0.1 Andrews - Number theory

0.1.1 Problem 1.1

Lets cut the chase

$$\begin{aligned}
\frac{n(n+1)(2n+1)}{6} + (n+1)^2 &= (n+1)\frac{n(2n+1)+6(n+1)}{6} && (1)\\
&= \frac{(n+1)}{6}(2n^2+7n+6) && (2)\\
&= \frac{(n+1)}{6}(n+2)(2n+3) && (3)\\
&= \frac{(n+1)}{6}(n+2)(2(n+1)+1) && (4)\\
&= \frac{(n+1)(n+2)(2(n+1)+1)}{6} && (5)\\
& && (6)
\end{aligned}$$

0.2 Morris - Georgi - Lie Algebras in Particle Physics 2nd ed.

0.2.1 Problem 1.A

We call the elements a, b, e - as we know a unique neutral element must exist

$$\begin{array}{c|ccc}
\circ & e & a & b \\
\hline
e & e & a & b \\
a & a & a^2 & b\circ a \\
b & b & a\circ b & b^2
\end{array} \tag{7}$$

We have 4 fields to fill

- a needs an inverse - only element left is b meaning $b = a^{-1}$ and therefore $a \circ b = b \circ a = e$
- a^2 can't be e (because $e^2 = e$), a^2 can't be a (because $a \circ e = a$) therefore $a^2 = b$

$$\begin{array}{c|ccc}
\circ & e & a & b \\
\hline
e & e & a & b \\
a & a & b & e \\
b & b & e & a
\end{array} \tag{8}$$

0.3 Morris - Topology without tears

0.3.1 Problem 1.1.7

(a)

$$\tau_{X1} = \{X, \emptyset\} \tag{9}$$
$$\tau_{X2} = \{X, \emptyset, \{a\}\} \tag{10}$$
$$\tau_{X3} = \{X, \emptyset, \{b\}\} \tag{11}$$
$$\tau_{X4} = \{X, \emptyset, \{a\}, \{b\}\} \tag{12}$$

(b)

$$\tau_{Y1} = \{Y, \emptyset\}, \tag{13}$$
$$\tau_{Y2} = \{Y, \emptyset, \{a\}\}, \tau_{Y3} = \{Y, \emptyset, \{b\}\}, \tau_{Y4} = \{Y, \emptyset, \{c\}\}, \tag{14}$$
$$\tau_{Y5} = \{Y, \emptyset, \{a,b\}\}, \tau_{Y6} = \{Y, \emptyset, \{b,c\}\}, \tau_{Y7} = \{Y, \emptyset, \{a,c\}\}, \tag{15}$$
$$\tau_{Y8} = \{Y, \emptyset, \{a\}, \{b\}, \{a,b\}\}, \tau_{Y9} = \{Y, \emptyset, \{a\}, \{c\}, \{a,c\}\}, \tau_{Y10} = \{Y, \emptyset, \{b\}, \{c\}, \{b,c\}\}, \tag{16}$$
$$\tau_{Y11} = \{Y, \emptyset, \{a\}, \{b,c\}\}, \tau_{Y12} = \{Y, \emptyset, \{b\}, \{a,c\}\}, \tau_{Y13} = \{Y, \emptyset, \{c\}, \{a,b\}\}, \tag{17}$$
$$\tau_{Y14} = \{Y, \emptyset, \{a\}, \{a,b\}\}, \tau_{Y15} = \{Y, \emptyset, \{b\}, \{a,b\}\}, \tau_{Y16} = \{Y, \emptyset, \{a\}, \{a,c\}\}, \tag{18}$$
$$\tau_{Y17} = \{Y, \emptyset, \{c\}, \{a,c\}\}, \tau_{Y18} = \{Y, \emptyset, \{b\}, \{b,c\}\}, \tau_{Y19} = \{Y, \emptyset, \{c\}, \{b,c\}\}, \tag{19}$$
$$\tau_{Y20} = \{Y, \emptyset, \{a\}, \{a,c\}, \{a,b\}\}, \tau_{Y21} = \{Y, \emptyset, \{b\}, \{a,b\}, \{b,c\}\}, \tau_{Y22} = \{Y, \emptyset, \{c\}, \{b,c\}, \{a,c\}\}, \tag{20}$$
$$\tau_{Y23} = \{Y, \emptyset, \{a\}, \{b\}, \{a,c\}, \{a,b\}\}, \tau_{Y24} = \{Y, \emptyset, \{a\}, \{c\}, \{a,c\}, \{a,b\}\}, \tag{21}$$
$$\tau_{Y25} = \{Y, \emptyset, \{a\}, \{b\}, \{b,c\}, \{a,b\}\}, \tau_{Y26} = \{Y, \emptyset, \{b\}, \{c\}, \{b,c\}, \{a,b\}\} \tag{22}$$
$$\tau_{Y27} = \{Y, \emptyset, \{a\}, \{c\}, \{a,c\}, \{a,b\}\}, \tau_{Y28} = \{Y, \emptyset, \{a\}, \{b\}, \{a,c\}, \{a,b\}\}, \tag{23}$$
$$\tau_{Y29} = \{Y, \emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}\} \tag{24}$$

0.4 Barton - Elements of Greens functions and propagation

0.4.1 Problem 1.1 - Delta function

(i) $\int_{-1}^{2} dx \delta(x) \cos(2x) = \cos(0) = 1$

(ii) $\int_{-1}^{2} dx \delta(2x) \cos(x) = \int_{-2}^{4} (dy/2) \delta(y) \cos(y/2) = \cos(0)/2 = 1/2$

(iii) $\int_{-\infty}^{\infty} dx \delta'(x) \exp(ix) = 0 - i \int_{-\infty}^{\infty} \delta(x) \exp(ix) = -i$

(iv) $\int_{0}^{\infty} dx \delta'(\sqrt{2}x - 1) \tan^{-1}(x) = \int_{0}^{\infty} \frac{\delta(x-1/\sqrt{2})}{|\sqrt{2}|} \tan^{-1}(x) = \frac{\tan^{-1}(1/\sqrt{2})}{\sqrt{2}}$

0.4.2 Problem 1.3 - Delta function

$$\int_0^\infty dx\ \delta(\cos(x)) e^{-x} = \sum_{x_n \in \{\pi/2 + n\pi\}} \int_0^\infty dx\ \frac{\delta(x - x_n)}{|\sin x_n|} e^{-x} \tag{25}$$
$$= e^{-\pi/2} \left(e^{-0\pi} + e^{-1\pi} + e^{-2\pi} + ... \right) \tag{26}$$
$$= \frac{e^{-\pi/2}}{1 - e^{-\pi}} \tag{27}$$
$$= \frac{1}{e^{\pi/2} - e^{-\pi/2}} \tag{28}$$
$$\tag{29}$$

0.5 Wyld - Mathematical methods for physics

0.5.1 Problem 10.2 - Bernoulli numbers

a) Rewriting

$$\frac{z}{e^z - 1} = \frac{z}{z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + ...} \tag{30}$$

$$= \frac{1}{1 + \frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4!} + ...} \tag{31}$$

$$= B_0 + \frac{B_1 z}{1!} + \frac{B_2 z^2}{2!} + \frac{B_3 z^3}{3!} + \frac{B_4 z^4}{4!} ... \tag{32}$$

then

$$1 = \left(B_0 + \frac{B_1 z}{1!} + \frac{B_2 z^2}{2!} + \frac{B_3 z^3}{3!} + \frac{B_4 z^4}{4!} + ...\right)\left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4!} + ...\right) \tag{33}$$

and we compare the polynomial coefficients in LHS and RHS for each order

$$z^0: \quad 1 = B_0 \cdot 1 \quad \to \quad B_0 = 1 \tag{34}$$

$$z^1: \quad 0 = B_0 \frac{1}{2!} + B_1 \quad \to \quad B_1 = -\frac{1}{2} \tag{35}$$

$$z^2: \quad 0 = B_0 \frac{1}{3!} + B_1 \frac{1}{2!} + \frac{1}{2!} B_2 \quad \to \quad B_2 = 2\left(-\frac{1}{3!} + \frac{1}{4}\right) = \frac{1}{6} \tag{36}$$

$$z^3: \quad 0 = B_0 \frac{1}{4!} + B_1 \frac{1}{3!} + \frac{1}{2!2!} B_2 + \frac{1}{4!} B_3 \quad \to \quad B_3 = 0 \tag{37}$$

$$z^4: \quad 0 = B_0 \frac{1}{5!} + B_1 \frac{1}{4!} + \frac{1}{3!2!} B_2 + \frac{1}{2!3!} B_3 + \frac{1}{4!} B_4 \quad \to \quad B_4 = \frac{1}{30} \tag{38}$$

$$\tag{39}$$

b) Rewriting

$$\frac{z}{e^z - 1} + \frac{z}{2} = z \frac{2 + (e^z - 1)}{2(e^z - 1)} \tag{40}$$

$$= z \frac{2 + e^{z/2}(e^{z/2} - e^{-z/2})}{2e^{z/2}(e^{z/2} - e^{-z/2})} \tag{41}$$

$$= z \frac{2e^{-z/2} + (e^{z/2} - e^{-z/2})}{2(e^{z/2} - e^{-z/2})} \tag{42}$$

$$= \frac{z}{2} \frac{e^{z/2} + e^{-z/2}}{e^{z/2} - e^{-z/2}} \tag{43}$$

and now it is obvious. For c) we can rewrite this as via $z \to 2iz$

$$\frac{2iz}{e^{2iz} - 1} + \frac{2iz}{2} = iz \frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} \tag{44}$$

c)

$$\cot z = \frac{\cos z}{\sin z} = \frac{e^{iz} + e^{-iz}}{2} \frac{2i}{e^{iz} - e^{-iz}} = i \frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} \tag{45}$$

$$z \cot z = (iz) \frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} \tag{46}$$

$$= \frac{2iz}{e^{2iz} - 1} + iz \tag{47}$$

$$= iz + \left(1 + B_1(2iz) + \frac{B_2}{2!}(2iz)^2 + \frac{B_4}{4!}(2iz)^4 + ...\right) \tag{48}$$

$$= 1 - \frac{1}{3}z^2 + \underbrace{\frac{16i^4}{24 \cdot (-30)}}_{=-2/90} z^4 - ... \tag{49}$$

0.5.2 Problem 11.1 - Integral $\int_0^\infty dx \frac{x^2}{(x^2+a^2)^2}$

The zero of $\frac{x^2}{(x^2+a^2)^2}$ are $\pm ia$ (lets assume a is positive) so we can decompose into the common partial fractions

$$\frac{x^2}{(x^2 + a^2)^2} = \left(\frac{x}{(x - ia)(x + ia)}\right)^2 = \frac{1}{4}\left(\frac{1}{x - ia} + \frac{1}{x + ia}\right)^2. \tag{50}$$

Using the residual theorem (closing the loop above as $f(x) \sim z^{-2}$) gives

$$\int_0^\infty \frac{x^2}{(x^2 + a^2)^2} = \frac{1}{2}\int_{-\infty}^\infty \frac{x^2}{(x^2 + a^2)^2} = \frac{1}{2}\left[2\pi i \operatorname{Res}(f(ia)) - \int_{\text{C above}} f(x)dx\right] \tag{51}$$

$$= i\pi \operatorname{Res}(f(ia)) \tag{52}$$

There are two methods to calculate the residum

1. Direct: As ia is a second order pole we have

$$\operatorname{Res}(f(ia)) = \frac{1}{(2-1)!} \frac{d^{2-1}}{dx^{2-1}} (x - ia)^2 \frac{x^2}{(x + ia)^2 (x - ia)^2}\Bigg|_{x=ia} = \frac{2x(x + ia)^2 - x^2 2(x + ia)}{(x + ia)^4} = \frac{1}{4ia} \tag{53}$$

2. Laurent series at ai:

$$\frac{x^2}{(x^2 + a^2)^2} = \left(\frac{x}{(x - ia)(x + ia)}\right)^2 = \frac{1}{4}\left(\frac{1}{x - ia} + \frac{1}{x + ia}\right)^2 \tag{54}$$

$$= \frac{1}{4}\left(\frac{1}{x - ia} + \frac{1}{2ia}\frac{1}{1 - \frac{x - ia}{-2ia}}\right)^2 \quad \text{(using geometric series trick)} \tag{55}$$

$$= \frac{1}{4}\left(\frac{1}{x - ia} + \frac{1}{2ia}\left[1 + \frac{x - ia}{-2ia} + \left(\frac{x - ia}{-2ia}\right)^2 + \left(\frac{x - ia}{-2ia}\right)^3 + ...\right]\right)^2 \tag{56}$$

$$= -\frac{1}{16a^2}\left(\frac{2ia}{x - ia} + 1 + \frac{x - ia}{-2ia} + \left(\frac{x - ia}{-2ia}\right)^2 + \left(\frac{x - ia}{-2ia}\right)^3 + ...\right)^2 \tag{57}$$

$$= -\frac{1}{16a^2}\left(\frac{(2ia)^2}{(x - ia)^2} + 2\frac{2ia}{x - ia} + \left(1 + 2\frac{x - ia}{-2ia}\frac{2ia}{x - ia} + \right) + ...\right) \tag{58}$$

then

$$\text{Res}(f(ia)) = -\frac{1}{16a^2} 4ia = \frac{1}{4ai} \tag{59}$$

and finally

$$\int_0^\infty \frac{x^2}{(x^2+a^2)^2} = \frac{\pi}{4a} \tag{60}$$

0.5.3 Problem 11.2 - Integral $\int_0^\infty dx \frac{1}{x^4+5x^2+6}$

Rewriting and utilizing the residual theorem

$$\begin{aligned}
\int_0^\infty dx \frac{1}{x^4+5x^2+6} &= \frac{1}{2} \int_{-\infty}^\infty dx \frac{1}{x^4+5x^2+6} && (61)\\
&= \frac{1}{2} \int_{-\infty}^\infty dx \frac{1}{(x-i\sqrt{3})(x+i\sqrt{3})(x-i\sqrt{2})(x+i\sqrt{2})} && (62)\\
&= \frac{1}{2} \left(2\pi i \sum_{a_i=\{i\sqrt{2}, i\sqrt{3}\}} \text{Res}(f(a_i)) - \int_{\text{C above}} \right) && (63)
\end{aligned}$$

then

$$\begin{aligned}
\text{Res}(f(i\sqrt{2})) &= \lim_{x \to i\sqrt{2}} (x-i\sqrt{2}) \frac{1}{(x-i\sqrt{3})(x+i\sqrt{3})(x-i\sqrt{2})(x+i\sqrt{2})} && (64)\\
&= \lim_{x \to i\sqrt{2}} \frac{1}{(x-i\sqrt{3})(x+i\sqrt{3})(x+i\sqrt{2})} && (65)\\
&= -\frac{i}{2\sqrt{2}} && (66)\\
\text{Res}(f(i\sqrt{3})) &= \frac{i}{2\sqrt{3}} && (67)
\end{aligned}$$

and

$$\int_0^\infty dx \frac{1}{x^4+5x^2+6} = \frac{\pi}{12}(3\sqrt{2} - 2\sqrt{3}) \tag{68}$$

0.5.4 Problem 11.3 - Integral $\int_0^\infty dx \frac{1}{x^4+1}$

Same idea as above -residue theorem, closing the half circle above (integral vanishes because $f(x)\ x^{-4}$),

$$\begin{aligned}
\int_0^\infty dx \frac{1}{x^4+1} &= \frac{1}{2} \int_{-\infty}^\infty dx \frac{1}{(x-e^{i\pi/4})(x-e^{i3\pi/4})(x-e^{-i\pi/4})(x-e^{-i3\pi/4})} && (69)\\
&= \frac{1}{2} 2\pi i \left[\text{Res}(f(e^{i\pi/4})) + \text{Res}(f(e^{i3\pi/4})) \right] && (70)\\
&= \frac{1}{2} 2\pi i \left[\frac{1}{4} e^{-3i\pi/4} + \frac{1}{4} e^{-i\pi/4} \right] && (71)\\
&= \frac{\pi}{2\sqrt{2}} && (72)
\end{aligned}$$

0.5.5 Problem 11.4 - Integral $\int d^3k \frac{e^{i\vec{k}\cdot\vec{r}}}{k^2+a^2}$

$$\int d^3k \frac{e^{i\vec{k}\cdot\vec{r}}}{k^2+a^2} = 2\pi \int k^2 dk \frac{e^{ikr\cos\theta}}{k^2+a^2} \sin\theta \, d\theta \tag{73}$$
$$= -\frac{2\pi}{ir} \int \frac{k^2}{k} dk \frac{e^{-ikr} - e^{ikr}}{k^2+a^2} \tag{74}$$
$$= -\frac{2\pi}{2ir} \int_0^\infty dk \left(\frac{1}{k-ia} + \frac{1}{k+ia} \right) \left(e^{-ikr} - e^{ikr} \right) \tag{75}$$
$$= -\frac{\pi}{2ir} \int_{-\infty}^\infty dk \left(\frac{1}{k-ia} + \frac{1}{k+ia} \right) \left(e^{-ikr} - e^{ikr} \right) \tag{76}$$

Using the residue theorem - for e^{ikr} we close the loop above and for e^{-ikr} below (the way the integral along the loops vanish)

$$\int d^3k \frac{e^{i\vec{k}\cdot\vec{r}}}{k^2+a^2} = -\frac{\pi}{2ir} \int_{-\infty}^\infty dk \left(\frac{e^{-ikr}}{k-ia} + \frac{-e^{ikr}}{k-ia} + \frac{e^{-ikr}}{k+ia} + \frac{-e^{ikr}}{k+ia} \right) \tag{77}$$
$$= -\frac{\pi}{2ir} \int_{-\infty}^\infty dk \left(0 + \frac{-e^{ikr}}{k-ia} + \frac{e^{-ikr}}{k+ia} + 0 \right) \tag{78}$$
$$= -\frac{\pi}{2ir} 2\pi i \left(-e^{ik(ia)} + (-1) e^{-ik(-ia)} \right) \quad \text{(Curve below and negative winding)} \tag{79}$$
$$= \frac{2\pi^2}{r} e^{-ka} \tag{80}$$

0.5.6 Problem 11.5 - Integral $\int d^3k \frac{e^{i\vec{k}\cdot\vec{r}}}{k^2-a^2-i\varepsilon}$

Same as in 11.4

$$\int d^3k \frac{e^{i\vec{k}\cdot\vec{r}}}{k^2-a^2-i\varepsilon} = -\frac{2\pi}{ir} \int_0^\infty \frac{k^2}{k} dk \frac{e^{-ikr} - e^{ikr}}{k^2-a^2-i\varepsilon} \tag{81}$$
$$= -\frac{2\pi}{4ir} \int_{-\infty}^\infty dk \left(\frac{1}{k+(a+i\frac{\varepsilon}{2a})} + \frac{1}{k-(a+i\frac{\varepsilon}{2a})} \right) \left(e^{-ikr} - e^{ikr} \right) \tag{82}$$
$$= -\frac{2\pi}{4ir} \int dk \left(\frac{e^{-ikr}}{k+(a+i\frac{\varepsilon}{2a})} + \frac{-e^{ikr}}{k-(a+i\frac{\varepsilon}{2a})} \right) \tag{83}$$
$$= -\frac{2\pi}{4ir} 2\pi i \left(-e^{-i(-1)(a+i\frac{\varepsilon}{2a})r} - e^{i(a+i\frac{\varepsilon}{2a})r} \right) \tag{84}$$
$$= \frac{2\pi^2}{2r} \left(e^{i(a+i\frac{\varepsilon}{2a})r} + e^{i(a+i\frac{\varepsilon}{2a})r} \right) \tag{85}$$
$$= \frac{2\pi^2}{r} e^{i(a+i\frac{\varepsilon}{2a})r} \tag{86}$$
$$= \frac{2\pi^2}{r} e^{iar} \tag{87}$$

0.6 Stone, Goldbart - Mathematics for physics: A guided tour for graduate students (2009)

0.6.1 Problem 1.1

$$\frac{\partial L}{\partial \dot{x}} = \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \tag{88}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\ddot{x}\sqrt{\dot{x}^2+\dot{y}^2} - \dot{x}\frac{\dot{x}\ddot{x}+\dot{y}\ddot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}}{\dot{x}^2+\dot{y}^2} = 0 \tag{89}$$

$$\to \ddot{x}(\dot{x}^2+\dot{y}^2) - \dot{x}(\dot{x}\ddot{x}+\dot{y}\ddot{y}) = 0 \tag{90}$$

$$\to \ddot{x}\dot{y}^2 - \dot{x}\dot{y}\ddot{y} = 0 \tag{91}$$

$$\to \dot{y}(\ddot{x}\dot{y} - \dot{x}\ddot{y}) = 0 \tag{92}$$

$$\tag{93}$$

0.7 Bender, Orszag - Advanced Mathematical Methods for Scientists and Engineers

0.7.1 Problem 1.1

1. $y' = e^{x+y}$

$$\int \frac{dy}{e^y} = \int e^x dx \tag{94}$$

$$-e^{-y} = e^x + c \tag{95}$$

$$y = -\log(-e^x + c) \tag{96}$$

2. $y' = xy + x + y + 1$

$$\frac{dy}{y+1} = x + 1 \tag{97}$$

$$\log y + 1 = \frac{x^2}{2} + x + c \tag{98}$$

$$y = c' e^{x/2(x+2)} - 1 \tag{99}$$

0.7.2 Problem 1.2

$y'' = yy'/x$

1. Equidimensional-in-s equation

$$x = e^t \tag{100}$$

$$\frac{d}{dx} = \frac{dt}{dx}\frac{d}{dt} \tag{101}$$

$$= \frac{1}{x}\frac{d}{dt} \tag{102}$$

$$\frac{d^2}{dx^2} = \frac{dt}{dx}\frac{d}{dt}\left(\frac{1}{x}\frac{d}{dt}\right) \tag{103}$$

$$= \frac{1}{x}\left(-\frac{1}{x^2}x\frac{d}{dt} + \frac{1}{x}\frac{d^2}{dx^2}\right) \tag{104}$$

$$= \frac{1}{x^2}\left(-\frac{d}{dt} + \frac{d^2}{dt^2}\right) \tag{105}$$

now with $y = y(t)$

$$-y' + y'' = yy' \tag{106}$$

2. Autonomous equation

$$y' \equiv u(y) \tag{107}$$

$$y'' = \frac{du}{dy}\frac{dy}{dt} = \dot{u}y' \tag{108}$$

now with $u = u(y)$

$$-u + \dot{u}u = yu \tag{109}$$

$$\dot{u} = y + 1 \tag{110}$$

3. integration

$$u = \frac{y^2}{2} + y + c_0 \tag{111}$$

4. resubstitution I (with $\tan z = i\frac{e^{-iz}-e^{iz}}{e^{-iz}+e^{iz}}$)

$$y' = \frac{y^2}{2} + y + c_0 \tag{112}$$

$$t + c_3 = \int \frac{dy}{y^2/2 + y + c_0} \tag{113}$$

$$= 2\frac{1}{2\sqrt{1-2c_0}} \int dy \left(-\frac{1}{y+1+\sqrt{1-2c_0}} + \frac{1}{y+1-\sqrt{1-2c_0}} \right) \tag{114}$$

$$= \frac{1}{\sqrt{1-2c_0}} \left(-\log\left[y+1+\sqrt{1-2c_0}\right] + \log\left[y+1-\sqrt{1-2c_0}\right] \right) \tag{115}$$

$$= \frac{1}{\sqrt{1-2c_0}} \log \frac{y+1-\sqrt{1-2c_0}}{y+1+\sqrt{1-2c_0}} \tag{116}$$

$$= \frac{1}{\sqrt{1-2c_0}} \log \frac{-i\sqrt{1-2c_0}\left(-i + \frac{i(y+1)}{\sqrt{1-2c_0}}\right)}{i\sqrt{1-2c_0}\left(-i - \frac{i(y+1)}{\sqrt{1-2c_0}}\right)} \tag{117}$$

$$= \frac{1}{\sqrt{1-2c_0}} \log \frac{-\left(-i + \frac{i(y+1)}{\sqrt{1-2c_0}}\right)}{\left(-i - \frac{i(y+1)}{\sqrt{1-2c_0}}\right)} \tag{118}$$

$$= \frac{2}{\sqrt{1-2c_0}} \log \sqrt{-\frac{-i + \frac{i(y+1)}{\sqrt{1-2c_0}}}{-i - \frac{i(y+1)}{\sqrt{1-2c_0}}}} \tag{119}$$

$$= \frac{2}{i\sqrt{1-2c_0}} \arctan\left(-\frac{i(y+1)}{\sqrt{1-2c_0}} \right) \tag{120}$$

5. resubstitution II

$$\log x + c_3 = \frac{2}{i\sqrt{1-2c_0}} \arctan \frac{y+1}{i\sqrt{1-2c_0}} \tag{121}$$

$$\tan\left[\frac{\sqrt{2c_0-1}}{2}(\log x + c_3)\right] = \frac{y+1}{\sqrt{2c_0-1}} \tag{122}$$

$$y = \sqrt{2c_0-1}\tan\left[\frac{\sqrt{2c_0-1}}{2}(\log x + c_3)\right] - 1 \tag{123}$$

$$y = 2c_1 \tan\left[c_1 \log x + c_2\right] - 1 \tag{124}$$

This solution has poles at

$$\log x_P = \frac{\pi/2 + k\pi - c_2}{c_1} \tag{125}$$

while the special solution $-2/(c_4 + \log x) - 1$ has a pole at

$$\log x_P = -c_4 \tag{126}$$

???

0.7.3 Problem 1.10

With $y = e^{rx}$ the equation $y''' - 3y'' + 3y' - y = 0$ becomes

$$r^3 - 3r^2 + 3r - 1 = 0 \tag{127}$$
$$(r-1)^3 = 0 \tag{128}$$

then $y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x$.

0.7.4 Problem 1.11

We guess $y_1 = e^{-x}$ and have another guess $y_2 = e^{-x}u(x)$ we see

$$r^{-x}\left(u'' + xu'\right) = 0 \tag{129}$$
$$v' + xv = 0 \tag{130}$$
$$v = c_0 e^{-x^2/2} \tag{131}$$
$$u = c_1 \mathrm{erf}\left(\frac{x}{\sqrt{2}}\right) + c_2 \tag{132}$$

and therefore $y = c_3 e^{-x} + c_4 e^{-x}\left[\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right) + c_5\right]$

0.7.5 Problem 1.23

Calculating the gradient

$$\nabla z = e^{-(x^4+4y^2)}(-4x^3, -8y) \tag{133}$$
$$= -4e^{-(x^4+4y^2)}(x^3, 2y) \tag{134}$$

Equation of motions $\ddot{\vec{x}} = -\nabla V$ are

$$\ddot{x} = 4e^{-(x^4+4y^2)}x^3 \tag{135}$$
$$\ddot{y} = 4e^{-(x^4+4y^2)}2y \tag{136}$$

with the initial conditions $x_0 = 0 = y_0$. To make this simpler to solve we rescale ($\tilde{t} = \alpha t$) the time variable

$$\frac{\partial}{\partial t} = \frac{\partial \tilde{t}}{\partial t}\frac{\partial}{\partial \tilde{t}} \tag{137}$$
$$= \alpha \frac{\partial}{\partial \tilde{t}} \tag{138}$$
$$\frac{\partial^2}{\partial t^2} = \frac{\partial^2 \tilde{t}}{\partial t^2}\frac{\partial}{\partial \tilde{t}} + \left(\frac{\partial \tilde{t}}{\partial t}\right)^2 \frac{\partial^2}{\partial \tilde{t}^2} \tag{139}$$
$$= \alpha^2 \frac{\partial^2}{\partial \tilde{t}^2}. \tag{140}$$

0.7.6 Problem 1.31

(a) Multiply by y and observe $yy' \sim (y^2)'$ and substitute $z = y^2$

$$y' = \frac{y}{x} + \frac{1}{y} \tag{141}$$
$$yy' - \frac{1}{x}y^2 - 1 = 0 \tag{142}$$
$$\frac{1}{2}(y^2)' - \frac{1}{x}y^2 - 1 = 0 \tag{143}$$
$$\frac{1}{2}z' - \frac{1}{x}z - 1 = 0 \qquad (z = y^2) \tag{144}$$
$$z' - \frac{2}{x}z - 2 = 0 \tag{145}$$

General solution of the homogeneous equation

$$\frac{z'}{z} = \frac{2}{x} \quad \to \quad z_H = cx^2 \tag{146}$$

For special solution of the inhomogeneous equation - varying constants

$$z_I = C(x)x^2 \tag{147}$$
$$\to C'x^2 + 2xC - \frac{2}{x}Cx^2 - 2 = 0 \tag{148}$$
$$\to C' = \frac{2}{x^2} \tag{149}$$
$$\to C = -\frac{2}{x} \tag{150}$$

therefore

$$z = z_H + z_I \tag{151}$$
$$= x(cx - 2) \tag{152}$$
$$y = \pm\sqrt{x(cx - 2)} \tag{153}$$

(b) Nothing obvious pops into the eye so we make a desperate try $z = y/x$

$$z' = \frac{y'x - y}{x^2} \to y' = z'x + z \tag{154}$$

then

$$y' = \frac{xy}{x^2 + y^2} \tag{155}$$
$$z'x + z = \frac{zx^2}{x^2 + z^2x^2} \tag{156}$$
$$= \frac{z}{1 + z^2}{}' \tag{157}$$
$$z'x = \frac{z - z(1 + z^2)}{1 + z^2} \tag{158}$$
$$= \frac{-z^3}{1 + z^2} \tag{159}$$

Now we can separate and integrate on both sides

$$\frac{1+z^2}{z^3}dz = -\frac{dx}{x} \tag{160}$$

$$\int \left(\frac{1}{z^2}+\frac{1}{z}\right) dz = \int \frac{dx}{x} \tag{161}$$

$$-\frac{1}{z} + \log z = \log x + c \tag{162}$$

$$-\frac{x}{y} + \log \frac{y}{x} = \log x + c \tag{163}$$

$$-\frac{x}{y} + \log y = 2\log x + c \tag{164}$$

(c) Try the obvious $z = x + y$

$$y' = x^2 + 2xy + y^2 \tag{165}$$

$$y' = (x+y)^2 \tag{166}$$

$$\to z' - 1 = z^2 \tag{167}$$

Now separate and integrate (subs $z = \tan t$)

$$\frac{dz}{z^2+1} = dx \tag{168}$$

$$\arctan z = x + c \tag{169}$$

$$y = \tan(x+c) - x \tag{170}$$

(d) Rewriting the ODE we see similarities to the quotient rule

$$\frac{yy''}{(y')^2} = 2 \tag{171}$$

Let's guess

$$\left(\frac{y}{y'}\right)' = \frac{y'y' - yy''}{(y')^2} = 1 - \frac{yy''}{(y')^2} \tag{172}$$

so we can rewrite the ODE

$$\frac{yy''}{(y')^2} = 1 - \left(\frac{y}{y'}\right)' = 2 \tag{173}$$

then we can solve

$$\left(\frac{y}{y'}\right)' = -1 \tag{174}$$

$$\frac{y}{y'} = -x + c_1 \tag{175}$$

$$\frac{y'}{y} = \frac{1}{-x+c_1} \tag{176}$$

$$\log y = -\log(-x+c_1) + c_2 \tag{177}$$

$$y = \frac{c_3}{c_1 - x} \tag{178}$$

(e)

$$\frac{y'}{y^2} = \frac{1}{x^2} + \frac{1}{x} \tag{179}$$
$$-\frac{1}{y} = -\frac{1}{x} + \log x + c \tag{180}$$
$$y = \frac{1}{\frac{1}{x} - \log x + c} = \frac{1}{1 - x\log x + xc} \tag{181}$$

(f) With $f = xy$

$$x^2y' + xy + y^2 = 0 \tag{182}$$
$$xy' + y + \frac{y^2}{x} = 0 \tag{183}$$
$$f' + \frac{f^2}{x^3} = 0 \tag{184}$$
$$\frac{f'}{f^2} + \frac{1}{x^3} = 0 \quad \to \quad -\frac{1}{f} - \frac{1}{2}x^{-2} + c = 0 \tag{185}$$
$$xy = f = -\frac{1}{\frac{1}{2x^2} + c} \tag{186}$$
$$y = -\frac{1}{\frac{1}{2x} + xc} = -\frac{2x}{1 + 2x^2c} \tag{187}$$

(g)

$$xy' = y(1 - \log x + \log y) \tag{188}$$
$$\frac{xy'}{y} = (1 - \log\frac{y}{x}) \tag{189}$$

(n) Observe $\left(\frac{y}{x}\right)' = \frac{xy'-y}{x^2}$ then

$$xy' - y = xe^{y/x} \tag{190}$$
$$x^2\left(\frac{y}{x}\right)' = xe^{y/x} \tag{191}$$
$$\frac{f'}{e^{-f}} = \frac{1}{x} \quad \to \quad -e^{-f} = \log x + c \tag{192}$$
$$-f = \log(-\log x - c) \tag{193}$$
$$y = -x\log(-\log x - c) \tag{194}$$
$$\tag{195}$$

(o) Lets try $(x^my^n)' = mx^{m-1}y^n + nx^my^{n-1}$ and rewrite

$$y' = \frac{x^4 - 3x^2y^2 - y^3}{2x^3y + 3y^2x} \tag{196}$$
$$2x^3yy' + 3y^2xy' = x^4 - 3x^2y^2 - y^3 \tag{197}$$

then with

$$(x^3y^2)' = 2x^3yy' + 3x^2y^2 \quad \to \quad 2x^3yy' = (x^3y^2)' - 3x^2y^2 \tag{198}$$
$$(xy^3)' = 3xy^2y' + y^3 \quad \to \quad 3xy^2y' = (xy^3)' - y^3 \tag{199}$$

we can rewrite the LHS

$$2x^3yy' + 3y^2xy' = (x^3y^2)' + (xy^3)' - 3x^2y^2 - y^3 \tag{200}$$
$$= (x^3y^2 + xy^3)' - 3x^2y^2 - y^3 \tag{201}$$

putting it back into the ODE

$$(x^3y^2 + xy^3)' - 3x^2y^2 - y^3 = x^4 - 3x^2y^2 - y^3 \tag{202}$$
$$(x^3y^2 + xy^3)' = x^4 \tag{203}$$
$$\to x^3y^2 + xy^3 = \frac{1}{5}x^5 + c \tag{204}$$
$$\to xy^2(x^2 + y) = \frac{1}{5}x^5 + c \tag{205}$$

(t) Observe $-\left(\frac{x}{y}\right)' = \frac{xy'-y}{y^2}$ then

$$xy' = y + \sqrt{xy} \tag{206}$$
$$\frac{xy'}{y^2} = \frac{y}{y^2} + \frac{\sqrt{xy}}{y^2} \tag{207}$$
$$-\left(\frac{x}{y}\right)' = \frac{\sqrt{xy}}{y^2} = \frac{x}{x}\frac{\sqrt{xy}}{y^2} = \frac{1}{x}\sqrt{\frac{x^3}{y^3}} \tag{208}$$
$$-f' = \frac{1}{x}f^{3/2} \quad \to \quad -2f^{-1/2} = \log x + c \tag{209}$$
$$y = \frac{x}{4}(\log x + c) \tag{210}$$

(x) First the homog. equations

$$\frac{y'}{y} + \frac{1}{(x-1)(x-2)} = 0 \tag{211}$$
$$\frac{y'}{y} - \left(\frac{1}{x-1} - \frac{1}{x-2}\right) = 0 \tag{212}$$
$$\log y_h - \log\frac{x-1}{x-2} = C \tag{213}$$
$$y_h = C\frac{x-1}{x-2} \tag{214}$$

Now variations of constants and resubstitude

$$y = C(x)\frac{x-1}{x-2} \tag{215}$$
$$\to (x-1)^2C'(x) = 2 \tag{216}$$
$$\to C(x) = -\frac{2}{x-1} + c \tag{217}$$
$$\to y = \left(-\frac{2}{x-1} + c\right)\frac{x-1}{x-2} \tag{218}$$
$$\to y = \frac{-2}{x-2} \tag{219}$$

(y) Playing around a bit we see $(xe^{-y})' = e^{-y} - xy'e^{-y}$ and then

$$y' = \frac{1}{x + e^y} \tag{220}$$
$$xy' + y'e^y = 1 \tag{221}$$
$$xy'e^{-y} - e^{-y} + y' = 0 \tag{222}$$
$$-(xe^{-y})' + y' = 0 \tag{223}$$
$$-xe^{-y} + y = c \tag{224}$$
$$ye^y = ce^y + x \tag{225}$$

and we recognize the productlog (Lambert W function).

(z) With substitution $f = xy$

$$xy' + y = y^2 x^4 \tag{226}$$
$$(xy)' = (xy)^2 x^2 \tag{227}$$
$$f' = f^2 x^2 \tag{228}$$
$$\to \frac{f'}{f^2} = x^2 \quad \to \quad -f^{-1} = \frac{x^3}{3} + c \tag{229}$$
$$\to y = -\frac{1}{x}\frac{3}{x^3 + \tilde{c}} \tag{230}$$

0.7.7 Problem 7.1

Inserting the series expansion into the equation and sorting by powers of ϵ

(a)

$$a_0 + a_0^2 = 0 \tag{231}$$
$$6 + a_1(1 + 2a_0) = 0 \tag{232}$$
$$a_1^2 + a_2(1 + 2a_0) = 0 \tag{233}$$

then coefficients upto second order (for both zeros) are

$$a_0 = -1 \quad \to \quad a_1 = 6 \quad \to \quad a_2 = 36 \tag{234}$$
$$\to x_- = -1 + 6\epsilon + 36\epsilon^2 \tag{235}$$
$$a_0 = 0 \quad \to \quad a_1 = -6 \quad \to \quad a_2 = -36 \tag{236}$$
$$\to x_+ = -6\epsilon - 36\epsilon^2 \tag{237}$$

which is consistent with the series expansion of the analytical roots

$$x_\pm = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - 6\epsilon} \tag{238}$$
$$= -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 24\epsilon} \tag{239}$$

(b)

$$1 + a_0^3 = 0 \tag{240}$$
$$-a_0 + 3a_0^2 a_1 = 0 \tag{241}$$
$$-a_1 + 3a_0 a_1^2 + 3a_0^2 a_2 = 0 \tag{242}$$

then

$$a_0 = 1 \quad \to \quad a_1 = 1/3 \quad \to \quad a_2 = 0 \tag{243}$$
$$\to x_0 = 1 + \frac{1}{3}\epsilon + 0\epsilon^2 \tag{244}$$
$$a_0 = e^{-2\pi i/3} \quad \to \quad a_1 = \frac{1}{3}e^{2\pi i/3} \quad \to \quad a_2 = \frac{i}{3\sqrt{3}} \tag{245}$$
$$\to x_1 = e^{-2\pi i/3} + \frac{1}{3}e^{2\pi i/3}\epsilon + \frac{i}{3\sqrt{3}}\epsilon^2 \tag{246}$$
$$a_0 = e^{-2\pi i/3} \quad \to \quad a_1 = \frac{1}{3}e^{2\pi i/3} \quad \to \quad a_2 = -\frac{i}{3\sqrt{3}} \tag{247}$$
$$\to x_2 = e^{2\pi i/3} + \frac{1}{3}e^{-2\pi i/3}\epsilon - \frac{i}{3\sqrt{3}}\epsilon^2 \tag{248}$$

(c)

0.7.8 Problem 7.3

With

$$x = a_0 + a_1\epsilon + a_2\epsilon^2 + ... \tag{249}$$

$$x^k = a_0^k + ka_0^{k-1}a_1\epsilon + \left[\binom{k}{2}a_0^{k-2}a_1^2 + ka_0^{k-1}a_2\right]\epsilon^2 + ... \tag{250}$$

$$(x+1)^n = \sum_k \binom{n}{k} x^k \tag{251}$$

$$= 1 + nx + \frac{n(n-1)}{2}x^2 + ... \tag{252}$$

we obtain for each power of ϵ

$$\sum_{k=0} \binom{n}{k} a_0^k = 0 \tag{253}$$

$$\sum_{k=1} \binom{n}{k} ka_0^{k-1}a_1 = a_0 \tag{254}$$

$$\sum_{k=2} \binom{n}{k} \left[\binom{k}{2}a_0^{k-2}a_1^2 + ka_0^{k-1}a_2\right] = a_1 \tag{255}$$

which we can solve

$$0 = \sum_{k=0} \binom{n}{k} a_0^k = (a_0+1)^n \quad \rightarrow \quad a_0 = -1$$

then

$$a_1 = \frac{a_0}{\sum_{k=1}\binom{n}{k}ka_0^{k-1}} = \frac{a_0}{n(1+a_0)^{n-1}} \quad \rightarrow \quad a_0 = -\infty \tag{256}$$

0.8 Arfken, Weber - Mathematical Methods for physicists 7th ed

0.8.1 6.5.19

(a) Lets generalize the problem a bit ($k, m \rightarrow k_1, k_2, k_3, m_1, m_2$)

$$L = T - V \tag{257}$$

$$= \frac{m_1}{2}\dot{x}_1^2 + \frac{m_1}{2}\dot{x}_2^2 - \frac{k_1}{2}(x_1 - 0 - l_1)^2 - \frac{k_2}{2}(x_2 - x_1 - l_2)^2 - \frac{k_3}{2}(L - x_2 - l_3)^2 \tag{258}$$

Using the Euler-Lagrange equations for x_1 and x_2

$$-k_1(x_1 - l_1) + k_2(x_2 - x_1 - l_2) - m_1\ddot{x}_1 = 0 \tag{259}$$

$$-k_2(x_2 - x_1 - l_2) + k_3(L - x_2 - l_3) - m_2\ddot{x}_2 = 0 \tag{260}$$

and simplifying

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 - k_1l_1 + k_2l_2 = 0 \tag{261}$$

$$m_2\ddot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 - l_2l_2 - k_3L + k_3l_3 = 0 \tag{262}$$

(b) Finding the eigenvalues of the Hessian

$$\begin{pmatrix} (k_1+k_2)/m_1 & -k_2/m_1 \\ -k_2/m_2 & (k_2+k_3)/m_2 \end{pmatrix} \tag{263}$$

we get

$$\omega_A^2 = \frac{k_2+k_3}{m_2} + \frac{k_1+k_2}{m_1} - \frac{1}{2}\sqrt{\frac{(k_2+k_3)^3}{m_2^2} - 2\frac{(k_2(k_3-k_2)+k_1(k_2+k_2))}{m_1 m_2} + \frac{(k_1+k_2)^2}{m_1^2}} \tag{264}$$

$$\to \frac{k}{m} \tag{265}$$

$$\omega_B^2 = \frac{k_2+k_3}{m_2} + \frac{k_1+k_2}{m_1} + \frac{1}{2}\sqrt{\frac{(k_2+k_3)^3}{m_2^2} - 2\frac{(k_2(k_3-k_2)+k_1(k_2+k_2))}{m_1 m_2} + \frac{(k_1+k_2)^2}{m_1^2}} \tag{266}$$

$$\to 3\frac{k}{m} \tag{267}$$

(c) The associated eigenvectors are

$$X_A = (1,1) \tag{268}$$

$$X_B = (-1,1) \tag{269}$$

0.9 ARNOL'D - Ordinary differential equations

0.9.1 Sample Examination Problem 2

$$\ddot{x} = 1 + 2\sin x \quad \to \quad \begin{array}{l} \dot{x} = y \\ \dot{y} = 1 + 2\sin x \end{array} \tag{270}$$

0.10 ARNOL'D - A mathematical trivium

0.10.1 Problem 4

Calculate the 100th derivative of the function $\frac{x^2+1}{x^3-x}$.

Rewrite the function as

$$\frac{x^2+1}{x^3-x} = \frac{x^2+1}{x(x+1)(x-1)} \tag{271}$$

$$= -\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x-1} \tag{272}$$

$$\frac{d}{dx}(x+a)^{-1} = -(x+a)^{-2} \tag{273}$$

$$\frac{d^{100}}{dx^{100}}(x+a)^{-1} = 100!(x+a)^{-101} \tag{274}$$

$$\tag{275}$$

Then

$$\frac{d^{100}}{dx^{100}}\left(\frac{x^2+1}{x^3-x}\right) = 100!\left(-\frac{1}{x^{101}} + \frac{1}{(x+1)^{101}} + \frac{1}{(x-1)^{101}}\right) \tag{276}$$

0.10.2 Problem 13

Calculate with 5% relative error $\int_1^{10} x^x dx$.

Analytic integration seems not possible

$$\int_1^{10} x^x dx < \int_1^{10} 10^x dx = \int_1^{10} e^{x \log 10} dx = \frac{1}{\log 10} e^{x \log 10}|_1^{10} = \frac{1}{\log 10} 10^x|_1^{10} \approx 4.35 \cdot 10^9 \quad (277)$$

0.10.3 Problem 20

$$\ddot{x} = x + A\dot{x}^2 \qquad x(0) = 1, \dot{x}(0) = 0 \quad (278)$$

Using the standard perturbation theory approach we assume $x(t) = x_0(t) + Ax_1(t) + A^2x_2(t) + \ldots$. Inserting into the ODE gives

$$\ddot{x}_0 + A\ddot{x}_1 + A^2\ddot{x}_2 + \ldots = x_0 + Ax_1 + A^2x_2 + \ldots + A\left(\dot{x}_0 + A\dot{x}_1 + A^2\dot{x}_2 + \ldots\right)^2. \quad (279)$$

Sorting by powers of A we obtain a set of ODEs

$$A^0: \qquad \ddot{x}_0 = x_0 \quad (280)$$
$$A^1: \qquad \ddot{x}_1 = x_1 + \dot{x}_0^2 \quad (281)$$
$$A^2: \qquad \ddot{x}_2 = x_2 + 2\dot{x}_0\dot{x}_1. \quad (282)$$

The first ODE can be solved directly

$$x_0 = c_1e^t + c_2e^{-t}. \quad (283)$$

The second ODE then transforms into

$$\ddot{x}_1 = x_1 + c_1^2e^{2t} + c2^2e^{-2t} - 2c_1c_2 \quad (284)$$

with the homogeneous solution

$$x_{1H} = c_3e^t + c_4e^{-t}. \quad (285)$$

For the particular solution we try the ansatz (inspired by the inhomogeneity)

$$x_{1S} = \alpha + \beta e^{2t} + \gamma e^{-2t} \quad (286)$$
$$= 2c_1c_2 + \frac{c_1^2}{3}e^{2t} + \frac{c_2^2}{3}e^{-2t} \quad (287)$$

then

$$x_1 = x_{1H} + x_{1S} \quad (288)$$
$$= c_3e^t + c_4e^{-t} + 2c_1c_2 + \frac{c_1^2}{3}e^{2t} + \frac{c_2^2}{3}e^{-2t} \quad (289)$$

Imposing initial conditions on x_0 gives

$$c_1 = c_2 = \frac{1}{2} \quad \to \quad x_0 = \cosh t \quad (290)$$
$$c_3 = c_4 = -\frac{1}{3} \quad \to \quad x_1 = -\frac{2}{3}\cosh t + \frac{1}{2} + \frac{1}{6}\cosh 2t \quad (291)$$

and therefore

$$\left.\frac{dx(t)}{dA}\right|_{A=0} = \frac{1}{2} - \frac{2}{3}\cosh t + \frac{1}{6}\cosh 2t \quad (292)$$

0.10.4 Problem 23

Solve the quasi-homogeneous equation $y' = x + \frac{x^3}{y}$.
Sharp look

$$\left(\frac{y}{x}\right)' = \frac{y'x - y}{x^2} \tag{293}$$
$$= \frac{y'}{x} - \frac{y}{x^2} \tag{294}$$

then

$$y' = x + \frac{x^3}{y} \tag{295}$$
$$\frac{y'}{x} = 1 + \frac{x^2}{y} \tag{296}$$

0.10.5 Problem 50

Assume real and $k > 0$. Using the residual theorem we obtain

$$\int_{-\infty}^{\infty} \frac{e^{ikx}}{1+x^2} = \int_{-\infty}^{\infty} \frac{e^{ikx}}{(x+i)(x-i)} \tag{297}$$
$$= \frac{1}{-2i} \int_{-\infty}^{\infty} \left(\frac{1}{x+i} - \frac{1}{x-i}\right) e^{ikx} \tag{298}$$
$$= -\frac{1}{-2i} \int_{-\infty}^{\infty} \frac{e^{ikx}}{x-i} \tag{299}$$
$$= -\frac{1}{-2i}(2\pi i)e^{iki} \tag{300}$$
$$= \pi e^{-k} \tag{301}$$

0.10.6 Problem 85

In three dimensions we have

$$x^2 + y^2 + z^2 + xy + yz + zx = 1 \tag{302}$$

which can be written as

$$\vec{x}^T A \vec{x} = 1 \tag{303}$$
$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1 \tag{304}$$

With an orthorgonal matrix S ($S^{-1} = S^T$) we can rotate the ellipsoid to line it up with the coordinate axes (choose S such that $D_A = S^{-1}AS$ is diagonal)

$$1 = \vec{x}^T A \vec{x} \tag{305}$$
$$= \vec{x}^T (SS^{-1}) A (SS^{-1}\vec{x} \tag{306}$$
$$= (\vec{x}^T S) S^{-1} A S (S^{-1}\vec{x}) \tag{307}$$
$$= (\vec{x}^T S) S^{-1} A S (S^T \vec{x}) \tag{308}$$
$$= (S^T \vec{x})^T S^{-1} A S (S^T \vec{x}) \tag{309}$$
$$= (S^T \vec{x})^T D_A (S^T \vec{x}) \tag{310}$$

For this we need to find the eigensystem $\{\vec{v}_i, \lambda_i\}$ of A. The characteristic polynomial is given by

$$\lambda^3 - 3\lambda^2 + \frac{9}{4}\lambda - \frac{1}{2} = 0. \tag{311}$$

Then

$$S = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{312}$$

$$D_A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \tag{313}$$

the length of the principal axes are therefore 4, 1 and 1.

0.11 NEEDHAM - Visual Complex Analysis

0.11.1 Exercise 1.7

$$|z-a| = |z-b| \tag{314}$$
$$(z-a)(z-a)^* = (z-b)(z-b)^* \tag{315}$$
$$zz^* - az^* - za^* - aa^* = zz^* - bz^* - zb^* - bb^* \tag{316}$$
$$(-a+b)z^* + (-a^* + b^*)z = aa^* - bb^* \tag{317}$$
$$(-a+b-a^*+b^*)x + (a-b-a^*+b^*)iy = aa^* - bb^* \tag{318}$$
$$2\Re(b-a)x + 2\Im(b-a)yi = aa^* - bb^* \tag{319}$$

Looking at it from a vector space perspective - set of all points which have same distance from a and b. So its the perpendicular bisector (Mittelsenkrechte).

0.11.2 Exercise 1.33

1. It is a polynomial of ninth order $(z-1)^{10} = z^{10} \to -10z^9 + 45z^8 + ... + 1 = 0$. We can rewrite it as $(z-1)^{10} = (z-0)^{10}$

2.

$$w^{10} = 1 \quad \to \quad w_k = e^{2\pi ik/10} \qquad k \in \{0, ..., 9\} \tag{320}$$
$$\to \quad z_k = \frac{1}{1-z_k} = \frac{1}{1-e^{2\pi ik/10}} \tag{321}$$

3.

$$z_k = \frac{1}{1-e^{\pi ik/5}} = \frac{1}{e^{\pi ik/10}(e^{-\pi ik/10} - e^{\pi ik/10})} = \frac{e^{-\pi ik/10}}{-2i\sin(\pi k/10)} \tag{322}$$
$$= \frac{\cos(-\pi k/10) + i\sin(-\pi k/10)}{-2i\sin(\pi k/10)} = \frac{i}{2}\tan[\pi k/10] + \frac{1}{2} \tag{323}$$

0.11.3 Exercise 9.1

Recognizing the Law of Cosine - we can rewrite

$$\int_0^{2\pi} \frac{dt}{1+a^2-2a\cos t} = \int_0^{2\pi} \frac{dt}{|1-ae^{it}|^2} \tag{324}$$
$$= \int_0^{2\pi} \frac{dt}{(1-ae^{it})(1-ae^{-it})} \qquad \frac{dz}{dt} = ie^{it} = iz \tag{325}$$
$$= \oint_C \frac{dz}{iz(1-az)(1-a\bar{z})} \tag{326}$$
$$= \oint_C \frac{-idz}{(1-az)(z-az\bar{z})} \tag{327}$$
$$= \oint_C \frac{-idz}{(1-az)(z-a)} \tag{328}$$
$$= \oint_C \frac{idz}{(az-1)(z-a)} \tag{329}$$

then using the residuum theorem we get

$$\oint_C \frac{idz}{(az-1)(z-a)} = \frac{i}{1-a^2}\oint_C \frac{1}{z-1/a} - \frac{1}{z-a}dz \tag{330}$$
$$= \frac{i}{1-a^2}\left(\oint_C \frac{dz}{z-1/a} - \oint_C \frac{dz}{z-a}dz\right) \tag{331}$$
$$= \frac{i}{1-a^2}(0-2\pi i) \tag{332}$$
$$= \frac{2\pi}{1-a^2} \tag{333}$$

0.12 Tall, Steward - Complex Analysis 2018

0.12.1 Problem 11.1 - Laurent expansion

(i) Using the common geometric series trick ($|z/3|<1$)

$$\frac{1}{z-3} = -\frac{1}{3}\frac{1}{1-z/3} \overset{\text{GS}}{=} -\frac{1}{3}\left(1+\frac{z}{3}+\frac{z^2}{9}+\frac{z^3}{27}...\right) = -\frac{1}{3}\sum_{k=0}^{\infty}\frac{z^k}{3^k} = -\sum_{k=0}^{\infty}\frac{z^k}{3^{k+1}} \tag{334}$$

(ii)

$$\frac{1}{(z-a)^k} = \frac{1}{(-a)^k}\frac{1}{(1-z/a)^k} = \frac{1}{(-a)^k}\left(1+\frac{z}{a}+\frac{z^2}{a^2}+\frac{z^3}{a^3}+...\right)^k \tag{335}$$
$$= \frac{1}{(-a)^k}\left(1+k\frac{z}{a}+\left[\binom{k}{2}+k\right]\frac{z^2}{a^2}+\left[\binom{k}{3}+\binom{k}{2}(k-2)+k\right]\frac{z^3}{a^3}\right. \tag{336}$$
$$\left.+\left[\binom{k}{4}+\binom{k}{3}(k-2)+k(k-1)+k\right]\frac{z^4}{a^4}+...\right) \tag{337}$$

0.13 Ahlfors - Complex Calculus

0.13.1 Chap 1.1

1. (a)

$$(1+2i)^3 = 1+3(2i)^2+3\cdot 2i+(2i)^3 \tag{338}$$
$$= 1-12+6i-8i \tag{339}$$
$$= -11-2i \tag{340}$$

(b)

$$\frac{5}{-3+4i} = \frac{5(-3-4i)}{(-3+4i)(-3-4i)} \tag{341}$$
$$= \frac{-15-20i}{25} \tag{342}$$
$$= -\frac{3}{5}-\frac{4}{5}i \tag{343}$$

(c)

$$\left(\frac{2+i}{3-2i}\right)^2 = \frac{3+4i}{5-12i} \tag{344}$$
$$= \frac{(3+4i)(5+12i)}{169} \tag{345}$$
$$= \frac{15-48+20i+36i}{169} \tag{346}$$
$$= -\frac{33}{169}+\frac{56}{169}i \tag{347}$$

(d)

$$(1+i)^n+(1-i)^n = \sqrt{2}^n\left(e^{i\pi n/4}+e^{-i\pi n/4}\right) \tag{348}$$
$$= 2^{(n+1)/2}\cos\frac{n\pi}{4} \tag{349}$$

2. (a)

$$z^4 = (x+iy)^4 \tag{350}$$
$$= x^4+4x^3(iy)+6x^2(iy)^2+4x(iy)^3+(iy)^4 \tag{351}$$
$$= x^4-6x^2y^2+y^4+(4x^3y-4xy^3)i \tag{352}$$

(b)

$$1/z = \frac{x-iy}{x^2+y^2} \tag{353}$$

(c)

$$\frac{z-1}{z+1} = \frac{(x-1)+iy}{(x+1)+iy} \tag{354}$$
$$= \frac{x^2+y^2-1+2xyi}{(x+1)^2+y^2} \tag{355}$$

(d)

$$1/z^2 = \frac{1}{x^2-y^2+2xyi} \tag{356}$$
$$= \frac{x^2-y^2-2xyi}{(x^2-y^2+2xyi)(x^2-y^2-2xyi)} \tag{357}$$
$$= \frac{x^2-y^2-2xyi}{(x^2+y^2)^2} \tag{358}$$

3. (a) With $\alpha = \pm 1$

$$(-1+i\alpha\sqrt{3})^2 = 1-3\alpha^2-i2\sqrt{3}\alpha \tag{359}$$
$$(-1+i\alpha\sqrt{3})^3 = -1+9\alpha^2+3\sqrt{3}\alpha(1-\alpha^2)i \tag{360}$$

then we see $(-1+i\alpha\sqrt{3})^3 = 9$ for $\alpha = \pm 1$.

(b) With $\alpha, \beta = \pm 1$

$$(-\beta + i\alpha\sqrt{3})^6 = (-\beta + i\alpha\sqrt{3})^{3\cdot 2} \tag{361}$$
$$= \beta^{3\cdot 2}\left(\underbrace{\left(-1 + i\frac{\alpha}{\beta}\sqrt{3}\right)^3}_{=1 \quad \text{see (a)}}\right)^2 \tag{362}$$
$$= \beta^6 \cdot 1^6 \tag{363}$$
$$= 1 \tag{364}$$

0.13.2 Chap 1.2

1. (a)

$$i = (x+iy)^2 \tag{365}$$
$$= x^2 - y^2 + 2xyi \tag{366}$$

then

$$x^2 - y^2 = 0 \qquad 2xy = 1 \quad \to \quad \frac{1}{4y^2} - y^2 = 0 \tag{367}$$
$$z_1 = \frac{1+i}{\sqrt{2}} \tag{368}$$
$$z_2 = \frac{-1-i}{\sqrt{2}} \tag{369}$$

(b)

$$-i = (x+iy)^2 \tag{370}$$
$$= x^2 - y^2 + 2xyi \tag{371}$$

then

$$x^2 - y^2 = 0 \qquad 2xy = -1 \quad \to \quad \frac{1}{4y^2} - y^2 = 0 \tag{372}$$
$$z_1 = \frac{-1+i}{\sqrt{2}} \tag{373}$$
$$z_2 = -\frac{1-i}{\sqrt{2}} \tag{374}$$

(c)

$$1+i = (x+iy)^2 \tag{375}$$
$$= x^2 - y^2 + 2xyi \tag{376}$$

then

$$x^2 - y^2 = 1 \qquad 2xy = 1 \quad \to \quad \frac{1}{4y^2} - y^2 = 1 \tag{377}$$
$$z_1 = \frac{1}{2\sqrt{\frac{1}{\sqrt{2}} - \frac{1}{2}}} + \sqrt{\frac{1}{\sqrt{2}} - \frac{1}{2}}i \tag{378}$$
$$z_2 = -\frac{1}{2\sqrt{\frac{1}{\sqrt{2}} - \frac{1}{2}}} - \sqrt{\frac{1}{\sqrt{2}} - \frac{1}{2}}i \tag{379}$$

(d)

$$\sqrt{\frac{1-i\sqrt{3}}{2}} = (x+iy)^2 \tag{380}$$
$$= x^2 - y^2 + 2xyi \tag{381}$$

then

$$x^2 - y^2 = \frac{1}{2} \qquad 2xy = -\frac{\sqrt{3}}{2} \quad \to \quad \frac{1}{4y^2} - y^2 = \frac{1}{2} \tag{382}$$
$$z_1 ... \tag{383}$$
$$z_2 ... \tag{384}$$

0.14 Stein, Shakarchi - Princeton Lectures in Analysis - Vol 1 Fourier-Analysis

0.14.1 Problem 1.1

$$u(x,y) = v(x)w(y) \tag{385}$$
$$\triangle u = v_{xx}w + vw_{xx} = 0 \tag{386}$$

Using a separation constant c^2 gives

$$v_{xx} + c^2 v = 0 \quad \to \quad v_c(x) = C\sin cx + D\cos cx \tag{387}$$
$$w_{yy} - c^2 w = 0 \quad \to \quad w_c(x) = E\sinh cy + F\cosh cy \tag{388}$$

then the general solution is (setting $F = 1$)

$$u_c(x,y) = (C\sin cx + D\cos cx)(E\sinh cy + \cosh cy) \tag{389}$$

Now we can look at the boundary conditions

$$u_c(x,0) = (C\sin cx + D\cos cx) \stackrel{!}{=} A_k \sin kx \tag{390}$$
$$u_c(x,1) = (C\sin cx + D\cos cx)(E\sinh c + \cosh c) \stackrel{!}{=} B_k \sin kx \tag{391}$$
$$u_c(0,y) = D(E\sinh cy + \cosh cy) \stackrel{!}{=} 0 \tag{392}$$
$$u_c(\pi,y) = (C\sin c\pi + D\cos c\pi)(E\sinh cy + \cosh cy) \stackrel{!}{=} 0 \tag{393}$$

then we see (using $\sin, \cos$ being a complete orth. system)

$$\to c = k, D = 0, C = A_k \tag{394}$$
$$\to B_k = C(E\sinh c + \cosh c) \tag{395}$$
$$\to E = \left(\frac{B_k}{A_k} - \cosh k\right)\frac{1}{\sinh k} \tag{396}$$

And therefore

$$\begin{aligned}
u(x,y) &= \sum_c u_c && (397)\\
&= \sum_k u_k && (398)\\
&= \sum_k A_k \sin kx \left[\left(\frac{B_k}{A_k} - \cosh k\right)\frac{1}{\sinh k}\sinh ky + \cosh ky\right] && (399)\\
&= \sum_k \sin kx \left[(B_k - A_k \cosh k)\frac{1}{\sinh k}\sinh ky + A_k \cosh ky\right] && (400)\\
&= \sum_k \sin kx \left[(B_k \sinh ky - A_k \cosh k \sinh ky)\frac{1}{\sinh k} + A_k \cosh ky \sinh k \frac{1}{\sinh k}\right] && (401)\\
&= \sum_k \sin kx \left[B_k \frac{\sinh ky}{\sinh k} + A_k \frac{-\cosh k \sinh ky + \cosh ky \sinh k}{\sinh k}\right] && (402)\\
&= \sum_k \sin kx \left[B_k \frac{\sinh ky}{\sinh k} + A_k \frac{\cosh k \sinh(-ky) + \cosh(-ky) \sinh k}{\sinh k}\right] && (403)\\
&= \sum_k \sin kx \left[B_k \frac{\sinh ky}{\sinh k} + A_k \frac{\sinh k(1-y)}{\sinh k}\right] && (404)
\end{aligned}$$

0.15 Spivak - Calculus on Manifolds

0.16 O'Neill - Elementary Differential Geometry

0.16.1 Problem 1.1 - 1

(a) $x^2y^3\sin[z]^2$

(b) $x^2y\sin[z] + 2xy^2\sin[z]$

(c) $2x^2y\cos[z]$

(d) $x^2\cos[x^2y]$

0.16.2 Problem 1.1 - 2

(a) 0

(b) $-19/2$

(c) $a^2 + a - 1$

(d) $t^4 - t^7$

0.16.3 Problem 1.1 - 3

(a) $xy\cos[xy] + \sin[xy] - yz\sin[xz]$

(b) $xe^{x^2+y^2+z^2}\cos(e^{x^2+y^2+z^2})$

0.16.4 Problem 1.1 - 4

(a) $-y^2 + 2(x+y)$

(b) $-2e^{2x+y}$

(c) $4x$

0.17 Boothby - An Introduction to Differential Mannifolds and Riemannian Geometry

0.18 Burke - Applied Differential Geometry

0.19 O'Neill - Semi-Riemannian Geometry - With Applications to Relativity

0.20 Hubbert - Vector Calculus, Linear Algebra, and Differential Forms

0.21 Flanders - Differential Forms with Applications to the Physical Sciences

0.22 Morse, Feshbach - Methods of mathematical physics

0.22.1 Problem 1.1

With

$$\cot^2\psi = \frac{\cos^2\psi}{\sin^2\psi} = \frac{\cos^2\psi}{1-\cos^2\psi} \tag{405}$$

we can obtain a quadratic equation

$$(x^2+y^2)\cos^2\psi(1-\cos^2\psi) + z^2\cos^2\psi = a^2(1-\cos^2\psi) \tag{406}$$

$$\cos^4\psi - \frac{x^2+y^2+z^2+a^2}{x^2+y^2}\cos^2\psi + \frac{a^2}{x^2+y^2} = 0 \tag{407}$$

with the solution

$$\cos^2\psi = \frac{x^2+y^2+z^2+a^2}{2(x^2+y^2)} \pm \sqrt{\frac{(x^2+y^2+z^2+a^2)^2}{4(x^2+y^2)^2} - \frac{4a^2(x^2+y^2)}{4(x^2+y^2)^2}} \tag{408}$$

$$= \frac{x^2+y^2+z^2+a^2 \pm \sqrt{(x^2+y^2+z^2+a^2)^2 - 4a^2(x^2+y^2)}}{2(x^2+y^2)} \tag{409}$$

To obtain the gradient we differentiate the surface equation implicitly with respect to x, y and z

$$2x\cos^2\psi - 2(x^2+y^2)\cos\psi\sin\psi\frac{\partial\psi}{\partial x} - 2z^2\cot\psi\csc^2\psi\frac{\partial\psi}{\partial x} = 0 \tag{410}$$

$$\rightarrow \frac{\partial\psi}{\partial x} = \psi_x = \frac{x\cos^2\psi}{z^2\cot\psi\csc^2\psi + (x^2+y^2)\sin\psi\cos\psi} \tag{411}$$

$$2y\cos^2\psi - 2(x^2+y^2)\cos\psi\sin\psi\frac{\partial\psi}{\partial x} - 2z^2\cot\psi\csc^2\psi\frac{\partial\psi}{\partial x} = 0 \tag{412}$$

$$\rightarrow \frac{\partial\psi}{\partial y} = \psi_y = \frac{y\cos^2\psi}{z^2\cot\psi\csc^2\psi + (x^2+y^2)\sin\psi\cos\psi} \tag{413}$$

$$-2(x^2+y^2)\cos\psi\sin\psi\frac{\partial\psi}{\partial z} + 2z\cot^2\psi - 2z^2\cot\psi\csc^2\psi\frac{\partial\psi}{\partial z} = 0 \tag{414}$$

$$\rightarrow \frac{\partial\psi}{\partial z} = \psi_z = \frac{z\cot^2\psi}{z^2\cot\psi\csc^2\psi + (x^2+y^2)\cos\psi\sin\psi} \tag{415}$$

The direction cosines are then given by

$$\cos\alpha = \frac{\psi_x}{\sqrt{\psi_x^2+\psi_y^2+\psi_z^2}} = \frac{2\sqrt{2}x\sin^2\psi}{\sqrt{8z^2+(x^2+y^2)(3-4\cos 2\psi+\cos 4\psi)}} \tag{416}$$

$$\cos\beta = \frac{\psi_y}{\sqrt{\psi_x^2+\psi_y^2+\psi_z^2}} = \frac{2\sqrt{2}y\sin^2\psi}{\sqrt{8z^2+(x^2+y^2)(3-4\cos 2\psi+\cos 4\psi)}} \tag{417}$$

$$\cos\gamma = \frac{\psi_z}{\sqrt{\psi_x^2+\psi_y^2+\psi_z^2}} = \frac{2\sqrt{2}z}{\sqrt{8z^2+(x^2+y^2)(3-4\cos 2\psi+\cos 4\psi)}}. \tag{418}$$

The second derivatives (for the Laplacian) can again be calculated via (lengthy) implicit differentiation and substituting the first derivatives from above. Adding them up gives zero which implies $\triangle\psi = 0$.

The surface equations $\psi = \text{const}$ can be written in form of an ellipsoid

$$\frac{x^2}{a^2\sec^2\psi} + \frac{y^2}{a^2\sec^2\psi} + \frac{z^2}{a^2\tan^2\psi} = 1 \tag{419}$$

which degenerates to a flat pancake for $\psi = 0, \pi$.

0.22.2 Problem 4.1 - NOT DoNE yet

Standard trick

$$x = \tan\vartheta/2 \rightarrow d\theta = \frac{2dx}{1+x^2},\ \sin\vartheta = \frac{2x}{1+x^2},\ \cos\vartheta = \frac{1-x^2}{1+x^2} \tag{420}$$

$$\int_0^{2\pi}\frac{\sin^2\vartheta d\vartheta}{a+b\cos\vartheta} = \int_?^? \frac{8x^3\cdot dx}{(1+x^2)^3(a+b\frac{1-x^2}{1+x^2})} \tag{421}$$

0.22.3 Problem 6.3 - NOT DoNE yet

Fourier series of initial condition on the interval $[0,\pi]$

$$\psi(t,0) = \psi_0(x) = \frac{b_0}{2} + \sum_{k=1}^{\infty}(a_k\sin 2kx + b_k\cos 2kx) \tag{422}$$

$$a_k = \frac{2}{\pi}\int_0^\pi \psi_0(y)\sin 2ky\,dy \tag{423}$$

$$b_k = \frac{2}{\pi}\int_0^\pi \psi_0(y)\cos 2ky\,dy \tag{424}$$

0.23 Woit - Quantum Theory, Groups and Representations

0.23.1 Problem B.1-3

Rotations of the 2D-plane

$$D^2_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \tag{425}$$

$$D^2_\phi D^2_\theta = \begin{pmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -\cos\phi\sin\theta - \cos\theta\sin\phi \\ \cos\phi\sin\theta + \cos\theta\sin\phi & \cos\theta\cos\phi - \sin\theta\sin\phi \end{pmatrix} \tag{426}$$

$$= \begin{pmatrix} \cos(\phi+\theta) & -\sin(\phi+\theta) \\ \sin(\phi+\theta) & \cos(\phi+\theta) \end{pmatrix} \tag{427}$$

$$= D^2_{\phi+\theta} \tag{428}$$

can also be represented by

$$D^1_\phi = e^{i\phi} \tag{429}$$

$$D^1_\phi D^1_\theta = e^{i\phi}e^{i\theta} = e^{i(\phi+\theta)} \tag{430}$$

$$= D^1_{\phi+\theta}. \tag{431}$$

Furthermore there is also the trivial representation

$$D^{1'}_\phi = 1 \tag{432}$$

$$D^{1'}_\phi D^1_\theta = 1 \cdot 1 = 1 \tag{433}$$

$$= D^{1'}_{\phi+\theta} \tag{434}$$

0.23.2 Problem B.1-4

The time evolution is given by

$$|\Psi(t)\rangle = e^{-iHt}|\Psi(0)\rangle \tag{435}$$

$$= \left(\sum_{k=0}^{\infty} \frac{(-iHt)^k}{k!}\right)|\Psi(0)\rangle \tag{436}$$

We see

$$H = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad H^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \qquad H^3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix} \tag{437}$$

and calculate

$$\sum_{k=0}^{\infty} \frac{(-it)^{2k}}{(2k)!} = \sum_{k=0}^{\infty}(-1)^k \frac{t^{2k}}{(2k)!} = \cos(t) \tag{438}$$

$$\sum_{k=0}^{\infty} \frac{(-it)^{2k+1}}{(2k+1)!} = (-i)\sum_{k=0}^{\infty}(-1)^k \frac{t^{2k+1}}{(2k+1)!} = -i\sin(t) \tag{439}$$

$$\sum_{k=0}^{\infty} \frac{(-i2t)^k}{k!} = \cos(2t) - i\sin(2t) = e^{-i2t} \tag{440}$$

which gives

$$e^{-iHt} = \begin{pmatrix} \cos(t) & -i\sin(t) & 0 \\ -i\sin(t) & \cos(t) & 0 \\ 0 & 0 & e^{-2it} \end{pmatrix} \tag{441}$$

and therefore

$$|\Psi(t)\rangle = \begin{pmatrix} \psi_1 \cos(t) - \psi_2 i \sin(t) \\ -\psi_1 i \sin(t) + \psi_2 \cos(t) \\ \psi_3 e^{-2it} \end{pmatrix} \tag{442}$$

. To check the result one can calculate both sides of $i\partial_t|\Psi(t)\rangle = H|\Psi(t)\rangle$.

0.23.3 Problem B.2-1

1. With $M = PDP^{-1}$ we have $M^2 = PDP^{-1}PDP^{-1} = PDDP^{-1}$ and see

$$e^{tM} = \sum_{k=0}^{\infty} \frac{(tM)^k}{k!} = \sum_{k=0}^{\infty} \frac{(tPDP^{-1})^k}{k!} = \sum_{k=0}^{\infty} \frac{P(tD)^k P^{-1}}{k!} \tag{443}$$

$$= P\left(\sum_{k=0}^{\infty} \frac{(tD)^k}{k!}\right) P^{-1} = Pe^{tD}P^{-1}. \tag{444}$$

The eigenvalues of M are given by

$$-\lambda^3 - (-\lambda)(-\pi^2) = 0 \quad \rightarrow \quad \lambda_1 = i\pi,\ \lambda_2 = -i\pi,\ \lambda_3 = 0 \tag{445}$$

with the eigenvectors

$$\vec{v}_1 = (-i, 1, 0) \tag{446}$$

$$\vec{v}_2 = (i, 1, 0) \tag{447}$$

$$\vec{v}_3 = (0, 0, 1) \tag{448}$$

we obtain

$$M = PDP^{-1} \tag{449}$$

$$= \begin{pmatrix} -i & i & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} i\pi & 0 & 0 \\ 0 & -i\pi & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} i/2 & 1/2 & 0 \\ -i/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{450}$$

With

$$\sum_{k=0}^{\infty} \frac{(i\pi)^k}{k!} = e^{i\pi} \tag{451}$$

$$\sum_{k=0}^{\infty} \frac{(-i\pi)^k}{k!} = e^{-i\pi} \tag{452}$$

we see

$$tD^k = \begin{pmatrix} (i\pi t)^k & 0 & 0 \\ 0 & (-i\pi t)^k & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{453}$$

$$e^{tD} = \sum_{k=0}^{\infty} \frac{(tD)^k}{k!} = \begin{pmatrix} e^{i\pi t} & 0 & 0 \\ 0 & e^{-i\pi t} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{454}$$

and therefore

$$e^{tM} = Pe^{tD}P^{-1} \tag{455}$$

$$= \begin{pmatrix} \frac{1}{2}(e^{-i\pi t} + e^{i\pi t}) & -\frac{1}{2}i(e^{i\pi t} - e^{-i\pi t}) & 0 \\ -\frac{1}{2}i(e^{-i\pi t} - e^{i\pi t}) & \frac{1}{2}(e^{-i\pi t} + e^{i\pi t}) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{456}$$

$$= \begin{pmatrix} \cos(\pi t) & \sin(\pi t) & 0 \\ -\sin(\pi t) & \cos(\pi t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{457}$$

2. Brute force calculation of the matrix powers reveals

$$(tM)^2 = \begin{pmatrix} -(t\pi)^2 & 0 & 0 \\ 0 & -(t\pi)^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (tM)^3 = \begin{pmatrix} 0 & -(t\pi)^3 & 0 \\ (t\pi)^3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{458}$$

$$(tM)^4 = \begin{pmatrix} (t\pi)^4 & 0 & 0 \\ 0 & (t\pi)^4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (tM)^5 \begin{pmatrix} 0 & (t\pi)^5 & 0 \\ -(t\pi)^5 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{459}$$

With

$$1 - \frac{1}{2!}(\pi t)^2 + \frac{1}{4!}(\pi t)^4 + ... = \cos(\pi t) \tag{460}$$

$$\pi t - \frac{1}{3!}(\pi t)^3 + \frac{1}{5!}(\pi t)^5 + ... = \sin(\pi t) \tag{461}$$

$$-\pi t + \frac{1}{3!}(\pi t)^3 - \frac{1}{5!}(\pi t)^5 + ... = (-\pi t) + \frac{1}{3!}(-\pi t)^3 - \frac{1}{5!}(-\pi t)^5 + ... \tag{462}$$

$$= \sin(-\pi t) \tag{463}$$

$$= -\sin(\pi t) \tag{464}$$

we obtain

$$e^{tM} = \begin{pmatrix} \cos(\pi t) & \sin(\pi t) & 0 \\ -\sin(\pi t) & \cos(\pi t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{465}$$

Problem B.2-2

For the Hamiltonian

$$H = -B_x\sigma_1 = \begin{pmatrix} 0 & -B_x \\ -B_x & 0 \end{pmatrix} \tag{466}$$

we find the eigensystem

$$E_1 = -B_x \quad |\psi_1\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{467}$$

$$E_2 = +B_x \quad |\psi_2\rangle = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{468}$$

The Hamiltonian (with full units) is given by

$$H = -g\frac{q\hbar}{2m}\frac{\sigma_1}{2}B_x \tag{469}$$

which translates into energies of

$$E_1 = -g\frac{q\hbar}{4m}B_x \tag{470}$$

$$E_2 = g\frac{q\hbar}{4m}B_x. \tag{471}$$

The time evolution is them given by

$$
\begin{aligned}
|\psi(t)\rangle &= e^{-\frac{i}{\hbar}Ht}|\psi(0)\rangle && (472)\\
&= e^{-i\frac{gq}{4m}\sigma_1 t}|\psi(0)\rangle && (473)\\
&= \left[\cos\left(\frac{gq}{4m}\sigma_1 t\right) - i\sin\left(\frac{gq}{4m}\sigma_1 t\right)\right]|\psi(0)\rangle && (474)\\
&= \left[\cos\left(\frac{gq}{4m}t\right)\mathbb{I}_2 - i\sin\left(\frac{gq}{4m}t\right)\sigma_1\right]|\psi(0)\rangle && (475)\\
&= \begin{pmatrix} \cos\left(\frac{gqt}{4m}\right) & -i\sin\left(\frac{gqt}{4m}\right) \\ -i\sin\left(\frac{gqt}{4m}\right) & \cos\left(\frac{gqt}{4m}\right) \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} && (476)\\
&= \begin{pmatrix} \cos\left(\frac{gqt}{4m}\right) \\ -i\sin\left(\frac{gqt}{4m}\right) \end{pmatrix} && (477)
\end{aligned}
$$

where we used $\sigma_1^{2n} = \mathbb{I}^n = \mathbb{I}$.

0.24 Baez, Muniain - Gauge Fields, Knots and Gravity

0.24.1 Problem I.1 - Plane waves in vacuum

With

$$\vec{\mathcal{E}} = \vec{E}e^{-i(\omega t - \vec{k}\vec{x})} \quad (478)$$

we calculate in cartesian coordinates

1. $\nabla \cdot \vec{\mathcal{E}} = 0$

$$
\begin{aligned}
\nabla \cdot \vec{\mathcal{E}} &= \partial_a \mathcal{E}_a && (479)\\
&= \partial_a(e^{-i(\omega t - \vec{k}\vec{x})})E_a\vec{e}^a && (480)\\
&= \delta_{ab}ik_bE_ae^{-i(\omega t - \vec{k}\vec{x})}\vec{e}^a && (481)\\
&= ik_bE_be^{-i(\omega t - \vec{k}\vec{x})}\vec{e}^a && (482)\\
&= 0 && (483)
\end{aligned}
$$

   where we assumed $E_a = \text{const}$ and used

$$
\begin{aligned}
0 &= \vec{k}\cdot\vec{E} && (484)\\
&= k_a\vec{e}^a E_a\vec{e}^a && (485)\\
&= k_aE_a && (486)
\end{aligned}
$$

2. $\nabla \times \vec{\mathcal{E}} = i\frac{\partial\vec{\mathcal{E}}}{\partial t}$

$$
\begin{aligned}
\nabla \times \vec{\mathcal{E}} &= \epsilon_{abc}\partial_b\mathcal{E}_c\vec{e}_a && (487)\\
&= \epsilon_{abc}E_c\vec{e}_a\partial_b(e^{-i(\omega t - \vec{k}\vec{x})}) && (488)\\
&= \epsilon_{abc}E_c\vec{e}_a\delta_{bd}ik_de^{-i(\omega t - \vec{k}\vec{x})} && (489)\\
&= i(\epsilon_{abc}k_bE_c\vec{e}_a)e^{-i(\omega t - \vec{k}\vec{x})} && (490)\\
&= i(-i\omega E_a\vec{e}^a)e^{-i(\omega t - \vec{k}\vec{x})} && (491)\\
&= i(E_a\vec{e}^a)(-i\omega)e^{-i(\omega t - \vec{k}\vec{x})} && (492)\\
&= i\vec{E}\frac{\partial}{\partial t}e^{-i(\omega t - \vec{k}\vec{x})} && (493)\\
&= i\frac{\partial\vec{\mathcal{E}}}{\partial t} && (494)
\end{aligned}
$$

where we used (typo in the book!)

$$-i\omega\vec{E} = \vec{k} \times \vec{E} \tag{495}$$
$$= \epsilon_{abc} k_b E_c \vec{e}_a \tag{496}$$

0.24.2 Problem I.7 - Adding and multiplying vector fields

1. With $(v+w)f \equiv (f) + w(f)$

   (a) $(v+w)(f+g) = v(f+g) + w(f+g) = vf + vg + wf + wg = (v+w)f + (v+w)g$

   (b) $(v+w)(\alpha f) = v(\alpha f) + w(\alpha f) = \alpha vf + \alpha wf = \alpha(v+w)f$

   (c) $(v+w)(fg) = v(fg)+w(fg) = v(f)g+fv(g)+w(f)g+fw(g) = [(v+w)f]g+f[(v+w)g]$

2. With $(gv)(f) \equiv gv(f)$

   (a) $(gv)(f+h) = gv(f+h) = gv(f) + gv(h) = g(v(f)+v(h)) = gv(f) + gv(h)$

   (b) $gv(\alpha f) = gv(\alpha f) = g\alpha v(f) = \alpha gv(f)$

   (c) $(gv)(fh) = gv(fh) = g(v(f)h + fv(h)) = (gv)(f)h + f(gv)(h)$

0.25 Kreyszig - Introduction to functional analysis

0.25.1 Problem 1.1 Problem 1

Real line: $x \in \mathbb{R}$ with $d(x,y) = |x-y|$

M1 d is real finite, nonnegative: obvious

M2 $d(x,y) = 0$ iff $x = y$: obvious

M3 $d(x,y) = d(y,x)$: obvious

M4 $x < z < y: \; d(x,y) = d(x,z) + d(z,y)$

0.25.2 Problem 11.3 Problem 3

Physicist: Ground state of the harmonic osci. - time-independent Schroedinger equation for harmonic oscillator

$$\psi_0' = -se^{-s^2/2} \tag{497}$$
$$= -s\psi_0 \tag{498}$$
$$\psi_0'' = -e^{-s^2/2} + s^2 e^{-s^2/2} \tag{499}$$
$$= -\psi_0 + s^2\psi_0 \tag{500}$$
$$= -(1-s^2)\psi_0 \tag{501}$$
$$\to \psi_0'' + (1-s^2)\psi_0 = 0 \tag{502}$$

0.26 Garrity et al. - Algebraic Geometry: A Problem Solving Approach

0.27 Garrity, Neumann-Chun - Electricity and magnetism for mathematicians - A guided path from Maxwell's equations to Yang-Mills

0.28 Guidry - Symmetry, Broken Symmetry, and Topology in Modern Physics

0.28.1 Problem 15.1 - Poincaré transformation I

$$\begin{aligned} g(b,\Lambda) \quad &\rightarrow \quad x' = \Lambda x + b && (503)\\ g(b',\Lambda')\circ g(b,\Lambda) \quad &\rightarrow \quad x'' = \Lambda' x' + b' && (504)\\ &\qquad\quad = \Lambda'(\Lambda x + b) + b' && (505)\\ &\qquad\quad = \Lambda'\Lambda x + \Lambda' b + b' \quad \rightarrow \quad g(\Lambda' b + b', \Lambda'\Lambda) && (506)\end{aligned}$$

0.28.2 Problem 15.2 - Poincaré transformation II

$$g(b', I')\circ g(0,\Lambda) = g(b',\Lambda) \tag{507}$$
$$\Lambda x + b = T(b)\circ \Lambda x \tag{508}$$

0.28.3 Problem 15.3 - Poincaré transformation III

0.29 Boltyanskii, Efremovich - Intuitive Combinatorial Topology

0.30 Nakahara - Geometry, Topology and Physics

0.31 Frankel - The Geometry of Physics

0.32 Sexl, Urbantke - Relativity, Groups, Particles

0.33 Scherer - Symmetrien und Gruppen in der Teilchenphysik

0.33.1 Problem 3.11 - Taylor series

$$e^{tC}e^{tD}e^{-tC}e^{-tD} \tag{509}$$

$$\simeq \left(1+tC+\frac{t^2}{2}C^2+...\right)\left(1+tD+\frac{t^2}{2}D^2+...\right)\left(1-tC+\frac{t^2}{2}C^2+...\right)\left(1-tD+\frac{t^2}{2}D^2+...\right) \tag{510}$$

$$= 1+t(C+D-C-D)+\frac{t^2}{2}(C^2+D^2+C^2+D^2)+t^2(CD-C^2-CD) \tag{511}$$

$$+t^2(-DC-D^2)+t^2(CD)+\mathcal{O}(t^3) \tag{512}$$

$$= 1+t\cdot 0+t^2(C^2+D^2)+t^2(-C^2-DC-D^2+CD)+\mathcal{O}(t^3) \tag{513}$$

$$= 1+t^2[C,D]+\mathcal{O}(t^3) \tag{514}$$

$$\tag{515}$$