Solutions - Christian Thierfelder

1 Quantum Field Theory II – Exercise sheet 2 (2025-05-14)

1.1 Exercise 1 - Berezin Integral

Let $\theta_i, i = 1, ..., N$, be complex Grassmann variables, i.e., they obey $\theta_i \theta_j = -\theta_j \theta_i$. We consider unitary transformations

$$\theta_i \to \theta'_i = U_i^{\ j} \theta_j, \qquad \text{where } \ UU^\dagger = 1 \tag{1}$$

where the unitarity condition reads in indices $U_i^{\ k} (U^\dagger)_k^{\ j} = U_i^{\ k} (U^*)^j_{\ k} = \delta_i^{\ j}$.

1. Invariance of the pairing under unitary transformations Complex conjugation raises and lowers indices, so that one should write θ_i^*. This means that the contraction of θ_i^* with a second set of complex Grassmann variables η_i, transforming as in (1), is invariant under unitary transformations. Verify this by showing that the pairing defined by

$$\langle \theta, \eta \rangle := (\theta^*)^T \eta = \theta_i^* \eta_i \tag{2}$$

is invariant.

2. Self-adjointness of Hermitian matrices with respect to the pairing

Show that a Hermitian $N \times N$ matrix $A = (A_i^{\ j})$ is self-adjoint with respect to the above pairing:

$$\langle \theta, A\eta \rangle = \langle A\theta, \eta \rangle. \tag{3}$$

Show that $\langle \theta, A\theta \rangle$ for self-adjoint A is real and bosonic.

3. Berezin integration and generating functional

Denoting the Berezin integration measure introduced in the lecture by

$$d^{2N}\theta \equiv d\theta^{*1} d\theta_1 \cdots d\theta^{*N} d\theta_N, \tag{4}$$

compute:

$$\int d^{2N}\theta \, e^{-\langle \theta, A\theta \rangle}. \tag{5}$$

Then generalize this to the generating functional:

$$Z[\eta, \eta^*] := \int d^{2N}\theta \, e^{-\langle \theta, A\theta \rangle + \langle \eta, \theta \rangle + \langle \theta, \eta \rangle}. \tag{6}$$

4. Two-point function under Gaussian integral

Compute:

$$\int d^{2N}\theta \, \theta_i \theta^{*j} \, e^{-\langle \theta, A\theta \rangle}. \tag{7}$$

Notation summary

$$U = U_i^{\ j} \tag{2}$$

$$U^\dagger = (U^\dagger)_i^{\ j} = (U^{T*})_i^{\ j} = (U^T)^{*i}_{\ \ j} = (U^*)^j_{\ i} \tag{3}$$

$$\to UU^\dagger = 1 \to U_i^{\ k} (U^\dagger)_k^{\ j} = U_i^{\ k} (U^*)^j_{\ k} = \delta_i^{\ j} \tag{4}$$

$$\to U^\dagger U = 1 \to (U^*)^j_{\ k} U_i^{\ k} = \delta^j_{\ i} \tag{5}$$

$$\to A = A^\dagger \to A_i^{\ j} = (A^*)^j_{\ i} \tag{6}$$

1. Now

$$\langle \theta', \eta' \rangle = \langle U\theta, U\eta \rangle \tag{7}$$

$$= (U_i^{\ k} \theta_k)^{*T} \, (U_i^{\ j} \eta_j) \tag{8}$$

$$= ((U^*)^i_{\ k} \theta^{*k})^T \, (U_i^{\ j} \eta_j) \tag{9}$$

$$= \theta^{*k} \delta_k^{\ j} \eta_j \tag{10}$$

$$= \theta^{*j} \eta_j \tag{11}$$

2. Now with $A = A^\dagger$ meaning $A_i^{\ j} = (A^*)^j_{\ i}$

   - Then

$$\begin{aligned}\langle\theta, A\eta\rangle &= \theta^{*j}(A\eta)_j &(12)\\ &= \theta^{*j}(A_j^{\ k}\eta_k) &(13)\\ &= (A_j^{\ k}\theta^{*j})\eta_k &(14)\\ &= ((A^*)^k_{\ j}\theta^{*j})\eta_k &(15)\\ &= (A\theta)^{*k}\eta_k &(16)\\ &= ((A\theta)^*)^T\eta &(17)\\ &= \langle A\theta, \eta\rangle &(18)\end{aligned}$$

   - We see (by splitting a complex Grassmann variable into a real and an imaginary part)

$$\begin{aligned}(\alpha\beta)^* &= [(\alpha_1 + i\alpha_2)(\beta_1 + i\beta_2)]^* &(19)\\ &= [(\alpha_1\beta_1 - \alpha_2\beta_2) + i(\alpha_1\beta_2 + \alpha_2\beta_1)]^* &(20)\\ &= (\beta_1\alpha_1 - \beta_2\alpha_2) - i(\beta_2\alpha_1 + \beta_1\alpha_2) &(21)\\ &= (\beta_1 - i\beta_2)(\alpha_1 - i\alpha_2) &(22)\\ &= (\beta_1 + i\beta_2)^*(\alpha_1 + i\alpha_2)^* &(23)\\ &= \beta^*\alpha^* &(24)\end{aligned}$$

   as well as (the anticommuting goes through the (linear) sum)

$$\begin{aligned}\langle\alpha, \beta\rangle &= \alpha^{*k}\beta_k &(25)\\ &= ((\alpha^{*k}\beta_k)^*)^* &(26)\\ &= (\beta^{*k}\alpha_k)^* &(27)\\ &= \langle\beta, \alpha\rangle^* &(28)\end{aligned}$$

   then using this results in $\langle A\theta, \theta\rangle = \langle\theta, A\theta\rangle = \langle A\theta, \theta\rangle^*$ implies $\langle\theta, A\theta\rangle$ is real.
   It is also bosonic (commutes with other Grassmann variables) - because

3. The Berezin integration is defined as

$$\int d\theta = 0, \qquad \int d\theta\,\theta = 1 \tag{29}$$

(we observe that the this rules actually look more like differentiation than integration). For an analytic function f which can be written as a finite series ($\theta_k^2 = 0$)

$$f(\theta_1, ..., \theta_n) = f^{(0)} + f_j^{(1)}\theta_j + f_{jl}^{(2)}\theta_j\theta_l + ... + f_{12...n}^{(n)}\theta_1\theta_2...\theta_n \tag{30}$$

with the graded Leibnitz rule

$$\frac{d}{d\theta_i}(\theta_k f) = f\delta_{ik} - \theta_k\frac{d}{d\theta_i}f \tag{31}$$

we obtain the interesting result

$$\int d\theta_k f = f_k^{(1)} + f_{kl}^{(2)}\theta_l - f_{lk}^{(2)}\theta_l + ... = \frac{d}{d\theta_k}f \tag{32}$$

meaning differentiation and integration regarding a Grassmann variable are identical. Then we see

$$\int d\theta_k d\theta_l f = -\int d\theta_l d\theta_k f \tag{33}$$

$$\int d\theta_n...d\theta_1 f = f^{(n)} \tag{34}$$

For a hermitian matrix A we can do the standard trick - performing a change of variables which diagonalizes A BUT the sheet did not explicitly make this restriction. So we need to try another way

With $f(\theta_1, ..., \theta_N, \theta^{*1}, ..., \theta^{*N}) = e^{-\langle\theta, A\theta\rangle}$

$$Z[0,0] = \int d^{2N}\theta\; e^{-\langle\theta, A\theta\rangle} \tag{35}$$

$$= \left(\prod_{k=1}^{N} \int d\theta^{*k} d\theta_k\right) \left(1 - \langle\theta, A\theta\rangle + \frac{1}{2!}\langle\theta, A\theta\rangle\langle\theta, A\theta\rangle - \frac{1}{3!}...\right) \tag{36}$$

$$= \left(\prod_{k=1}^{N} \int d\theta^{*k} d\theta_k\right) \left(1 - \theta^{*i} A_i^{\,j}\theta_j + \frac{1}{2!}(\theta^{*i} A_i^{\,j}\theta_j)(\theta^{*l} A_l^{\,m}\theta_m) - \frac{1}{3!}...\right) \tag{37}$$

the last term is the finite (see above) series is

$$f^{2N}\theta_1...\theta_N\theta^{*1}...\theta^{*N} = \frac{1}{N!}(\theta^{*i} A_i^{\,j}\theta_j)^N \tag{38}$$

$$= \frac{1}{N!}(\theta^{*i_1} A_{i_1}^{\,j_1}\theta_{j_1})...(\theta^{*i_N} A_{i_N}^{\,j_N}\theta_{j_N}) \tag{39}$$

$$= \frac{1}{N!}\epsilon_{i_1...i_N}\epsilon_{j_1...j_N}\theta^{*1}\theta_1...\theta^{*N}\theta_N A_{i_1}^{\,j_1}...A_{i_N}^{\,j_N} \tag{40}$$

$$= \frac{1}{N!}\epsilon_{j_1...j_N} A_1^{\,j_1}...A_N^{\,j_N}\theta^{*1}\theta_1...\theta^{*N}\theta_N \tag{41}$$

$$= \det A\,\theta^{*1}\theta_1...\theta^{*N}\theta_N \tag{42}$$

$$= \det A\,\theta^{*N}\theta_N...\theta^{*1}\theta_1 \tag{43}$$

as shown above - the integration over all $2N$ Grassmann variables is only survived by the last term - so

$$Z[0,0] = \det(A). \tag{44}$$

Now we can calculate

$$Z[\eta, \eta^*] = \int d^{2N}\theta\; e^{-\langle\theta, A\theta\rangle + \langle\eta, \theta\rangle + \langle\theta, \eta\rangle} \tag{45}$$

by completing the square (**now we require that A is also invertible**)

$$-\langle\theta, A\theta\rangle + \langle\eta, \theta\rangle + \langle\theta, \eta\rangle = -\langle(\theta - A^{-1}\eta), A(\theta - A^{-1}\eta)\rangle + \langle\eta, A^{-1}\eta\rangle \tag{46}$$

we can split the exponential and pull the η part in front of the integral - shifting (offset) of the integration variables does not change the result and we obtain with above

$$Z[\eta, \eta^*] = e^{\langle\eta, A^{-1}\eta\rangle}\int d^{2N}\theta\; e^{-\langle(\theta - A^{-1}\eta), A(\theta - A^{-1}\eta)\rangle} \tag{47}$$

$$= \det(A) e^{\langle\eta, A^{-1}\eta\rangle} \tag{48}$$

4. Being a reasonably lax with commuting of integral and derivative we can write

$$\int d^{2N}\theta\; \theta_i\theta^{*j} e^{-\langle\theta, A\theta\rangle} = -\frac{d}{d\eta^{*j}}\frac{d}{d\eta_i}\int d^{2N}\theta\; e^{-\langle\theta, A\theta\rangle + \langle\eta, \theta\rangle + \langle\theta, \eta\rangle}\Big|_{\eta_i = 0 = \eta^{*j}} \tag{49}$$

$$= -\frac{d}{d\eta^{*j}}\frac{d}{d\eta_i}\det(A) e^{\langle\eta, A^{-1}\eta\rangle}\Big|_{\eta_i = 0 = \eta^{*j}} \tag{50}$$

$$= -\det(A)\,\frac{d}{d\eta^{*j}}\frac{d}{d\eta_i}(1 + \langle\eta, A^{-1}\eta\rangle)\Big|_{\eta_i = 0 = \eta^{*j}} \tag{51}$$

$$= -\det(A)(A_{ij}^{-1}) \tag{52}$$