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1 Advanced Topics in Gravity – Exercise sheet 3 - 2025-05-27

1.1 Exercise 1 - Killing vectors in Kerr spacetime

1.) Check that Kerr spacetime has two Killing vector fields that can be expressed as

$$\xi = \partial_t \quad \text{and} \quad \psi = \partial_\varphi . \tag{1}$$

2.) Show that the hypersurfaces defined by $r = r_\pm$ are Killing horizons of the Killing vectors

$$\chi_\pm = \xi + \Omega_H \psi, \tag{2}$$

where $\Omega_H = \frac{a}{2mr_\pm}$ is the angular velocity.

3.) Show that the Killing vector $\chi_+ = \xi + \Omega_{H+}\psi$ is spacelike in the ergosphere region $r_+ < r < r_{E+}$.

Note: A Killing horizon is a null hypersurface endowed with a normal Killing vector, which is null at the hypersurface.

1. First lets make some general statement about Killing vectors:

   Let x^μ be a coordinate chart on a spacetime manifold $\mathcal{M}$. A general Killing vector field ξ needs to satisfy the Killing equation:

   $$(\mathcal{L}_\xi g)_{\mu\nu} = \xi^\sigma \partial_\sigma g_{\mu\nu} + g_{\sigma\nu}\partial_\mu \xi^\sigma + g_{\mu\sigma}\partial_\nu \xi^\sigma = 0 \tag{3}$$

   where $\mathcal{L}_\xi$ denotes the Lie derivative with respect to ξ. In case of a coordinate vector $\xi = \partial_\lambda$, we have $\xi^\sigma = \delta^\sigma_\lambda$, and thus:

   $$\partial_\mu \xi^\sigma = \partial_\nu \xi^\sigma = 0. \tag{4}$$

   Therefore,

   $$(\mathcal{L}_\xi g)_{\mu\nu} = \partial_\lambda g_{\mu\nu}. \tag{5}$$

   So we conclude: if a coordinate x^λ do not appear in the metric components, the associated coordinate vector field $\xi^\sigma = \delta^\sigma_\lambda$ is a Killing vector field.

   The Kerr solution in the Boyer–Lindquist (I might use a different signature the the lecture) form is given by:

   $$ds^2 = -\left(1 - \frac{2mr}{\rho^2}\right) dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2mra^2 \sin^2\theta}{\rho^2}\right) \sin^2\theta \, d\phi^2 - \frac{4Mra \sin^2\theta}{\rho^2} dt d\phi \tag{6}$$

   $$\rho^2 = r^2 + a^2 \cos^2\theta \tag{7}$$

   $$\Delta = r^2 - 2mr + a^2. \tag{8}$$

Since the coordinates t and ϕ do not appear in the Kerr metric, the coordinate vector fields $\xi = \partial_t$, and $\psi = \partial_\phi$ are Killing vector fields.

2. Assuming $a^2 < m^2$.

   (a) Then the normal co-vector field is given by

$$\begin{aligned} n_\mu &= (0,1,0,0) && (9)\\ \rightarrow n^\nu &= g^{\nu\mu} n_\mu && (10)\\ &= (0, g^{rr}, 0, 0) && (11)\\ \rightarrow n^2 &= n^\mu n_\mu = g^{rr} && (12) \end{aligned}$$

   This means the hypersurfaces $r =$ constant becomes null, when g^{rr} (of the inverse metric) vanishes.
   From the Boyer–Lindquist form, we find

$$\begin{aligned} g^{rr} &= \frac{\Delta}{\rho^2} = \frac{r^2 - 2mr + a^2}{r^2 + a^2 \cos^{\theta}} \overset{!}{=} 0 && (13)\\ \rightarrow \Delta &= r^2 - 2mr + a^2 = 0 && (14)\\ \rightarrow r_\pm &= m \pm \sqrt{m^2 - a^2} && (15) \end{aligned}$$

   (b) Now we are calculating with Mathematica

$$\begin{aligned} & g_{\mu\nu}\chi^\mu\chi^\nu && (16)\\ &= \left(1, 0, \frac{a}{2mr_\pm}, 0\right) g_{\mu\nu}|_{r=r_\pm} \begin{pmatrix} 1 \\ 0 \\ \frac{a}{2mr_\pm} \\ 0 \end{pmatrix} && (17)\\ &= g_{tt} + 2g_{t\phi}\frac{a}{2mr_\pm} + g_{\phi\phi}\frac{a^2}{4m^2r_\pm^2} && (18)\\ &= 0 && (19) \end{aligned}$$

   so the Killing vector is null at the hypersurface.

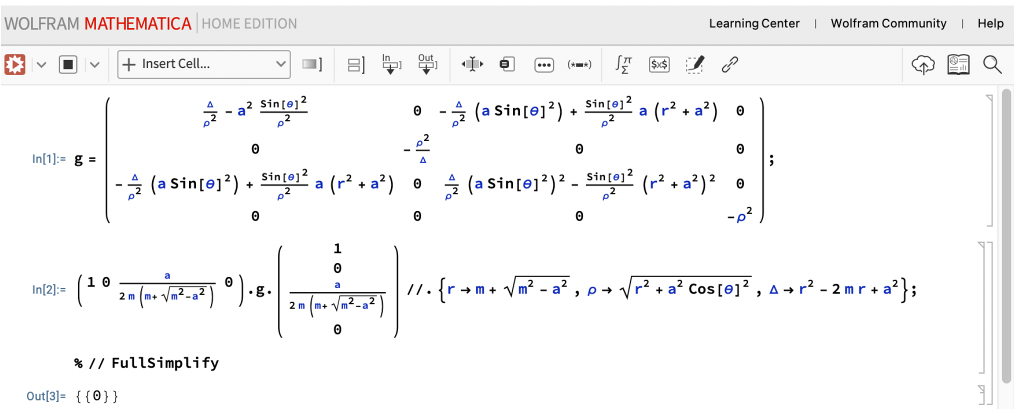

Figure 1: Mathematica calculations

3. With

$$\begin{aligned} r_+ &= m + \sqrt{m^2 - a^2} && (20)\\ r_{E+} &= m + \sqrt{m^2 - a^2\cos^2\theta} && (21) \end{aligned}$$

For χ_+ to be spacelike we calculate

$$g_{\mu\nu}\chi_+^\mu\chi_+^\nu = g_{tt}\cdot 1 + 2\Omega_{H+}g_{t\phi} + \Omega_{H+}^2 g_{\phi\phi} \tag{22}$$

$$= \frac{1}{2}\left(\frac{a\sin^2(\theta)\left(\frac{2a^2mr\sin^2(\theta)}{a^2\cos^2(\theta)+r^2}+a^2+r^2\right)}{m\left(\sqrt{m^2-a^2}+m\right)} + \frac{4mr}{a^2\cos^2(\theta)+r^2} - \frac{a^3r\sin^2(\theta)}{m\left(\sqrt{m^2-a^2}+m\right)^2\left(a^2\cos^2(\theta)+r^2\right)} - 2\right) \tag{23}$$

This looks quite ugly - therefore lets check the components

- Checking the terms

$$g_{tt} = -\frac{r^2+a^2\cos^2\theta-2mr}{r^2+a^2\cos^2\theta} \tag{24}$$

$$\to g_{tt} \overset{!}{=} 0 \tag{25}$$

$$\to r = m \pm \sqrt{m^2-a^2\cos^2\theta} = r_{E+} \tag{26}$$

$$g_{tt}(r_+) > 0 \tag{27}$$

  So g_{tt} is positive is ergosphere and vanishes at r_{E+}. This implies $\xi = \partial_t$ is spacelike.

- Next is $g_{\phi\phi}$.

$$g_{\phi\phi} = \left(r^2+a^2+\frac{2mra^2\sin^2\theta}{r^2+a^2\cos^2\theta}\right)\sin^2\theta \tag{28}$$

$$> 0 \tag{29}$$

  This implies $\psi = \partial_\phi$ is spacelike (because $\Omega_{H+} > 0$).

- And

$$g_{t\phi} = -\frac{2mar\sin^2\theta}{r^2+a^2\cos^2\theta} \tag{30}$$

$$< 0 \tag{31}$$

  which makes the whole makes the analysis inconclusive.

But with the use of Mathematica I could show analytically that $g_{\mu\nu}\chi_+^\mu\chi^\nu$ is spacelike in the ergoshperes equatorial plane - so there might be some argument (which I do not have) to conclude that this holds in the whole ergoshpere.