0.1 Ryden - Introduction to Cosmology, 2016

0.1.1 Exercise 2.1

The power emitted by a surface dA under the angle θ ($\theta = 0$ is perpendicular to the surface dA) into the solid angle $d\Omega$ is $B_\nu \cos\theta \, dA \, d\Omega \, d\nu$ with

$$B_\nu(\nu, T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1} \tag{1}$$

this is related to the spectral energy density (eqn 2.27) of the photon field by

$$\varepsilon(\nu) = u_\nu = \frac{4\pi}{c} B_\nu. \tag{2}$$

The angular integration (for dA in the xy plane) gives

$$\int \cos\theta \, d\Omega = \int_0^{2\pi} \int_0^{\pi/2} \cos\theta \, \sin\theta \, d\theta \, d\phi \tag{3}$$

and therefore with $\rho = m/V$ and $V = 4/3\pi R^3$ we obtain

$$\begin{aligned}
P &= \int B_\nu \cos\theta \, dA \, d\Omega \, d\nu && (4)\\
&= \pi \int B_\nu \, dA \, d\nu && (5)\\
&= \frac{2\pi^5 k^4}{15c^2h^3} T^4 \int dA && (6)\\
&= \frac{2\pi^5 k^4}{15c^2h^3} T^4 \cdot 4\pi R^2 && (7)\\
&= \frac{2\pi^5 k^4}{15c^2h^3} T^4 \cdot \left(\frac{6\sqrt{\pi} m}{\rho}\right)^{2/3} && (8)
\end{aligned}$$

which gives for $m = 75$kg a power of 440W. The net power is obviously smaller $P_{\text{net}} = \sigma A(T^4_{\text{body}} - T^4_{\text{ambient}})$.

0.1.2 Exercise 2.2

The photon number density is given by (eqn 2.30)

$$n(\nu) = \frac{\varepsilon(\nu)}{h\nu} = \frac{8\pi\nu^2}{c^3} \frac{1}{e^{h\nu/kT} - 1} \tag{9}$$

then the flux across the projected surface of the sphere πR^2 from each direction is given by

$$\begin{aligned}
N &= \int d\Omega \int d\nu \, \pi R^2 c n(\nu) && (10)\\
&= 4\pi^2 c R^2 \int d\nu \, n(\nu) && (11)\\
&= \zeta(3) \frac{64\pi^3 R^2}{c^2 h^3} (kT)^3 && (12)\\
&= 3.2 \cdot 10^{17} && (13)
\end{aligned}$$

assuming $m = \rho V = 75\text{kg}$. Analog

$$
\begin{aligned}
P &= \int d\Omega \int d\nu \, \pi R^2 c\varepsilon(\nu) && (14)\\
&= 4\pi^2 c R^2 \int d\nu \, varepsilon(\nu) && (15)\\
&= \frac{32\pi^7 R^2}{15c^2h^3}(kT)^4 && (16)\\
&= 3.3 \cdot 10^{-5}\text{W} && (17)
\end{aligned}
$$

0.1.3 Exercise 2.3

Combining the results

$$
\begin{aligned}
P_{\text{rad}} &= -440\text{W} && (18)\\
P_{\text{absCMB}} &= 3.3 \cdot 10^{-5}\text{W} && (19)\\
P_{\text{tot}} &= -440W && (20)
\end{aligned}
$$

So the astronaut is loosing heat and can not overheat. The astronauts energy loss is given by

$$
\begin{aligned}
\Delta E_{\text{heat}} = mC\Delta T &\equiv P_{\text{tot}}\Delta t && (21)\\
\rightarrow \frac{\Delta t}{\Delta T} = \frac{mC}{P_{\text{tot}}} &= 716\text{s/K} = 12\text{min/K} && (22)
\end{aligned}
$$

therefore so the lack of oxygen seems to be most likely.

0.1.4 Exercise 2.4

$$
\begin{aligned}
z(r) &= \frac{\lambda(r) - \lambda_{\text{em}}}{\lambda_{\text{em}}} \quad \rightarrow \quad \lambda(r) = [1 + z(r)]\lambda_{\text{em}} && (23)\\
E(r) &= hc\frac{1}{\lambda(r)} = \frac{hc}{[1 + z(r)]\lambda_{\text{em}}} && (24)\\
\frac{dE}{dr} &= -kE \quad \rightarrow \quad z' - k(1 + z) = 0 && (25)\\
&\rightarrow \quad z(r) = ce^{kr} - 1 \quad (z(0) = 0) && (26)\\
&\rightarrow \quad z(r) = e^{kr} - 1 && (27)\\
&\rightarrow \quad z(r) \approx kr && (28)
\end{aligned}
$$

with $k = H_0/c$

0.1.5 Exercise 2.5

With

$$
\begin{aligned}
n(\nu) &= \frac{\varepsilon(\nu)}{h\nu} = \frac{8\pi\nu^2}{c^3}\frac{1}{e^{h\nu/kT} - 1} && (29)\\
n_\gamma &= \int d\nu \, n(\nu) && (30)\\
&= \frac{16\zeta(3)\pi}{c^3h^3}(kT)^3 = \frac{2\zeta(3)}{\pi^2c^3\hbar^3}(kT)^3 && (31)
\end{aligned}
$$

then

$$n(h\nu > E_0) = \frac{8\pi}{c^3}\int_{E_0/h}^{\infty}\frac{\nu^2}{e^{h\nu/kT}-1}d\nu \tag{32}$$

$$\overset{h\nu>E_0\gg kT}{\simeq} \frac{8\pi}{c^3}\int_{E_0/h}^{\infty} e^{-h\nu/kT}\nu^2 d\nu \tag{33}$$

$$= \frac{8\pi kT}{c^3h^3}e^{-E_0/kT}(E_0^2 + 2E_0kT + 2(kT)^2) \tag{34}$$

$$\simeq \frac{8\pi kT}{c^3h^3}e^{-E_0/kT}E_0^2 \tag{35}$$

then

$$\frac{n(h\nu > E_0)}{n_\gamma} = \frac{1}{2\zeta(3)}e^{-E_0/kT}\left(\frac{E_0^2}{kT}\right)^2 \tag{36}$$

Using this result we obtain 5.8% of infrared photons. Exact numerical integration gives $6 \cdot 10^{-4}\%$ radio waves (or longer), 91.6% microwaves, 8.4% infrared, 0% optical (and shorter).

0.1.6 Exercise 2.6

Now

$$n(h\nu < E_0) = \frac{8\pi}{c^3}\int_0^{E_0/h}\frac{\nu^2}{e^{h\nu/kT}-1}d\nu \tag{37}$$

$$\overset{h\nu<E_0\ll kT}{\simeq} \frac{8\pi}{c^3}\int_{E_0/h}^{\infty}\frac{kT}{h\nu}\nu^2 d\nu \tag{38}$$

$$= \frac{4E_0^2\pi kT}{c^3h^3} \tag{39}$$

then

$$\frac{n(h\nu < E_0)}{n_\gamma} = \frac{E_0^2}{4\zeta(3)k^2T^2} \tag{40}$$

For $\lambda > 3$cm ($hc/\lambda = h\nu < E_0 = hc/\lambda_0$) we obtain 0.6%.

0.1.7 Exercise 3.2

We replace $d\theta$ by $d\varphi$! Calculating the size of the object as distance on the sphere of radius R

$$dl^2 = R^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{41}$$

$$= R^2\sin^2\theta d\varphi^2 \tag{42}$$

with $\theta = r/R$. Then

$$dl = R\sin\frac{r}{R}\cdot d\varphi \tag{43}$$

$$\rightarrow d\varphi = \frac{dl}{R\sin\frac{r}{R}} \tag{44}$$

For $r \rightarrow \pi R$ $d\varphi$ increases to 2π

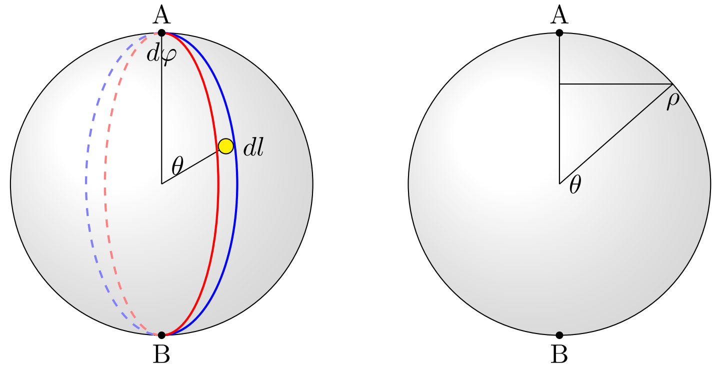


Figure 1: (left) Ex 3.2. Spherical universe, Observer at A, object of size dl at distance r, (right) Ex 3.3. Spherical universe, Observer at A

0.1.8 Exercise 3.3

Simple geometry

$$\begin{aligned} \theta &= 2\pi \frac{r}{2\pi R} = \frac{r}{R} && (45)\\ \sin\theta &= \frac{\rho}{R} && (46)\\ C &= 2\pi\rho && (47) \end{aligned}$$

gives

$$\begin{aligned} C &= 2\pi\rho && (48)\\ &= 2\pi R \sin\theta && (49)\\ &= 2\pi R \sin\frac{r}{R}. && (50) \end{aligned}$$

For the euclidean case we get of course $C_{\text{Euclid}} = 2\pi r$. Then

$$\begin{aligned} \Delta s &= C_{\text{Euclid}} - C && (51)\\ &= 2\pi(r - R\sin\frac{r}{R}) && (52)\\ &\simeq \frac{\pi r^3}{3R^2} - \frac{\pi r^5}{60R^2} && (53)\\ &\simeq 33.8 \text{ km} && (54) \end{aligned}$$

0.1.9 Exercise 3.4

1. $\kappa = +1$ With

$$\alpha + \beta + \gamma = \frac{A}{R^2} + \pi \tag{55}$$

we see that each angle can be maximally π. So

$$A_{\max} = (3\pi - \pi)R^2 = 2\pi R^2. \tag{56}$$

It is easy to see that such a (degenerated) triangle (half sphere) can be realized.

A bit more formal - integrating over a triangle-shape slice

$$A = \int_0^\alpha \int_0^\alpha R^2 \sin\theta \, d\theta \, d\phi \tag{57}$$
$$= R^2(\alpha - \alpha\cos\alpha) \tag{58}$$
$$A_{\max} = A(\alpha = \pi) = 2\pi R^2 \tag{59}$$

2. $\kappa = 0$ There is no limited to the triangle size.
3. $\kappa = -1$ With

$$A = (\pi - \alpha - \beta - \gamma)R^2 \tag{60}$$

we see that the potential maximum is $A_{\max} = \pi R^2$. Now we need to show that such a triangle exists.

0.1.10 Exercise 3.5

With

$$dx = \frac{x}{r}dr + \frac{x}{\sin\theta}\cos\theta \, d\theta + \frac{x}{\cos\phi}(-\sin\phi)\, d\phi \tag{61}$$
$$dy = \frac{y}{r}dr + \frac{y}{\sin\theta}\cos\theta \, d\theta + \frac{y}{\sin\phi}\cos\phi \, d\phi \tag{62}$$
$$dz = \frac{z}{r}dr + \frac{z}{\cos\theta}(-\sin\theta)d\theta \tag{63}$$

we obtain

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{64}$$
$$= dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{65}$$

0.1.11 Exercise 4.1

$$E_{\text{sun}} = M_{\text{sun}}c^2 = 1.79 \cdot 10^{47}\,\text{J} \tag{66}$$
$$E_\Lambda = \varepsilon_\Lambda \frac{4}{3}\pi R^3 = 1.1 \cdot 10^{25}\,\text{J} \tag{67}$$

0.2 Boerner - The Early Universe - Facts and Fiction (4th edition)

0.2.1 1.1 Friedman equations

1. The Friedman equations in book contain a small typo ($\rho = \varrho$)

$$(A) \qquad \ddot{R} = -\frac{4\pi}{3}(\varrho + 3p)GR + \frac{1}{3}\Lambda R \tag{68}$$
$$(B) \qquad \dot{R}^2 = \frac{8\pi}{3}G\varrho R^2 + \frac{1}{3}\Lambda R^2 - K \tag{69}$$
$$(C) \qquad 0 = (\varrho R^3)^{\cdot} + p(R^3)^{\cdot} \tag{70}$$

Calculating the time derivative of (B)

$$2\dot{R}\ddot{R} = \frac{8\pi}{3}G(\dot{\varrho}R^2 + 2\varrho R\dot{R}) + \frac{2}{3}\Lambda R\dot{R} \tag{71}$$
$$\ddot{R} = \frac{R}{3}\left(4\pi G\dot{\varrho}\frac{R}{\dot{R}} + 8\pi G\varrho + \Lambda\right) \tag{72}$$

and simplifying (A)

$$\ddot{R} = \frac{R}{3}\left(-4\pi G(\varrho + 3p) + \Lambda\right) \tag{73}$$

Combining both yields

$$\dot{\varrho}\frac{R}{\dot{R}} + 2\varrho = -(\varrho + 3p) \tag{74}$$

$$\dot{\varrho}R = -3(\varrho + p)\dot{R} \tag{75}$$

which is (C). Rearranging the order of the steps gives the other two cases.

2. From (C) we have

$$\dot{\varrho} = -3(\varrho + p)\frac{\dot{R}}{R} \tag{76}$$

$$= -3\varrho\left(1 + k\varrho^{\gamma-1}\right)\frac{\dot{R}}{R} \tag{77}$$

which can be rearranged and integrated

$$\frac{\dot{R}}{R} = \frac{\dot{\varrho}}{-3\varrho\left(1 + k\varrho^{\gamma-1}\right)} \tag{78}$$

$$\to -\frac{1}{3(1-\gamma)}\log(k + \varrho^{1-\gamma}) = \log R + c \tag{79}$$

$$\to \log(k + \varrho^{1-\gamma}) = -3(1-\gamma)\log R + c' \tag{80}$$

$$\to k + \varrho^{1-\gamma} = e^{-3(1-\gamma)\log R + c'} \tag{81}$$

$$\to k + \varrho^{1-\gamma} = c'' R^{-3(1-\gamma)} \tag{82}$$

$$\to \varrho = \left(c'' R^{3(\gamma-1)} - k\right)^{1/(1-\gamma)} \tag{83}$$

with

$$c'' = \frac{k + \varrho_0^{1-\gamma}}{R_0^{3(\gamma-1)}} \tag{84}$$

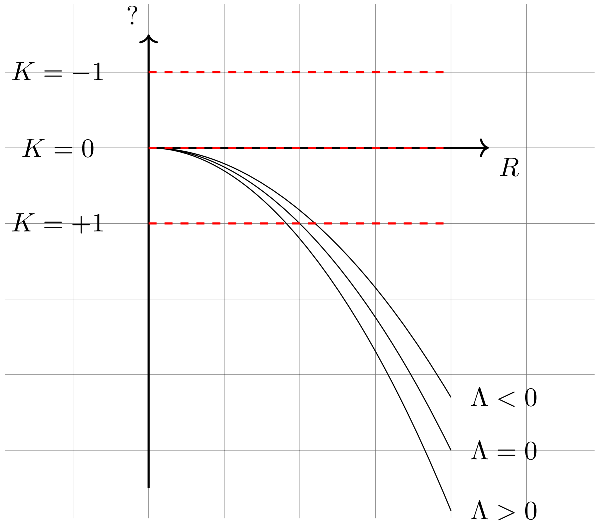
$$\to \quad \varrho = \left([k + \varrho_0^{1-\gamma}]\frac{R}{R_0}^{3(\gamma-1)} - k\right)^{1/(1-\gamma)} \tag{85}$$

$$\to \quad \varrho = \left(k\left[\frac{R}{R_0}^{3(\gamma-1)} - 1\right] + \left[\frac{R^3}{\varrho_0 R_0^3}\right]^{\gamma-1}\right)^{1/(1-\gamma)} \tag{86}$$

We obtain from (B)

$$\dot{R}^2 - \left(\frac{8\pi}{3}G\varrho + \frac{1}{3}\Lambda\right)R^2 = -K \tag{87}$$

which we can interpret as motion of a particle in a changing $-R^2$ potential.



0.3 Baumann - Cosmology (1nd edition)

0.3.1 Problem 1.1 Length scales

object	size	size in m
pepper corn	5mm	0.005m
basketball size 7 (75 cm circumference)	24cm	0.24m
basketball court	30.62yds	28m

1. $R_{\text{Moon}} = 6.5\text{cm}$, d_{ME}=11.3m
2. $R_{\text{Sun}} = 55\text{cm}$, $r_{\text{Earth orbit}}$=118m, $r_{\text{Neptune orbit}}$=3544m
3. $d_{\text{Solar system}}$=0.2mm
4. $R_{\text{Solar neigh}} = 18\text{mm}$
5. $R_{\text{Galaxy}} = 28\text{cm}$
6. $R_{\text{Local Group}} = 56\text{cm}$
7. $R_{\text{Super Cluster}} = 30\text{cm}$

0.3.2 Problem 1.2 Hubble constant

1. $t_{H_0} = 14 \cdot 10^9\text{a}$
2. $d_{H_0} = 140 \cdot 10^9\text{ly}$
3. $\rho_0 = 9 \cdot 10^{-27}\text{kg/m}^3$
4. $n_{\text{H universe}} = \frac{\rho_0 d^3}{m_{\text{H}}} = 10^{79}$, $n_{\text{H brain}} = \frac{m_{\text{brain}}}{m_{H_2O}} = 10^{26}$
5. $l_{\text{min}} = \frac{hc}{E_{\text{max}}} = 1.23 \cdot 10^{-18}\text{m}$

0.3.3 Exercise 2.1

Using the Euler Lagrange equations we obtain

$$\frac{\partial L}{\partial r} = mr\dot{\phi}^2, \quad \frac{\partial L}{\partial \dot{r}} = mr\dot{r} \quad \rightarrow \quad \ddot{r} = r\dot{\phi}^2 \tag{88}$$

$$\frac{\partial L}{\partial \phi} = 0, \quad \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} \quad \rightarrow \quad \ddot{\phi} = -2\frac{\dot{r}}{r}\dot{\phi} \tag{89}$$

0.3.4 Exercise 2.2

Calculating

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}g^{\mu\lambda}(g_{\beta\lambda,\alpha} + g_{\alpha\lambda,\beta} - g_{\alpha\beta,\lambda}) \tag{90}$$

we need the FRW metric which is given by

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & a^2\gamma \end{pmatrix} \qquad g^{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & \frac{1}{a^2}\gamma^{-1} \end{pmatrix} \tag{91}$$

then

$$\Gamma^{i}_{0j} = \frac{1}{2}g^{i\lambda}(g_{j\lambda,0} + g_{0\lambda,j} - g_{0j,\lambda}) \tag{92}$$

$$= \frac{1}{2}g^{il}(g_{jl,0} + g_{0l,j} - g_{0j,l}) \tag{93}$$

$$= \frac{1}{2}g^{il}g_{jl,0} = \frac{1}{2}g^{il}\frac{1}{c}\partial_t g_{jl} \tag{94}$$

$$= \frac{1}{2}\frac{1}{a^2}\gamma^{il}\frac{1}{c}\partial_t(a^2\gamma_{jl}) = \frac{1}{2}\frac{1}{a^2}\gamma^{il}\frac{1}{c}2a\dot{a}\gamma_{jl} \tag{95}$$

$$= \frac{\dot{a}}{a}\frac{1}{c}\gamma^{il}\gamma_{jl} = \frac{\dot{a}}{a}\frac{1}{c}\delta^i_j \tag{96}$$

and

$$\Gamma^{i}_{jk} = \frac{1}{2}g^{i\lambda}(g_{k\lambda,j} + g_{j\lambda,k} - g_{jk,\lambda}) \tag{97}$$

$$= \frac{1}{2}g^{il}(g_{kl,j} + g_{j\lambda,k} - g_{jk,l}) \tag{98}$$

0.3.5 Exercise 2.3

With $P^{\mu} = (E/c, P^i)$

$$-m^2c^2 = g_{\mu\nu}P^{\mu}P^{\nu} \tag{99}$$

$$= g_{00}(P^0)^2 + g_{ij}P^iP^j \tag{100}$$

$$= -\frac{E^2}{c^2} + a^2\gamma_{ij}P^iP^j \tag{101}$$

$$\rightarrow \vec{p}^2 = a^2\gamma_{ij}P^iP^j = \left(\frac{E^2}{c^2} - m^2c^2\right) \tag{102}$$

then

$$\frac{E}{c^3}\frac{dE}{dt} = -\frac{1}{c}a\dot{a}\gamma_{ij}P^iP^j \tag{103}$$

$$= -\frac{1}{c}\frac{\dot{a}}{a}\left(\frac{E^2}{c^2} - m^2c^2\right) \tag{104}$$

$$\frac{E}{E^2 - m^2c^4}dE = -\frac{da}{a} \tag{105}$$

Integrating on both sides

$$\frac{1}{2}\log(E^2 - m^2c^4) = -\log a + k_1 \tag{106}$$

$$\sqrt{E^2 - m^2c^4} = \frac{k_2}{a} \tag{107}$$

$$pc = \frac{k_2}{a} \tag{108}$$

meaning $p \sim a^{-1}$.

0.3.6 Exercise 2.4

With

$$\frac{dU}{dt} = (c^2\dot{\rho})V + (\rho c^2)\dot{V} \tag{109}$$

$$= c^2ka^3\dot{\rho} + 3ka^2(\rho c^2)\dot{a} \tag{110}$$

$$-P\frac{dV}{dt} = -P \cdot 3ka^2\dot{a} \tag{111}$$

then

$$c^2ka^3\dot{\rho} + 3ka^2(\rho c^2)\dot{a} + P \cdot 3ka^2\dot{a} = 0 \tag{112}$$

$$\dot{\rho} + 3\rho\frac{\dot{a}}{a} + \frac{P}{c^2} \cdot 3\frac{\dot{a}}{a} = 0 \tag{113}$$

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) = 0 \tag{114}$$

0.3.7 Problem 2.1 - Robertson-Walker metric

1.
   - t proper time measured along the world lines of the galaxies or fluid elements: $g_{00} = \text{const} = -1$
   - Spacial part - isometry at every point means none of the γ_{ij} have preferred time dependency - which can be ultimately factored out

$$\gamma_{ij} = \gamma_{ij}(t, x^k) = a(t)^2\gamma_{ij}(x^k) \tag{115}$$

   - Weyl postulate: The world lines of the fluid elements, that model the universe's matter content, are orthogonal to hypersurfaces of constant time: $g_{0i} \equiv \mathbf{g}(\mathbf{e}_0, \mathbf{e}_1) = 0$

   Therefore

$$ds^2 = -dt^2 + a(t)^2\gamma_{ij}(x^k)dx^i dx^j \tag{116}$$

2.
   - Spherical symmetry around a point means the proper distance between two points does not change under rotations this means the angular part is $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$
   - θ and ϕ mirror symmetry implies $g_{\hat{r}\phi} = 0$ and $g_{\hat{r}\theta} = 0$ so we are left with

$$ds^2 = -dt^2 + a(t)^2\left[C(\hat{r})d\hat{r}^2 + D(\hat{r})d\Omega^2\right] \tag{117}$$

   at this moment $\hat{r}$ is an arbitrary radial coordinate with $D(\hat{r}) > 0$
   - Defining new radial coordinate $r = \sqrt{D(\hat{r})}$ then

$$ds^2 = -dt^2 + a(t)^2\left[\tilde{C}(r)dr^2 + r^2d\Omega^2\right] \tag{118}$$

- Now we just rewrite $\tilde{C}(r) > 0$ in a more convenient way

$$ds^2 = -dt^2 + a(t)^2\left[e^{2\alpha(r)}dr^2 + r^2 d\Omega^2\right] \tag{119}$$

Now we calculate the connection coefficients - the non-vanishing ones are

$$\Gamma^r_{rr} = \alpha' \quad \Gamma^r_{\theta\theta} = -re^{-2\alpha} \quad \Gamma^r_{\phi\phi} = -re^{-2\alpha}\sin^2\theta \tag{120}$$
$$\Gamma^\theta_{\theta r} = 1/r \quad \Gamma^\theta_{r\theta} = 1/r \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \tag{121}$$
$$\Gamma^\phi_{\phi r} = 1/r \quad \Gamma^\phi_{\phi\theta=\cot\theta} \quad \Gamma^\phi_{r\phi} = 1/r \quad \Gamma^\phi_{\theta\phi} = \cot\theta \tag{122}$$

then

$$R_{ij} = \begin{pmatrix} \frac{2\alpha'}{r} & 0 & 0 \\ 0 & e^{-2\alpha}(-1+e^{2\alpha}+r\alpha') & 0 \\ 0 & 0 & e^{-2\alpha}\sin^2\theta(-1+e^{2\alpha}+r\alpha') \end{pmatrix} \tag{123}$$
$$R_{(3)} = R_{ij}\gamma^{ij} \tag{124}$$
$$= \frac{2e^{-2\alpha}(-1+e^{2\alpha}+2r\alpha')}{r^2} \tag{125}$$
$$= \frac{2}{r^2}(1-e^{-2\alpha}+2r\alpha' e^{-2\alpha}) \tag{126}$$
$$= \frac{2}{r^2}(1-\partial_r[re^{-2\alpha}]) \tag{127}$$

3. Solving the differential equation for the constant curvature $\hat{K}$

$$\frac{2}{r^2}(1-\partial_r[re^{-2\alpha}]) = K \tag{128}$$
$$\partial_r[re^{-2\alpha}] = 1 - \frac{\hat{K}r^2}{2} \tag{129}$$
$$re^{-2\alpha} = r - \frac{Kr^3}{6} - b \tag{130}$$
$$\alpha = -\frac{1}{2}\log\left(1-\frac{\hat{K}r^2}{6}-\frac{b}{r}\right) \tag{131}$$
$$\alpha = \frac{1}{2}\log\left(1-\frac{\hat{K}r^2}{6}-\frac{b}{r}\right)^{-1} \tag{132}$$
$$e^{2\alpha} = \frac{1}{1-Kr^2-br^{-1}} \quad (K = \hat{K}/6) \tag{133}$$

Locally flat means

$$e^{2\alpha}|_{r=0} = 1 \quad \rightarrow \quad b = 0. \tag{134}$$

Now we rewrite

$$\frac{1}{1-Kr^2} = \frac{1}{1-k\frac{r^2}{R_0^2}} \tag{135}$$

where R_0 is a scaling parameter and k determines the sign of the constant 3-curvature $R_{(3)}$.

4. Using the coordinate transformation

$$d\rho = \dot{a}r\,dt + a\,dr \tag{136}$$
$$dT = dt + \frac{1}{2}(\ddot{a}a+\dot{a}^2)r^2\,dt + \dot{a}ar\,dt \tag{137}$$

we see

$$dt \simeq \left(1 + \frac{\dot{a}^2 - a\ddot{a}}{2a^2}\rho^2\right) dT - \frac{\dot{a}}{a}\rho\, d\rho \tag{138}$$
$$dr \simeq -\frac{\dot{a}}{a^2}\rho\, dT + \frac{1}{a}\left(1 + \frac{\dot{a}^2}{a^2}\rho^2\right) d\rho \tag{139}$$

then with $\frac{1}{1-Kr^2} \simeq 1 + Kr^2 = 1 + k\frac{r^2}{R_0^2} = 1 + k\frac{\rho^2}{a^2R_0^2}$ we obtain

$$ds^2 = -dt^2 + a(t)^2\left[\frac{1}{1-Kr^2}dr^2 + r^2 d\Omega^2\right] \tag{140}$$
$$= -dT^2\left(1 - \frac{\ddot{a}}{a}\rho^2\right) + \left(\frac{\dot{a}^2}{a^2}\rho^2 + 1 + \frac{k}{a^2R_0^2}\rho^2\right) d\rho^2 + \rho^2 d\Omega^2 \tag{141}$$

0.3.8 Problem 2.2 - Geodesics from a simple Lagrangian

1. Calculating every term individually

$$\frac{\partial \mathcal{L}}{\partial x^\alpha} = -\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\dot{x}^\mu \dot{x}^\nu \tag{142}$$
$$\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} = -g_{\mu\nu}\left(\frac{\partial \dot{x}^\mu}{\partial \dot{x}^\alpha}\dot{x}^\nu + \dot{x}^\mu\frac{\partial \dot{x}^\nu}{\partial \dot{x}^\alpha}\right) \tag{143}$$
$$= -g_{\mu\nu}\left(\delta^\mu_\alpha \dot{x}^\nu + \dot{x}^\mu \delta^\nu_\alpha\right) \tag{144}$$
$$= -(g_{\alpha\nu}\dot{x}^\nu + g_{\mu\alpha}\dot{x}^\mu) = -2g_{\alpha\nu}\dot{x}^\nu \tag{145}$$
$$\frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} = -\left(\frac{\partial g_{\alpha\nu}}{\partial x^\beta}\dot{x}^\beta \dot{x}^\nu + g_{\alpha\nu}\ddot{x}^\alpha + \frac{\partial g_{\mu\alpha}}{\partial x^\beta}\dot{x}^\beta \dot{x}^\mu + g_{\mu\alpha}\ddot{x}^\mu\right) \tag{146}$$
$$= -\left(\frac{\partial g_{\alpha\nu}}{\partial x^\mu}\dot{x}^\mu \dot{x}^\nu + \frac{\partial g_{\mu\alpha}}{\partial x^\nu}\dot{x}^\nu \dot{x}^\mu + 2g_{\mu\alpha}\ddot{x}^\mu\right) \tag{147}$$

then the equations of motion are

$$g_{\mu\alpha}\ddot{x}^\mu + \frac{1}{2}\left(\frac{\partial g_{\alpha\nu}}{\partial x^\mu}\dot{x}^\mu \dot{x}^\nu + \frac{\partial g_{\mu\alpha}}{\partial x^\nu}\dot{x}^\nu \dot{x}^\mu - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\dot{x}^\mu \dot{x}^\nu\right) = 0 \tag{148}$$
$$g_{\mu\alpha}\ddot{x}^\mu + \frac{1}{2}\left(\frac{\partial g_{\alpha\nu}}{\partial x^\mu} + \frac{\partial g_{\mu\alpha}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right)\dot{x}^\mu \dot{x}^\nu = 0 \tag{149}$$

Now we multiply with $g^{\alpha\beta}$ and use $g_{\mu\alpha}g^{\alpha\beta}\ddot{x}^\mu = \delta^\beta_\mu \ddot{x}^\mu = \ddot{x}^\beta$

$$\ddot{x}^\beta + \frac{1}{2}g^{\alpha\beta}\left(\frac{\partial g_{\alpha\nu}}{\partial x^\mu} + \frac{\partial g_{\mu\alpha}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right)\dot{x}^\mu \dot{x}^\nu = 0 \tag{150}$$
$$\ddot{x}^\beta + \Gamma^\beta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0 \tag{151}$$

2. Calculating the λ derivative of $\mathcal{H}$ along the geodesic (substituting)

$$
\begin{aligned}
\frac{\partial \mathcal{H}}{\partial \lambda} &= \frac{\partial \mathcal{L}}{\partial \lambda} - \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\gamma}\dot{x}^\gamma\right) && (152)\\
&= \frac{\partial \mathcal{L}}{\partial \lambda} - \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\gamma}\right)\dot{x}^\gamma - \frac{\partial \mathcal{L}}{\partial \dot{x}^\gamma}\ddot{x}^\gamma && (153)\\
&= \frac{\partial \mathcal{L}}{\partial \lambda} - \underbrace{\frac{\partial \mathcal{L}}{\partial x^\gamma}}_{-\frac{\partial g_{\mu\nu}}{\partial x^\gamma}\dot{x}^\mu\dot{x}^\nu}\dot{x}^\gamma - \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{x}^\gamma}}_{-2g_{\gamma\varepsilon}\dot{x}^\varepsilon}\underbrace{\ddot{x}^\gamma}_{-\Gamma^\gamma_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} && (154)\\
&= \frac{\partial \mathcal{L}}{\partial \lambda} + \frac{\partial g_{\mu\nu}}{\partial x^\gamma}\dot{x}^\mu\dot{x}^\nu\dot{x}^\gamma - 2g_{\gamma\varepsilon}\dot{x}^\varepsilon\Gamma^\gamma_{\mu\nu}\dot{x}^\mu\dot{x}^\nu && (155)\\
&= \frac{\partial \mathcal{L}}{\partial \lambda} + g_{\mu\nu,\varepsilon}\dot{x}^\mu\dot{x}^\nu\dot{x}^\varepsilon - g_{\gamma\varepsilon}g^{\gamma\sigma}(g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma})\dot{x}^\varepsilon\dot{x}^\mu\dot{x}^\nu && (156)\\
&= \frac{\partial \mathcal{L}}{\partial \lambda} + g_{\mu\nu,\varepsilon}\dot{x}^\mu\dot{x}^\nu\dot{x}^\varepsilon - \delta^\sigma_\varepsilon(g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma})\dot{x}^\varepsilon\dot{x}^\mu\dot{x}^\nu && (157)\\
&= \frac{\partial \mathcal{L}}{\partial \lambda} - (-g_{\mu\nu,\varepsilon} + g_{\mu\varepsilon,\nu} + g_{\nu\varepsilon,\mu} - g_{\mu\nu,\varepsilon})\dot{x}^\varepsilon\dot{x}^\mu\dot{x}^\nu \quad \text{(reindex)} && (158)\\
&= \frac{\partial \mathcal{L}}{\partial \lambda} && (159)\\
&= 0 && (160)
\end{aligned}
$$

then

$$
\begin{aligned}
\mathcal{H} &= \mathcal{L} - \frac{\partial \mathcal{L}}{\partial \dot{x}^\gamma}\dot{x}^\gamma && (161)\\
&= -g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - (-2g_{\alpha\gamma}\dot{x}^\alpha)\dot{x}^\gamma && (162)\\
&= g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu && (163)
\end{aligned}
$$

0.3.9 Problem 2.3 - Christoffel symbols from a Lagrangian

$$
\begin{aligned}
\mathcal{L} &= -g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu && (164)\\
&= \dot{t}^2 - a(t)^2(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) && (165)
\end{aligned}
$$

Now $\mu = 0, x^\mu = t$

$$
\begin{aligned}
\frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{t}} &= \frac{\partial \mathcal{L}}{\partial t} && (166)\\
2\ddot{t} &= -2a\dot{a}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) && (167)\\
&\rightarrow \ddot{t} + a\dot{a}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = 0 && (168)\\
&\rightarrow \Gamma^0_{11} = \Gamma^0_{22} = \Gamma^0_{33} = a\dot{a} && (169)
\end{aligned}
$$

Now $\mu = 1, x^\mu = x$

$$
\begin{aligned}
\frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{x}} &= \frac{\partial \mathcal{L}}{\partial x} && (170)\\
-2\left(\ddot{x}a^2 + \dot{x}2a\frac{\partial a}{\partial t}\frac{\partial t}{\partial \lambda}\right) &= 0 && (171)\\
&\rightarrow \ddot{x} + 2\frac{\dot{a}}{a}\dot{x}\dot{t} = 0 && (172)\\
&\rightarrow \Gamma^1_{01} = \Gamma^1_{10} = \frac{\dot{a}}{a} && (173)
\end{aligned}
$$

Analog for $\mu = 2, 3$

$$\Gamma^2_{02} = \Gamma^1_{20} = \frac{\dot{a}}{a} \tag{174}$$
$$\Gamma^3_{03} = \Gamma^1_{30} = \frac{\dot{a}}{a} \tag{175}$$

0.3.10 Problem 2.4 - Geodesics in de Sitter

1.) To derive the conserved quantities we need to find the Killing vectors ξ^α defined by

$$\mathcal{L}_\xi g_{\mu\nu} = 0 \tag{176}$$
$$\rightarrow \quad g_{\mu\nu,\alpha}\xi^\alpha + g_{\alpha\nu}\xi^\alpha_{\ ,\mu} + g_{\mu\alpha}\xi^\alpha_{\ ,\nu} = 0 \tag{177}$$

which for $ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\Omega^2$ is a system of 10 coupled PDEs

$$-\frac{\partial A}{\partial r}\xi^r - 2A\xi^t_{,t} = 0 \qquad (\mu = 1, \nu = 1) \tag{178}$$
$$-A\xi^t_{,r} + B\xi^r_{,t} = 0 \qquad (\mu = 1, \nu = 2) \tag{179}$$
$$-A\xi^t_{,\theta} + r^2\xi^\theta_{,t} = 0 \qquad (\mu = 1, \nu = 3) \tag{180}$$
$$-A\xi^t_{,\phi} + r^2\sin^2\theta\xi^\phi_{,t} = 0 \qquad (\mu = 1, \nu = 4) \tag{181}$$
$$\xi^r B' + 2B\xi^r_{,r} = 0 \qquad (\mu = 2, \nu = 2) \tag{182}$$
$$B\xi^r_{,\theta} + r^2\xi^\theta_{,r} = 0 \qquad (\mu = 2, \nu = 3) \tag{183}$$
$$B\xi^r_{,\phi} + r^2\sin^2\theta\xi^\phi_{,r} = 0 \qquad (\mu = 2, \nu = 4) \tag{184}$$
$$\frac{1}{r}\xi^r + \xi^\theta_{,\theta} = 0 \qquad (\mu = 3, \nu = 3) \tag{185}$$
$$\xi^\theta_{,\phi} + \sin^2\theta\xi^\phi_{,\theta} = 0 \qquad (\mu = 3, \nu = 4) \tag{186}$$
$$\frac{1}{r}\xi^r + \cot\theta\xi^\theta + \xi^\phi_{,\phi} = 0 \qquad (\mu = 4, \nu = 4) \tag{187}$$

We guess some solutions

$$\xi^\alpha_{(t)} = (1, 0, 0, 0) \rightarrow \partial_t \tag{188}$$
$$\xi^\alpha_{(\phi)} = (0, 0, 0, 1) \rightarrow \partial_\phi \tag{189}$$
$$\xi^\alpha_{(1)} = (0, 0, \sin\phi, \cos\phi\cot\theta) \rightarrow \sin\phi\partial_\theta + \cos\phi\cot\theta\partial_\phi \tag{190}$$
$$\xi^\alpha_{(2)} = (0, 0, \cos\phi, -\sin\phi\cot\theta) \rightarrow \cos\phi\partial_\theta - \sin\phi\cot\theta\partial_\phi \tag{191}$$

With the geodesic equation

$$u^\alpha_{;\beta}u^\beta = 0 \tag{192}$$
$$\rightarrow \quad (u^\alpha_{,\beta} + \Gamma^\alpha_{\beta\gamma}u^\gamma)u^\beta = 0 \tag{193}$$
$$\rightarrow \quad \underbrace{u^\alpha_{,\beta}u^\beta}_{\frac{\partial u^\alpha}{\partial x^\beta}\frac{\partial x^\beta}{\partial\lambda}} + \Gamma^\alpha_{\beta\gamma}u^\gamma u^\beta = 0 \tag{194}$$
$$\rightarrow \quad \frac{du^\alpha}{d\lambda} + \Gamma^\alpha_{\beta\gamma}u^\gamma u^\beta = 0 \tag{195}$$

we see with the Killing equation $\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0$

$$\xi_\alpha (u^\alpha_{\ ;\beta} u^\beta) = 0 \tag{196}$$
$$(\xi_\alpha u^\alpha)_{\ ;\beta} u^\beta - \xi_{\alpha;\beta} u^\alpha u^\beta = 0 \tag{197}$$
$$(\xi_\alpha u^\alpha)_{\ ,\beta} u^\beta - \xi_{\alpha;\beta} u^\alpha u^\beta = 0 \qquad (\xi_\alpha u^\alpha \text{ is a scalar}) \tag{198}$$
$$\frac{\partial(\xi_\alpha u^\alpha)}{\partial x^\beta}\frac{dx^\beta}{d\lambda} - \xi_{\alpha;\beta} u^\alpha u^\beta = 0 \qquad \text{symmetry of Killing equation} \tag{199}$$
$$\frac{d}{d\lambda}(\xi_\alpha u^\alpha) = 0 \tag{200}$$

which means $\xi_\alpha u^\alpha$ is constant along the geodesic. Therefore we find

$$L_1 = g_{\theta\theta}\xi^\theta_{(1)} u^\theta + g_{\phi\phi}\xi^\phi_{(1)} u^\phi \tag{201}$$
$$= r^2 \sin\phi \cdot \dot\theta + r^2 \sin^2\theta \cos\phi \cot\theta \cdot \dot\phi \tag{202}$$
$$= \text{const} \tag{203}$$
$$L_2 = r^2 \cos\phi \cdot \dot\theta - r^2 \sin^2\theta \sin\phi \cot\theta \cdot \dot\phi \tag{204}$$
$$= \text{const} \tag{205}$$
$$\to L_1 \sin\phi + L_2 \cos\phi = r^2\dot\theta \tag{206}$$
$$\to \dot\theta = \frac{1}{r^2}(L_1 \sin\phi + L_2 \cos\phi) \tag{207}$$
$$\to \dot\theta = \frac{\dot\phi \sin^2\theta}{L}(L_1 \sin\phi + L_2 \cos\phi) \tag{208}$$

From here we should?!? to conclude $\dot\theta = 0$ ... and therefore $\theta = \pi/2$=const

$$L = g_{\alpha\beta}\xi^\beta_{(\phi)} u^\alpha \tag{209}$$
$$= g_{\phi\phi} u^\phi \tag{210}$$
$$= r^2 \sin^2\theta \cdot \dot\phi \tag{211}$$
$$= r^2\dot\phi \qquad (\theta = \pi/2 = \text{const}) \tag{212}$$

and

$$E = g_{\alpha\beta}\xi^\beta_{(t)} u^\alpha \tag{213}$$
$$= g_{tt} u^t \tag{214}$$
$$= -\left(1 - \frac{r^2}{R^2}\right)\dot t \tag{215}$$

The conserved Hamiltonian is given by

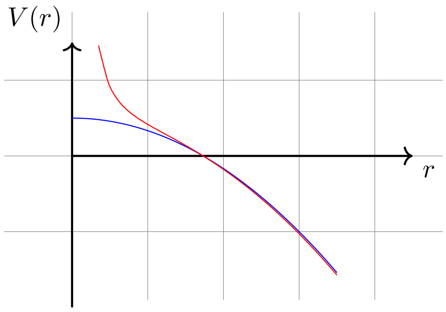
$$\mathcal{H} = g_{\mu\nu}\dot x^\mu \dot x^\nu \tag{216}$$
$$-1 = -\left(1 - \frac{r^2}{R^2}\right)\dot t^2 + \left(1 - \frac{r^2}{R^2}\right)^{-1}\dot r^2 + r^2(\dot\theta^2 + \sin^2\theta\,\dot\phi^2) \tag{217}$$
$$-1 = -\left(1 - \frac{r^2}{R^2}\right)\dot t^2 + \left(1 - \frac{r^2}{R^2}\right)^{-1}\dot r^2 + r^2\dot\phi^2 \tag{218}$$
$$-1 = -\left(1 - \frac{r^2}{R^2}\right)^{-1} E^2 + \left(1 - \frac{r^2}{R^2}\right)^{-1}\dot r^2 + \frac{L^2}{r^2} \tag{219}$$

which gives an ODE for $\dot{r}$

$$\left(1-\frac{r^2}{R^2}\right)^{-1}(\dot{r}^2-E^2)+\left(1+\frac{L^2}{r^2}\right)=0 \tag{220}$$

$$\dot{r}^2=E^2-\left(1-\frac{r^2}{R^2}\right)\left(1+\frac{L^2}{r^2}\right) \tag{221}$$

$$\dot{r}^2=E^2-\left(1-\frac{L^2}{R^2}+\frac{L^2}{r^2}-\frac{r^2}{R^2}\right) \tag{222}$$



3.) Small radial velocity means $E\approx 1$ and $L=0$

$$\dot{r}=\sqrt{E^2-1+\frac{r^2}{R^2}} \tag{223}$$

$$=\sqrt{E^2-1}\sqrt{1+\frac{r^2}{R^2(E^2-1)}} \tag{224}$$

$$\frac{\dot{r}}{\sqrt{E^2-1}}=\sqrt{1+\frac{r^2}{R^2(E^2-1)}} \tag{225}$$

$$\dot{y}=\sqrt{1+\frac{y^2}{R^2}}\qquad (y=r/\sqrt{E^2-1}) \tag{226}$$

Now set $\lambda=\tau/R$ then

$$\frac{\partial y}{\partial\lambda}=\frac{\partial y}{\partial\tau}\frac{\partial\tau}{\partial\lambda}=\frac{1}{R}\frac{\partial y}{\partial\tau} \tag{227}$$

and with $z=y/R$

$$\frac{\dot{y}}{R}=\sqrt{1+\frac{y^2}{R^2}} \tag{228}$$

$$z'=\sqrt{1+z^2} \tag{229}$$

with the solutions $z=\sinh(\lambda+c)$ and resubstitution we obtain

$$r(\lambda)=R\sqrt{E^2-1}\sinh\lambda/R \tag{230}$$

and

$$\Delta\lambda=R\cdot\text{arcsinh}\frac{1}{\sqrt{E^2-1}} \tag{231}$$

$$\frac{dr}{d\lambda} = \sqrt{E^2-1}\cosh\lambda/R \tag{232}$$
$$= \sqrt{E^2-1}\sqrt{1+\sinh^2\lambda/R} \tag{233}$$
$$= \sqrt{E^2-1}\sqrt{1+\frac{r^2}{R^2\sqrt{E^2-1}}} \tag{234}$$
$$\rightarrow t = \int_0^R \frac{dt}{d\lambda}d\lambda = \int dr\frac{-E}{1-\frac{r^2}{R^2}}\frac{1}{\sqrt{E^2-1+\frac{r^2\sqrt{E^2-1}}{R^2}}} \tag{235}$$

0.3.11 Problem 2.5 - Distances

Metric distance d_M, luminosity distance d_L

$$d_M = S_k(\chi) \tag{236}$$
$$d_L(z) = (1+z)d_M(z) \tag{237}$$
$$d_A(z) = \frac{d_M(z)}{1+z} \tag{238}$$

0.3.12 Problem 2.6 - Friedmann universes

0.3.13 Problem 2.7 - Einsteins biggest blunder

0.3.14 Problem 2.8 - The accelerating universe

0.3.15 Problem 2.9 - Phantom Dark energy

0.3.16 Exercise 3.1

Let's first rewrite the Zeta function as an integral starting with the common definitions

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt \tag{239}$$
$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} \tag{240}$$

Then with $t/n = x$ and $dx = dt/n$

$$\zeta(s)\Gamma(s) = \sum_{n=1}^\infty \int_0^\infty \frac{1}{n^s}t^{s-1}e^{-t}dt \tag{241}$$
$$= \sum_{n=1}^\infty \int_0^\infty \frac{1}{n^s}t^{s-1}e^{-t}n\,dx \tag{242}$$
$$= \sum_{n=1}^\infty \int_0^\infty \frac{t^{s-1}}{n^{s-1}}e^{-nx}dx \tag{243}$$
$$= \int_0^\infty x^{s-1}\sum_{n=1}^\infty e^{-nx}dx \tag{244}$$
$$= \int_0^\infty \frac{x^{s-1}}{e^x-1}dx \tag{245}$$

we obtain

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx. \tag{246}$$

Now

$$\begin{aligned} J_-(0) &= \int_0^\infty \frac{\xi^3}{e^\xi - 1} d\xi && (247)\\ &= \Gamma(4)\zeta(4) && (248)\\ &= 3!\zeta(4) && (249) \end{aligned}$$

Furthermore we can write

$$\begin{aligned} J_-(0) &= \int_0^\infty \frac{\xi^3}{e^\xi - 1} d\xi && (250)\\ &= \int_0^\infty \frac{\xi^3}{(e^{\xi/2} - 1)(e^{\xi/2} + 1)} d\xi && (251)\\ &= \frac{1}{2} \int_0^\infty \frac{\xi^3}{e^{\xi/2} - 1} d\xi - \frac{1}{2} \int_0^\infty \frac{\xi^3}{e^{\xi/2} + 1} d\xi && (252)\\ &= 8 \int_0^\infty \frac{x^3}{e^x - 1} dx - 8 \int_0^\infty \frac{x^3}{e^x + 1} dx && (253)\\ &= 8J_-(0) - 8J_+(0) && (254)\\ &\to J_+ = \frac{7}{8} J_-(0) && (255) \end{aligned}$$

0.4 Dodelson, Schmidt - Cosmology (2nd edition)

0.4.1 1.2

We start with

$$\rho_{\rm cr} = \frac{3H_0^2}{8\pi G} \tag{256}$$
$$H(t) = \frac{1}{a}\frac{da}{dt} \tag{257}$$
$$H(t)^2 = \frac{8\pi G}{3}\left[\varrho(t) + \frac{\Lambda}{3} - \frac{k}{a^2}\right] \tag{258}$$
$$= \frac{8\pi G}{3}\left[\rho(t) + \frac{\rho_{\rm cr} - \rho(t_0)}{a^2}\right] \tag{259}$$
$$= \frac{8\pi G}{3}\left[\Omega_m\left(\frac{a_0}{a}\right)^3\rho_{\rm cr} + \Omega_\Lambda\rho_{\rm cr} + \frac{\rho_{\rm cr} - \rho(t_0)}{a^2}\right] \tag{260}$$
$$= H_0^2\left[\Omega_m\left(\frac{a_0}{a}\right)^3 + \Omega_\Lambda + \frac{\rho_{\rm cr} - \rho(t_0)}{\rho_{\rm cr}a^2}\right] \tag{261}$$
$$\tag{262}$$

and assume $\rho_{\rm cr} = \rho(t_0)$ (same as Euclidean $k = 0$?!?) and $\Omega_\Lambda + \Omega_m = 1$ and $a_0 = 1$

$$dt = \frac{da}{a}\frac{1}{H(t)} \tag{263}$$
$$= \frac{da}{a}\frac{1}{H_0\sqrt{\Omega_m\left(\frac{a_0}{a}\right)^3 + \Omega_\Lambda}} \tag{264}$$
$$= \frac{1}{H_0}\frac{da}{a}\left[\frac{1-\Omega_\Lambda}{a^3} + \Omega_\Lambda\right]^{-1/2} \tag{265}$$

(a) Now with $\Omega_\Lambda = 0$

$$dt = \frac{1}{H_0}\frac{da}{a}a^{3/2} = \frac{1}{H_0}da\ a^{1/2} \tag{266}$$
$$\rightarrow t - t_i = \frac{2}{3H_0}(a^{3/2} - a_i^{3/2}) \tag{267}$$
$$\rightarrow a(t) = \left(\frac{3H_0}{2}(t-t_i) + a_i^{3/2}\right)^{2/3} \tag{268}$$

with $a(t=0) = 0$

$$a(t) = \left(\frac{3H_0}{2}t\right)^{2/3} \tag{269}$$
$$\rightarrow T = \frac{2}{3H_0} \tag{270}$$

(b) ...

0.4.2 1.3 Lyman-α splitting in hydrogen isotopes

The energy eigenvalues are

$$E_n = -\frac{1}{2}\mu c^2\frac{\alpha^2}{n^2} \tag{271}$$
$$= -\frac{1}{2}\frac{m_e M_{\rm nuc}}{m_e + M_{\rm nuc}}c^2\frac{\alpha^2}{n^2} \tag{272}$$

then

$$\begin{aligned}
\Delta E_{2\to1} &= -\frac{1}{2}\frac{m_e M_{\rm nuc}}{m_e + M_{\rm nuc}} c^2\alpha^2 \left(\frac{1}{2^2} - \frac{1}{1^2}\right) && (273)\\
&= \frac{3}{8}\frac{m_e M_{\rm nuc}}{m_e + M_{\rm nuc}} c^2\alpha^2 && (274)\\
&= \frac{3}{8}\frac{m_e M_{\rm nuc}}{M_{\rm nuc}(1 + m_e/M_{\rm nuc})} c^2\alpha^2 && (275)\\
&= \frac{3}{8}\frac{m_e}{1 + m_e/M_{\rm nuc}} c^2\alpha^2 && (276)
\end{aligned}$$

and

$$\begin{aligned}
\Delta E^{\rm D}_{2\to1} &= \frac{3}{8}\frac{m_e}{1 + m_e/2m_p} c^2\alpha^2 && (277)\\
\Delta E^{\rm H}_{2\to1} &= \frac{3}{8}\frac{m_e}{1 + m_e/m_p} c^2\alpha^2 && (278)\\
\to \Delta E^{\rm D}_{2\to1} &= \Delta E^{\rm H}_{2\to1}\frac{1 + m_e/m_p}{1 + m_e/2m_p} && (279)
\end{aligned}$$

and with $E = hc/\lambda$

$$\begin{aligned}
\lambda^{\rm D}_{2\to1} &= \frac{hc}{\Delta E^{\rm D}_{2\to1}} && (280)\\
&= \frac{hc}{\Delta E^{\rm H}_{2\to1}}\frac{1 + m_e/2m_p}{1 + m_e/m_p} && (281)\\
&= \lambda^{\rm H}_{2\to1}\frac{1 + m_e/2m_p}{1 + m_e/m_p} && (282)\\
&= \lambda^{\rm H}_{2\to1}\left(1 + \frac{m_e}{2m_p}\right)\left(1 - \frac{m_e}{m_p}\right) && (283)\\
&\simeq \lambda^{\rm H}_{2\to1}\left(1 - \frac{1}{2}\frac{m_e}{m_p}\right) && (284)\\
&= 1215.67\text{Å} && (285)
\end{aligned}$$

furthermore

$$\begin{aligned}
c\frac{\Delta\lambda}{\lambda} &= c\frac{\lambda^{\rm D}_{2\to1} - \lambda^{\rm H}_{2\to1}}{\lambda^{\rm H}_{2\to1}} && (286)\\
&= \left(1 - \frac{1}{2}\frac{m_e}{m_p}\right) && (287)\\
&= 0.999727c && (288)
\end{aligned}$$

0.4.3 1.4 Planck law for CMB

Insider hint $1\text{MJy} = 10^6\text{Jansky} = 10^6 \cdot 10^{-26}\text{J} \cdot \text{s}^{-1} \cdot \text{Hz}^{-1} \cdot \text{m}^{-2}$. We start with $c = \lambda\nu = 2\pi\nu/k$

$$I_\nu = \frac{4\pi\hbar\nu^3}{c^2}\frac{1}{e^{2\pi\hbar\nu/k_{\rm B}T} - 1} \qquad (289)$$

which has the unit energy per area (per frequency per time are cancelling)

$$\frac{\text{Js} \cdot \text{s}^{-3}}{\text{m}^2/\text{s}^2} = \text{J} \cdot \text{m}^{-2} \qquad (290)$$

then

$$\frac{I_\nu d\nu}{d\Omega} \qquad (291)$$

0.5 Mukhanov - Physical foundations of cosmology, 2005