0.1 RH

- Numbers as Functions - Yuri Manin
    - Many different way to think about numbers as functions
    - Certain numbers called periods appear in number theory and QFT $\left\{ \sqrt[3]{5}, \pi, \frac{\pi^2}{6}, \Gamma\left(\frac{3}{7}\right)^7 \right\}$
- Using the imaginary parts of the non-trivial zeros of the Zeta function

$$f(x) = -\sum_k \cos(\text{Im}(\zeta_k)\log x) \tag{1}$$

we see peaks at the prices and smaller peaks at their powers $2, 2^2, 2^3, ..., 3, 3^2, 3^3, ..., 5, 5^2, ...$

- Riemann's Hypothesis - Brian Conrey
- SageMathCell

0.2 Linear algebra

0.2.0 Basic Concepts

How to write up mathematics

> DEFINITION: <u>Set</u>

> DEFINITION: <u>Structure on $\mathbb{Q}$</u>

> DEFINITION: <u>Group</u>

> DEFINITION: <u>Field</u>

The Complex Numbers

> DEFINITION: <u>Complex numbers</u>

> REMARK: $i^2 = -1$

> FACT 1:

> (i) $\mathbb{C}$ with operations $+/\cdot$ is a field
> (ii) $\mathbb{R} \to \mathbb{C}$

> REMARK: Fundamental theorem of algebra: Every polynom of order n: $P(z) = \sum_k^n a_k z^k$ has exactly n zeros

> DEFINITION: <u>Complex conjugation</u>

0.2.1 Vector Spaces

Vector Space

0.3 Classical Mechanics

0.3.1 Lagrangian Mechanics

$$L = T - V, \qquad S = \int L(q, \dot{q}, t)dt \tag{2}$$

Integration by parts - neglecting boundary terms

$$\delta S = \int \delta L \, dt = 0 \tag{3}$$

$$\delta L = \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \tag{4}$$

$$= \left[\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right] \delta q \tag{5}$$

Canonical momentum

$$p = \frac{\partial L}{\partial \dot{q}} \tag{6}$$

Cyclic coordinates

$$\frac{\partial L}{\partial q} = 0 \quad \rightarrow \quad \frac{\partial L}{\partial \dot{q}} = p = \text{const} \tag{7}$$

0.4 Classical Field Theory

The physics - to derive the equations of motion

$$S = \int L \, dt = \int \mathcal{L}(\psi, \partial_\mu \psi) \, d^4 x \tag{8}$$

$$0 = \delta S = \int d^4 x \, \delta \mathcal{L} \tag{9}$$

Adding a four-divergence to the Lagrangian $\mathcal{L}' = \mathcal{L} + \partial_\mu K^\mu(\psi)$ results in

$$\int d^4 x \, \partial_\mu K^\mu = \int dA \, n_\mu K^\mu \tag{10}$$

which should vanish for well behaved fields and therefore should not change anything.

$$\delta \mathcal{L} = \sum_a \frac{\partial \mathcal{L}}{\partial \psi_a} \delta \psi_a + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi_a)} \overbrace{\delta (\partial_\mu \psi_a)}^{= \partial_\mu (\delta \psi_a)} \tag{11}$$

$$= \sum_a \underbrace{\left[\frac{\partial \mathcal{L}}{\partial \psi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi_a)} \right]}_{\text{equations of motion}} \delta \psi_a + \underbrace{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi_a)} \delta \psi_a \right)}_{= \partial_\mu K^\mu} \tag{12}$$

Internal symmetry: $\psi_a \rightarrow \psi_a' = \psi_a + \delta \psi_a$ if $\delta \mathcal{L} = 0$

$$\rightarrow j^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi_a)} \delta \psi_a - K^\mu \qquad \partial_\mu j^\mu = 0 \tag{13}$$

$$\rightarrow Q = \int d^3 x \, j_0 \qquad \frac{d}{dt} Q = \int d^3 x \frac{\partial j_0}{\partial t} = \int d^3 x \nabla \vec{j} = \int d\vec{A} \cdot \vec{j} = 0 \tag{14}$$

Consider spacetime translation: $x^\nu \rightarrow x'^\nu = x^\nu - \epsilon^\nu$ implying $\psi(x) \rightarrow \psi(x') = \psi(x) + \epsilon^\nu \partial_\nu \psi(x)$ and $\mathcal{L} \rightarrow \mathcal{L}' = \mathcal{L} + \epsilon^\nu \partial_\nu \mathcal{L} = \mathcal{L} + \epsilon^\nu \partial_\mu (\delta_\nu^\mu \mathcal{L})$ results in four Noether currents $\nu = 0, 1, 2, 3$

$$\rightarrow T_\nu^\mu \equiv (j^\mu)_\nu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} \partial_\nu \psi - \delta_\nu^\mu \mathcal{L} \tag{15}$$

$$\rightarrow T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} \partial^\nu \psi - \eta^{\mu\nu} \mathcal{L} \tag{16}$$

$$\rightarrow \Theta^{\mu\nu} = -\frac{2}{\sqrt{-g}} \left. \frac{\partial (\sqrt{-g} \mathcal{L})}{\partial g_{\mu\nu}} \right|_{g_{\mu\nu} = \eta_{\mu\nu}} \tag{17}$$

Also

$$\pi_a = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_a)} \tag{18}$$

$$\mathcal{H} = \sum_a \pi_a \partial_o \psi_a - \mathcal{L} \tag{19}$$

0.4.1 Lagrangian Lookup Table

Real scalar field	x	x	•
$\mathcal{L}[\phi] = \frac{1}{2}\eta^{\mu\nu}(\partial_\mu \phi)(\partial_\nu \phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{n!}\phi^n$	•	•	•
$(\Box + m^2)\phi + \frac{\lambda}{(n-1)!}\phi^{n-1} = 0$	•	•	•
•	•	•	•

- Real scalar field $\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}(\partial_\mu \phi)(\partial_\nu \phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{n!}\phi^n$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi - \frac{\lambda}{(n-1)!}\phi^{n-1} \tag{20}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi)} = \eta^{\mu\nu}(\partial_\mu \phi)\delta^\alpha_\nu = \partial^\alpha \phi \tag{21}$$

$$\rightarrow (\Box + m^2)\phi + \frac{\lambda}{(n-1)!}\phi^{n-1} = 0 \tag{22}$$

Hamiltonian

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial^0 \phi \tag{23}$$

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} \tag{24}$$

$$= \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{n!}\phi^n \tag{25}$$

$$H = \frac{1}{2}\int d^3y \left(\pi^2 + (\nabla\phi)^2 + m^2\phi^2 + \frac{2\lambda}{n!}\phi^n\right) \tag{26}$$

Heisenberg equations with $[\phi(\vec{x}, t), \pi(\vec{y}, t)] = i\delta^3(\vec{x} - \vec{y})$

$$\int d^3y [\pi(y)^2, \phi(x)] = \int d^3y \left(\pi(y)^2\phi(x) - \phi(x)\pi(y)^2\right) \tag{27}$$

$$= \int d^3y \left(\pi(y)^2\phi(x) - \pi(y)\phi(x)\pi(y) - i\pi(y)\delta^3(\vec{x} - \vec{y})\right) \tag{28}$$

$$= \int d^3y \left(\pi(y)^2\phi(x) - \pi(y)^2\phi(x) - 2i\pi(y)\delta^3(\vec{x} - \vec{y})\right) \tag{29}$$

$$= -2i\pi(x) \tag{30}$$

$$\rightarrow \dot{\phi} = i[H, \phi] = \pi(x) \tag{31}$$

$$\int d^3y[(\nabla_y\phi(y))^2, \pi(x)] = \int d^3y\left((\nabla_y\phi(y))^2\pi(x) - \pi(x)(\nabla_y\phi(y))^2\right) \tag{32}$$

$$= \int d^3y\left(\nabla_y\phi(y)(\nabla_y\phi(y)\pi(x)) - (\pi(x)\nabla_y\phi(y))\nabla_y\phi(y)\right) \tag{33}$$

$$= \int d^3y\left(\nabla_y\phi(y)\nabla_y[\phi(y), \pi(x)]\nabla_y\phi(y)\right) \tag{34}$$

$$= i\int d^3y\left(\nabla_y\phi(y)\right)^2 \nabla_y\delta^{(3)}(\vec{x} - \vec{y}) \tag{35}$$

$$= -2i\int d^3y\left(\nabla_y^2\phi(y)\right)\delta^{(3)}(\vec{x} - \vec{y}) \tag{36}$$

$$= -2i\nabla_x^2\phi(x) \tag{37}$$

$$\rightarrow \dot{\pi} = i[H, \pi] = \nabla^2\pi(x) - m^2\phi - \frac{\lambda}{(n-1)!}\phi^{n-1} \tag{38}$$

- Maxwell field $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + j^\mu A_\mu = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)\eta^{\mu\sigma}\eta^{\nu\rho}(\partial_\sigma A_\rho - \partial_\rho A_\sigma) + j^\mu A_\mu$

$$\frac{\partial\mathcal{L}}{\partial A_\alpha} = j^\mu\delta_\mu^\alpha = j^\alpha \tag{39}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\beta A_\alpha)} = -\frac{2}{4}(\delta_\mu^\beta\delta_\nu^\alpha - \delta_\nu^\beta\delta_\mu^\alpha)F^{\mu\nu} = -F^{\alpha\beta} \tag{40}$$

$$\rightarrow \partial_\beta F^{\alpha\beta} + j^\alpha = 0 \tag{41}$$

$$\rightarrow T^{\mu\nu}_{\text{free}} = -F^{\alpha\mu}\partial^\nu A_\alpha + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \tag{42}$$

$$\rightarrow T^{\mu\nu}_{\text{free,sym}} = T^{\mu\nu}_{\text{free}} + F^{\alpha\mu}\partial_\alpha A^\nu = -F^{\alpha\mu}F^\nu_\alpha + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \tag{43}$$

- Dirac field $\mathcal{L} = \overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi$

$$\frac{\partial\mathcal{L}}{\partial\overline{\psi}} = (i\gamma^\mu\partial_\mu - m)\psi \tag{44}$$

$$\frac{\partial\mathcal{L}}{\partial\psi} = -m\overline{\psi} \tag{45}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\overline{\psi})} = 0 \tag{46}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\psi)} = \overline{\psi}i\gamma^\mu\delta_\mu^\alpha = i\overline{\psi}\gamma^\alpha \tag{47}$$

$$\rightarrow (i\gamma^\mu\partial_\mu - m)\psi = 0 \tag{48}$$

$$\rightarrow \partial_\alpha(i\overline{\psi}\gamma^\alpha) + m\overline{\psi} = 0 \tag{49}$$

- Massive vector field $\mathcal{L} = -\frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{1}{2}m^2 B_\mu B^\mu$

$$\frac{\partial\mathcal{L}}{\partial B_\alpha} = m^2 B^\alpha \tag{50}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\beta B_\alpha)} = -\frac{2}{4}(\delta_\mu^\beta\delta_\nu^\alpha - \delta_\nu^\beta\delta_\mu^\alpha)G^{\mu\nu} = G^{\alpha\beta} \tag{51}$$

$$\rightarrow \partial_\beta G^{\alpha\beta} - m^2 B^\alpha = 0 \tag{52}$$

0.5 Classical Electrodynamics

Notation

$$\eta_{ab} = \eta^{ab} = \text{diag}(1, -1, -1, -1) \tag{53}$$

$$\mathbf{A} \to A^i = \begin{pmatrix} A^0 \\ \vec{A} \end{pmatrix} \qquad A_i = \begin{pmatrix} A^0 \\ -\vec{A} \end{pmatrix} \tag{54}$$

$$\mathbf{E} = -\nabla A^0 - \partial_t \mathbf{A} \tag{55}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{56}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{57}$$

$$F_{10} = \partial_x A_0 - \partial_t A_x = \partial_x A^0 + \partial_t A^x = -E_x \tag{58}$$

$$F_{21} = \partial_y A_x - \partial_x A_y = -\partial_y A^x + \partial_x A^y = B_z \tag{59}$$

$$F_{31} = \partial_z A_x - \partial_x A_z = -\partial_z A^x + \partial_x A^z = -B_y \tag{60}$$

$$F_{\mu\nu} = F_{\downarrow\downarrow} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \qquad F^{\mu\nu} = F_{\uparrow\uparrow} = \eta F_{\downarrow\downarrow} \eta^T = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{61}$$

$$F_{\mu\nu} F^{\mu\nu} = -\text{tr}(F_{\downarrow\downarrow}.F_{\uparrow\uparrow}) = 2(\mathbf{B}^2 - \mathbf{E}^2) \qquad F^{\mu\lambda} F_{\lambda\nu} = ... \tag{62}$$

0.5.1 Multipole expansion

Spherical Harmonics

$$Y_{00} = \frac{1}{2\sqrt{\pi}} \tag{63}$$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\vartheta\, e^{i\varphi} = -\sqrt{\frac{3}{8\pi}} \frac{x+iy}{r} \tag{64}$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\vartheta = \sqrt{\frac{3}{4\pi}} \frac{z}{r} \tag{65}$$

$$Y_{1,-1} = \sqrt{\frac{3}{8\pi}} \sin\vartheta\, e^{-i\varphi} = \sqrt{\frac{3}{8\pi}} \frac{x-iy}{r} \tag{66}$$

$$Y_{22} = \sqrt{\frac{15}{32\pi}} \sin^2\vartheta\, e^{2i\varphi} = \sqrt{\frac{15}{32\pi}} \frac{(x+iy)^2}{r^2} \tag{67}$$

$$Y_{21} = -\sqrt{\frac{15}{8\pi}} \sin\vartheta \cos\vartheta\, e^{i\varphi} = -\sqrt{\frac{15}{32\pi}} \frac{(x+iy)z}{r^2} \tag{68}$$

$$Y_{20} = \sqrt{\frac{5}{16\pi}} (3\cos^2\vartheta - 1) = \sqrt{\frac{5}{16\pi}} \frac{3z^2 - r^2}{r^2} \tag{69}$$

$$Y_{2,-1} = \sqrt{\frac{15}{8\pi}} \sin\vartheta \cos\vartheta\, e^{-i\varphi} = \sqrt{\frac{15}{32\pi}} \frac{(x-iy)z}{r^2} \tag{70}$$

$$Y_{2,-2} = \sqrt{\frac{15}{32\pi}} \sin^2\vartheta\, e^{-2i\varphi} = \sqrt{\frac{15}{32\pi}} \frac{(x-iy)^2}{r^2} \tag{71}$$

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Cartesian

With $|\mathbf{x}| \gg |\mathbf{x}'|$

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{\sqrt{|\mathbf{x}|^2 - 2\mathbf{x} \cdot \mathbf{x}' + |\mathbf{x}'|^2}} \tag{72}$$

$$= \frac{1}{|\mathbf{x}|} \frac{1}{\sqrt{1 - 2\frac{\mathbf{x} \cdot \mathbf{x}'}{|\mathbf{x}|^2} + \frac{|\mathbf{x}'|^2}{|\mathbf{x}^2|}}} \tag{73}$$

$$= \frac{1}{|\mathbf{x}|} \frac{1}{\sqrt{1 - \underbrace{\left(2\dfrac{\mathbf{x} \cdot \mathbf{x}'}{|\mathbf{x}|^2} - \dfrac{|\mathbf{x}'|^2}{|\mathbf{x}^2|}\right)}_{=y}}} \tag{74}$$

$$= \frac{1}{|\mathbf{x}|} \left(1 + \frac{1}{2}y + \frac{3}{8}y^2 + \frac{5}{16}y^3 + ...\right) \tag{75}$$

$$= \frac{1}{|\mathbf{x}|} \left(1 + \frac{1}{2}\left(2\frac{\mathbf{x} \cdot \mathbf{x}'}{|\mathbf{x}|^2} - \frac{|\mathbf{x}'|^2}{|\mathbf{x}^2|}\right) + \frac{3}{8}\left(2\frac{\mathbf{x} \cdot \mathbf{x}'}{|\mathbf{x}|^2} - \frac{|\mathbf{x}'|^2}{|\mathbf{x}^2|}\right)^2 + \frac{5}{16}\left(2\frac{\mathbf{x} \cdot \mathbf{x}'}{|\mathbf{x}|^2} - \frac{|\mathbf{x}'|^2}{|\mathbf{x}^2|}\right)^3 + ...\right) \tag{76}$$

$$= \frac{1}{|\mathbf{x}|} + \frac{\mathbf{x} \cdot \mathbf{x}'}{|\mathbf{x}|^3} + \frac{1}{2}\underbrace{\frac{3(\mathbf{x} \cdot \mathbf{x}')^2 - |\mathbf{x}|^2|\mathbf{x}'|^2}{|\mathbf{x}|^5}}_{=\frac{[3x'^i x'^j - \delta_{ij}(x'^j x'^j)]x^i x^j}{|\mathbf{x}|^5}} + \frac{1}{2}\frac{5(\mathbf{x} \cdot \mathbf{x}')^3 - 3(\mathbf{x} \cdot \mathbf{x}')|\mathbf{x}|^2|\mathbf{x}'|^2}{|\mathbf{x}|^7} + ... \tag{77}$$

Then

$$4\pi\epsilon_0\Phi(\mathbf{x}) = \int d^3x' \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \tag{78}$$

$$= \frac{1}{|\mathbf{x}|} \int d^3x' \rho(\mathbf{x}') + \frac{1}{|\mathbf{x}|^3}\mathbf{x} \cdot \int d^3x' \, \mathbf{x}'\rho(\mathbf{x}') + \frac{1}{2|\mathbf{x}|^5}x^i x^j \int d^3x' \, (3x'_i x'_j - \delta_{ij}|\mathbf{x}'|^2)\rho(\mathbf{x}') + ... \tag{79}$$

$$= \frac{q}{|\mathbf{x}|} + \frac{\mathbf{x} \cdot \mathbf{p}}{|\mathbf{x}|^3} + \frac{(\mathbf{x}, \mathbf{Q}\mathbf{x})}{2|\mathbf{x}|^5} + ... \tag{80}$$

Spherical

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{|\mathbf{x}|} + \frac{\mathbf{x} \cdot \mathbf{x}'}{|\mathbf{x}|^3} + \frac{1}{2}\frac{[3x'^i x'^j - \delta_{ij}(x'^j x'^j)]x^i x^j}{|\mathbf{x}|^5} + \frac{1}{2}\frac{5(\mathbf{x} \cdot \mathbf{x}')^3 - 3(\mathbf{x} \cdot \mathbf{x}')|\mathbf{x}|^2|\mathbf{x}'|^2}{|\mathbf{x}|^7} + ... \tag{81}$$

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r}\sum_{l=0} \frac{4\pi}{2l+1} \sum_{m=-l}^{l} \left(\frac{r'}{r}\right)^l Y_{lm}(\vartheta, \varphi)Y_{lm}^*(\vartheta', \varphi') \tag{82}$$

Then

$$4\pi\epsilon_0\Phi(\mathbf{x}) = \int d^3x' \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \tag{83}$$

$$= \sum_{l,m} \frac{4\pi}{2l+1} \frac{q_{lm}}{r^{l+1}} Y_{lm}(\vartheta,\varphi) \tag{84}$$

$$= 4\pi \frac{q_{00}}{r} Y_{00} + \frac{4\pi}{3} \frac{q_{11}Y_{11} + q_{10}Y_{10} + q_{1,-1}Y_{1,-1}}{r^2} \tag{85}$$

$$= \frac{q}{r} + \frac{4\pi}{3} \frac{-\sqrt{\frac{3}{8\pi}}(-p_x+ip_y)\sqrt{\frac{3}{8\pi}}\frac{x+iy}{r} + \sqrt{\frac{3}{4\pi}}p_z\sqrt{\frac{3}{4\pi}}\frac{z}{r} + \sqrt{\frac{3}{8\pi}}(p_x+ip_y)\sqrt{\frac{3}{8\pi}}\frac{x-iy}{r}}{r^2} + ... \tag{86}$$

$$= \frac{q}{r} + \frac{1}{2} \frac{(p_xx+p_yy+i(p_xy-p_yx)) + 2p_zz + (p_xx+p_yy-i(p_xy-p_yx))}{r^3} + ... \tag{87}$$

with

$$q_{lm} = \int d^3r'\, r'^l Y^*_{lm}(\vartheta',\varphi')\rho(\mathbf{r}') \tag{88}$$

$$q_{00} = \frac{1}{2\sqrt{\pi}} \int d^3r'\, \rho(\mathbf{r}')r'^2\sin\vartheta' = \frac{q}{2\sqrt{\pi}} \tag{89}$$

$$q_{11} = -\frac{1}{2}\sqrt{\frac{3}{2\pi}} \int d^3r'\, r'\rho(\mathbf{r}')(\sin\vartheta'\, e^{-i\varphi'})r'^2\sin\vartheta' \tag{90}$$

$$= -\frac{1}{2}\sqrt{\frac{3}{2\pi}} \int d^3r'\, \rho(\mathbf{r}')(r'\sin\vartheta'[\cos\varphi' - i\sin\varphi'])r'^2\sin\vartheta' \tag{91}$$

$$= -\frac{1}{2}\sqrt{\frac{3}{2\pi}} \int d^3r'\, \rho(\mathbf{r}')(x'-iy')r'^2\sin\vartheta' \tag{92}$$

$$= \sqrt{\frac{3}{8\pi}}(-p_x+ip_y) \tag{93}$$

$$q_{10} = \sqrt{\frac{3}{4\pi}}p_z \tag{94}$$

$$q_{1,-1} = \sqrt{\frac{3}{8\pi}}(p_x+ip_y) \tag{95}$$

0.5.2 Radiation

Starting with

$$\nabla\cdot\mathbf{D} = \rho \qquad \nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t} \tag{96}$$

$$\nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \tag{97}$$

then

$$\mathbf{B} = \nabla\times\mathbf{A} \tag{98}$$

$$\to \nabla\times\left(\mathbf{E} + \frac{\partial\mathbf{A}}{\partial t}\right) = 0 = \nabla\times(-\nabla\phi) \tag{99}$$

$$\to \mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} \tag{100}$$

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in vacuum we find

$$\nabla \cdot \mathbf{E} \quad \rightarrow \quad \nabla^2 \phi + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho}{\epsilon_0} \tag{101}$$

$$\nabla \times \mathbf{H} \quad \rightarrow \quad \nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla\left(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t}\right) = -\mu_0 \mathbf{J} \tag{102}$$

using the Lorenz condition

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0 \tag{103}$$

we get

$$\nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon_0} \tag{104}$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} \tag{105}$$

with the solution

$$\phi(\mathbf{x}, t) = \frac{1}{4\pi\epsilon_0} \int dt' \int d^3\mathbf{x}' \frac{\rho(\mathbf{x}', t')}{|\mathbf{x} - \mathbf{x}'|} \delta\left(t' + \frac{|\mathbf{x} - \mathbf{x}'|}{c} - t\right) \tag{106}$$

$$\mathbf{A}(\mathbf{x}, t) = \frac{\mu_0}{4\pi} \int dt' \int d^3\mathbf{x}' \frac{\mathbf{J}(\mathbf{x}', t')}{|\mathbf{x} - \mathbf{x}'|} \delta\left(t' + \frac{|\mathbf{x} - \mathbf{x}'|}{c} - t\right) \tag{107}$$

with $\rho(\mathbf{x}, t) = \rho(\mathbf{x})e^{-i\omega t}$ and $\mathbf{J}(\mathbf{x}, t) = \mathbf{J}(\mathbf{x})e^{-i\omega t}$

$$\mathbf{A}(\mathbf{x}, t) = \frac{\mu_0}{4\pi}e^{-i\omega t} \int d^3\mathbf{x}' \, \mathbf{J}(\mathbf{x}') \frac{e^{ik|\mathbf{x} - \mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|} \tag{108}$$

$$= \mathbf{A}(\mathbf{x})e^{-i\omega t} \tag{109}$$

where $k = \omega/c$ then the fields are given by

$$\mathbf{H} = \frac{1}{\mu_0}\nabla \times \mathbf{A} \tag{110}$$

$$= e^{-i\omega t}\frac{1}{\mu_0}\nabla \times \mathbf{A}(\mathbf{x}) \tag{111}$$

and outside the source

$$\frac{\partial}{\partial t}\mathbf{D} = \epsilon_0\frac{\partial}{\partial t}\mathbf{E} = \nabla \times \mathbf{H} = e^{-i\omega t}\nabla \times \mathbf{H}(\mathbf{x}) \tag{112}$$

$$\rightarrow \mathbf{E} = \frac{1}{-i\omega}\frac{1}{\epsilon_0}e^{-i\omega t}\nabla \times \mathbf{H}(\mathbf{x}) \tag{113}$$

$$= \frac{i}{k}\frac{1}{\epsilon_0 c}\nabla \times \mathbf{H} \tag{114}$$

$$= \frac{i}{k}\frac{\sqrt{\mu_0 \epsilon_0}}{\epsilon_0}\nabla \times \mathbf{H} \tag{115}$$

$$= \frac{i}{k}\sqrt{\frac{\mu_0}{\epsilon_0}}\nabla \times \mathbf{H} \tag{116}$$

Expressing this directly via the vector potential gives

$$\mathbf{E} = \frac{i}{k}\frac{1}{\epsilon_0 c}\nabla \times \mathbf{H} \tag{117}$$

$$= \frac{i}{k}\frac{1}{\epsilon_0\mu_0 c}\nabla \times (\nabla \times \mathbf{A}) \tag{118}$$

$$= \frac{ic}{k}[\nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}] \tag{119}$$

$$= \frac{ic}{k}[\nabla(-\frac{1}{c^2}\frac{\partial\phi}{\partial t}) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A}] \tag{120}$$

$$= -\frac{i}{kc}\frac{\partial}{\partial t}[\nabla\phi + \frac{\partial}{\partial t}\mathbf{A}] \tag{121}$$

$$= -\frac{i}{kc}(-i\omega)[\nabla\phi + \frac{\partial}{\partial t}\mathbf{A}] \tag{122}$$

$$= -[\nabla\phi + \frac{\partial}{\partial t}\mathbf{A}] \tag{123}$$

0.5.3 Multipole Radiation

$$\mathbf{A}(\mathbf{x}, t) = \frac{\mu_0}{4\pi}\int dt'\int d^3\mathbf{x}'\frac{\mathbf{J}(\mathbf{x}', t')}{|\mathbf{x} - \mathbf{x}'|}\delta\left(c(t' - t) + |\mathbf{x} - \mathbf{x}'|\right) \tag{124}$$

$$= \frac{\mu_0}{4\pi}\int dt'\int d^3\mathbf{x}'\left(\frac{1}{|\mathbf{x}|} + \frac{\mathbf{x}\cdot\mathbf{x}'}{|\mathbf{x}|^3} + ...\right)\mathbf{J}(\mathbf{x}', t')\delta\left(c(t' - t) + |\mathbf{x}| - \mathbf{n}\cdot\mathbf{x}' + ...\right) \tag{125}$$

$$= \frac{\mu_0}{4\pi r}\int d^3\mathbf{x}'\left(1 + \mathbf{n}\cdot\frac{\mathbf{x}'}{|\mathbf{x}|} + ...\right)\mathbf{J}(\mathbf{x}', t - \frac{1}{c}|\mathbf{x}| + \frac{1}{c}\mathbf{n}\cdot\mathbf{x}' + ...) \tag{126}$$

$$= \frac{\mu_0}{4\pi r}\int d^3\mathbf{x}'\left(1 + \mathbf{n}\cdot\frac{\mathbf{x}'}{|\mathbf{x}|} + ...\right)\left[\mathbf{J}(\mathbf{x}', t - \frac{1}{c}|\mathbf{x}|) + \frac{1}{c}(\mathbf{n}\cdot\mathbf{x}')\partial_t\mathbf{J}(\mathbf{x}', t - \frac{1}{c}|\mathbf{x}| + ..)\right] \tag{127}$$

$$= \frac{\mu_0}{4\pi r}\int d^3\mathbf{x}'\left[\mathbf{J}(\mathbf{x}', t - \frac{1}{c}|\mathbf{x}|) + \frac{1}{c}(\mathbf{n}\cdot\mathbf{x}')\partial_t\mathbf{J}(\mathbf{x}', t - \frac{1}{c}|\mathbf{x}|) + ..\right] + \mathbf{n}\cdot\frac{\mathbf{x}'}{|\mathbf{x}|}[...] + ... \tag{128}$$

$$= \frac{\mu_0}{4\pi r}\int d^3\mathbf{x}'\,\mathbf{J}(\mathbf{x}', t - \frac{1}{c}|\mathbf{x}|) + \frac{\mu_0}{4\pi cr}\int d^3\mathbf{x}'(\mathbf{n}\cdot\mathbf{x}')\partial_t\mathbf{J}(\mathbf{x}', t - \frac{1}{c}|\mathbf{x}|) + ... \tag{129}$$

$$= \mathbf{A}_{\text{ED}}(\mathbf{x}, t) + \mathbf{A}_{\text{MD/EQ}}(\mathbf{x}, t) \tag{130}$$

Treating each dimension individually we can integrate by parts

$$\mathbf{A}_{\text{ED}}(\mathbf{x}, t) = \frac{\mu_0}{4\pi r}\int d^3\mathbf{x}'\,\mathbf{J}(\mathbf{x}', t - \frac{1}{c}|\mathbf{x}|) \tag{131}$$

$$= -\frac{\mu_0}{4\pi r}\int d^3\mathbf{x}'\,\mathbf{x}'\nabla\cdot\mathbf{J}(\mathbf{x}', t - \frac{1}{c}|\mathbf{x}|) \tag{132}$$

$$= -\frac{\mu_0}{4\pi r}\int d^3\mathbf{x}'\,\mathbf{x}'\dot{\rho}(\mathbf{x}', t - \frac{1}{c}|\mathbf{x}|) \tag{133}$$

$$= -\frac{\mu_0}{4\pi r}\dot{\mathbf{p}}(t - \frac{1}{c}|\mathbf{x}|) \tag{134}$$

with

$$\mathbf{J}(\mathbf{x}, t) = \mathbf{J}(\mathbf{x})e^{-i\omega t} \tag{135}$$

$$\mathbf{J}(\mathbf{x}, t - \frac{1}{c}|\mathbf{x} - \mathbf{x}'|) = \mathbf{J}(\mathbf{x})e^{-i\omega t}e^{ik|\mathbf{x} - \mathbf{x}'|} \tag{136}$$

$$\mathbf{A}_{\text{MD/EQ}}(\mathbf{x}, t) = \frac{\mu_0}{4\pi cr}\int d^3\mathbf{x}'(\mathbf{n}\cdot\mathbf{x}')\partial_t\mathbf{J}(\mathbf{x}', t - \frac{1}{c}|\mathbf{x}|) \tag{137}$$

0.6 Light Scattering

1. Thomson

2. Rayleigh

3. Rayleigh-Gans

4. Anomalous diffraction approximation of van de Hulst

5. Mie scattering

6. Compton

0.7 Quantum Mechanics

0.7.1 Pictures

Prelims - at $t = t_0$

$$|\psi_H\rangle = |\psi(t_0)\rangle \tag{138}$$

and obviously

$$U(t, t_0) = U^{-1}(t_0, t) \tag{139}$$

$$U^\dagger(t, t_0)U(t, t_0) = 1 \qquad \text{(probability conservation)} \tag{140}$$

1. Schroedinger - time dependency in the states

$$i\partial_t|\psi(t)\rangle = H|\psi(t)\rangle \tag{141}$$

$$|\psi(t)\rangle = U(t, t_0)|\psi(t_0)\rangle \tag{142}$$

$$i\partial_t U(t, t_0) = HU(t, t_0) \tag{143}$$

$$\frac{\partial H}{\partial t} = 0 \quad \to \quad U(t, t_0) = e^{-iH(t-t_0)} \tag{144}$$

Time evolution with $i\partial_t|\psi\rangle = H|\psi\rangle$

$$|\psi(t)\rangle = U(t, t_0)|\psi(t_0)\rangle \tag{145}$$

$$\simeq (1 - i(t - t_0)H)|\psi(t_0)\rangle \tag{146}$$

$$\simeq (1 - i(t - t_0)i\partial_t)|\psi(t_0)\rangle \tag{147}$$

$$\simeq |\psi(t_0)\rangle + \frac{\partial|\psi(t_0)\rangle}{\partial t}(t - t_0) \tag{148}$$

Time evolution with $H|E_k\rangle = E_k|E_k\rangle$

$$|\psi(t)\rangle = U(t, t_0)|\psi(t_0)\rangle \tag{149}$$

$$= U(t, t_0) \sum_k |E_k\rangle\langle E_k|\psi(t_0)\rangle \tag{150}$$

$$= \sum_k e^{-iH(t-t_0)}|E_k\rangle\langle E_k|\psi(t_0)\rangle \tag{151}$$

$$= \sum_k e^{-iE_k(t-t_0)}|E_k\rangle\langle E_k|\psi(t_0)\rangle \tag{152}$$

Measurement

$$\langle A(t)\rangle = \langle \psi(t)|A_S|\psi(t)\rangle \tag{153}$$

2. Heisenberg - time dependency in the operators

$$\langle A(t) \rangle = \langle \psi(t) | A_S | \psi(t) \rangle \tag{154}$$

$$= \langle \psi(t_0) | U^\dagger(t, t_0) A_S U(t, t_0) | \psi(t_0) \rangle \tag{155}$$

$$= \langle \psi(t_0) | A_H(t) | \psi(t_0) \rangle \tag{156}$$

$$\rightarrow A_H(t) = U^\dagger(t, t_0) A_S U(t, t_0) \tag{157}$$

Time derivative

$$\frac{d}{dt} A_H(t) = \left(\frac{d}{dt} U^\dagger(t, t_0) \right) A_S U(t, t_0) + U^\dagger(t, t_0) \left(\frac{d}{dt} A_S \right) U(t, t_0) + U^\dagger(t, t_0) A_S \left(\frac{d}{dt} U(t, t_0) \right)$$

$$\tag{158}$$

$$= U^\dagger(t, t_0) i (H A_S - A_S H) U(t, t_0) + U^\dagger(t, t_0) \frac{\partial A_S}{\partial t} U(t, t_0) \tag{159}$$

$$= i[H, A_H] + \underbrace{U^\dagger(t, t_0) \frac{\partial A_S}{\partial t} U(t, t_0)}_{\equiv \frac{\partial A_H}{\partial t}} \tag{160}$$

$$= i[H, A_H] + \frac{\partial A_H}{\partial t} \tag{161}$$

3. Dirac - $H = H_0 + H_{\text{int}}$

$$|\psi(t)\rangle_D = e^{iH_0 t} |\psi(t)\rangle_S \tag{162}$$

$$A_D(t) = e^{iH_0 t} A_S e^{-iH_0 t} \tag{163}$$

then

$$\langle A(t) \rangle = \langle \psi(t) | A_S | \psi(t) \rangle \tag{164}$$

$$= \langle \psi(t_0) | U^\dagger(t, t_0) A_S U(t, t_0) | \psi(t_0) \rangle \tag{165}$$

$$= \langle \psi(t_0) | U^\dagger(t, t_0) \underbrace{U_0(t, t_0) U_0^\dagger(t, t_0)}_{=1} A_S \underbrace{U_0(t, t_0) U_0^\dagger(t, t_0)}_{=1} U(t, t_0) | \psi(t_0) \rangle \tag{166}$$

$$\rightarrow A_D = U_0^\dagger(t, t_0) A_S U_0(t, t_0) \tag{167}$$

$$\rightarrow |\psi_D(t)\rangle = U_0^\dagger(t, t_0) U(t, t_0) | \psi(t_0) \rangle = U_0^\dagger(t, t_0) | \psi(t) \rangle \tag{168}$$

Now calc evolution between the TWO Dirac states $|\psi_D(t_1)\rangle$ and $|\psi_D(t_2)\rangle$

$$|\psi_D(t_1)\rangle = U_0^\dagger(t_1, t_0) U(t_1, t_0) | \psi(t_0) \rangle \tag{169}$$

$$|\psi_D(t_2)\rangle = U_0^\dagger(t_2, t_0) U(t_2, t_0) | \psi(t_0) \rangle \tag{170}$$

$$= U_0^\dagger(t_2, t_0) U(t_2, t_0) \left(U_0^\dagger(t_1, t_0) U(t_1, t_0) \right)^{-1} | \psi_D(t_1) \rangle \tag{171}$$

$$= U_0^\dagger(t_2, t_0) U(t_2, t_0) U^{-1}(t_1, t_0) \left(U_0^\dagger(t_1, t_0) \right)^{-1} | \psi_D(t_1) \rangle \tag{172}$$

$$= U_0^\dagger(t_2, t_0) U(t_2, t_0) U(t_0, t_1) U_0^\dagger(t_1, t_0) | \psi_D(t_1) \rangle \tag{173}$$

$$= U_0^\dagger(t_2, t_0) U(t_2, t_1) U_0^\dagger(t_1, t_0) | \psi_D(t_1) \rangle \tag{174}$$

with $t_0 = 0$ and H_0 time-independent

$$U_D(t_2, t_1) = U_0^\dagger(t_2, 0) U(t_2, t_1) U_0^\dagger(t_1, 0) | \psi_D(t_1) \rangle \tag{175}$$

$$= e^{iH_0 t_2} U(t_2, t_1) e^{iH_0 t_1} \tag{176}$$

picture	equation	state	operator
Schroedinger	$i\partial_t|\psi(t)\rangle_S = H_0|\psi(t)\rangle_S$	$|\psi(t)\rangle_S = e^{-iH_0(t-t_0)}|\psi(t_0)\rangle_S$	$A_S(t) = A_S$
Heisenberg	$\frac{d}{dt}A_H = \partial_t A_H + i[H_0, A_H]$	$|\psi(t)\rangle_H = |\psi(t_0)\rangle_S$	$A_H(t) = e^{iH_0(t-t_0)}A_H(t_0)e^{-iH_0(t-t_0)}$
Dirac	$i\partial_t|\psi(t)\rangle_D = H_I|\psi(t)\rangle_D$	$|\psi(t)\rangle_D = e^{+iH_0(t-t_0)}|\psi(t_0)\rangle_D$	$A_D(t) = e^{iH_0(t-t_0)}A_S e^{-iH_0(t-t_0)}$

where

$$|\psi(t_0)\rangle_S = |\psi\rangle_H = |\psi(t_0)\rangle_D \tag{177}$$

$$A_S = A_H(t_0) = A_D(t_0) \tag{178}$$

$$H = H_0 + H_{\text{int}} \qquad H_I = (H_{\text{int}})_D = e^{iH_0(t-t_0)}H_{\text{int}}e^{-iH_0(t-t_0)} \tag{179}$$

0.7.2 3D Spherical well

$$\left\{-\frac{\hbar^2}{2m}\triangle + V(r)\right\}\psi = E\psi \tag{180}$$

$$\left\{-\frac{\hbar^2}{2m}\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \frac{1}{r^2}\triangle_{\phi\theta}\right] + V(r)\right\}\psi = E\psi \tag{181}$$

$$\left\{\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \frac{1}{r^2}\triangle_{\phi\theta} - \frac{2m[V(r)-E]}{\hbar^2}\right\}\psi = 0 \tag{182}$$

Separation $\psi = R(r)Y(\phi,\theta)$

$$\frac{r^2\left\{\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{2m[V(r)-E]}{\hbar^2}\right\}R(r)}{R(r)} = l(l+1) = -\frac{\triangle_{\phi,\theta}Y(\phi,\theta)}{Y(\phi,\theta)} \tag{183}$$

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} - \frac{2m[V(r)-E]}{\hbar^2}\right)R(r) = 0 \tag{184}$$

With the definition of the well potential

$$V(r) = \begin{cases} -V_0 & r < a \\ 0 & r > a \end{cases} \tag{185}$$

With $-V_0 < E < 0$

$$k = \frac{\sqrt{2m[E+V_0]}}{\hbar} \tag{186}$$

$$\kappa = \frac{\sqrt{2m(-E)}}{\hbar} \tag{187}$$

be have with $\rho = kr$ and $\rho = i\kappa r$

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} + \binom{k^2}{-\kappa^2}\right]R(r) = 0 \tag{188}$$

$$\left[\frac{d^2}{d\rho^2} + \frac{2}{\rho}\frac{d}{d\rho} - \frac{l(l+1)}{\rho^2} + 1\right]R(\rho) = 0 \tag{189}$$

$$\left[\rho^2\frac{d^2}{d\rho^2} + 2\rho\frac{d}{d\rho} + \rho^2 - l(l+1)\right]R(\rho) = 0 \tag{190}$$

Independent solutions

$$R(\rho) = Aj_l(\rho) + By_l(\rho) \tag{191}$$

$$= A\sqrt{\frac{\pi}{2\rho}}J_{l+1/2}(\rho) + B\sqrt{\frac{\pi}{2\rho}}Y_{l+1/2}(\rho) \tag{192}$$

Here the requirements

- regular at the origin with $R(r) \sim r^l$

- continous and differentiable at $r = a$

- exponential decay outside to ensure normalizability

and here a quick overview of the two functions and a special linear combination

$$j_l(x) = (-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\sin x}{x} \qquad y_l(x) = -(-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\cos x}{x} \qquad h_0^{(1)}(x) = j_l(ix) + iy_l(ix)$$

$$j_0(x) = \frac{\sin x}{x} \qquad\qquad y_0(x) = -\frac{\cos x}{x} \qquad\qquad h_0^{(1)}(x) = -\frac{e^{-x}}{x}$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} \qquad y_1(x) = -\frac{\cos x}{x} - \frac{\sin x}{x} \qquad h_1^{(1)}(x) = i(1+x)\frac{e^{-x}}{x^2}$$

$$J_2(x) = ... \qquad\qquad\qquad y_l(x) = ... \qquad\qquad\qquad h_2^{(1)}(x) = (x^2 + 3x + 3)\frac{e^{-x}}{x^3}$$

We see that j_l is suitable for the inside and $h_l^{(1)}$ for the outside.

$$R(\rho) = \begin{cases} Aj_l(\rho) & r < a \\ Ch_l^{(1)}(\rho) & r > a \end{cases} \tag{193}$$

Now $l = 0$

$$Aj_0(\rho = ka) = Ch_0^{(1)}(\rho = \kappa a) \quad \rightarrow \quad A\frac{\sin ka}{ka} = -C\frac{e^{-\kappa a}}{\kappa a} \tag{194}$$

$$A\partial_r j_0(\rho = ka) = C\partial_r h_0^{(1)}(\rho = \kappa a) \quad \rightarrow \quad A\frac{\sin ak}{a}\left(\cot ka - \frac{1}{ka}\right) = C\frac{e^{-\kappa a}}{a}\left(1 + \frac{1}{\kappa a}\right) \tag{195}$$

By substituting first into the second equation we kick out A and C and obtain

$$\cot ka = -\frac{\kappa}{k} \tag{196}$$

$$\cot \sqrt{\frac{2ma^2}{\hbar^2}[E + V_0]} = -\sqrt{\frac{-E}{E + V_0}} \tag{197}$$

Now $l = 1$

$$Aj_1(\rho = ka) = Ch_1^{(1)}(\rho = \kappa a) \quad \rightarrow \quad A\left(-\frac{\cos ka}{ka} + \frac{\sin ka}{k^2a^2}\right) = iC\frac{e^{-\kappa a}}{\kappa^2 a^2}(1 + \kappa a) \tag{198}$$

$$A\partial_r j_1(\rho = ka) = C\partial_r h_1^{(1)}(\rho = \kappa a) \quad \rightarrow \quad A\left(2\frac{\cos ka}{ka^2} + \frac{\sin ka}{k^2a^3}(a^2k^2 - 2)\right) = -iC\frac{e^{-\kappa a}}{\kappa^2 a^3}(\kappa^2 a^2 + 2\kappa a + 2) \tag{199}$$

Then

$$\cot ka = \frac{k^2 + ak^2\kappa + \kappa^2}{ak\kappa^2} \tag{200}$$

0.8 Quantum statistics

Quick thermodynamics review

$$\text{1st law} \quad dU = \delta Q + \delta W \tag{201}$$

$$\text{2nd law} \quad dS = dS_i + \frac{\delta Q}{T}, \qquad dS_i > 0 \tag{202}$$

$$\text{Gibbs Fund.Form} \to dS = \frac{1}{T}dU - \frac{1}{T}\delta W = \frac{1}{T}dU + \frac{1}{T}\sum_i y_i dX_i \tag{203}$$

$$\to \left.\frac{dS}{dU}\right|_{X_i} = \frac{1}{T} \qquad \to \qquad U = U(T, X_i) \tag{204}$$

$$\to \left.\frac{dS}{dX_i}\right|_{U,X_j} = \frac{y_i}{T} \qquad \to \qquad y_i = y_i(T, X_j) \tag{205}$$

0.8.1 Microcanonical ensemble

Macroscopic equilibrium state is defined by E, N, V:

$$\text{Sirling Formula} \qquad n! \simeq (n/e)^n \sqrt{2\pi n} \tag{206}$$

$$\ln n! \simeq (n + \tfrac{1}{2})\ln n - n + \tfrac{1}{2}\ln(2\pi) \tag{207}$$

$$\text{phasespace element} \qquad D\Gamma = \frac{1}{h^{3N}}\prod_\alpha dp_\alpha dq_\alpha \tag{208}$$

$$\text{phasespace element (identical part)} \qquad D\Gamma = \frac{1}{N! h^{3N}}\prod_\alpha dp_\alpha dq_\alpha \tag{209}$$

$$\text{phasespace volume} \qquad \Gamma(E, V, N) = \int_{H(q_\alpha, q_\alpha) \leq E} D\Gamma \tag{210}$$

$$\text{micro states in}[E, E + \Delta E] \qquad \Omega = \left(\frac{\partial\Gamma}{\partial E}\right)_{N,V}\Delta E \tag{211}$$

$$\text{phasespace, prob. density} \qquad \Omega^{-1} = \rho = \frac{\delta(H(q_\alpha, p_\alpha) - E)}{\int D\Gamma \delta(H(q_\alpha, p_\alpha) - E)} \tag{212}$$

$$\text{entropy of the system} \quad S = k\ln\Omega = -k\overline{\ln\rho} = -k\int D\Gamma \rho\ln\rho \tag{213}$$

$$\text{inner Energy} \qquad U = E \tag{214}$$

$$\text{temperature} \qquad \frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{E,N} \tag{215}$$

$$\text{pressure} \qquad \frac{p}{T} = \left(\frac{\partial S}{\partial V}\right)_{E,N} \tag{216}$$

$$\text{chemical potential} \qquad -\frac{\mu}{T} = \left(\frac{\partial S}{\partial N}\right)_{E,N} \tag{217}$$

$$\tag{218}$$

0.8.2 Canonical ensemble

Macroscopic equilibrium state is defined by T, N, V (system can exchange energy with external reservoir - but system + reservoir is microcanonical ensemble):

$$\text{State integral} \qquad Z = \int D\Gamma \exp[-\tfrac{H}{kT}] \tag{219}$$

$$\text{phasespace, prob. density} \qquad \rho = \tfrac{1}{Z}\exp[-\tfrac{H}{kT}] \tag{220}$$

$$\text{discrete} \qquad Z = \sum_i \exp[-\tfrac{E_i}{kt}], \quad p_i = \tfrac{1}{Z}\sum_i \exp[-\tfrac{E_i}{kt}] \tag{221}$$

$$\text{entropy} \quad S = -k\int D\Gamma \rho\ln\rho = -k\int D\Gamma \rho(-\tfrac{H}{kT} - \ln Z) = \tfrac{1}{T}\bar{H} + k\ln Z \tag{222}$$

$$\text{free energy} \qquad F = U - TS = -kT\ln Z \tag{223}$$

0.8.3 Great canonical ensemble

$$\text{...} \qquad \mathcal{Z} = \sum_N \int D\Gamma \exp[-\tfrac{H_N - \mu N}{kT}] \tag{224}$$

$$\text{...} \qquad \rho_N = \tfrac{1}{\mathcal{Z}} \int D\Gamma \exp[-\tfrac{H_N - \mu N}{kT}] \tag{225}$$

$$\text{discrete} \quad \mathcal{Z} = \sum_N \sum_i \exp[-\tfrac{E_i - \mu N}{kT}], \quad p_{N,i} = \tfrac{1}{\mathcal{Z}} \exp[-\tfrac{E_i - \mu N}{kT}] \tag{226}$$

$$\text{entropy} \qquad S = -k\overline{\ln \rho_N} = -k \sum_N \int D\Gamma \rho_N \ln \rho_N \tag{227}$$

$$\text{great canonical potential} \qquad \mathcal{F} = U - TS - \mu\overline{N} = -kT \ln \mathcal{Z} \tag{228}$$

0.8.4 Density matrix - statistical operator

Using the principle of equal probability

$$\hat{\varrho} = \sum_k p_k |\Psi_k\rangle\langle\Psi_k| \tag{229}$$

$$= \frac{1}{\Omega} \sum_k |\Psi_k\rangle\langle\Psi_k| \tag{230}$$

$$\text{Tr}\hat{\varrho} = 1 \tag{231}$$

$$S = -k\langle\hat{\varrho}\rangle \tag{232}$$

$$= -k\text{Tr}(\hat{\varrho}\log\hat{\varrho}) \tag{233}$$

$$\tag{234}$$

0.8.5 Canonical ensemble

Represents all states of a system in thermodynamic equilibrium. Meaning the temperature T and therefore the mean energy $\bar{E} = U$ is fixed but the total energy can fluctuate

$$Z = \text{Tr}\left[\exp\left(-\frac{\hat{H}}{kT}\right)\right] \tag{235}$$

$$\hat{\varrho} = \frac{1}{Z(T)}\exp\left(-\frac{\hat{H}}{kT}\right) = \frac{1}{Z(T)}\sum_k |\Psi_k\rangle\exp\left(-\frac{E_k}{kT}\right)\langle\Psi_k| \tag{236}$$

$$F = -kT\log Z \tag{237}$$

$$\frac{\partial F}{\partial T} = -S \tag{238}$$

$$U = F + TS \tag{239}$$

0.8.6 Great Canonical ensemble

Represents all states of a system in thermodynamic equilibrium. Meaning the temperature T and therefore the mean energy $\bar{E} = U$ is fixed but the total energy can fluctuate

$$\mathcal{Z} = \text{Tr}\left[\exp\left(-\frac{\hat{H} - \mu\hat{N}}{kT}\right)\right] \tag{240}$$

$$\hat{\varrho} = \frac{1}{\mathcal{Z}(T)}\exp\left(-\frac{\hat{H} - \mu\hat{N}}{kT}\right) \tag{241}$$

$$\mathcal{F} = -kT\log\mathcal{Z} \tag{242}$$

$$\left(\frac{\partial\mathcal{F}}{\partial T}\right)_\mu = -S \qquad \left(\frac{\partial\mathcal{F}}{\partial\mu}\right)_T = -\bar{N} = -\langle\hat{N}\rangle \tag{243}$$

0.9 Special relativity

Definition of line element

$$ds^2 = dx^\mu dx_\nu = \eta_{\mu\nu} dx^\mu dx^\nu \tag{244}$$
$$= dx^T \eta dx \tag{245}$$

Definition of Lorentz transformation

$$dx^\mu = \Lambda^\mu_{\ \nu} dx^\nu \tag{246}$$

By postulate the line element ds is invariant under Lorentz transformation

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \tag{247}$$
$$\overset{!}{=} \eta_{\alpha\beta} \Lambda^\alpha_{\ \mu} dx^\mu \Lambda^\beta_{\ \nu} dx^\nu \quad \rightarrow \quad \eta_{\mu\nu} = \eta_{\alpha\beta} \Lambda^\alpha_{\ \mu} \Lambda^\beta_{\ \nu} \tag{248}$$

or analog

$$ds^2 = dx^T \eta dx \tag{249}$$
$$\overset{!}{=} (\Lambda dx)^T \eta (\Lambda dx) \tag{250}$$
$$= dx^T \Lambda^T \eta \Lambda dx \quad \rightarrow \quad \eta = \Lambda^T \eta \Lambda \tag{251}$$

Observation with the eigentime $d\tau = ds/c$ and 3-velocity $dx^i = v^i dt$

$$\frac{ds^2}{d\tau^2} = c^2 = c^2 \frac{dt^2}{d\tau^2} - \frac{dx^i}{dt} \frac{dx_i}{dt} \left(\frac{dt}{d\tau}\right)^2 \tag{252}$$
$$1 = \frac{dt^2}{d\tau^2} \left(1 - \frac{v^i v_i}{c^2}\right) \quad \rightarrow \quad \frac{dt}{d\tau} \equiv \gamma = \left(\sqrt{1 - \frac{v^2}{c^2}}\right)^{-1} \tag{253}$$

0.9.1 Definition 4-velocity

with 3-velocity $d\vec{x} = \vec{v} dt$

$$u^\mu \equiv \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt} \frac{dt}{d\tau} = \quad \rightarrow \quad u^\mu u_\mu = \eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = \frac{ds^2}{d\tau^2} = c^2 \tag{254}$$
$$= (c, \vec{v}) \gamma \tag{255}$$

Object moving in x direction with v meaning $dx = v \cdot dt$ compared to rest frame $dx' = 0$

$$c^2 dt'^2 = ds^2 = c^2 dt^2 - v^2 dt^2 \tag{256}$$
$$= c^2 dt^2 \left(1 - \frac{v^2}{c^2}\right) \tag{257}$$
$$dt' = \frac{ds}{c} \equiv d\tau = dt \sqrt{1 - \frac{v^2}{c^2}} = \frac{dt}{\gamma} \tag{258}$$

0.9.2 Definition 4-momentum

using the 3-momentum $\vec{p} = \gamma m \vec{v}$

$$p^\mu \equiv m u^\mu = (\gamma m c, \gamma m \vec{v}) = \left(\frac{E_p}{c}, \vec{p} \right) \quad \rightarrow \quad p^\mu p_\mu = m^2 u^\mu u_\mu = m^2 c^2 \tag{259}$$

$$\rightarrow \quad (p^0)^2 - p^i p_i = m^2 c^2 \tag{260}$$

$$\rightarrow \quad p^0 = \sqrt{m^2 c^2 + \vec{p}^2} \tag{261}$$

$$\rightarrow \quad E_p = \sqrt{m^2 c^4 + \vec{p}^2 c^2} \tag{262}$$

$$= \frac{mc^2}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}} \tag{263}$$

0.9.3 Definition 4-acceleration

First observe

$$u^\mu u_\mu = c^2 \tag{264}$$

$$\frac{d}{d\tau}(u^\mu u_\mu) = 0 \tag{265}$$

$$\rightarrow \alpha^\mu u_\mu = 0 \tag{266}$$

meaning

$$\alpha^0 u_0 - \vec{\alpha} \cdot \vec{u} = 0 \tag{267}$$

$$\gamma(\alpha^0 c - \vec{\alpha} \cdot \vec{v}) = 0 \tag{268}$$

$$\rightarrow \alpha^0 = \frac{\vec{\alpha} \cdot \vec{v}}{c} \tag{269}$$

$$\frac{d^2 x^\mu}{d\tau^2} = \frac{d}{d\tau} \frac{dx^\mu}{d\tau} \tag{270}$$

$$= \frac{d}{d\tau} \left(\frac{dx^\mu}{dt} \frac{dt}{d\tau} \right) \tag{271}$$

$$\vec{\alpha} = \frac{d^2 x^k}{d\tau^2} = \frac{d^2 x^k}{dt^2} \left(\frac{dt}{d\tau} \right)^2 + \frac{dx^k}{dt} \frac{d^2 t}{d\tau^2} \tag{272}$$

$$\equiv a^k \gamma^2 + v^k \frac{d\gamma}{d\tau} \tag{273}$$

$$= a^k \gamma^2 + v^k \frac{d\gamma}{dt} \frac{dt}{d\tau} \tag{274}$$

$$= a^k \gamma^2 + v^k \left(-\frac{1}{2} \right) \gamma^3 \frac{-2 v^\alpha \frac{dv^\alpha}{dt}}{c^2} \frac{dt}{d\tau} \tag{275}$$

$$= a^k \gamma^2 + v^k \gamma^4 (\vec{v} \cdot \vec{a}) \frac{1}{c^2} \tag{276}$$

$$\alpha^0 = \frac{d^2 x^0}{d\tau^2} = \frac{d^2 x^0}{dt^2} \left(\frac{dt}{d\tau} \right)^2 + \frac{dx^0}{dt} \frac{d^2 t}{d\tau^2} \tag{277}$$

$$= 0 \cdot \gamma^2 + c \gamma^4 (\vec{v} \cdot \vec{a}) \frac{1}{c^2} \tag{278}$$

$$= \gamma^4 (\vec{v} \cdot \vec{a}) \frac{1}{c} \tag{279}$$

we see after a short calculation (in the initial restframe) $\alpha^\mu \alpha_\mu = -\vec{a}^2 \equiv a_0^2$ (a_0 proper acceleration in the restframe) where

$$\text{4-velocity} \quad u^\mu = \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt}\frac{dt}{d\tau} \tag{280}$$

$$\text{3-velocity} \quad v^k = \frac{dx^k}{dt} \tag{281}$$

$$\text{4-acceleration} \quad \alpha^\mu = \frac{d^2x^\mu}{d\tau^2} = \frac{du^\mu}{d\tau} = \frac{du^\mu}{dt}\frac{dt}{d\tau} \tag{282}$$

$$\text{3-acceleration} \quad a^k = \frac{d^2x^k}{dt} = \frac{dv^k}{dt} \tag{283}$$

First we observe

$$\eta_{\mu\nu} = \eta_{\alpha\beta}\Lambda^\alpha_{\ \mu}\Lambda^\beta_{\ \nu} \tag{284}$$

$$\det(\eta) = \det(\Lambda)^2 \det(\eta) \tag{285}$$

$$1 = \det(\Lambda)^2. \tag{286}$$

Now we see

$$\Lambda_\gamma^{\ \nu}\Lambda^\gamma_{\ \mu} = \eta_{\alpha\gamma}\eta^{\nu\beta}\Lambda^\alpha_{\ \beta}\Lambda^\gamma_{\ \mu} \tag{287}$$

$$= \eta^{\nu\beta}(\eta_{\alpha\gamma}\Lambda^\alpha_{\ \beta}\Lambda^\gamma_{\ \mu}) \tag{288}$$

$$= \eta^{\nu\beta}\eta_{\beta\mu} \tag{289}$$

$$= \delta^\nu_{\ \mu} \tag{290}$$

which means in matrix notation $\Lambda_\gamma^{\ \nu} = (\Lambda^{-1})^\nu_{\ \gamma}$. General transformation laws for tensors of first order

$$V'^\alpha = \Lambda^\alpha_{\ \beta}V^\beta \tag{291}$$

$$\eta_{\alpha\mu}V'^\alpha = \eta_{\alpha\mu}\Lambda^\alpha_{\ \beta}V^\beta = \eta_{\alpha\mu}\Lambda^\alpha_{\ \beta}(\eta^{\nu\beta}V_\nu) \tag{292}$$

$$V'_\mu = \Lambda_\mu^{\ \nu}V_\nu \tag{293}$$

$$\rightarrow \quad \Lambda_\mu^{\ \nu} = \eta_{\alpha\mu}\eta^{\nu\beta}\Lambda^\alpha_{\ \beta} \tag{294}$$

and second order

$$T'^{\alpha\beta} = \Lambda^\alpha_{\ \mu}\Lambda^\beta_{\ \nu}T^{\mu\nu} \tag{295}$$

$$\eta_{\alpha\delta}\eta_{\beta\gamma}T'^{\alpha\beta} = \eta_{\alpha\delta}\eta_{\beta\gamma}\Lambda^\alpha_{\ \mu}\Lambda^\beta_{\ \nu}T^{\mu\nu} = \eta_{\alpha\delta}\eta_{\beta\gamma}\Lambda^\alpha_{\ \mu}\Lambda^\beta_{\ \nu}(\eta^{\mu\rho}\eta^{\nu\sigma}T_{\rho\sigma}) \tag{296}$$

$$T'_{\delta\gamma} = \Lambda_\delta^{\ \rho}\Lambda_\gamma^{\ \sigma}T_{\rho\sigma}. \tag{297}$$

The general transformation is therefore given by

$$T'_{\mu_1\mu_2\ldots}^{\quad\ \nu_1\nu_2\cdots} = \Lambda_{\mu_1}^{\ \rho_1}\Lambda_{\mu_2}^{\ \rho_2}\ldots\Lambda^{\nu_1}_{\ \sigma_1}\Lambda^{\nu_2}_{\ \sigma_2}\ldots T'_{\rho_1\rho_2\ldots}^{\quad\ \sigma_1\sigma_2\cdots} \tag{298}$$

There exist two invariant tensors

$$\eta'_{\mu\nu} = \eta_{\alpha\beta}\Lambda^\alpha_{\ \mu}\Lambda^\beta_{\ \nu} = \Lambda_{\beta\mu}\Lambda^\beta_{\ \nu} = \eta_{\mu\sigma}\Lambda^\sigma_{\ \beta}\Lambda^\beta_{\ \nu} = \eta_{\mu\sigma}\delta^\sigma_{\ \nu} = \eta_{\mu\nu} \tag{299}$$

$$\epsilon'^{\mu\nu\rho\sigma} = \Lambda^\mu_{\ \alpha}\Lambda^\nu_{\ \beta}\Lambda^\rho_{\ \gamma}\Lambda^\sigma_{\ \delta}\epsilon'^{\alpha\beta\gamma\delta} \equiv \epsilon^{\mu\nu\rho\sigma}\det(\Lambda) = \pm\epsilon^{\mu\nu\rho\sigma} \tag{300}$$

Due to the possibility of the minus sign the Levi-Civita symbol ϵ is sometimes called pseudo-tensor.

0.10 Hydrodynamics

With $\rho = m/V$ we use mass conservation

$$\frac{\partial}{\partial t} m_V = \frac{\partial}{\partial t} \int_V \rho \, dV = -\oint_{\partial V} \mathbf{j} \cdot d\mathbf{A} \tag{301}$$

$$= -\oint_{\partial V} \rho \mathbf{u} \cdot d\mathbf{A} \tag{302}$$

$$= -\int_V \nabla \cdot (\rho \mathbf{u}) \cdot dV \tag{303}$$

$$\to \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{304}$$

$$\to \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u} = 0 \tag{305}$$

$$\overset{\rho = \text{const}}{\to} \nabla \cdot \mathbf{u} = 0 \tag{306}$$

We use Newtons 3. law

$$\frac{d\mathbf{p}}{dt} = \mathbf{F} \tag{307}$$

$$m \frac{d\mathbf{u}}{dt} + \mathbf{u}\frac{dm}{dt} = -\oint p \, d\mathbf{A} \tag{308}$$

$$m \left(\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial x^i}{\partial t} \frac{\partial \mathbf{u}}{\partial x^i} \right) + \mathbf{u}\frac{dm}{dt} = -\int \nabla p \, dV \tag{309}$$

$$m \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right) + \mathbf{u}\frac{dm}{dt} = -\nabla p \, V \tag{310}$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right) + \frac{1}{V}\mathbf{u}\frac{dm}{dt} = -\nabla p \tag{311}$$

0.11 Nonrelativistiv Magnetohydrodynamics

Ingredience

- Maxwell equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{312}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{313}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{314}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \tag{315}$$

- Ohms law in fluid local rest (usually accelerated) frame

$$\mathbf{j}' = \kappa \mathbf{E}' \tag{316}$$

- Lorentz transformation with $\hat{\mathbf{v}} = \mathbf{v}/v$

$$\mathbf{E}' = \gamma \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} \right) - (\gamma - 1)(\mathbf{E} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}} \tag{317}$$

$$\mathbf{B}' = \gamma \left(\mathbf{B} - \frac{1}{c^2}\mathbf{v} \times \mathbf{E} \right) - (\gamma - 1)(\mathbf{B} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}} \tag{318}$$

$$\mathbf{j}' = \mathbf{j} - \gamma \rho \mathbf{v} + (\gamma - 1)(\mathbf{j} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}} \tag{319}$$

$$\rho' = \gamma \left(\rho - \frac{1}{c^2}\mathbf{j} \cdot \mathbf{v} \right) \tag{320}$$

- Assumptions $v/c \ll 1$ meaning $\gamma = 1$ and κ is high

Conclusion using $v/c \ll 1$

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B} \tag{321}$$

$$\mathbf{B}' = \mathbf{B} - \frac{1}{c^2}\mathbf{v} \times \mathbf{E} \tag{322}$$

$$\mathbf{j}' = \mathbf{j} - \rho\mathbf{v} \tag{323}$$

$$\rho' = \rho - \frac{1}{c^2}\mathbf{j} \cdot \mathbf{v} \tag{324}$$

High κ implies $E' \ll E$ and therefore

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} \simeq 0 \quad \rightarrow \quad E \sim vB \tag{325}$$

$$\mathbf{B}' \simeq \mathbf{B} - \frac{1}{c^2}\mathbf{v} \times \mathbf{E} \overset{E \sim vB}{=} \mathbf{B} - \mathcal{O}(v^2/c^2) \tag{326}$$

as well as $\rho' \ll \rho$.

From Amprere Law

$$\nabla \times \mathbf{B} - \mu_0\mathbf{j} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \sim \frac{1}{c^2}\frac{E}{T} \sim \frac{1}{c^2}E\frac{v}{L} \overset{E \sim vB}{\sim} \frac{1}{c^2}vB\frac{v}{L} \sim \mathcal{O}(v^2/c^2) \tag{327}$$

$$= 0 \tag{328}$$

$$\nabla \times \mathbf{B} = \mu_0\mathbf{j} \tag{329}$$

$$\frac{B}{L} \sim \mu_0 j \tag{330}$$

then

$$\rho \overset{\text{Gauss}}{\simeq} \epsilon_0\frac{E}{L} \overset{E \sim vB}{\simeq} \epsilon_0\frac{vB}{L} \overset{\text{Ampere}}{\simeq} \epsilon_0\mu_0 vj \simeq \frac{v}{c^2}j \tag{331}$$

therefore

$$\mathbf{j}' = \mathbf{j} - \rho\mathbf{v} \overset{\rho \sim jv/c^2}{\simeq} \mathbf{j} - \mathcal{O}(v^2/c^2) \tag{332}$$

and with

$$\mathbf{j}' = \kappa\mathbf{E}' \tag{333}$$

$$= \kappa(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{334}$$

we have

$$\mathbf{j} = \kappa(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{335}$$

And

$$\nabla \times \mathbf{B} = \mu_0\mathbf{j} = \mu_0\kappa(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{336}$$

$$\rightarrow \mathbf{E} = \frac{1}{\mu_o\kappa}\nabla \times \mathbf{B} - \mathbf{v} \times \mathbf{B} \tag{337}$$

$$\frac{\rho}{\epsilon_0} = \nabla \cdot \mathbf{E} = -\nabla \cdot (\mathbf{v} \times \mathbf{B}) \tag{338}$$

$$\rightarrow \rho = -\epsilon_0\nabla \cdot (\mathbf{v} \times \mathbf{B}) \tag{339}$$

$$\tag{340}$$

Now

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \tag{341}$$

$$= -\nabla \times \left(\frac{1}{\mu_o \kappa} \nabla \times \mathbf{B} - \mathbf{v} \times \mathbf{B} \right) \tag{342}$$

$$= -\frac{1}{\mu_o \kappa} \nabla \times \nabla \times \mathbf{B} + \nabla \times (\mathbf{v} \times \mathbf{B}) \tag{343}$$

$$= \frac{1}{\mu_o \kappa} \triangle \mathbf{B} + \nabla \times (\mathbf{v} \times \mathbf{B}) \tag{344}$$

0.12 Perturbation theory

1. Find a hard problem

2. Introduce an ϵ

3. Assume the solution can be expressed as a power series $x_s = \sum_k a_k \epsilon^k$

4. Find all a_k and sum them up

5. Set $\epsilon = 1$

Now consider solving $x^5 + x = 1$

$$x^5 + \epsilon x = 1 \tag{345}$$

$$\rightarrow x = 1 - \frac{1}{5}\epsilon - \frac{1}{25}\epsilon^2 - \frac{1}{125}\epsilon^3 + 0\epsilon^4 + \frac{21}{15625}\epsilon^5 + ... \tag{346}$$

or

$$\epsilon x^5 + x = 1 \tag{347}$$

$$\rightarrow x = 1 - \epsilon + 5\epsilon^2 - 35\epsilon^3 + 285\epsilon^4 - 2530\epsilon^5 + ... \tag{348}$$

Method of dominant balance

- Asymptotics $f(x)$ $g(x)$ for $x \rightarrow x_0$

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = 1 \tag{349}$$

- Neglectable $f(x) \ll g(x)$ for $x \rightarrow x_0$

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = 0 \tag{350}$$

0.12.1 Series summation

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - ... = \frac{\pi}{4} \tag{351}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - ... = \log 2 \tag{352}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} - ... = \frac{\pi^2}{6} \tag{353}$$

- Consider converging series, meaning

$$A_n = \sum_{m=0}^{n} a_m, \qquad A = \sum_{m=0}^{\infty} a_m, \tag{354}$$

- Shanks summation

$$S(A_n) = \frac{A_{n+1}A_{n-1} - A_n^2}{A_{n+1} - 2A_n + A_{n-1}} \tag{355}$$

usually $S(A_n)$ converges faster than A_n. Further speed-up $S(S(...(A_n))$