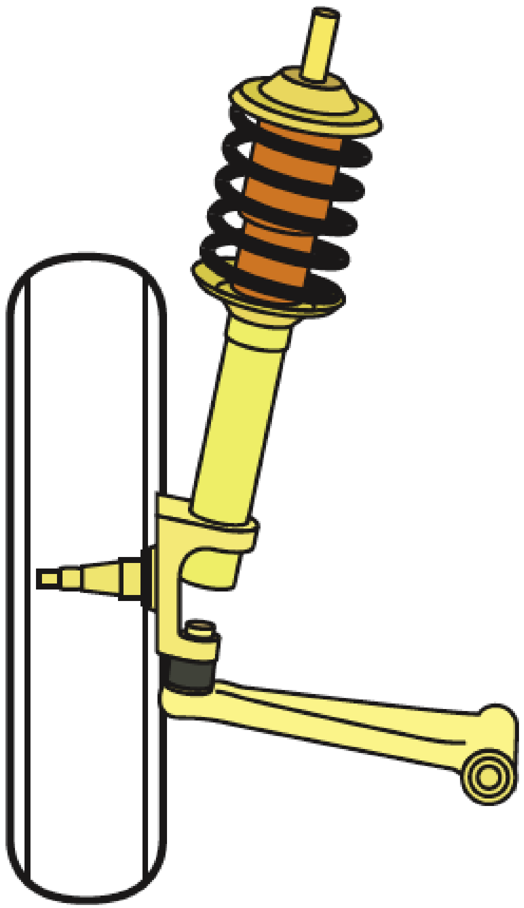
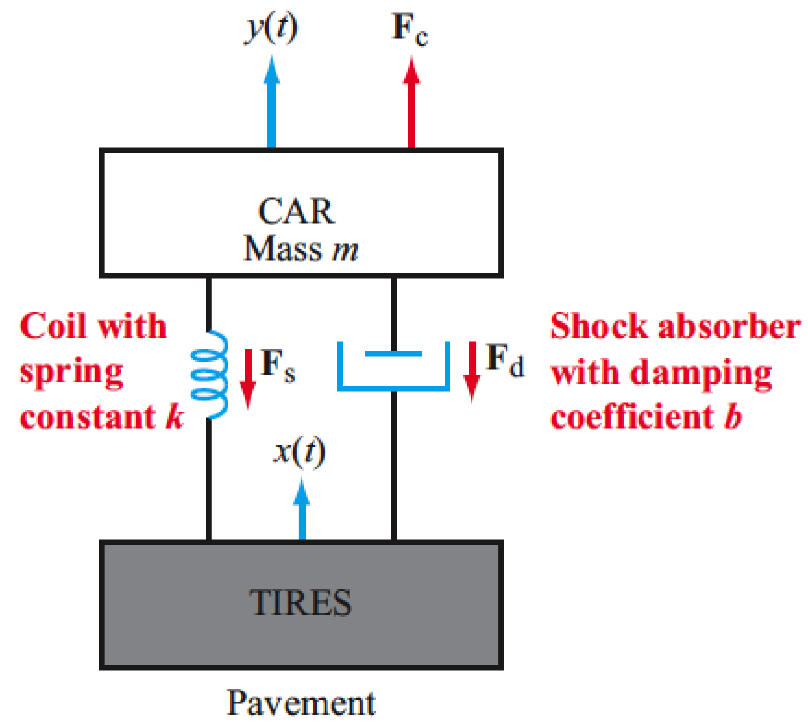


(a) Damping system



(b) Model

2. LTI SYSTEMS

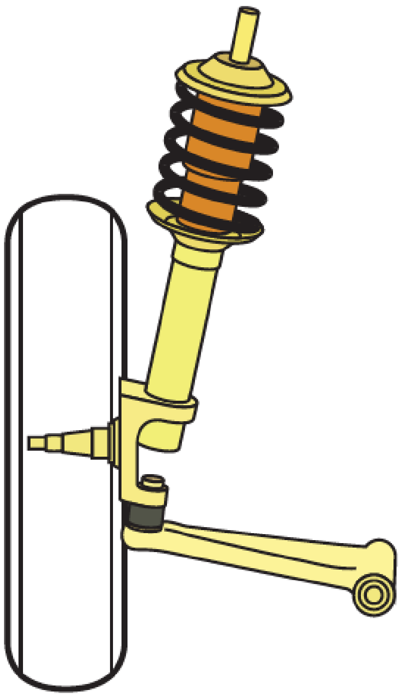
2 Linear Time-Invariant Systems

Contents

Objectives

Learn to:

- Describe the properties of LTI systems.
- Determine the impulse and step responses of LTI systems.
- Perform convolution of two functions.
- Determine causality and stability of LTI systems.
- Determine the overdamped, underdamped, and critically damped responses of second-order systems.
- Determine a car's response to various pavement profiles.



By modeling a car suspension system in terms of a differential equation, we can determine ***the response*** of the car's body to any pavement profile it is made to drive over. The same approach can be used to compute the response of any ***linear system*** to any ***input excitation***. This chapter provides the language, the mathematical models, and the tools for characterizing linear, time-invariant systems.

LTI Systems

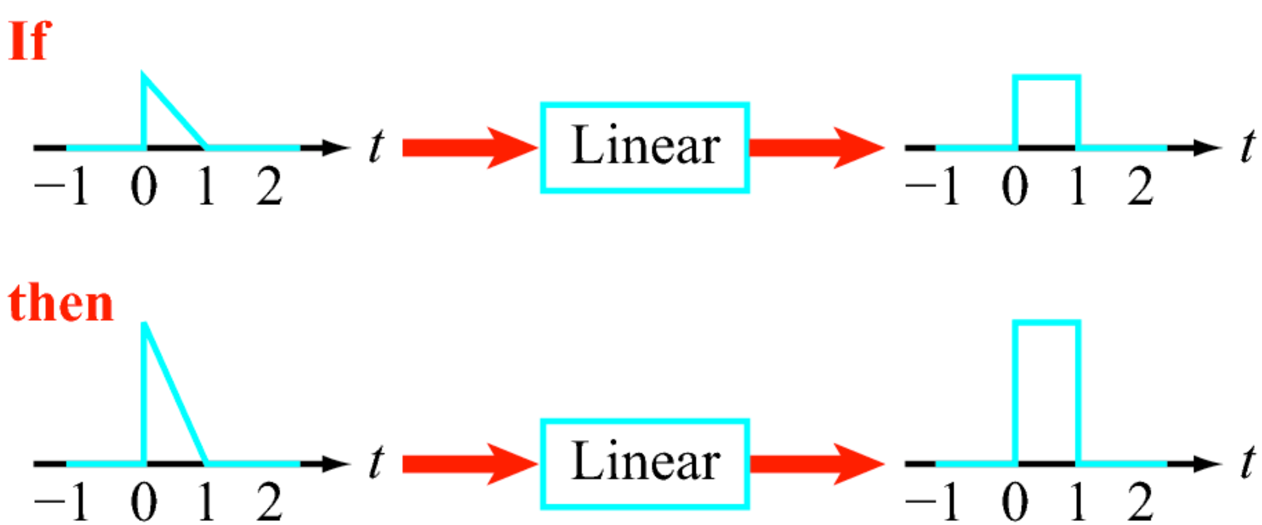
- A system is LTI if it is linear and time invariant.
- A system is linear if it has the scaling + additivity properties.

Scaling Property

Given: $x(t) \rightarrow$ System $\rightarrow y(t)$,

then the system is ***scalable*** (has the scaling property) if

$c\,x(t) \rightarrow$ System $\rightarrow c\,y(t)$.



Scaling Property (cont.)

- Example 1: System described by Diff. Eq.

$$\frac{d^2y}{dt^2} + 2\,\frac{dy}{dt} + 3y = 4\,\frac{dx}{dt} + 5x. \qquad (2.1)$$

Upon replacing $x(t)$ with $c\,x(t)$ and $y(t)$ with $c\,y(t)$ in all terms, we end up with

$$\frac{d^2}{dt^2}\,(cy) + 2\,\frac{d}{dt}\,(cy) + 3(cy) = 4\,\frac{d}{dt}\,(cx) + 5(cx).$$

Since c is constant, we can rewrite the expression as

$$c\left[\frac{d^2y}{dt^2} + 2\,\frac{dy}{dt} + 3y\right] = c\left[4\,\frac{dx}{dt} + 5x\right], \qquad (2.2)$$

which is identical to the original equation, but multiplied by the constant c. Hence, since the response to $c\,x(t)$ is $c\,y(t)$, the system is scalable and has the scaling property.

Scaling Property (cont.)

- Example 2: System described by Diff. Eq.

$$\frac{d^2y}{dt^2} + 2\,\frac{dy}{dt} + 3y = 4\,\frac{dx}{dt} + 5x + 6$$

Because of the constant term on the far right-hand side of the equation, this system is NOT scalable, and therefore NOT linear.

Additivity Property

If the system responses to N inputs $x_1(t), x_2(t), \ldots, x_N(t)$ are respectively $y_1(t), y_2(t), \ldots, y_N(t)$, then the system is ***additive*** if

$$\sum_{i=1}^{N} x_i(t) \longrightarrow \boxed{\text{System}} \longrightarrow \sum_{i=1}^{N} y_i(t). \qquad (2.4)$$

That is, *the response to the sum is the sum of the responses*.

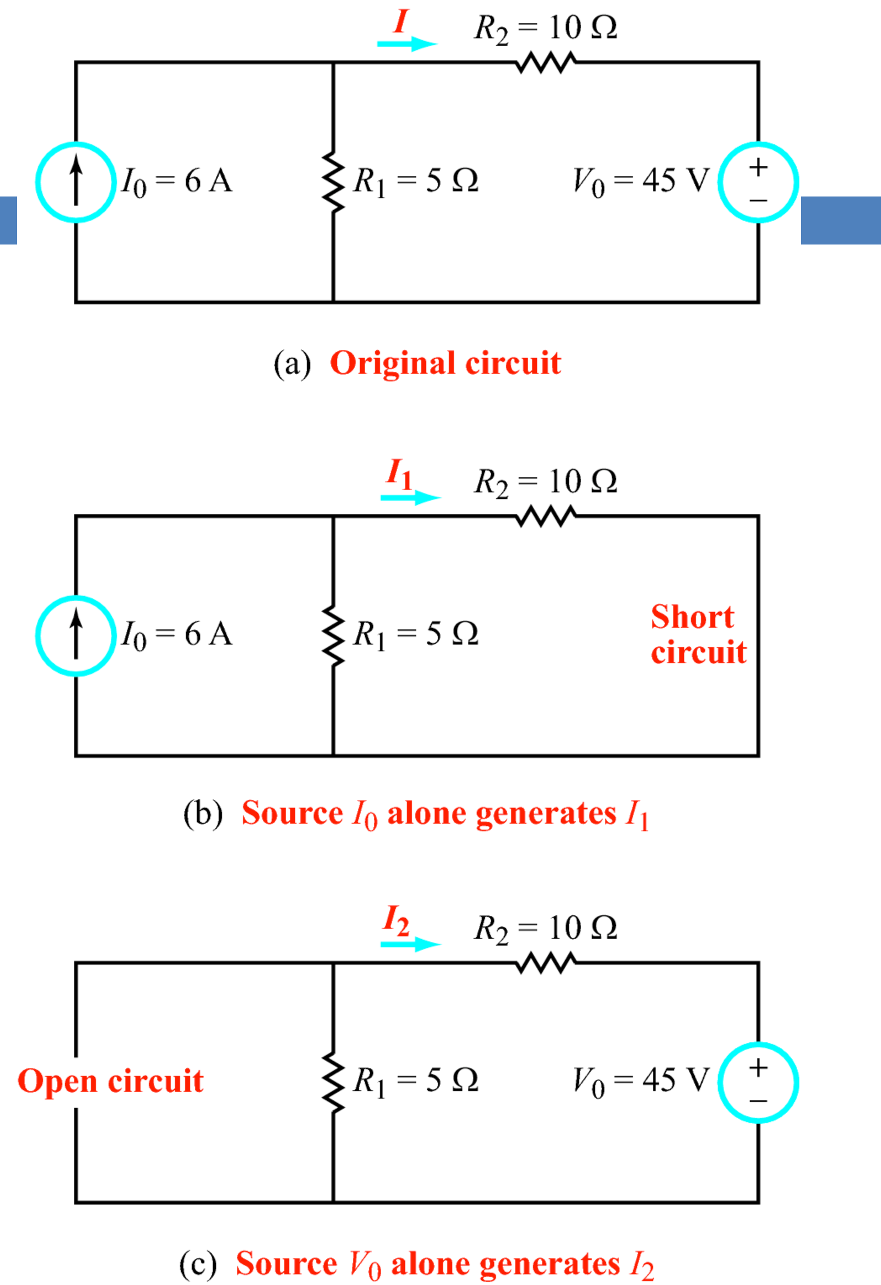
▶ The combination of scalability and additivity is also known as the ***superposition principle***. ◀

Superposition Example

$$I_1 = \frac{I_0 R_1}{R_1 + R_2} = \frac{6 \times 5}{5 + 10} = 2\,\text{A}.$$

$$I_2 = \frac{-V_0}{R_1 + R_2} = \frac{-45}{5 + 10} = -3\,\text{A}.$$

$$I = I_1 + I_2 = 2 - 3 = -1\,\text{A}.$$



Time-Invariant Systems

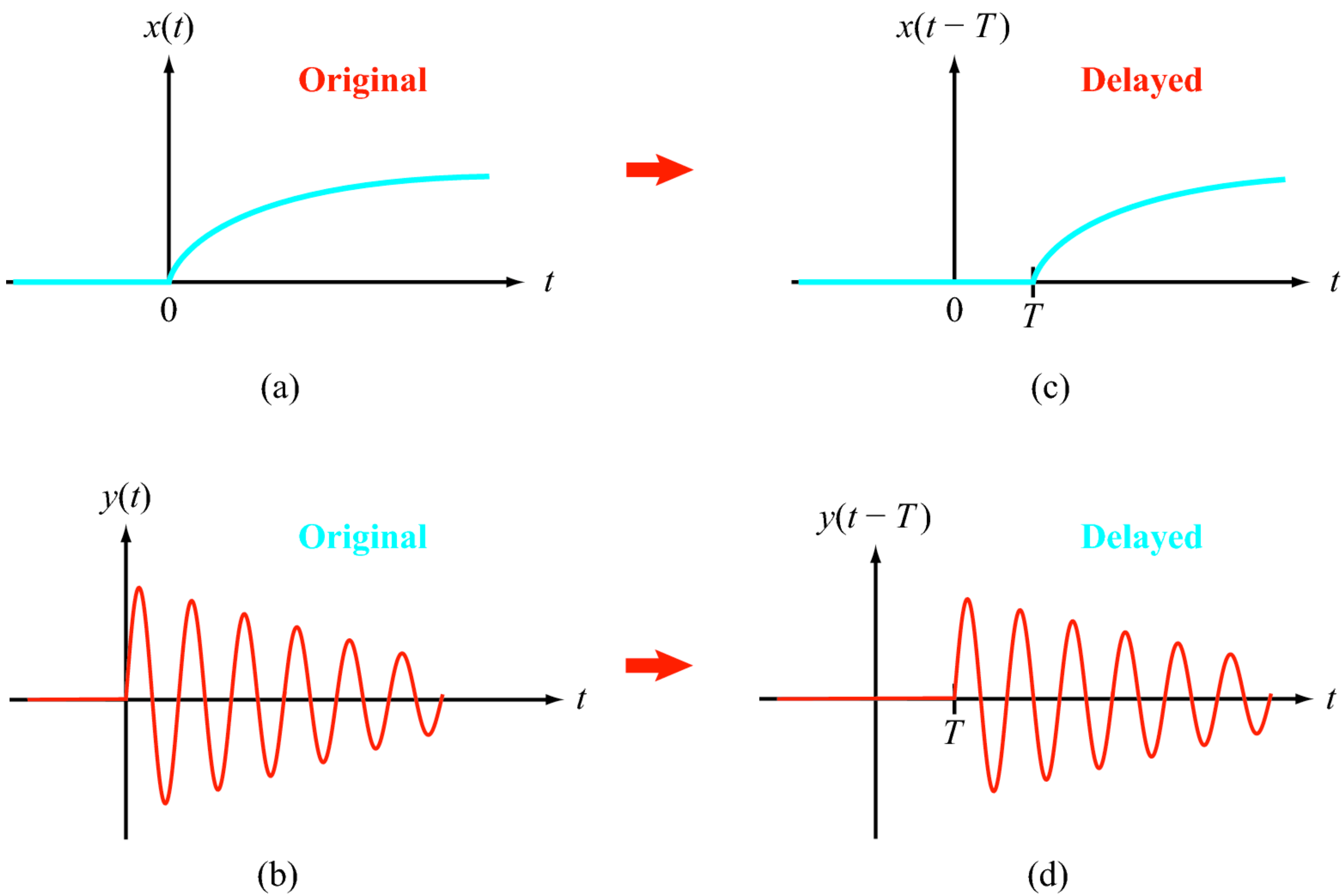
► A system is ***time-invariant*** if delaying the input signal $x(t)$ by any constant T generates the same output $y(t)$, but delayed by exactly T. ◄

$$\text{Given: } x(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t),$$

then the system is ***time-invariant*** if

$$x(t-T) \longrightarrow \boxed{\text{System}} \longrightarrow y(t-T).$$

Examples of Time-Invariant Systems



More Examples

- Time-Invariant

(a) $y_1(t) = 3\,\dfrac{d^2x}{dt^2}$,

(b) $y_2(t) = \sin[x(t)]$

(c) $y_3(t) = \dfrac{x(t+2)}{x(t-1)}$

Not Time-Invariant

(d) $y_4(t) = t\;x(t)$,

(e) $y_5(t) = x(t^2)$,

(f) $y_6(t) = x(-t)$.

Note1: Systems b and c are time-invariant, but not linear.

Note2: Systems d to f are linear, but not time-invariant.

Impulse & Step Responses

The ***impulse response*** $h(t)$ of a system is (logically enough) the response of the system to an impulse $\delta(t)$. Similarly, the ***step response*** $y_{\text{step}}(t)$ is the response of the system to a unit step $u(t)$. In symbolic form:

$$\delta(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow h(t) \tag{2.7a}$$

and

$$u(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow y_{\text{step}}(t). \tag{2.7b}$$

Static and Dynamic Systems

- Static System (Memoryless)

▶ A system for which the output $y(t)$ at time t depends only on the input $x(t)$ at time t is called a ***static*** or ***memoryless*** system. ◀

An example of such a system is

$$y(t) = \frac{\sin[x(t)]}{x^2(t) + 1}.$$

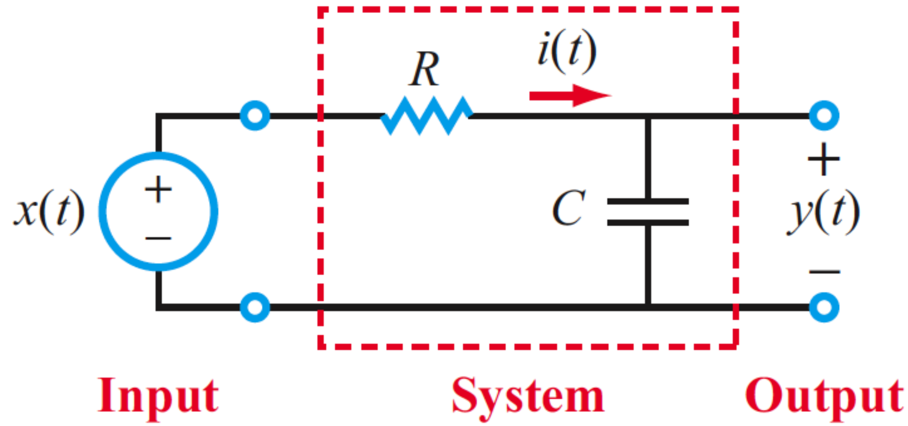
- Dynamic System

  Output *y(t)* depends on past (or future) as well as present values of input *x(t)*.

Computing $h(t)$ and $y_{\text{step}}(t)$ of RC Circuit

$$R\,i(t) + y(t) = x(t)$$

$$i(t) = C\,\frac{dy}{dt}$$



(a) RC circuit

$$\frac{dy}{dt} + \frac{1}{RC}\,y(t) = \frac{1}{RC}\,x(t)$$

Impulse Response

$$\frac{dy}{dt} + \frac{1}{RC}\, y(t) = \frac{1}{RC}\, x(t)$$

To compute the ***impulse response***, we label $x(t) = \delta(t)$ and $y(t) = h(t)$ and obtain

$$\frac{dh}{dt} + \frac{1}{RC}\, h(t) = \frac{1}{RC}\, \delta(t). \tag{2.11}$$

Next, we introduce the ***time constant*** $\tau_c = RC$ and multiply both sides of the differential equation by the ***integrating factor*** e^{t/τ_c}. The result is

$$\frac{dh}{dt}\, e^{t/\tau_c} + \frac{1}{\tau_c}\, e^{t/\tau_c}\, h(t) = \frac{1}{\tau_c}\, e^{t/\tau_c}\, \delta(t). \tag{2.12}$$

The left side of Eq. (2.12) is recognized as

$$\frac{dh}{dt}\, e^{t/\tau_c} + \frac{1}{\tau_c}\, e^{t/\tau_c}\, h(t) = \frac{d}{dt}\,[h(t)\, e^{t/\tau_c}], \tag{2.13a}$$

and the sampling property of the impulse function given by Eq. (1.27) reduces the right-hand side of Eq. (2.12) to

$$\frac{1}{\tau_c}\, e^{t/\tau_c}\, \delta(t) = \frac{1}{\tau_c}\, \delta(t). \tag{2.13b}$$

Incorporating these two modifications in Eq. (2.12) leads to

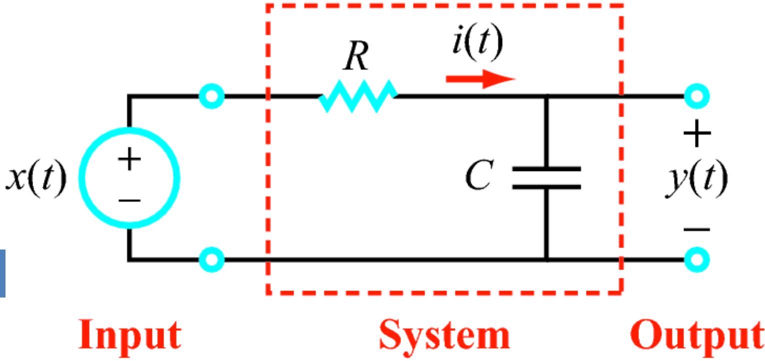
$$\frac{d}{dt}\,[h(t)\, e^{t/\tau_c}] = \frac{1}{\tau_c}\, \delta(t). \tag{2.14}$$

Integrating both sides from 0^- to t gives

$u(t)$

$$\int_{0^-}^{t} \frac{d}{d\tau}\,[h(\tau)\, e^{\tau/\tau_c}]\, d\tau = \frac{1}{\tau_c} \int_{0^-}^{t} \delta(\tau)\, d\tau, \tag{2.15}$$

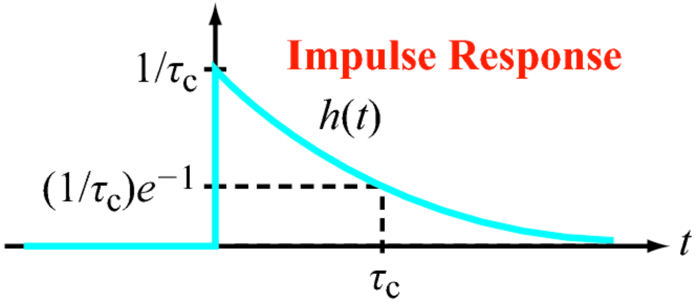
Impulse & Step Responses



(a) RC circuit

$$h(t) = \frac{1}{\tau_c} e^{-t/\tau_c} u(t).$$

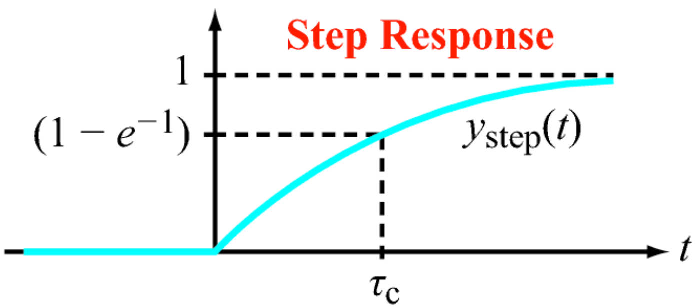
(impulse response of the RC circuit)



(b) $h(t)$

$$y_{\text{step}}(t) = [1 - e^{-t/\tau_c}]\, u(t).$$

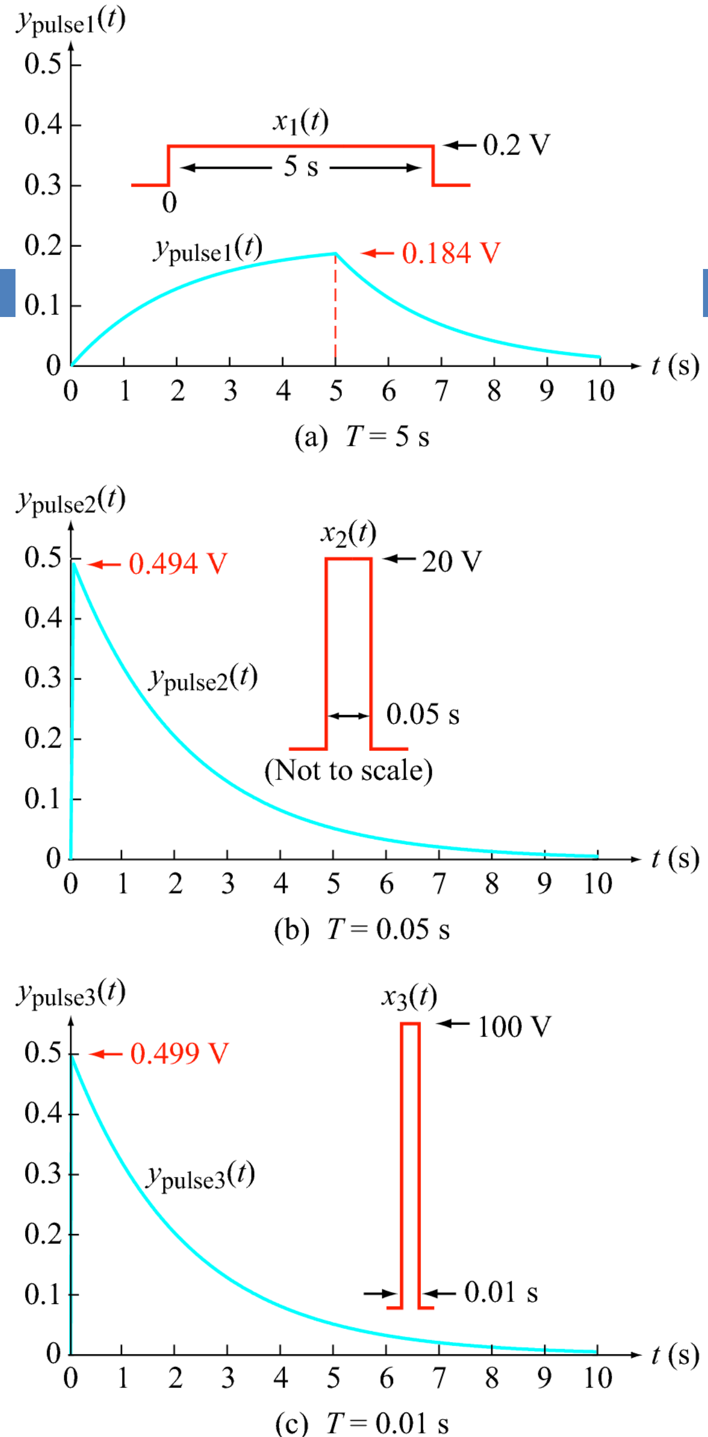
(step response of the RC circuit)



(c) $y_{\text{step}}(t)$

Simulating Impulse Response for Circuit with RC=2s

For a perfect impulse at the input, h(t)=0.5 at t=0.



Impulse Response From Step Response

Step 1: Physically *measure* the step response $y_{\text{step}}(t)$.

Step 2: Differentiate it to obtain

$$h(t) = \frac{dy_{\text{step}}}{dt}.$$

$$\frac{du}{dt} = \delta(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow h(t) = \frac{dy_{\text{step}}}{dt}$$

Ramp Response From Step Response

$$r(t) = \int_{-\infty}^{t} u(\tau)\, d\tau$$

$$r(t) = \int_{-\infty}^{t} u(\tau)\, d\tau \longrightarrow \boxed{\text{LTI}} \longrightarrow y_{\text{ramp}}(t)$$

RC Circuit Example

$$y_{\text{step}}(t) = [1 - e^{-t/\tau_c}]\, u(t).$$

(step response of the RC circuit)

$$\begin{aligned}
y_{\text{ramp}}(t) &= \int_{-\infty}^{t} y_{\text{step}}(\tau)\, d\tau \\
&= \int_{-\infty}^{t} (1 - e^{-\tau/\tau_c})\, u(\tau)\, d\tau \\
&= \int_{0}^{t} (1 - e^{-\tau/\tau_c})\, d\tau \\
&= [t - \tau_c(1 - e^{-t/\tau_c})]\, u(t).
\end{aligned}$$

Convolution

LTI System with Zero Initial Conditions

$$\delta(t) \rightarrow \boxed{\textbf{LTI}} \rightarrow y(t) = h(t)$$

$$\downarrow$$

$$\delta(t-\tau) \rightarrow \boxed{\textbf{LTI}} \rightarrow y(t) = h(t-\tau)$$

$$\downarrow$$

$$x(\tau)\,\delta(t-\tau) \rightarrow \boxed{\textbf{LTI}} \rightarrow y(t) = x(\tau)\,h(t-\tau)$$

$$\downarrow$$

$$\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau \rightarrow \boxed{\textbf{LTI}} \rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

$$\downarrow$$

$$x(t) \rightarrow \boxed{\textbf{LTI}} \rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

Use of Convolution

► The response $y(t)$ of an LTI system with impulse response $h(t)$ to *any* input $x(t)$ can be computed *explicitly* using the ***convolution integral***

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = h(t) * x(t). \tag{2.30}$$

(convolution integral)

All initial conditions must be zero. ◄

- Convolution is Commutative:

$$x(t) * h(t) = h(t) * x(t)$$

- For Causal Signals and Systems

$$y(t) = u(t) \int_0^t x(\tau)\, h(t-\tau)\, d\tau$$

$$= u(t) \int_0^t x(t-\tau)\, h(\tau)\, d\tau.$$

(causal signals and systems)

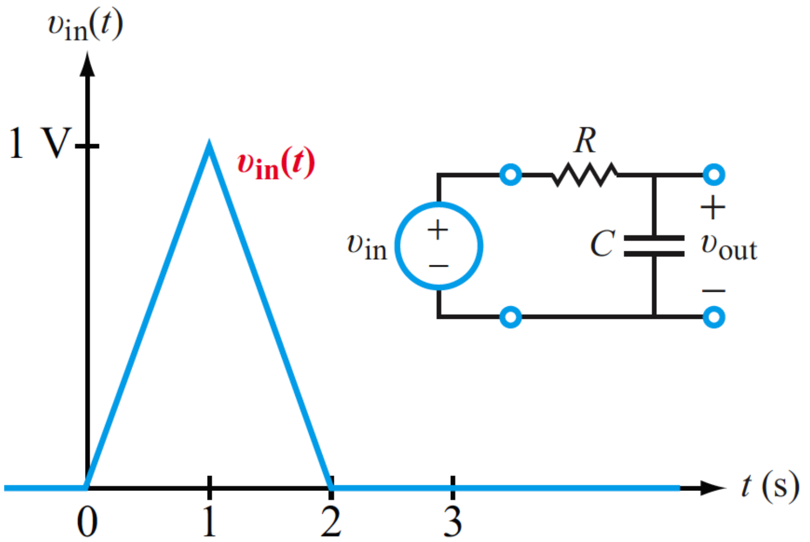
Response of Circuit with RC=1s to Triangular Pulse

Solution:
The input signal, measured in volts, is given by

$$\upsilon_{\text{in}}(t) = \begin{cases} 0 & \text{for } t \le 0, \\ t & \text{for } 0 \le t \le 1 \text{ s}, \\ 2-t & \text{for } 1 \le t \le 2 \text{ s}, \\ 0 & \text{for } t \ge 2 \text{ s}, \end{cases}$$

and according to Eq. (2.17), the impulse response for $\tau_c = 1$ is

$$h(t) = \frac{1}{\tau_c}\, e^{-t/\tau_c}\, u(t) = e^{-t}\, u(t).$$



(a) Triangular pulse

$$\upsilon_{\text{out}}(t) = \upsilon_{\text{in}}(t) * h(t) = \int_0^t \upsilon_{\text{in}}(\tau)\, h(t-\tau)\, d\tau$$

with

$$h(t-\tau) = e^{-(t-\tau)}\, u(t-\tau) = \begin{cases} 0 & \text{for } t < \tau, \\ e^{-(t-\tau)} & \text{for } t > \tau. \end{cases}$$

(1) $t < 0$:

The lowest value that the integration variable τ can assume is zero. Therefore, when $t < 0$, $t < \tau$ and $h(t-\tau) = 0$. Consequently,

$$\upsilon_{\text{out}}(t) = 0 \qquad \text{for } t < 0.$$

(2) $0 \leq t \leq 1$ s:

$$h(t-\tau) = e^{-(t-\tau)}, \qquad \upsilon_{\text{in}}(\tau) = \tau,$$

and

$$\upsilon_{\text{out}}(t) = \int_0^t \tau e^{-(t-\tau)}\, d\tau$$

$$= e^{-t} + t - 1, \qquad \text{for } 0 \leq t \leq 1 \text{ s}.$$

(3) $1 \text{ s} \leq t \leq 2$ s:

$$\upsilon_{\text{in}}(\tau) = \begin{cases} \tau & \text{for } 0 \leq \tau \leq 1 \text{ s}, \\ 2-\tau & \text{for } 1 \text{ s} \leq \tau \leq 2 \text{ s}, \end{cases}$$

and

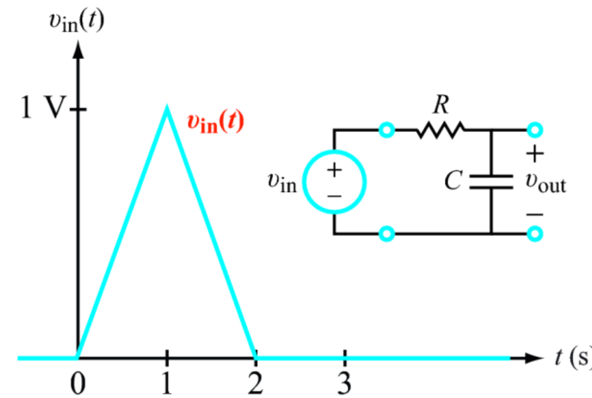
$$\upsilon_{\text{out}}(t) = \int_0^1 \tau e^{-(t-\tau)}\, d\tau + \int_1^t (2-\tau) e^{-(t-\tau)}\, d\tau$$

$$= (1-2e)e^{-t} - t + 3, \qquad \text{for } 1 \text{ s} \leq t \leq 2 \text{ s}.$$

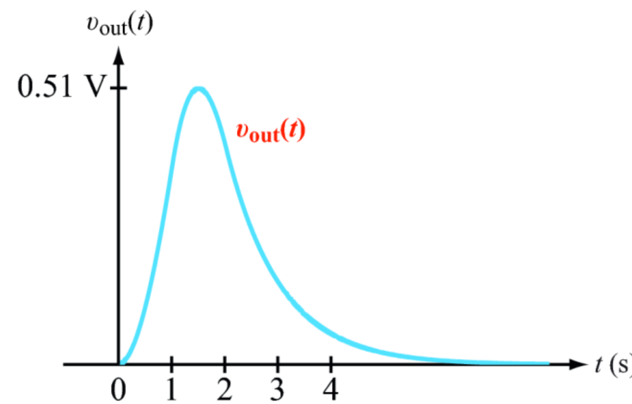
(4) $t \geq 2$ s:

$$\upsilon_{\text{out}}(t) = \int_0^1 \tau e^{-(t-\tau)}\, d\tau + \int_1^2 (2-\tau) e^{-(t-\tau)}\, d\tau$$

$$= (1-2e+e^2)e^{-t} \qquad \text{for } t \geq 2 \text{ s}.$$



(a) Triangular pulse



(b) Output response

Useful Convolution Formula

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Convolution Integral

For functions $x(t)$ and $h(t)$ given by

$$x(t) = f_1(t)\, u(t - T_1) \tag{2.51a}$$

and

$$h(t) = f_2(t)\, u(t - T_2), \tag{2.51b}$$

where $f_1(t)$ and $f_2(t)$ are any constants or time-dependent signals and T_1 and T_2 are any non-negative numbers, their convolution is

$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= u(t) \int_0^t x(t-\tau)\, h(\tau)\, d\tau \\ &= \int_0^t f_1(t-\tau)\, f_2(\tau)\, u(t - T_1 - \tau)\, u(\tau - T_2)\, d\tau \\ &= \left[\int_{T_2}^{t-T_1} f_1(t-\tau)\, f_2(\tau)\, d\tau \right] u(t - T_1 - T_2). \end{aligned} \tag{2.52}$$

RC Circuit Response to Rectangular Pulse

Solution:

The time constant of the RC circuit is $\tau_c = RC = (0.5 \times 10^6) \times 10^{-6} = 0.5$ s. In view of Eq. (2.17), the impulse response of the circuit is

$$h(t) = \frac{1}{\tau_c}\, e^{-t/\tau_c}\, u(t) = 2e^{-2t}\, u(t). \qquad (2.53)$$

The input voltage is

$$\upsilon_{in}(t) = [u(t) - u(t-1)] \text{ V}. \qquad (2.54)$$

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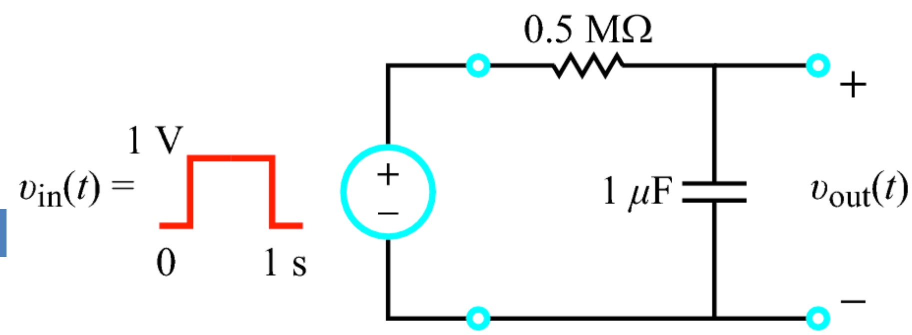
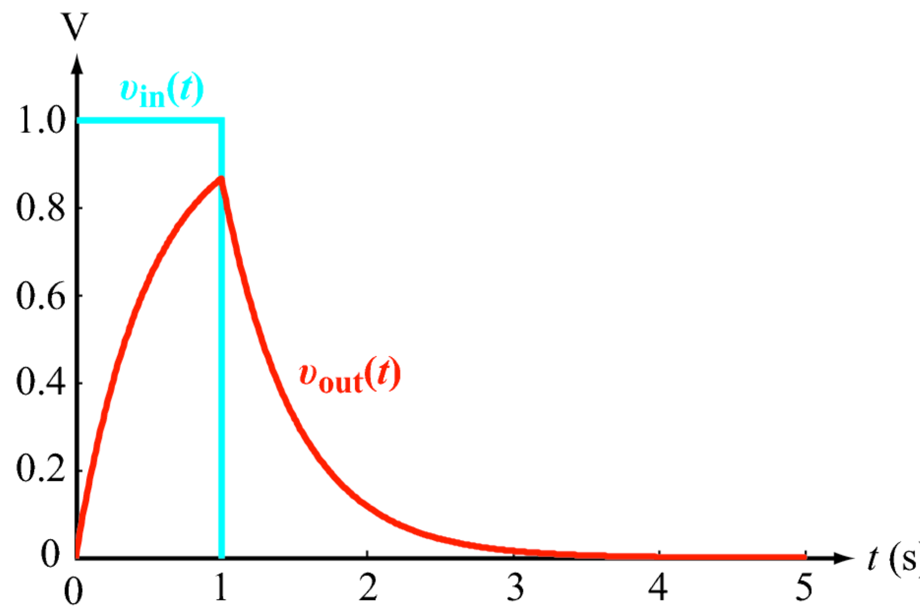
Solution: Next slide.



(a) RC lowpass filter

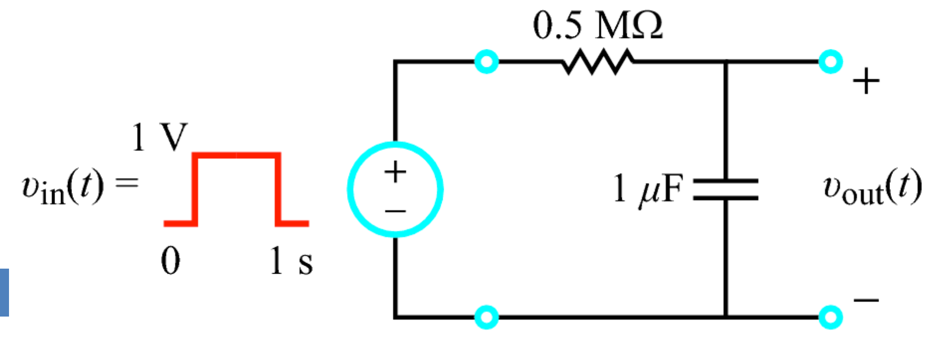


(b) Output response

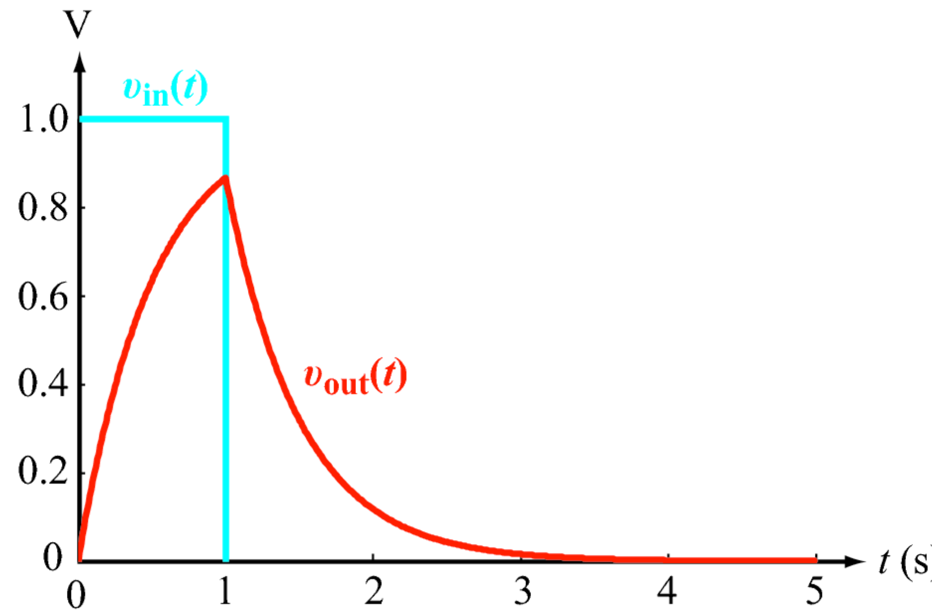
$$
\begin{aligned}
\upsilon_{\text{out}}(t) &= \upsilon_{\text{in}}(t) * h(t) \\
&= u(t) \int_0^t \upsilon_{\text{in}}(\tau)\, h(t-\tau)\, d\tau \\
&= u(t) \int_0^t [u(\tau) - u(\tau - 1)] \\
&\qquad\qquad \times 2e^{-2(t-\tau)}\, u(t-\tau)\, d\tau \\
&= u(t) \int_0^t 2e^{-2(t-\tau)}\, u(\tau)\, u(t-\tau)\, d\tau \\
&\quad - u(t) \int_0^t 2e^{-2(t-\tau)}\, u(\tau - 1)\, u(t-\tau)\, d\tau.
\end{aligned}
$$

Skip

$$
\begin{aligned}
&= \left[\int_0^t 2e^{-2(t-\tau)}\, d\tau \right] u(t) \\
&\quad - \left[\int_1^t 2e^{-2(t-\tau)}\, d\tau \right] u(t-1) \\
&= \frac{2}{2}\, e^{-2(t-\tau)} \Big|_0^t\, u(t) - \frac{2}{2}\, e^{-2(t-\tau)} \Big|_1^t\, u(t-1) \\
&= [1 - e^{-2t}]\, u(t) - [1 - e^{-2(t-1)}]\, u(t-1) \text{ V},
\end{aligned}
$$



(a) RC lowpass filter



(b) Output response

Graphical Convolution Technique

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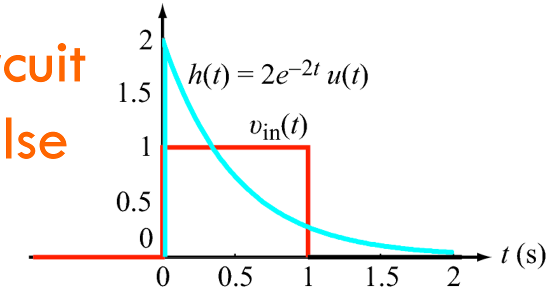
Step 1: On the τ-axis, display $x(\tau)$ and $h(-\tau)$ with the latter being a folded image of $h(\tau)$ about the vertical axis.

Step 2: Shift $h(-\tau)$ to the right by a small increment t to obtain $h(t-\tau) = h(-(\tau - t))$.

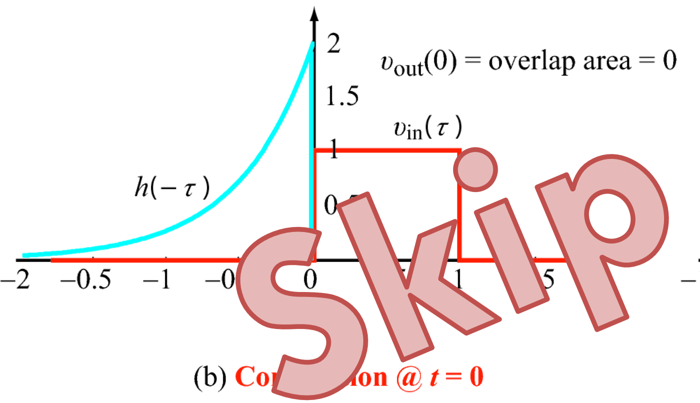
Step 3: Determine the product of $x(\tau)$ and $h(t-\tau)$ and integrate it over the τ-domain from $\tau = 0$ to $\tau = t$ to get $y(t)$. The integration is equal to the area overlapped by the two functions.

Step 4: Repeat steps 2 and 3 for each of many successive values of t to generate the complete response $y(t)$.

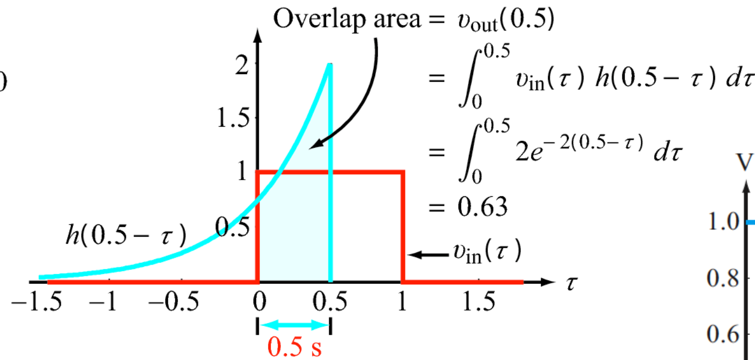
Example: RC Circuit Excited by a Pulse



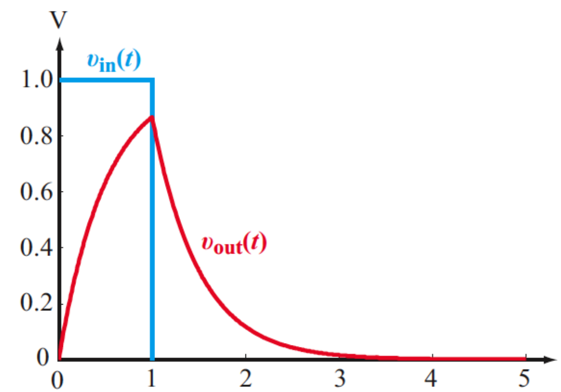
(a) Time domain



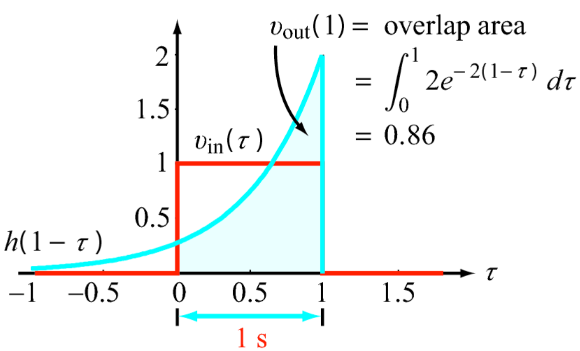
(b) Co… ion @ $t = 0$



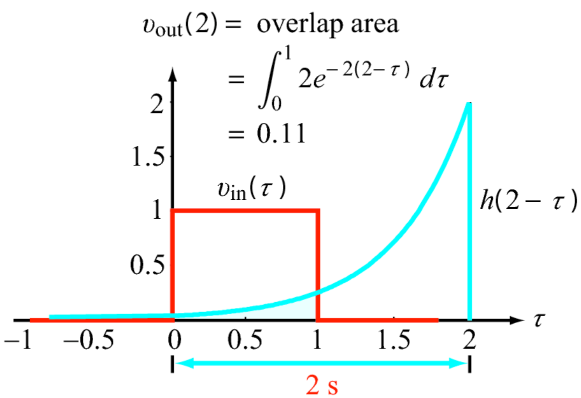
(c) Convolution @ $t = 0.5$ s



(b) Output response



(d) Convolution @ $t = 1$ s



(e) Convolution @ $t = 2$ s

- Convolution of 1 cycle of square wave with Triangle

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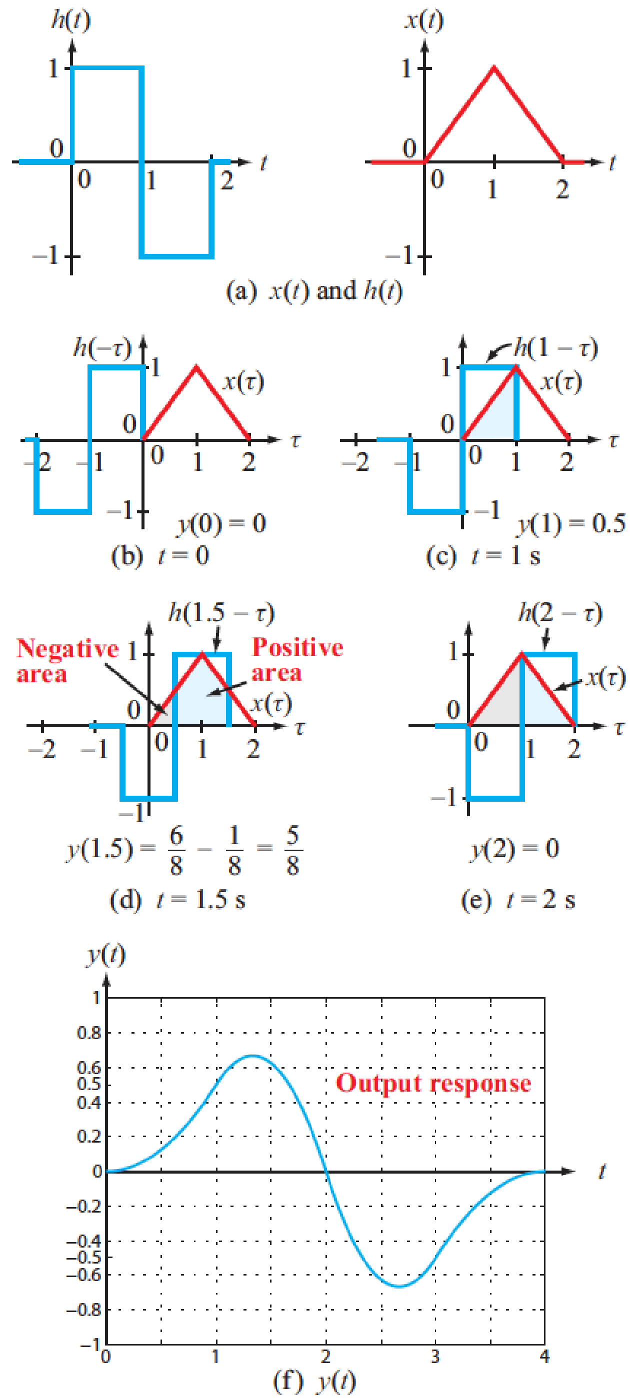
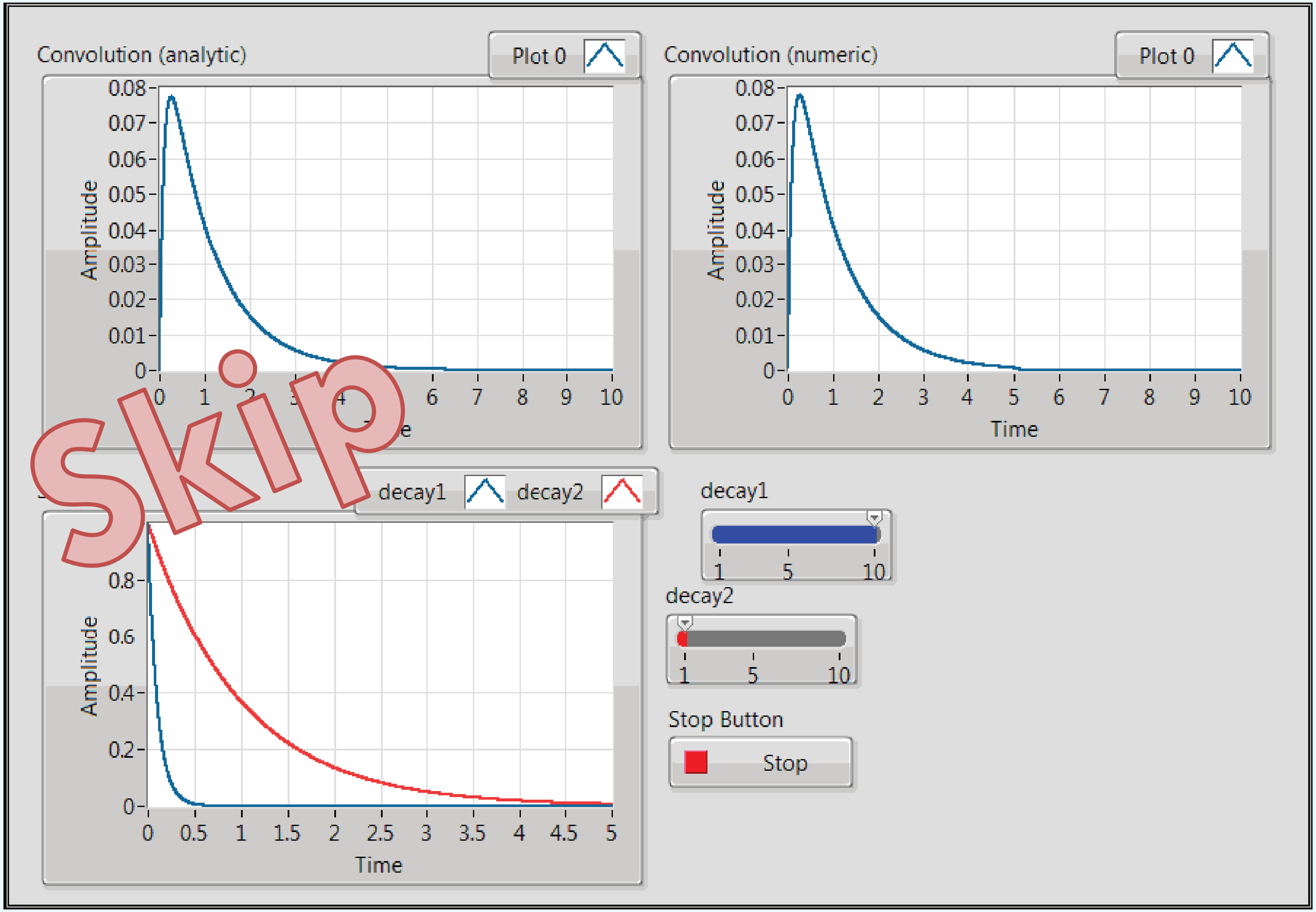
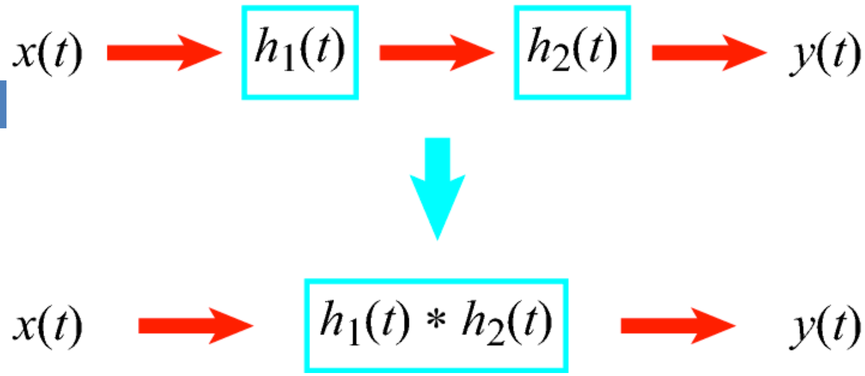


Module 2.1 Convolution of Exponential Functions This module computes the convolution of $e^{-at}\,u(t)$ and $e^{-bt}\,u(t)$. The values of exponents a and b are selectable.



Systems Connected In-Series



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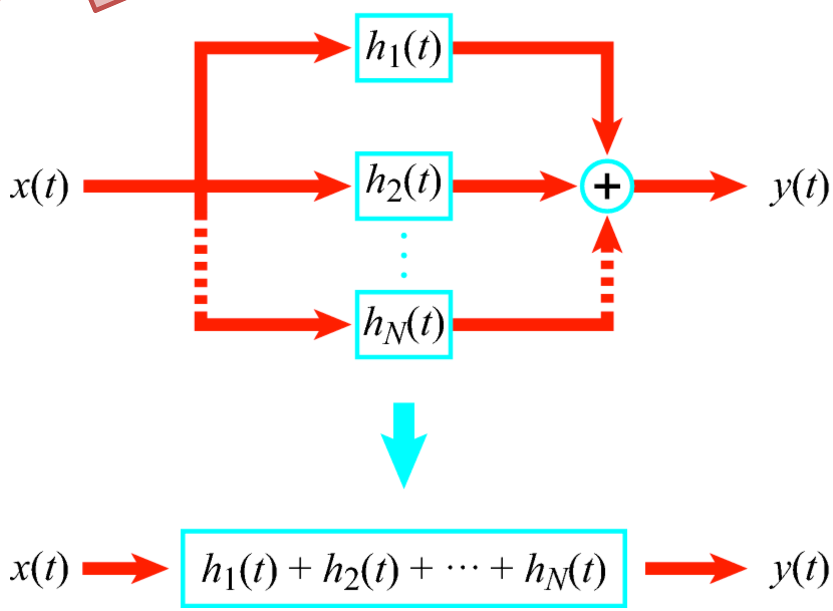
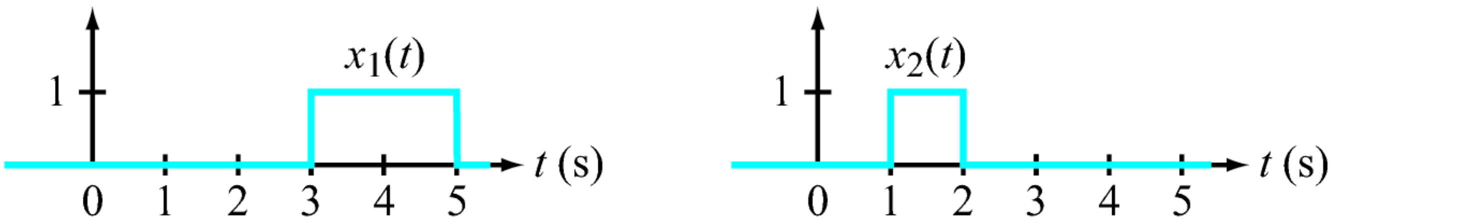
Systems Connected In-Parallel

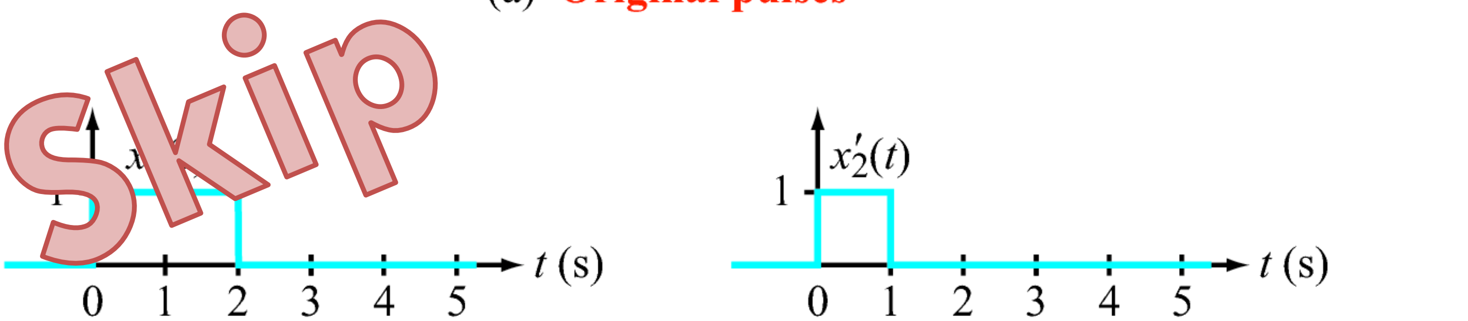


$$h(t - T_1) * x(t - T_2) = y(t - T_1 - T_2)$$

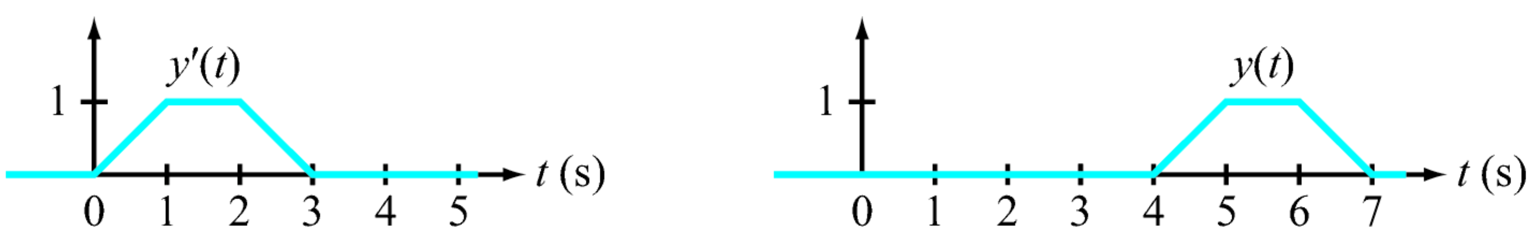
(time-shift property)



(a) **Original pulses**



(b) **Pulses shifted to start at $t = 0$**



(c) **Convolution of $x_1'(t)$ with $x_2'(t)$**

(d) **$y'(t)$ delayed by 4 s gives $y(t)$**

Table 2-1: Convolution properties.

Convolution Integral	$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$
• **Causal Systems:**	Replace lower limit with 0
Property	**Description**
1. Commutative	$x(t) * h(t) = h(t) * x(t)$
2. Associative	$[g(t) * h(t)] * x(t) = g(t) * [h(t) * x(t)]$
3. Distributive	$x(t) * [h_1(t) + \cdots + h_N(t)] = x(t) * h_1(t) + \cdots + x(t) * h_N(t)$
4. Causal $*$ Causal = Causal	$y(t) = u(t) \int_{0}^{t} h(\tau)\, x(t-\tau)\, d\tau$
5. Time-shift	$h(t-T_1) * x(t-T_2) = y(t-T_1-T_2)$
6. Convolution with Impulse	$x(t) * \delta(t-T) = x(t-T)$
7. Width	Width of $y(t)$ = width of $x(t)$ + width of $h(t)$
8. Area	Area of $y(t)$ = area of $x(t)$ $\times$ area of $h(t)$
9. Convolution with $u(t)$	$y(t) = x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$ (Ideal integrator)
10a. Differentiation	$\left(\frac{d^m x}{dt^m}\right) * \left(\frac{d^n h}{dt^n}\right) = \frac{d^{m+n} y}{dt^{m+n}}$
10b. Integration	$\int_{-\infty}^{t} y(\tau)\, d\tau = x(t) * \left[\int_{-\infty}^{t} h(\tau)\, d\tau\right] = \left[\int_{-\infty}^{t} x(\tau)\, d\tau\right] * h(t)$

2-6.1 Causality

We define a ***causal system*** as a system for which the present value of the output $y(t)$ can only depend on present and past values of the input $\{x(\tau), \quad \tau \leq t\}$. For a noncausal system, the present output could depend on future inputs. Noncausal systems are also called ***anticipatory*** systems, since they anticipate the future.

A physical system must be causal, because a noncausal system must have the ability to see into the future! For example, the noncausal system $y(t) = x(t+2)$ must know the input two seconds into the future to deliver its output at the present time. This is clearly impossible in the real world.

► An LTI system is causal ***if and only if*** its impulse response is a causal function: $h(t) = 0$ for $t < 0$. ◄

Stability

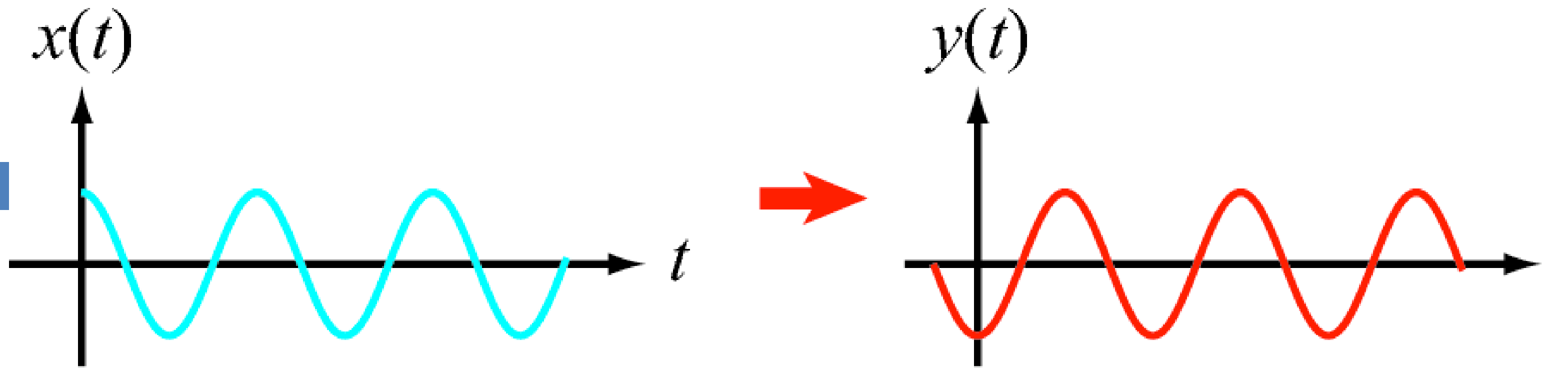
A signal $x(t)$ is ***bounded*** if there is a constant C so that $|x(t)| \leq C$ for all t. Examples of bounded signals include:

- $\cos(3t)$, $\quad 7e^{-2t}\,u(t)$, $\quad$ and $\quad e^{2t}\,u(1-t)$.

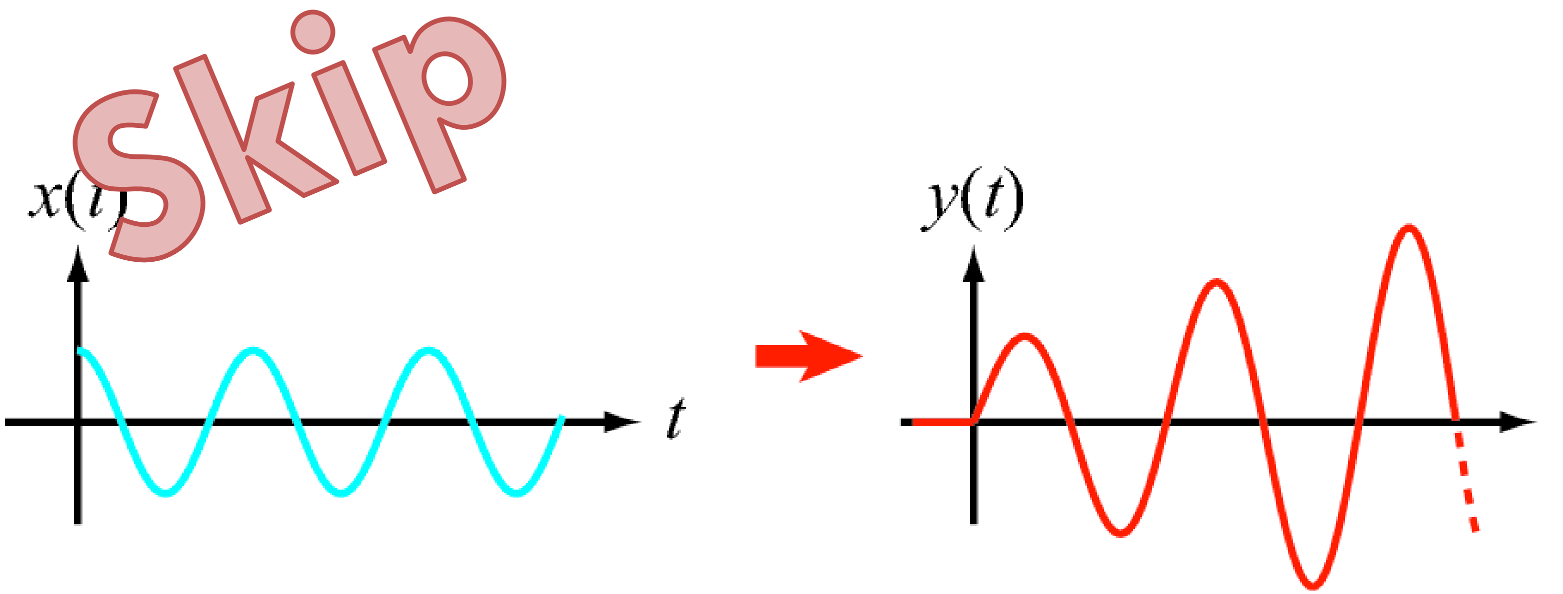
Examples of unbounded signals include:

- t^2, $\quad e^{2t}\,u(t)$, $\quad e^{-t}$, $\quad$ and $\quad 1/t$.

A system is ***BIBO (bounded input/bounded output) stable*** every bounded input $x(t)$ results in a bounded output $y(t)$,



(a) BIBO-stable system

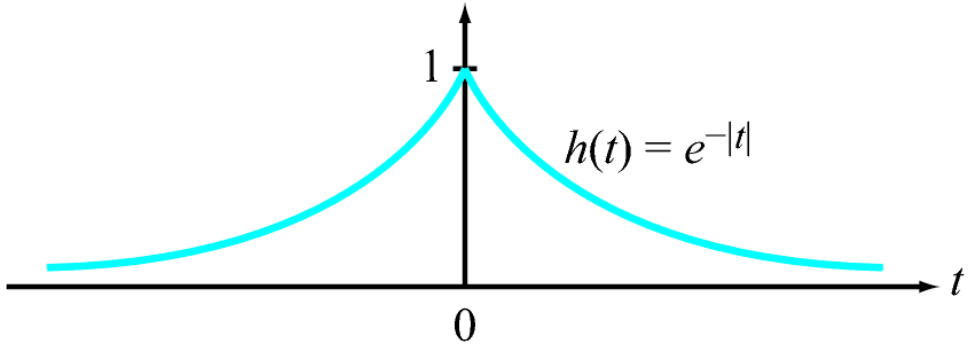


(b) Non-BIBO-stable system

▶ An LTI system is BIBO stable *if and only if* its impulse response $h(t)$ is ***absolutely integrable*** (i.e., if $\int_{-\infty}^{\infty} |h(t)|\, dt$ is finite). ◀

Example

Skip



$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{-\infty}^{\infty} e^{-|t|}\, dt = 2\int_{0}^{\infty} e^{-t}\, dt = 2.$$

Hence, the system is BIBO stable.

2-6.4 BIBO Stability of System with Decaying Exponentials

Consider a causal system with an impulse response

$$h(t) = Ce^{\gamma t}\, u(t), \tag{2.93}$$

where C is a finite constant and γ is, in general, a finite complex coefficient given by

$$\gamma = \alpha + j\beta, \quad \alpha = \Re\mathfrak{e}[\gamma], \quad \text{and} \quad \beta = \Im\mathfrak{m}[\gamma]. \tag{2.94}$$

Skip

Such a system is BIBO stable if and only if $\alpha < 0$ (i.e., $h(t)$ is a one-sided exponential with an exponent coefficient whose real part is negative). To verify the validity of this statement, we test to see if $h(t)$ is absolutely integrable. Since $|e^{j\beta t}| = 1$ and $e^{\alpha t} > 0$,

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{0}^{\infty} |Ce^{\alpha t} e^{j\beta t}|\, dt$$
$$= |C| \int_{0}^{\infty} e^{\alpha t}\, dt.$$

(a) $\alpha < 0$

If $\alpha < 0$, we can rewrite it as $\alpha = -|\alpha|$ in the exponential, which leads to

$$\int_{-\infty}^{\infty} |h(t)|\, dt = |C| \int_{0}^{\infty} e^{-|\alpha| t}\, dt = \frac{|C|}{|\alpha|} < \infty. \tag{2.96}$$

Hence, $h(t)$ is absolutely integrable and the system is BIBO stable.

(b) $\alpha \geq 0$

If $\alpha \geq 0$, Eq. (2.95) becomes

$$\int_{-\infty}^{\infty} |h(t)|\, dt = |C| \int_{0}^{\infty} e^{\alpha t}\, dt \to \infty,$$

thereby proving that the system is not BIBO stable when $\alpha \geq 0$.

By extension, for any positive integer N, an impulse response composed of a linear combination of N exponential signals

$$h(t) = \sum_{i=1}^{N} C_i e^{\gamma_i t}\, u(t) \tag{2.97}$$

Skip

is absolutely integrable, and its LTI system is BIBO stable, if and only if all of the exponential coefficients γ_i have negative real parts. This is a fundamental attribute of LTI system theory.

2-7.1 LTI System Response to Complex Exponential Signals

Let us examine the response of an LTI system to a complex exponential $x(t) = Ae^{j\omega t}$. Since the system is LTI, we may (without loss of generality) set $A = 1$. Hence,

$$
\begin{aligned}
y(t) = h(t) * x(t) = h(t) * e^{j\omega t} &= \int_{-\infty}^{\infty} h(\tau)\, e^{j\omega(t-\tau)}\, d\tau \\
&= e^{j\omega t} \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau \\
&= \mathbf{H}(\omega)\, e^{j\omega t}, \qquad (2.100)
\end{aligned}
$$

where the ***frequency response function*** $\mathbf{H}(\omega)$ is defined as

$$
\mathbf{H}(\omega) = \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau. \qquad (2.101)
$$

That is:

$$
e^{j\omega t} \longrightarrow \boxed{h(t)} \longrightarrow \mathbf{H}(\omega)\, e^{j\omega t}.
$$

LTI Response to Sinusoid

$$e^{j\omega t} \rightarrow \boxed{h(t)} \rightarrow \mathbf{H}(\omega)\, e^{j\omega t}.$$

$$e^{-j\omega t} \rightarrow \boxed{h(t)} \rightarrow \mathbf{H}(-\omega)\, e^{-j\omega t}.$$

$$x(t) = \cos\omega t = \frac{1}{2}[e^{j\omega t} + e^{-j\omega t}]. \tag{2.115}$$

Using the additivity property of LTI systems, we can determine the system's response to $\cos\omega t$ by adding the responses expressed by Eqs. (2.102) and (2.105):

$$\begin{array}{ccccc} \frac{1}{2}\, e^{j\omega t} & \rightarrow & \boxed{h(t)} & \rightarrow & \frac{1}{2}\,\mathbf{H}(\omega)\, e^{j\omega t} \\ + & & & & + \\ \frac{1}{2}\, e^{-j\omega t} & \rightarrow & \boxed{h(t)} & \rightarrow & \frac{1}{2}\,\mathbf{H}(-\omega)\, e^{-j\omega t}. \end{array} \tag{2.116}$$

Since $h(t)$ is real, Eq. (2.107) stipulates that

$$\mathbf{H}(-\omega) = \mathbf{H}^*(\omega).$$

If we define

$$\mathbf{H}(\omega) = |\mathbf{H}(\omega)|e^{j\theta},$$
$$\mathbf{H}(-\omega) = |\mathbf{H}(\omega)|e^{-j\theta},$$

where $\theta = \underline{/\mathbf{H}(\omega)}$, the sum of the two outputs in
becomes

$$\begin{aligned} y(t) &= \frac{1}{2}\,[\mathbf{H}(\omega)\,e^{j\omega t} + \mathbf{H}(-\omega)\,e^{-j\omega t}] \\ &= \frac{1}{2}\,|\mathbf{H}(\omega)|[e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}] \\ &= |\mathbf{H}(\omega)|\cos(\omega t + \theta). \end{aligned}$$

In summary,

$$y(t) = |\mathbf{H}(\omega)|\cos(\omega t + \theta)$$
$$(\text{for } x(t) = \cos\omega t).$$

In symbolic form:

$$A\cos(\omega t + \phi)$$
$$\downarrow$$
$$\boxed{h(t)}$$
$$\downarrow$$
$$A|\mathbf{H}(\omega)|\cos(\omega t + \theta + \phi).$$

Example 2-13: Response of an LCCDE to a Sinusoidal Input

$$5\cos(4t+30^\circ) \rightarrow \boxed{\frac{dy}{dt}+3y = 3\frac{dx}{dt}+5x} \rightarrow ?$$

Solution: Inserting $x(t) = e^{j\omega t}$ and $y(t) = \mathbf{H}(\omega)\, e^{j\omega t}$ in the specified LCCDE gives

$$j\omega\, \mathbf{H}(\omega)\, e^{j\omega t} + 3\mathbf{H}(\omega)\, e^{j\omega t} = 3\, j\omega e^{j\omega t} + 5e^{j\omega t}.$$

Solving for $\mathbf{H}(\omega)$ gives

$$\mathbf{H}(\omega) = \frac{5+j3\omega}{3+j\omega}\,.$$

The angular frequency ω of the input signal is 4 rad/s. Hence,

$$\mathbf{H}(4) = \frac{5+j12}{3+j4} = \frac{13e^{j67.38^\circ}}{5e^{j53.13^\circ}} = \frac{13}{5}\, e^{j14.25^\circ}.$$

Application of Eq. (2.120) with $A = 5$ and $\phi = 30^\circ$ gives

$$y(t) = 5 \times \frac{13}{5}\cos(4t + 14.25^\circ + 30^\circ) = 13\cos(4t + 44.25^\circ).$$

2-8 Impulse Response of Second-Order LCCDEs

Many physical systems are described by second-order LCCDEs of the form

$$\frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_2\, y(t) = b_1 \frac{dx}{dt} + b_2\, x(t), \qquad (2.121)$$

where a_1, a_2, b_1, and b_2 are constant coefficients.

2-8.1 LCCDE with No Input Derivatives

For simplicity, we start by considering a version of Eq. (2.121) without the dx/dt term, and then we use the result to treat the more general case in the next subsection.

For $b_1 = 0$ and $b_2 = 1$, Eq. (2.121) becomes

$$\frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_2\, y(t) = x(t). \qquad (2.122)$$

2-8.1 LCCDE with No Input Derivatives

For simplicity, we start by considering a version of Eq. (2.121) without the dx/dt term, and then we use the result to treat the more general case in the next subsection.

For $b_1 = 0$ and $b_2 = 1$, Eq. (2.121) becomes

$$\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_2\, y(t) = x(t). \tag{2.122}$$

Step 1: Roots of Characteristic Equation

Assuming $y(t)$ has a general solution of the form $y(t) = Ae^{st}$, substitution in the homogeneous form of Eq. (2.122)—i.e., with $x(t) = 0$—leads to the ***characteristic equation***:

$$s^2 + a_1 s + a_2 = 0. \tag{2.123}$$

If p_1 and p_2 are the roots of Eq. (2.123), then

$$s^2 + a_1 s + a_2 = (s - p_1)(s - p_2), \tag{2.124}$$

which leads to

$$p_1 + p_2 = -a_1, \qquad p_1 p_2 = a_2,$$

and

$$p_1 = -\frac{a_1}{2} + \sqrt{\left(\frac{a_1}{2}\right)^2 - a_2}\,,$$

$$p_2 = -\frac{a_1}{2} - \sqrt{\left(\frac{a_1}{2}\right)^2 - a_2}\,.$$

Roots p_1 and p_2 are

(a) real if $a_1^2 > 4a_2$,

(b) complex conjugates if $a_1^2 < 4a_2$, or

(c) identical if $a_1^2 = 4a_2$.

Step 2: Two Coupled First-Order LCCDEs

The original differential equation given by Eq. (2.122) now can be rewritten as

$$\frac{d^2 y}{dt^2} - (p_1 + p_2)\,\frac{dy}{dt} + (p_1 p_2)\,y(t) = x(t), \qquad (2.127a)$$

which can in turn be cast in the form

$$\left[\frac{d}{dt} - p_1\right]\left[\frac{d}{dt} - p_2\right] y(t) = x(t). \qquad (2.127b)$$

Furthermore, we can split the ***second-order differential equation*** into ***two coupled first-order equations*** by introducing an intermediate variable $z(t)$:

$$\frac{dz}{dt} - p_1\,z(t) = x(t) \qquad (2.128a)$$

and

$$\frac{dy}{dt} - p_2\,y(t) = z(t). \qquad (2.128b)$$

These coupled first-order LCCDEs represent a ***series (or cascade) connection*** of LTI systems, each described by a first-order LCCDE. In symbolic form, we have

$$\begin{aligned} x(t) &\longrightarrow \boxed{h_1(t)} \longrightarrow z(t) \\ z(t) &\longrightarrow \boxed{h_2(t)} \longrightarrow y(t), \end{aligned} \qquad (2.129)$$

where $h_1(t)$ and $h_2(t)$ are the impulse responses corresponding to Eqs. (2.128a and b), respectively.

Step 3: Impulse Response of Cascaded LTI Systems

By comparison with Eq. (2.10) and its corresponding impulse response, Eq. (2.17), we conclude that

$$h_1(t) = e^{p_1 t}\, u(t) \tag{2.130a}$$

and

$$h_2(t) = e^{p_2 t}\, u(t). \tag{2.130b}$$

Using convolution property #2 in Table 2-1, the impulse response of the series connection of two LTI systems is the convolution of their impulse responses. Utilizing property #3 in Table 2-2, the combined impulse response becomes

$$\begin{aligned} h_c(t) &= h_1(t) * h_2(t) \\ &= e^{p_1 t}\, u(t) * e^{p_2 t}\, u(t) \\ &= \left[\frac{1}{p_1 - p_2}\right] [e^{p_1 t} - e^{p_2 t}]\, u(t). \end{aligned} \tag{2.131}$$

Summary for LCCDE with No Input Derivative

- Diff. Eq.:

$$\frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_2\, y(t) = x(t).$$

- Impulse Response:

$$\begin{aligned} h_c(t) &= h_1(t) * h_2(t) \\ &= e^{p_1 t}\, u(t) * e^{p_2 t}\, u(t) \\ &= \left[\frac{1}{p_1 - p_2}\right] [e^{p_1 t} - e^{p_2 t}]\, u(t). \end{aligned}$$

- Roots:

$$p_1 = -\frac{a_1}{2} + \sqrt{\left(\frac{a_1}{2}\right)^2 - a_2}\,,$$

$$p_2 = -\frac{a_1}{2} - \sqrt{\left(\frac{a_1}{2}\right)^2 - a_2}\,.$$

2-8.2 LCCDE with Input Derivative

We now consider the more general case of a second-order LCCDE that contains a first-order derivative on the input side of the equation

$$\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_2\, y(t) = b_1\frac{dx}{dt} + b_2\, x(t). \qquad (2.132)$$

By defining the right-hand side of Eq. (2.132) as an intermediate variable $w(t)$, the system can be represented as

$$x(t) \longrightarrow \boxed{w(t) = b_1\frac{dx}{dt} + b_2\, x(t)} \longrightarrow w(t)$$

$$w(t) \longrightarrow \boxed{h_c(t)} \longrightarrow y(t), \qquad (2.133)$$

Convolution is commutative, so we can reverse the order of the two systems

$$x(t) \longrightarrow \boxed{h_c(t)} \longrightarrow \upsilon(t)$$

$$\upsilon(t) \longrightarrow \boxed{y(t) = b_1\frac{d\upsilon}{dt} + b_2\, \upsilon(t)} \longrightarrow y(t),$$

Where $\mathcal{V}(t)$ is a new intermediate variable

$$x(t) \longrightarrow \boxed{h_c(t)} \longrightarrow \upsilon(t)$$

$$\upsilon(t) \longrightarrow \boxed{y(t) = b_1 \frac{d\upsilon}{dt} + b_2\, \upsilon(t)} \longrightarrow y(t),$$

By definition, when $x(t) = \delta(t)$, the output $y(t)$ becomes the impulse response $h(t)$ of the overall system. That is, if we set $x(t) = \delta(t)$, which results in $\upsilon(t) = h_c(t)$ and $y(t) = h(t)$, the system becomes

$$\delta(t) \longrightarrow \boxed{h_c(t)} \longrightarrow h_c(t)$$

$$h_c(t) \longrightarrow \boxed{h(t) = b_1 \frac{dh_c}{dt} + b_2\, h_c(t)} \longrightarrow h(t).$$

where $h_c(t)$ is the impulse response of the LCCDE with no input derivative

$$h_c(t) = \left[\frac{1}{p_1 - p_2}\right] [e^{p_1 t} - e^{p_2 t}]\, u(t)$$

Solution for Impulse Response of LCCDE With Input Derivative

Finally, the impulse response $h(t)$ of the overall system is

$$\begin{aligned} h(t) &= b_1 \frac{dh_c}{dt} + b_2\, h_c(t) \\ &= \left[b_1 \frac{d}{dt} + b_2 \right] \left[\frac{1}{p_1 - p_2} \right] [e^{p_1 t} - e^{p_2 t}]\, u(t) \\ &= \frac{b_1 p_1 + b_2}{p_1 - p_2}\, e^{p_1 t}\, u(t) - \frac{b_1 p_2 + b_2}{p_1 - p_2}\, e^{p_2 t}\, u(t). \end{aligned} \quad (2.136)$$

Having established in the form of Eq. (2.136) an explicit expression for the impulse response of the general LCCDE given by Eq. (2.132), we can now determine the response $y(t)$ to any causal input excitation $x(t)$ by evaluating

$$y(t) = u(t) \int_0^t h(\tau)\, x(t - \tau)\, d\tau. \quad (2.137)$$

Parameters of Second-Order LCCDE

$$
\begin{aligned}
h(t) &= b_1 \frac{dh_c}{dt} + b_2\, h_c(t) \\
&= \left[b_1 \frac{d}{dt} + b_2 \right] \left[\frac{1}{p_1 - p_2} \right] \left[e^{p_1 t} - e^{p_2 t} \right] u(t) \\
&= \frac{b_1 p_1 + b_2}{p_1 - p_2}\, e^{p_1 t}\, u(t) - \frac{b_1 p_2 + b_2}{p_1 - p_2}\, e^{p_2 t}\, u(t)
\end{aligned}
$$

Recall from an earlier section the conclusion:

► By extension, for any positive integer N, an impulse response composed of a linear combination of N exponential signals

$$h(t) = \sum_{i=1}^{N} C_i e^{\gamma_i t}\, u(t) \qquad (2.97)$$

is absolutely integrable, and its LTI system is BIBO stable, if and only if all of the exponential coefficients γ_i have negative real parts. This is a fundamental attribute of LTI system theory. ◄

Hence, p_1 and p_2 should have negative real parts in order for the system to be BIBO stable

Parameters of Second-Order LCCDE

$$p_1 = -\frac{a_1}{2} + \sqrt{\left(\frac{a_1}{2}\right)^2 - a_2}\,, \qquad (2.138a)$$

$$p_2 = -\frac{a_1}{2} - \sqrt{\left(\frac{a_1}{2}\right)^2 - a_2}\,. \qquad (2.138b)$$

(a) If both p_1 and p_2 are real, distinct, and negative, Eq. (2.138) leads to the conclusion that $a_1^2 > 4a_2$, $a_1 > 0$, and $a_2 > 0$.

(b) If p_1 and p_2 are complex conjugates with negative real parts, it follows that $a_1^2 < 4a_2$, $a_1 > 0$, and $a_2 > 0$.

(c) If p_1 and p_2 are real, equal, and negative, then $a_1^2 = 4a_2$, $a_1 > 0$, and $a_2 > 0$.

▶ The LTI system described by the LCCDE Eq. (2.132) is BIBO stable if and only if $a_1 > 0$ and $a_2 > 0$. ◀

We now introduce three new non-negative, physically meaningful parameters:

$$\alpha = \frac{a_1}{2} = \textit{attenuation coefficient} \quad \text{(Np/s)}, \tag{2.139a}$$

$$\omega_0 = \sqrt{a_2} = \textit{undamped natural frequency} \quad \text{(rad/s)}, \tag{2.139b}$$

$$\xi = \frac{\alpha}{\omega_0} = \frac{a_1}{2\sqrt{a_2}} = \textit{damping coefficient} \quad \text{(unitless)}. \tag{2.139c}$$

(a) $\xi > 1$ → *overdamped* response,

(b) $\xi = 1$ → *critically damped* response,

(c) $\xi < 1$ → *underdamped* response.

Step response

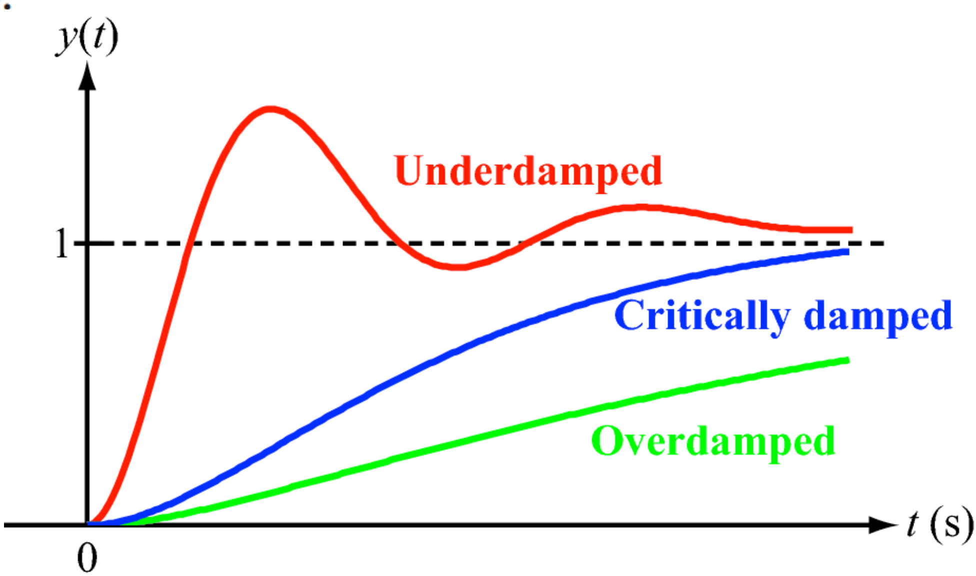


Table 2-3: Impulse and step responses of second-order LCCDE.

LCCDE $$\frac{d^2y}{dt^2} + a_1\,\frac{dy}{dt} + a_2 y = b_1\,\frac{dx}{dt} + b_2 x$$

$$\alpha = \frac{a_1}{2}\,, \qquad \omega_0 = \sqrt{a_2}\,, \qquad \xi = \frac{\alpha}{\omega_0}\,, \qquad p_1 = \omega_0[-\xi + \sqrt{\xi^2 - 1}], \qquad p_2 = \omega_0[-\xi - \sqrt{\xi^2 - 1}]$$

Overdamped Case $\xi > 1$

$$h(t) = A_1 e^{p_1 t}\,u(t) + A_2 e^{p_2 t}\,u(t) \qquad y_{\text{step}}(t) = \left[\frac{A_1}{p_1}\,(e^{p_1 t} - 1) + \frac{A_2}{p_2}\,(e^{p_2 t} - 1)\right] u(t)$$

$$A_1 = \frac{b_1 p_1 + b_2}{p_1 - p_2}\,, \qquad A_2 = \frac{-(b_1 p_2 + b_2)}{p_1 - p_2}$$

Underdamped Case $\xi < 1$

$$h(t) = [B_1 \cos\omega_{\text{d}} t + B_2 \sin\omega_{\text{d}} t]e^{-\alpha t}\,u(t)$$

$$y_{\text{step}}(t) = \frac{1}{\alpha^2 + \omega_{\text{d}}^2}\left\{[-(B_1\alpha + B_2\omega_{\text{d}})\cos\omega_{\text{d}} t + (B_1\omega_{\text{d}} + B_2\alpha)\sin\omega_{\text{d}} t]e^{-\alpha t} + (B_1\alpha + B_2\omega_{\text{d}})\right\} u(t)$$

$$B_1 = b_1, \qquad B_2 = \frac{b_2 - b_1\alpha}{\omega_{\text{d}}}\,, \qquad \omega_{\text{d}} = \omega_0\sqrt{1 - \xi^2}$$

Critically Damped Case $\xi = 1$

$$h(t) = (C_1 + C_2 t)e^{-\alpha t}\,u(t) \qquad y_{\text{step}}(t) = \left[\left(\frac{C_1}{\alpha} + \frac{C_2}{\alpha^2}\right)(1 - e^{-\alpha t}) - \frac{C_2}{\alpha}\,t e^{-\alpha t}\right] u(t)$$

$$C_1 = b_1, \qquad C_2 = b_2 - \alpha b_1$$

Car Suspension System

- $x(t)$ = input = vertical displacement of the pavement, defined relative to a reference ground level.
- $y(t)$ = output = vertical displacement of the car chassis from its equilibrium position.

- m = *one-fourth* of the car's mass, because the car has four wheels.
- k = ***spring constant*** or ***stiffness*** of the coil.
- b = ***damping coefficient*** of the shock absorber.

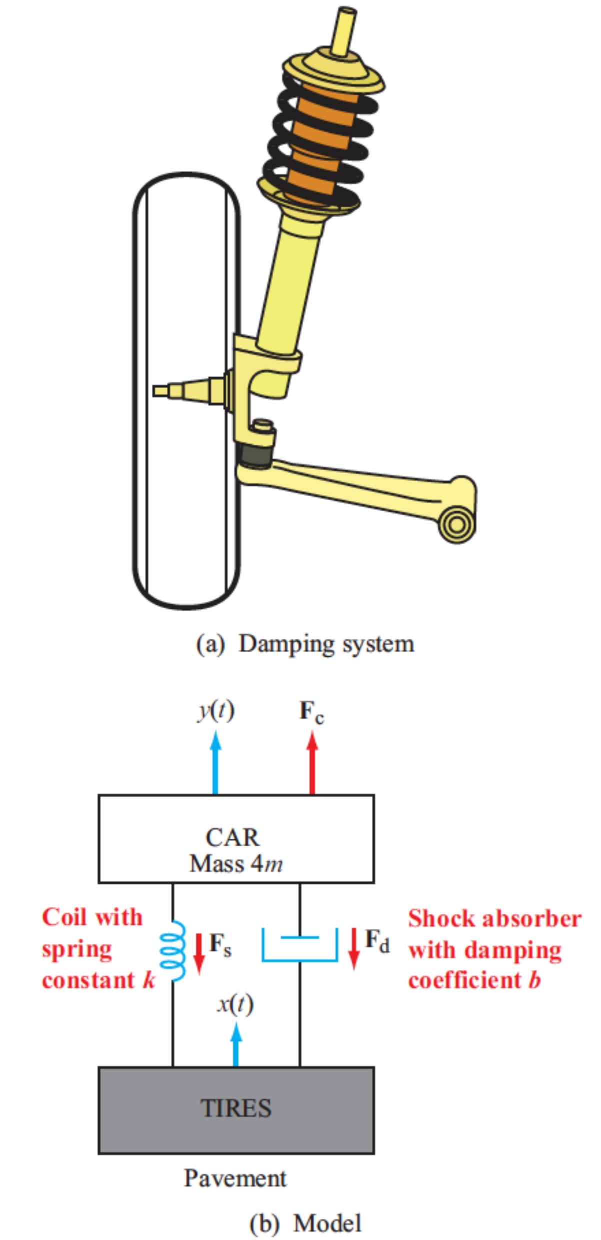


Figure 2-26: Car suspension system model.

Force Equation

$$\mathbf{F}_s = -k(y - x)$$

$$\mathbf{F}_d = -b \frac{d}{dt}(y - x). \quad (2.156)$$

Using Newton's law, $\mathbf{F}_c = ma = m(d^2y/dt^2)$, the force equation is

$$\mathbf{F}_c = \mathbf{F}_s + \mathbf{F}_d \quad (2.157)$$

or

$$m\frac{d^2y}{dt^2} = -k(y - x) - b\frac{d}{dt}(y - x),$$

which can be recast as

$$\frac{d^2y}{dt^2} + \frac{b}{m}\frac{dy}{dt} + \frac{k}{m}y = \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x.$$

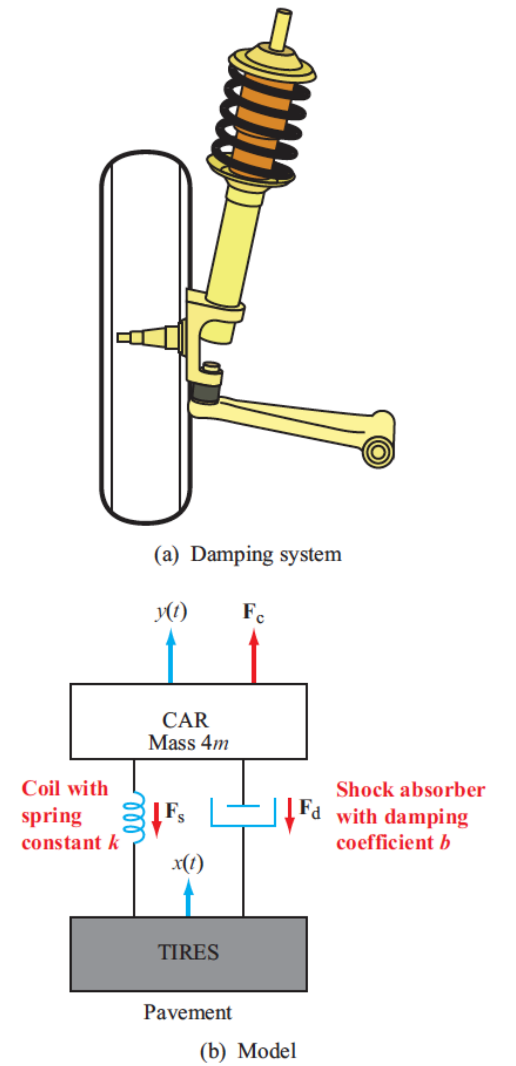


Figure 2-26: Car suspension system model.

Typical Car Parameters

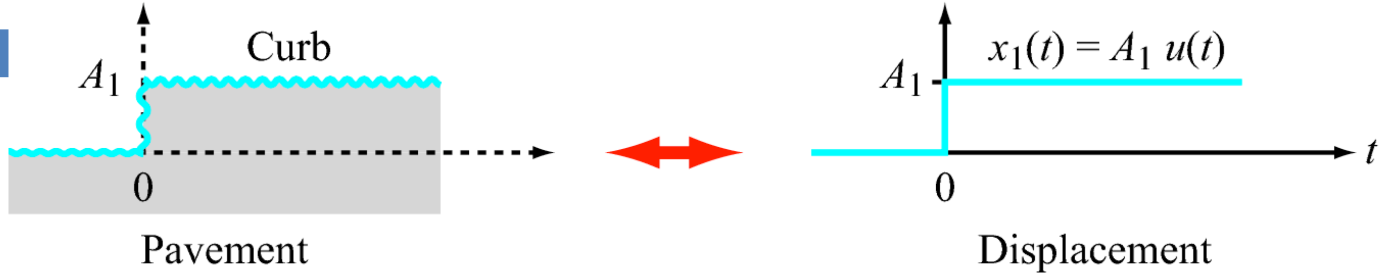
Typical values for a small automobile are:

- $m = 250$ kg for a car with a total mass of one metric ton (1000 kg); each wheel supports one-fourth of the car's mass.

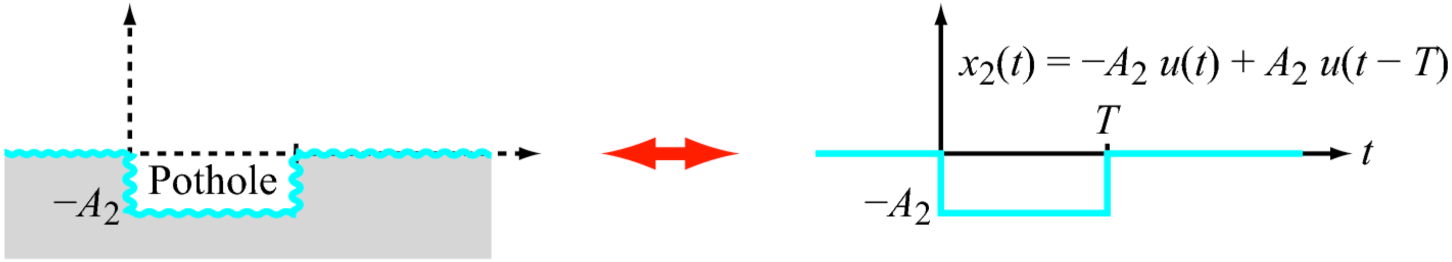
- $k = 10^5$ N/m; it takes a force of 1000 N to compress the spring by 1 cm.

- $b = 10^4$ N·s/m; a vertical motion of 1 m/s incurs a resisting force of 10^4 N.

Pavement Models



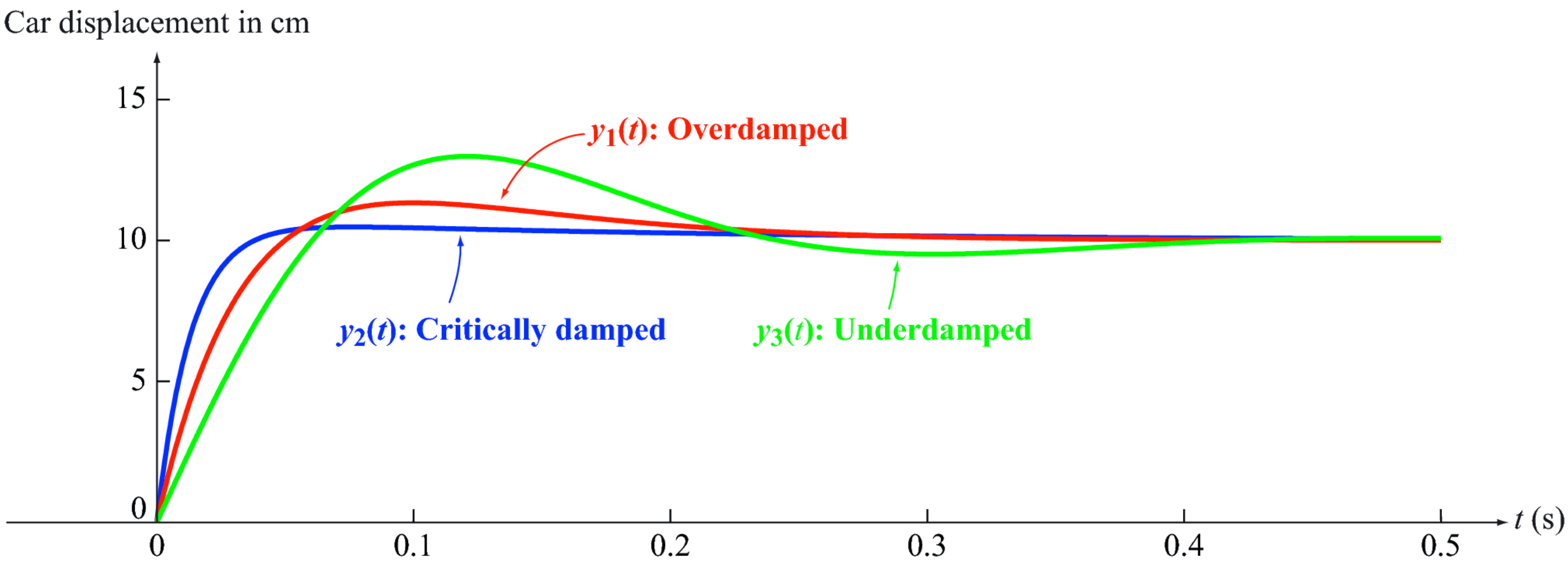
(a) Curb



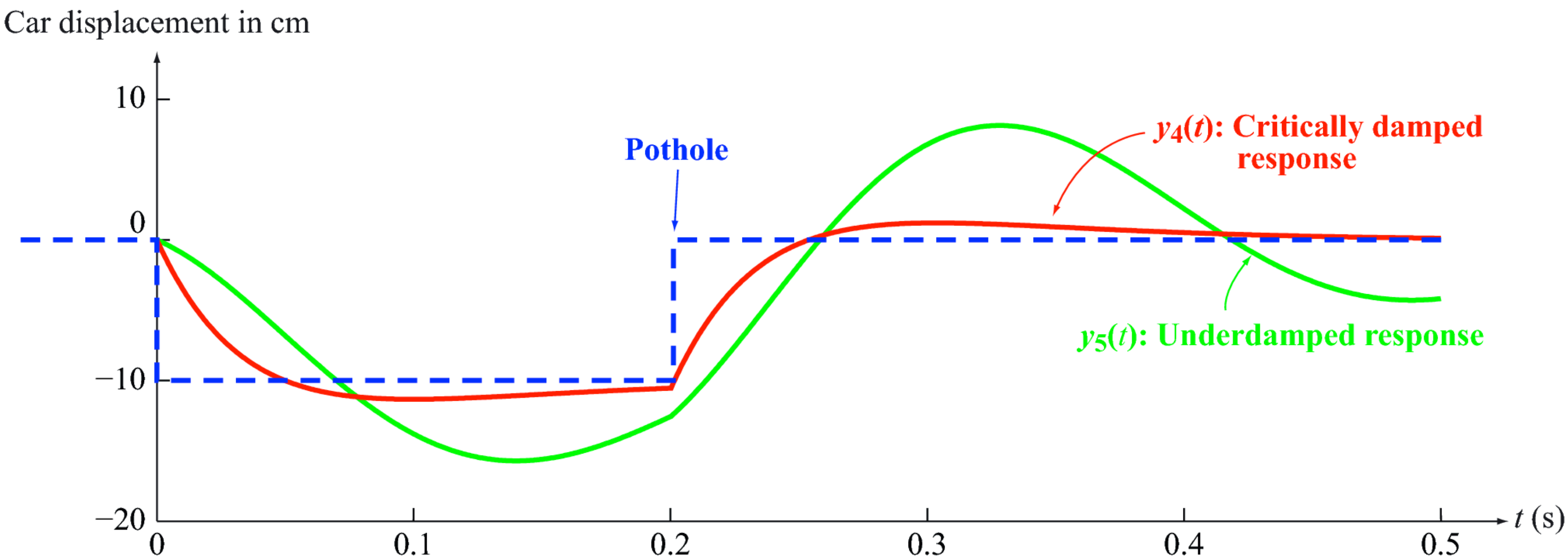
(b) Pothole

(c) Wavy pavement

Car's Response to Driving over 10-cm Curb



Car's Response to Driving over 10-cm Deep Pothole



Module 2.2 **Automobile Suspension Response** Select curb, pothole, or wavy pavement. Then, select the pavemen characteristics, the automobile's mass, and its suspension's spring constant and damping coefficient.