\vec{y},t)\psi(\vec{y},0) \tag{1009}$$

Causality of the theory is guaranteed if the commutator of two operators/observables (associated with points x and y in space time) commute if the points are space-like separated

$$|x-y| < 0 \quad \rightarrow \quad [\mathcal{O}_i, \mathcal{O}_j] = 0. \tag{1010}$$

Localizing a particle in a small region L means

$$p \sim \frac{\hbar}{L} \tag{1011}$$

$$E = \sqrt{m^2c^4 + p^2c^2} = pc\sqrt{1 + \frac{m^2c^2}{p^2}} \tag{1012}$$

The L at which the momentum contribution becomes comparable to the rest energy of the particle

$$mc^2 = pc = \frac{\hbar c}{L} \quad \rightarrow \quad L_c = \frac{\hbar}{mc} \tag{1013}$$

is called Compton wavelength at which a relativistic theory is required and creation of particles and antiparticles appears.

This is therefore the method of choice to produce particles. A collision of two particles localizes a large amount of energy in a small region - creating particles

$$p\bar{p} \rightarrow X\bar{X} + ... \tag{1014}$$

Important general principles

- CPT invariance
- Spin-statistic theorem
- Interactions of particles with higher spin rather quite constrainted
  1. for lower spins $s = 0, 1/2$ the only restrictions are locality and Lorentz invariance
  2. the constrains are so restrictive that there are no relativistic quantum particle with $s > 2$

12 Companion for Banks book

1. Obtaining (1.2)

$$
\begin{aligned}
p &= (\omega_p, \vec{p}) \\
\langle \vec{p} | \vec{q} \rangle &= N_p^2 \cdot \delta^3(\vec{p} - \vec{q}) \\
\mathbb{I} &= C \int d^3\vec{p} |\vec{p}\rangle\langle\vec{p}| \quad \rightarrow |q\rangle = C \int d^3\vec{p}|p\rangle\langle p|q\rangle \\
|\vec{y}\rangle &= C \int d^3\vec{p}|\vec{p}\rangle\langle\vec{p}|\vec{y}\rangle = C \int d^3\vec{p}|\vec{p}\rangle e^{-i\vec{p}\cdot\vec{y}} \\
H|\vec{p}\rangle &= \omega_p|\vec{p}\rangle \\
A_{\mathrm{AE}} &= \int d^4x d^4y J_{\mathrm{A}}(x) J_{\mathrm{B}}(y) \cdot C^2 \int d^3\vec{p} \int d^3\vec{q} \langle\vec{p}| e^{-H(x^0-y^0)} |\vec{q}\rangle e^{i\vec{q}\cdot\vec{x}} e^{-i\vec{p}\cdot\vec{y}} \\
&= \int d^4x d^4y J_{\mathrm{A}}(x) J_{\mathrm{B}}(y) \cdot C^2 \int d^3\vec{p} \int d^3\vec{q} \langle\vec{p}| e^{-\omega_q(x^0-y^0)} |\vec{q}\rangle e^{i(\vec{q}\cdot\vec{x} - \vec{p}\cdot\vec{y})} \\
&= \int d^4x d^4y J_{\mathrm{A}}(x) J_{\mathrm{B}}(y) \cdot C^2 \int d^3\vec{p} \int d^3\vec{q} \langle\vec{p}|\vec{q}\rangle e^{-\omega_q(x^0-y^0)} e^{i(\vec{q}\cdot\vec{x} - \vec{p}\cdot\vec{y})} \\
&= \int d^4x d^4y J_{\mathrm{A}}(x) J_{\mathrm{B}}(y) \cdot C^2 \int d^3\vec{p} N_p^2 e^{-\omega_p(x^0-y^0)} e^{i\vec{p}(\vec{x} - \cdot\vec{y})} \\
&= \int d^4x d^4y J_{\mathrm{A}}(x) J_{\mathrm{B}}(y) \cdot C^2 \int d^3\vec{p} N_p^2 e^{-ip(x-y))}
\end{aligned}
$$

2.

13 Some stuff for later

1. QFT on Riemann sphere with $g : S^2 \to G$ consider the action

$$\mathcal{S}_0 = \frac{1}{4\lambda^2} \int_{S^2} d^2z \; \mathrm{tr}(g^{-1}\partial_\mu g g^{-1}\partial^\mu g) \tag{1015}$$

   then $g^{-1}\partial_\mu g$ defines and element of the Lie algebra and $g^{-1}dg$ is the pullback of the Maurer-Cartan form to S^2 under the map defined by g.

2.
   - Baez review octonions HTTPS://ARXIV.ORG/ABS/MATH/0105155V4
   - Complex quaternions, octonions HTTPS://ARXIV.ORG/ABS/1611.09182
   - Conway, Smith - On quaternions and octonions

14 Representations CheatSheet

14.1 Preliminaries

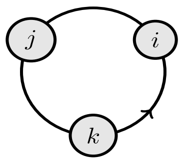
Definition 14.1. Number spaces $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$

- A **complex number** is an objects of the form $a + bi$ with $a, b \in \mathbb{R}$ and

$$i^2 = -1. \tag{1016}$$

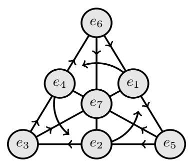
- A **quaternion** is an objects of the form $a + bi + cj + dk$ with $a, b, c, d \in \mathbb{R}$ and

$$i^2 = j^2 = k^2 = ijk = -1. \tag{1017}$$



- An **octonion** is an objects of the form $a+bi+cj+dk+el+fm+gn+ho$ with $a, \ldots, h \in \mathbb{R}$ and $e_0 = 1, e_1 = i, \ldots, e_7 = o$

$$e_i e_j = \begin{cases} e_j, & \text{if } i = 0 \\ e_i, & \text{if } j = 0 \\ -\delta_{ij} e_0 + \varepsilon_{ijk} e_k & \text{otherwise} \end{cases} \tag{1018}$$



Remark 14.1. *$\mathbb{C}$ forms a field, $\mathbb{H}$ forms a non-commutative ring*

Definition 14.2. The **conjugates** are defined by

$$\begin{aligned} \bar{z} &= a - bi & (1019) \\ \bar{q} &= a - bi - cj - dk & (1020) \\ &= -\frac{1}{2}\left[q + iqi + jqj + kqk\right] & (1021) \\ \bar{x} &= a - bi - cj - dk - el - fm - gn - ho & (1022) \\ &= -\frac{1}{6}\left[x + (ix)i + (jq)j + (kq)k) + (le)l + (mf)m + (ng)n + (oh)o\right] & (1023) \end{aligned}$$

14.2 Groups theory

Definition 14.3. For a subgroup H of a group G **a left-coset** of the subgroup H in G is defined as the set formed by a distinct $g \in G$

$$gH = \{gh : \forall h \in H\} \tag{1024}$$

G/H denotes the set of left cosets $\{gH : g \in G\}$ of H in G (called coset-space).

Definition 14.4. A subgroup N of a group G is called **normal subgroup (Normalteiler)** $N \lhd G$ if it is invariant under conjugation by members of G. Meaning

$$gng^{-1} \in N \quad \forall g \in G \tag{1025}$$
$$gN = Ng \quad \forall g \in G \tag{1026}$$
$$gNg^{-1} = N \quad \forall g \in G \tag{1027}$$

Definition 14.5. A **simple group** is a nontrivial group whose only normal subgroups are the trivial group and the group itself.

Theorem 14.1. Every finite simple group is isomorphic to one of the following groups:

1. Z_p cyclic group of prime order
2. A_n alternating group of degree $n > 4$
3. groups of Lie type (names derived from Lie algebras with $q = p^k, m \in \mathbb{N}$
   - $A_n(q)$ Special projective linear group
   - $B_n(q), n > 1$ Commutator subgroup of $SO(2n + 1)$
   - $C_n(q), n > 2$ projective symplectic group
   - $D_n(q), n > 1$ Commutator subgroup of $SO(2n)$
   - $E_6(q), E_7(q), E_8(q), F_4(q), G_2(q)$ Chevalley group
   - ${}^2A_n(q^2), n > 1$ Special unitary group $SU(n)$
   - ${}^2B_2(2^{2m+1}))$ Suzuki Groups $Sz(2^{2m+1})$
   - ${}^2D_n(q^2),{}^3 D_4(q^3),{}^2 E_6(q^2)$ Steinberg group
   - ${}^2F_4(2^{2m+1}),{}^2 G_2(2^{2m+1})$ Ree group
4. 26 sporadic groups
   - Mathieu groups $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}$
   - Janko groups J_1, J_2, J_3, J_4
   - Conway groups Co_1, Co_2, Co_3
   - Fischer groups Fi_{22}, Fi_{23}, F_{3+}
   - Higman–Sims group HS
   - McLaughlin group McL
   - Held group F_7
   - Rudvalis group Ru
   - Suzuki group F_{3-}
   - O'Nan group $O'N$
   - Harada–Norton group F_5
   - Lyons group Ly
   - Thompson group F_3
   - Baby Monster group F_2
   - Fischer–Griess Monster group F_1
5. ${}^2F_4(2)'$ Tits group (order $2^11 \cdot 3^3 \cdot 5^2 \cdot 13 = 17,971,200$)
   - sometimes called the 27th sporadic group - but belongs for $m = 0$ to the family ${}^2F_4(2^{2m+1})'$ of commutator subgroups of ${}^2F_4(2^{2m+1})$

The Periodic Table Of Finite Simple Groups

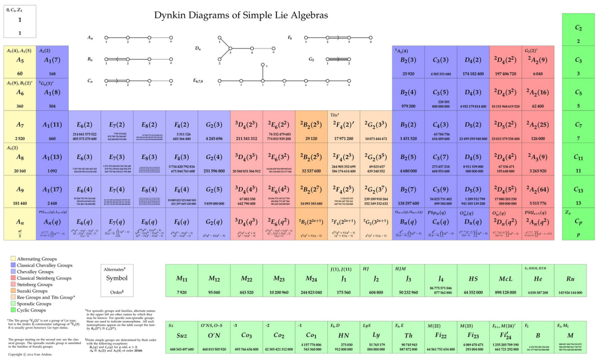


Figure 2: Periodic table of finite simple groups

Definition 14.6. Exceptional Lie groups

- G_2 (order 14)
- F_4 (order 52)
- E_6 (order 78)
- E_7 (order 133)
- E_8 (order 248)

Theorem 14.2. (Frobenius theorem, Hurwitz theorem) Any real finite-dimensional normed division algebra over the reals must be

- isomorphic to $\mathbb{R}$ or $\mathbb{C}$ if unitary and commutative (equivalently: associative and commutative)
- isomorphic to the quaternions $\mathbb{H}$ if noncommutative but associative
- isomorphic to the octonions $\mathbb{O}$ if non-associative but alternative.

Remark 14.2. *Projective spaces*

- $\mathfrak{so}(n+1)$ *is infinitesimal isometry of the real projective spaces* $\mathbb{RP}^n$
- $\mathfrak{su}(n+1)$ *is infinitesimal isometry of the complex projective spaces* $\mathbb{CP}^n$
- $\mathfrak{sp}(n+1)$ *is infinitesimal isometry of the quaternionic projective spaces* $\mathbb{HP}^n$
- *octonionic projective line* $\mathbb{OP}^1$ *reproduces* $\mathfrak{so}(8)$ *(already accomodated by* $\mathbb{RP}^7$*)*
- *Cayley projective plane* $\mathbb{OP}^2$ *reproduces* $\mathfrak{f}_4$*)*
- $\mathbb{OP}^n$ *for* $n > 2$ *gives nothing due to non-associativity of* $\mathbb{O}$

Remark 14.3. *Freudenthal-Rosenfeld-Tits magic square of Lie algebras*

$$\begin{array}{c||cccc}
\mathbb{A}_1/\mathbb{A}_2 & \mathbb{R} & \mathbb{C} & \mathbb{H} & \mathbb{O} \\
\hline\hline
\mathbb{R} & \mathfrak{so}(3) & \mathfrak{su}(3) & \mathfrak{sp}(3) & \mathfrak{f}_4 \\
\mathbb{C} & \mathfrak{su}(3) & \mathfrak{su}(3)\otimes\mathfrak{su}(3) & \mathfrak{su}(6) & \mathfrak{e}_6 \\
\mathbb{H} & \mathfrak{sp}(3) & \mathfrak{su}(6) & \mathfrak{so}(12) & \mathfrak{e}_7 \\
\mathbb{O} & \mathfrak{f}_4 & \mathfrak{e}_6 & \mathfrak{e}_7 & \mathfrak{e}_8
\end{array} \tag{1028}$$

14.3 Representation theory

Definition 14.7. A **representation** of a group $G = (\{g_i\}, \circ)$ is a mapping D of the elements of G onto a set of linear operators with

1. $D(e) = \mathbb{I}$
2. $D(g_1)D(g_2) = D(g_1 \circ g_2)$.

This obviously implies $D(g^{-1}) = D(g)^{-1}$.

Remark 14.4. *A bit more formal - let G a group and V be a $\mathbb{K}$-vector space then a linear representation is a group homomorphism with $D : G \to \mathrm{GL}(V) \overset{!}{=} \mathrm{Aut}(V)$. V is then called representation space with $\dim V$ being the dimension of the representation and $D(g) \in \mathrm{GL}(V)$*

Definition 14.8. An **equivalent representation** D' of a representation D is defined by

$$D(g) \to D'(g) = S^{-1}D(g)S \qquad \forall g \in G \tag{1029}$$

Definition 14.9. A representation D is called **unitary representation** if

$$D(g)^\dagger = D(g)^{-1} \qquad \forall g \in G \tag{1030}$$

Remark 14.5. *For a unitary representation $D(g)^\dagger D(g) = \mathbb{I}$ an equivalent representation $D'(g) = S^{-1}D(g)S$ is only unitary*

$$\begin{aligned} D'(g)^\dagger D'(g) &= \left(S^{-1}D(g)S\right)^\dagger S^{-1}D(g)S && (1031)\\ &= S^\dagger D(g)^\dagger (S^{-1})^\dagger S^{-1}D(g)S && (1032)\\ &= S^\dagger D(g)^\dagger (S^\dagger)^{-1} S^{-1}D(g)S && (1033)\\ &= S^\dagger D(g)^\dagger (SS^\dagger)^{-1} D(g)S && (1034)\end{aligned}$$

iff S is unitary itself $SS^\dagger = \mathbb{I}$

$$D'(g)^\dagger D'(g) = S^{-1}D(g)^\dagger D(g)S = S^{-1}S = \mathbb{I}. \tag{1035}$$

Definition 14.10. A representation is called a **reducible representation** if V has an invariant subspace meaning that the action of any $D(g)$ on any vector of the subspace V_P is still in the subspace. If the projection operator $P : V \to V_P$ projects to this subspace then

$$PD(g)P = D(g)P \qquad \forall g \in G \tag{1036}$$

Remark 14.6. *$\forall |v\rangle \in V$ we have $P|v\rangle \in V_P$. If the subspace is invariant then any group action can not move it outside $D(g)P|v\rangle \in V_P$. But this means projecting it again would not change anything $PD(g)P|v\rangle = D(g)P|v\rangle$*

Definition 14.11. A representation is called an **irreducible representation** if it is not reducible.

Definition 14.12. A representation is called a **completely reducible representation** if it is equivalent to a representation whose matrix elements have the form

$$D(g) = \begin{pmatrix} D_1(g) & 0 & \cdots \\ 0 & D_2(g) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \tag{1037}$$

where all $D_j(g)$ are irreducible. Representation D is is said to be the direct sum of subrepresentation D_j

$$D = D_1(g) \oplus D_2(g) \oplus \cdots \tag{1038}$$

Definition 14.13. For a group of order n the n-dimensional representation D defined by

$$g_k \to |e_k\rangle \tag{1039}$$

$$D(g_j)|e_k\rangle \stackrel{!}{=} |e_m\rangle \qquad \text{with } g_j \circ g_k = g_m \to |e_m\rangle \tag{1040}$$

(where $\{|e_i\rangle\}$ is the ordinary n-dimensional cartesian basis) is called the **regular representation**. The matrices are then constructed by

$$[D(g_j)]_{ik} = \langle e_i|D(g_j)|e_k\rangle = \langle e_i|e_m\rangle. \tag{1041}$$

Theorem 14.3. Every representation of a finite group is equivalent to a unitary representation.

Theorem 14.4. Every representation of a finite group is complete reducible.

Definition 14.14. Given two representations D_1 and D_2 acting on V_1 and V_2, an intertwiner between D_1 and D_2 is a linear operator $F : D_1 \to D_2$ which "commutes with G" in the sense that

$$FD_1(g) = D_2(g)F \quad \forall g \in G. \tag{1042}$$

15 Lie algebras

Remark 15.1. *Killing classification of simple Lie groups*

- *SO(2n), SO(2n+1) - Lie algebra:* $J^T = -J$ *(skew-hermitian, trace free matrices* $GL(n,\mathbb{R})$
- *SU(n) - Lie algebra:* $J^\dagger = -J$ *(skew-hermitian, trace free matrices in* $GL(n,\mathbb{C})$
- *Sp(2n) - Lie algebra:* $J^\dagger = -J$ *(skew-hermitian matrices in* $GL(n,\mathbb{H})$

16 Example representations

16.1 Cyclic group Z_2

$$\begin{array}{c||cc} Z_2 & e & p \\ \hline\hline e & e & p \\ p & p & e \end{array} \tag{1043}$$

1d

$$D(e) = 1, \quad D(p) = 1 \tag{1044}$$

$$D'(e) = 1, \quad D'(p) = -1 \tag{1045}$$

16.2 Cyclic group Z_3

$$\begin{array}{c||ccc} Z_3 & e & a & b \\ \hline\hline e & e & a & b \\ a & a & b & e \\ b & b & e & a \end{array} \tag{1046}$$

1d

$$D(e) = 1, \quad D(a) = 1, \quad D(b) = 1 \tag{1047}$$

$$D'(e) = 1, \quad D'(a) = e^{i\frac{2\pi}{3}}, \quad D'(b) = e^{i\frac{4\pi}{3}} \tag{1048}$$

3d - regular representation

$$|e\rangle = (1,0,0)^T, \quad |a\rangle = (0,1,0)^T, \quad |b\rangle = (0,0,1)^T \tag{1049}$$

$$D(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad D(b) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \tag{1050}$$

16.3 Group S_3

S_3	e	a_1	a_2	a_3	a_4	a_5
e	e	a_1	a_2	a_3	a_4	a_5
a_1	a_1	a_2	e	a_5	a_3	a_4
a_2	a_2	e	a_1	a_4	a_5	a_3
a_1	a_3	a_4	a_5	e	a_1	a_2
a_1	a_4	a_5	a_3	a_2	e	a_1
a_1	a_5	a_3	a_4	a_1	a_2	e

(1051)

$$a_1 = (1,2,3), \quad a_2 = (3,2,1), \quad a_3 = (1,2), \quad a_4 = (2,3), \quad a_5 = (3,1) \tag{1052}$$

2d

$$D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D(a_1) = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}, \quad D(a_2) = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}, \tag{1053}$$

$$D(a_3) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D(a_4) = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}, \quad D(a_5) = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix} \tag{1054}$$

17 Fun with names

- Gordon vs Gordan
  - Paul Gordan (1837-1912) - Clebsch-Gordan decomposition
  - Walter Gordon (1893-1939) - Klein-Gordon equation
- Lorentz vs Lorenz
  - Hendrik Lorentz (1853-1928) - Lorentz transformation, Lorentz force
  - Ludvig Lorenz (1829-1891) - Lorenz gauge
- Hertz vs Hertz
  - Heinrich Hertz (1857-1894) - Hertzian dipole antenna
  - Gustav Hertz (1887-1975) - Franck-Hertz experiment
- Bragg vs Bragg
  - William Henry Bragg (1862-1942) - Bragg equation
  - William Lawrence Bragg (1890-1971) - Bragg equation
- Klein vs Klein
  - Oskar Klein (1894-1977) - Klein-Gordon equation, Kaluza-Klein theory
  - Felix Klein (1849-1925) - Klein bottle
- Euler vs Euler
  - Hans Heinrich Euler (1909-1941) - Euler–Heisenberg Lagrangian
  - Leonhard Euler (1707-1783) - Euler's formula
- Weyl vs Weil
  - Hermann Weyl (1885-1955) - Weyl spinor, Weyl group
  - Andre Weil (1906-1998) - Weil group, Chern–Weil homomorphism
- Jordan vs Jordan vs Jordan
  - Camille Jordan (1838-1922) - Jordan normal, Jordan-Hoelder theorem
  - Wilhelm Jordan (1842-1899) - Gauss-Jordan elimination
  - Pascual Gordon (1902-1980) - Jordan algebra, Jordan Wigner transformation
- Kac vs Kac
  - Victor Kac (1943-...) - Kac–Moody algebra
  - Mark Kac (1904-1984) - Feynman–Kac formula