Appendix C Notes

C.1 Theoretical Background

- Any signal can be written as $$y(t) = \sum_{j=1}^{5} y_j sin(2\pi f_j t + \phi_j)$$ which expresses the summation of sin waves for a function y(t), though most signals are more complicated, yet it guarantees the existence
- To use both sines and cosines, it can be expressed as
  $$y(t) = \int_{-\infty}^{\infty} Y(f) e^{-2\pi i f t} df = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\frac{\omega}{2\pi}) e^{-i\omega t} d\omega$$
  This equation being a fourier transform making it complex where y(t) is a function of time and Y(f) is the transform in a frequency domain
- These equations can also be expressed in a forward and inverse transform
  - A forward transform, when followed by an inverse transform, will result in the original function
  $$Y(f) = \int_{-\infty}^{\infty} y(t) e^{2\pi i f t} dt = \int_{-\infty}^{\infty} y(t) e^{i\omega t} dt$$
- Using this transform we can separate a particular signal which is whats used in high fidelity systems

C.2 Discrete Fourier Transform

- To actually compute the transform, if we are given a function y(t), we can find Y(f) by specifically analyzing the given function
- Numerically, we are never given the analytic form of a signal but we typically know the amplitude at discrete times such that
  $$y_m = \frac{1}{N} \sum_{n=0}^{N-1} Y_n \, e^{-2\pi i m n/N}$$
  $$Y_n = \sum_{m=0}^{N-1} y_m \, e^{2\pi i m n/N} \, ,$$
- Where index m in y relates to discrete times, $t_m = m\Delta t$
  The index n in Y relates the discrete frequencies, $f_n = \frac{n}{N\Delta t}$
  Both of which go from 0 to N-1 where N is the number of data points

- Therefore the transforms can be related using $\sum_{n=0}^{N-1} e^{2\pi i n \frac{(m-m')}{N}} = N\delta_{m,m'}$ where $\delta_{m,m'}$ equals 1 if m = m', otherwise it equals 0
- Represented is the Kronecker delta function which proves the previous statement where performing a forward transform, then an inverse transform returns the original function
- Since the data points are in the same collection, we can represent it either in the time domain of y_m or the frequency domain Y_n as they are equivalent
- That being said, y_m and Y_n should have the same number of data points but because Y_n is complex, it may appear that there are 2 times the number of y_m
    - Considering that there are real and imaginary parts of Y_n that are needed, there are still N pieces of information where the real part of the transform gives y_m correctly
    - As for the imaginary part, it comes out to be nonzero which is incorrect, showing that N pieces in the frequency domain cant correctly reconstruct 2N pieces of information in the time domain
- For real functions y_m, where $Y_n \geq \frac{N}{2}$ are redundant, making the highest frequency Fourier component $Y_{N/2-1}$ $n \geq \frac{N}{2}$ are redundant, making the highest frequency Fourier component $Y_{N/2-1}$
    - This is known as the Nyquist frequency, $f = \frac{N/2-1}{N\Delta t} \approx \frac{1}{2\Delta t}$ meaning that if a signal is measured in intervals of Δt, then the spectral components can be recovered with a Fourier transform and have a frequency below the Nyquist frequency
    - A sine wave at the Nyquist frequency would only be sampled twice during each period of oscillation

C.3 Fast Fourier Transform (FFT)

- There are specific algorithms that evaluate transformations at a major time reduction as brute forcing it, even with a very powerful computer is very resource intensive and therefore takes a long time for the typical values
    - Consider a small set of data points where you separated values of m into groups of even and odd values:
    $$\begin{aligned} Y_n &= Y_n^e + w^n Y_n^o \\ &= \sum_{m'=0}^{3} y_{2m'} \, w^{2m'n} + w^n \sum_{m'=0}^{3} y_{2m'+1} \, w^{2m'n} \end{aligned}$$
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    - Y^e_n uses y_m where the original indices of m has the least significant binary bit value equal to 0 meanwhile the odd part, m equals 1

- The significance is the amount of calculations needed for the Fourier transform, Y_n, where the number of operations proportional to N where there are $\log_2 N$ levels
- For example, given N=8 transforms, then it can be done in 24 steps (where $\log_2 N = 3$ therefore $N \log_2 N = 24$) instead of 64 steps (N^2)
- This shows that as N increases, the bigger the difference in steps
- The process of FFT finds the power and bit reverses the indices of the data, looping through indices starting from 1 through the power of the even-odd decompositions
- Beginning with the final decomposition of the individual y_m through the first decomposition of the even-odd values

C.4 Sampling Interval & Number of Data Points

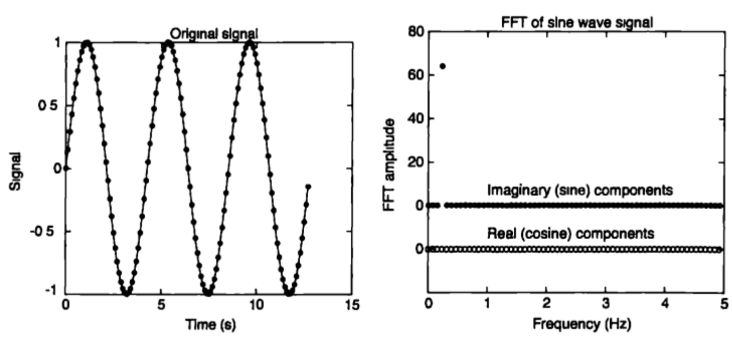
- Two parameters are under control in a Fourier transform
  - Sampling Interval
    - Determines range of spectral frequencies to be represented
  - Number of points to sample
    - Determines the details to be obtained
  - Sampling duration also corresponds to the whole periods of the signal and if the number of data points match the specific FFT algorithm designed for it
- Best example is a pure sine wave where the signal period is 4.3 seconds with a chosen 0.1 second sampling period over 12.8s, the total recorded signal spans three complete periods
  - Results in 128 Fourier amplitudes, half of which are sine, the other half cosine, all of which result in 0 except the initial sin wave
  - This FFT result says that there is a signal composed of one Fourier component to a sine wave with a frequency of 0.23 Hz
  - 

FIGURE C.2: Left Pure sine wave signal, $y(t) = \sin(2\pi f t)$ These 128 points are the values of the signal that are Fourier analyzed Right FFT of this signal The real (cosine) and imaginary (sine) parts of the transform are shown separately We have plotted them as discrete points to emphasize that the number of Fourier amplitudes yielded by FFT is equal to the number of original data points Hence we have 64 cosine and 64 sine components Only one of the points in the transform is nonzero

- If we take the same signal but shift it by a phase factor of $\frac{\pi}{4}$, sampling 128 times at 0.1 s intervals, the FFT comes out more complicated

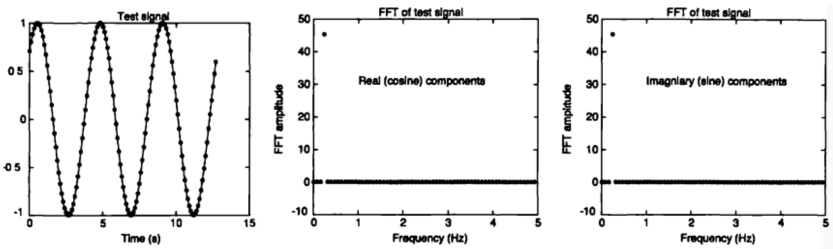


FIGURE C.3: Left Test signal that is just a sine wave that is phase shifted by $\pi/4$ The dots are the data points that were used in the FFT, center and right FFT of this test signal

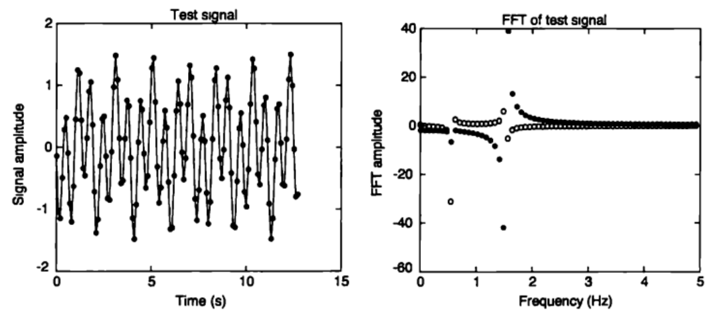


FIGURE C.4: Left Test signal which is the sum of two sine waves with different frequencies, amplitudes, and phases, right FFT of this test signal The filled circles show the real (cosine) components, while the open circles are the imaginary (sine) amplitudes

  - Now we see that there is one instance of a nonzero component for both sine and cosine, therefore writing it as a sum of the two as they are needed to describe a signal using some arbitrary phase factor

C.5 Aliasing

- Aliasing is the folding back of the frequencies greater than the Nyquist frequency, resulting in higher frequencies being represented as lower frequencies causing inaccuracies

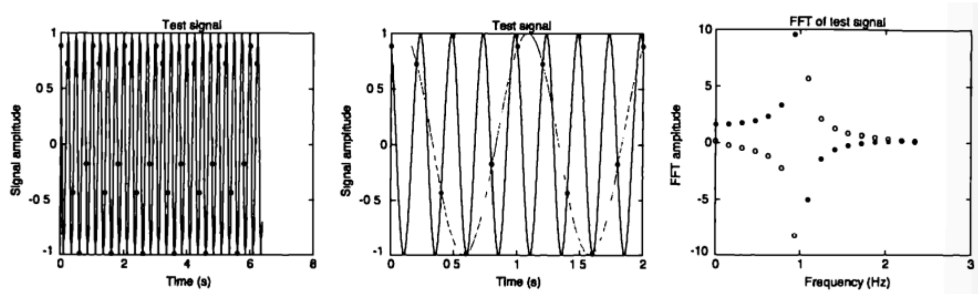


FIGURE C.6: Left test signal that is a single sine wave The solid curve is what the signal would look like if it were sampled at a very large number of points, while the solid symbols show the 32 data points that were analyzed in the FFT Center an expanded view of the signal The dotted curve shows a second sine wave, which has a much lower frequency (1 Hz) and which is seen to also pass through all of the data points Right FFT. The filled circles are the real (cosine) components, while the open circles are the imaginary (sine) components

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- Its generally more preferable to arrange for the Nyquist frequency to be higher than any of the Fourier components that are expected to be present
    - Typically done by using filters, usually low pass filters to remove the higher frequencies to remove aliasing

C.6 Power Spectrum

- Another form of displaying an FFT is the power spectrum, or a function y(t) defined as the Fourier transform of its autocorrelation
    - Autocorrelation: $Corr[y](r) = \int_{-\infty}^{\infty} y(t) * y(t+r)dr$
        - Measures how well the signal y is correlated across times separated by periodicity of r
        - If there are peaks at certain points of r, then the Fourier transform, the power spectrum PS[y](f) also have peaks at the conjugate or corresponding values of frequencies f = 1/r
    - Power Spectrum: $PS[y](f) = \int_{-\infty}^{\infty} y(t) * y(t+r)e^{2\pi ifr}dr = |Y(f)|^2$
        - In a practical sense, suppose y(t) is an electric signal where the voltage is a function of time across a resistor. The average power dissipated in the resistor over a long interval of time is described as

$$P = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |y(t)|^2 dt .$$

Since

$$\int_{-\infty}^{\infty} |Y(f)|^2 df = 2\pi \int_{-\infty}^{\infty} |y(t)|^2 dt ,$$

- The power spectrum can be seen visualized in the graph below, relating to figure C.4 (figure just above section C.5)

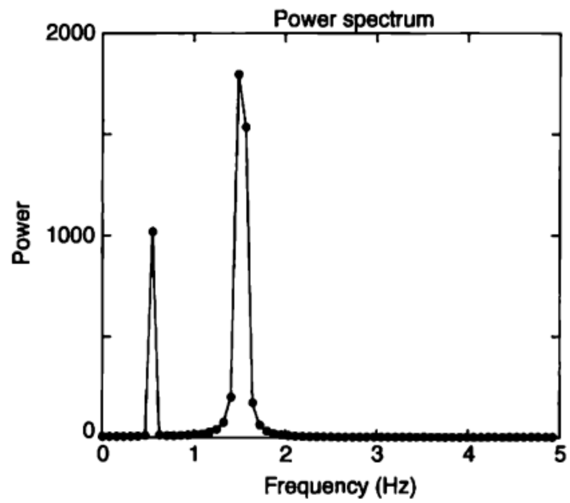

FIGURE C.7: Power spectrum of signal in Figure C 4 The solid symbols are the calculated values of the power

- We see the average power dissipated per unit of frequency interval around f is proportional to $|Y(f)|^2$
- Now assigning the real and imaginary components we get: $P_j = Y_j(\text{real})^2 + Y_j(\text{imaginary})^2$
- Where the real components are the amplitudes of cosine and the imaginary are the amplitudes of sine at f_j
- In the graph there are two peaks proportional to the squares using the corresponding Fourier amplitudes where the phase information is disregarded since its relative magnitude cannot be determined from power alone
- That being said the power spectrum alone cannot reconstruct the original signal