Oscillatory Motion and Chaos

Simple Harmonic Motion - best described by a pendulum

- Perpendicular force is described as $F_\theta =- mg \sin(\theta)$ where the minus represents force is in opposite direction

- From Newton's second law, we know F=ma along a circular arc in a projectile's trajectory: $F_\theta = m\frac{d^2s}{dt^2}$ where the displacement along the arc is $s = l * \theta$. Then assuming theta is always small s.t. sin(theta) = 0, we can obtain the equation of motion: $\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta$
- Then by the general solution $\theta = \theta_0 \sin(\Omega t + \phi)$ where $\Omega = \sqrt{\frac{g}{l}}$ as well as initial theta and phi are constants dependent on initial displacement, velocity and momentum
- Basic equation of motion as first order DE: $\frac{d\omega}{dt} = -\frac{g}{l}\theta \ \ and \ \ \frac{d\theta}{dt} = \omega$

  Applying these we get $\omega_{i+1} = \omega_i - \frac{g}{l}\theta_i\Delta t \ \ and \ \ \theta_{i+1} = \theta_i + \omega_i\Delta t$
- Though an issue still exists with the Euler method as energy is still not conserved
- Euler-Cromer method utilizes the previous values of omega and theta to calculate the new value of omega but the new value of omega is then used to calculate the new theta
- Adding friction: damping force proportional to velocity
- q represents strength of dampening: $\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta - q\frac{d\theta}{dt}$
- Adding an electric field as a driving force: $\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta - q\frac{d\theta}{dt} + F_D\sin(\Omega_D t)$

  This new force adds energy into (or out of) the system and the externally imposed frequency will compete with the frequency of the pendulum
- To then analytically solve: $\theta(t) = \theta_0 \sin(\Omega_D t + \phi)$

  Where $\theta_0 = \frac{F_D}{\sqrt{(\Omega^2 - \Omega_D{}^2)^2 + (q\Omega_D)^2}}$
- In the event the driving frequency Ω_D is equal to that of the natural frequency Ω, resonance occurs and the amplitude becomes large, even more so if the friction is small

- Putting everything together, the equation of motion: $\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta - q\frac{d\theta}{dt} + F_D\sin(\Omega_D t)$

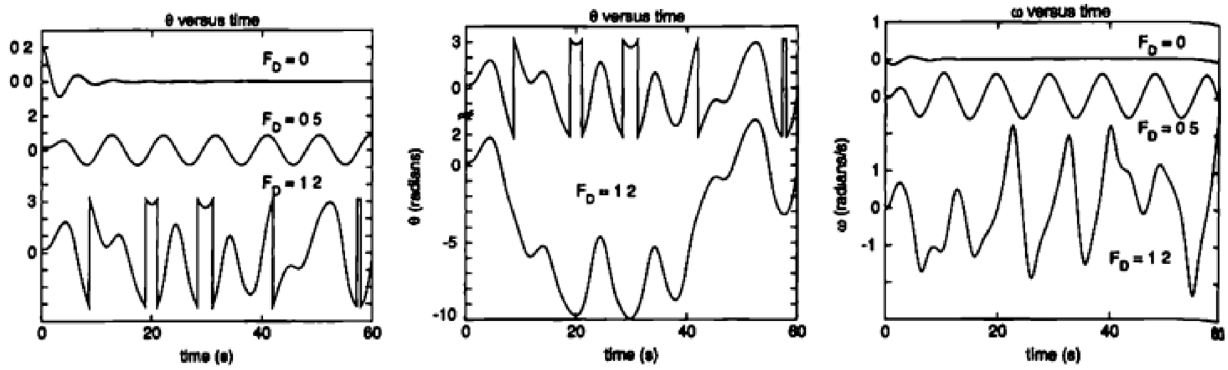
  $\omega_{i+1} = \omega_i - [(\frac{g}{l})sin(\theta_i) - q\omega_i + F_D\sin(\Omega_D t)]\Delta t$

  $\theta_{i+1} = \theta_i + \omega_{i+1}\Delta t$

  $t_{i+1} = t_i + \Delta t$

  Adding or subtracting 2*pi will keep theta in range

- The behavior can change drastically with the driving force, making it no longer simple even with longer times, resulting in Chaotic behavior



- If data is unpredictable and extremely sensitive to initial conditions, then its labeled chaotic
- However by plotting omega as a function of theta instead of theta as a function of t, that can allow for a more straightforward plot
- Chaotic plots have fractal structure (strange attractors) or a fuzziness