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1 Advanced Topics in Gravity – Exercise sheet 5 - 2025-06-24

1.1 Exercise 1 - Energy Conditions

Analyze the different energy conditions: whether they hold, are violated, and which conditions must be satisfied if needed.

1. Consider a cosmological constant such that:

$$T^{(\Lambda)}_{\alpha\beta} = -\rho_\Lambda g_{\alpha\beta} \tag{1.1}$$

   where

$$\rho_\Lambda = -p_\Lambda = \frac{\Lambda}{8\pi G}, \quad \text{and} \quad \left(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T\right) = \rho_\Lambda g_{\alpha\beta}.$$

2. Consider a perfect fluid

$$T_{\alpha\beta} = (\rho + p)u_\alpha u_\beta + p g_{\alpha\beta}, \quad \text{with,} \tag{1.2}$$

   with $g^{\alpha\beta}u_\alpha u_\beta = -1$ and

$$T^\alpha_{\ \alpha} = -\rho + 3p, \quad \text{so} \quad \left(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T\right) = p_\Lambda g_{\alpha\beta}$$

   Note:

   These two exercises may be substituted for one of the analysis topics mentioned during the lecture today, such as:

   - Misner spacetime geodesics and lightcones
   - Exotic matter and safety requirements for travellers in Morris-Thorne wormholes

Taking the trace of the Einstein field equations (signature $[-,+,+,+]$)

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu} \tag{1}$$

$$\to R - \frac{1}{2}4R = \kappa T \tag{2}$$

$$\to -R = \kappa T \tag{3}$$

$$R_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right) \tag{4}$$

where we used $g_{\mu\nu}g^{\nu\rho} = \delta^\rho_\mu$ and $g_{\mu\nu}g^{\mu\nu} = \delta^\mu_\mu = 4$.

1. With EFE:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa\left(\underbrace{T_{\mu\nu}}_{\equiv 0} - \frac{\Lambda}{\kappa}g_{\mu\nu}\right) \tag{5}$$

$$= \underbrace{\kappa\left(-\frac{\Lambda}{\kappa}g_{\mu\nu}\right)}_{=T^{\Lambda}} \tag{6}$$

So we see that the cosmological constant can be understood as a special fluid with $p = -\Lambda/\kappa$ and $\rho = -p = \Lambda/\kappa$

- *WEC - weak energy condition $T_{\alpha\beta}t^\alpha t^\beta \geq 0$ for all timelike t^α ($g_{\alpha\beta}t^\alpha t^\beta < 0$)*

$$T^{\Lambda}_{\alpha\beta}t^\alpha t^\beta = -\frac{\Lambda}{\kappa}\underbrace{g_{\alpha\beta}t^\alpha t^\beta}_{<0} \tag{7}$$

$$= \frac{\Lambda}{\kappa} \tag{8}$$

  Holds if $\Lambda \geq 0$.

- *NEC - null energy condition $T_{\alpha\beta}l^\alpha l^\beta \geq 0$ for all null l^α*

$$T^{\Lambda}_{\alpha\beta}l^\alpha l^\beta = -\frac{\Lambda}{\kappa}\underbrace{g_{\alpha\beta}l^\alpha l^\beta}_{=0} \tag{9}$$

$$= 0 \tag{10}$$

  Holds.

- *SEC - strong energy condition $(T_{\alpha\beta} - \frac{1}{2}Tg_{\alpha\beta})u^\alpha u^\beta \geq 0$ for all unit timelike u^α*

$$(T^{\Lambda}_{\alpha\beta} - \frac{1}{2}T^{\Lambda}g_{\alpha\beta})u^\alpha u^\beta = \left[-\frac{\Lambda}{\kappa} - \frac{1}{2}4(-\frac{\Lambda}{\kappa})\right]g_{\alpha\beta}u^\alpha u^\beta \tag{11}$$

$$= -\frac{\Lambda}{\kappa}(-1)g_{\alpha\beta}u^\alpha u^\beta \tag{12}$$

$$= \frac{\Lambda}{\kappa}g_{\alpha\beta}u^\alpha u^\beta \tag{13}$$

$$= 0 \tag{14}$$

  Holds if $\Lambda \geq 0$

- *DEC - Dominant energy condition $-T^\alpha_\beta u^\beta$ is a future directed timelike or null vector for all future directed timelike vectors u^α*
  For a perfect fluid the DEC holds when $\rho \geq |p|$. This means re require $\rho_\Lambda \geq |p_\Lambda| = \rho_\Lambda$ - so it holds.

2. **The problem is a bit unclear - do we consider just a ideal fluid or fluid plus cosmological constant (there is a p_Λ in the last equation)**. We therefore consider the more complicated case and can set $\Lambda = 0$ if needed

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa\left(T^{\mathrm{PF}}_{\mu\nu} - \frac{\Lambda}{\kappa}g_{\mu\nu}\right) \tag{15}$$

$$= \kappa\left((\rho + p)u_\mu u_\nu + pg_{\mu\nu} - \frac{\Lambda}{\kappa}g_{\mu\nu}\right) \tag{16}$$

$$= \kappa\left((\rho + p)u_\mu u_\nu + \left[p - \frac{\Lambda}{\kappa}\right]g_{\mu\nu}\right) \tag{17}$$

$$\tag{18}$$

then

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + \left[p - \frac{\Lambda}{\kappa}\right] g_{\mu\nu} \tag{19}$$

$$\to T^\mu_\mu = (\rho + p)\underbrace{u^\mu u_\mu}_{=-1} + 4\left[p - \frac{\Lambda}{\kappa}\right] \tag{20}$$

$$= 3p - \rho - 4\frac{\Lambda}{\kappa} \tag{21}$$

$$\to T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T = (\rho + p)u_\mu u_\nu + \left[p - \frac{\Lambda}{\kappa}\right] g_{\mu\nu} - \frac{1}{2}(3p - \rho - 4\frac{\Lambda}{\kappa})g_{\mu\nu} \tag{22}$$

$$= (\rho + p)u_\mu u_\nu + \left[-\frac{1}{2}p + \frac{1}{2}\rho + \frac{\Lambda}{\kappa}\right] g_{\mu\nu} \tag{23}$$

- *WEC - weak energy condition $T_{\alpha\beta}t^\alpha t^\beta \geq 0$ for all timelike t^α ($g_{\alpha\beta}t^\alpha t^\beta < 0$)*

$$T_{\alpha\beta}t^\alpha t^\beta = (\rho + p)\underbrace{u_\alpha u_\beta t^\alpha t^\beta}_{=(u_\alpha t^\alpha)^2 \geq 0} + \left[p - \frac{\Lambda}{\kappa}\right]\underbrace{g_{\alpha\beta}t^\alpha t^\beta}_{<0} \tag{24}$$

  Holds if $p + \rho \geq 0$ and $p - \frac{\Lambda}{\kappa} \geq 0$.
- *NEC - null energy condition $T_{\alpha\beta}l^\alpha l^\beta \geq 0$ for all null l^α*
  See above - holds if $p + \rho \geq 0$
- *SEC - strong energy condition $(T_{\alpha\beta} - \frac{1}{2}Tg_{\alpha\beta})u^\alpha u^\beta \geq 0$ for all unit timelike u^α*

$$(T_{\alpha\beta} - \frac{1}{2}Tg_{\alpha\beta})u^\alpha u^\beta = (\rho + p)\underbrace{u_\alpha u_\beta u^\alpha u^\beta}_{=(u^2)^2=1} + \left[-\frac{1}{2}p + \frac{1}{2}\rho + \frac{\Lambda}{\kappa}\right]\underbrace{g_{\alpha\beta}u^\alpha u^\beta}_{=-1} \tag{25}$$

$$= (\rho + p) + \left[\frac{1}{2}p - \frac{1}{2}\rho - \frac{\Lambda}{\kappa}\right] \tag{26}$$

$$= \frac{1}{2}\left(3p + \rho + 2\frac{\Lambda}{\kappa}\right) \tag{27}$$

  Holds if $3p + \rho + 2\frac{\Lambda}{\kappa} \geq 0$
- *DEC - Dominant energy condition $-T^\alpha_\beta u^\beta$ is a future directed timelike or null vector for all future directed timelike vectors u^α*
  For $\Lambda = 0$ holds if $\rho \geq |p|$.

3. Safety requirements for travelers in Morris-Thorne wormholes

   The general safety requirements (derived from what we know about Schwarzschild/Kerr BHs) are (independent of specific solution/shape of the wormhole)

   - Small tidal forces - to avoid squeezing/streching
   - Small acceleration - to avoid trauma
   - No even horizon I - to avoid signal loss between brain and rest of the body - or weird quantum effects on atoms (nucleus inside EH - electrons outside)
   - No even horizon II - to be able to go back and forth (no sure if this counts as safety requirement)
   - Solution must be stable with respect to external perturbations - wormhole should not collapse when traveler passes through
   - Short transition time to other side - to avoid starvation/suffocation
   - Large enough passage - so whole body/ship fits through
   - Low radiation levels (Doppler shift of surrounding particles)

   I just found the paper - but haven't read it yet.