# Chapter 4 : The Solar System

## 4.1 Kepler's Laws

- We must consider the force of gravity of planets acting on each other within the solar system : $F_G = \frac{G M_S M_E}{r^2}$
- We can assume from this equation that the Sun's mass is sufficiently large enough to where the motion can be neglected

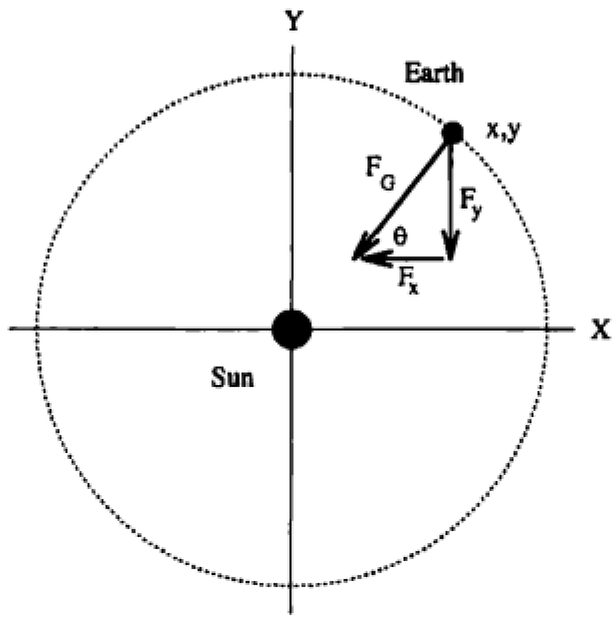


**FIGURE 4.1: Coordinate system for describing the motion of Earth in orbit around the Sun. The Sun is at the origin and Earth is located at coordinates $(x, y)$.**

- From Newton's Second Law, we can express the x and y components of the force as $\frac{d^2x}{dt^2} = \frac{F_{G,x}}{M_E}$ and $\frac{d^2y}{dt^2} = \frac{F_{G,y}}{M_E}$

  Therefore, we can rewrite the original equation: $F_{G,x} = -\frac{G M_S M_E x}{r^3}$ and $F_{G,y} = -\frac{G M_S M_E y}{r^3}$
- Its important to note that the negative sign represents the force being in the direction of the sun
- We can now also write the second-order differential equations as: $\frac{dv_x}{dt} = -\frac{G M_S x}{r^3}$ and $\frac{dv_x}{dt} = -\frac{G M_S x}{r^3}$ where $\frac{dx}{dt} = v_x$ and $\frac{dy}{dt} = v_y$
- The units used must also be taken into consideration as the radius of Earth's orbit is about $1.5 \times 10^{11}$ meters or 1 au. Then we can use 1 year as the time it takes to complete one orbit

- Now to derive our system of units: $\frac{M_E v^2}{r} = F_G = \frac{G M_S M_E}{r^2}$

  $G M_S = v^2 r = 4\pi^2 \frac{au^3}{yr^2}$
- Using these equations we can build our computational solutions using the Euler-Cromer method that way we can utilize conservation of energy:

  $v_x[i] = v_x[i - 1] - \frac{4\pi^2 x[i-1]}{r^3[i-1]}\Delta t$

  $v_y[i] = v_y[i - 1] - \frac{4\pi^2 y[i-1]}{r^3[i-1]}\Delta t$

  $x[i] = x[i - 1] + v_x[i]\Delta t$

  $y[i] = y[i - 1] + v_y[i]\Delta t$
- When plotting planetary motion, there are many small details that are worth paying attention to as they can affect aspects of the planet's orbit. Even something like the screen aspect ratio can cause the orbit to appear more circular than it truly is
- Then looking at Kepler's third law, $\frac{T^2}{a^3}$, which represents a constant which should help confirm non circular orbits

# 4.2 Inverse Square Law and Stability of Orbits

- There is a slight issue concerning some analytic results. The trajectory of these orbits assume there is only the Sun and the body being plotted
- To fix some of this, we can define the max velocity at perihelion and minimum velocity at aphelion

  $v_{max} = \sqrt{GM_S}\sqrt{\frac{(1+e)}{a(1-e)}\left(1 + \frac{M_P}{M_S}\right)}$ and $v_{min} = \sqrt{GM_S}\sqrt{\frac{(1-e)}{a(1+e)}\left(1 + \frac{M_P}{M_S}\right)}$

  e represents the eccentricity of the orbit, or how circular it is, with e=0 being the most circular, a represents the focus at the origin
- We can use these equations as they follow the inverse-square functional form of gravitational force
- Suppose the gravitational force has the form: $F_G = \frac{G M_S M_E}{r^\beta}$ where β represents some inverse law, such that β=2 is the inverse square and β=3 would be the inverse cube law

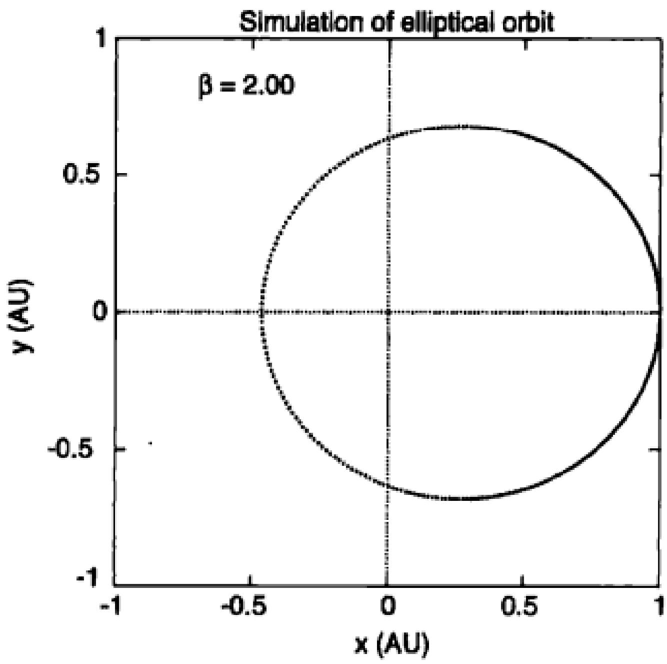


**FIGURE 4.4: Elliptical orbit calculated for a force law with $\beta = 2$. The time step here, and in all of the calculations shown in this section, was 0.001 yr We also used the same initial conditions in all of these simulations The sun is at the origin.**

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- By changing this value, we can better improve the accuracy of the computations

# 4.3 Precession of Perihelion of Mercury

- Considering several planets have near circular orbits, Mercury and Pluto deviate the most and how Mercury's axes of its ellipse that describes its orbit rotate with time
- This shows a deviation from Kepler's law showing that there is a precession within some of the orbits of planets, most of which is due to Jupiter being so massive
- Going by general relativity, the force law for Mercury can be described as $F_G = \frac{G M_S M_M}{r^2}(1 + \frac{\alpha}{r^2})$
- To determine numerical errors, the points will deviate from the ideal result, therefore fitting a line to noisy data

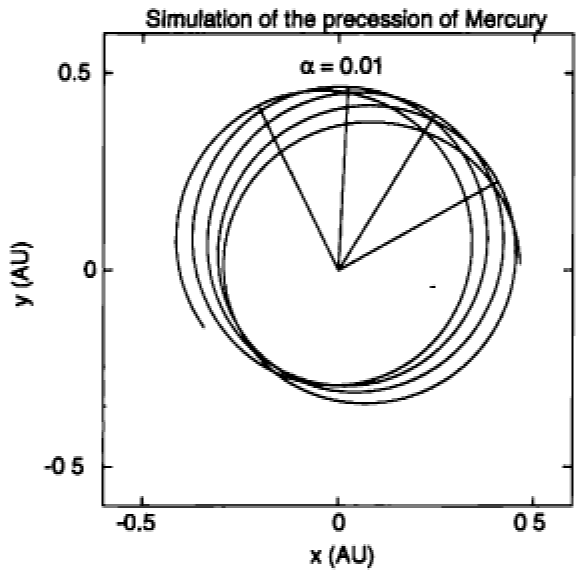


**FIGURE 4.8:** Simulated orbit for Mercury orbiting the Sun. The force law (4.13) was used, with $\alpha = 0.01$ The time step was 0.0001 yr The program was stopped after several orbits The solid lines emanating from the Sun (i e , the origin) are drawn to the points on the orbit that are farthest from the Sun, so as to show the precession of the orientation of the orbit

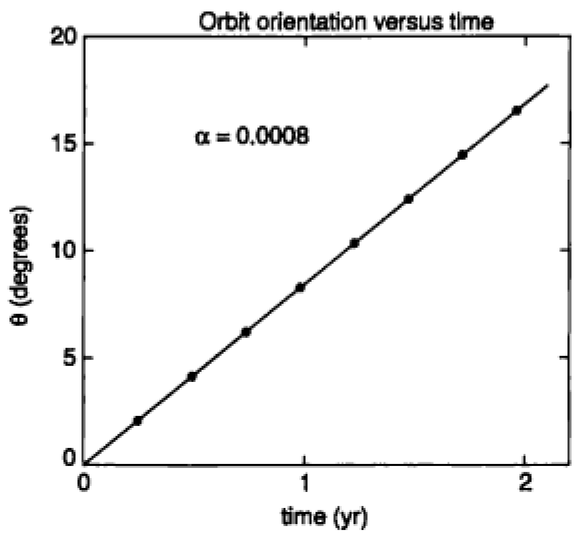


**FIGURE 4.9:** Precession of the axis of Mercury's orbit as a function of time, calculated for $\alpha = 0.0008$. The time step was 0.0001 yr. The solid line is a least-squares fit

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- Repeating the process with different values of $\alpha$, to find the best-fit precession rate
- Through the use of extrapolation, it deals with effects of interest too small to directly estimate using a numerical approach. The other method being the least squares

# 4.4 Three-Body Problem

- The problem now is having multiple planetary bodies interacting with each other within the calculation. Thats because Jupiter has some gravitational effect on Earth's orbit, so our equation for the force can be changed to be between Earth and Jupiter:

$$F_{E,J} = \frac{G\,M_J\,M_E}{r_{EJ}^2}$$

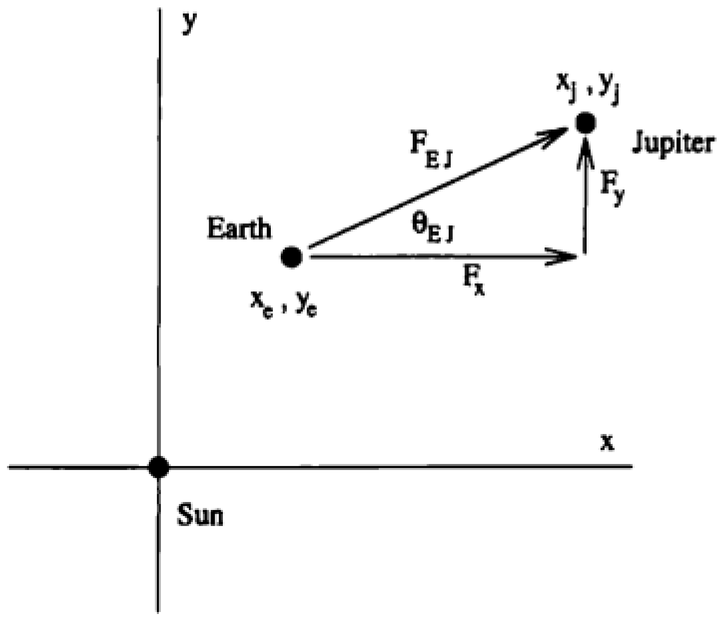


**FIGURE 4.11: Components of the gravitational force due to Jupiter, located at $x_j, y_j$, with Earth at $x_e, y_e$ The Sun is at the origin.**

- The equation must then be broken down into its x and y components

$$F_{EJ,x} = -\frac{G\,M_J\,M_E}{r_{EJ}^2}\cos\theta_{EJ} = -\frac{G\,M_J\,M_E\,(x_e - x_j)}{r_{EJ}^3}$$

- Converting this to a differential equation we get:

$$\frac{dv_{x,e}}{dt} = -\frac{G\,M_S\,x_e}{r^3} - \frac{G\,M_J\,(x_e - x_j)}{r_{EJ}^3}$$

- Notice that we still use the distance, r, representing the distance from Earth to the Sun. This is due to the fact that we still take into account the Sun with the added effect of Jupiter
- We can also see the effect of increasing the mass of Jupiter to see what happens to Earth's orbit as a result. Because Jupiter would gain more gravitational pull, it would

cause Earth to leave it’s standard orbit

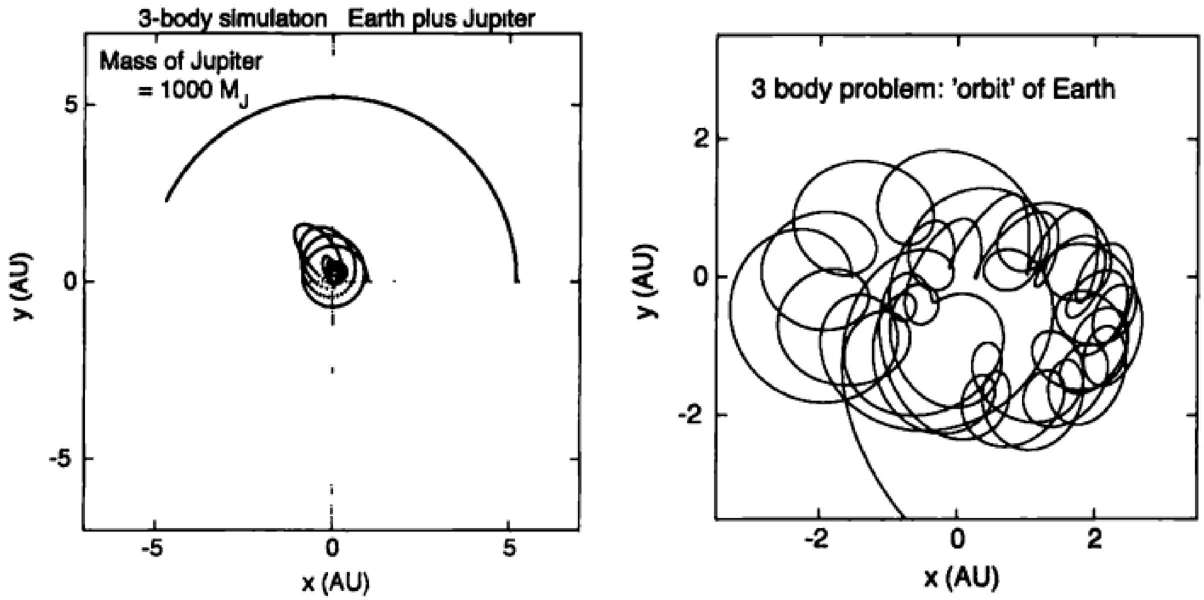


**FIGURE 4.13:** Simulation of a solar system with two planets, Earth and Jupiter Left: the mass of Jupiter has been set to 1000 times its true mass, and we have used the routine `jupiter-earth`, which does *not* take into account the motion of the Sun Here we stopped the simulation before Jupiter had completed even half an orbit, as the motion of Earth was unstable Right: Typical results for a true 3-body simulation, in which the motions of Earth, Jupiter, and the Sun were all computed. Here we show only the motion of the Earth. The origin is now the center of mass of the 3-body system.

- This process can be continued further to add the other planets, increasing the general accuracy of the model