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1 Intro to LFT – Exercise sheet 2 (2025-05-08)

In the following problems, $A_\mu(x)$ is a smooth $\mathrm{SU}(N)$ gauge field in Euclidean spacetime. In particular, this means that $A_\mu(x)$ is an element of the Lie algebra $\mathfrak{su}(N)$ associated to the Lie group $\mathrm{SU}(N)$; that is, it is an $N \times N$ traceless Hermitian matrix. Even though $A_\mu(x)$ can be decomposed with respect to a basis of generators, you will not need such a representation.

- The field tensor is defined as:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]. \tag{1}$$

  In particular $F_{\mu\nu}(x)$ is also an element of the Lie algebra $\mathfrak{su}(N)$.

- A gauge transformation $\Omega(x)$ is an element of the gauge group $\mathrm{SU}(N)$. Under a gauge transformation, the gauge field transforms as:

$$A_\mu \to A_\mu^{[\Omega]} = \Omega A_\mu \Omega^\dagger + i(\partial_\mu \Omega)\Omega^\dagger. \tag{2}$$

- Using the definition of the field tensor, one easily proves that under a gauge transformation:

$$F_{\mu\nu} \to F_{\mu\nu}^{[\Omega]} = \Omega F_{\mu\nu} \Omega^\dagger. \tag{3}$$

1.1 Exercise 1 - Path-ordered exponential

Consider a smooth parametric curve $[s_0, s_1] \ni s \mapsto \gamma(s) \in \mathbb{R}^4$ in Euclidean spacetime. The following Cauchy problem:

$$\begin{cases} \dfrac{d}{ds} P(s) = i P(s) A_\mu(\gamma(s)) \gamma'^\mu(s), \\ P(s_0) = \mathbb{I}_N, \end{cases} \tag{4}$$

where $P(s)$ is an $N \times N$ matrix, admits a unique solution for $s \in [s_0, s_1]$. The solution $P(s)$ is referred to as the *path-ordered exponential* of the gauge field A_μ along the curve γ, and it is denoted by

$$P(s) = \mathcal{P} \exp\left(i \int_{s_0}^{s} d\sigma\, \gamma'^\mu(\sigma) A_\mu(\gamma(\sigma)) \right). \tag{5}$$

In particular, $P(s_1)$ is referred to as the *parallel transport* along the curve γ, and it is often denoted by $W(\gamma)$. The following notation is also often used:

$$W(\gamma) = \mathcal{P} \exp\left(i \int_\gamma dx^\mu A_\mu(x) \right). \tag{6}$$

If γ parametrizes a segment, then $W(\gamma)$ is referred to as the *Wilson line* along γ. If γ is a closed curve, then $W(\gamma)$ is referred to as the *Wilson loop* along γ. We stress that the expressions given on the right-hand side of Eqs. (5) and (6) are just symbols, whose definition is given in terms of the solution of the Cauchy problem (4).

We now want to study some properties of the solution $P(s)$.

a.) **Special unitarity.** Using the Cauchy problem (4), prove that $P(s)$ is a special unitary matrix for every s, i.e., that $P(s)P^\dagger(s) = \mathbb{I}_N$ and $\det P(s) = 1$.

Hint: Write differential equations for $P(s)P^\dagger(s)$ and $\det P(s)$, and solve them.

b.) **Parametrization invariance.** Let $\tilde\gamma$ be a reparametrization of the curve γ, which preserves orientation. This is defined by providing an invertible function $[\tilde s_0, \tilde s_1] \ni \tilde s \mapsto h(\tilde s) \in [s_0, s_1]$, with the property that $h(\tilde s_0) = s_0$ and $h(\tilde s_1) = s_1$. In terms of this function, we write

$$\tilde\gamma(\tilde s) = \gamma(h(\tilde s)). \tag{7}$$

Let $\tilde P(\tilde s)$ be the unique solution of the Cauchy problem

$$\begin{cases} \dfrac{d}{d\tilde s}\tilde P(\tilde s) = i\tilde P(\tilde s) A_\mu(\tilde\gamma(\tilde s))\tilde\gamma'^\mu(\tilde s), \\ \tilde P(\tilde s_0) = \mathbb{I}_N, \end{cases} \tag{8}$$

Prove that $\tilde P(\tilde s) = P(h(\tilde s))$.

Note: This equation expresses the fact that the path-ordered exponential does not depend on the particular parametrization chosen for the curve γ.

Hint: Write the differential equation for $P(h(\tilde s))$ and compare it with the differential equation for $\tilde P(\tilde s)$.

c.) **Dependence on initial condition.** Given a generic $N \times N$ matrix M, let $R(s)$ be the unique solution of the Cauchy problem

$$\begin{cases} \dfrac{d}{ds}R(s) = iR(s) A_\mu(\gamma(s))\gamma'^\mu(s), \\ R(s_0) = M, \end{cases} \tag{9}$$

Prove that the solution of this equation is given by

$$R(s) = M\,\mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\, \gamma'^\mu(\sigma) A_\mu(\gamma(\sigma))\right). \tag{10}$$

Hint: Write the differential equation satisfied by the right-hand side of the above equation, and compare it with the differential equation for $R(s)$.

d.) **Group property.** Consider $s_0 \le \bar s \le s \le s_1$. Prove that

$$\mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\, \gamma'^\mu(\sigma) A_\mu(\gamma(\sigma))\right) = \mathcal{P}\exp\left(i\int_{s_0}^{\bar s} d\sigma\, \gamma'^\mu(\sigma) A_\mu(\gamma(\sigma))\right) \cdot \mathcal{P}\exp\left(i\int_{\bar s}^{s} d\sigma\, \gamma'^\mu(\sigma) A_\mu(\gamma(\sigma))\right). \tag{11}$$

Provide a geometrical interpretation of this equation.

Hint: Show that both sides satisfy the same Cauchy problem.

e.) **Gauge transformation.** Prove that

$$\mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\, \gamma'^\mu(\sigma) A_\mu^{[\Omega]}(\gamma(\sigma))\right) = \Omega(\gamma(s_0)) \cdot \mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\, \gamma'^\mu(\sigma) A_\mu(\gamma(\sigma))\right) \cdot \Omega^\dagger(\gamma(s)). \tag{12}$$

Hint: Show that both sides of the above equation satisfy the same Cauchy problem.

f.) **Approximation as product of exponentials.** Prove that, if Δs is small,

$$\mathcal{P}\exp\left(i\int_{s}^{s+\Delta s} d\sigma\, \gamma'^\mu(\sigma) A_\mu(\gamma(\sigma))\right) = e^{i\gamma'^\mu(s)A_\mu(\gamma(s))\Delta s} + \mathcal{O}(\Delta s^2). \tag{13}$$

Use this fact to argue that the path-ordered exponential can be approximated by a product of standard exponentials of the gauge field, as in

$$\mathcal{P}\exp\left(i\int_{s_0}^{s_1} d\sigma\,\gamma'^{\mu}(\sigma)A_\mu(\gamma(\sigma))\right) = \lim_{N\to\infty}\prod_{k=0}^{N-1} e^{i\gamma'^{\mu}(\sigma_k)A_\mu(\gamma(\sigma_k))\Delta\sigma}, \tag{14}$$

where

$$\sigma_k = s_0 + k\Delta\sigma, \quad \Delta\sigma = \frac{s_1 - s_0}{N}. \tag{15}$$

Note: A full proof of this fact is beyond the scope of this exercise sheet; a plausible argument is sufficient.

a.) Using $\frac{d}{ds}P(s)$ and the fact that $A_\mu(\gamma(x))$ is hermitian (because SU(N) matrix $e^{i\mathfrak{su}(2)}$ is unitary)

$$\begin{aligned}
\frac{d}{ds}P(s)P^\dagger(s) &= \frac{dP(s)}{ds}P^\dagger(s) + P(s)\frac{dP^\dagger(s)}{ds} &(4)\\
&= [iP(s)A_\mu(\gamma(x))\gamma'^{\mu}(s)]P^\dagger(s) + P(s)[iP(s)A_\mu(\gamma(x))\gamma'^{\mu}(s)]^\dagger &(5)\\
&= iP(s)A_\mu(\gamma(x))\gamma'^{\mu}(s)P^\dagger(s) - iP(s)\gamma'^{\mu}(s)\underbrace{A_\mu^\dagger(\gamma(x))}_{=A_\mu(\gamma(x))}P^\dagger(s) &(6)\\
&= 0 &(7)
\end{aligned}$$

and the initial condition

$$\begin{aligned}
P(s_0) = \mathbb{I}_N &\to P^\dagger(s_0) = \mathbb{I} &(8)\\
&\to P(s_0)P^\dagger(s_0) = \mathbb{I} &(9)
\end{aligned}$$

we conclude

$$P(s)P^\dagger(s) = \mathbb{I}_N \qquad \forall s. \tag{10}$$

Now using the fact that A_μ is trace free

$$1 = \det \mathrm{SU}(N) = \det e^{\mathfrak{su}(N)} = e^{\mathrm{tr}(\mathfrak{su}(N))} \to \mathrm{tr}(\mathfrak{su}(N)) = 0 \tag{11}$$

and the Jacobi formula

$$\begin{aligned}
\frac{d}{ds}\det P(s) &= \det P(s)\cdot \mathrm{tr}\left(P^{-1}(s)\frac{dP(s)}{ds}\right) &(12)\\
&= \det P(s)\cdot \mathrm{tr}\left(P^{-1}(s)[iP(s)A_\mu(\gamma(x))\gamma'^{\mu}(s)]\right) &(13)\\
&= i\,\gamma'^{\mu}(s)\det P(s)\cdot \underbrace{\mathrm{tr}\left(A_\mu(\gamma(x))\right)}_{=0} &(14)\\
&= 0 &(15)
\end{aligned}$$

and the initial condition

$$P(s_0) = \mathbb{I}_N \to \det P(s_0) = 1 \tag{16}$$

to conclude that $\det P(s) = 1$

b.) Initial conditions

$$P(h(\tilde{s}_0)) = P(s_0) \tag{17}$$

and the differential equation

$$\frac{d}{d\tilde{s}}P(h(\tilde{s})) = \frac{dP(h(\tilde{s}))}{dh}\frac{dh}{d\tilde{s}} \tag{18}$$

$$= iP(h)A_\mu(\gamma(h))\underbrace{\underbrace{\gamma'^\mu(h)}_{=\frac{d\tilde{\gamma^\mu}}{dh}}\frac{dh}{d\tilde{s}}}_{=\frac{d\tilde{\gamma^\mu}}{d\tilde{s}}} \tag{19}$$

$$= iP(h(\tilde{s}))A_\mu(\gamma(h(\tilde{s})))\frac{d\tilde{\gamma^\mu}}{d\tilde{s}} \tag{20}$$

so we see that the differential equations for $\tilde{P}(\tilde{s})$ and $P(h(\tilde{s}))$ are identical.

c.) Using

$$\frac{d}{ds}\int_{g(s)}^{h(s)} f(t,s)dt = \frac{d}{ds}\left(F[h(s),s] - F[g(s),s]\right) \tag{21}$$

$$= \frac{\partial F}{\partial h}\frac{\partial h}{\partial s} + \frac{\partial F[h,s]}{\partial s} - \frac{\partial F}{\partial g}\frac{\partial g}{\partial s} - \frac{\partial F[g,s]}{\partial s} \tag{22}$$

$$= f(h(s))h'(s) - f(g(s))g'(s) + \int_{g(s)}^{h(s)}\frac{d}{ds}f(t,s)dt \tag{23}$$

and substituting the solution into the differential equations

$$\frac{dR}{ds} = \frac{d}{ds}M\mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\,\gamma'^\mu(\sigma)A_\mu(\gamma(\sigma))\right) \tag{24}$$

$$= \underbrace{M\mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\,\gamma'^\mu(s)A_\mu(\gamma(s))\right)}_{=R(s)}(i\gamma'^\mu(\sigma)A_\mu(\gamma(\sigma)))\frac{ds}{ds} \tag{25}$$

$$= iR(s)A_\mu(\gamma(s))\gamma'^\mu(s) \tag{26}$$

so we see the the solution obeys the differential equation.

d.) LHS

$$\frac{d}{ds}\mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\,\gamma'^\mu(\sigma)A_\mu(\gamma(\sigma))\right) = \mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\,\gamma'^\mu(\sigma)A_\mu(\gamma(\sigma))\right)iA_\mu(\gamma(s))\gamma'^\mu(s) \tag{27}$$

$$\rightarrow \qquad \frac{d}{ds}Q(s) = Q(s)\cdot iA_\mu(\gamma(s))\gamma'^\mu(s) \tag{28}$$

RHS

$$\frac{d}{ds}\mathcal{P}\exp\left(i\int_{s_0}^{\bar{s}} d\sigma\,\gamma'^\mu(\sigma)A_\mu(\gamma(\sigma))\right)\cdot\mathcal{P}\exp\left(i\int_{\bar{s}}^{s} d\sigma\,\gamma'^\mu(\sigma)A_\mu(\gamma(\sigma))\right) \tag{29}$$

$$= \exp\left(i\int_{s_0}^{\bar{s}} d\sigma\,\gamma'^\mu(\sigma)A_\mu(\gamma(\sigma))\right)\cdot\mathcal{P}\exp\left(i\int_{\bar{s}}^{s} d\sigma\,\gamma'^\mu(\sigma)A_\mu(\gamma(\sigma))\right)iA_\mu(\gamma(s))\gamma'^\mu(s) \tag{30}$$

$$\rightarrow \qquad \frac{d}{ds}Q(s) = Q(s)\cdot iA_\mu(\gamma(s))\gamma'^\mu(s) \tag{31}$$

e.) LHS

$$\frac{d}{ds}\left(\mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\,\gamma'^\mu(\sigma)A_\mu^{[\Omega]}(\gamma(\sigma))\right)\right) = \mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\,\gamma'^\mu(\sigma)A_\mu^{[\Omega]}(\gamma(\sigma))\right)iA_\mu^{[\Omega]}(\gamma(s))\gamma'^\mu(s) \tag{32}$$

RHS with $A_\mu^{[\Omega]} = \Omega A_\mu \Omega^\dagger + i(\partial_\mu \Omega)\Omega^\dagger$ and the adjoint $A_\mu^{[\Omega]} = \Omega A_\mu \Omega^\dagger - i(\Omega \partial_\mu \Omega^\dagger)$

$$\frac{d}{ds}\left(\Omega(\gamma(s_0)) \cdot \mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\, \gamma'^\mu(\sigma) A_\mu(\gamma(\sigma))\right) \cdot \Omega^\dagger(\gamma(s))\right) \tag{33}$$

$$= \Omega(\gamma(s_0))\left(\frac{d}{ds}\mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\, \gamma'^\mu(\sigma) A_\mu(\gamma(\sigma))\right)\right)\Omega^\dagger(\gamma(s)) \tag{34}$$

$$+ \Omega(\gamma(s_0))\mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\, \gamma'^\mu(\sigma) A_\mu(\gamma(\sigma))\right)\left(\frac{d}{ds}\Omega^\dagger(\gamma(s))\right) \tag{35}$$

$$= \Omega(\gamma(s_0))\mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\, \gamma'^\mu(\sigma) A_\mu(\gamma(\sigma))\right)(iA_\mu(\gamma(s))\gamma'^\mu(s))\,\Omega^\dagger(\gamma(s)) \tag{36}$$

$$+ \Omega(\gamma(s_0))\mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\, \gamma'^\mu(\sigma) A_\mu(\gamma(\sigma))\right)\underbrace{\left(\frac{d}{ds}\Omega^\dagger(\gamma(s))\right)}_{=\partial_\nu\Omega^\dagger\gamma'^\nu(s)} \tag{37}$$

$$= \Omega(\gamma(s_0))\mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\, \gamma'^\mu(\sigma) A_\mu(\gamma(\sigma))\right)(iA_\mu(\gamma(s))\gamma'^\mu(s))\,\Omega^\dagger(\gamma(s)) \tag{38}$$

$$+ \Omega(\gamma(s_0))\mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\, \gamma'^\mu(\sigma) A_\mu(\gamma(\sigma))\right)\gamma'^\nu(s)\partial_\nu\Omega^\dagger \tag{39}$$

$$= \Omega(\gamma(s_0))\mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\, \gamma'^\mu(\sigma) A(\gamma^\mu(s))\right)\gamma'^\mu(\sigma)\underbrace{\left[iA_\mu(\gamma(s))\Omega^\dagger(\gamma(a)) + \partial_\mu\Omega^\dagger\right]}_{=i\Omega^\dagger(\gamma(s))A_\mu^{[\Omega]}} \tag{40}$$

$$= \Omega(\gamma(s_0))\mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\, \gamma'^\mu(\sigma) A_\mu(\gamma'^\mu(\sigma))\right)\Omega^\dagger(\gamma(s)) i\gamma'^\mu(s) A_\mu^{[\Omega]} \tag{41}$$

and therefore comparing LHS and RHS

$$\mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\, \gamma'^\mu(\sigma) A_\mu^{[\Omega]}(\gamma'^\mu(s))\right) = \Omega(\gamma(s_0))\mathcal{P}\exp\left(i\int_{s_0}^{s} d\sigma\, \gamma'^\mu(\sigma) A_\mu(\gamma'^\mu(s))\right)\Omega^\dagger(\gamma(s)) \tag{42}$$

f.) With $\gamma'^\mu(\sigma)A_\mu(\gamma'^\mu(\sigma)) = \gamma'^\mu(s)A_\mu(\gamma'^\mu(s))$ and the fact that the path-ordering becomes obsolete if the integrand it constant

$$\mathcal{P}\exp\left(i\int_{s}^{s+\Delta s} d\sigma\, \gamma'^\mu(\sigma) A_\mu(\gamma'^\mu(\sigma))\right) \simeq \mathcal{P}\exp\left(i\int_{s}^{s+\Delta s} d\sigma\, \gamma'^\mu(s) A_\mu(\gamma'^\mu(s))\right) \tag{43}$$

$$\simeq \exp\left(i\,\gamma'^\mu(s) A_\mu(\gamma'^\mu(s))\int_{s}^{s+\Delta s} d\sigma\right) \tag{44}$$

$$\simeq \exp\left(i\,\gamma'^\mu(s) A_\mu(\gamma'^\mu(s))\Delta s + \mathcal{O}(\Delta s^2)\right) \tag{45}$$

For the next part we use the group property (we showed above) - splitting this into N equal sized sections $\Delta\sigma$

$$\mathcal{P}\exp\left(i\int_{s_0}^{s_1} d\sigma\, \gamma'^\mu(\sigma) A_\mu(\gamma'^\mu(\sigma))\right) \tag{46}$$

$$= \mathcal{P}\exp\left(i\int_{s_0}^{s_0+\Delta\sigma} d\sigma\, \gamma'^\mu(\sigma) A_\mu(\gamma'^\mu(\sigma))\right) ...\mathcal{P}\exp\left(i\int_{s_0+(N-1)\Delta\sigma}^{s_1} d\sigma\, \gamma'^\mu(\sigma) A_\mu(\gamma'^\mu(\sigma))\right) \tag{47}$$

$$\simeq \lim_{N\to\infty} \exp\left(i\,\gamma'^\mu(s_0) A_\mu(\gamma'^\mu(s_0))\Delta\sigma\right) ... \exp\left(i\,\gamma'^\mu(s_0 + (N-1)\Delta\sigma) A_\mu(\gamma'^\mu(s_0 + (N-1)\Delta\sigma))\Delta\sigma\right) \tag{48}$$

$$\simeq \lim_{N\to\infty} \exp\left(i\,\gamma'^\mu(s_0) A_\mu(\gamma'^\mu(s_0))\Delta\sigma\right) ... \exp\left(i\,\gamma'^\mu(s_{N-1}) A_\mu(\gamma'^\mu(s_{N-1})\Delta\sigma\right) \tag{49}$$

$$\tag{50}$$

I think the limit is the handwavy part.

1.2 Exercise 2 - Temporal gauge

Given a gauge field $A_\mu(x)$, prove that it is always possible to find a gauge transformation $\Omega(x)$ such that $A_0^{[\Omega]}(x) = 0$ for every x (temporal gauge condition). Derive a formula for $\Omega(x)$ in terms of the path-ordered exponential.

We require (using $\Omega^{-1} = \Omega^\dagger$)

$$A_0^{[\Omega]} = \Omega A_0 \Omega^\dagger + i(\partial_0 \Omega)\Omega^\dagger \stackrel{!}{=} 0 \tag{51}$$

$$\rightarrow \Omega A_0 + i(\partial_0 \Omega) = 0 \tag{52}$$

$$\rightarrow \partial_0 \Omega = i\Omega A_0(\vec{x}, t) \tag{53}$$

so we obtained a homogeneous linear first order ODE which we can solve as usual by separation of variables

$$\Omega(\vec{x}, t) = \mathcal{P} \exp\left(i \int_{t_0}^{t} A_0(\vec{x}, s) ds \right) \tag{54}$$