## 0.1 RH

- Numbers as Functions - Yuri Manin
  - Many different way to think about numbers as functions
  - Certain numbers called periods appear in number theory and QFT $\{\sqrt[3]{5}, \pi, \frac{\pi^2}{6}, \Gamma\left(\frac{3}{7}\right)^7\}$
- Using the imaginary parts of the non-trivial zeros of the Zeta function

$$f(x) = -\sum_k \cos(\mathrm{Im}(\zeta_k) \log x) \tag{1}$$

  we see peaks at the prices and smaller peaks at their powers $2, 2^2, 2^3, ..., 3, 3^2, 3^3, ..., 5, 5^2, ...$
- Riemann's Hypothesis - Brian Conrey
- SageMathCell

## 0.2 Linear algebra

### 0.2.0 Basic Concepts

#### How to write up mathematics

DEFINITION: Set

DEFINITION: Structure on $\mathbb{Q}$

DEFINITION: Group

DEFINITION: Field

#### The Complex Numbers

DEFINITION: Complex numbers

REMARK: $i^2 = -1$

FACT 1:

(i) $\mathbb{C}$ with operations $+/\cdot$ is a field

(ii) $\mathbb{R} \to \mathbb{C}$

REMARK: Fundamental theorem of algebra: Every polynom of order $n$: $P(z) = \sum_k^n a_k z^k$ has exactly $n$ zeros

DEFINITION: Complex conjugation

### 0.2.1 Vector Spaces

#### Vector Space

## 0.3 Classical Mechanics

### 0.3.1 Lagrangian Mechanics

$$L = T - V, \qquad S = \int L(q, \dot{q}, t)dt \tag{2}$$

Integration by parts - neglecting boundary terms

$$\delta S = \int \delta L dt = 0 \tag{3}$$

$$\delta L = \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} \tag{4}$$

$$= \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right]\delta q \tag{5}$$

Canonical momentum

$$p = \frac{\partial L}{\partial \dot{q}} \tag{6}$$

Cyclic coordinates

$$\frac{\partial L}{\partial q} = 0 \quad \rightarrow \quad \frac{\partial L}{\partial \dot{q}} = p = \text{const} \tag{7}$$

## 0.4 Classical Field Theory

The physics - to derive the equations of motion

$$S = \int L dt = \int \mathcal{L}(\psi, \partial_\mu \psi)\, d^4x \tag{8}$$

$$0 = \delta S = \int d^4x \delta \mathcal{L} \tag{9}$$

Adding a four-divergence to the Lagrangian $\mathcal{L}' = \mathcal{L} + \partial_\mu K^\mu(\psi)$ results in

$$\int d^4x\, \partial_\mu K^\mu = \int dA\, n_\mu K^\mu \tag{10}$$

which should vanish for well behaved fields and therefore should not change anything.

$$\delta \mathcal{L} = \sum_a \frac{\partial \mathcal{L}}{\partial \psi_a}\delta\psi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_a)}\overbrace{\delta(\partial_\mu \psi_a)}^{=\partial_\mu(\delta\psi_a)} \tag{11}$$

$$= \sum_a \underbrace{\left[\frac{\partial \mathcal{L}}{\partial \psi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_a)}\right]}_{\text{equations of motion}}\delta\psi_a + \underbrace{\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_a)}\delta\psi_a\right)}_{=\partial_\mu K^\mu} \tag{12}$$

Internal symmetry: $\psi_a \rightarrow \psi'_a = \psi_a + \delta\psi_a$ if $\delta\mathcal{L} = 0$

$$\rightarrow j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_a)}\delta\psi_a - K^\mu \qquad \partial_\mu j^\mu = 0 \tag{13}$$

$$\rightarrow Q = \int d^3x\, j_0 \qquad \frac{d}{dt}Q = \int d^3x \frac{\partial j_0}{\partial t} = \int d^3x \nabla\vec{j} = \int d\vec{A}\cdot\vec{j} = 0 \tag{14}$$

Consider spacetime translation: $x^\nu \rightarrow x'^\nu = x^\nu - \epsilon^\nu$ implying $\psi(x) \rightarrow \psi(x') = \psi(x) + \epsilon^\nu \partial_\nu \psi(x)$ and $\mathcal{L} \rightarrow \mathcal{L}' = \mathcal{L} + \epsilon^\nu \partial_\nu \mathcal{L} = \mathcal{L} + \epsilon^\nu \partial_\mu(\delta^\mu_\nu \mathcal{L})$ results in four Noether currents $\nu = 0, 1, 2, 3$

$$\rightarrow T^\mu_{\ \nu} \equiv (j^\mu)_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\partial_\nu \psi - \delta^\mu_\nu \mathcal{L} \tag{15}$$

$$\rightarrow T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\partial^\nu \psi - \eta^{\mu\nu}\mathcal{L} \tag{16}$$

$$\rightarrow \Theta^{\mu\nu} = -\frac{2}{\sqrt{-g}}\left.\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g_{\mu\nu}}\right|_{g_{\mu\nu}=\eta_{\mu\nu}} \tag{17}$$

Also

$$\pi_a = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_a)} \tag{18}$$
$$\mathcal{H} = \sum_a \pi_a \partial_o \psi_a - \mathcal{L} \tag{19}$$

### 0.4.1 Lagrangian Lookup Table

| | | | |
|---|---|---|---|
| Real scalar field | x | x | • |
| $\mathcal{L}[\phi] = \frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{n!}\phi^n$ | • | • | • |
| $(\Box + m^2)\phi + \frac{\lambda}{(n-1)!}\phi^{n-1} = 0$ | • | • | • |
| • | • | • | • |

- Real scalar field $\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{n!}\phi^n$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi - \frac{\lambda}{(n-1)!}\phi^{n-1} \tag{20}$$
$$\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi)} = \eta^{\mu\nu}(\partial_\mu\phi)\delta^\alpha_\nu = \partial^\alpha\phi \tag{21}$$
$$\rightarrow (\Box + m^2)\phi + \frac{\lambda}{(n-1)!}\phi^{n-1} = 0 \tag{22}$$

Hamiltonian

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0\phi)} = \partial^0\phi \tag{23}$$
$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} \tag{24}$$
$$= \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{n!}\phi^n \tag{25}$$
$$H = \frac{1}{2}\int d^3y \left(\pi^2 + (\nabla\phi)^2 + m^2\phi^2 + \frac{2\lambda}{n!}\phi^n\right) \tag{26}$$

Heisenberg equations with $[\phi(\vec{x},t), \pi(\vec{y},t)] = i\delta^3(\vec{x}-\vec{y})$

$$\int d^3y[\pi(y)^2, \phi(x)] = \int d^3y \left(\pi(y)^2\phi(x) - \phi(x)\pi(y)^2\right) \tag{27}$$
$$= \int d^3y \left(\pi(y)^2\phi(x) - \pi(y)\phi(x)\pi(y) - i\pi(y)\delta^3(\vec{x}-\vec{y})\right) \tag{28}$$
$$= \int d^3y \left(\pi(y)^2\phi(x) - \pi(y)^2\phi(x) - 2i\pi(y)\delta^3(\vec{x}-\vec{y})\right) \tag{29}$$
$$= -2i\pi(x) \tag{30}$$
$$\rightarrow \dot{\phi} = i[H, \phi] = \pi(x) \tag{31}$$

$$\int d^3y[(\nabla_y\phi(y))^2,\pi(x)] = \int d^3y\left((\nabla_y\phi(y))^2\pi(x) - \pi(x)(\nabla_y\phi(y))^2\right) \tag{32}$$

$$= \int d^3y\,(\nabla_y\phi(y)(\nabla_y\phi(y)\pi(x)) - (\pi(x)\nabla_y\phi(y))\nabla_y\phi(y)) \tag{33}$$

$$= \int d^3y\,(\nabla_y\phi(y)\nabla_y[\phi(y),\pi(x)]\nabla_y\phi(y)) \tag{34}$$

$$= i\int d^3y\,(\nabla_y\phi(y))^2\,\nabla_y\delta^{(3)}(\vec{x}-\vec{y}) \tag{35}$$

$$= -2i\int d^3y\left(\nabla_y^2\phi(y)\right)\delta^{(3)}(\vec{x}-\vec{y}) \tag{36}$$

$$= -2i\nabla_x^2\phi(x) \tag{37}$$

$$\to \dot{\pi} = i[H,\pi] = \nabla^2\pi(x) - m^2\phi - \frac{\lambda}{(n-1)!}\phi^{n-1} \tag{38}$$

- Maxwell field $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + j^\mu A_\mu = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\nu)\eta^{\mu\sigma}\eta^{\nu\rho}(\partial_\sigma A_\rho - \partial_\rho A_\sigma) + j^\mu A_\mu$

$$\frac{\partial\mathcal{L}}{\partial A_\alpha} = j^\mu\delta_\mu^\alpha = j^\alpha \tag{39}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\beta A_\alpha)} = -\frac{2}{4}(\delta_\mu^\beta\delta_\nu^\alpha - \delta_\nu^\beta\delta_\mu^\alpha)F^{\mu\nu} = -F^{\alpha\beta} \tag{40}$$

$$\to \partial_\beta F^{\alpha\beta} + j^\alpha = 0 \tag{41}$$

$$\to T^{\mu\nu}_{\text{free}} = -F^{\alpha\mu}\partial^\nu A_\alpha + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \tag{42}$$

$$\to T^{\mu\nu}_{\text{free,sym}} = T^{\mu\nu}_{\text{free}} + F^{\alpha\mu}\partial_\alpha A^\nu = -F^{\alpha\mu}F^\nu_{\ \alpha} + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \tag{43}$$

- Dirac field $\mathcal{L} = \overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi$

$$\frac{\partial\mathcal{L}}{\partial\overline{\psi}} = (i\gamma^\mu\partial_\mu - m)\psi \tag{44}$$

$$\frac{\partial\mathcal{L}}{\partial\psi} = -m\overline{\psi} \tag{45}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\overline{\psi})} = 0 \tag{46}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\psi)} = \overline{\psi}i\gamma^\mu\delta_\mu^\alpha = i\overline{\psi}\gamma^\alpha \tag{47}$$

$$\to (i\gamma^\mu\partial_\mu - m)\psi = 0 \tag{48}$$

$$\to \partial_\alpha(i\overline{\psi}\gamma^\alpha) + m\overline{\psi} = 0 \tag{49}$$

- Massive vector field $\mathcal{L} = -\frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{1}{2}m^2B_\mu B^\mu$

$$\frac{\partial\mathcal{L}}{\partial B_\alpha} = m^2B^\alpha \tag{50}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\beta B_\alpha)} = -\frac{2}{4}(\delta_\mu^\beta\delta_\nu^\alpha - \delta_\nu^\beta\delta_\mu^\alpha)G^{\mu\nu} = G^{\alpha\beta} \tag{51}$$

$$\to \partial_\beta G^{\alpha\beta} - m^2B^\alpha = 0 \tag{52}$$

## 0.5 Classical Electrodynamics

**Notation**

$$\eta_{ab} = \eta^{ab} = \mathrm{diag}(1, -1, -1, -1) \tag{53}$$

$$\mathbf{A} \to A^i = \begin{pmatrix} A^0 \\ \vec{A} \end{pmatrix} \qquad A_i = \begin{pmatrix} A^0 \\ -\vec{A} \end{pmatrix} \tag{54}$$

$$\mathbf{E} = -\nabla A^0 - \partial_t \mathbf{A} \tag{55}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{56}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{57}$$

$$F_{10} = \partial_x A_0 - \partial_t A_x = \partial_x A^0 + \partial_t A^x = -E_x \tag{58}$$

$$F_{21} = \partial_y A_x - \partial_x A_y = -\partial_y A^x + \partial_x A^y = B_z \tag{59}$$

$$F_{31} = \partial_z A_x - \partial_x A_z = -\partial_z A^x + \partial_x A^z = -B_y \tag{60}$$

$$F_{\mu\nu} = F_{\downarrow\downarrow} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \qquad F^{\mu\nu} = F_{\uparrow\uparrow} = \eta F_{\downarrow\downarrow} \eta^T = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{61}$$

$$F_{\mu\nu} F^{\mu\nu} = -\mathrm{tr}(F_{\downarrow\downarrow}.F_{\uparrow\uparrow}) = 2(\mathbf{B}^2 - \mathbf{E}^2) \qquad F^{\mu\lambda} F_{\lambda\nu} = ... \tag{62}$$

### 0.5.1 Multipole expansion

**Spherical Harmonics**

$$Y_{00} = \frac{1}{2\sqrt{\pi}} \tag{63}$$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\vartheta \, e^{i\varphi} = -\sqrt{\frac{3}{8\pi}} \frac{x + iy}{r} \tag{64}$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\vartheta = \sqrt{\frac{3}{4\pi}} \frac{z}{r} \tag{65}$$

$$Y_{1,-1} = \sqrt{\frac{3}{8\pi}} \sin\vartheta \, e^{-i\varphi} = \sqrt{\frac{3}{8\pi}} \frac{x - iy}{r} \tag{66}$$

$$Y_{22} = \sqrt{\frac{15}{32\pi}} \sin^2\vartheta \, e^{2i\varphi} = \sqrt{\frac{15}{32\pi}} \frac{(x + iy)^2}{r^2} \tag{67}$$

$$Y_{21} = -\sqrt{\frac{15}{8\pi}} \sin\vartheta \cos\vartheta \, e^{i\varphi} = -\sqrt{\frac{15}{32\pi}} \frac{(x + iy)z}{r^2} \tag{68}$$

$$Y_{20} = \sqrt{\frac{5}{16\pi}} (3\cos^2\vartheta - 1) = \sqrt{\frac{5}{16\pi}} \frac{3z^2 - r^2}{r^2} \tag{69}$$

$$Y_{2,-1} = \sqrt{\frac{15}{8\pi}} \sin\vartheta \cos\vartheta \, e^{-i\varphi} = \sqrt{\frac{15}{32\pi}} \frac{(x - iy)z}{r^2} \tag{70}$$

$$Y_{2,-2} = \sqrt{\frac{15}{32\pi}} \sin^2\vartheta \, e^{-2i\varphi} = \sqrt{\frac{15}{32\pi}} \frac{(x - iy)^2}{r^2} \tag{71}$$

**Cartesian**

With $|\mathbf{x}| \gg |\mathbf{x}'|$

$$
\begin{aligned}
\frac{1}{|\mathbf{x}-\mathbf{x}'|} &= \frac{1}{\sqrt{|\mathbf{x}|^2 - 2\mathbf{x}\cdot\mathbf{x}' + |\mathbf{x}'|^2}} && (72)\\
&= \frac{1}{|\mathbf{x}|}\frac{1}{\sqrt{1 - 2\frac{\mathbf{x}\cdot\mathbf{x}'}{|\mathbf{x}|^2} + \frac{|\mathbf{x}'|^2}{|\mathbf{x}^2|}}} && (73)\\
&= \frac{1}{|\mathbf{x}|}\frac{1}{\sqrt{1 - \underbrace{\left(2\frac{\mathbf{x}\cdot\mathbf{x}'}{|\mathbf{x}|^2} - \frac{|\mathbf{x}'|^2}{|\mathbf{x}^2|}\right)}_{=y}}} && (74)\\
&= \frac{1}{|\mathbf{x}|}\left(1 + \frac{1}{2}y + \frac{3}{8}y^2 + \frac{5}{16}y^3 + ...\right) && (75)\\
&= \frac{1}{|\mathbf{x}|}\left(1 + \frac{1}{2}\left(2\frac{\mathbf{x}\cdot\mathbf{x}'}{|\mathbf{x}|^2} - \frac{|\mathbf{x}'|^2}{|\mathbf{x}^2|}\right) + \frac{3}{8}\left(2\frac{\mathbf{x}\cdot\mathbf{x}'}{|\mathbf{x}|^2} - \frac{|\mathbf{x}'|^2}{|\mathbf{x}^2|}\right)^2 + \frac{5}{16}\left(2\frac{\mathbf{x}\cdot\mathbf{x}'}{|\mathbf{x}|^2} - \frac{|\mathbf{x}'|^2}{|\mathbf{x}^2|}\right)^3 + ...\right) && (76)\\
&= \frac{1}{|\mathbf{x}|} + \frac{\mathbf{x}\cdot\mathbf{x}'}{|\mathbf{x}|^3} + \frac{1}{2}\underbrace{\frac{3(\mathbf{x}\cdot\mathbf{x}')^2 - |\mathbf{x}|^2|\mathbf{x}'|^2}{|\mathbf{x}|^5}}_{=\frac{[3x'^i x'^j - \delta_{ij}(x'^j x'^j)]x^i x^j}{|\mathbf{x}|^5}} + \frac{1}{2}\frac{5(\mathbf{x}\cdot\mathbf{x}')^3 - 3(\mathbf{x}\cdot\mathbf{x}')|\mathbf{x}|^2|\mathbf{x}'|^2}{|\mathbf{x}|^7} + ... && (77)
\end{aligned}
$$

Then

$$
\begin{aligned}
4\pi\epsilon_0\Phi(\mathbf{x}) &= \int d^3x' \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} && (78)\\
&= \frac{1}{|\mathbf{x}|}\int d^3x' \rho(\mathbf{x}') + \frac{1}{|\mathbf{x}|^3}\mathbf{x}\cdot\int d^3x'\,\mathbf{x}'\rho(\mathbf{x}') + \frac{1}{2|\mathbf{x}|^5}x^i x^j \int d^3x'\,(3x'_i x'_j - \delta_{ij}|\mathbf{x}'|^2)\rho(\mathbf{x}') + ... && (79)\\
&= \frac{q}{|\mathbf{x}|} + \frac{\mathbf{x}\cdot\mathbf{p}}{|\mathbf{x}|^3} + \frac{(\mathbf{x},\mathbf{Q}\mathbf{x})}{2|\mathbf{x}|^5} + ... && (80)
\end{aligned}
$$

**Spherical**

$$
\begin{aligned}
\frac{1}{|\mathbf{x}-\mathbf{x}'|} &= \frac{1}{|\mathbf{x}|} + \frac{\mathbf{x}\cdot\mathbf{x}'}{|\mathbf{x}|^3} + \frac{1}{2}\frac{[3x'^i x'^j - \delta_{ij}(x'^j x'^j)]x^i x^j}{|\mathbf{x}|^5} + \frac{1}{2}\frac{5(\mathbf{x}\cdot\mathbf{x}')^3 - 3(\mathbf{x}\cdot\mathbf{x}')|\mathbf{x}|^2|\mathbf{x}'|^2}{|\mathbf{x}|^7} + ... && (81)\\
\frac{1}{|\mathbf{r}-\mathbf{r}'|} &= \frac{1}{r}\sum_{l=0}\frac{4\pi}{2l+1}\sum_{m=-l}^{l}\left(\frac{r'}{r}\right)^l Y_{lm}(\vartheta,\varphi)Y^*_{lm}(\vartheta',\varphi') && (82)
\end{aligned}
$$

Then

$$
\begin{aligned}
4\pi\epsilon_0\Phi(\mathbf{x}) &= \int d^3x' \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} && (83)\\
&= \sum_{l,m} \frac{4\pi}{2l+1} \frac{q_{lm}}{r^{l+1}} Y_{lm}(\vartheta,\varphi) && (84)\\
&= 4\pi \frac{q_{00}}{r} Y_{00} + \frac{4\pi}{3} \frac{q_{11}Y_{11} + q_{10}Y_{10} + q_{1,-1}Y_{1,-1}}{r^2} && (85)\\
&= \frac{q}{r} + \frac{4\pi}{3} \frac{-\sqrt{\frac{3}{8\pi}}(-p_x+ip_y)\sqrt{\frac{3}{8\pi}}\frac{x+iy}{r} + \sqrt{\frac{3}{4\pi}}p_z\sqrt{\frac{3}{4\pi}}\frac{z}{r} + \sqrt{\frac{3}{8\pi}}(p_x+ip_y)\sqrt{\frac{3}{8\pi}}\frac{x-iy}{r}}{r^2} + \ldots && (86)\\
&= \frac{q}{r} + \frac{1}{2} \frac{(p_x x + p_y y + i(p_x y - p_y x)) + 2p_z z + (p_x x + p_y y - i(p_x y - p_y x))}{r^3} + \ldots && (87)
\end{aligned}
$$

with

$$
\begin{aligned}
q_{lm} &= \int d^3r'\, {r'}^l Y^*_{lm}(\vartheta',\varphi')\rho(\mathbf{r}') && (88)\\
q_{00} &= \frac{1}{2\sqrt{\pi}} \int d^3r'\, \rho(\mathbf{r}') r'^2 \sin\vartheta' = \frac{q}{2\sqrt{\pi}} && (89)\\
q_{11} &= -\frac{1}{2}\sqrt{\frac{3}{2\pi}} \int d^3r'\, r' \rho(\mathbf{r}')(\sin\vartheta'\, e^{-i\varphi'}) r'^2 \sin\vartheta' && (90)\\
&= -\frac{1}{2}\sqrt{\frac{3}{2\pi}} \int d^3r'\, \rho(\mathbf{r}')(r'\sin\vartheta'[\cos\varphi' - i\sin\varphi']) r'^2 \sin\vartheta' && (91)\\
&= -\frac{1}{2}\sqrt{\frac{3}{2\pi}} \int d^3r'\, \rho(\mathbf{r}')(x' - iy') r'^2 \sin\vartheta' && (92)\\
&= \sqrt{\frac{3}{8\pi}}(-p_x + ip_y) && (93)\\
q_{10} &= \sqrt{\frac{3}{4\pi}} p_z && (94)\\
q_{1,-1} &= \sqrt{\frac{3}{8\pi}}(p_x + ip_y) && (95)
\end{aligned}
$$

### 0.5.2 Radiation

Starting with

$$
\begin{aligned}
\nabla\cdot\mathbf{D} &= \rho \qquad \nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t} && (96)\\
\nabla\cdot\mathbf{B} &= 0 \qquad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} && (97)
\end{aligned}
$$

then

$$
\begin{aligned}
\mathbf{B} &= \nabla\times\mathbf{A} && (98)\\
&\to \nabla\times\left(\mathbf{E} + \frac{\partial\mathbf{A}}{\partial t}\right) = 0 = \nabla\times(-\nabla\phi) && (99)\\
&\to \mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} && (100)
\end{aligned}
$$

in vacuum we find

$$\nabla \cdot \mathbf{E} \quad \rightarrow \quad \nabla^2 \phi + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho}{\epsilon_0} \tag{101}$$

$$\nabla \times \mathbf{H} \quad \rightarrow \quad \nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla \left( \nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} \right) = -\mu_0 \mathbf{J} \tag{102}$$

using the Lorenz condition

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0 \tag{103}$$

we get

$$\nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon_0} \tag{104}$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} \tag{105}$$

with the solution

$$\phi(\mathbf{x}, t) = \frac{1}{4\pi\epsilon_0} \int dt' \int d^3\mathbf{x}' \frac{\rho(\mathbf{x}', t')}{|\mathbf{x} - \mathbf{x}'|} \delta \left( t' + \frac{|\mathbf{x} - \mathbf{x}'|}{c} - t \right) \tag{106}$$

$$\mathbf{A}(\mathbf{x}, t) = \frac{\mu_0}{4\pi} \int dt' \int d^3\mathbf{x}' \frac{\mathbf{J}(\mathbf{x}', t')}{|\mathbf{x} - \mathbf{x}'|} \delta \left( t' + \frac{|\mathbf{x} - \mathbf{x}'|}{c} - t \right) \tag{107}$$

with $\rho(\mathbf{x}, t) = \rho(\mathbf{x}) e^{-i\omega t}$ and $\mathbf{J}(\mathbf{x}, t) = \mathbf{J}(\mathbf{x}) e^{-i\omega t}$

$$\mathbf{A}(\mathbf{x}, t) = \frac{\mu_0}{4\pi} e^{-i\omega t} \int d^3\mathbf{x}'\, \mathbf{J}(\mathbf{x}') \frac{e^{ik|\mathbf{x} - \mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|} \tag{108}$$

$$= \mathbf{A}(\mathbf{x}) e^{-i\omega t} \tag{109}$$

where $k = \omega / c$ then the fields are given by

$$\mathbf{H} = \frac{1}{\mu_0} \nabla \times \mathbf{A} \tag{110}$$

$$= e^{-i\omega t} \frac{1}{\mu_0} \nabla \times \mathbf{A}(\mathbf{x}) \tag{111}$$

and outside the source

$$\frac{\partial}{\partial t}\mathbf{D} = \epsilon_0 \frac{\partial}{\partial t}\mathbf{E} = \nabla \times \mathbf{H} = e^{-i\omega t} \nabla \times \mathbf{H}(\mathbf{x}) \tag{112}$$

$$\rightarrow \mathbf{E} = \frac{1}{-i\omega} \frac{1}{\epsilon_0} e^{-i\omega t} \nabla \times \mathbf{H}(\mathbf{x}) \tag{113}$$

$$= \frac{i}{k} \frac{1}{\epsilon_0 c} \nabla \times \mathbf{H} \tag{114}$$

$$= \frac{i}{k} \frac{\sqrt{\mu_0 \epsilon_0}}{\epsilon_0} \nabla \times \mathbf{H} \tag{115}$$

$$= \frac{i}{k} \sqrt{\frac{\mu_0}{\epsilon_0}} \nabla \times \mathbf{H} \tag{116}$$

Expressing this directly via the vector potential gives

$$\begin{aligned}
\mathbf{E} &= \frac{i}{k}\frac{1}{\epsilon_0 c}\nabla\times\mathbf{H} &(117)\\
&= \frac{i}{k}\frac{1}{\epsilon_0\mu_0 c}\nabla\times(\nabla\times\mathbf{A}) &(118)\\
&= \frac{ic}{k}[\nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}] &(119)\\
&= \frac{ic}{k}[\nabla(-\frac{1}{c^2}\frac{\partial\phi}{\partial t}) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A}] &(120)\\
&= -\frac{i}{kc}\frac{\partial}{\partial t}[\nabla\phi + \frac{\partial}{\partial t}\mathbf{A}] &(121)\\
&= -\frac{i}{kc}(-i\omega)[\nabla\phi + \frac{\partial}{\partial t}\mathbf{A}] &(122)\\
&= -[\nabla\phi + \frac{\partial}{\partial t}\mathbf{A}] &(123)
\end{aligned}$$

### 0.5.3 Multipole Radiation

$$\begin{aligned}
\mathbf{A}(\mathbf{x},t) &= \frac{\mu_0}{4\pi}\int dt'\int d^3\mathbf{x}'\frac{\mathbf{J}(\mathbf{x}',t')}{|\mathbf{x}-\mathbf{x}'|}\delta\left(c(t'-t)+|\mathbf{x}-\mathbf{x}'|\right) &(124)\\
&= \frac{\mu_0}{4\pi}\int dt'\int d^3\mathbf{x}'\left(\frac{1}{|\mathbf{x}|}+\frac{\mathbf{x}\cdot\mathbf{x}'}{|\mathbf{x}|^3}+...\right)\mathbf{J}(\mathbf{x}',t')\delta\left(c(t'-t)+|\mathbf{x}|-\mathbf{n}\cdot\mathbf{x}'+...\right) &(125)\\
&= \frac{\mu_0}{4\pi r}\int d^3\mathbf{x}'\left(1+\mathbf{n}\cdot\frac{\mathbf{x}'}{|\mathbf{x}|}+...\right)\mathbf{J}(\mathbf{x}',t-\frac{1}{c}|\mathbf{x}|+\frac{1}{c}\mathbf{n}\cdot\mathbf{x}'+...) &(126)\\
&= \frac{\mu_0}{4\pi r}\int d^3\mathbf{x}'\left(1+\mathbf{n}\cdot\frac{\mathbf{x}'}{|\mathbf{x}|}+...\right)\left[\mathbf{J}(\mathbf{x}',t-\frac{1}{c}|\mathbf{x}|)+\frac{1}{c}(\mathbf{n}\cdot\mathbf{x}')\partial_t\mathbf{J}(\mathbf{x}',t-\frac{1}{c}|\mathbf{x}|+..)\right] &(127)\\
&= \frac{\mu_0}{4\pi r}\int d^3\mathbf{x}'\left[\mathbf{J}(\mathbf{x}',t-\frac{1}{c}|\mathbf{x}|)+\frac{1}{c}(\mathbf{n}\cdot\mathbf{x}')\partial_t\mathbf{J}(\mathbf{x}',t-\frac{1}{c}|\mathbf{x}|)+..\right]+\mathbf{n}\cdot\frac{\mathbf{x}'}{|\mathbf{x}|}[...]+... &(128)\\
&= \frac{\mu_0}{4\pi r}\int d^3\mathbf{x}'\,\mathbf{J}(\mathbf{x}',t-\frac{1}{c}|\mathbf{x}|)+\frac{\mu_0}{4\pi cr}\int d^3\mathbf{x}'(\mathbf{n}\cdot\mathbf{x}')\partial_t\mathbf{J}(\mathbf{x}',t-\frac{1}{c}|\mathbf{x}|)|+... &(129)\\
&= \mathbf{A}_{\mathrm{ED}}(\mathbf{x},t)+\mathbf{A}_{\mathrm{MD/EQ}}(\mathbf{x},t) &(130)
\end{aligned}$$

Treating each dimension individually we can integrate by parts

$$\begin{aligned}
\mathbf{A}_{\mathrm{ED}}(\mathbf{x},t) &= \frac{\mu_0}{4\pi r}\int d^3\mathbf{x}'\,\mathbf{J}(\mathbf{x}',t-\frac{1}{c}|\mathbf{x}|) &(131)\\
&= -\frac{\mu_0}{4\pi r}\int d^3\mathbf{x}'\,\mathbf{x}'\nabla\cdot\mathbf{J}(\mathbf{x}',t-\frac{1}{c}|\mathbf{x}|) &(132)\\
&= -\frac{\mu_0}{4\pi r}\int d^3\mathbf{x}'\,\mathbf{x}'\dot{\rho}(\mathbf{x}',t-\frac{1}{c}|\mathbf{x}|) &(133)\\
&= -\frac{\mu_0}{4\pi r}\dot{\mathbf{p}}(t-\frac{1}{c}|\mathbf{x}|) &(134)
\end{aligned}$$

with

$$\begin{aligned}
\mathbf{J}(\mathbf{x},t) &= \mathbf{J}(\mathbf{x})e^{-i\omega t} &(135)\\
\mathbf{J}(\mathbf{x},t-\frac{1}{c}|\mathbf{x}-\mathbf{x}'|) &= \mathbf{J}(\mathbf{x})e^{-i\omega t}e^{ik|\mathbf{x}-\mathbf{x}'|} &(136)
\end{aligned}$$

$$\mathbf{A}_{\mathrm{MD/EQ}}(\mathbf{x},t) = \frac{\mu_0}{4\pi cr}\int d^3\mathbf{x}'(\mathbf{n}\cdot\mathbf{x}')\partial_t\mathbf{J}(\mathbf{x}',t-\frac{1}{c}|\mathbf{x}|) \qquad (137)$$

## 0.6 Light Scattering

1. Thomson
2. Rayleigh
3. Rayleigh-Gans
4. Anomalous diffraction approximation of van de Hulst
5. Mie scattering
6. Compton

## 0.7 Quantum Mechanics

### 0.7.1 Mathematical

Linear algebra

$$\langle x, y\rangle \equiv \bar{x}^T y \equiv (\bar{x}_1, ..., \bar{x}_n) \begin{pmatrix} y_1 \\ .. \\ y_n \end{pmatrix} \equiv \bar{x}_1 y_1 + ... + \bar{x}_n y_n \tag{138}$$

$$\langle Ax, y\rangle = (\overline{Ax})^T y = (\overline{A_{11}x_1 + ... + A_{1n}x_n})y_1 + ... + (\overline{A_{n1}x_1 + ... + A_{nn}x_n})y_n \tag{139}$$

$$= \bar{A}_{11}\bar{x}_1 y_1 + \bar{A}_{12}\bar{x}_2 y_1 + ... \tag{140}$$

$$\langle x, Ay\rangle = \bar{x}^T(Ay) = \bar{x}_1(A_{11}y_1 + ... + A_{1n}y_n) + ... + \bar{x}_n(A_{n1}y_1 + ... + A_{nn}y_n) \tag{141}$$

$$= A_{11}\bar{x}_1 y_1 + A_{21}\bar{x}_2 y_1 + ... \tag{142}$$

then with adjoint matrix $A^T$ (transpose + complex conjugated matrix of $A$)

$$\langle Ax, y\rangle = \langle x, A^T y\rangle \tag{143}$$

If $A = A^T$ we call the matrix (complex)-symmetric, hermitean or selfadjoint.
Unbounded (has no finite operator norm) Operators $A : D(A) \subseteq \mathcal{H} \to \mathcal{H}$

- symmetric, hermitean

$$\langle Ax, y\rangle = \langle x, A^T y\rangle \qquad \forall x, y \in \mathcal{H} \tag{144}$$

- selfadjoint is a stronger requirement because

$$A = A^T \qquad \to \qquad D(A) = D(A^T) \tag{145}$$

### 0.7.2 Pictures

Prelims - at $t = t_0$

$$|\psi_H\rangle = |\psi(t_0)\rangle \tag{146}$$

and obviously

$$U(t, t_0) = U^{-1}(t_0, t) \tag{147}$$

$$U^\dagger(t, t_0)U(t, t_0) = 1 \qquad \text{(probability conservation)} \tag{148}$$

1. Schroedinger - time dependency in the states

$$i\partial_t|\psi(t)\rangle = H|\psi(t)\rangle \tag{149}$$
$$|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle \tag{150}$$
$$i\partial_t U(t,t_0) = HU(t,t_0) \tag{151}$$
$$\frac{\partial H}{\partial t} = 0 \quad \rightarrow \quad U(t,t_0) = e^{-iH(t-t_0)} \tag{152}$$

Time evolution with $i\partial_t|\psi\rangle = H|\psi\rangle$

$$|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle \tag{153}$$
$$\simeq (1 - i(t-t_0)H)|\psi(t_0)\rangle \tag{154}$$
$$\simeq (1 - i(t-t_0)i\partial_t)|\psi(t_0)\rangle \tag{155}$$
$$\simeq |\psi(t_0)\rangle + \frac{\partial|\psi(t_0)\rangle}{\partial t}(t-t_0) \tag{156}$$

Time evolution with $H|E_k\rangle = E_k|E_k\rangle$

$$|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle \tag{157}$$
$$= U(t,t_0)\sum_k |E_k\rangle\langle E_k|\psi(t_0)\rangle \tag{158}$$
$$= \sum_k e^{-iH(t-t_0)}|E_k\rangle\langle E_k|\psi(t_0)\rangle \tag{159}$$
$$= \sum_k e^{-iE_k(t-t_0)}|E_k\rangle\langle E_k|\psi(t_0)\rangle \tag{160}$$

Measurement

$$\langle A(t)\rangle = \langle\psi(t)|A_S|\psi(t)\rangle \tag{161}$$

2. Heisenberg - time dependency in the operators

$$\langle A(t)\rangle = \langle\psi(t)|A_S|\psi(t)\rangle \tag{162}$$
$$= \langle\psi(t_0)|U^\dagger(t,t_0)A_S U(t,t_0)|\psi(t_0)\rangle \tag{163}$$
$$= \langle\psi(t_0)|A_H(t)|\psi(t_0)\rangle \tag{164}$$
$$\rightarrow A_H(t) = U^\dagger(t,t_0)A_S U(t,t_0) \tag{165}$$

Time derivative

$$\frac{d}{dt}A_H(t) = \left(\frac{d}{dt}U^\dagger(t,t_0)\right)A_S U(t,t_0) + U^\dagger(t,t_0)\left(\frac{d}{dt}A_S\right)U(t,t_0) + U^\dagger(t,t_0)A_S\left(\frac{d}{dt}U(t,t_0)\right) \tag{166}$$

$$= U^\dagger(t,t_0)i(HA_S - A_S H)U(t,t_0) + U^\dagger(t,t_0)\frac{\partial A_S}{\partial t}U(t,t_0) \tag{167}$$
$$= i[H,A_H] + \underbrace{U^\dagger(t,t_0)\frac{\partial A_S}{\partial t}U(t,t_0)}_{\equiv\frac{\partial A_H}{\partial t}} \tag{168}$$
$$= i[H,A_H] + \frac{\partial A_H}{\partial t} \tag{169}$$

3. Dirac - $H = H_0 + H_{\text{int}}$

$$|\psi(t)\rangle_D = e^{iH_0 t}|\psi(t)\rangle_S \tag{170}$$
$$A_D(t) = e^{iH_0 t} A_S e^{-iH_0 t} \tag{171}$$

then

$$\langle A(t)\rangle = \langle\psi(t)|A_S|\psi(t)\rangle \tag{172}$$
$$= \langle\psi(t_0)|U^\dagger(t,t_0)A_S U(t,t_0)|\psi(t_0)\rangle \tag{173}$$
$$= \langle\psi(t_0)|U^\dagger(t,t_0)\underbrace{U_0(t,t_0)U_0^\dagger(t,t_0)}_{=1} A_S \underbrace{U_0(t,t_0)U_0^\dagger(t,t_0)}_{=1} U(t,t_0)|\psi(t_0)\rangle \tag{174}$$
$$\to A_D = U_0^\dagger(t,t_0)A_S U_0(t,t_0) \tag{175}$$
$$\to |\psi_D(t)\rangle = U_0^\dagger(t,t_0)U(t,t_0)|\psi(t_0)\rangle = U_0^\dagger(t,t_0)|\psi(t)\rangle \tag{176}$$

Now calc evolution between the TWO Dirac states $|\psi_D(t_1)\rangle$ and $|\psi_D(t_2)\rangle$

$$|\psi_D(t_1)\rangle = U_0^\dagger(t_1,t_0)U(t_1,t_0)|\psi(t_0)\rangle \tag{177}$$
$$|\psi_D(t_2)\rangle = U_0^\dagger(t_2,t_0)U(t_2,t_0)|\psi(t_0)\rangle \tag{178}$$
$$= U_0^\dagger(t_2,t_0)U(t_2,t_0)\left(U_0^\dagger(t_1,t_0)U(t_1,t_0)\right)^{-1}|\psi_D(t_1)\rangle \tag{179}$$
$$= U_0^\dagger(t_2,t_0)U(t_2,t_0)U^{-1}(t_1,t_0)\left(U_0^\dagger(t_1,t_0)\right)^{-1}|\psi_D(t_1)\rangle \tag{180}$$
$$= U_0^\dagger(t_2,t_0)U(t_2,t_0)U(t_0,t_1)U_0^\dagger(t_1,t_0)|\psi_D(t_1)\rangle \tag{181}$$
$$= U_0^\dagger(t_2,t_0)U(t_2,t_1)U_0^\dagger(t_1,t_0)|\psi_D(t_1)\rangle \tag{182}$$

with $t_0 = 0$ and $H_0$ time-independent

$$U_D(t_2,t_1) = U_0^\dagger(t_2,0)U(t_2,t_1)U_0^\dagger(t_1,0)|\psi_D(t_1)\rangle \tag{183}$$
$$= e^{iH_0 t_2}U(t_2,t_1)e^{iH_0 t_1} \tag{184}$$

| picture | equation | state | operator |
|---|---|---|---|
| Schroedinger | $i\partial_t|\psi(t)\rangle_S = H_0|\psi(t)\rangle_S$ | $|\psi(t)\rangle_S = e^{-iH_0(t-t_0)}|\psi(t_0)\rangle_S$ | $A_S(t) = A_S$ |
| Heisenberg | $\frac{d}{dt}A_H = \partial_t A_H + i[H_0, A_H]$ | $|\psi(t)\rangle_H = |\psi(t_0)\rangle_S$ | $A_H(t) = e^{iH_0(t-t_0)}A_H(t_0)e^{-iH_0(t-t_0)}$ |
| Dirac | $i\partial_t|\psi(t)\rangle_D = H_I|\psi(t)\rangle_D$ | $|\psi(t)\rangle_D = e^{+iH_0(t-t_0)}|\psi(t_0)\rangle_D$ | $A_D(t) = e^{iH_0(t-t_0)}A_S e^{-iH_0(t-t_0)}$ |

where

$$|\psi(t_0)\rangle_S = |\psi\rangle_H = |\psi(t_0)\rangle_D \tag{185}$$
$$A_S = A_H(t_0) = A_D(t_0) \tag{186}$$
$$H = H_0 + H_{\text{int}} \qquad H_I = (H_{\text{int}})_D = e^{iH_0(t-t_0)}H_{\text{int}}e^{-iH_0(t-t_0)} \tag{187}$$

### 0.7.3 3D Spherical well

$$\left\{-\frac{\hbar^2}{2m}\triangle + V(r)\right\}\psi = E\psi \tag{188}$$
$$\left\{-\frac{\hbar^2}{2m}\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \frac{1}{r^2}\triangle_{\phi\theta}\right] + V(r)\right\}\psi = E\psi \tag{189}$$
$$\left\{\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \frac{1}{r^2}\triangle_{\phi\theta} - \frac{2m[V(r)-E]}{\hbar^2}\right\}\psi = 0 \tag{190}$$

Separation $\psi = R(r)Y(\phi,\theta)$

$$\frac{r^2\left\{\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{2m[V(r)-E]}{\hbar^2}\right\}R(r)}{R(r)} = l(l+1) = -\frac{\triangle_{\phi,\theta}Y(\phi,\theta)}{Y(\phi,\theta)} \tag{191}$$

$$\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{l(l+1)}{r^2}-\frac{2m[V(r)-E]}{\hbar^2}\right)R(r) = 0 \tag{192}$$

With the definition of the well potential

$$V(r) = \begin{cases} -V_0 & r < a \\ 0 & r > a \end{cases} \tag{193}$$

With $-V_0 < E < 0$

$$k = \frac{\sqrt{2m[E+V_0]}}{\hbar} \tag{194}$$

$$\kappa = \frac{\sqrt{2m(-E)}}{\hbar} \tag{195}$$

be have with $\rho = kr$ and $\rho = i\kappa r$

$$\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{l(l+1)}{r^2}+\begin{pmatrix} k^2 \\ -\kappa^2 \end{pmatrix}\right]R(r) = 0 \tag{196}$$

$$\left[\frac{d^2}{d\rho^2}+\frac{2}{\rho}\frac{d}{d\rho}-\frac{l(l+1)}{\rho^2}+1\right]R(\rho) = 0 \tag{197}$$

$$\left[\rho^2\frac{d^2}{d\rho^2}+2\rho\frac{d}{d\rho}+\rho^2-l(l+1)\right]R(\rho) = 0 \tag{198}$$

Independent solutions

$$R(\rho) = Aj_l(\rho) + By_l(\rho) \tag{199}$$

$$= A\sqrt{\frac{\pi}{2\rho}}J_{l+1/2}(\rho) + B\sqrt{\frac{\pi}{2\rho}}Y_{l+1/2}(\rho) \tag{200}$$

Here the requirements

- regular at the origin with $R(r) \sim r^l$
- continous and differentiable at $r = a$
- exponential decay outside to ensure normalizability

and here a quick overview of the two functions and a special linear combination

$$j_l(x) = (-x)^l\left(\frac{1}{x}\frac{d}{dx}\right)^l\frac{\sin x}{x} \qquad y_l(x) = -(-x)^l\left(\frac{1}{x}\frac{d}{dx}\right)^l\frac{\cos x}{x} \qquad h_0^{(1)}(x) = j_l(ix) + iy_l(ix)$$

$$j_0(x) = \frac{\sin x}{x} \qquad y_0(x) = -\frac{\cos x}{x} \qquad h_0^{(1)}(x) = -\frac{e^{-x}}{x}$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} \qquad y_1(x) = -\frac{\cos x}{x} - \frac{\sin x}{x} \qquad h_1^{(1)}(x) = i(1+x)\frac{e^{-x}}{x^2}$$

$$J_2(x) = ... \qquad y_l(x) = ... \qquad h_2^{(1)}(x) = (x^2+3x+3)\frac{e^{-x}}{x^3}$$

We see that $j_l$ is suitable for the inside and $h_l^{(1)}$ for the outside.

$$R(\rho) = \begin{cases} Aj_l(\rho) & r < a \\ Ch_l^{(1)}(\rho) & r > a \end{cases} \tag{201}$$

Now $l = 0$

$$A j_0(\rho = ka) = C h_0^{(1)}(\rho = \kappa a) \quad \rightarrow \quad A\frac{\sin ka}{ka} = -C\frac{e^{-\kappa a}}{\kappa a} \tag{202}$$

$$A \partial_r j_0(\rho = ka) = C \partial_r h_0^{(1)}(\rho = \kappa a) \quad \rightarrow \quad A\frac{\sin ak}{a}\left(\cot ka - \frac{1}{ka}\right) = C\frac{e^{-\kappa a}}{a}\left(1 + \frac{1}{\kappa a}\right) \tag{203}$$

By substituting first into the second equation we kick out $A$ and $C$ and obtain

$$\cot ka = -\frac{\kappa}{k} \tag{204}$$

$$\cot\sqrt{\frac{2ma^2}{\hbar^2}[E + V_0]} = -\sqrt{\frac{-E}{E + V_0}} \tag{205}$$

Now $l = 1$

$$A j_1(\rho = ka) = C h_1^{(1)}(\rho = \kappa a) \quad \rightarrow \quad A\left(-\frac{\cos ka}{ka} + \frac{\sin ka}{k^2a^2}\right) = iC\frac{e^{-\kappa a}}{\kappa^2 a^2}(1 + \kappa a) \tag{206}$$

$$A \partial_r j_1(\rho = ka) = C \partial_r h_1^{(1)}(\rho = \kappa a) \quad \rightarrow \quad A\left(2\frac{\cos ka}{ka^2} + \frac{\sin ka}{k^2a^3}(a^2k^2 - 2)\right) = -iC\frac{e^{-\kappa a}}{\kappa^2 a^3}(\kappa^2 a^2 + 2\kappa a + 2) \tag{207}$$

Then

$$\cot ka = \frac{k^2 + ak^2\kappa + \kappa^2}{ak\kappa^2} \tag{208}$$

## 0.8 Quantum statistics

Quick thermodynamics review

$$\text{1st law} \quad dU = \delta Q + \delta W \tag{209}$$

$$\text{2nd law} \quad dS = dS_i + \frac{\delta Q}{T}, \qquad dS_i > 0 \tag{210}$$

$$\text{Gibbs Fund.Form} \rightarrow dS = \frac{1}{T}dU - \frac{1}{T}\delta W = \frac{1}{T}dU + \frac{1}{T}\sum_i y_i dX_i \tag{211}$$

$$\rightarrow \left.\frac{dS}{dU}\right|_{X_i} = \frac{1}{T} \qquad \rightarrow \qquad U = U(T, X_i) \tag{212}$$

$$\rightarrow \left.\frac{dS}{dX_i}\right|_{U, X_j} = \frac{y_i}{T} \qquad \rightarrow \qquad y_i = y_i(T, X_j) \tag{213}$$

### 0.8.1 Microcanonical ensemble

Macroscopic equilibrium state is defined by $E, N, V$:

| | | |
|---|---|---|
| Sirling Formula | $n! \simeq (n/e)^n \sqrt{2\pi n}$ | (214) |
| | $\ln n! \simeq (n+\frac{1}{2})\ln n - n + \frac{1}{2}\ln(2\pi)$ | (215) |
| phasespace element | $D\Gamma = \frac{1}{h^{3N}} \prod_\alpha dp_\alpha dq_\alpha$ | (216) |
| phasespace element (identical part) | $D\Gamma = \frac{1}{N! h^{3N}} \prod_\alpha dp_\alpha dq_\alpha$ | (217) |
| phasespace volume | $\Gamma(E,V,N) = \int_{H(q_\alpha,q_\alpha)\leq E} D\Gamma$ | (218) |
| micro states in$[E, E+\Delta E]$ | $\Omega = \left(\frac{\partial\Gamma}{\partial E}\right)_{N,V} \Delta E$ | (219) |
| phasespace, prob. density | $\Omega^{-1} = \rho = \frac{\delta(H(q_\alpha,p_\alpha)-E)}{\int D\Gamma\delta(H(q_\alpha,p_\alpha)-E)}$ | (220) |
| entropy of the system | $S = k\ln\Omega = -k\overline{\ln\rho} = -k\int D\Gamma\rho\ln\rho$ | (221) |
| inner Energy | $U = E$ | (222) |
| temperature | $\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{E,N}$ | (223) |
| pressure | $\frac{p}{T} = \left(\frac{\partial S}{\partial V}\right)_{E,N}$ | (224) |
| chemical potential | $-\frac{\mu}{T} = \left(\frac{\partial S}{\partial N}\right)_{E,N}$ | (225) |
| | | (226) |

### 0.8.2 Canonical ensemble

Macroscopic equilibrium state is defined by $T, N, V$ (system can exchange energy with external reservoir - but system + reservoir is microcanonical ensemble):

| | | |
|---|---|---|
| State integral | $Z = \int D\Gamma \exp[-\frac{H}{kT}]$ | (227) |
| phasespace, prob. density | $\rho = \frac{1}{Z}\exp[-\frac{H}{kT}]$ | (228) |
| discrete | $Z = \sum_i \exp[-\frac{E_i}{kt}], \quad p_i = \frac{1}{Z}\sum_i \exp[-\frac{E_i}{kt}]$ | (229) |
| entropy | $S = -k\int D\Gamma\rho\ln\rho = -k\int D\Gamma\rho(-\frac{H}{kT} - \ln Z) = \frac{1}{T}\bar{H} + k\ln Z$ | (230) |
| free energy | $F = U - TS = -kT\ln Z$ | (231) |

### 0.8.3 Great canonical ensemble

| | | |
|---|---|---|
| ... | $\mathcal{Z} = \sum_N \int D\Gamma \exp[-\frac{H_N - \mu N}{kT}]$ | (232) |
| ... | $\rho_N = \frac{1}{\mathcal{Z}}\int D\Gamma \exp[-\frac{H_N-\mu N}{kT}]$ | (233) |
| discrete | $\mathcal{Z} = \sum_N \sum_i \exp[-\frac{E_i-\mu N}{kT}], \quad p_{N,i} = \frac{1}{\mathcal{Z}}\exp[-\frac{E_i-\mu N}{kT}]$ | (234) |
| entropy | $S = -k\overline{\ln\rho_N} = -k\sum_N \int D\Gamma\rho_N\ln\rho_N$ | (235) |
| great canonical potential | $\mathcal{F} = U - TS - \mu\overline{N} = -kT\ln\mathcal{Z}$ | (236) |

### 0.8.4 Density matrix - statistical operator

Using the principle of equal probability

$$\hat{\varrho} = \sum_k p_k |\Psi_k\rangle\langle\Psi_k| \tag{237}$$

$$= \frac{1}{\Omega}\sum_k |\Psi_k\rangle\langle\Psi_k| \tag{238}$$

$$\mathrm{Tr}\hat{\varrho} = 1 \tag{239}$$

$$
\begin{aligned}
S &= -k\langle \hat{\varrho} \rangle && (240)\\
&= -k\mathrm{Tr}(\hat{\varrho} \log \hat{\varrho}) && (241)\\
& && (242)
\end{aligned}
$$

### 0.8.5 Canonical ensemble

Represents all states of a system in thermodynamic equilibrium. Meaning the temperature $T$ and therefore the mean energy $\bar{E} = U$ is fixed but the total energy can fluctuate

$$
\begin{aligned}
Z &= \mathrm{Tr}\left[\exp\left(-\frac{\hat{H}}{kT}\right)\right] && (243)\\
\hat{\varrho} &= \frac{1}{Z(T)} \exp\left(-\frac{\hat{H}}{kT}\right) = \frac{1}{Z(T)} \sum_k |\Psi_k\rangle \exp\left(-\frac{E_k}{kT}\right) \langle \Psi_k| && (244)\\
F &= -kT \log Z && (245)\\
\frac{\partial F}{\partial T} &= -S && (246)\\
U &= F + TS && (247)
\end{aligned}
$$

### 0.8.6 Great Canonical ensemble

Represents all states of a system in thermodynamic equilibrium. Meaning the temperature $T$ and therefore the mean energy $\bar{E} = U$ is fixed but the total energy can fluctuate

$$
\begin{aligned}
\mathcal{Z} &= \mathrm{Tr}\left[\exp\left(-\frac{\hat{H} - \mu\hat{N}}{kT}\right)\right] && (248)\\
\hat{\varrho} &= \frac{1}{\mathcal{Z}(T)} \exp\left(-\frac{\hat{H} - \mu\hat{N}}{kT}\right) && (249)\\
\mathcal{F} &= -kT \log \mathcal{Z} && (250)\\
\left(\frac{\partial \mathcal{F}}{\partial T}\right)_\mu &= -S \qquad \left(\frac{\partial \mathcal{F}}{\partial \mu}\right)_T = -\bar{N} = -\langle \hat{N} \rangle && (251)
\end{aligned}
$$

## 0.9 Special relativity

Definition of line element

$$
\begin{aligned}
ds^2 &= dx^\mu dx_\nu = \eta_{\mu\nu} dx^\mu dx^\nu && (252)\\
&= dx^T \eta dx && (253)
\end{aligned}
$$

Definition of Lorentz transformation

$$
dx^\mu = \Lambda^\mu{}_\nu dx^\nu \qquad (254)
$$

By postulate the line element $ds$ is invariant under Lorentz transformation

$$
\begin{aligned}
ds^2 &= \eta_{\mu\nu} dx^\mu dx^\nu && (255)\\
&\stackrel{!}{=} \eta_{\alpha\beta} \Lambda^\alpha{}_\mu dx^\mu \Lambda^\beta{}_\nu dx^\nu \quad \to \quad \eta_{\mu\nu} = \eta_{\alpha\beta} \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu && (256)
\end{aligned}
$$

or analog

$$
\begin{aligned}
ds^2 &= dx^T \eta dx && (257)\\
&\stackrel{!}{=} (\Lambda dx)^T \eta (\Lambda dx) && (258)\\
&= dx^T \Lambda^T \eta \Lambda dx \quad \rightarrow \quad \eta = \Lambda^T \eta \Lambda && (259)
\end{aligned}
$$

Observation with the eigentime $d\tau = ds/c$ and 3-velocity $dx^i = v^i dt$

$$
\begin{aligned}
\frac{ds^2}{d\tau^2} = c^2 &= c^2 \frac{dt^2}{d\tau^2} - \frac{dx^i}{dt}\frac{dx_i}{dt}\left(\frac{dt}{d\tau}\right)^2 && (260)\\
1 &= \frac{dt^2}{d\tau^2}\left(1 - \frac{v^i v_i}{c^2}\right) \quad \rightarrow \quad \frac{dt}{d\tau} \equiv \gamma = \left(\sqrt{1-\frac{v^2}{c^2}}\right)^{-1} && (261)
\end{aligned}
$$

### 0.9.1 Definition 4-velocity

with 3-velocity $d\vec{x} = \vec{v} dt$

$$
\begin{aligned}
u^\mu \equiv \frac{dx^\mu}{d\tau} &= \frac{dx^\mu}{dt}\frac{dt}{d\tau} = \quad \rightarrow \quad u^\mu u_\mu = \eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = \frac{ds^2}{d\tau^2} = c^2 && (262)\\
&= (c, \vec{v})\gamma && (263)
\end{aligned}
$$

Object moving in $x$ direction with $v$ meaning $dx = v \cdot dt$ compared to rest frame $dx' = 0$

$$
\begin{aligned}
c^2 dt'^2 = ds^2 &= c^2 dt^2 - v^2 dt^2 && (264)\\
&= c^2 dt^2 \left(1 - \frac{v^2}{c^2}\right) && (265)\\
dt' = \frac{ds}{c} \equiv d\tau &= dt\sqrt{1-\frac{v^2}{c^2}} = \frac{dt}{\gamma} && (266)
\end{aligned}
$$

### 0.9.2 Definition 4-momentum

using the 3-momentum $\vec{p} = \gamma m \vec{v}$

$$
\begin{aligned}
p^\mu \equiv m u^\mu = (\gamma m c, \gamma m \vec{v}) = \left(\frac{E_p}{c}, \vec{p}\right) \quad &\rightarrow \quad p^\mu p_\mu = m^2 u^\mu u_\mu = m^2 c^2 && (267)\\
&\rightarrow \quad (p^0)^2 - p^i p_i = m^2 c^2 && (268)\\
&\rightarrow \quad p^0 = \sqrt{m^2c^2 + \vec{p}^2} && (269)\\
&\rightarrow \quad E_p = \sqrt{m^2c^4 + \vec{p}^2c^2} && (270)\\
&\qquad\quad = \frac{mc^2}{\sqrt{1-\frac{\vec{v}^2}{c^2}}} && (271)
\end{aligned}
$$

### 0.9.3 Definition 4-acceleration

First observe

$$
\begin{aligned}
u^\mu u_\mu &= c^2 && (272)\\
\frac{d}{d\tau}(u^\mu u_\mu) &= 0 && (273)\\
\rightarrow \alpha^\mu u_\mu &= 0 && (274)
\end{aligned}
$$

meaning

$$\alpha^0 u_0 - \vec{\alpha}\cdot\vec{u} = 0 \tag{275}$$
$$\gamma(\alpha^0 c - \vec{\alpha}\cdot\vec{v}) = 0 \tag{276}$$
$$\rightarrow \alpha^0 = \frac{\vec{\alpha}\cdot\vec{v}}{c} \tag{277}$$

$$\frac{d^2x^\mu}{d\tau^2} = \frac{d}{d\tau}\frac{dx^\mu}{d\tau} \tag{278}$$
$$= \frac{d}{d\tau}\left(\frac{dx^\mu}{dt}\frac{dt}{d\tau}\right) \tag{279}$$
$$\vec{\alpha} = \frac{d^2x^k}{d\tau^2} = \frac{d^2x^k}{dt^2}\left(\frac{dt}{d\tau}\right)^2 + \frac{dx^k}{dt}\frac{d^2t}{d\tau^2} \tag{280}$$
$$\equiv a^k\gamma^2 + v^k\frac{d\gamma}{d\tau} \tag{281}$$
$$= a^k\gamma^2 + v^k\frac{d\gamma}{dt}\frac{dt}{d\tau} \tag{282}$$
$$= a^k\gamma^2 + v^k\left(-\frac{1}{2}\right)\gamma^3\frac{-2v^\alpha\frac{dv^\alpha}{dt}}{c^2}\frac{dt}{d\tau} \tag{283}$$
$$= a^k\gamma^2 + v^k\gamma^4(\vec{v}\cdot\vec{a})\frac{1}{c^2} \tag{284}$$
$$\alpha^0 = \frac{d^2x^0}{d\tau^2} = \frac{d^2x^0}{dt^2}\left(\frac{dt}{d\tau}\right)^2 + \frac{dx^0}{dt}\frac{d^2t}{d\tau^2} \tag{285}$$
$$= 0\cdot\gamma^2 + c\gamma^4(\vec{v}\cdot\vec{a})\frac{1}{c^2} \tag{286}$$
$$= \gamma^4(\vec{v}\cdot\vec{a})\frac{1}{c} \tag{287}$$

we see after a short calculation (in the initial restframe) $\alpha^\mu\alpha_\mu = -\vec{a}^2 \equiv a_0^2$ ($a_0$ proper acceleration in the restframe) where

$$\text{4-velocity} \quad u^\mu = \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt}\frac{dt}{d\tau} \tag{288}$$
$$\text{3-velocity} \quad v^k = \frac{dx^k}{dt} \tag{289}$$
$$\text{4-acceleration} \quad \alpha^\mu = \frac{d^2x^\mu}{d\tau^2} = \frac{du^\mu}{d\tau} = \frac{du^\mu}{dt}\frac{dt}{d\tau} \tag{290}$$
$$\text{3-acceleration} \quad a^k = \frac{d^2x^k}{dt} = \frac{dv^k}{dt} \tag{291}$$

First we observe

$$\eta_{\mu\nu} = \eta_{\alpha\beta}\Lambda^\alpha{}_\mu\Lambda^\beta{}_\nu \tag{292}$$
$$\det(\eta) = \det(\Lambda)^2\det(\eta) \tag{293}$$
$$1 = \det(\Lambda)^2. \tag{294}$$

Now we see

$$\Lambda_\gamma{}^\nu\Lambda^\gamma{}_\mu = \eta_{\alpha\gamma}\eta^{\nu\beta}\Lambda^\alpha{}_\beta\Lambda^\gamma{}_\mu \tag{295}$$
$$= \eta^{\nu\beta}(\eta_{\alpha\gamma}\Lambda^\alpha{}_\beta\Lambda^\gamma{}_\mu) \tag{296}$$
$$= \eta^{\nu\beta}\eta_{\beta\mu} \tag{297}$$
$$= \delta^\nu{}_\mu \tag{298}$$

which means in matrix notation $\Lambda_\gamma^{\ \nu} = (\Lambda^{-1})^\nu_{\ \gamma}$. General transformation laws for tensors of first order

$$V'^\alpha = \Lambda^\alpha_{\ \beta} V^\beta \tag{299}$$
$$\eta_{\alpha\mu} V'^\alpha = \eta_{\alpha\mu}\Lambda^\alpha_{\ \beta} V^\beta = \eta_{\alpha\mu}\Lambda^\alpha_{\ \beta}(\eta^{\nu\beta}V_\nu) \tag{300}$$
$$V'_\mu = \Lambda_\mu^{\ \nu} V_\nu \tag{301}$$
$$\rightarrow \quad \Lambda_\mu^{\ \nu} = \eta_{\alpha\mu}\eta^{\nu\beta}\Lambda^\alpha_{\ \beta} \tag{302}$$

and second order

$$T'^{\alpha\beta} = \Lambda^\alpha_{\ \mu}\Lambda^\beta_{\ \nu} T^{\mu\nu} \tag{303}$$
$$\eta_{\alpha\delta}\eta_{\beta\gamma} T'^{\alpha\beta} = \eta_{\alpha\delta}\eta_{\beta\gamma}\Lambda^\alpha_{\ \mu}\Lambda^\beta_{\ \nu} T^{\mu\nu} = \eta_{\alpha\delta}\eta_{\beta\gamma}\Lambda^\alpha_{\ \mu}\Lambda^\beta_{\ \nu}(\eta^{\mu\rho}\eta^{\nu\sigma}T_{\rho\sigma}) \tag{304}$$
$$T'_{\delta\gamma} = \Lambda_\delta^{\ \rho}\Lambda_\gamma^{\ \sigma} T_{\rho\sigma}. \tag{305}$$

The general transformation is therefore given by

$$T'_{\mu_1\mu_2\ldots}{}^{\nu_1\nu_2\ldots} = \Lambda_{\mu_1}{}^{\rho_1}\Lambda_{\mu_2}{}^{\rho_2}\ldots\Lambda^{\nu_1}{}_{\sigma_1}\Lambda^{\nu_2}{}_{\sigma_2}\ldots T'_{\rho_1\rho_2\ldots}{}^{\sigma_1\sigma_2\ldots} \tag{306}$$

There exist two invariant tensors

$$\eta'_{\mu\nu} = \eta_{\alpha\beta}\Lambda^\alpha_{\ \mu}\Lambda^\beta_{\ \nu} = \Lambda_{\beta\mu}\Lambda^\beta_{\ \nu} = \eta_{\mu\sigma}\Lambda_\beta^{\ \sigma}\Lambda^\beta_{\ \nu} = \eta_{\mu\sigma}\delta^\sigma_{\ \nu} = \eta_{\mu\nu} \tag{307}$$
$$\epsilon'^{\mu\nu\rho\sigma} = \Lambda^\mu_{\ \alpha}\Lambda^\nu_{\ \beta}\Lambda^\rho_{\ \gamma}\Lambda^\sigma_{\ \delta}\epsilon'^{\alpha\beta\gamma\delta} \equiv \epsilon^{\mu\nu\rho\sigma}\det(\Lambda) = \pm\epsilon^{\mu\nu\rho\sigma} \tag{308}$$

Due to the possibility of the minus sign the Levi-Civita symbol $\epsilon$ is sometimes called pseudo-tensor.

## 0.10 Hydrodynamics

With $\rho = m/V$ we use mass conservation

$$\frac{\partial}{\partial t}m_V = \frac{\partial}{\partial t}\int_V \rho\, dV = -\oint_{\partial V} \mathbf{j}\cdot d\mathbf{A} \tag{309}$$
$$= -\oint_{\partial V} \rho\mathbf{u}\cdot d\mathbf{A} \tag{310}$$
$$= -\int_V \nabla\cdot(\rho\mathbf{u})\cdot dV \tag{311}$$
$$\rightarrow \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0 \tag{312}$$
$$\rightarrow \frac{\partial\rho}{\partial t} + \mathbf{u}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{u} = 0 \tag{313}$$
$$\overset{\rho=\text{const}}{\rightarrow} \nabla\cdot\mathbf{u} = 0 \tag{314}$$

We use Newtons 3. law

$$\frac{d\mathbf{p}}{dt} = \mathbf{F} \tag{315}$$
$$m\frac{d\mathbf{u}}{dt} + \mathbf{u}\frac{dm}{dt} = -\oint p\, d\mathbf{A} \tag{316}$$
$$m\left(\frac{\partial\mathbf{u}}{\partial t} + \frac{\partial x^i}{\partial t}\frac{\partial\mathbf{u}}{\partial x^i}\right) + \mathbf{u}\frac{dm}{dt} = -\int \nabla p\, dV \tag{317}$$
$$m\left(\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) + \mathbf{u}\frac{dm}{dt} = -\nabla p\, V \tag{318}$$
$$\rho\left(\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) + \frac{1}{V}\mathbf{u}\frac{dm}{dt} = -\nabla p \tag{319}$$

## 0.11 Nonrelativistiv Magnetohydrodynamics

Ingredience

- Maxwell equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{320}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{321}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{322}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \tag{323}$$

- Ohms law in fluid local rest (usually accelerated) frame

$$\mathbf{j}' = \kappa \mathbf{E}' \tag{324}$$

- Lorentz transformation with $\hat{\mathbf{v}} = \mathbf{v}/v$

$$\mathbf{E}' = \gamma\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) - (\gamma - 1)(\mathbf{E} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}} \tag{325}$$

$$\mathbf{B}' = \gamma\left(\mathbf{B} - \frac{1}{c^2}\mathbf{v} \times \mathbf{E}\right) - (\gamma - 1)(\mathbf{B} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}} \tag{326}$$

$$\mathbf{j}' = \mathbf{j} - \gamma\rho\mathbf{v} + (\gamma - 1)(\mathbf{j} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}} \tag{327}$$

$$\rho' = \gamma\left(\rho - \frac{1}{c^2}\mathbf{j} \cdot \mathbf{v}\right) \tag{328}$$

- Assumptions $v/c \ll 1$ meaning $\gamma = 1$ and $\kappa$ is high

Conclusion using $v/c \ll 1$

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B} \tag{329}$$

$$\mathbf{B}' = \mathbf{B} - \frac{1}{c^2}\mathbf{v} \times \mathbf{E} \tag{330}$$

$$\mathbf{j}' = \mathbf{j} - \rho\mathbf{v} \tag{331}$$

$$\rho' = \rho - \frac{1}{c^2}\mathbf{j} \cdot \mathbf{v} \tag{332}$$

High $\kappa$ implies $E' \ll E$ and therefore

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} \simeq 0 \quad \rightarrow \quad E \sim vB \tag{333}$$

$$\mathbf{B}' \simeq \mathbf{B} - \frac{1}{c^2}\mathbf{v} \times \mathbf{E} \overset{E \sim vB}{\simeq} \mathbf{B} - \mathcal{O}(v^2/c^2) \tag{334}$$

as well as $\rho' \ll \rho$.

From Amprere Law

$$\nabla \times \mathbf{B} - \mu_0 \mathbf{j} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \sim \frac{1}{c^2}\frac{E}{T} \sim \frac{1}{c^2}E\frac{v}{L} \overset{E \sim vB}{\sim} \frac{1}{c^2}vB\frac{v}{L} \sim \mathcal{O}(v^2/c^2) \tag{335}$$

$$= 0 \tag{336}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} \tag{337}$$

$$\frac{B}{L} \sim \mu_0 j \tag{338}$$

then

$$\rho \overset{\text{Gauss}}{\simeq} \epsilon_0 \frac{E}{L} \overset{E\sim vB}{\simeq} \epsilon_0 \frac{vB}{L} \overset{\text{Ampere}}{\simeq} \epsilon_0 \mu_0 v j \simeq \frac{v}{c^2} j \tag{339}$$

therefore

$$\mathbf{j}' = \mathbf{j} - \rho \mathbf{v} \overset{\rho\sim jv/c^2}{\simeq} \mathbf{j} - \mathcal{O}(v^2/c^2) \tag{340}$$

and with

$$\mathbf{j}' = \kappa \mathbf{E}' \tag{341}$$
$$= \kappa(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{342}$$

we have

$$\mathbf{j} = \kappa(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{343}$$

And

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} = \mu_0 \kappa(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{344}$$
$$\to \mathbf{E} = \frac{1}{\mu_o \kappa} \nabla \times \mathbf{B} - \mathbf{v} \times \mathbf{B} \tag{345}$$
$$\frac{\rho}{\epsilon_0} = \nabla \cdot \mathbf{E} = -\nabla \cdot (\mathbf{v} \times \mathbf{B}) \tag{346}$$
$$\to \rho = -\epsilon_0 \nabla \cdot (\mathbf{v} \times \mathbf{B}) \tag{347}$$
$$\tag{348}$$

Now

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \tag{349}$$
$$= -\nabla \times \left( \frac{1}{\mu_o \kappa} \nabla \times \mathbf{B} - \mathbf{v} \times \mathbf{B} \right) \tag{350}$$
$$= -\frac{1}{\mu_o \kappa} \nabla \times \nabla \times \mathbf{B} + \nabla \times (\mathbf{v} \times \mathbf{B}) \tag{351}$$
$$= \frac{1}{\mu_o \kappa} \triangle \mathbf{B} + \nabla \times (\mathbf{v} \times \mathbf{B}) \tag{352}$$

## 0.12 Perturbation theory

1. Find a hard problem
2. Introduce an $\epsilon$
3. Assume the solution can be expressed as a power series $x_s = \sum_k a_k \epsilon^k$
4. Find all $a_k$ and sum them up
5. Set $\epsilon = 1$

Now consider solving $x^5 + x = 1$

$$x^5 + \epsilon x = 1 \tag{353}$$
$$\to x = 1 - \frac{1}{5}\epsilon - \frac{1}{25}\epsilon^2 - \frac{1}{125}\epsilon^3 + 0\epsilon^4 + \frac{21}{15625}\epsilon^5 + ... \tag{354}$$

or

$$\epsilon x^5 + x = 1 \tag{355}$$
$$\rightarrow x = 1 - \epsilon + 5\epsilon^2 - 35\epsilon^3 + 285\epsilon^4 - 2530\epsilon^5 + ... \tag{356}$$

Method of dominant balance

- Asymptotics $f(x)$ $g(x)$ for $x \to x_0$

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1 \tag{357}$$

- Neglectable $f(x) \ll g(x)$ for $x \to x_0$

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0 \tag{358}$$

### 0.12.1 Series summation

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - ... = \frac{\pi}{4} \tag{359}$$
$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - ... = \log 2 \tag{360}$$
$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} - ... = \frac{\pi^2}{6} \tag{361}$$

- Consider converging series, meaning

$$A_n = \sum_{m=0}^{n} a_m, \qquad A = \sum_{m=0}^{\infty} a_m, \tag{362}$$

- Shanks summation

$$S(A_n) = \frac{A_{n+1}A_{n-1} - A_n^2}{A_{n+1} - 2A_n + A_{n-1}} \tag{363}$$

usually $S(A_n)$ converges faster than $A_n$. Further speed-up $S(S(...(A_n))$