Solutions - Christian Thierfelder

1 Quantum Field Theory II – Exercise sheet 1 2025-04-29

1.1 Exercise 1 - Two-point function in interacting QFT

Consider ϕ^3 theory with action

$$S = \frac{1}{2}\int d^4x\, \phi(x)\left[m^2 - \Box\right]\phi(x) - \frac{\lambda}{3!}\int d^4x\, \phi^3(x) \tag{1}$$

1. Compute the two-point functions $\langle\phi(x_1)\phi(x_2)\rangle$ to second order in perturbation theory (to order λ^2) by use of the Gell-Mann-Low formula.
2. Compute the two-point functions $\langle\phi(x_1)\phi(x_2)\rangle$ to order λ^2 by use of the Schwinger-Dyson equations.

1. Gell-Man-Low formula

$$\mathcal{L}_{int} = -\frac{\lambda}{3!}\int d^4x\, \phi^3(x) \tag{1}$$

$$\mathcal{H}_{int} = +\frac{\lambda}{3!}\int d^4x\, \phi^3(x) \tag{2}$$

$$\langle\phi(x_1)\phi(x_2)\rangle = \frac{\langle 0|T\{\phi(x_1)\phi(x_2)\exp(-i\int d^4x\mathcal{H}_{int})\}|0\rangle}{\langle 0|T\{\exp(-i\int d^4x\mathcal{H}_{int})\}|0\rangle} \tag{3}$$

$$= \frac{\langle 0|T\{\phi(x_1)\phi(x_2)\exp(-i\frac{\lambda}{3!}\int d^4x\,\phi^3(x))\}|0\rangle}{\langle 0|T\{\exp(-i\frac{\lambda}{3!}\int d^4x\,\phi^3(x))\}|0\rangle} \tag{4}$$

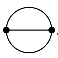
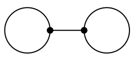
Using Wicks theorem we see that due to the ϕ^3 interaction the first orders of numerator (power: $2+3=5$) and denominator (power: 3) vanish.

- Denominator

$$\langle 0|T\{\exp\left(-i\frac{\lambda}{3!}\int d^4x\,\phi^3(x)\right)\}|0\rangle \tag{5}$$

$$\simeq \langle 0|T\{\left(1 - i\frac{\lambda}{3!}\int d^4x\,\phi^3(x) + \frac{i^2}{2}\frac{\lambda^2}{(3!)^2}\iint d^4xd^4y\,\phi^3(x)\phi^3(y)\right)\}|0\rangle \tag{6}$$

$$\simeq \langle 0|0\rangle - i\frac{\lambda}{3!}\int d^4x\,\cancel{\langle 0|\phi^3(x)|0\rangle} - \frac{\lambda^2}{2(3!)^2}\iint d^4xd^4y\,\langle 0|T\{\phi^3(x)\phi^3(y)\}|0\rangle \tag{7}$$

$\simeq 1 - \frac{\lambda^2}{2(3!)^2}\cdot \Big[\, 6\times$ , $9\times$  $\Big]$ (8)

$$\simeq 1 - \frac{\lambda^2}{2(3!)^2}\iint d^4x\,d^4y\left[6\cdot(\Delta_F(x-y))^3 + 9\cdot(\Delta_F(0))^2\Delta_F(x-y)\right] \tag{9}$$

  - Each $\phi(x)$ contracts with a $\phi(y)$: $3\cdot 2\cdot 1 = 6\ \times\ \Delta_F(x-y)$
  - Two $\phi(x)$ and two $\phi(y)$ contract and the remaining $\phi(x)$ contracts with $\phi(y)$: $3\cdot 3 = 9\times(\Delta_F(0))^2\Delta_F(x-y)$
  - Double check combination count: $(6-1)!! = \frac{6!}{2^{6/2}\cdot 3!} = 15 = 6+9$

- Numerator

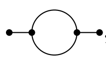
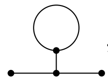
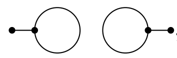
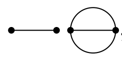
$$\langle 0|T\{\phi(x_1)\phi(x_2)\exp\left(-i\frac{\lambda}{3!}\int d^4x\,\phi^3(x)\right)\}|0\rangle \tag{10}$$

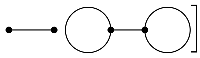
$$\simeq \langle 0|T\{\phi(x_1)\phi(x_2)\left(1 - i\frac{\lambda}{3!}\int d^4x\,\phi^3(x) + \frac{i^2}{2}\frac{\lambda^2}{(3!)^2}\iint d^4xd^4y\,\phi^3(x)\phi^3(y)\right)\}|0\rangle \tag{11}$$

$$\simeq \langle 0|T\{\phi(x_1)\phi(x_2)\}|0\rangle - i\frac{\lambda}{3!}\int d^4x\,\cancel{\langle 0|T\{\phi(x_1)\phi(x_2)\phi^3(x)\}|0\rangle} \tag{12}$$

$$-\frac{\lambda^2}{2(3!)^2}\iint d^4xd^4y\,\langle 0|T\{\phi(x_1)\phi(x_2)\phi^3(x)\phi^3(y)\}|0\rangle \tag{13}$$

$$\langle 0|T\{\phi(x_1)\phi(x_2)\exp\left(-i\frac{\lambda}{3!}\int d^4x\,\phi^3(x)\right)\}|0\rangle \tag{14}$$

$\simeq 1\cdot[$ $] - \frac{\lambda^2}{2(3!)^2}\cdot\Big[\,36\times$  $,\ 36\times$  $,\ 18\times$  $,\ 6\times$  $,$ (15)

$9\times$  $\Big]$ (16)

$$\simeq \Delta_F(x_1-x_2) - \frac{\lambda^2}{2(3!)^2}\iint d^4x\,d^4y\left[36\cdot\Delta_F(x_1-x)\Delta_F(x_2-y)(\Delta_F(x-y))^2\right] \tag{17}$$

$$- \frac{\lambda^2}{2(3!)^2}\iint d^4x\,d^4y\,[36\cdot\Delta_F(x_1-x)\Delta_F(x-y)\Delta_F(0)\Delta_F(x-x_2)] \tag{18}$$

$$- \frac{\lambda^2}{2(3!)^2}\iint d^4x\,d^4y\left[18\cdot\Delta_F(x_1-x)(\Delta_F(0))^2\Delta_F(x_2-y)\right] \tag{19}$$

$$+ \text{(disc. contr.- see denominator)} \tag{20}$$

  - Double check combination count: $(8-1)!! = \frac{8!}{2^{8/2}\cdot 4!} = 105 = 36+36+18+6+9$

- Using the Taylor expansion of the full expression

$$\frac{D_F + \lambda^2(a^{(2)}_{\text{conn}} + D_F a^{(2)}_{\text{disc.}})}{1+\lambda^2 a^{(2)}_{\text{disc.}}} \simeq F + \lambda^2 a^{(2)}_{\text{conn}} + \mathcal{O}(\lambda^3) \tag{21}$$

  we see that the disconnected contributions cancel

- Result

$$\langle\phi(x_1)\phi(x_2)\rangle = \Delta_F(x_1-x_2) - \frac{\lambda^2}{2}\iint d^4x\,d^4y\,\Delta_F(x_1-x)\Delta_F(x_2-y)(\Delta_F(x-y))^2 \tag{22}$$

$$- \frac{\lambda^2}{2}\iint d^4x\,d^4y\,[\Delta_F(x_1-x)\Delta_F(x-y)\Delta_F(0)\Delta_F(x-x_2)] \tag{23}$$

$$- \frac{\lambda^2}{4}\iint d^4x\,d^4y\left[\Delta_F(x_1-x)(\Delta_F(0))^2\Delta_F(x_2-y)\right] \tag{24}$$

2. With

$$S[\phi] = \int d^4x\left[\frac{1}{2}\phi(x)(m^2-\Box)\phi(x) - \frac{\lambda}{3!}\phi^3(x)\right] \tag{25}$$

$$\to \delta S = \int d^4x\,\delta\phi\left[(\Box - m^2)\phi - \frac{\lambda^2}{2}\phi^2\right] \tag{26}$$

$$\to \frac{\delta S}{\delta\phi} = (\Box - m^2)\phi(x) - \frac{\lambda^2}{2}\phi^2(x) \tag{27}$$

we enter the Schwinger-Dyson equation

$$\left\langle\frac{\delta S}{\delta\phi}\phi(x_1)\right\rangle = i\delta^{(4)}(x-x_1) \tag{28}$$

$$(\Box - m^2)\langle\phi(x)\phi(x_1)\rangle - \frac{\lambda}{2}\langle\phi^2(x)\phi(x_1)\rangle = i\delta^{(4)}(x-x_1) \tag{29}$$

we rewrite the λ expansion with $G = \langle\phi(x)\phi(x_1)\rangle$

$$(\Box - m^2)G = \frac{\lambda}{2}\langle\phi^2(x)\phi(x_1)\rangle + i\delta^{(4)}(x-x_1) \tag{30}$$

$$G = G^{(0)} + \lambda G^{(1)} + \lambda^2 G^{(2)} + \cdots \tag{31}$$

$$\langle\phi(x)\phi(x_1)\rangle = \underbrace{\langle\phi(x)\phi(x_1)\rangle^{(0)}}_{=G^{(0)}} + \lambda\underbrace{\langle\phi(x)\phi(x_1)\rangle^{(1)}}_{=G^{(1)}} + \lambda^2\underbrace{\langle\phi(x)\phi(x_1)\rangle^{(2)}}_{=G^{(2)}} + \cdots \tag{32}$$

as well as for the perturbation

$$\langle\phi^2(x)\phi(x_1)\rangle = \cancel{\langle\phi^2(x)\phi(x_1)\rangle^{(0)}} + \lambda\langle\phi^2(x)\phi(x_1)\rangle^{(1)} + \lambda^2\langle\phi^2(x)\phi(x_1)\rangle^{(2)} + \cdots \tag{33}$$

$$= \frac{\langle 0|T\{\phi^2(x)\phi(x_2)\exp(+i\frac{\lambda}{3!}\int d^4x\,\phi^3(x))\}|0\rangle}{\langle 0|T\{\exp(+i\frac{\lambda}{3!}\int d^4x\,\phi^3(x))\}|0\rangle} \tag{34}$$

$$= \langle 0|T\{\phi^2(x)\phi(x_1)\}|0\rangle + \lambda\frac{i}{3!}\int d^4y\,\langle 0|T\{\phi^2(x)\phi(x_1)\phi^3(y)\}|0\rangle + \cdots \tag{35}$$

$$= \lambda\frac{i}{3!}\int d^4y\,\langle 0|T\{\phi^2(x)\phi(x_1)\phi^3(y)\}|0\rangle + \cdots \tag{36}$$

$$= \lambda\frac{i}{3!}\int d^4y\,6\Delta_F(x_1-y)\Delta_F(x-y)^2 + 3\Delta_F(x_1-y)\Delta_F(0)^2 \tag{37}$$

$$+ \lambda\frac{i}{3!}\int d^4y\,6\Delta_F(x_1-x)\Delta_F(x-y)\Delta_F(0) \tag{38}$$

$$= i\lambda\int d^4y\,\Delta_F(x_1-y)\Delta_F(x-y)^2 + \frac{i\lambda}{2}\int d^4y\,\Delta_F(x_1-y)\Delta_F(0)^2 \tag{39}$$

$$+ i\lambda\int d^4y\,\Delta_F(x_1-x)\Delta_F(x-y)\Delta_F(0) \tag{40}$$

(we recover all $(6-1)!! = 15$ combinations) and insert

$$(\Box - m^2)(G^{(0)} + \lambda G^{(1)} + \lambda^2 G^{(2)} + \cdots) = \frac{\lambda}{2}\left(\langle\phi^2(x)\phi(x_1)\rangle^{(0)} + \lambda\langle\phi^2(x)\phi(x_1)\rangle^{(1)} + \cdots\right) \tag{41}$$

$$+ i\delta^{(4)}(x-x_1) \tag{42}$$

and ordering by powers of λ we can solve the equations successively

$$\to \lambda^0 : (\Box - m^2)G^{(0)} = i\delta^{(4)}(x-x_1) \tag{43}$$

$$\to G^{(0)} = G^{(0)}(x-x_1) = \Delta_F(x-x_1) \tag{44}$$

$$\to \lambda^1 : (\Box - m^2)G^{(1)} = \frac{1}{2}\langle\phi^2(x)\phi(x_1)\rangle^{(0)} \tag{45}$$

$$\to G^{(1)} = 0 \qquad \text{(Wick theorem)} \tag{46}$$

$$\to \lambda^2 : (\Box - m^2)G^{(2)} = \frac{1}{2}\langle\phi^2(x)\phi(x_1)\rangle^{(1)} \tag{47}$$

$$(\Box_x - m^2)G^{(2)}(x-x_1) = \frac{i\lambda}{2}\int dy\,\Delta_F(x_1-y)\Delta_F(x-y)^2 \tag{48}$$

$$+ \text{(two other terms contaning } \Delta_F(0)) \tag{49}$$

We see that the λ^0 for $G^{(0)}$ delivers the Greens function of the Klein-Gordon equation. The λ^2 equation is the KG with a source term - so we use the Greens function to solve for $G^{(2)}$

$$\to G^{(2)}(x-x_1) = \int d^4z\,G^{(0)}(z-x-x_1)\left(\frac{i\lambda}{2}\int d^4y\,\Delta_F(x_1-y)\Delta_F(z-y)^2 + \cdots\right) \tag{50}$$

$$= -\frac{\lambda}{2}\iint d^4y d^4z\,\Delta_F(z-x)\Delta_F(x_1-y)\Delta_F(z-y)^2 \tag{51}$$

$$+ \text{(two other terms contaning } \Delta_F(0)) \tag{52}$$

resulting in

$$\langle\phi(x)\phi(x_1)\rangle = \Delta_F(x-x_1) - \frac{\lambda^2}{2}\iint d^4y d^4z\,\Delta_F(z-x)\Delta_F(x_1-y)\Delta_F(z-y)^2 \tag{53}$$

$$+ \text{(two other terms contaning } \Delta_F(0)) \tag{54}$$

which is the same is in part 1.

2 Quantum Field Theory II – Exercise sheet 2 2025-05-14

2.1 Exercise 1 - Berezin Integral

Let $\theta_i, i = 1, ..., N$, be complex Grassmann variables, i.e., they obey $\theta_i \theta_j = -\theta_j \theta_i$. We consider unitary transformations

$$\theta_i \to \theta_i' = U_i^{\ j} \theta_j, \qquad \text{where } \ UU^\dagger = 1 \tag{55}$$

where the unitarity condition reads in indices $U_i^{\ k}(U^\dagger)_k^{\ j} = U_i^{\ k}(U^*)^j_{\ k} = \delta_i^{\ j}$.

1. Invariance of the pairing under unitary transformations Complex conjugation raises and lowers indices, so that one should write θ^{*i}. This means that the contraction of θ^{*i} with a second set of complex Grassmann variables η_i, transforming as in (1), is invariant under unitary transformations. Verify this by showing that the pairing defined by

$$\langle \theta, \eta \rangle := (\theta^*)^T \eta = \theta_i^* \eta_i \tag{2}$$

   is invariant.

2. Self-adjointness of Hermitian matrices with respect to the pairing

   Show that a Hermitian $N \times N$ matrix $A = (A_i^{\ j})$ is self-adjoint with respect to the above pairing:

$$\langle \theta, A\eta \rangle = \langle A\theta, \eta \rangle. \tag{3}$$

   Show that $\langle \theta, A\theta \rangle$ for self-adjoint A is real and bosonic.

3. Berezin integration and generating functional

   Denoting the Berezin integration measure introduced in the lecture by

$$d^{2N}\theta \equiv d\theta^{*1} d\theta_1 \cdots d\theta^{*N} d\theta_N, \tag{4}$$

   compute:

$$\int d^{2N}\theta \, e^{-\langle \theta, A\theta \rangle}. \tag{5}$$

   Then generalize this to the generating functional:

$$Z[\eta, \eta^*] := \int d^{2N}\theta \, e^{-\langle \theta, A\theta \rangle + \langle \eta, \theta \rangle + \langle \theta, \eta \rangle}. \tag{6}$$

4. Two-point function under Gaussian integral

   Compute:

$$\int d^{2N}\theta \, \theta_i \theta^{*j} \, e^{-\langle \theta, A\theta \rangle}. \tag{7}$$

Notation summary

$$U = U_i^{\ j} \tag{56}$$

$$U^\dagger = (U^\dagger)_i^{\ j} = (U^{T*})_i^{\ j} = (U^T)^{*i}_{\ \ j} = (U^*)^j_{\ i} \tag{57}$$

$$\to UU^\dagger = 1 \to U_i^{\ k}(U^\dagger)_k^{\ j} = U_i^{\ k}(U^*)^j_{\ k} = \delta_i^{\ j} \tag{58}$$

$$\to U^\dagger U = 1 \to (U^*)^j_{\ k} U_i^{\ k} = \delta^j_{\ i} \tag{59}$$

$$\to A = A^\dagger \to A_i^{\ j} = (A^*)^j_{\ i} \tag{60}$$

1. Now

$$\begin{aligned}
\langle \theta', \eta' \rangle &= \langle U\theta, U\eta \rangle && (61)\\
&= (U_i^{\ k}\theta_k)^{*T} \, (U_i^{\ j}\eta_j) && (62)\\
&= ((U^*)^i_{\ k}\theta^{*k})^T \, (U_i^{\ j}\eta_j) && (63)\\
&= \theta^{*k}\delta_k^{\ j}\eta_j && (64)\\
&= \theta^{*j}\eta_j && (65)
\end{aligned}$$

2. Now with $A = A^\dagger$ meaning $A_i^{\ j} = (A^*)^j_{\ i}$

- Then

$$\begin{aligned}
\langle \theta, A\eta \rangle &= \theta^{*j}(A\eta)_j && (66)\\
&= \theta^{*j}(A_j^{\ k}\eta_k) && (67)\\
&= (A_j^{\ k}\theta^{*j})\eta_k && (68)\\
&= ((A^*)^k_{\ j}\theta^{*j})\eta_k && (69)\\
&= (A\theta)^{*k}\eta_k && (70)\\
&= ((A\theta)^*)^T\eta && (71)\\
&= \langle A\theta, \eta \rangle && (72)
\end{aligned}$$

- We see (by splitting a complex Grassmann variable into a real and an imaginary part)

$$\begin{aligned}
(\alpha\beta)^* &= [(\alpha_1 + i\alpha_2)(\beta_1 + i\beta_2)]^* && (73)\\
&= [(\alpha_1\beta_1 - \alpha_2\beta_2) + i(\alpha_1\beta_2 + \alpha_2\beta_1)]^* && (74)\\
&= (\beta_1\alpha_1 - \beta_2\alpha_2) - i(\beta_2\alpha_1 + \beta_1\alpha_2) && (75)\\
&= (\beta_1 - i\beta_2)(\alpha_1 - i\alpha_2) && (76)\\
&= (\beta_1 + i\beta_2)^*(\alpha_1 + i\alpha_2)^* && (77)\\
&= \beta^*\alpha^* && (78)
\end{aligned}$$

  as well as (the anticommuting goes through the (linear) sum)

$$\begin{aligned}
\langle \alpha, \beta \rangle &= \alpha^{*k}\beta_k && (79)\\
&= ((\alpha^{*k}\beta_k)^*)^* && (80)\\
&= (\beta^{*k}\alpha_k)^* && (81)\\
&= \langle \beta, \alpha \rangle^* && (82)
\end{aligned}$$

  then using this results in $\langle A\theta, \theta \rangle = \langle \theta, A\theta \rangle = \langle A\theta, \theta \rangle^*$ implies $\langle \theta, A\theta \rangle$ is real.
  It is also bosonic (commutes with other Grassmann variables) - because

3. The Berezin integration is defined as

$$\int d\theta = 0, \qquad \int d\theta\, \theta = 1 \tag{83}$$

(we observe that the this rules actually look more like differentiation than integration). For an analytic function f which can be written as a finite series ($\theta_k^2 = 0$)

$$f(\theta_1, ..., \theta_n) = f^{(0)} + f_j^{(1)}\theta_j + f_{jl}^{(2)}\theta_j\theta_l + ... + f_{12...n}^{(n)}\theta_1\theta_2...\theta_n \tag{84}$$

with the graded Leibnitz rule

$$\frac{d}{d\theta_i}(\theta_k f) = f\delta_{ik} - \theta_k\frac{d}{d\theta_i}f \tag{85}$$

we obtain the interesting result

$$\int d\theta_k f = f_k^{(1)} + f_{kl}^{(2)}\theta_l - f_{lk}^{(2)}\theta_l + ... = \frac{d}{d\theta_k}f \tag{86}$$

meaning differentiation and integration regarding a Grassmann variable are identical. Then we see

$$\int d\theta_k d\theta_l f = -\int d\theta_l d\theta_k f \tag{87}$$

$$\int d\theta_n ... d\theta_1 f = f^{(n)} \tag{88}$$

For a hermitian matrix A we can do the standard trick - performing a change of variables which diagonalizes A BUT the sheet did not explicitly make this restriction. So we need to try another way

With $f(\theta_1, ..., \theta_N, \theta^{*1}, ..., \theta^{*N}) = e^{-\langle\theta, A\theta\rangle}$

$$Z[0,0] = \int d^{2N}\theta \; e^{-\langle\theta, A\theta\rangle} \tag{89}$$

$$= \left(\prod_{k=1}^{N} \int d\theta^{*k} d\theta_k\right) \left(1 - \langle\theta, A\theta\rangle + \frac{1}{2!}\langle\theta, A\theta\rangle\langle\theta, A\theta\rangle - \frac{1}{3!}...\right) \tag{90}$$

$$= \left(\prod_{k=1}^{N} \int d\theta^{*k} d\theta_k\right) \left(1 - \theta^{*i} A_i^{\,j} \theta_j + \frac{1}{2!}(\theta^{*i} A_i^{\,j} \theta_j)(\theta^{*l} A_l^{\,m} \theta_m) - \frac{1}{3!}...\right) \tag{91}$$

the last term is the finite (see above) series is

$$f^{2N}\theta_1 ... \theta_N \theta^{*1} ... \theta^{*N} = \frac{1}{N!}(\theta^{*i} A_i^{\,j} \theta_j)^N \tag{92}$$

$$= \frac{1}{N!}(\theta^{*i_1} A_{i_1}^{\,j_1} \theta_{j_1}) ... (\theta^{*i_N} A_{i_N}^{\,j_N} \theta_{j_N}) \tag{93}$$

$$= \frac{1}{N!}\epsilon_{i_1 ... i_N}\epsilon_{j_1 ... j_N}\theta^{*1}\theta_1 ... \theta^{*N}\theta_N A_{i_1}^{\,j_1} ... A_{i_N}^{\,j_N} \tag{94}$$

$$= \frac{1}{N!}\epsilon_{j_1 ... j_N} A_1^{\,j_1} ... A_N^{\,j_N} \theta^{*1}\theta_1 ... \theta^{*N}\theta_N \tag{95}$$

$$= \det A \, \theta^{*1}\theta_1 ... \theta^{*N}\theta_N \tag{96}$$

$$= \det A \, \theta^{*N}\theta_N ... \theta^{*1}\theta_1 \tag{97}$$

as shown above - the integration over all $2N$ Grassmann variables is only survived by the last term - so

$$Z[0,0] = \det(A). \tag{98}$$

Now we can calculate

$$Z[\eta, \eta^*] = \int d^{2N}\theta \; e^{-\langle\theta, A\theta\rangle + \langle\eta, \theta\rangle + \langle\theta, \eta\rangle} \tag{99}$$

by completing the square (**now we require that A is also invertible**)

$$-\langle\theta, A\theta\rangle + \langle\eta, \theta\rangle + \langle\theta, \eta\rangle = -\langle(\theta - A^{-1}\eta), A(\theta - A^{-1}\eta)\rangle + \langle\eta, A^{-1}\eta\rangle \tag{100}$$

we can split the exponential and pull the η part in front of the integral - shifting (offset) of the integration variables does not change the result and we obtain with above

$$Z[\eta, \eta^*] = e^{\langle\eta, A^{-1}\eta\rangle} \int d^{2N}\theta \; e^{-\langle(\theta - A^{-1}\eta), A(\theta - A^{-1}\eta)\rangle} \tag{101}$$

$$= \det(A) e^{\langle\eta, A^{-1}\eta\rangle} \tag{102}$$

4. Being a reasonably lax with commuting of integral and derivative we can write

$$\int d^{2N}\theta \; \theta_i \theta^{*j} e^{-\langle\theta, A\theta\rangle} = -\frac{d}{d\eta^{*j}}\frac{d}{d\eta_i}\int d^{2N}\theta \; e^{-\langle\theta, A\theta\rangle + \langle\eta, \theta\rangle + \langle\theta, \eta\rangle}\bigg|_{\eta_i = 0 = \eta^{*j}} \tag{103}$$

$$= -\frac{d}{d\eta^{*j}}\frac{d}{d\eta_i}\det(A) e^{\langle\eta, A^{-1}\eta\rangle}\bigg|_{\eta_i = 0 = \eta^{*j}} \tag{104}$$

$$= -\det(A)\;\frac{d}{d\eta^{*j}}\frac{d}{d\eta_i}(1 + \langle\eta, A^{-1}\eta\rangle)\bigg|_{\eta_i = 0 = \eta^{*j}} \tag{105}$$

$$= -\det(A)(A^{-1}_{ij}) \tag{106}$$

3 Quantum Field Theory II – Exercise sheet 3 (2025-05-28)

3.1 Exercise 1 - Tree-level= Classical Field Theory

We want to understand the claim that tree-level diagrams correspond to classical field theory (which is equivalently stated by saying that $\hbar$ is the loop counting parameter). We consider ϕ^3 theory with action:

$$S[\phi] = \int d^4x \left(-\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{3!}\phi^3 + J\phi \right), \tag{1}$$

where J is a fixed (non-dynamical) external source. Our main goal is to solve the field equations as a perturbation theory in λ, making the power series ansatz:

$$\phi = \phi_0 + \lambda\phi_1 + \lambda^2\phi_2 + \lambda^3\phi_3 + \cdots .$$

1) Determine the Euler-Lagrange equations of (1) and use this to write the field equations for ϕ_0, ϕ_1, and ϕ_2.
2) Solve the above equations for ϕ_0, ϕ_1, and ϕ_2 in terms of J by use of a suitable Greens function like the Feynman propagator $D_F(x-y)$.
3) Find a graphical notation to represent the above solutions and use these Feynman diagrams to determine the solution to order λ^3, i.e., to determine ϕ_3. Convince yourself directly that the equations hold.

   Hint: Nobody can stop you from reading section 3.5 of the book by Schwartz.
4) Consider now the 'on-shell action' obtained by substituting the solution $\phi(J)$ into (1):

$$S_{\text{on-shell}}[J] := S[\phi(J)]. \tag{2}$$

   The claim is that the n-point tree-level amplitude can be obtained from the on-shell action via:

$$(2\pi)^4\delta^{(4)}(k_1+\cdots+k_n)M_n^{\text{tree}}(k_1,\ldots,k_n) = (-1)^n \prod_{i=1}^{n} \int d^4x_i\, e^{ik_i\cdot x_i} \left(\Box_{x_i} - m^2\right) \frac{\delta}{\delta J(x_i)} S_{\text{on-shell}}[J]\bigg|_{J=0}. \tag{3}$$

   Check this claim by looking at the $2 \to 2$ scattering.

1) From $\delta S = 0$ we obtain the Euler-Lagrange equations - so we calculate the terms

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi)} &= -\frac{1}{2}\frac{\partial g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi}{\partial(\partial_\alpha\phi)} && (107)\\
&= -\frac{1}{2}(g^{\mu\nu}\delta^\alpha_\mu\partial_\nu\phi + g^{\mu\nu}\partial_\mu\phi\delta^\alpha_\nu) && (108)\\
&= -\partial^\alpha\phi && (109)\\
\frac{\partial\mathcal{L}}{\partial\phi} &= -m^2\phi - \frac{\lambda}{2}\phi^2 + J && (110)\\
&\to -\partial_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)} + \frac{\partial\mathcal{L}}{\partial\phi} = 0 && (111)\\
&\to \partial^\alpha\partial_\alpha\phi - m^2\phi - \frac{\lambda}{2}\phi^2 + J = 0 && (112)\\
&\to (\Box - m^2)\phi - \frac{\lambda}{2}\phi^2 + J = 0 && (113)
\end{aligned}$$

As we are working in $g^{\mu\nu} = \text{diag}(-1,1,1,1)$ (meaning $\Box = -\partial_t^2 + \triangle$) the signs of the KG equation are ok. The substituting in the series expansion

$$(\Box - m^2)(\phi_0 + \lambda\phi_1 + \lambda^2\phi_2 + \cdots) - \frac{\lambda}{2}(\phi_0 + \lambda\phi_1 + \lambda^2\phi_2 + \cdots)^2 + J = 0$$

$$(\Box - m^2)(\phi_0 + \lambda\phi_1 + \lambda^2\phi_2 + \cdots) - \frac{\lambda}{2}(\phi_0^2 + 2\phi_0\phi_1\lambda + (\phi_1^2 + 2\phi_0\phi_2)\lambda^2 + (2\phi_1\phi_2 + 2\phi_0\phi_3)\lambda^3 + \cdots) + J = 0$$

we obtain

$$\begin{aligned}
\lambda^0 :&\quad (\Box - m^2)\phi_0 = -J && (114)\\
\lambda^1 :&\quad (\Box - m^2)\phi_1 = \frac{1}{2}\phi_0^2 && (115)\\
\lambda^2 :&\quad (\Box - m^2)\phi_2 = \phi_0\phi_1 && (116)\\
\lambda^3 :&\quad (\Box - m^2)\phi_3 = \frac{1}{2}(\phi_1^2 + 2\phi_0\phi_2) && (117)
\end{aligned}$$

2) With the definition of the Feynman propagator D_F

$$\begin{aligned}
D_F(x,y) &= i\int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2+m^2-i\epsilon}e^{ip(x-y)} &(118)\\
\to (\Box_x - m^2)D_F(x,y) &= -\int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2+m^2-i\epsilon}e^{ip(x-y)}[i^2(-(p_0)^2+\vec{p}^2)-m^2] &(119)\\
&= -i\int \frac{d^4p}{(2\pi)^4}\frac{-(p^2+m^2)}{p^2+m^2-i\epsilon}e^{ip(x-y)} &(120)\\
&= +i\int \frac{d^4p}{(2\pi)^4}e^{ip(x-y)} &(121)\\
&= i\delta^{(4)}(x-y) &(122)
\end{aligned}$$

we find

$$\begin{aligned}
\phi_0(x) &= \int d^4y\, D_F(x-y)[iJ(y)] &(123)\\
&= i\int d^4y\, D_F(x-y)J(y) &(124)
\end{aligned}$$

and

$$\begin{aligned}
\phi_1(x) &= (-i)\frac{1}{2}\int d^4y\, D_F(x-y)\left(\underbrace{i\int d^4z_1\, D_F(y-z_1)J(z_1)}_{=\phi_0(y)}\cdot\underbrace{i\int d^4z_2\, D_F(y-z_2)J(z_2)}_{=\phi_0(y)}\right) &(125)\\
&= \frac{i}{2}\int d^4y\, D_F(x-y)\left(\int d^4z_1\, D_F(y-z_1)J(z_1)\cdot\int d^4z_2\, D_F(y-z_2)J(z_2)\right) &(126)\\
&= \frac{i}{2}\iiint d^4y\, d^4z_1\, d^4z_2\, D_F(x-y)D_F(y-z_1)J(z_1)D_F(y-z_2)J(z_2) &(127)
\end{aligned}$$

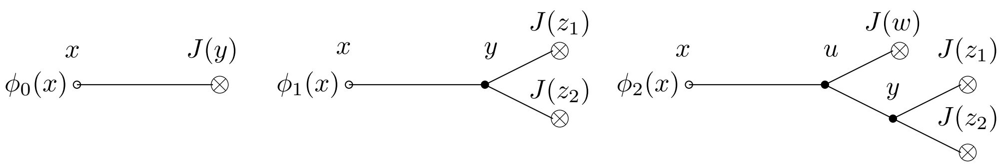
and

$$\phi_2(x) = (-i)\int d^4u\, D_F(x-u) \tag{128}$$

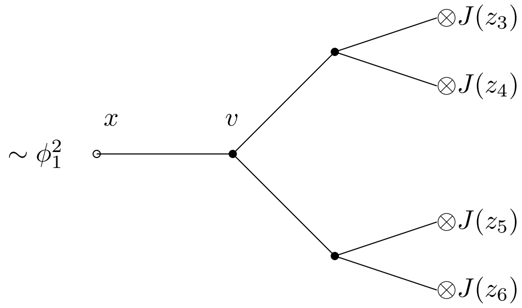
$$\left(\underbrace{i\int d^4wD_F(u-w)J(w)}_{=\phi_0(u)}\cdot\underbrace{\frac{i}{2}\iiint d^4y\, d^4z_1\, d^4z_2\, D_F(u-y)D_F(y-z_1)J(z_1)D_F(y-z_2)J(z_2)}_{=\phi_1(u)}\right) \tag{129}$$

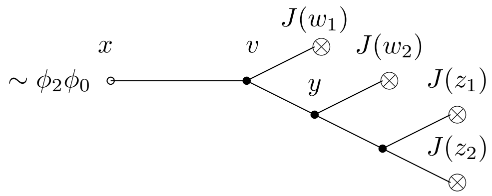
$$= \frac{i}{2}\int\!\!\int\!\!\int\!\!\int\!\!\int d^4u\, d^4w\, d^4y\, d^4z_1\, d^4z_2\, D_F(x-u)D_F(u-w)J(w)D_F(u-y)D_F(y-z_1)J(z_1)D_F(y-z_2)J(z_2) \tag{130}$$

3) Graphical representation



Now constructing the λ^3 term $\phi_3(x)$ (three black nodes)





We can also calculate - and obtain the same result

$$\phi_3(x) = (-i)\int dv\, D_F(x-v)\left(2\,\underbrace{i\int dw_1\, D_F(v-w_1)J(w_1)}_{=\phi_0(v)}\cdot\right. \tag{131}$$

$$\cdot\underbrace{\frac{i}{2}\iiiint\!\!\int du\, dw_2\, dy\, dz_1\, dz_2\, D_F(v-u)D_F(u-w_2)J(w_2)D_F(u-y)D_F(y-z_1)J(z_1)D_F(y-z_2)J(z_2)}_{=\phi_2(v)}+ \tag{132}$$

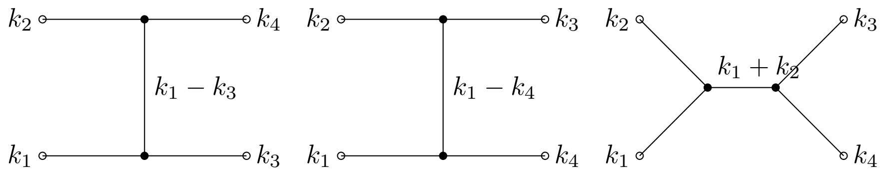
$$+\underbrace{\frac{i}{2}\iiint dy_1\, dz_3\, dz_4\, D_F(v-y_1)D_F(y_1-z_3)J(z_3)D_F(y_1-z_4)J(z_4)}_{\phi_1(v)}\cdot \tag{133}$$

$$\left.\cdot\underbrace{\frac{i}{2}\iiint dy_2\, dz_5\, dz_6\, D_F(v-y_2)D_F(y_2-z_5)J(z_5)D_F(y_2-z_6)J(z_6)}_{\phi_1(v)}\right) \tag{134}$$

$$= i\int dv\, dw_1\, du\, dw_2\, dy\, dz_1\, dz_2\, D_{xv}D_{vw_1}J(w_1)D_{vu}D_{uw_2}J(w_2)D_{uy}D_{yz_1}J(z_1)D_{yz_2}J(z_2) \tag{135}$$

$$+\frac{i}{4}\int dv\, dy_1\, dz_3\, dz_4\, dy_2\, dz_5\, dz_6\, D_{xv}D_{vy_1}D_{y_1z_3}J(z_3)D_{y_1z_4}J(z_4)D_{vy_2}D_{y_2z_5}J(z_5)D_{y_2z_6}J(z_6) \tag{136}$$

4) Intuitively we expect only stuff to happen at λ^2 order because the only relevant (low order) tree level diagrams are. We calculate anyway - so substituting (shortening the notation $D_F(x-y) \equiv D_{xy}$)



$$S_{\text{on-shell}}[J] = S[\phi(J)] \tag{137}$$

$$= S[\phi_0 + \lambda\phi_1 + \lambda^2\phi_2 + ...] \tag{138}$$

$$= \int d^4x \left(-\frac{1}{2}(\Box - m^2)(\phi_0 + \lambda\phi_1 + \lambda^2\phi_2 + ...) - \frac{\lambda}{3!}(\phi_0 + \lambda\phi_1 + \lambda^2\phi_2 + ...)^3 + J(\phi_0 + \lambda\phi_1 + \lambda^2\phi_2 + ...) \right) \tag{139}$$

$$= \int d^4x \left(-\frac{1}{2}(-J + \lambda\frac{1}{2}\phi_0^2 + \lambda^2\phi_0\phi_1 + ...) - \frac{\lambda}{3!}(\phi_0^3 + 3\phi_1\phi_0^2\lambda + 3\phi_0(\phi_1^2 + \phi_0\phi_2)\lambda^2 + ...) + J(\phi_0 + \lambda\phi_1 + \lambda^2\phi_2 + ...) \right) \tag{140}$$

$$= \int d^4x\, J\left(\phi_0 + \frac{1}{4}\right) + \lambda \int d^4x \left(-\frac{1}{6}\phi_0^2\left[\phi_0 + \frac{3}{2}\right] + J\phi_1 \right) + \lambda^2 \int d^4x \left(-\frac{1}{2}\phi_0\phi_1(\phi_0 + 1) + J\phi_2 \right) + ... \tag{141}$$

Now we look at the individual contributions (shortening notation) - and doing to first functional integral in baby steps

$$S_0 = \int d^4x\, J\left(\phi_0 + \frac{1}{4}\right) \tag{142}$$

$$= \int dx\, J(x)\left(i\int dy\, D_F(x-y)J(y) + \frac{1}{4} \right) \tag{143}$$

$$= \frac{1}{4}\int dx\, J(x) + i\int dx\, dy J(x)D_{xy}J(y) \tag{144}$$

$$\to \frac{\delta S_0[J]}{\delta J(x_i)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\frac{1}{4}\int dx\, \left[J(x) + \epsilon\delta(x - x_i)\right] - \frac{1}{4}\int dx\, J(x)\right] \tag{145}$$

$$+ i\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[i\int dx\, dy (J(x) + \epsilon\delta(x-x_i))D_{xy}(J(y) + \epsilon\delta(y - x_i)) - i\int dx\, dy J(x)D_{xy}J(y)\right] \tag{146}$$

$$= \frac{1}{4}\int dx\, \delta(x - x_j) + i\int dx\, dy\, [\delta(x - x_i)D_{xy}J(y) + J(x)D_{xy}\delta(y - x_i)] \tag{147}$$

$$= \frac{1}{4} + i\int dy\, D_{x_i y}J(y) + i\int dx\, D_{x x_i}J(x) \tag{148}$$

this terms contains only one J - performing the other three functional derivatives $\delta/\delta J(x_k)$ will result in a zero. Now we can calculate a bit faster

$$S_1 = -\frac{1}{6}\int dx\, \phi_0^3 - \frac{1}{4}\int dx\, \phi_0^2 + \int dx\, J\phi_1 \tag{149}$$

$$= -\frac{i^3}{6}\int dx\, dy_1\, dy_2\, dy_3\, D_{xy_3}J(y_3)D_{xy_2}J(y_2)D_{xy_1}J(y_1) - \frac{i^2}{4}\int dx\, dy_1\, dy_2 D_{xy_2}J(y_2)D_{xy_1}J(y_1) \tag{150}$$

$$+ \frac{i}{2}\int dx\, dy\, dz_1\, dz_2\, J(x)\, D_{xy}D_{yz_1}J(z_1)D_{yz_2}J(z_2) \tag{151}$$

$$\to \frac{\delta S_1[J]}{\delta J(x_i)} = -\frac{i^3}{6}\int dx\, dy_1\, dy_2\, D_{xx_i}D_{xy_2}J(y_2)D_{xy_1}J(y_1) - \frac{i^3}{6}\int dx\, dy_1\, dy_3\, D_{xy_3}J(y_3)D_{xx_i}D_{xy_1}J(y_1) \tag{152}$$

$$- \frac{i^3}{6}\int dx\, dy_2\, dy_3\, D_{xy_3}J(y_3)D_{xy_2}J(y_2)D_{xx_1} \tag{153}$$

$$- \frac{i^2}{4}\int dx\, dy_1\, D_{xx_i}D_{xy_1}J(y_1) - \frac{i^2}{4}\int dx\, dy_2 D_{xy_2}J(y_2)D_{xx_i} \tag{154}$$

$$+ \frac{i}{2}\int dy\, dz_1\, dz_2\;\; D_{x_1 y}D_{yz_1}J(z_1)D_{yz_2}J(z_2) + \frac{i}{2}\int dx\, dy\, dz_2\, J(x)\, D_{xy}D_{yx_1}D_{yz_2}J(z_2) \tag{155}$$

$$+ \frac{i}{2}\int dx\, dy\, dz_1\, J(x)\, D_{xy}D_{yz_1}J(z_1)D_{yx_i} \tag{156}$$

this terms contains only two J - performing the other three functional derivatives $\delta/\delta J(x_k)$ will result in a zero. And

$$S_2 = -\frac{1}{2}\int dx\,\phi_0^2\phi_1 - \frac{1}{2}\int dx\,\phi_0\phi_1 + \int dx\,J(x)\phi_2 \tag{157}$$

$$= -\frac{i^2}{2}\int dx\,dy_1 d\,y_2\,D_{xy_2}J(y_2)D_{xy_1}J(y_1)\frac{i}{2}\iiint d^4y\,d^4z_1\,d^4z_2\,D_{xy}D_{yz_1}J(z_1)D_{yz_2}J(z_2) \tag{158}$$

$$-\frac{i^2}{2}\int dx\,dy_1 d\,D_{xy_2}J(y_2)\frac{i}{2}\iiint d^4y\,d^4z_1\,d^4z_2\,D_{xy}D_{yz_1}J(z_1)D_{yz_2}J(z_2) \tag{159}$$

$$+\int dx\,J(x)\frac{i}{2}\int du\,dw\,dy\,dz_1\,dz_2\,D_{xu}D_{uw}J(w)D_{uy}D_{yz_1}J(z_1)D_{yz_2}J(z_2) \tag{160}$$

$$= -\frac{i^3}{4}\int dx\,dy_1 d\,y_2\,dy\,dz_1\,dz_2\,D_{xy_2}J(y_2)D_{xy_1}J(y_1)D_{xy}D_{yz_1}J(z_1)D_{yz_2}J(z_2) \tag{161}$$

$$+\frac{i}{2}\int dx\,du\,dw\,dy\,dz_1\,dz_2\;J(x)D_{xu}D_{uw}J(w)D_{uy}D_{yz_1}J(z_1)D_{yz_2}J(z_2) + \mathcal{O}(J^3) \tag{162}$$

Now first integral - one derivative

$$\frac{\delta}{\delta J(x_1)}\int dx\,dy_1\,dy_2\,dy\,dz_1\,dz_2\,D_{xy_2}J(y_2)D_{xy_1}J(y_1)D_{xy}D_{yz_1}J(z_1)D_{yz_2}J(z_2) \tag{163}$$

$$= \int dx\,dy_1\,dy\,dz_1\,dz_2\,D_{xx_1}D_{xy_1}J(y_1)D_{xy}D_{yz_1}J(z_1)D_{yz_2}J(z_2) \tag{164}$$

$$+\int dx\,dy_2\,dy\,dz_1\,dz_2\,D_{xy_2}J(y_2)D_{xx_1}D_{xy}D_{yz_1}J(z_1)D_{yz_2}J(z_2) \tag{165}$$

$$+\int dx\,dy_1\,dy_2\,dy\,dz_2\,D_{xy_2}J(y_2)D_{xy_1}J(y_1)D_{xy}D_{yx_1}D_{yz_2}J(z_2) \tag{166}$$

$$+\int dx\,dy_1\,dy_2\,dy\,dz_1\,D_{xy_2}J(y_2)D_{xy_1}J(y_1)D_{xy}D_{yz_1}J(z_1)D_{yz_1} \tag{167}$$

two derivatives

$$\frac{\delta}{\delta J(x_2)}\frac{\delta}{\delta J(x_1)}\int dx\,dy_1\,dy_2\,dy\,dz_1\,dz_2\,D_{xy_2}J(y_2)D_{xy_1}J(y_1)D_{xy}D_{yz_1}J(z_1)D_{yz_2}J(z_2) \tag{168}$$

$$= +\int dx\,dy\,dz_1\,dz_2\,D_{xx_1}D_{xx_2}D_{xy}D_{yz_1}J(z_1)D_{yz_2}J(z_2) \tag{169}$$

$$+\int dx\,dy_1\,dy\,dz_2\,D_{xx_1}D_{xy_1}J(y_1)D_{xy}D_{yx_2}D_{yz_2}J(z_2) \tag{170}$$

$$+\int dx\,dy_1\,dy\,dz_1\,D_{xx_1}D_{xy_1}J(y_1)D_{xy}D_{yz_1}J(z_1)D_{yx_2} \tag{171}$$

$$+\int dx\,dy_2\,dy\,dz_2\,D_{xy_2}J(y_2)D_{xx_1}D_{xy}D_{yx_2}D_{yz_2}J(z_2) \tag{172}$$

$$+\int dx\,dy\,dz_1\,dz_2\,D_{xx_2}D_{xx_1}D_{xy}D_{yz_1}J(z_1)D_{yz_2}J(z_2) \tag{173}$$

$$+\int dx\,dy_2\,dy\,dz_1\,D_{xy_2}J(y_2)D_{xx_1}D_{xy}D_{yz_1}J(z_1)D_{yx_2} \tag{174}$$

$$+\int dx\,dy_1\,dy\,dz_2\,D_{xx_2})D_{xy_1}J(y_1)D_{xy}D_{yx_1}D_{yz_2}J(z_2) \tag{175}$$

$$+\int dx\,dy_2\,dy\,dz_2\,D_{xy_2}J(y_2)D_{xx_2}D_{xy}D_{yx_1}D_{yz_2}J(z_2) \tag{176}$$

$$+\int dx\,dy_1\,dy_2\,dy\,D_{xy_2}J(y_2)D_{xy_1}J(y_1)D_{xy}D_{yx_1}D_{yx_2} \tag{177}$$

$$+\int dx\,dy_1\,dy\,dz_1\,D_{xx_2}D_{xy_1}J(y_1)D_{xy}D_{yz_1}J(z_1)D_{yz_1} \tag{178}$$

$$+\int dx\,dy_2\,dy\,dz_1\,D_{xy_2}J(y_2)D_{xx_2}D_{xy}D_{yz_1}J(z_1)D_{yz_1} \tag{179}$$

$$+\int dx\,dy_1\,dy_2\,dy\,D_{xy_2}J(y_2)D_{xy_1}J(y_1)D_{xy}D_{yx_2}D_{yz_1} \tag{180}$$

three derivatives

$$\frac{\delta}{\delta J(x_3)}\frac{\delta}{\delta J(x_2)}\frac{\delta}{\delta J(x_1)}\int dx\,dy_1\,dy_2\,dy\,dz_1\,dz_2\,D_{xy_2}J(y_2)D_{xy_1}J(y_1)D_{xy}D_{yz_1}J(z_1)D_{yz_2}J(z_2) \tag{181}$$

$$= +\int dx\,dy\,dz_2\,D_{xx_1}D_{xx_2}D_{xy}D_{yx_3}D_{yz_2}J(z_2) + \int dx\,dy\,dz_1\,D_{xx_1}D_{xx_2}D_{xy}D_{yz_1}J(z_1)D_{yx_3} \tag{182}$$

$$+\int dx\,dy\,dz_2\,D_{xx_1}D_{xx_3}D_{xy}D_{yx_2}D_{yz_2}J(z_2) + \int dx\,dy_1\,dy\,D_{xx_1}D_{xy_1}J(y_1)D_{xy}D_{yx_2}D_{yx_3} \tag{183}$$

$$+\int dx\,dy\,dz_1\,D_{xx_1}D_{xx_3}D_{xy}D_{yz_1}J(z_1)D_{yx_2} + \int dx\,dy_1\,dy\,D_{xx_1}D_{xy_1}J(y_1)D_{xy}D_{yx_3}D_{yx_2} \tag{184}$$

$$+\int dx\,dy\,dz_2\,D_{xx_3}D_{xx_1}D_{xy}D_{yx_2}D_{yz_2}J(z_2) + \int dx\,dy_2\,dy\,D_{xy_2}J(y_2)D_{xx_1}D_{xy}D_{yx_2}D_{yx_3} \tag{185}$$

$$+\int dx\,dy\,dz_2\,D_{xx_2}D_{xx_1}D_{xy}D_{yx_3}D_{yz_2}J(z_2) + \int dx\,dy\,dz_1\,D_{xx_2}D_{xx_1}D_{xy}D_{yz_1}J(z_1)D_{yx_3} \tag{186}$$

$$+\int dx\,dy\,dz_1\,D_{xx_3}D_{xx_1}D_{xy}D_{yz_1}J(z_1)D_{yx_2} + \int dx\,dy_2\,dy\,D_{xy_2}J(y_2)D_{xx_1}D_{xy}D_{yx_3}D_{yx_2} \tag{187}$$

$$+\int dx\,dy\,dz_2\,D_{xx_2})D_{xx_3}D_{xy}D_{yx_1}D_{yz_2}J(z_2) + \int dx\,dy_1\,dy\,D_{xx_2})D_{xy_1}J(y_1)D_{xy}D_{yx_1}D_{yx_3} \tag{188}$$

$$+\int dx\,dy\,dz_2\,D_{xx_3}D_{xx_2}D_{xy}D_{yx_1}D_{yz_2}J(z_2) + \int dx\,dy_2\,dy\,D_{xy_2}J(y_2)D_{xx_2}D_{xy}D_{yx_1}D_{yx_3} \tag{189}$$

$$+\int dx\,dy_1\,dy\,D_{xx_3}D_{xy_1}J(y_1)D_{xy}D_{yx_1}D_{yx_2} + \int dx\,dy_2\,dy\,D_{xy_2}J(y_2)D_{xx_3}D_{xy}D_{yx_1}D_{yx_2} \tag{190}$$

$$+\int dx\,dy\,dz_1\,D_{xx_2}D_{xx_3}D_{xy}D_{yz_1}J(z_1)D_{yz_1} + \int dx\,dy_1\,dy\,D_{xx_2}D_{xy_1}J(y_1)D_{xy}D_{yx_3}D_{yz_1} \tag{191}$$

$$+\int dx\,dy\,dz_1\,D_{xx_3}D_{xx_2}D_{xy}D_{yz_1}J(z_1)D_{yz_1} + \int dx\,dy_2\,dy\,D_{xy_2}J(y_2)D_{xx_2}D_{xy}D_{yx_3}D_{yz_1} \tag{192}$$

$$+\int dx\,dy_2\,dy\,D_{xx_3}D_{xy_1}J(y_1)D_{xy}D_{yx_2}D_{yz_1} + \int dx\,dy_1\,dy\,D_{xy_2}J(y_2)D_{xx_3}D_{xy}D_{yx_2}D_{yz_1} \tag{193}$$

four derivatives

$$\frac{\delta}{\delta J(x_4)}\frac{\delta}{\delta J(x_3)}\frac{\delta}{\delta J(x_2)}\frac{\delta}{\delta J(x_1)}\int dx\,dy_1\,dy_2\,dy\,dz_1\,dz_2\,D_{xy_2}J(y_2)D_{xy_1}J(y_1)D_{xy}D_{yz_1}J(z_1)D_{yz_2}J(z_2) \tag{194}$$

$$= +\int dx\,dy\,D_{xx_1}D_{xx_2}D_{xy}D_{yx_3}D_{yx_4} + \int dx\,dy\,D_{xx_1}D_{xx_2}D_{xy}D_{yx_4}D_{yx_3} \tag{195}$$

$$+\int dx\,dy\,D_{xx_1}D_{xx_3}D_{xy}D_{yx_2}D_{yx_4} + \int dx\,dy\,D_{xx_1}D_{xx_4}D_{xy}D_{yx_2}D_{yx_3} \tag{196}$$

$$+\int dx\,dy\,D_{xx_1}D_{xx_3}D_{xy}D_{yx_4}D_{yx_2} + \int dx\,dy\,D_{xx_1}D_{xx_4}D_{xy}D_{yx_3}D_{yx_2} \tag{197}$$

$$+\int dx\,dy\,D_{xx_3}D_{xx_1}D_{xy}D_{yx_2}D_{yx_4} + \int dx\,dy\,D_{xx_4}D_{xx_1}D_{xy}D_{yx_2}D_{yx_3} \tag{198}$$

$$+\int dx\,dy\,D_{xx_2}D_{xx_1}D_{xy}D_{yx_3}D_{yx_4} + \int dx\,dy\,D_{xx_2}D_{xx_1}D_{xy}D_{yx_4}D_{yx_3} \tag{199}$$

$$+\int dx\,dy\,D_{xx_3}D_{xx_1}D_{xy}D_{yx_4}D_{yx_2} + \int dx\,dy\,D_{xx_4}D_{xx_1}D_{xy}D_{yx_3}D_{yx_2} \tag{200}$$

$$+\int dx\,dy\,D_{xx_2}D_{xx_3}D_{xy}D_{yx_1}D_{yx_4} + \int dx\,dy\,D_{xx_2}D_{xx_4}D_{xy}D_{yx_1}D_{yx_3} \tag{201}$$

$$+\int dx\,dy\,D_{xx_3}D_{xx_2}D_{xy}D_{yx_1}D_{yx_4} + \int dx\,dy\,D_{xx_4}D_{xx_2}D_{xy}D_{yx_1}D_{yx_3} \tag{202}$$

$$+\int dx\,dy\,D_{xx_3}D_{xx_4}D_{xy}D_{yx_1}D_{yx_2} + \int dx\,dy\,D_{xx_4}D_{xx_3}D_{xy}D_{yx_1}D_{yx_2} \tag{203}$$

$$+\int dx\,dy\,D_{xx_2}D_{xx_3}D_{xy}D_{yx_4}D_{yz_1} + \int dx\,dy\,D_{xx_2}D_{xx_4}D_{xy}D_{yx_3}D_{yz_1} \tag{204}$$

$$+\int dx\,dy\,D_{xx_3}D_{xx_2}D_{xy}D_{yx_4}D_{yz_1} + \int dx\,dy\,D_{xx_4}D_{xx_2}D_{xy}D_{yx_3}D_{yz_1} \tag{205}$$

$$+\int dx\,dy\,D_{xx_3}D_{xx_4}D_{xy}D_{yx_2}D_{yz_1} + \int dx\,dy\,D_{xx_4}D_{xx_3}D_{xy}D_{yx_2}D_{yz_1} \tag{206}$$

$$\frac{\delta}{\delta J(x_4)}\frac{\delta}{\delta J(x_3)}\frac{\delta}{\delta J(x_2)}\frac{\delta}{\delta J(x_1)}\int dx\, dy_1\, dy_2\, dy\, dz_1\, dz_2\, D_{xy_2}J(y_2)D_{xy_1}J(y_1)D_{xy}D_{yz_1}J(z_1)D_{yz_2}J(z_2)$$
$$= 4\int dx\, dy\, D_{xx_1}D_{xx_2}D_{xy}D_{yx_3}D_{yx_4} + 4\int dx\, dy\, D_{xx_1}D_{xx_3}D_{xy}D_{yx_2}D_{yx_4} + 4\int dx\, dy\, D_{xx_1}D_{xx_4}D_{xy}D_{yx_2}D_{yx_3}$$
$$+4\int dx\, dy\, D_{xx_2}D_{xx_3}D_{xy}D_{yx_1}D_{yx_4} + 4\int dx\, dy\, D_{xx_2}D_{xx_4}D_{xy}D_{yx_1}D_{yx_3} + 4\int dx\, dy\, D_{xx_3}D_{xx_4}D_{xy}D_{yx_3}D_{yx_4}$$

Second integral - analog calculation

$$\frac{\delta}{\delta J(x_4)}\frac{\delta}{\delta J(x_3)}\frac{\delta}{\delta J(x_2)}\frac{\delta}{\delta J(x_1)}\int dx\, du\, dw\, dy\, dz_1\, dz_2\;\; J(x)D_{xu}D_{uw}J(w)D_{uy}D_{yz_1}J(z_1)D_{yz_2}J(z_2) \tag{207}$$

$$= 2\int du\, dy\;\; D_{x_1u}D_{ux_2}D_{uy}D_{yx_3}D_{yx_4} + 2\int du\, dy\;\; D_{x_2u}D_{ux_2}D_{uy}D_{yx_3}D_{yx_4} + 2\int du\, dy\;\; D_{x_1u}D_{ux_3}D_{uy}D_{yx_2}D_{yx_4} \tag{208}$$

$$+2\int du\, dy\;\; D_{x_3u}D_{ux_1}D_{uy}D_{yx_2}D_{yx_4} + 2\int du\, dy\;\; D_{x_1u}D_{ux_4}D_{uy}D_{yx_2}D_{yx_3} + 2\int du\, dy\;\; D_{x_4u}D_{ux_1}D_{uy}D_{yx_2}D_{yx_3} \tag{209}$$

$$+2\int du\, dy\;\; D_{x_2u}D_{ux_3}D_{uy}D_{yx_1}D_{yx_4} + 2\int du\, dy\;\; D_{x_3u}D_{ux_2}D_{uy}D_{yx_1}D_{yx_4} + 2\int du\, dy\;\; D_{x_2u}D_{ux_4}D_{uy}D_{yx_1}D_{yx_3} \tag{210}$$

$$+2\int du\, dy\;\; D_{x_4u}D_{ux_2}D_{uy}D_{yx_1}D_{yx_3} + 2\int du\, dy\;\; D_{x_3u}D_{ux_4}D_{uy}D_{yx_1}D_{yx_2} + 2\int du\, dy\;\; D_{x_4u}D_{ux_3}D_{uy}D_{yx_1}D_{yx_2} \tag{211}$$

This last two expression is all that survives the funtional derivatives and setting $J \to 0$.

Now we can calculate one example term of the first integral using the Klein-Gordon Greens function property of the Feynman propagator $(\Box_x - m^2)D_F(x-y) = i\delta^{(4)}(x-y)$

$$\int d^4x_1\, e^{i(k_1x_1)}(\Box_{x_1} - m^2)\int dx\, dy\, D_{xx_1}D_{xx_2}D_{xy}D_{yx_3}D_{yx_4} = \int dx\, dy\, e^{i(k_1x_1)}\, (-i)\delta^{(4)}(x-x_1)D_{xx_2}D_{xy}D_{yx_3}D_{yx_4} \tag{212}$$
$$= -i\int dx\, dy\, e^{i(k_1x)}\, D_{xx_2}D_{xy}D_{yx_3}D_{yx_4} \tag{213}$$

so we can generalize to the x_1, x_2, x_3, x_4 integration - and do the Fourier trafo of the Feynman propagator (via substitution $u = x - y, v = y$, meaning $x = u + v, y = v, dxdy = dudv$)

$$\int\prod_{i=1}^{4} d^4x_i\, e^{i(k_ix_i)}(\Box_{x_i} - m^2)\int dx\, dy\, D_{xx_1}D_{xx_2}D_{xy}D_{yx_3}D_{yx_4} = (-i)^4\int\; dx\, dy\, e^{i(k_1x+k_2x+k_3y+k_4y)}D_{xy} \tag{214}$$
$$= \int\; dx\, dy\, e^{i(k_1+k_2)x+i(k_3+k_4)y}D_F(x-y) \tag{215}$$
$$= \int\; du\, dv\, e^{i(k_1+k_2)(u+v)+i(k_3+k_4)v}D_F(u) \tag{216}$$
$$= \int dv\, e^{i(k_1+k_2+k_3+k_4)v}\cdot\int\; du\, e^{i(k_1+k_2)u}D_F(u) \tag{217}$$
$$= (2\pi)^4\delta^{(4)}(k_1+k_2+k_3+k_4)\frac{i}{(k_1+k_2)^2 - m^2 + i\epsilon} \tag{218}$$

we see that all the 6 different terms end up in the similar expression.

Now we can do one term of the second integral

$$\int \prod_{i=1}^{4} d^4x_i \, e^{i(k_i x_i)} (\Box_{x_i} - m^2) \int du\, dy\ D_{x_1 u} D_{u x_2} D_{uy} D_{y x_3} D_{y x_4} = (-i)^4 \int du\, dy\, e^{i(k_1 u + k_2 u + k_3 y + k_4 y)} D_F(u-y) \tag{219}$$

$$= (-i)^4 \int du\, dy\, e^{i(k_1+k_2)u} e^{i(k_3+k_4)y} D_F(u-y) \tag{220}$$

$$= (2\pi)^4 \delta^{(4)}(k_1+k_2+k_3+k_4) \frac{i}{(k_1+k_2)^2 - m^2 + i\epsilon} \tag{221}$$

So in total we should get three types of terms for the pairs (s, t, u-channels) k_1, k_2 and k_1, k_3 and k_1, k_4 (I would need some more time to collect all prefactors)

$$\int \prod_{i=1}^{4} d^4x_i \, e^{i(k_i x_i)} (\Box_{x_i} - m^2) \ \frac{\delta}{\delta J(x_i)} S[(J)]\bigg|_{J=0} = \lambda^2 \int \prod_{i=1}^{4} d^4x_i \, e^{i(k_i x_i)} (\Box_{x_i} - m^2) \ \frac{\delta}{\delta J(x_i)} S_2[(J)]\bigg|_{J=0} \tag{222}$$

$$\sim \lambda^2 (2\pi)^4 \delta^{(4)}(k_1+k_2+k_3+k_4) \left(\frac{1}{(k_1+k_2)^2 - m^2} + \frac{1}{(k_1+k_3)^2 - m^2} + \frac{1}{(k_1+k_4)^2 - m^2} \right) \tag{223}$$

4 Quantum Field Theory II – Exercise sheet 2 (2025-06-11)

4.1 Exercise 1 - Dimensional Regularization in QED

We consider the 1-loop vacuum polarization discussed in the lecture, for which we found

$$\Pi_2^{\mu\nu} = -4ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{2k^\mu k^\nu - \eta^{\mu\nu}(k^2 - p\cdot k + m^2)}{[(k-p)^2 + m^2][k^2 + m^2]} \tag{1}$$

Our goal is to compute this 1-loop integral in dimensional regularization.

1. Use the Feynman parameter trick to write the integral with a denominator that is a complete square.
2. Prove
$$\int \frac{d^Dk}{(2\pi)^D} \frac{k^{2a}}{(k^2-\Delta)^b} = i(-1)^{a-b} \frac{1}{(4\pi)^{D/2}} \frac{1}{\Delta^{b-a-\frac{D}{2}}} \frac{\Gamma\left(a+\frac{D}{2}\right)\Gamma\left(b-a-\frac{D}{2}\right)}{\Gamma(b)\Gamma\left(\frac{D}{2}\right)} \tag{2}$$
where Γ is the Euler gamma function, and write out the special cases $a=0, b=2$ and $a=1, b=2$.
3. Compute (1) in dimensional regularization setting $D = 4-\epsilon$. Give the result in the limit $p^2 \gg m^2$.

1. 1.) With the observation

$$\frac{1}{AB} = \int_0^1 \frac{1}{[At + B(1-t)]^2} dt \tag{224}$$

we can write

$$\begin{aligned}
\to \frac{2k^\mu k^\nu - \eta^{\mu\nu}(k^2 - p\cdot k + m^2)}{[(k-p)^2 + m^2][k^2 + m^2]} &= \int_0^1 dt \frac{2k^\mu k^\nu - \eta^{\mu\nu}(k^2 - p\cdot k + m^2)}{([(k-p)^2 + m^2]t + [k^2 + m^2](1-t))^2} && (225)\\
&= \int_0^1 dt \frac{2k^\mu k^\nu - \eta^{\mu\nu}(k^2 - p\cdot k + m^2)}{([k^2 - 2kp + p^2 + m^2]t + [k^2 + m^2](1-t))^2} && (226)\\
&= \int_0^1 dt \frac{2k^\mu k^\nu - \eta^{\mu\nu}(k^2 - p\cdot k + m^2)}{(k^2 - 2kpt + p^2t + m^2)^2} && (227)\\
&= \int_0^1 dt \frac{2k^\mu k^\nu - \eta^{\mu\nu}(k^2 - p\cdot k + m^2)}{([k-pt]^2 - p^2t^2 + p^2t + m^2)^2} && (228)\\
&= \int_0^1 dt \frac{2k^\mu k^\nu - \eta^{\mu\nu}(k^2 - p\cdot k + m^2)}{([k-pt]^2 + p^2t(1-t) + m^2)^2} && (229)
\end{aligned}$$

then with $q^\mu = k^\mu - p^\mu t$ and $\Delta = -(p^2t(1-t) + m^2) = p^2t(t-1) - m^2$

$$\begin{aligned}
\to \frac{2k^\mu k^\nu - \eta^{\mu\nu}(k^2 - p\cdot k + m^2)}{[(k-p)^2 + m^2][k^2 + m^2]} &= \int_0^1 dt \frac{2(q^\mu + p^\mu t)(q^\nu + p^\nu t) - \eta^{\mu\nu}[(q+pt)^2 - p(q+pt) + m^2]}{(q^2-\Delta)^2} && (230)\\
= \int_0^1 dt \frac{2(q^\mu q^\nu + (p^\mu q^\nu + p^\nu q^\mu)t + p^\mu p^\nu t^2) - \eta^{\mu\nu}[q^2 + 2q\cdot pt + p^2t^2 - p\cdot q - p^2t + m^2]}{(q^2-\Delta)^2} & && (231)\\
= \int_0^1 dt \frac{2(q^\mu q^\nu + (p^\mu q^\nu + p^\nu q^\mu)t + p^\mu p^\nu t^2) - \eta^{\mu\nu}[q^2 + q\cdot p(2t-1) + p^2(t-1)t + m^2]}{(q^2-\Delta)^2} & && (232)
\end{aligned}$$

we have with $d^4q = d^4k$ (the momentum shift does not change the integral measure)

$$\begin{aligned}
\Pi_2^{\mu\nu} &= -4ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{2k^\mu k^\nu - \eta^{\mu\nu}(k^2 - p\cdot k + m^2)}{[(k-p)^2 + m^2][k^2 + m^2]} && (233)\\
&= -4ie^2 \int_0^1 dt \int \frac{d^4q}{(2\pi)^4} \frac{2(q^\mu q^\nu + (p^\mu q^\nu + p^\nu q^\mu)t + p^\mu p^\nu t^2) - \eta^{\mu\nu}[q^2 + q\cdot p(2t-1) + p^2(t-1)t + m^2]}{(q^2-\Delta)^2} && (234)
\end{aligned}$$

Since the denominator is rotationally symmetric in q so the linear terms are vanishing (see substitution $q \to -q$: $d^4q\, q f(q^2) = 0$)

$$\begin{aligned}
\Pi_2^{\mu\nu} &= -4ie^2 \int_0^1 dt \int \frac{d^4q}{(2\pi)^4} \frac{2(q^\mu q^\nu + \cancel{(p^\mu q^\nu + p^\nu q^\mu)}t + p^\mu p^\nu t^2) - \eta^{\mu\nu}[q^2 + \cancel{q\cdot p}(2t-1) + p^2(t-1)t + m^2]}{(q^2-\Delta)^2} && (235)\\
&= -4ie^2 \int_0^1 dt \int \frac{d^4q}{(2\pi)^4} \frac{2(q^\mu q^\nu + p^\mu p^\nu t^2) - \eta^{\mu\nu}[q^2 + p^2(t-1)t + m^2]}{(q^2-\Delta)^2} && (236)
\end{aligned}$$

now we can split-off the q^2-part of the integrand (for the dimensional regularization it is important to leave the $q^\mu q^\nu$-part untouched for now)

$$\Pi_2^{\mu\nu} = -4ie^2 \int_0^1 dt \int \frac{d^4q}{(2\pi)^4} \frac{2p^\mu p^\nu t^2 + 2q^\mu q^\nu - \eta^{\mu\nu}[q^2 + p^2(t-1)t + m^2]}{(q^2-\Delta)^2} \tag{237}$$

$$= -4ie^2 \int_0^1 dt \int \frac{d^4q}{(2\pi)^4} \frac{2p^\mu p^\nu t^2 - \eta^{\mu\nu}[p^2(t-1)t + m^2]}{(q^2-\Delta)^2} - 4ie^2 \int_0^1 dt \int \frac{d^4q}{(2\pi)^4} \frac{2q^\mu q^\nu - \eta^{\mu\nu} q^2}{(q^2-\Delta)^2} \tag{238}$$

Now we can split-off the elementary t-integration (marking the $p^\mu p^\nu$ part in red)

$$\Pi_2^{\mu\nu} = -4ie^2(2p^\mu p^\nu) \int_0^1 dt\, t^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2-\Delta)^2} - 4ie^2 \eta^{\mu\nu} \int_0^1 dt\, (p^2(1-t)t - m^2) \int \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2-\Delta)^2} \tag{239}$$

$$- 4ie^2 \int_0^1 dt \int \frac{d^4q}{(2\pi)^4} \frac{2q^\mu q^\nu - \eta^{\mu\nu} q^2}{(q^2-\Delta)^2} \tag{240}$$

2.) The surface and volume of D-dimensional unit sphere are given by $S_{D-1} = D \cdot V_D = \frac{2\pi^{D/2}}{\Gamma(\frac{D}{2})}$ (from the old the standard trick converting of D-dimensional Gauss integral spherical coordinates and recognizing the Gamma function in the radial integration).

$$\int \frac{d^Dk}{(2\pi)^D} \frac{k^{2a}}{(k^2-\Delta)^b} = \frac{1}{(2\pi)^D} \int d^{D-1}\Omega \int_0^\infty dk\, k^{D-1} \frac{k^{2a}}{(k^2-\Delta)^b} \tag{241}$$

$$= \frac{1}{(2\pi)^D} \frac{1}{(-\Delta)^b} \frac{2\pi^{D/2}}{\Gamma(\frac{D}{2})} \int_0^\infty dk \frac{k^{2a+D-1}}{(1-k^2/\Delta)^b} \tag{242}$$

$$= \frac{1}{2^{D-1}\pi^{D/2}} \frac{1}{(-1)^b \Delta^b} \frac{1}{\Gamma(\frac{D}{2})} \int_0^\infty dk \frac{k^{2a+D-1}}{(1-k^2/\Delta)^b} \tag{243}$$

$$\overset{q^2=k^2/\Delta}{=} \frac{1}{2^{D-1}\pi^{D/2}} \frac{1}{(-1)^b \Delta^b} \frac{1}{\Gamma(\frac{D}{2})} \Delta^{(2a+D-1)/2+1/2} \int_0^\infty dq \frac{q^{2a+D-1}}{(1-q^2)^b} \tag{244}$$

$$= \frac{1}{2^{D-1}\pi^{D/2}} \frac{1}{(-1)^b \Delta^{b-a-D/2}} \frac{1}{\Gamma(\frac{D}{2})} \int_0^\infty dq \frac{q^{2a+D-1}}{(1-q^2)^b} \tag{245}$$

$$\overset{q=iy}{=} \frac{1}{2^{D-1}\pi^{D/2}} \frac{1}{\Delta^{b-a-D/2}} \frac{1}{\Gamma(\frac{D}{2})} (-1)^a(-1)^{-b} i \int_0^\infty dy \frac{y^{2a+D-1}}{(1+y^2)^b} \tag{246}$$

$$= \frac{1}{2^{D-1}\pi^{D/2}} \frac{1}{\Delta^{b-a-D/2}} \frac{1}{\Gamma(\frac{D}{2})} (-1)^a(-1)^{-b} i \frac{\Gamma(a+\frac{D}{2})\Gamma(b-a-\frac{D}{2})}{2\Gamma(b)} \tag{247}$$

$$= \frac{1}{(4\pi)^{D/2}} \frac{1}{\Delta^{b-a-D/2}} (-1)^{a-b} i \frac{\Gamma(a+\frac{D}{2})\Gamma(b-a-\frac{D}{2})}{\Gamma(b)\Gamma(\frac{D}{2})} \tag{248}$$

Using $\Gamma(2) = 1$ and $\Gamma\left(1+\frac{D}{2}\right) = \frac{D}{2}\Gamma\left(\frac{D}{2}\right)$

- Special case $a=0, b=2$

$$\int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2-\Delta)^2} = \frac{i}{(4\pi)^{D/2}} \frac{1}{\Delta^{2-\frac{D}{2}}} \Gamma\left(2-\frac{D}{2}\right) \tag{249}$$

- Special case $a=1, b=2$

$$\int \frac{d^Dk}{(2\pi)^D} \frac{k^2}{(k^2-\Delta)^2} = -\frac{i}{(4\pi)^{D/2}} \frac{1}{\Delta^{1-\frac{D}{2}}} \frac{\Gamma\left(1+\frac{D}{2}\right)\Gamma\left(1-\frac{D}{2}\right)}{\Gamma\left(\frac{D}{2}\right)} \tag{250}$$

$$= -\frac{i}{(4\pi)^{D/2}} \frac{1}{\Delta^{1-\frac{D}{2}}} \frac{D}{2} \Gamma\left(1-\frac{D}{2}\right) \tag{251}$$

3.) With

$$\int \frac{d^{4-\epsilon}k}{(2\pi)^{4-\epsilon}} \frac{1}{(k^2-\Delta)^2} = \frac{i}{(4\pi)^{2-\epsilon/2}} \frac{1}{\Delta^{2-\frac{4-\epsilon}{2}}} \Gamma\left(2-\frac{4-\epsilon}{2}\right) \tag{252}$$

$$= \frac{i}{(4\pi)^{2-\epsilon/2}} \frac{1}{\Delta^{\frac{\epsilon}{2}}} \Gamma\left(\frac{\epsilon}{2}\right) \tag{253}$$

$$\int \frac{d^{4-\epsilon}k}{(2\pi)^{4-\epsilon}} \frac{k^2}{(k^2-\Delta^2)^2} = -\frac{i}{(4\pi)^{2-\epsilon/2}} \frac{1}{\Delta^{1-\frac{4-\epsilon}{2}}} \left(\frac{4-\epsilon}{2}\right) \Gamma\left(1-\frac{4-\epsilon}{2}\right) \tag{254}$$

$$= -\frac{i}{(4\pi)^{2-\epsilon/2}} \frac{1}{\Delta^{\frac{\epsilon}{2}-1}} \left(2-\frac{\epsilon}{2}\right) \Gamma\left(\frac{\epsilon}{2}-1\right) \tag{255}$$

we can simplify with $q^\mu q^\nu = \frac{q^2}{D}\eta^{\mu\nu}$ and write the integrals in dimensional regularization and introducing a mass-dimension parameter μ and $D = 4 - \epsilon$

$$\Pi_2^{\mu\nu} = -4ie^2(2p^\mu p^\nu)\int_0^1 dt\, t^2 \int \frac{d^4q}{(2\pi)^4}\frac{1}{(q^2-\Delta)^2} - 4ie^2\eta^{\mu\nu}\int_0^1 dt\,(p^2(1-t)t - m^2)\int \frac{d^4q}{(2\pi)^4}\frac{1}{(q^2-\Delta)^2} \tag{256}$$

$$-4i\left(\frac{2}{D}-1\right)e^2\eta^{\mu\nu}\int_0^1 dt\int\frac{d^4q}{(2\pi)^4}\frac{q^2}{(q^2-\Delta)^2} \tag{257}$$

$$= -4ie^2\mu^\epsilon(2p^\mu p^\nu)\frac{i}{(4\pi)^{2-\epsilon/2}}\Gamma\left(\frac{\epsilon}{2}\right)\int_0^1 dt\,\frac{t^2}{\Delta^{\frac{\epsilon}{2}}} - 4ie^2\mu^\epsilon\eta^{\mu\nu}\frac{i}{(4\pi)^{2-\epsilon/2}}\Gamma\left(\frac{\epsilon}{2}\right)\int_0^1 dt\,\frac{p^2(1-t)t-m^2}{\Delta^{\frac{\epsilon}{2}}} \tag{258}$$

$$-4ie^2\mu^\epsilon\left(\frac{2}{4-\epsilon}-1\right)\eta^{\mu\nu}\frac{-i}{(4\pi)^{2-\epsilon/2}}\left(2-\frac{\epsilon}{2}\right)\Gamma\left(\frac{\epsilon}{2}-1\right)\int_0^1 dt\frac{1}{\Delta^{\frac{\epsilon}{2}-1}} \tag{259}$$

$$= (p^\mu p^\nu)\frac{8e^2\mu^\epsilon}{(4\pi)^{2-\epsilon/2}}\Gamma\left(\frac{\epsilon}{2}\right)\int_0^1 dt\,\frac{t^2}{\Delta^{\frac{\epsilon}{2}}} + \eta^{\mu\nu}\frac{4e^2\mu^\epsilon}{(4\pi)^{2-\epsilon/2}}\Gamma\left(\frac{\epsilon}{2}\right)\int_0^1 dt\,\frac{p^2(1-t)t-m^2}{\Delta^{\frac{\epsilon}{2}}} \tag{260}$$

$$-\eta^{\mu\nu}\frac{4e^2\mu^\epsilon}{(4\pi)^{2-\epsilon/2}}\left(\frac{\epsilon-2}{2}\right)\Gamma\left(\frac{\epsilon}{2}-1\right)\int_0^1 dt\frac{1}{\Delta^{\frac{\epsilon}{2}-1}} \tag{261}$$

For the limit we use the following series expansions

$$\mu^\epsilon = 1 + \log(\mu)\epsilon + \frac{1}{2}\log^2(\mu)\epsilon^2 + \dots \tag{262}$$

$$\frac{1}{(4\pi)^{2-\epsilon/2}} = \frac{1}{(4\pi)^2}\left(1 + \frac{1}{2}\log(4\pi)\epsilon + \frac{1}{8}\log^2(4\pi)\epsilon^2 + \dots\right) \tag{263}$$

$$\Gamma\left(\frac{\epsilon}{2}\right) = \frac{2}{\epsilon} - \gamma_{EM} + \frac{1}{24}(6\gamma_{EM}^2 + \pi^2)\epsilon + \frac{1}{24}[-\gamma_{EM}^3 - \gamma_{EM}\frac{\pi^2}{2} + \psi^{(2)}(1)]\epsilon^2 + \dots \tag{264}$$

$$\Gamma\left(\frac{\epsilon}{2}-1\right) = -\frac{2}{\epsilon} + (\gamma_{EM}-1) + \frac{1}{24}\left(-12 + 12\gamma_{EM} - 6\gamma_{EM}^2 - \pi^2\right)\epsilon + \dots \tag{265}$$

$$\frac{1}{\Delta^{\epsilon/2}} = 1 - \frac{1}{2}\log(\Delta)\epsilon + \frac{1}{8}\log^2(\Delta)\epsilon^2 + \dots \tag{266}$$

which gives combined

$$\frac{\mu^\epsilon}{(4\pi)^{-\epsilon/2}}\Gamma\left(\frac{\epsilon}{2}\right) = \frac{2}{\epsilon} + [2\log(\mu) - \gamma_{EM} + \log(4\pi)] + O(\epsilon) \tag{267}$$

$$= \frac{2}{\epsilon} + [\log(\mu^2) + \log(e^{-\gamma_{EM}}) + \log(4\pi)] + O(\epsilon) \tag{268}$$

$$= \frac{2}{\epsilon} + \log(4\pi\mu^2 e^{-\gamma_{EM}}) + O(\epsilon) \tag{269}$$

$$\frac{\mu^\epsilon}{(4\pi)^{-\epsilon/2}}\Gamma\left(\frac{\epsilon}{2}-1\right) = -\frac{2}{\epsilon} + (-2\log(\mu) + \gamma_{EM} - 1 - \log(4\pi)) + O(\epsilon) \tag{270}$$

$$= -\frac{2}{\epsilon} + (-\log(\mu^2) - \log(e^{-\gamma_{EM}}) - 1 - \log(4\pi)) + O(\epsilon) \tag{271}$$

$$= -\frac{2}{\epsilon} - 1 - \log(4\pi\mu^2 e^{-\gamma_{EM}}) + O(\epsilon) \tag{272}$$

then with $m^2 = p^2 t(t-1) - \Delta$

$$\Pi_2^{\mu\nu} = (p^\mu p^\nu)\frac{8e^2}{(4\pi)^2}\left[\frac{2}{\epsilon} + \log(4\pi\mu^2 e^{-\gamma_{EM}})\right]\int_0^1 dt\, t^2 \left(1 - \frac{1}{2}\log(\Delta)\epsilon + ...\right) \tag{273}$$

$$+ \eta^{\mu\nu}\frac{4e^2}{(4\pi)^2}\left[\frac{2}{\epsilon} + \log(4\pi\mu^2 e^{-\gamma_{EM}}) + ...\right]\int_0^1 dt\,[p^2(1-t)t - m^2]\left(1 - \frac{1}{2}\log(\Delta)\epsilon + ...\right) \tag{274}$$

$$- \eta^{\mu\nu}\frac{4e^2}{(4\pi)^2}(-1)\left[-\frac{2}{\epsilon} - 1 - \log(4\pi\mu^2 e^{-\gamma_{EM}}) + ...\right]\int_0^1 dt\,\Delta\left(1 - \frac{1}{2}\log^2(\Delta)\epsilon + ...\right) \tag{275}$$

$$= (p^\mu p^\nu)\frac{8e^2}{(4\pi)^2}\left[\frac{2}{\epsilon} + \log(4\pi\mu^2 e^{-\gamma_{EM}})\right]\int_0^1 dt\, t^2 \left(1 - \frac{1}{2}\log(\Delta)\epsilon + ...\right) \tag{276}$$

$$+ \eta^{\mu\nu}\frac{4e^2}{(4\pi)^2}\left[\frac{2}{\epsilon} + \log(4\pi\mu^2 e^{-\gamma_{EM}}) + ...\right]\int_0^1 dt\,[2p^2(1-t)t + \Delta]\left(1 - \frac{1}{2}\log(\Delta)\epsilon + ...\right) \tag{277}$$

$$- \eta^{\mu\nu}\frac{4e^2}{(4\pi)^2}\left[\frac{2}{\epsilon} + \log(4\pi\mu^2 e^{-\gamma_{EM}}) + 1 + ...\right]\int_0^1 dt\,\Delta\left(1 - \frac{1}{2}\log^2(\Delta)\epsilon + ...\right) \tag{278}$$

$$= (p^\mu p^\nu)\frac{8e^2}{(4\pi)^2}\left[\frac{2}{\epsilon} + \log(4\pi\mu^2 e^{-\gamma_{EM}})\right]\int_0^1 dt\, t^2 \left(1 - \frac{1}{2}\log(\Delta)\epsilon + ...\right) \tag{279}$$

$$+ \eta^{\mu\nu}\frac{4e^2}{(4\pi)^2}\left[\frac{2}{\epsilon} + \log(4\pi\mu^2 e^{-\gamma_{EM}}) + ...\right]\int_0^1 dt\,[2p^2(1-t)t + \Delta]\left(1 - \frac{1}{2}\log(\Delta)\epsilon + ...\right) \tag{280}$$

$$- \eta^{\mu\nu}\frac{4e^2}{(4\pi)^2}\left[\frac{2}{\epsilon} + \log(4\pi\mu^2 e^{-\gamma_{EM}}) + 1 + ...\right]\int_0^1 dt\,\Delta\left(1 - \frac{1}{2}\log^2(\Delta)\epsilon + ...\right) \tag{281}$$

Then taking $\epsilon \to 0$

$$\Pi_2^{\mu\nu} = (p^\mu p^\nu)\frac{8e^2}{(4\pi)^2}\int_0^1 dt\, t^2\left(1 - \frac{1}{2}\log(\Delta)\epsilon + ...\right)\left[\frac{2}{\epsilon} + \log(4\pi\mu^2 e^{-\gamma_{EM}})\right] \tag{282}$$

$$+ \eta^{\mu\nu}p^2\frac{8e^2}{(4\pi)^2}\int_0^1 dt\,(1-t)t\left(1 - \frac{1}{2}\log(\Delta)\epsilon + ...\right)\left[\frac{2}{\epsilon} + \log(4\pi\mu^2 e^{-\gamma_{EM}}) + ...\right] \tag{283}$$

$$- \eta^{\mu\nu}\frac{4e^2}{(4\pi)^2}\int_0^1 dt\,\Delta\left(1 - \frac{1}{2}\log^2(\Delta)\epsilon + ...\right) \tag{284}$$

$$= (p^\mu p^\nu)\frac{8e^2}{(4\pi)^2}\int_0^1 dt\, t^2\left[\frac{2}{\epsilon} + \log\left(\frac{4\pi\mu^2 e^{-\gamma_{EM}}}{\Delta}\right)\right] \tag{285}$$

$$+ \eta^{\mu\nu}p^2\frac{8e^2}{(4\pi)^2}\int_0^1 dt\,(1-t)t\left[\frac{2}{\epsilon} + \log\left(\frac{4\pi\mu^2 e^{-\gamma_{EM}}}{\Delta}\right)\right] \tag{286}$$

$$- \eta^{\mu\nu}\frac{4e^2}{(4\pi)^2}\int_0^1 dt\,\Delta \tag{287}$$

Leaving the $p^\mu p^\nu$ term as is we can t-integrate the other using $p^2 \gg m^2$ and $\int_0^1 dt\,(1-t)t\log[-(t-1)t] = -5/18$

$$\Pi_2^{\mu\nu} = (p^\mu p^\nu)\frac{8e^2}{(4\pi)^2}\int_0^1 dt\, t^2\left[\frac{2}{\epsilon} + \log\left(\frac{4\pi\mu^2 e^{-\gamma_{EM}}}{\Delta}\right)\right] + \eta^{\mu\nu}p^2\frac{8e^2}{(4\pi)^2}\int_0^1 dt\,(1-t)t\left[\frac{2}{\epsilon} + \log\left(\frac{4\pi\mu^2 e^{-\gamma_{EM}}}{p^2 t(t-1) - \cancel{m^2}}\right)\right] \tag{288}$$

$$= (p^\mu p^\nu)\frac{8e^2}{(4\pi)^2}\int_0^1 dt\, t^2\left[\frac{2}{\epsilon} + \log\left(\frac{4\pi\mu^2 e^{-\gamma_{EM}}}{\Delta}\right)\right] + \eta^{\mu\nu}p^2\frac{8e^2}{(4\pi)^2}\left[\frac{1}{6}\frac{2}{\epsilon} + \frac{1}{6}\log\left(\frac{4\pi\mu^2 e^{-\gamma_{EM}}}{-p^2}\right) + \frac{5}{18}\right] \tag{289}$$

$$= (p^\mu p^\nu)\frac{8e^2}{(4\pi)^2}\int_0^1 dt\, t^2\left[\frac{2}{\epsilon} + \log\left(\frac{4\pi\mu^2 e^{-\gamma_{EM}}}{\Delta}\right)\right] + \eta^{\mu\nu}p^2\frac{e^2}{12\pi^2}\left[\frac{2}{\epsilon} + \log\left(\frac{4\pi\mu^2 e^{-\gamma_{EM}}}{-p^2}\right) + \frac{5}{3}\right] \tag{290}$$

4.2 Exercise 2:* Ward identity

In the integral (1) we neglected contributions proportional to $p^\mu p^\nu$. Compute these missing contributions and verify the Ward identities.

Applying the the same substitution from above $q^\mu = k^\mu - p^\mu t$ and $\Delta = -(p^2 t(1-t) + m^2) = p^2 t(t-1) - m^2$ with the missing part (writing everything now a bit more condensed - using the results from above like cancellation of linear q-terms)

$$-4ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{-k^\mu p^\nu - k^\nu p^\mu}{[(k-p)^2+m^2][k^2+m^2]} = -4ie^2 \int_0^1 dt \int \frac{d^4q}{(2\pi)^4} \frac{-(q^\mu + tp^\mu)p^\nu - (q^\nu + tp^\nu)p^\mu}{(q^2-\Delta)^2} \tag{291}$$

$$= -4ie^2 \int_0^1 dt \int \frac{d^4q}{(2\pi)^4} \frac{-\cancel{q^\mu p^\nu} - tp^\mu p^\nu - \cancel{q^\nu p^\mu} - tp^\nu p^\mu}{(q^2-\Delta)^2} \tag{292}$$

$$= -4ie^2 \int_0^1 dt \int \frac{d^4q}{(2\pi)^4} \frac{-2tp^\mu p^\nu}{(q^2-\Delta)^2} \tag{293}$$

Now we can combine this with the red term (we use the from of the first appearance)

$$-4ie^2(2p^\mu p^\nu) \int_0^1 dt\, t^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2-\Delta)^2} - 4ie^2 \int_0^1 dt \int \frac{d^4q}{(2\pi)^4} \frac{-2tp^\mu p^\nu}{(q^2-\Delta)^2} \tag{294}$$

$$= -4ie^2(2p^\mu p^\nu) \int_0^1 dt\, t(t-1) \int \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2-\Delta)^2} \tag{295}$$

Here we see that the $p^\mu p^\nu$ t^2-term from the problem above is joined by an identical t-term - so we can reuse the calculation result from above and obtain for the $p^\mu p^\nu$ contribution

$$(p^\mu p^\nu) \frac{8e^2}{(4\pi)^2} \int_0^1 dt\, t(t-1) \left[\frac{2}{\epsilon} + \log\left(\frac{4\pi\mu^2 e^{-\gamma_{EM}}}{\Delta}\right)\right] = (p^\mu p^\nu) \frac{e^2}{12\pi^2} \left[\frac{2}{\epsilon} + \log\left(\frac{4\pi\mu^2 e^{-\gamma_{EM}}}{-p^2}\right) + \frac{5}{3}\right] \tag{296}$$

which implies that adding the missing contribution added to $\Pi_2^{\mu\nu}$ gives

$$\Pi_2'^{\mu\nu} = \Pi_2^{\mu\nu} + \text{neglected}\,(p^\mu p^\nu) = (-p^\mu p^\nu + \eta^{\mu\nu} p^2) \frac{e^2}{12\pi^2} \left[\frac{2}{\epsilon} + \log\left(\frac{4\pi\mu^2 e^{-\gamma_{EM}}}{-p^2}\right) + \frac{5}{3}\right]. \tag{297}$$

Now we can see

$$p_\mu \Pi_2'^{\mu\nu} \sim p_\mu(-p^\mu p^\nu + \eta^{\mu\nu} p^2) \tag{298}$$

$$\sim -p^2 p^\nu + p^\nu p^2 \tag{299}$$

$$= 0 \tag{300}$$

So we proved the Ward identity for this case.

5 Quantum Field Theory II – Exercise sheet 3 2024-04-24

5.1 Exercise 1: BRST Quantization of Yang-Mills Theory

In the lecture, we found the following Faddeev-Popov Lagrangian for the bosonic fields $A_\mu = A_\mu^a t_a, B = B^a t_a$ and the fermionic ghost fields $c = c^a t_a, \bar{c} = \bar{c}^a t_a$

$$\mathcal{L}_{\text{FP}} = -\frac{1}{4}\langle F^{\mu\nu} F_{\mu\nu}\rangle + \frac{\xi}{2}\langle B, B\rangle + \langle B, \partial_\mu A^\mu\rangle - \langle \partial^\mu \bar{c}, D_\mu c\rangle \tag{1}$$

where $\langle,\rangle$ denotes an invariant Cartan-Killing metric. We established that this theory is invariant under the global (infinitesimal) BRST transformations

$$\begin{aligned} \delta A_\mu &= D_\mu(\theta c), \qquad & \delta B &= 0, \\ \delta \bar{c} &= -\theta B, \qquad & \delta c &= \frac{1}{2}\theta[c, c] \end{aligned}$$

with fermionic (Grassmann odd) symmetry parameter θ.

1. Compute the Euler-Lagrange equations from (1) for all fields.
2. Apply Noether's theorem to compute the current j_μ that is conserved, satisfying $\partial_\mu j^\mu = 0$, as a consequence of BRST invariance.
   Hint: The Noether trick to promote $\theta \to \epsilon(x)$, with $\epsilon(x)$ a Grassmann odd scalar on spacetime is applicable.
3. Verify that the Noether current is indeed conserved on-shell, i.e., upon using the Euler-Lagrange equations.
   Hint: Use the integrability condition obtained by taking the divergence of the field equation for A_μ, using and proving the Bianchi identity $D_\nu D_\mu F^{\mu\nu} \equiv 0$.
4. In the free theory the BRST current reduces to the expression

   $$j^\mu = \langle B, \partial^\mu c\rangle - \langle c, \partial^\mu B\rangle.$$

   Consider the conserved charge

   $$\mathcal{Q} = \int d^3x\, j^0 \tag{2}$$

   and express it in terms of A_μ and c, using the equations of motion. Then writing A_μ and c in terms of creation and annihilation operators satisfying the familiar algebra

   $$\begin{aligned} A^\mu(x) &= \sum_{\lambda = >,<,+,-} \int dk \left[\varepsilon_\lambda^{\mu *}(k)\, a_\lambda(k)\, e^{ikx} + \varepsilon_\lambda^\mu(k)\, a_\lambda^\dagger(k)\, e^{-ikx}\right] \\ c(x) &= \int dk \left[c(k)\, e^{ikx} + c^\dagger(k)\, e^{-ikx}\right] \end{aligned}$$

   Show that the adjoint action of $\mathcal{Q}$ on field operators reproduces the action of the BRST operator introduced in the lecture.
5. We now view $\mathcal{Q}$ as an operator on the multi-particle Hilbert space defined by the creation and annihilation operators introduced above, satisfying the nilpotency condition $\mathcal{Q}^2 = 0$.

   There is the notion of **cohomology**, the space of $\mathcal{Q}$-*closed* vectors satisfying $\mathcal{Q}|\psi\rangle = 0$, modulo $\mathcal{Q}$-*exact* vectors of the form $\mathcal{Q}|\chi\rangle$:

   $$\mathcal{H} := \frac{\ker \mathcal{Q}}{\operatorname{im} \mathcal{Q}} = \{[|\psi\rangle] \mid \mathcal{Q}|\psi\rangle = 0\},$$

   that is, $\mathcal{H}$ consists of equivalence classes: $|\psi\rangle = [|\psi\rangle + \mathcal{Q}|\chi\rangle]$.

   Show that the cohomology $\mathcal{H}$ precisely encodes the physical states, i.e., the transverse gluon polarizations.

1. Simplifying the terms of the Lagrangian using the Lie algebra $[t_b, t_c] = f_{bc}{}^a t_a$ with $f_{bc}{}^a = -f_{ba}{}^c$ (this one is a bit of guess work) and normalization $\kappa_{ab} = \delta_{ab}$

- Yang-Mills term $-\frac{1}{4}\langle F^{\mu\nu}, F_{\mu\nu}\rangle$

$$
\begin{aligned}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu] && (301)\\
&= (\partial_\mu A^a_\nu)t_a - (\partial_\nu A^a_\mu)t_a - A^b_\mu A^c_\nu [t_b, t_c] && (302)\\
&= (\partial_\mu A^a_\nu)t_a - (\partial_\nu A^a_\mu)t_a - A^b_\mu A^c_\nu f_{bc}{}^a t_a && (303)\\
\to F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f_{bc}{}^a A^b_\mu A^c_\nu && (304)
\end{aligned}
$$

then

$$
\begin{aligned}
-\frac{1}{4}\langle F^{\mu\nu}, F_{\mu\nu}\rangle &= -\frac{1}{4}\mathrm{tr}[F^{\mu\nu}F_{\mu\nu}] && (305)\\
&= -\frac{1}{4}\kappa_{ab}F^{\mu\nu a}F^b_{\mu\nu} && (306)
\end{aligned}
$$

- Nakanishi-Lautrup term $\frac{\xi}{2}\langle B, B\rangle$

$$
\frac{\xi}{2}\langle B, B\rangle = \frac{\xi}{2}B^a B^a \qquad (307)
$$

- Gauge-fixing term $\langle B, \partial_\mu A^\mu\rangle$

$$
\langle B, \partial_\mu A^\mu\rangle = B^a(\partial^\mu A^a_\mu) \qquad (308)
$$

- Ghost term $\langle \partial^\mu \bar{c}, D_\mu c\rangle$

$$
\begin{aligned}
D_\mu c &= \partial_\mu c - [A_\mu, c] && (309)\\
&= (\partial_\mu c^a)t_a - A^b_\mu c^c [t_b, t_c] && (310)\\
&= (\partial_\mu c^a)t_a - A^b_\mu c^c f_{bc}{}^a t_a && (311)
\end{aligned}
$$

then

$$
\begin{aligned}
\langle \partial^\mu \bar{c}, D_\mu c\rangle &= \langle \partial^\mu \bar{c}, D_\mu c\rangle && (312)\\
&= \langle \partial^\mu \bar{c}, \partial_\mu c - [A_\mu, c]\rangle && (313)\\
&= \langle \partial^\mu \bar{c}, \partial_\mu c\rangle - \langle \partial^\mu \bar{c}, [A_\mu, c]\rangle && (314)\\
&= \kappa_{ab}(\partial^\mu \bar{c}^a)(\partial_\mu c^b) - \kappa_{ab}(\partial^\mu \bar{c}^a)A^c_\mu c^d f_{cd}{}^b && (315)\\
&= (\partial^\mu \bar{c}^a)(\partial_\mu c^a) - f_{cd}{}^a(\partial^\mu \bar{c}^a)A^c_\mu c^d && (316)
\end{aligned}
$$

(a) Gauge field $A_\mu = A^a_\mu t_a$

$$
\begin{aligned}
\frac{\partial \mathcal{L}_{\text{FP}}}{\partial(\partial_\beta A^b_\alpha)} &= -\frac{1}{4}2(F^{\beta\alpha b} - F^{\alpha\beta b}) + B^a\delta^\alpha_\mu \delta^\mu_\beta \delta^b_a = F^{\alpha\beta b} + B^b\delta^\beta_\alpha && (317)\\
\frac{\partial \mathcal{L}_{\text{FP}}}{\partial A^b_\alpha} &= \frac{\partial}{\partial A^b_\alpha}\langle \partial^\mu \bar{c}, [A_\mu, c]\rangle + \frac{1}{4}2F^{\mu\nu a}\frac{\partial}{\partial A^b_\alpha} f_{ef}{}^a A^e_\mu A^f_\nu && (318)\\
&= \frac{\partial}{\partial A^b_\alpha}(\partial^\mu \bar{c}^d)[A_\mu, c]^d + \frac{1}{2}F^{\mu\nu a} f_{ef}{}^a(\delta^e_b\delta^\alpha_\mu A^f_\nu + A^e_\mu \delta^f_b \delta^\alpha_\nu) && (319)\\
&= \frac{\partial}{\partial A^b_\alpha}(\partial^\mu \bar{c}^d)A^e_\mu c^f f_{ef}{}^d + \frac{1}{2}(F^{\alpha\nu a} f_{bf}{}^a A^f_\nu + F^{\mu\alpha a} f_{eb}{}^a A^e_\mu) && (320)\\
&= (\partial^\mu \bar{c}^d)\delta^\alpha_\mu \delta^e_b c^f f_{ef}{}^d + F^{\alpha\nu a} f_{bf}{}^a A^f_\nu && (321)\\
&= [(\partial^\alpha \bar{c}), c] + [A_\nu, F^{\alpha\nu a}] && (322)
\end{aligned}
$$

then

$$
\boxed{D^\mu F_{\mu\nu} - \partial_\nu B + [\partial_\nu \bar{c}, c] = 0} \qquad (323)
$$

(b) Nakanishi-Laudrup field $B = B^a t_a$

$$
\begin{aligned}
\frac{\partial \mathcal{L}_{\text{FP}}}{\partial(\partial_\mu B^b)} &= 0 && (324)\\
\frac{\partial \mathcal{L}_{\text{FP}}}{\partial B^b} &= \frac{\xi}{2}\cdot 2B^a\delta^b_a + \delta^b_a(\partial^\mu A^a_\mu) && (325)\\
&= \xi B^b + (\partial^\mu A^b_\mu) && (326)
\end{aligned}
$$

then

$$\boxed{B = -\frac{1}{\xi}\partial^\mu A_\mu} \tag{327}$$

(c) Ghost field $c = c^a t_a$

$$\frac{\partial \mathcal{L}_{\text{FP}}}{\partial(\partial_\nu c^b)} = -\delta^\nu_\mu \delta^b_a \partial^\mu \bar{c}^a \tag{328}$$

$$= -\partial^\nu \bar{c}^b \tag{329}$$

$$\frac{\partial \mathcal{L}_{\text{FP}}}{\partial c^b} = \frac{\partial}{\partial c^b}\langle \partial^\mu \bar{c}, [A_\mu, c]\rangle = \frac{\partial}{\partial c^b}(\partial^\mu \bar{c}^d)[A_\mu, c]^d = \frac{\partial}{\partial c^b}(\partial^\mu \bar{c}^d) A^e_\mu c^f f_{ef}^{\ \ d} \tag{330}$$

$$= (\partial^\mu \bar{c}^d) A^e_\mu f_{ef}^{\ \ d} \delta^f_b = (\partial^\mu \bar{c}^d) A^e_\mu f_{eb}^{\ \ d} \tag{331}$$

$$= -(\partial^\mu \bar{c}^d) A^e_\mu f_{ed}^{\ \ b} = -[A_\mu, (\partial^\mu \bar{c})] \tag{332}$$

then

$$\partial_\nu \partial^\nu \bar{c} + [A_\nu, (\partial^\nu \bar{c})] = 0 \tag{333}$$

$$\boxed{\to D_\nu \partial^\nu \bar{c} = 0} \tag{334}$$

(d) Anti-ghost field $\bar{c} = \bar{c}^a t_a$

$$\frac{\partial \mathcal{L}_{\text{FP}}}{\partial(\partial_\nu \bar{c}^b)} = -\delta^a_b \delta^\mu_\nu D^\mu c^b \tag{335}$$

$$= -D^\nu c^a \tag{336}$$

$$\frac{\partial \mathcal{L}_{\text{FP}}}{\partial \bar{c}^b} = 0 \tag{337}$$

then

$$\boxed{\partial_\nu D^\nu c = 0} \tag{338}$$

$$D_\nu D^\nu c = \partial_\nu D^\nu c - [A_\nu, D^\nu c] \tag{339}$$

2. Rederiving the Noether theorem (somehow I can never remember it):

- Equations of motion - $uv' = -u'v + (uv)'$

$$0 \stackrel{!}{=} \delta S = \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi_a}\delta\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta(\partial_\mu \phi_a)\right) \tag{340}$$

$$= \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi_a}\delta\phi_a - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\right]\delta\phi_a + \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta\phi_a\right]\right) \tag{341}$$

$$= \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\right]\right)\delta\phi_a + \int_{\partial\Omega} d^3S \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta\phi_a\right] \tag{342}$$

- Symmetry trafo of the fields (if EoM are not changing meaning $\leftrightarrow$ if δS only has changes in the boundary term $\leftrightarrow$ meaning $\mathcal{L}$ changes only by 4-divergence $\partial_\mu \mathcal{J}$)

$$\phi_a(x) \to \phi'_a(x) + \varepsilon\delta\phi_a(x) \quad \text{allowing} \quad \mathcal{L}(x) \to \mathcal{L}'(x) = \mathcal{L}(x) + \varepsilon\partial_\mu \mathcal{J}^\mu(x) \tag{343}$$

- calculating implied change $\delta\mathcal{L}$

$$\epsilon\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi_a}(\epsilon\delta\phi_a) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\partial_\mu(\epsilon\delta\phi_a) \tag{344}$$

$$= \epsilon\partial_\mu \underbrace{\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta\phi_a\right)}_{=\mathcal{J}^\mu} + \underbrace{\left(\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\right]\right)}_{=0}\delta\phi_a \tag{345}$$

With $\theta \to \epsilon(x)$

$$\delta_\theta A_\mu = \partial_\mu(\epsilon(x)c) + [A_\mu, \epsilon(x)c] \tag{346}$$

$$= \epsilon(x) D_\mu c + c(\partial_\mu \epsilon(x)) \tag{347}$$

$$\delta_\theta B = 0, \tag{348}$$

$$\delta_\theta \bar{c} = -\epsilon(x) B \tag{349}$$

$$\delta_\theta c = \frac{1}{2}\epsilon(x)[c, c] \tag{350}$$

Under this local change, the Lagrangian transforms as

$$\begin{align}
\delta\mathcal{L}_{\text{FP}} &= \frac{\partial\mathcal{L}_{\text{FP}}}{\partial(\partial_\nu A_\mu)}\delta A_\mu + \frac{\partial\mathcal{L}_{\text{FP}}}{\partial(\partial_\nu B)}\delta B + \frac{\partial\mathcal{L}_{\text{FP}}}{\partial(\partial_\nu c)}\delta c + \frac{\partial\mathcal{L}_{\text{FP}}}{\partial(\partial_\nu \bar{c})}\delta\bar{c} \tag{351}\\
&= (F^{\mu\nu}+B)[\epsilon D_\mu c + c(\partial_\nu\epsilon)] + 0\cdot 0 + (-D^\nu c)(-\epsilon B) + (-\partial_\nu\bar{c})\frac{1}{2}\epsilon[c,c] \tag{352}
\end{align}$$

then we can read off j^ν as the ϵ coefficient (up to some signs)

$$j^\nu_{\text{BRS}} = \langle F^{\mu\nu}, D_\mu c\rangle - \langle B, D^\nu c\rangle + \langle\partial^\mu\bar{c}, \frac{1}{2}[c,c]\rangle \tag{353}$$

there is also a scaling symmetry for the ghost fields $c\to e^\lambda c$, $\bar{c}\to e^{-\lambda}\bar{c}$ with current

$$j^\nu_{\text{gh}} = \langle\partial^\nu\bar{c}, c\rangle - \langle\bar{c}, D^\nu c\rangle \tag{354}$$

3. Bianchi identity

$$D_{[\lambda}F_{\mu\nu]} = 0 \quad \to \quad D_\nu D_\mu F^{\mu\nu} = 0 \tag{355}$$

and

$$\begin{align}
D_\mu D_\nu c - D_\nu D_\mu c &= D_\mu(\partial_\nu c - [A_\nu, c]) - D_\nu(\partial_\mu c - [A_\mu, c]) \tag{356}\\
&= (\cancel{\partial_\mu\partial_\nu c} - [A_\mu, \partial_\nu c] - D_\mu[A_\nu, c]) - (\cancel{\partial_\nu\partial_\mu c} - [A_\nu, \partial_\mu c] - D_\nu[A_\mu, c]) \tag{357}\\
&= -[A_\mu, \partial_\nu c] - \partial_\mu[A_\nu, c] + [A_\mu, [A_\nu, c]] + [A_\nu, \partial_\mu c] + \partial_\nu[A_\mu, c] - [A_\nu, [A_\mu, c]]) \tag{358}\\
&= [F_{\mu\nu}, c] \tag{359}\\
\to \langle F^{\mu\nu}, D_\nu D_\mu c\rangle &= \langle F^{\mu\nu}, D_\mu D_\nu c + [F_{\nu\mu}, c]\rangle \tag{360}\\
&= \langle F^{\mu\nu}, D_\mu D_\nu c\rangle + \langle F^{\mu\nu}, [F_{\nu\mu}, c]\rangle \tag{361}
\end{align}$$

Taking the 4-divergence and substituting the eom's - keeping in mind that total divergence vanish

$$\begin{align}
\partial_\nu j^\nu_{\text{BRS}} &= \partial_\nu\langle F^{\mu\nu}, D_\mu c\rangle - \partial_\nu\langle B, D^\nu c\rangle + \partial_\nu\langle\partial^\mu\bar{c}, \frac{1}{2}[c,c]\rangle \tag{362}\\
&= \langle D_\nu F^{\mu\nu}, D_\mu c\rangle + \langle F^{\mu\nu}, D_\nu D_\mu c\rangle + \underbrace{\langle\partial_\nu B, D^\nu c\rangle}_{=\langle D_\nu B, D^\nu c\rangle + \langle[A_\nu, B], D^\nu c\rangle} + \langle B, \underbrace{\partial_\nu D^\nu c}_{D_\nu D^\nu c - [A_\nu, D^\nu c]}\rangle + \ldots \tag{363}\\
&= \langle\underbrace{D_\nu F^{\mu\nu}}_{=\partial^\mu B - [\partial^\mu\bar{c}, c]}, D_\mu c\rangle + \langle F^{\mu\nu}, D_\nu D_\mu c\rangle + \underbrace{\langle D_\nu B, D^\nu c\rangle}_{=\frac{1}{2}\partial_\nu\langle B, D^\nu c\rangle} + \cancel{\langle[A_\nu, B], D^\nu c\rangle} + \langle B, \underbrace{D_\nu D^\nu c}_{=0}\rangle - \cancel{\langle B, [A_\nu, D^\nu c]\rangle} + \ldots \tag{364}\\
&= \langle\partial^\mu B, D_\mu c\rangle - \langle[\partial^\mu\bar{c}, c], D_\mu c\rangle + \ldots \tag{365}\\
&= \partial^\mu\langle B, D_\mu c\rangle + \langle B, \underbrace{D_\mu D^\mu c}_{=0}\rangle - \langle[\partial^\mu\bar{c}, c], D_\mu c\rangle + \ldots \tag{366}\\
&= -\langle\partial^\mu\bar{c}, [c, D_\mu c]\rangle \tag{367}\\
&= 0 \tag{368}
\end{align}$$

because last term is a 4-divergence again.

4. In the free theory the the gauge couple becomes $g\to 0$ meaning

$$\begin{align}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - g[A_\mu, A_\nu] \tag{369}\\
\to F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \tag{370}\\
\to D_\mu c &= \partial_\mu c \tag{371}
\end{align}$$

$$\begin{align}
j^\nu_{\text{BRS}} &= \langle F^{\mu\nu}, D_\mu c\rangle - \langle B, D^\nu c\rangle \tag{372}\\
&= \langle F^{\mu\nu}, \partial_\mu c\rangle - \langle B, \partial^\nu c\rangle \tag{373}\\
&= \ldots \tag{374}\\
&= \langle B, \partial^\nu c\rangle - \langle c, \partial^\nu B\rangle. \tag{375}
\end{align}$$

5. I run out of time ...

6 Quantum Field Theory II – Exercise sheet 3 2024-04-24

6.1 Exercise 1: Traces of Gamma matrices and Wick's theorem

Give a general prescription for computing the traces of gamma matrices

$$\mathrm{tr}(\gamma^{\nu_1}\gamma^{\nu_{2n}}) \tag{1}$$

in terms of the Wick contraction of two gamma matrices $\langle\gamma^\mu\gamma^\nu\rangle \equiv \mathrm{tr}(\gamma^\mu\gamma^\nu) = -4\eta^{\mu\nu}$.

6.2 Exercise 2: Lie groups and Yang-Mills theory

1) Compute the dimension of the Lie algebra of the group $\mathrm{SU}(N)$ and give an explicit set of generators.
2) Compute the dimension of the Lie algebra of the group $\mathrm{SO}(n)$ and give an explicit set of generators.
3) Prove that the bilinear form on a Lie algebra $\mathfrak{g}$ defined by

$$\langle\alpha,\beta\rangle = \mathrm{tr}(\mathrm{ad}_\alpha \cdot \mathrm{ad}_\beta) \tag{376}$$

   is invariant under the adjoint action of $\mathfrak{g}$.
4) Let $\kappa_{ab} = \langle t_a, t_b\rangle$ be the invariant form evaluated on a basis of generators t_a. Prove that $f_{abc} := \kappa_{cd} f_{ab}^{\;\;d}$, obtained by lowering one index with κ_{ab}, is totally antisymmetric.
5) In the lecture we found the infinitesimal Yang-Mills gauge transformation $\delta_\Lambda A_\mu = D_\mu\Lambda$ for a Lie algebra valued parameter $\Lambda \in \mathfrak{g}$. Find the finite gauge transformations $A_\mu \to A'_\mu$ under $g \in G$ that for $g \simeq 1 + \Lambda$ reproduce the infinitesimal ones.

1.
   - The SU(N) group matrices has N^2 complex entries (meaning $2N^2$ real parameters) and obey two conditions
     - $U^\dagger U = 1 \quad \to \quad U^*_{mj}U_{mk} = \delta_{jk}$ which gives N^2 independent restrictive equations (one for each matrix entry)
     - $\det U = 1$ which gives one restrictive equation

     so the number of real degrees of freedom is $2N^2 - N^2 - 1 = N^2 - 1$.
   - As the Lie-algebra $\mathfrak{su}(N)$ is the tangent space to the group manifold at the identity element and SU(N) is simply connected - the Lie algebra has the same dimension as the group.
   - The elements of the Lie algebra are (skew)-hermitian (depending on taste the i can be absorbed into S)

$$(e^{iS})^\dagger e^{iS} = (e^{-iS^\dagger})e^{iS} = e^{i(S-S^\dagger)} = I \tag{377}$$

$$\to S = S^\dagger \tag{378}$$

     as physicists prefer hermitian we use it like that.
   - Additionally $1 = \det e^{iS} = e^{\mathrm{tr}S} \to \mathrm{tr}S = 0$ - therefore the Lie algebra consists of traceless hermitian matrices
2.
   - The SO(n) group matrices has n^2 real entries (rotations in real n-dimensional space) and **seem** to obey two conditions
     - $O^TO = 1 \quad \to \quad O_{mj}O_{mk} = \delta_{jk}$ which gives $\frac{n(n+1)}{2}$ independent restrictive equations
     - $\det O = 1$ which gives no additional restrictive equations because $\det O^TO = \det 1 \to \det O = \pm 1$

     so the number of real degrees of freedom is $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$. As not all SO($n$) are simply connected we can not use the argument from above.
   - A rotations is defined as a single continuous parameter operation which transforms a plane into a plane (to think about rotations about an axis is a coincidence in 3d space). As a plane is defined by two vectors there are $\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}$ rotations and therefore $\frac{n(n-1)}{2}$ generators (dimensions) of the Lie algebra
   - So the generators (in this particular representation) have the shape of the 2d generator

$$\begin{pmatrix} & \vdots & & \vdots & \\ \cdots & 0 & \cdots & -1 & \cdots \\ & \vdots & & \vdots & \\ \cdots & 1 & \cdots & 0 & \cdots \\ & \vdots & & \vdots & \end{pmatrix} \tag{379}$$

from this we can see that the matrices are anti-symmetric - which makes sense because an anti-symmetric matrix $A = -A^T$ of the Lie algebra produces a orthogonal Lie group element e^A (we leave out the conventional physicists i so the generator are real)

$$(e^A)^T e^A = e^{A^T} e^A = e^{-A} e^A = I \tag{380}$$

- We can write the $\frac{n(n-1)}{2}$ generators $t^{(ab)}$ with $1 \leq a \leq b \leq n$ in a compact matrix form

$$[t^{(ab)}]_{kl} = \delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk} \tag{381}$$

$$\to [t^{(ab)}, t^{(cd)}] = \delta_{bc} t^{(ad)} - \delta_{ac} t^{(bd)} - \delta_{bd} t^{(ac)} + \delta_{ad} t^{(bc)} \tag{382}$$