0.1 Carroll, Ostlie - An Introduction to Modern Astrophysics

0.1.1 Problem 7.3 - Binary star

(a) With $a = R_1 + R_2$ and using the circular property of the system we find

$$m_1\omega^2 R_1 = G\frac{m_1 m_2}{a^2} = m_2\omega^2 R_2 = m_2\omega^2(a - R_1) \tag{1}$$
$$m_1 R_1 = m_2(a - R_1) \tag{2}$$
$$R_1 = \frac{m_2}{m_1 + m_2}a \tag{3}$$
$$R_2 = \frac{m_1}{m_1 + m_2}a \tag{4}$$
$$\to m_1 R_1 = m_2 R_2 \tag{5}$$

and using the geometry

$$\cos i = \frac{r_2}{R_2} = \frac{r_2}{a}\frac{m_1 + m_2}{m_1} \tag{6}$$
$$= \frac{r_1}{R_1} = \frac{r_1}{a}\frac{m_1 + m_2}{m_2} \tag{7}$$
$$\to m_1 r_1 = m_2 r_2 \tag{8}$$

we see that the $\sin i$ still contains the mass ratio. One more look at the geometry reveals $\cos i = \frac{r_1 + r_2}{a}$ which is the solution. But we lets just combine all results to if we can get some information about the masses

$$\cos i = \frac{r_1 + r_2}{a} \tag{9}$$

(b)

$$\cos i = \frac{11R_s}{2\text{AU}} = \frac{7,700,000\text{km}}{150,000,000\text{km}} \quad \to \quad i = 88.5^o \tag{10}$$

0.2 Binney, Tremaine - Galactic Dynamics (2008)

0.3 Weinberg - Lecture on Astrophysics

0.3.1 Problem 1 - Hydrostatics of spherical star

Gravitational force on a mass element must be balanced by the top and bottom pressure (buoyancy)

$$F_p^{\text{top}} - F_p^{\text{bottom}} = F_g \tag{11}$$
$$dA \cdot p\left(r + \frac{dr}{2}\right) - dA \cdot p\left(r - \frac{dr}{2}\right) = -g(r)\rho(r) \cdot dA \cdot dr \tag{12}$$
$$\frac{dp}{dr} = -g(r)\rho(r) \tag{13}$$
$$= -G\frac{\mathcal{M}(r)}{r^2}\rho(r) \tag{14}$$

and therefore

$$\rho(r)\mathcal{M}(r) = -\frac{dp}{dr}\frac{r^2}{G} \tag{15}$$

where

$$g(r) = G\frac{\mathcal{M}(r)}{r^2} = \frac{G}{r^2}\int_0^r 4\pi\rho(r')r'^2dr'. \tag{16}$$

The gravitational binding energy Ω is given by

$$\begin{aligned}
d\Omega &= -G\frac{m_{\text{shell}}\mathcal{M}}{r} &(17)\\
\Omega &= -G\int_0^R \frac{4\pi\rho(r)\mathcal{M}(r)}{r}r^2dr &(18)\\
&= -4\pi G\int_0^R r\rho(r)\mathcal{M}(r)dr &(19)\\
&= 4\pi\int_0^R \frac{dp}{dr}r^3dr &(20)\\
&= 4\pi pr^3|_0^R - 3\cdot 4\pi\int_0^R p(r)r^2dr &(21)\\
&= 4\pi p_0R^3 - 3\left(4\pi\int_0^R p(r)r^2dr\right) &(22)\\
&= 4\pi p_0R^3 - 3\int_{K_R} p(\vec{r})d^3r. &(23)
\end{aligned}$$

0.3.2 Problem 2 - CNO cycle

$$\begin{aligned}
\Gamma(ii) = \Gamma(iii) = \Gamma(iv) = \Gamma(v) = \Gamma(i) &\quad (24)\\
\Gamma(vi) = P\cdot\Gamma(i) &\quad (25)\\
\Gamma(vii) = \Gamma(viii) = \Gamma(ix) = \Gamma(x) = (1-P)\cdot\Gamma(i) &\quad (26)
\end{aligned}$$

Check result!

0.3.3 Problem 3

Not done yet

0.3.4 Problem 4

Not done yet

0.3.5 Problem 5 - Radial density expansion for a polytrope

For the polytrope equation

$$p = K\rho^\Gamma \tag{27}$$

we obtain

$$\begin{aligned}
\frac{dp}{d\rho} &= K\Gamma\rho^{\Gamma-1} &(28)\\
&= \Gamma\frac{p}{\rho} &(29)
\end{aligned}$$

With equations (1.1.4/5)

$$\frac{dp}{dr} = -\frac{G\mathcal{M}(r)\rho(r)}{r^2} \quad \rightarrow \quad \mathcal{M}(r) = -\frac{p'r^2}{G\rho} \tag{30}$$

$$\frac{d\mathcal{M}(r)}{dr} = 4\pi r^2 \rho(r) \tag{31}$$

we can obtain a second order ODE by differentiating the first one and substituting $\mathcal{M}'$

$$\mathcal{M}' = -\frac{1}{G}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{d}{dr}p\right) \tag{32}$$

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{d}{dr}p\right) + G\mathcal{M}' = 0 \tag{33}$$

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{d}{dr}p\right) + 4\pi G r^2 \rho = 0 \tag{34}$$

now we can substitute the $p = K\rho^\Gamma$ and obtain

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{d}{dr}\rho^\Gamma\right) + \frac{4\pi G}{K} r^2 \rho = 0. \tag{35}$$

The Taylor expansion

$$\rho(r) = \rho(0)\left[1 + ar^2 + br^4 + ...\right] \tag{36}$$

$$\rho(r)^\Gamma = \rho(0)^\Gamma \left[1 + ar^2 + br^4 + ...\right]^\Gamma \tag{37}$$

$$= \rho(0)^\Gamma \left[1 + a\Gamma r^2 + \left(b\Gamma + \frac{1}{2}a^2\Gamma(\Gamma-1)\right) r^4 + ...\right] \tag{38}$$

$$\frac{1}{\rho} = \frac{1}{\rho(0)}\left[1 - ar^2 + (a^2 - b)r^4 + ...\right] \tag{39}$$

can be substituted into the ODE

$$\rho(0)^{\Gamma-1}\frac{d}{dr}\left(r^2\left[1 - ar^2 + (a^2-b)r^4 + ...\right]\left[a\Gamma 2r + \left(b\Gamma + \frac{1}{2}a^2\Gamma(\Gamma-1)\right)4r^3 + ...\right]\right) \tag{40}$$

$$+\frac{4\pi G}{K}\rho(0)\left[r^2 + ar^4 + br^6 + ...\right] = 0. \tag{41}$$

and sort by powers of r

$$\rho(0)^{\Gamma-1}\frac{d}{dr}\left(2\Gamma a r^3 + \left[-2\Gamma a^2 + 4\left(b\Gamma + \frac{1}{2}a^2\Gamma(\Gamma-1)\right)\right] r^5 + ...\right) + \frac{4\pi G}{K}\rho(0)\left[r^2 + ar^4 + br^6 + ...\right] = 0. \tag{42}$$

In second order of r we obtain

$$\rho(0)^{\Gamma-1} 2\Gamma a 3 + \frac{4\pi G}{K}\rho(0) = 0 \tag{43}$$

which results in

$$a = -\frac{2\pi G}{3\Gamma K \rho(0)^{\Gamma-2}} \tag{44}$$

0.3.6 Problem 6

Not done yet

0.3.7 Problem 7

Not done yet

0.3.8 Problem 8

Not done yet

0.3.9 Problem 9

Not done yet

0.3.10 Problem 10

Not done yet

0.3.11 Problem 11 - Modified Newtonian gravity

The modified Poisson equation is given by

$$\left(\triangle + \mathcal{R}^{-2}\right)\phi = 4\pi G\rho \tag{45}$$

with the Greens function

$$\left(\triangle + \mathcal{R}^{-2}\right)G(\vec{r}) = -\delta^3(\vec{r}). \tag{46}$$

The Fourier transform of the Greens function

$$G(\vec{k}) = \int d^3\vec{r}\, G(\vec{r})e^{-i\vec{k}\vec{r}} \tag{47}$$

and the field equations are given by

$$\begin{aligned}
\left[k^2 + \mathcal{R}^{-2}\right]G(\vec{k}) &= -1 && (48)\\
G(\vec{k}) &= \frac{1}{k^2 + \mathcal{R}^{-2}} && (49)\\
G(\vec{x}) &= \frac{1}{(2\pi)^3}\int d^3\vec{k}\frac{e^{i\vec{k}\vec{r}}}{k^2+\mathcal{R}^{-2}} && (50)\\
&= \frac{1}{(2\pi)^3}2\pi\int_0^\infty\int_0^\pi \frac{e^{ik_r\cdot r\cos\theta}}{k_r^2+\mathcal{R}^{-2}}k_r^2\sin\theta\, d\theta dk_r && (51)\\
&= \frac{1}{(2\pi)^3}2\pi\int_0^\infty\left[-\frac{e^{ik_r r\cos\theta}}{ik_r r}\right]_0^\pi \frac{1}{k_r^2+\mathcal{R}^{-2}}k_r^2 dk_r && (52)\\
&= \frac{1}{2\pi^2 r}\int_0^\infty \frac{k_r\sin(k_r r)}{k_r^2+\mathcal{R}^{-2}}dk_r && (53)\\
& && (54)
\end{aligned}$$

The integral can be can be calculated using the residual theorem

$$\begin{aligned}
\int_0^\infty \frac{k_r\sin(k_r r)}{k_r^2+\mathcal{R}^{-2}}dk_r &= \frac{1}{2}\int_{-\infty}^\infty \frac{k_r\sin(k_r r)}{k_r^2+\mathcal{R}^{-2}}dk_r && (55)\\
&= \frac{1}{2}\int_{-\infty}^\infty \frac{k_r\sin(k_r r)}{(k_r+i\mathcal{R}^{-1})(k_r-i\mathcal{R}^{-1})}dk_r && (56)\\
&= \frac{1}{2}\int_{-\infty}^\infty \frac{k_r\sin(k_r r)}{2k_r}\left(\frac{1}{k_r+i\mathcal{R}^{-1}}+\frac{1}{k_r-i\mathcal{R}^{-1}}\right)dk_r && (57)\\
&= \frac{1}{4}\int_{-\infty}^\infty \frac{\sin(k_r r)}{k_r+i\mathcal{R}^{-1}}dk_r + \frac{1}{4}\int_{-\infty}^\infty \frac{\sin(k_r r)}{k_r-i\mathcal{R}^{-1}}dk_r && (58)
\end{aligned}$$

Not done yet

0.3.12 Problem 12

Not done yet