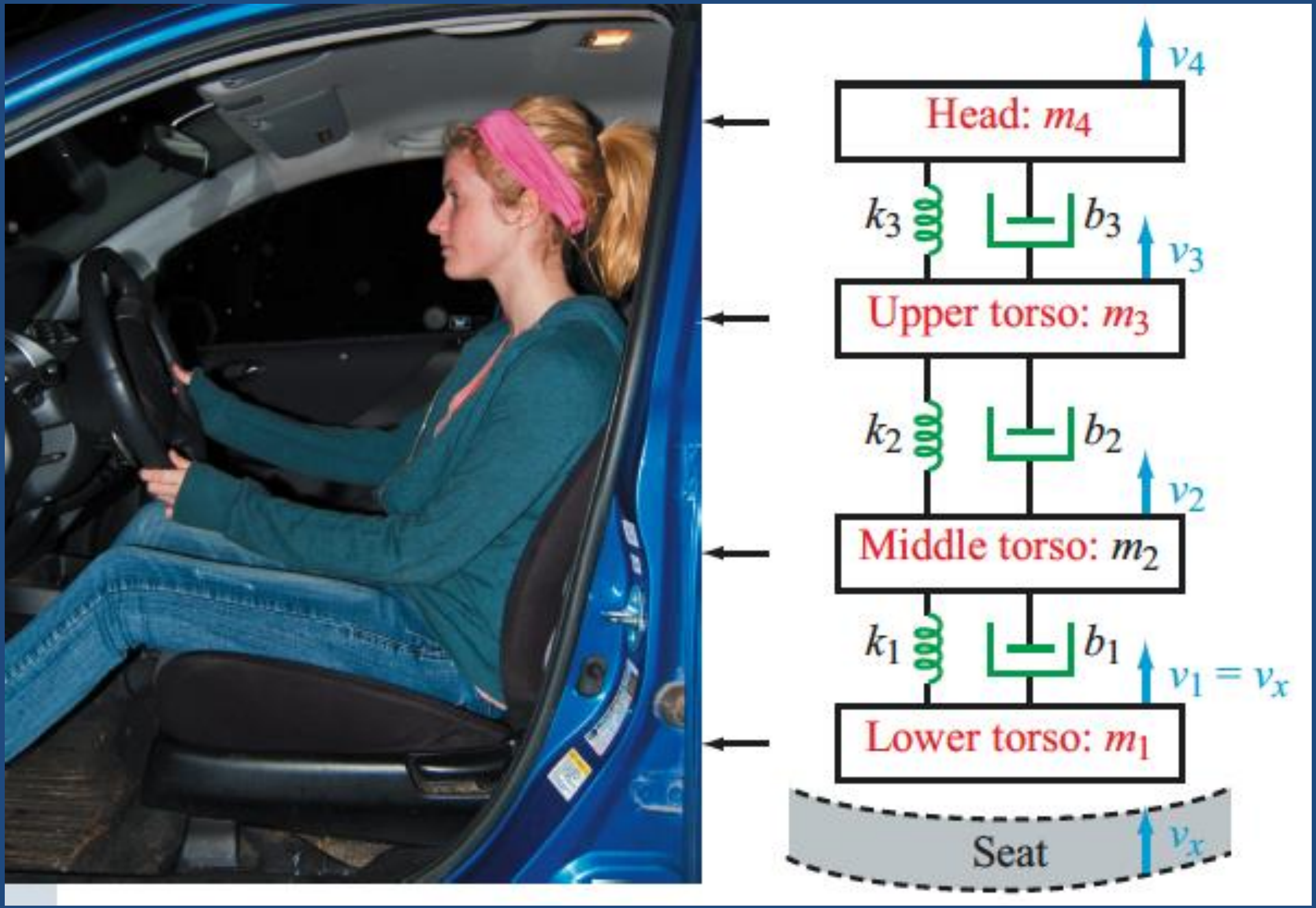


4.APPLICATIONS OF THE LAPLACE TRANSFORM

Signals & Systems, by Ulaby & Yagle

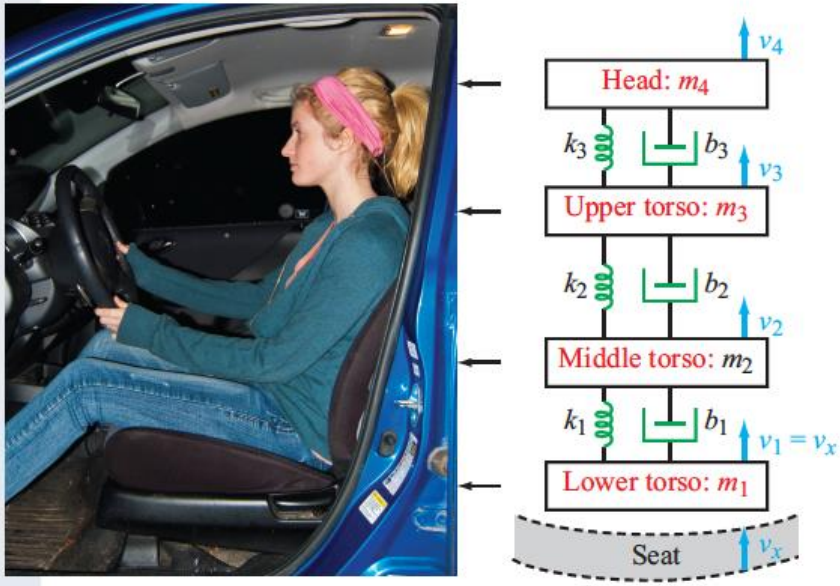
4 Applications of the Laplace Transform

Contents

Objectives

Learn to:

- Use s-domain circuit element models to analyze electric circuits.
- Use electromechanical analogues to simulate and analyze mechanical systems.
- Use op-amp circuits to implement systems.
- Develop system realizations that conform to specified transfer functions.
- Employ feedback control techniques to improve system performance and stability



The Laplace-transform tools learned in the previous chapter are now applied to model and solve a wide variety of ***mechanical and thermal systems***, including how to compute the movement of a passenger's head as the car moves over curbs and other types of pavements, and how to design ***feedback loops*** to control ***motors*** and heating systems.

s-Domain Circuit Element Models

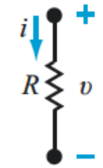
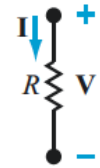
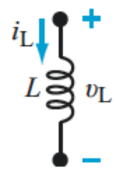
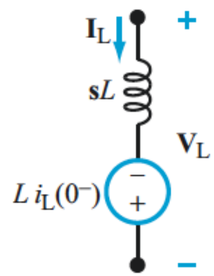
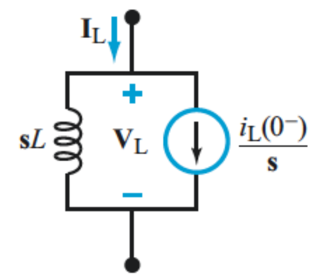
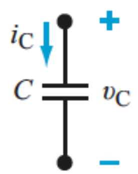
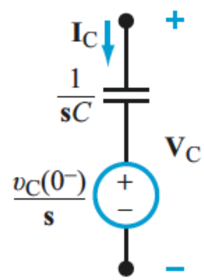
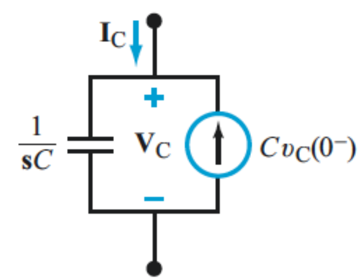
▶ The s-domain transformation of circuit elements incorporates initial conditions associated with any energy storage that may have existed in capacitors and inductors at $t = 0^-$. ◀

$$v = Ri \quad \longleftrightarrow \quad \mathbf{V} = R\mathbf{I}.$$

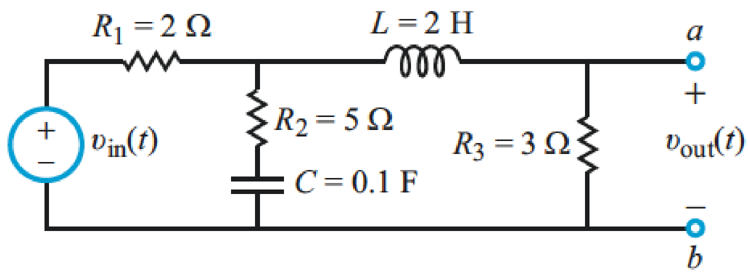
$$v = L\frac{di}{dt} \quad \longleftrightarrow \quad \mathbf{V} = sL\mathbf{I} - L\,i(0^-).$$

$$i = C\frac{dv}{dt} \quad \longleftrightarrow \quad \mathbf{I} = sC\mathbf{V} - C\,v(0^-),$$

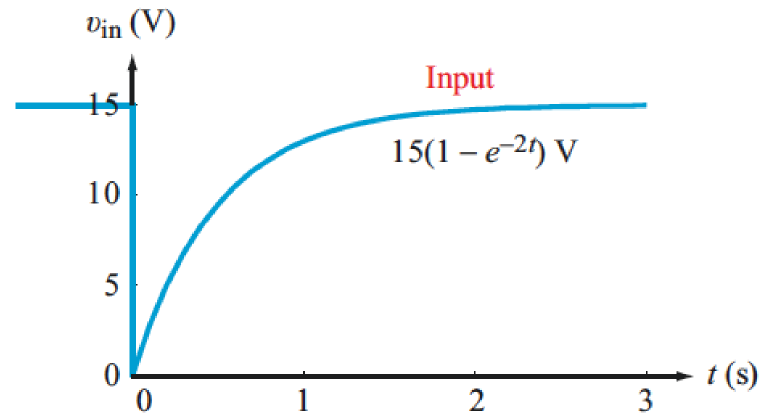
Table 4-1: Circuit models for R, L, and C in the s-domain.

Time-Domain	s-Domain		
Resistor  $\upsilon = Ri$	 $\mathbf{V} = R\mathbf{I}$		
Inductor  $\upsilon_L = L \dfrac{di_L}{dt}$ $i_L = \dfrac{1}{L}\int_{0^-}^{t} \upsilon_L \, dt' + i_L(0^-)$	 $\mathbf{V}_L = \mathbf{s}L\mathbf{I}_L - L\,i_L(0^-)$	**OR**	 $\mathbf{I}_L = \dfrac{\mathbf{V}_L}{\mathbf{s}L} + \dfrac{i_L(0^-)}{\mathbf{s}}$
Capacitor  $i_C = C \dfrac{d\upsilon_C}{dt}$ $\upsilon_C = \dfrac{1}{C}\int_{0^-}^{t} i_C \, dt' + \upsilon_C(0^-)$	 $\mathbf{V}_C = \dfrac{\mathbf{I}_C}{\mathbf{s}C} + \dfrac{\upsilon_C(0^-)}{\mathbf{s}}$	**OR**	 $\mathbf{I}_C = \mathbf{s}C\mathbf{V}_C - C\,\upsilon_C(0^-)$

Example 4-1: Interrupted Voltage Source

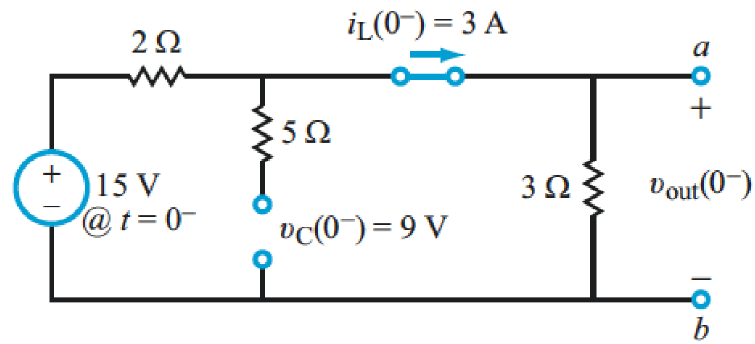


(a) Time domain



(b) Waveform of $v_{in}(t)$

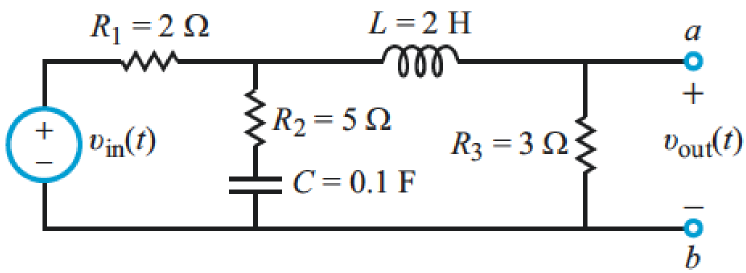
Initial Conditions:



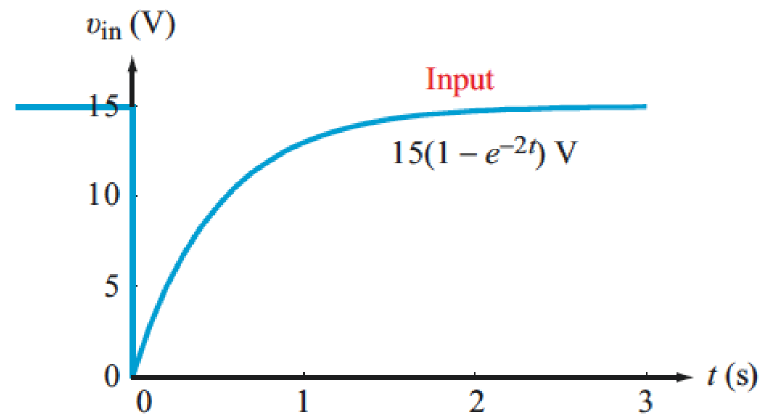
(d) At $t = 0^-$

$$v_C(0^-) = 9\text{ V},\ \ i_L(0^-) = 3\text{ A},\ \text{and } v_{out}(0^-) = 9\text{ V}$$

Example 4-1: Interrupted Voltage Source



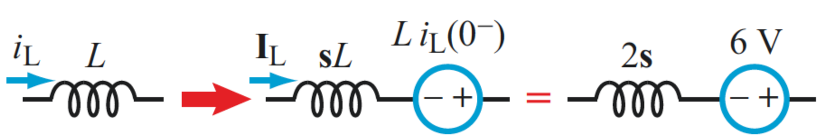
(a) Time domain



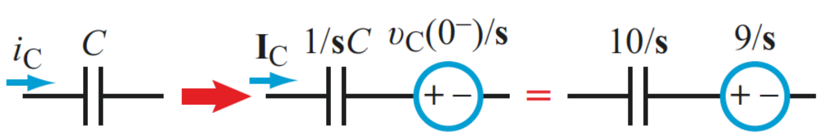
(b) Waveform of $v_{in}(t)$

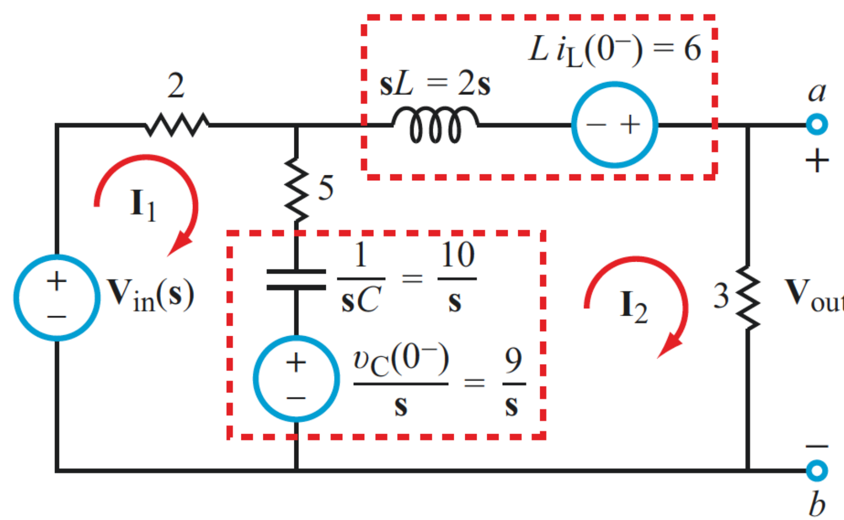
$$\mathbf{V}_{in}(\mathbf{s}) = \frac{15}{\mathbf{s}} - \frac{15}{\mathbf{s}+2}$$

Time Domain **s-Domain**



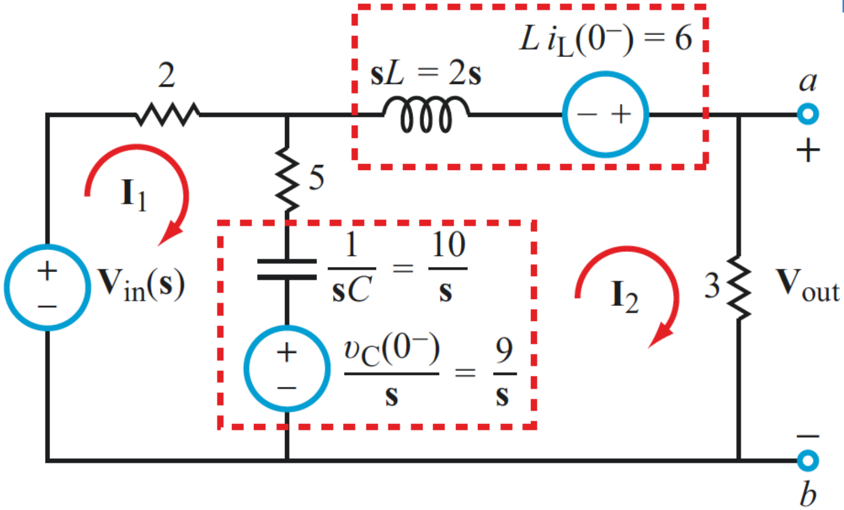
and





(e) s-domain

Example 4-1: Interrupted Voltage Source



(e) s-domain

$$\mathbf{V}_{in}(\mathbf{s}) = \frac{15}{\mathbf{s}} - \frac{15}{\mathbf{s}+2}$$

By inspection, the mesh-current equations for loops 1 and 2 are given by

$$\left(2 + 5 + \frac{10}{\mathbf{s}}\right)\mathbf{I}_1 - \left(5 + \frac{10}{\mathbf{s}}\right)\mathbf{I}_2 = \mathbf{V}_{in} - \frac{9}{\mathbf{s}} \qquad (4.13)$$

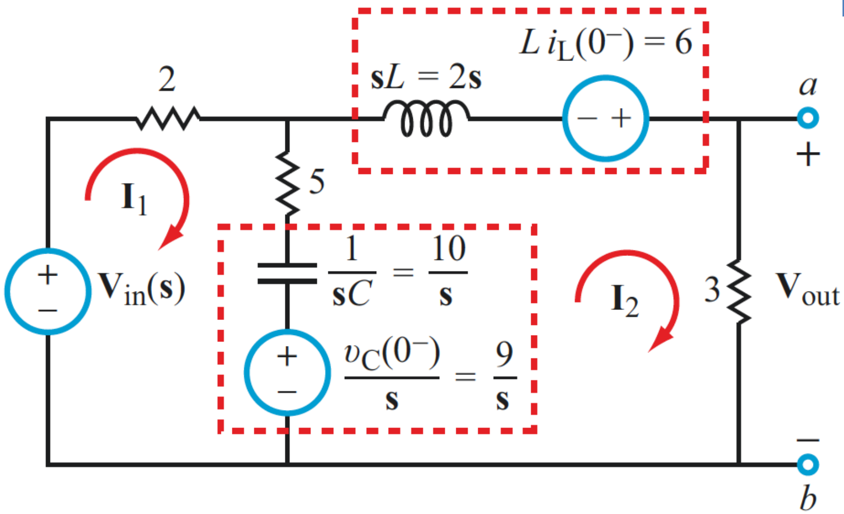
and

$$-\left(5 + \frac{10}{\mathbf{s}}\right)\mathbf{I}_1 + \left(3 + 5 + 2\mathbf{s} + \frac{10}{\mathbf{s}}\right)\mathbf{I}_2 = \frac{9}{\mathbf{s}} + 6. \qquad (4.14)$$

Simultaneous solution leads to:

$$\mathbf{I}_2 = \frac{42\mathbf{s}^3 + 162\mathbf{s}^2 + 306\mathbf{s} + 300}{\mathbf{s}(\mathbf{s}+2)(14\mathbf{s}^2 + 51\mathbf{s} + 50)}$$
$$= \frac{42\mathbf{s}^3 + 162\mathbf{s}^2 + 306\mathbf{s} + 300}{14\mathbf{s}(\mathbf{s}+2)(\mathbf{s}^2 + 51\mathbf{s}/14 + 50/14)} .$$

Example 4-1: Interrupted Voltage Source



(e) s-domain

$$\mathbf{I}_2 = \frac{42s^3 + 162s^2 + 306s + 300}{s(s+2)(14s^2 + 51s + 50)}$$

$$= \frac{42s^3 + 162s^2 + 306s + 300}{14s(s+2)(s^2 + 51s/14 + 50/14)}.$$

$$= \frac{42s^3 + 162s^2 + 306s + 300}{14s(s+2)(s+1.82+j0.5)(s+1.82-j0.5)}$$

Partial fraction expansion:

$$\mathbf{I}_2 = \frac{3}{s} + \frac{5.32e^{-j90^\circ}}{s+1.82+j0.5} + \frac{5.32e^{j90^\circ}}{s+1.82-j0.5}$$

Laplace Transform pairs:

$$\frac{3}{s} \leftrightarrow 3\,u(t),$$

and from property #3 of Table 3-3, we have

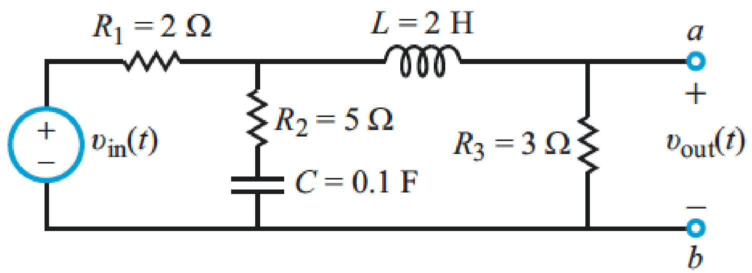
$$\frac{Ae^{j\theta}}{s+a+jb} + \frac{Ae^{-j\theta}}{s+a-jb} \leftrightarrow 2Ae^{-at}\cos(bt-\theta)\,u(t).$$

Time-domain current:

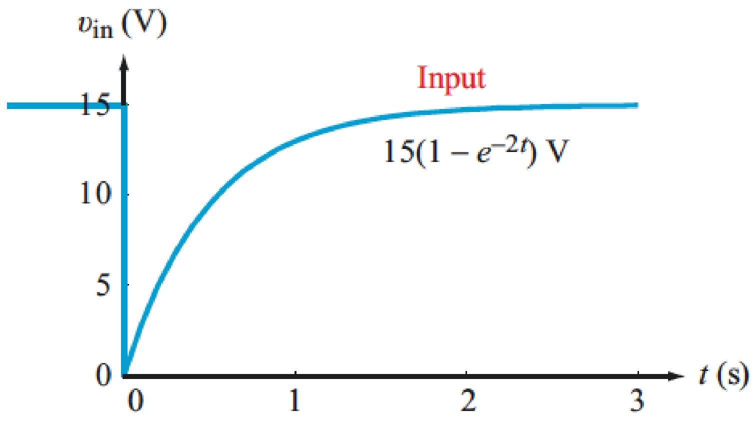
$$i_2(t) = [3 + 10.64e^{-1.82t}\cos(0.5t + 90^\circ)]\,u(t)$$

$$= [3 - 10.64e^{-1.82t}\sin 0.5t]\,u(t)\text{ A},$$

Example 4-1: Interrupted Voltage Source

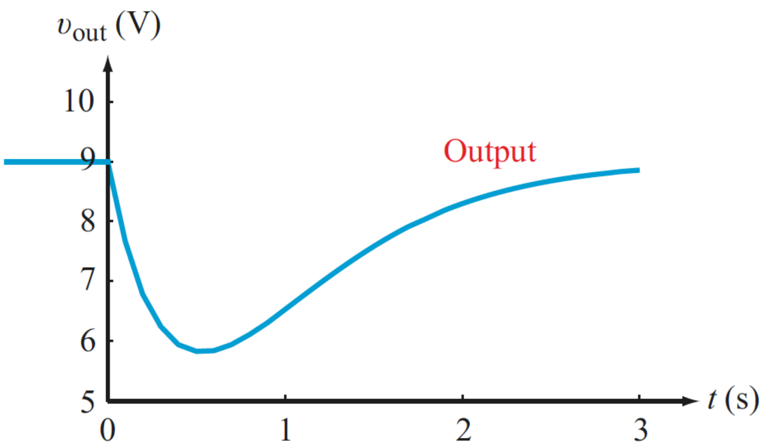


(a) Time domain



(b) Waveform of $v_{in}(t)$

$$
\begin{aligned}
\upsilon_{out}(t) &= 3i_2(t) \\
&= [9 - 31.92e^{-1.82t} \sin 0.5t]\, u(t) \text{ V}.
\end{aligned}
$$



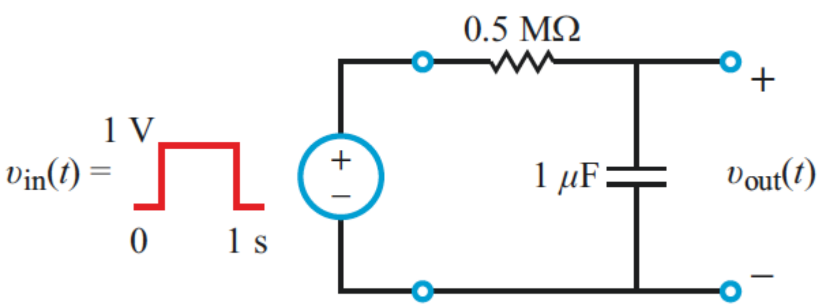
(c) Waveform of $v_{out}(t)$

Example 4-4: Lowpass Filter Response to a Rectangular Pulse

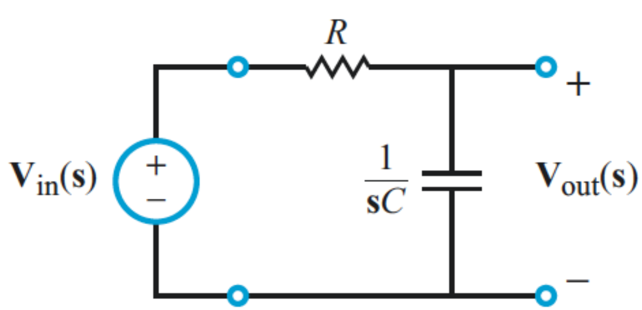
Solution:

With $R = 0.5$ MΩ and $C = 1$ μF, the product is $RC = 0.5$ s. Voltage division in the **s**-domain (Fig. 4-4(b)) leads to

$$\mathbf{H}(\mathbf{s}) = \frac{\mathbf{V}_{\text{out}}(\mathbf{s})}{\mathbf{V}_{\text{in}}(\mathbf{s})} = \frac{1/\mathbf{s}C}{R + 1/\mathbf{s}C} = \frac{1/RC}{\mathbf{s} + 1/RC} = \frac{2}{\mathbf{s}+2}. \tag{4.43}$$



(a) RC lowpass filter



(b) **s**-domain

The rectangular pulse is given by

$$\upsilon_{\text{in}}(t) = [u(t) - u(t-1)] \text{ V},$$

and with the help of Table 3-2, its **s**-domain counterpart

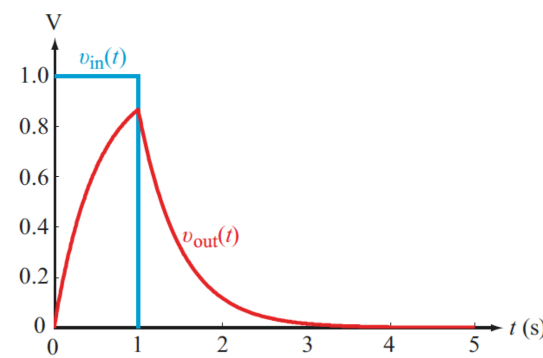
$$\mathbf{V}_{\text{in}}(\mathbf{s}) = \left[\frac{1}{\mathbf{s}} - \frac{1}{\mathbf{s}} e^{-\mathbf{s}}\right] \text{V}.$$

Hence,

$$\begin{aligned}\mathbf{V}_{\text{out}}(\mathbf{s}) &= \mathbf{H}(\mathbf{s})\ \mathbf{V}_{\text{in}}(\mathbf{s}) \\ &= 2(1 - e^{-\mathbf{s}})\left[\frac{1}{\mathbf{s}(\mathbf{s}+2)}\right].\end{aligned}$$

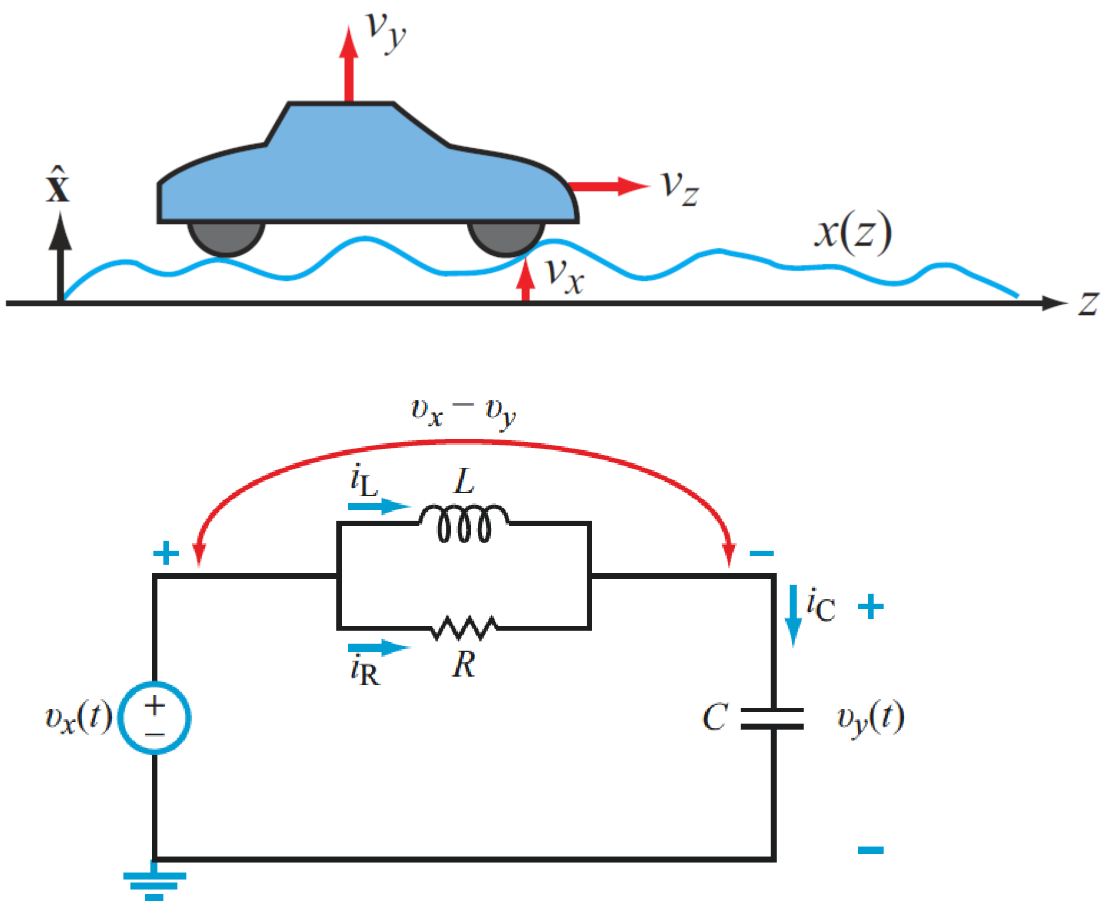
$$\mathbf{V}_{\text{out}}(\mathbf{s}) = \frac{1}{\mathbf{s}} - \frac{1}{\mathbf{s}+2} - \frac{1}{\mathbf{s}} e^{-\mathbf{s}} + \frac{1}{\mathbf{s}+2} e^{-\mathbf{s}}.$$

$$\upsilon_{\text{out}}(t) = \left[[1 - e^{-2t}]\, u(t) - [1 - e^{-2(t-1)}]\, u(t-1)\right] \text{V}.$$

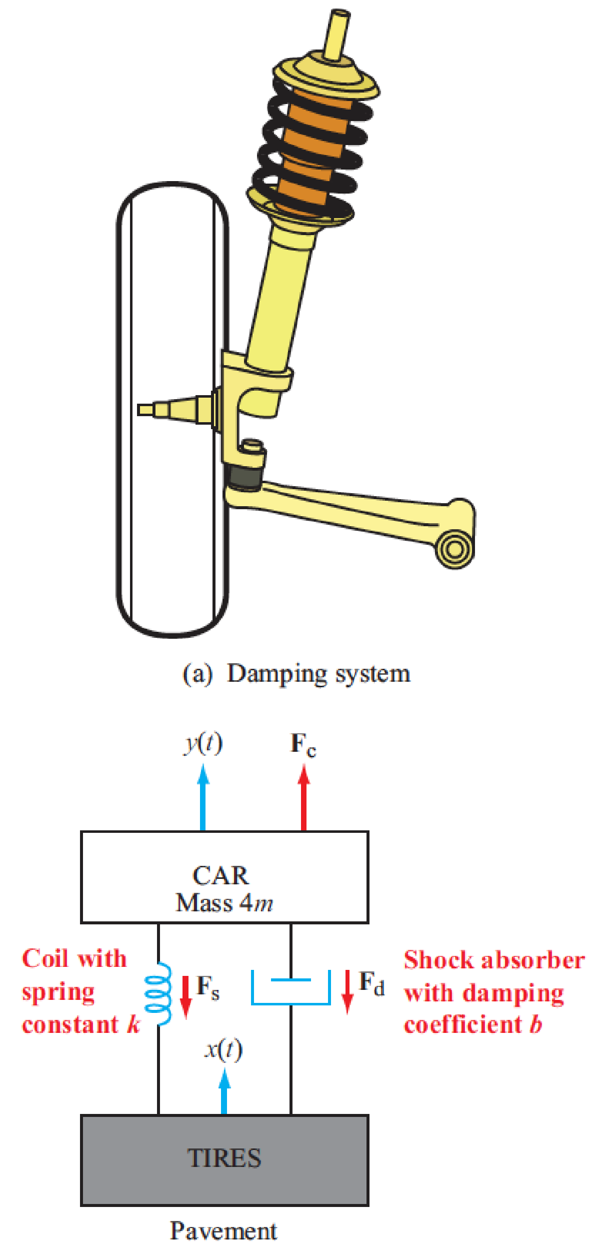


(c) Output response

Electromechanical Analogues



(b) Electrical system



(a) Damping system

(b) Model

Figure 2-26: Car suspension system model.

Table 4-2: Mechanical-electrical analogue.

Mechanical		Electrical
• Force **F** **F** is positive when pointing upwards	⟷	Current i i is positive when entering positive voltage terminal of device
• Vertical velocity v v is positive when car or tire is moving upwards	⟷	Voltage υ υ's positive terminal is where i enters device
• Mass m (1/4 of car's mass) $F_c = m \dfrac{dv_y}{dt}$	⟷	Capacitance C $i_C = C \dfrac{d\upsilon_y}{dt}$
• Spring constant k $F_s = k \int_0^t (v_x - v_y)\, d\tau$	⟷	$1/L$: Inverse of inductance $i_L = \dfrac{1}{L} \int_0^t (\upsilon_x - \upsilon_y)\, d\tau$
• Damping coefficient b $F_d = b(v_x - v_y)$	⟷	$1/R$: Inverse of resistance (conductance) $i_R = \dfrac{1}{R} (\upsilon_x - \upsilon_y)$
• $F_c = F_s + F_d$	⟷	$i_C = i_L + i_R$

SMD-RLC Analysis Procedure

Step 1: **Replace each mass with a capacitor** with one terminal connected to a node and the other to ground.

Step 2: **Replace each spring with an inductor** with $L = 1/k$, where k is the spring's stiffness coefficient.

- If the spring connects two masses, its equivalent inductor connects to their equivalent capacitors at their non-ground terminals.
- If the spring connects a mass to a stationary surface, its equivalent inductor should be connected between the capacitor's non-ground terminal and ground.
- If one end of the spring connects to a moving surface, the corresponding terminal of its equivalent inductor should be connected to a voltage source.

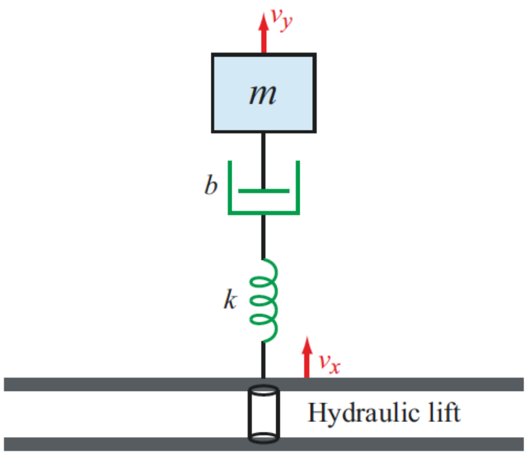
Step 3: **Replace each damper with a resistor** with $R = 1/b$. Connection rules are the same as for springs.

Step 4: **Analyze the RLC circuit** using the **s**-domain technique described in Section 4-2.

The solution of the RLC circuit provides expressions for the voltages across capacitors, corresponding to the velocities of their counterpart masses in the mechanical system. Displacement of a mass or its acceleration can be obtained by integrating or differentiating its velocity $v(t)$, respectively.

Hydraulic Lift Example

. The lift was used to raise the platform by 4 m at a constant speed of 0.5 m/s. Determine the corresponding vertical speed and displacement of the mass m, given that $m = 150$ kg, $k = 1200$ N/m, and $b = 200$ N·s/m.



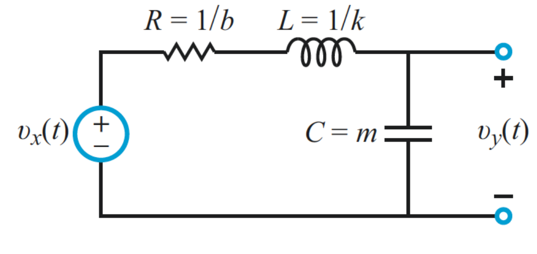
(a) Series system

0.5 m/s over a distance of 4 m, which corresponds to a travel time of $4/0.5 = 8$ s, $v_x(t)$ is a rectangle waveform given by

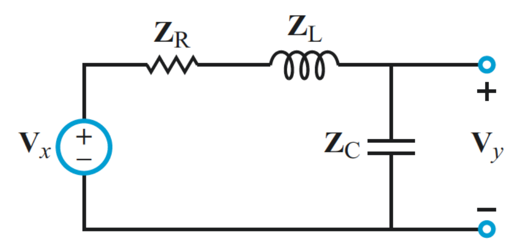
$$v_x(t) = 0.5[u(t) - u(t-8)] \text{ m/s.} \tag{4.60}$$

Using entries #2 and #2a in Table 3-2, the Laplace transform of $v_x(t)$ is

$$\mathbf{V}_x = \frac{0.5}{\mathbf{s}} - \frac{0.5}{\mathbf{s}} e^{-8\mathbf{s}}. \tag{4.61}$$



(b) Equivalent circuit



$\mathbf{Z}_R = 1/b$, $\mathbf{Z}_L = \mathbf{s}/k$, $\mathbf{Z}_C = 1/\mathbf{s}m$

(c) **s**-domain circuit

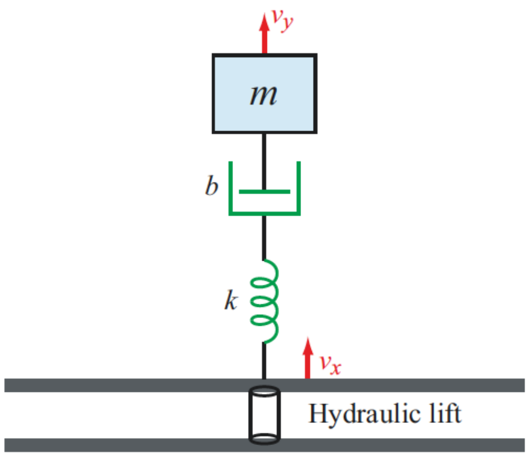
$$\mathbf{V}_y = \frac{\mathbf{V}_x \mathbf{Z}_C}{\mathbf{Z}_R + \mathbf{Z}_L + \mathbf{Z}} = \frac{1/\mathbf{s}m}{\frac{1}{b} + \frac{\mathbf{s}}{k} + \frac{1}{\mathbf{s}m}} \mathbf{V}_x$$

$$= \frac{k/m}{\mathbf{s}^2 + \frac{k}{b}\mathbf{s} + \frac{k}{m}} \mathbf{V}_x$$

$$= \frac{8}{\mathbf{s}^2 + 6\mathbf{s} + 8} \mathbf{V}_x.$$

Hydraulic Lift Example

The lift was used to raise the platform by 4 m at a constant speed of 0.5 m/s. Determine the corresponding vertical speed and displacement of the mass m, given that $m = 150$ kg, $k = 1200$ N/m, and $b = 200$ N·s/m.



(a) Series system

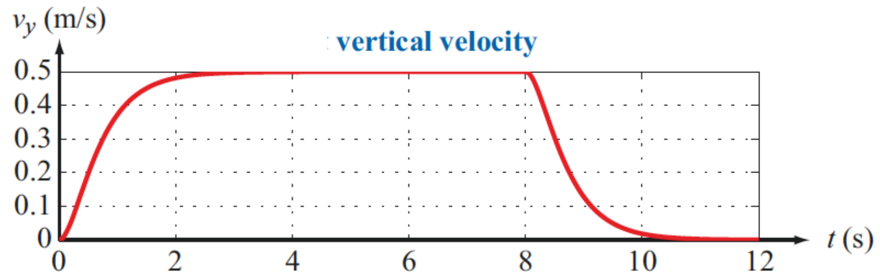
$$\mathbf{V}_y = \frac{4}{\mathbf{s}(\mathbf{s}^2 + 6\mathbf{s} + 8)} - \frac{4e^{-8\mathbf{s}}}{\mathbf{s}(\mathbf{s}^2 + 6\mathbf{s} + 8)}$$
$$= \frac{4}{\mathbf{s}(\mathbf{s}+2)(\mathbf{s}+4)} - \frac{4e^{-8\mathbf{s}}}{\mathbf{s}(\mathbf{s}+2)(\mathbf{s}+4)}$$

$$v_{y_1}(t) = [0.5 - e^{-2t} + 0.5e^{-4t}]\, u(t)$$

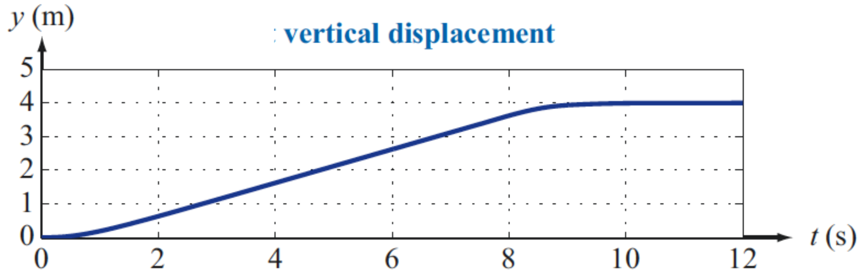
$$v_{y_2}(t) = -[0.5 - e^{-2(t-8)} + 0.5e^{-4(t-8)}]\, u(t-8),$$

$$v_y(t) = v_{y_1}(t) + v_{y_2}(t).$$

$$y(t) = \int_0^t v_y(t)\, d\tau.$$

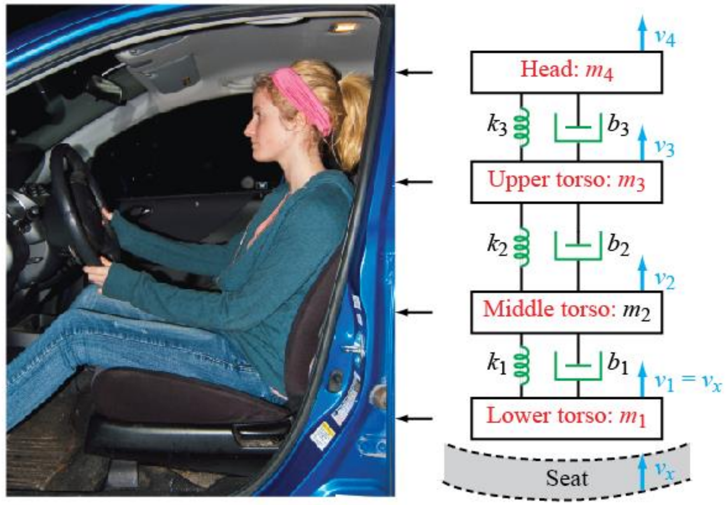


(d) Vertical velocity $v_y(t)$

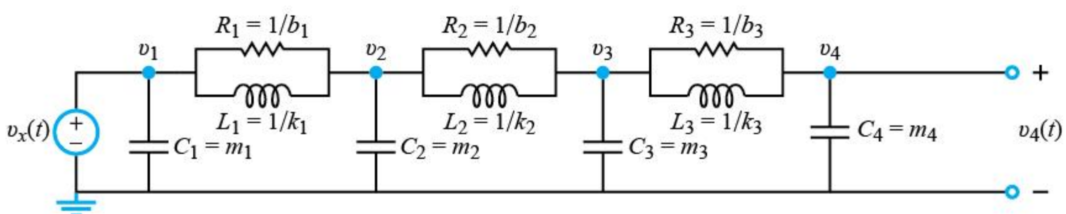


(e) Vertical displacement $y(t)$

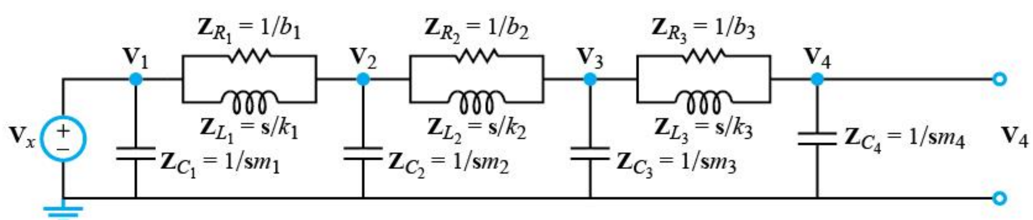
Biomechanical Model of a Person Sitting in a Moving Chair



(a) Series system

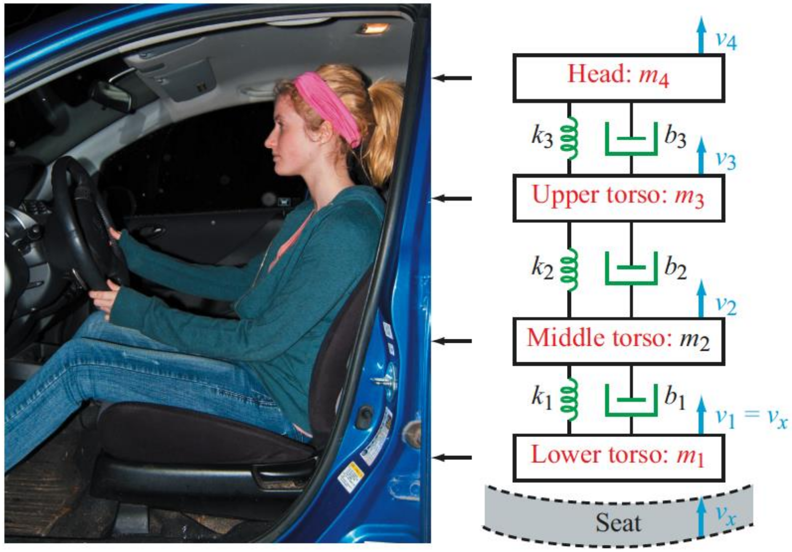


(b) **Electrical circuit analogue (time domain)**

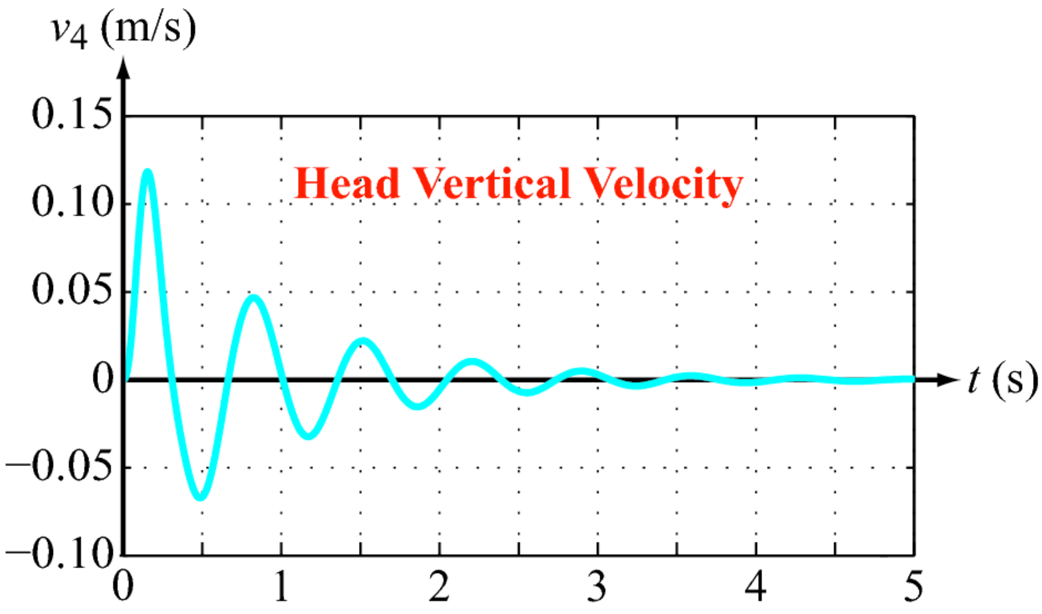


(c) **s-domain circuit**

Example: Car Driving over a 10-cm Curb



(a) Series system



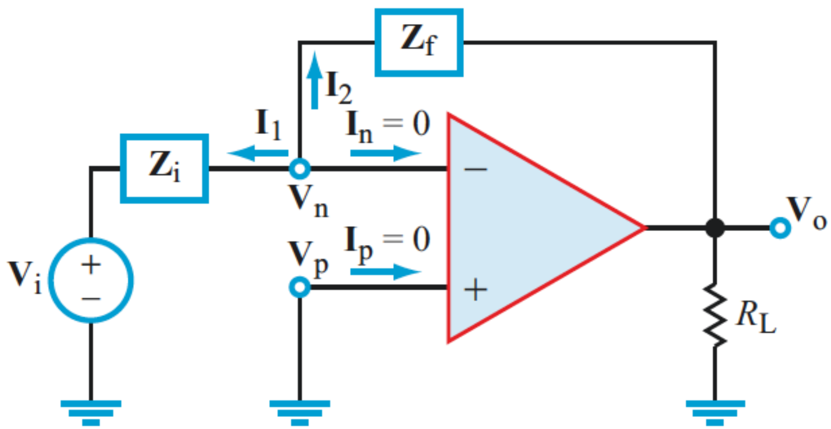
$$x(t) = 0.01\, u(t) \qquad \text{(m)},$$

and the associated vertical velocity is

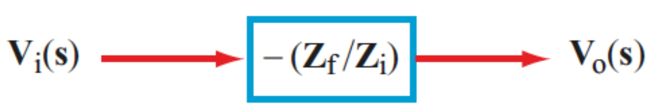
$$v_x(t) = \frac{dx}{dt} = 0.01\, \delta(t).$$

Review of Op-Amp Circuits

Inverting Amplifier



(a) Circuit



(b) Block diagram

$$\mathbf{H}(s) = -\frac{Z_f}{Z_i}$$

Op-Amp Differentiator and Integrator

RC Differentiator

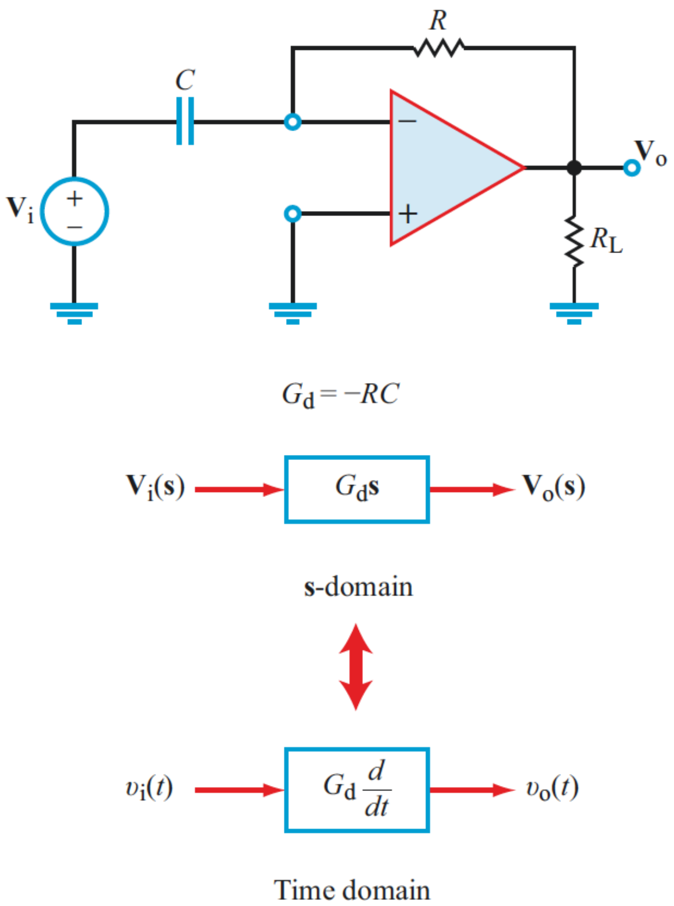


Figure 4-12: Differentiator circuit.

RC Integrator

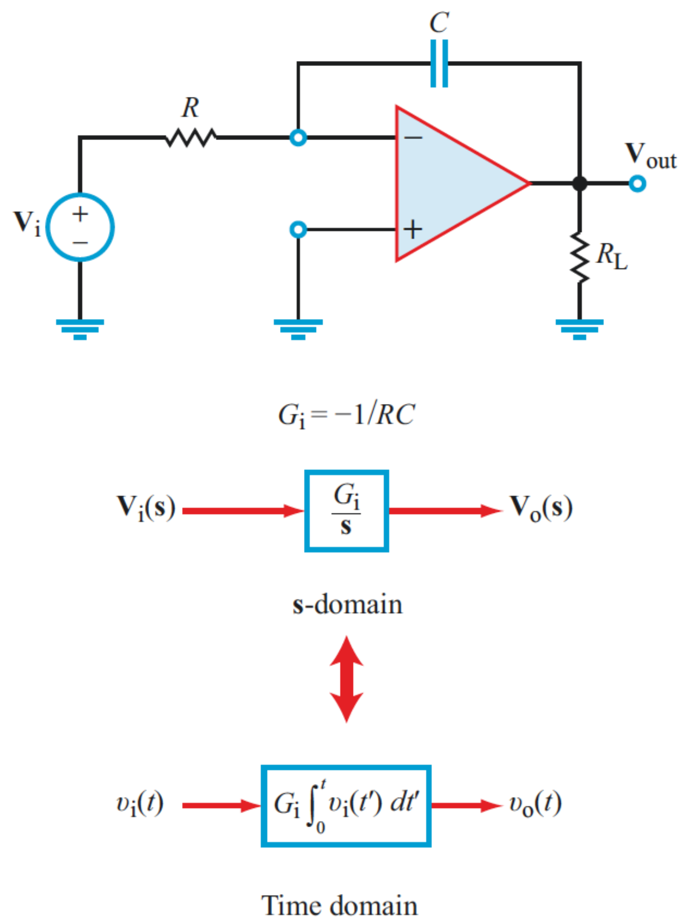
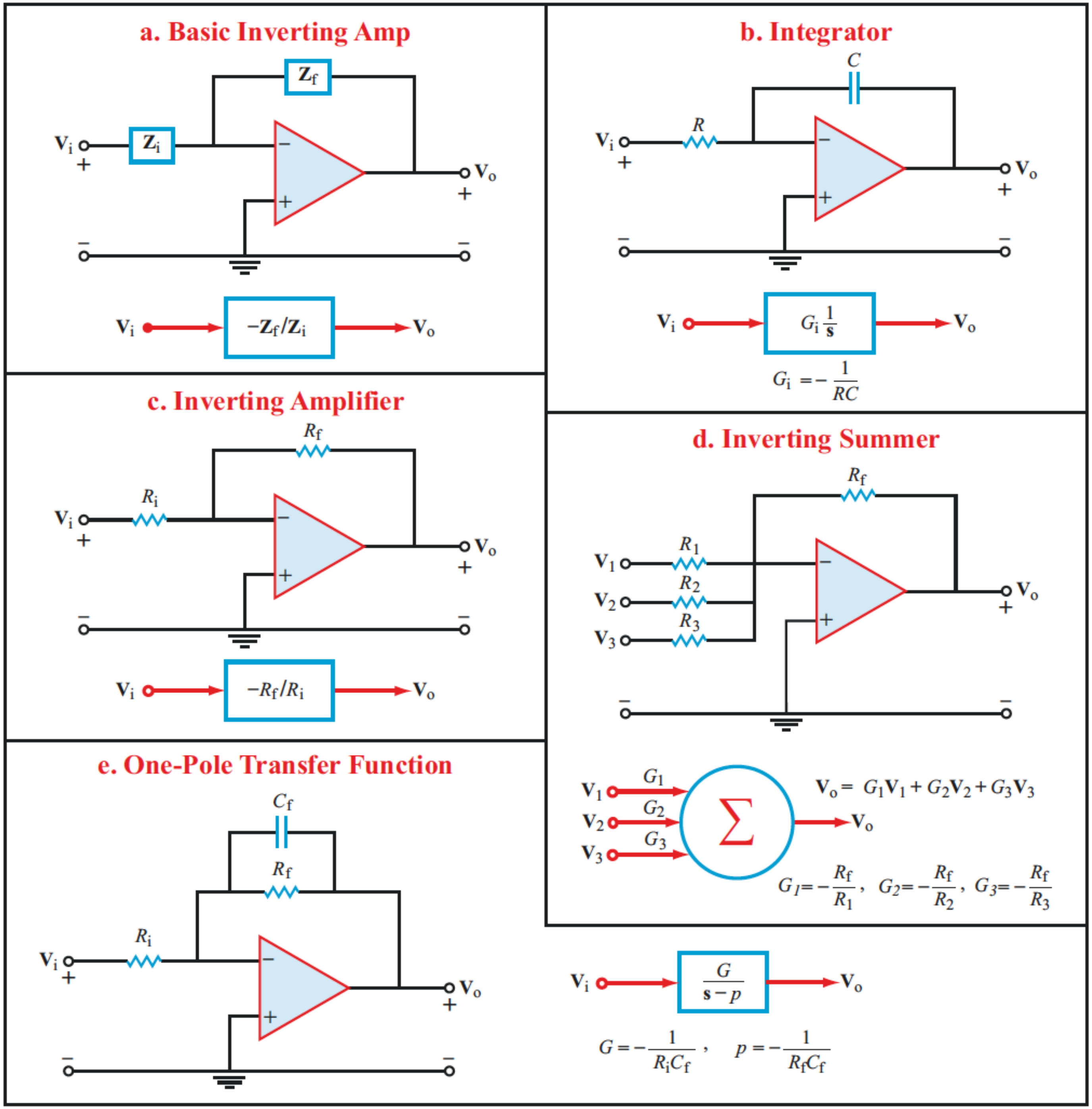
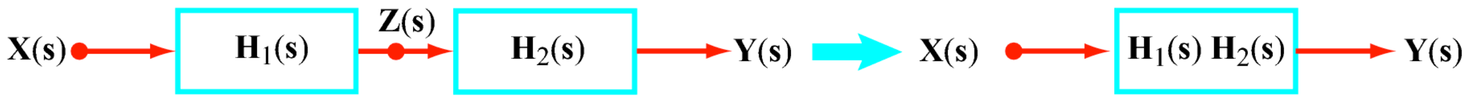


Figure 4-13: Integrator circuit.

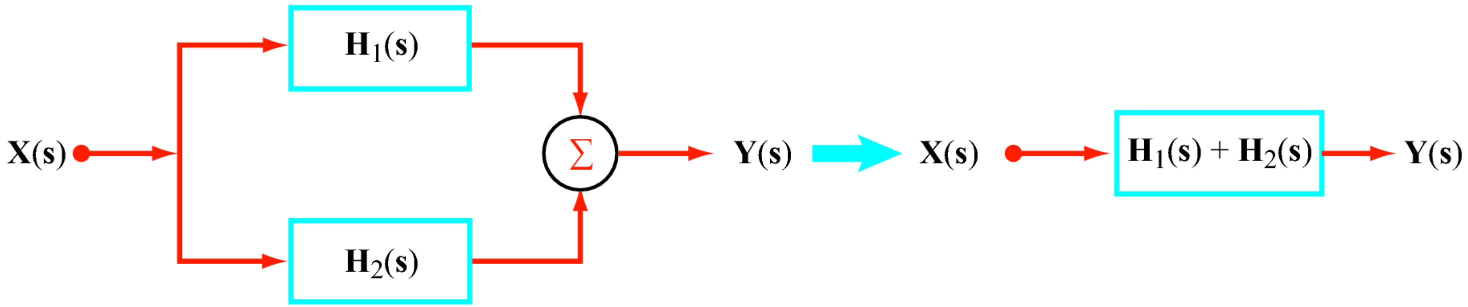
Table 4-3: Op-amp circuits and their block-diagram representations.



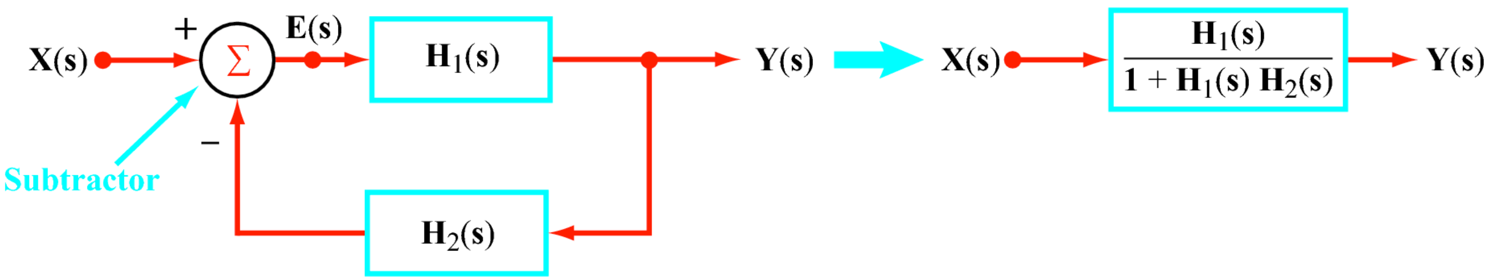
System Configurations



(a) Two cascaded systems



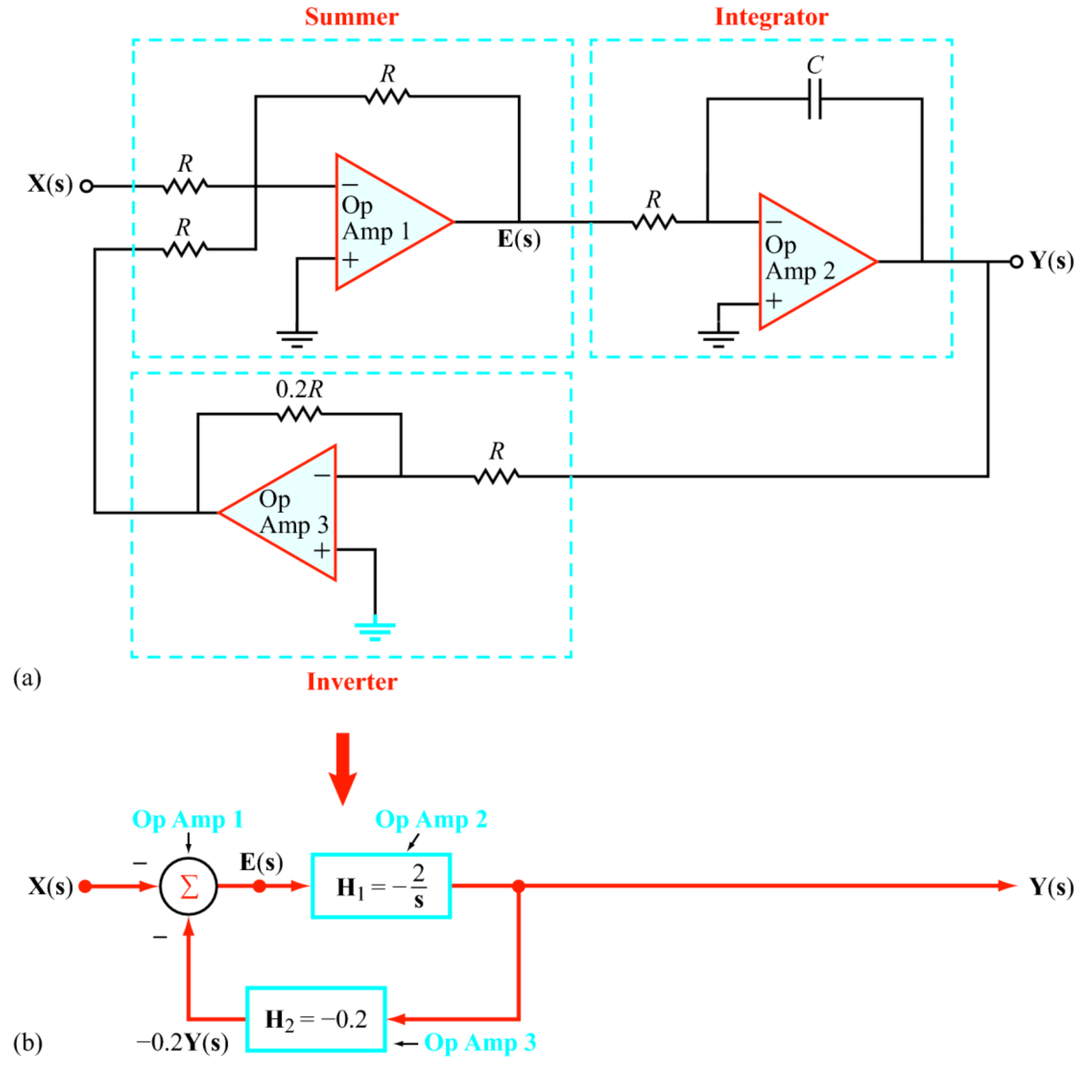
(b) Two parallel systems



(c) Negative feedback system

Example 4-9: Circuit Transfer Function

Determine the overall transfer function $\mathbf{H}(s) = \mathbf{Y}(s)/\mathbf{X}(s)$ of the circuit in Fig. 4-18. Element values are $R = 100\ \text{k}\Omega$ and $C = 5\ \mu\text{F}$.



Op Amp #2: Integrator

$$\mathbf{H}_1(s) = \left(-\frac{1}{RC}\right)\frac{1}{s} = \left(-\frac{1}{10^5 \times 5 \times 10^{-6}}\frac{1}{s}\right) = -\frac{2}{s}\,.$$

Op Amp #3:Inverter

$$\mathbf{H}_2(s) = -\frac{0.2R}{R} = -0.2.$$

Op Amp #1: Summer:

$$\mathbf{E}(s) = -\mathbf{X}(s) - [-0.2\mathbf{Y}(s)] = -\mathbf{X}(s) + 0.2\mathbf{Y}(s).$$

$$\mathbf{Y}(s) = \mathbf{H}_1(s)\,\mathbf{E}(s) = -\frac{2}{s}\,[-\mathbf{X}(s) + 0.2\mathbf{Y}(s)].$$

System Synthesis– Direct Forms I & II Topologies

Given:

$$\mathbf{H}(\mathbf{s}) = \frac{b_0\mathbf{s}^3 + b_1\mathbf{s}^2 + b_2\mathbf{s}}{\mathbf{s}^3 + a_1\mathbf{s}^2 + a_2\mathbf{s} + a_3}$$

procedure entails rewriting the expression in terms of inverse powers of s:

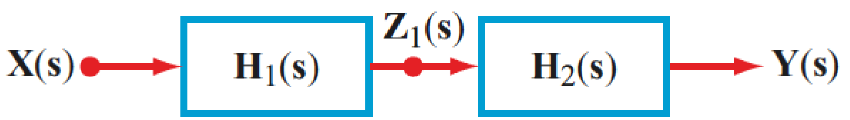
$$\begin{aligned}\mathbf{H}(\mathbf{s}) &= \frac{b_0\mathbf{s}^3 + b_1\mathbf{s}^2 + b_2\mathbf{s}}{\mathbf{s}^3 + a_1\mathbf{s}^2 + a_2\mathbf{s} + a_3} \cdot \frac{1/\mathbf{s}^3}{1/\mathbf{s}^3} \\ &= \left(b_0 + \frac{b_1}{\mathbf{s}} + \frac{b_2}{\mathbf{s}^2}\right)\left(\frac{1}{1 + \frac{a_1}{\mathbf{s}} + \frac{a_2}{\mathbf{s}^2} + \frac{a_3}{\mathbf{s}^3}}\right) \\ &= \mathbf{H}_1(\mathbf{s})\ \mathbf{H}_2(\mathbf{s}), \end{aligned} \tag{4.89}$$

with

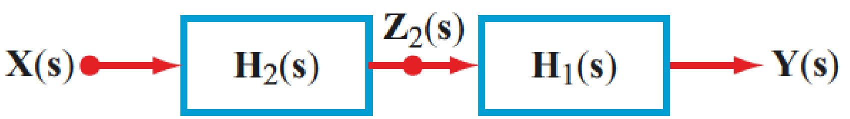
$$\mathbf{H}_1(\mathbf{s}) = b_0 + \frac{b_1}{\mathbf{s}} + \frac{b_2}{\mathbf{s}^2} \tag{4.90a}$$

and

$$\mathbf{H}_2(\mathbf{s}) = \left(1 + \frac{a_1}{\mathbf{s}} + \frac{a_2}{\mathbf{s}^2} + \frac{a_3}{\mathbf{s}^3}\right)^{-1}. \tag{4.90b}$$

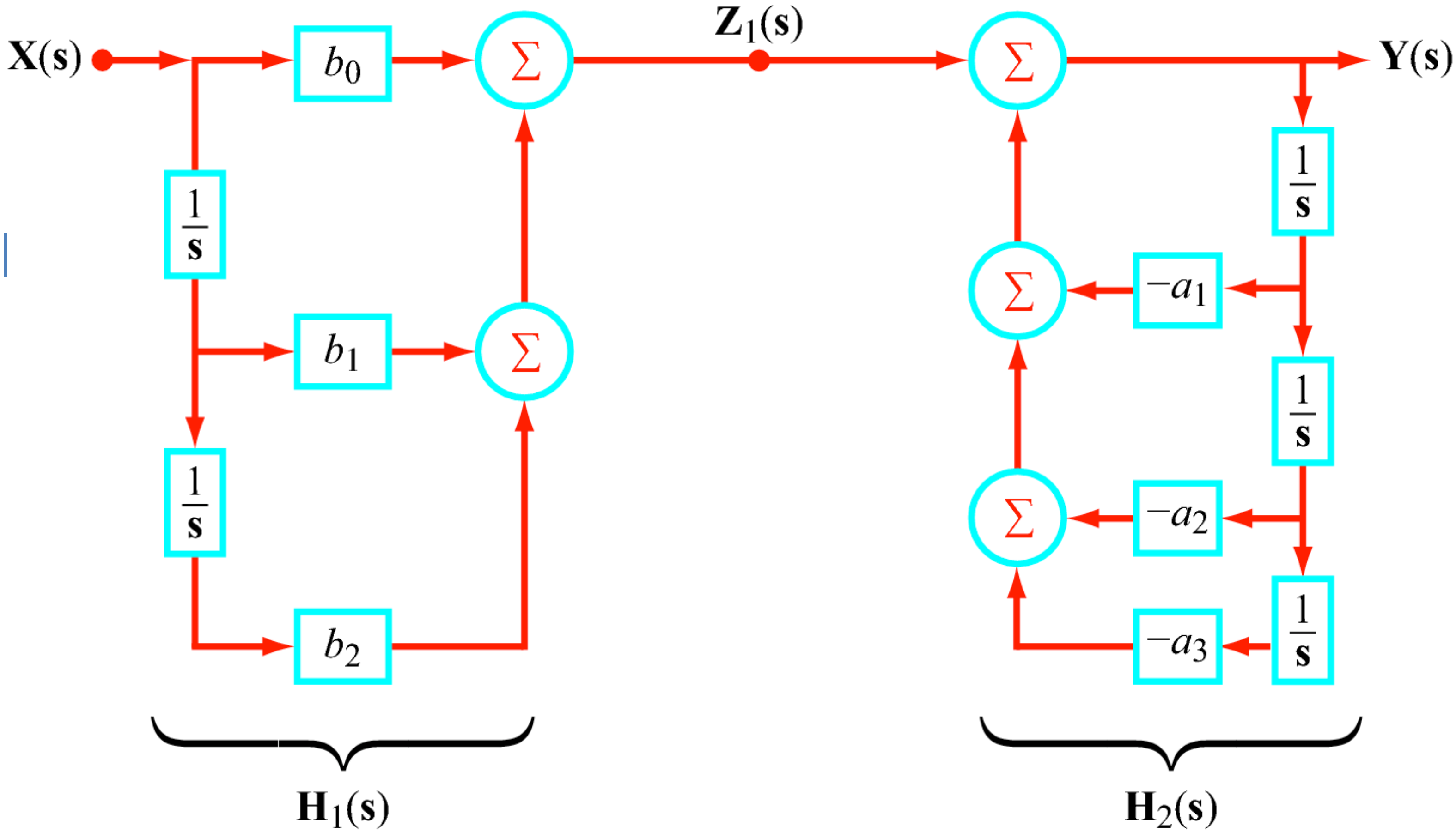


(a) **DFI** realization topology



(b) **DFII** realization topology

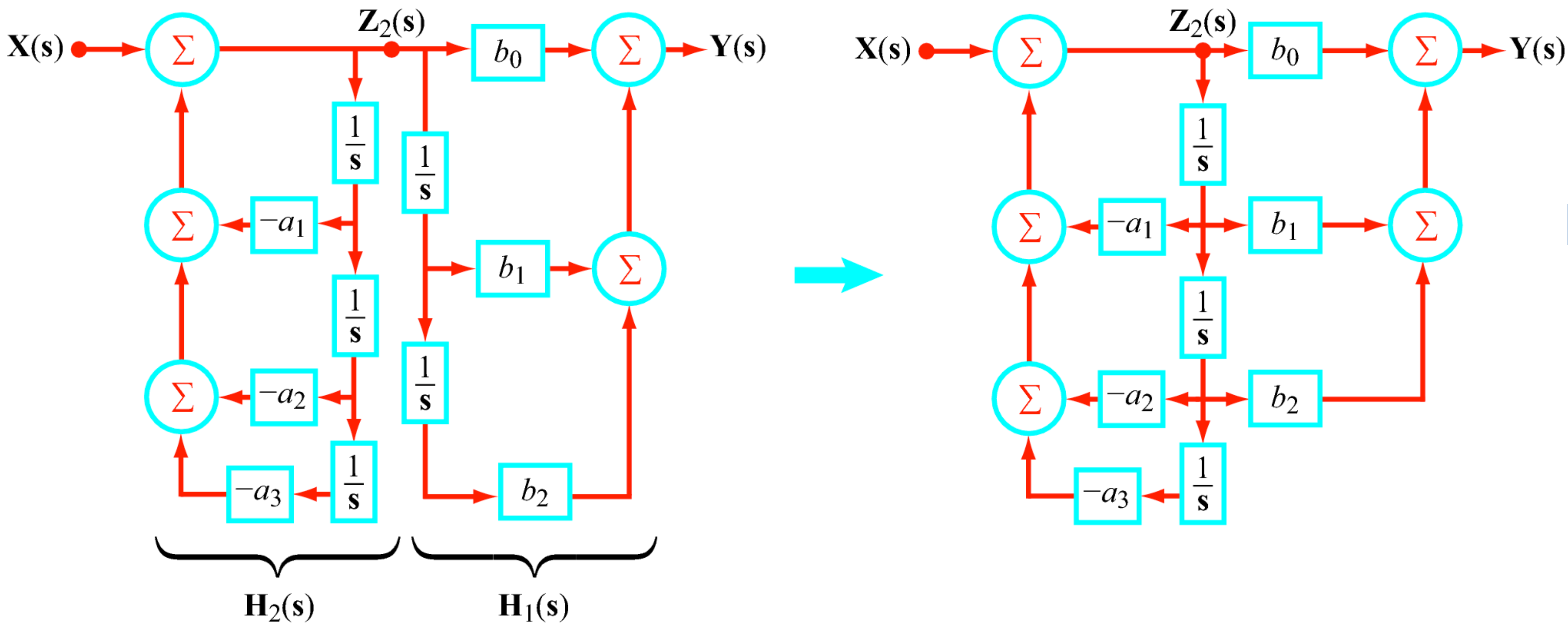
Figure 4-19: In the DFI process, $\mathbf{H}_1(\mathbf{s})$ is realized ahead of $\mathbf{H}_2(\mathbf{s})$, whereas the reverse is the case for the DFII process.



$$\mathbf{H}_1(\mathbf{s}) = b_0 + \frac{b_1}{\mathbf{s}} + \frac{b_2}{\mathbf{s}^2}$$

Direct Form I Topology

$$\mathbf{H}_2(\mathbf{s}) = \left(1 + \frac{a_1}{\mathbf{s}} + \frac{a_2}{\mathbf{s}^2} + \frac{a_3}{\mathbf{s}^3}\right)^{-1}$$



(a) **Separate integrators** (b) **Common integrators**

$$\mathbf{H}_1(\mathbf{s}) = b_0 + \frac{b_1}{\mathbf{s}} + \frac{b_2}{\mathbf{s}^2}$$

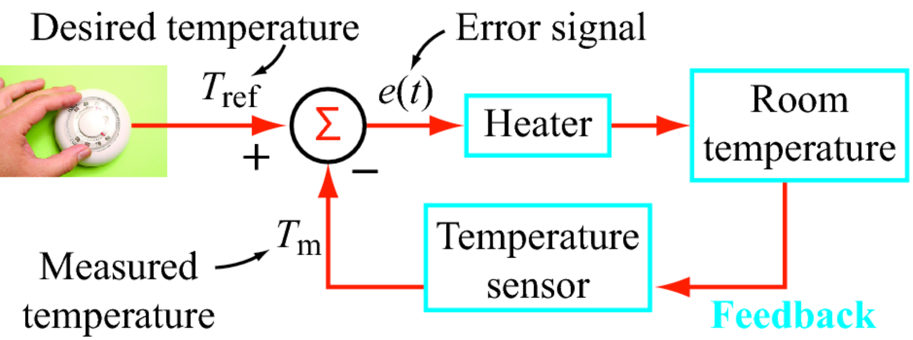
DFII topology requires fewer operations to implement than DFI

Direct Form II Topology

$$\mathbf{H}_2(\mathbf{s}) = \left(1 + \frac{a_1}{\mathbf{s}} + \frac{a_2}{\mathbf{s}^2} + \frac{a_3}{\mathbf{s}^3}\right)^{-1}$$

Control Theory

Temperature Control Example



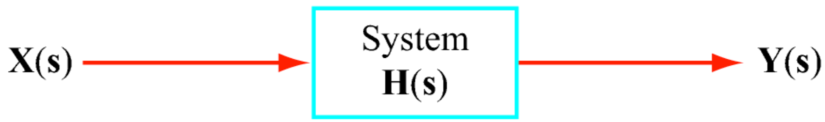
$$\mathbf{Y}(s) = \mathbf{H}(s)\ \mathbf{E}(s)$$
$$= \mathbf{H}(s)[\mathbf{X}(s) - \mathbf{G}(s)\ \mathbf{Y}(s)],$$

which leads to

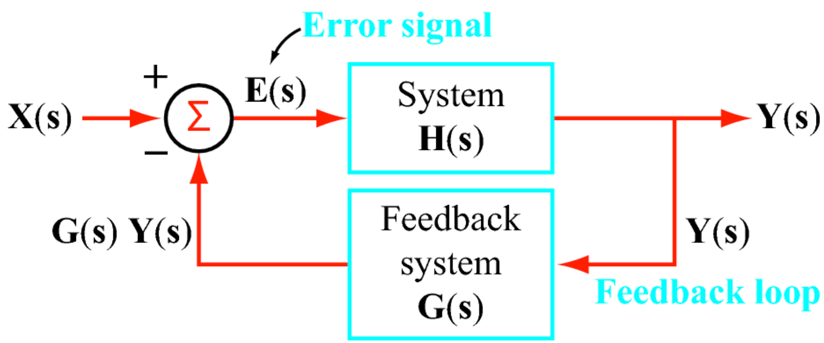
$$\mathbf{Y}(s) = \frac{\mathbf{H}(s)\ \mathbf{X}(s)}{1 + \mathbf{G}(s)\ \mathbf{H}(s)}\ . \qquad (4.100)$$

The output-to-input ratio of the closed-loop system is called the ***closed-loop transfer function*** $\mathbf{Q}(s)$. From Eq. (4.100), we obtain

$$\boxed{\mathbf{Q}(s) = \frac{\mathbf{Y}(s)}{\mathbf{X}(s)} = \frac{\mathbf{H}(s)}{1 + \mathbf{G}(s)\ \mathbf{H}(s)}\ .} \qquad (4.101)$$

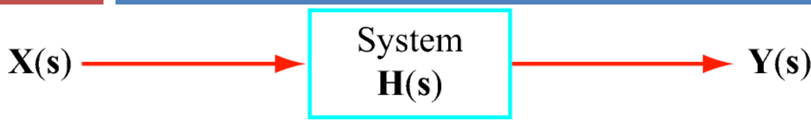


(a) **Open-loop system**

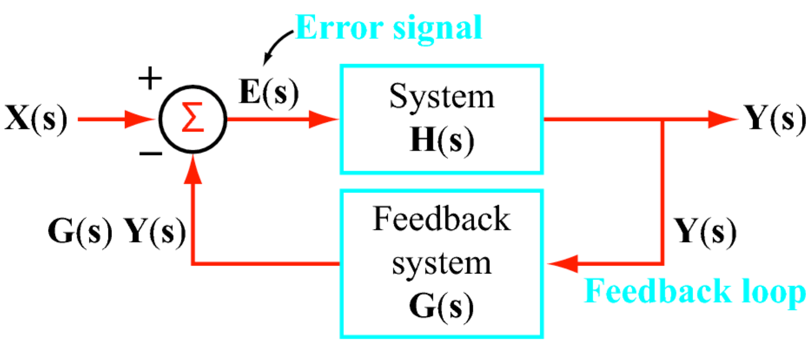


(b) **Closed-loop system**:

System Stabilization



(a) **Open-loop system**



(b) **Closed-loop system**:

Consider the first-order system:

$$\mathbf{H}(\mathbf{s}) = \frac{A}{\mathbf{s} - p_1} \qquad \text{with } p_1 > 0.$$

The open-loop transfer function $\mathbf{H}(\mathbf{s})$ has a single pole at $\mathbf{s} = p_1$, and since $p_1 > 0$, the pole resides in the RHP

Hence, the system is unstable.

Let's introduce Proportional Feedback with:

$$\mathbf{G}(\mathbf{s}) = K$$

Hence:

$$\mathbf{Q}(\mathbf{s}) = \frac{\mathbf{H}(\mathbf{s})}{1 + \mathbf{G}(\mathbf{s})\, \mathbf{H}(\mathbf{s})} = \frac{A/(\mathbf{s} - p_1)}{1 + \dfrac{KA}{\mathbf{s} - p_1}} = \frac{A}{\mathbf{s} - p_1 + KA}$$

The pole of Q(s) is at p_1-*KA*. *By choosing* $KA > p_1$, the pole location moves to the OLHP, thereby converting the unstable open-loop system into a stable closed-loop system.

(4.106)

Example 4-11: First-Order Stabilization

Choose the value of the feedback-loop constant K so that the pole of $\mathbf{H}(\mathbf{s}) = (\mathbf{s}+3)/(\mathbf{s}-2)$ moves from 2 to its diametrically opposite location in the OLHP, namely, -2.

Solution:
With ***proportional feedback***, $\mathbf{G}(\mathbf{s}) = K$, which leads to

$$\begin{aligned}\mathbf{Q}(\mathbf{s}) &= \frac{\mathbf{H}(\mathbf{s})}{1 + K\,\mathbf{H}(\mathbf{s})} \\ &= \frac{(\mathbf{s}+3)/(\mathbf{s}-2)}{1 + K\dfrac{(\mathbf{s}+3)}{(\mathbf{s}-2)}} \cdot \frac{\mathbf{s}-2}{\mathbf{s}-2} \\ &= \frac{\mathbf{s}+3}{(K+1)\left[\mathbf{s} + \dfrac{3K-2}{K+1}\right]} .\end{aligned}$$

To move the pole of $\mathbf{H}(\mathbf{s})$ from $+2$ to a pole at -2 (Fig. 4-28) for $\mathbf{Q}(\mathbf{s})$, it is necessary that

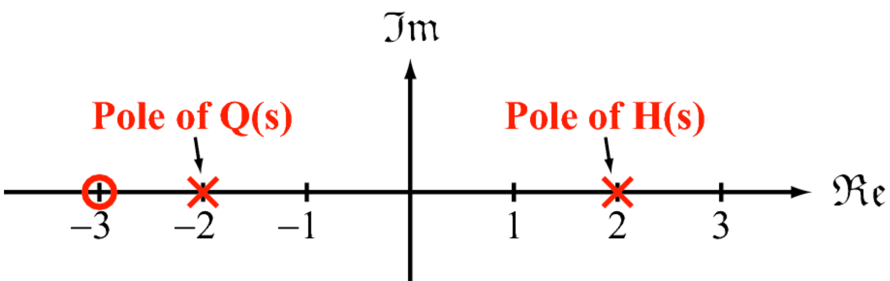
$$\frac{3K-2}{K+1} = 2.$$

Solving for K gives

$$K = 4,$$

which in turn leads to

$$\mathbf{Q}(\mathbf{s}) = 0.2\,\frac{\mathbf{s}+3}{\mathbf{s}+2} .$$



Without Feedback

For an input step function $x(t) = u(t)$ and corresponding Laplace transform $\mathbf{X}(\mathbf{s}) = 1/\mathbf{s}$, the step response is $y_1(t)$ and the Laplace transform of $y_1(t)$ is

$$\begin{aligned}\mathbf{Y}_1(\mathbf{s}) &= \mathbf{H}(\mathbf{s})\ \mathbf{X}(\mathbf{s}) \\ &= \frac{1}{\mathbf{s}}\,\frac{(\mathbf{s}+3)}{(\mathbf{s}-2)} .\end{aligned}$$

Through application of partial fraction expansion, $\mathbf{Y}_1(\mathbf{s})$ becomes

$$\mathbf{Y}_1(\mathbf{s}) = -\frac{1.5}{\mathbf{s}} + \frac{2.5}{\mathbf{s}-2} .$$

Conversion to the time domain, facilitated by entries #2 and 3 in Table 3-2, gives

$$y_1(t) = -1.5u(t) + 2.5e^{2t}\ u(t).$$

As $t \to \infty$, so does $y_1(t)$. This result confirms that $\mathbf{H}(\mathbf{s})$ represents a BIBO unstable system.

With Feedback

For a feedback system with closed-loop transfer function $\mathbf{Q}(\mathbf{s})$, repetition of the earlier steps leads to

$$\begin{aligned}\mathbf{Y}_2(\mathbf{s}) &= \mathbf{Q}(\mathbf{s})\ \mathbf{X}(\mathbf{s}) \\ &= \frac{1}{\mathbf{s}} \cdot 0.2\,\frac{\mathbf{s}+3}{\mathbf{s}+2} \\ &= \frac{0.3}{\mathbf{s}} - \frac{0.1}{\mathbf{s}+2} ,\end{aligned}$$

and

$$y_2(t) = 0.3u(t) - 0.1e^{-2t}\ u(t).$$

Because of the feedback loop, the exponent of the second term is negative, so as $t \to \infty$, $y_2(t) \to 0.3$.

Example 4-12: Second-Order Stabilization

Given a system with $\mathbf{H}(\mathbf{s}) = 3/(\mathbf{s}^2 + 4)$, determine if/how it can be stabilized by a feedback loop with (a) ***proportional feedback*** $[\mathbf{G}(\mathbf{s}) = K]$, and (b) ***proportional-plus-derivative (PD) feedback*** $[\mathbf{G}(\mathbf{s}) = K_1 + K_2\mathbf{s}]$.

Solution:

(a) For $\mathbf{G}(\mathbf{s}) = K$, we obtain

$$\begin{aligned}\mathbf{Q}(\mathbf{s}) &= \frac{\mathbf{H}(\mathbf{s})}{1 + K\ \mathbf{H}(\mathbf{s})} \\ &= \frac{3/(\mathbf{s}^2 + 4)}{1 + K\left(\dfrac{3}{\mathbf{s}^2 + 4}\right)} \\ &= \frac{3}{\mathbf{s}^2 + (3K + 4)}\ .\end{aligned}$$

The denominator of $\mathbf{Q}(\mathbf{s})$ is identical in form with that of Eq. (4.108). Hence, the use of a feedback loop with a constant-gain function K is insufficient to realize stabilization.

(b) Using PD feedback with $\mathbf{G}(\mathbf{s}) = K_1 + K_2\mathbf{s}$ leads to

$$\begin{aligned}\mathbf{Q}(\mathbf{s}) &= \frac{\mathbf{H}(\mathbf{s})}{1 + (K_1 + K_2\mathbf{s})\ \mathbf{H}(\mathbf{s})} \\ &= \frac{3/(\mathbf{s}^2 + 4)}{1 + (K_1 + K_2\mathbf{s})\left(\dfrac{3}{\mathbf{s}^2 + 4}\right)} \\ &= \frac{3}{\mathbf{s}^2 + 3K_2\mathbf{s} + (3K_1 + 4)}\ .\end{aligned}$$

The poles of $\mathbf{Q}(\mathbf{s})$ will have negative real parts (i.e., they will reside in the OLHP) only if all constant coefficients in the denominator of $\mathbf{Q}(\mathbf{s})$ are real and positive. Equivalently, K_1 and K_2 must satisfy the conditions

$$K_1 > -\frac{4}{3}\ , \qquad K_2 > 0,$$

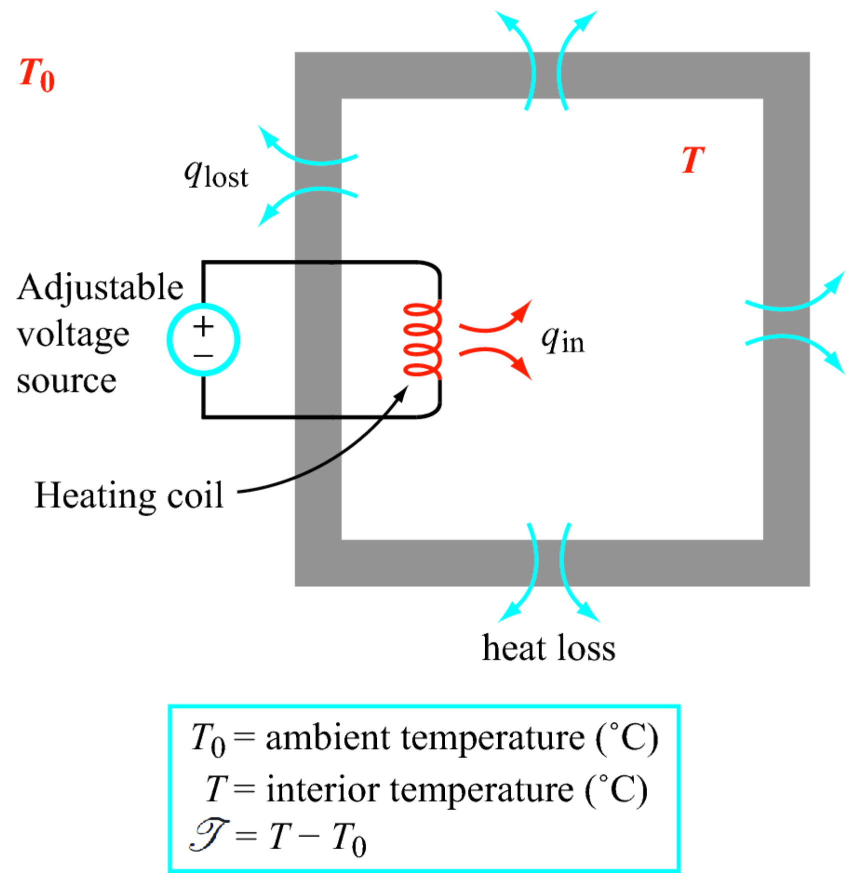
in order for the closed-loop system to be stable.

Temperature Control System

4-9.1 Heat Transfer Model

The heat transfer quantities of interest are the following:

- T_0 = ambient temperature of the box's exterior space, and also the initial temperature of its interior space.
- T_∞ = the final temperature to which the interior space will be raised (selected by the operator).
- $\Delta T = T_\infty - T_0$ = desired temperature rise.
- $T(t)$ = temperature of the container's interior space as a function of time with $T(0) = T_0$ and $T(\infty) = T_\infty$.
- $\mathcal{T}(t) = T(t) - T_0$ = temperature of container's interior ***relative*** to the ambient temperature. That is, $\mathcal{T}(t)$ is a ***temperature deviation*** with initial and final values $\mathcal{T}(0) = 0$ and $\mathcal{T}(\infty) = T_\infty - T_0$.
- Q = ***enthalpy*** (heat content in joules) of the interior space of the container (including its contents), defined ***relative*** to that of the exterior space. That is, $Q = 0$ represents the equilibrium condition when $T = T_0$ or, equivalently, $\mathcal{T} = 0$.
- C = ***heat capacity*** of the container's interior space, measured in joules/°C.
- R = thermal resistance of the interface (walls) between the container's interior and exterior measured in °C per watt (°C/W).
- $q = \frac{dQ}{dt}$ = the rate of heat flow in joules/s or, equivalently, watts (W).



Temperature Control System

The amount of heat required to raise the temperature of the container's interior by $\mathcal{T}$ is

$$Q = C\mathcal{T}. \qquad (4.114)$$

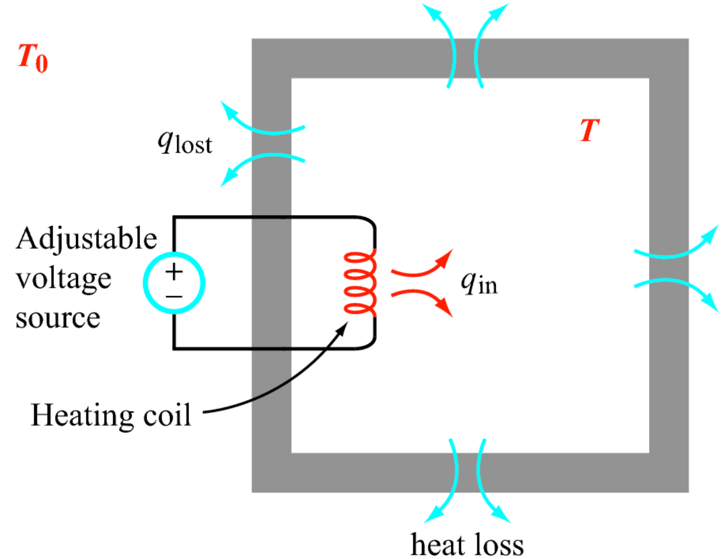
The rate of heat flow q is analogous to electric current i in electric circuits. For the scenario depicted in Fig. 4-31:

- q_{in} = rate of heat flow supplied by the source.
- q_{lost} = rate of heat flow loss to the outside through the walls.
- q_{abs} = rate of heat flow absorbed by the air and contents of the container's interior, raising their temperature.

For an interface with thermal resistance R,

$$q_{\text{lost}} = \frac{T - T_0}{R} = \frac{\mathcal{T}}{R} \quad \longleftrightarrow \quad i = \frac{v}{R}. \qquad (4.115)$$

$$q_{\text{abs}} = \frac{dQ}{dT} = C\,\frac{d\mathcal{T}}{dt} \quad \longleftrightarrow \quad i = C\,\frac{dv}{dt},$$

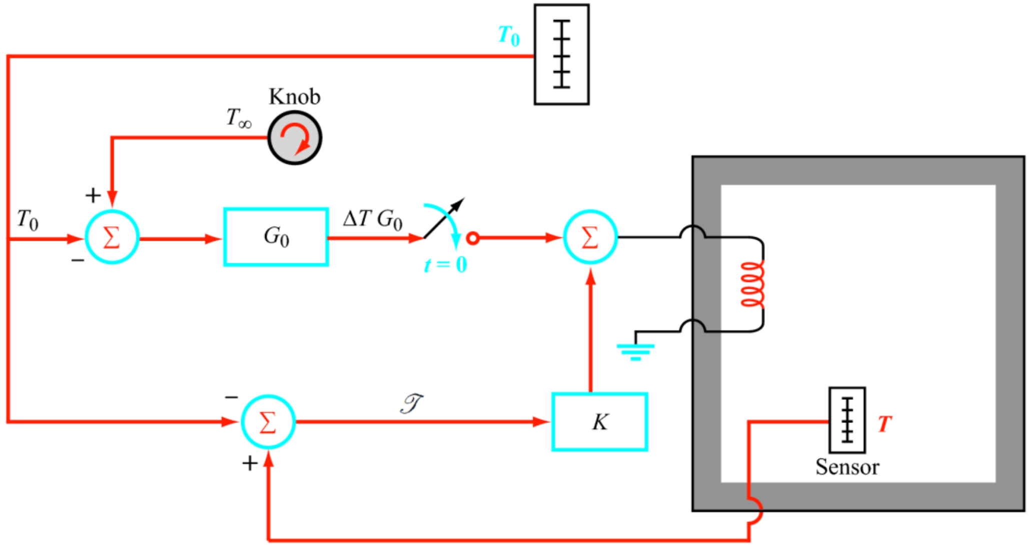


T_0 = ambient temperature (°C)
T = interior temperature (°C)
$\mathcal{T} = T - T_0$

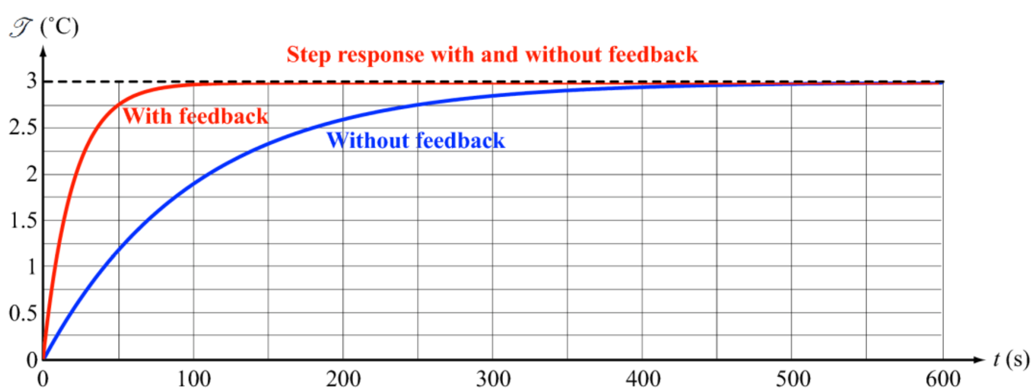
$$q_{\text{abs}} + q_{\text{lost}} = q_{\text{in}}$$

$$C\,\frac{d\mathcal{T}}{dt} + \frac{\mathcal{T}}{R} = q_{\text{in}}(t).$$

Step Response– Desired Temperature Rise=3C

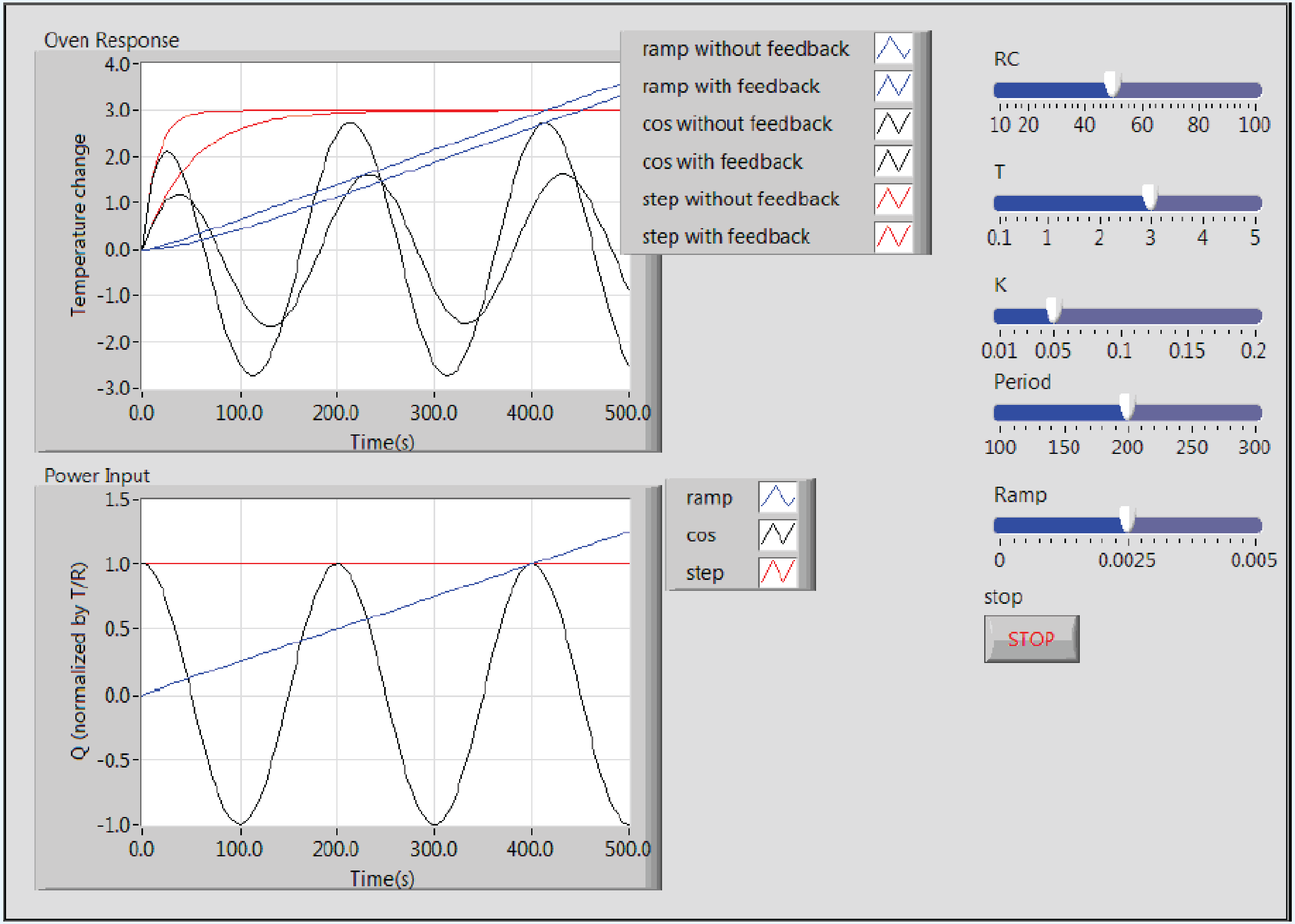


(a) Closed-loop mode heating system

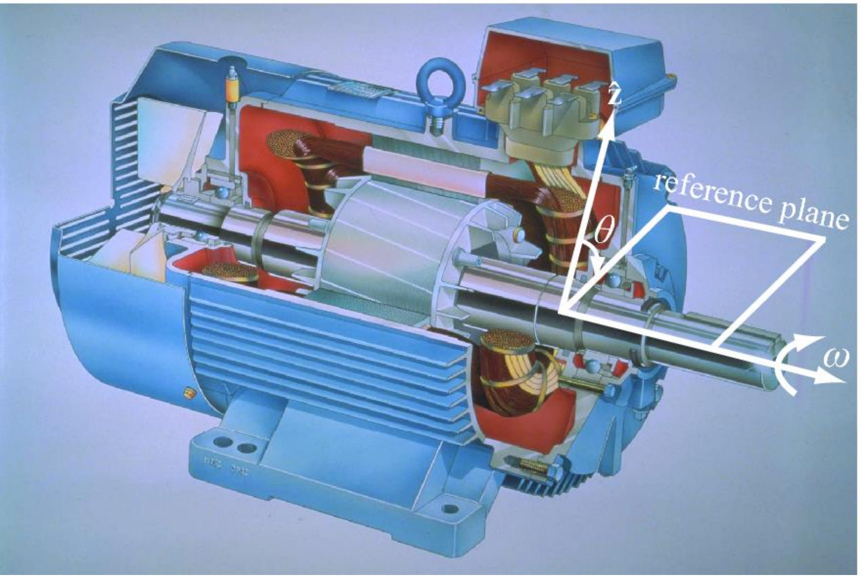


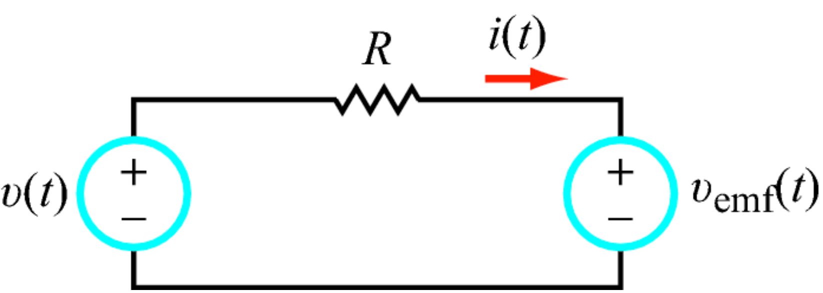
(b) Step response

Module 4.1 Oven Temperature Response The input can be a step, ramp, or sinusoid. The RC time constant of the oven, the desired temperature change T, the period of the sinusoidal input, the slope of the ramp input, and the gain K of the feedback are all selectable parameters.



Step Response of a Motor System





$$\upsilon(t) - \upsilon_{\mathrm{emf}} - R\, i(t) = 0$$

By Faraday's law, the induced emf is directly proportional to the angular velocity ω, which by definition is the time derivative of the ***shaft rotation angle*** θ. That is,

$$\upsilon_{\mathrm{emf}}(t) = c_1\omega = c_1 \frac{d\theta}{dt}, \tag{4.133}$$

where c_1 is a motor-specific constant.

By Ampere's law, the induced magnetic field is proportional to the current $i(t)$. Moreover, the torque is proportional to the magnetic-field strength. Hence, current and torque are linearly related as

$$\tau(t) = c_2\, i(t), \tag{4.134}$$

where c_2 is another motor-specific constant. Also, the rotational version of the force equation ($f = ma$) is

$$\tau(t) = J\, \frac{d^2\theta}{dt^2}, \tag{4.135}$$

where J is the ***moment of inertia*** of the load attached to the rotor shaft. Combining Eqs. (4.134) and (4.135) leads to

$$i(t) = \frac{J}{c_2}\, \frac{d^2\theta}{dt^2}. \tag{4.136}$$

Inserting Eqs. (4.133) and (4.136) into Eq. (4.132) leads to

$$\frac{d^2\theta}{dt^2} + a\, \frac{d\theta}{dt} = b\, \upsilon(t), \tag{4.137}$$

where $a = (c_1c_2)/(RJ)$ and $b = c_2/(RJ)$. From here on

Open-Loop Configuration

$$\frac{d^2\theta}{dt^2} + a\,\frac{d\theta}{dt} = b\,\upsilon(t)$$

$$\mathbf{s}^2\,\boldsymbol{\theta}(\mathbf{s}) + a\mathbf{s}\,\boldsymbol{\theta}(\mathbf{s}) = b\mathbf{V}(\mathbf{s}),$$

and the ***motor transfer function*** is

$$\mathbf{H}(\mathbf{s}) = \frac{\boldsymbol{\theta}(\mathbf{s})}{\mathbf{V}(\mathbf{s})} = \frac{b}{\mathbf{s}^2 + a\mathbf{s}} = \frac{b}{\mathbf{s}(\mathbf{s}+a)}$$

Motor's Step Response

step input $\upsilon(t) = u(t)$ or, equivalently, $\mathbf{V}(\mathbf{s}) = 1/\mathbf{s}$.

$$\begin{aligned}\boldsymbol{\theta}(\mathbf{s}) &= \mathbf{H}(\mathbf{s})\,\mathbf{V}(\mathbf{s}) \\ &= \frac{b}{\mathbf{s}(\mathbf{s}+a)} \cdot \frac{1}{\mathbf{s}} = \frac{b}{\mathbf{s}^2(\mathbf{s}+a)}.\end{aligned} \tag{4.140}$$

Partial fraction expansion gives

$$\boldsymbol{\theta}(\mathbf{s}) = \frac{b/a}{\mathbf{s}^2} - \frac{b/a^2}{\mathbf{s}} + \frac{b/a^2}{\mathbf{s}+a}, \tag{4.141}$$

and using Table 3-2, the time-domain counterpart of $\boldsymbol{\theta}(\mathbf{s})$ is

$$\theta(t) = \frac{b}{a^2}\,[at - 1 + e^{-at}]\,u(t) \qquad \text{(radians)}. \tag{4.142}$$

The rotational angular velocity of the shaft is

$$\omega_{\text{rad/s}} = \frac{d\theta}{dt} = \frac{b}{a}\,(1 - e^{-at})\,u(t) \qquad \text{(rad/s)}. \tag{4.143}$$

Converting ω in (rad/s) to revolutions per minute (with 1 revolution $= 2\pi$ radians) gives

$$\omega_{\text{rev/m}} = \frac{60}{2\pi}\,\omega_{\text{rad/s}} = \frac{30b}{\pi a}\,(1 - e^{-at})\,u(t). \tag{4.144}$$

If the motor is designed such that $a \approx 3$ or greater, the second term becomes negligible compared to the first within 1 second or less, thereby simplifying the result to

$$\omega_{\text{rev/m}} \approx \frac{30b}{\pi a} \qquad (t > 3/a \text{ seconds}). \tag{4.145}$$

Through proper choice of the motor physical parameters, the ratio b/a can be selected to yield the desired value for ω.

In summary, when a motor is operated in an open-loop configuration, the application of a step voltage causes the shaft to reach a constant angular velocity ω within a fraction of a second (if $a > 3$).

Closed-Loop Configuration

$$s^2\,\theta(s) + as\,\theta(s) = b\mathbf{V}(s),$$

and the ***motor transfer function*** is

$$\mathbf{H}(s) = \frac{\theta(s)}{\mathbf{V}(s)} = \frac{b}{s^2 + as} = \frac{b}{s(s+a)}$$

$$\mathbf{Q}(s) = \frac{\mathbf{H}(s)}{1 + K\,\mathbf{H}(s)} . \tag{4.147}$$

Use of Eq. (4.139) for $\mathbf{H}(s)$ leads to

$$\mathbf{Q}(s) = \frac{b/(s^2 + as)}{1 + Kb/(s^2 + as)} = \frac{b}{s^2 + as + bK} . \tag{4.148}$$

The expression for $\mathbf{Q}(s)$ can be rewritten in the form

$$\mathbf{Q}(s) = \frac{b}{(s - p_1)(s - p_2)} , \tag{4.149}$$

where poles p_1 and p_2 are given by

$$p_1 = -\frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 - bK} , \tag{4.150a}$$

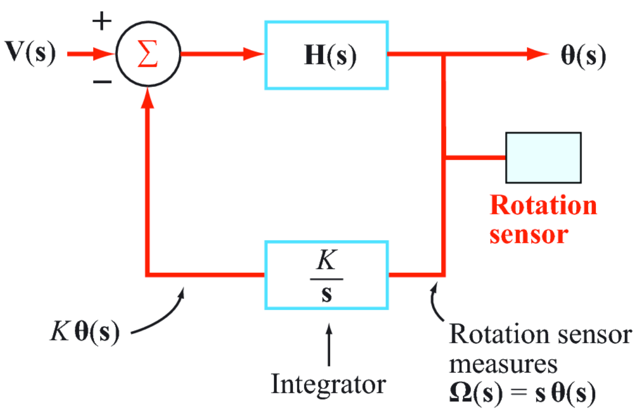
$$p_2 = -\frac{a}{2} - \sqrt{\left(\frac{a}{2}\right)^2 - bK} . \tag{4.150b}$$

By choosing $K > 0$, we ensure that the poles p_1 and p_2 are in the open left half-plane, thereby guaranteeing BIBO stability.

The step response is obtained by calculating $\theta(s)$ for $\upsilon(t) = u(t)$ or, equivalently, $\mathbf{V}(s) = 1/s$:

$$\theta(s) = \mathbf{Q}(s)\,\mathbf{V}(s) = \frac{b}{s(s - p_1)(s - p_2)} . \tag{4.151}$$

(a) Telescope system



(b) Block diagram

Steering Telescope to Desired Direction

$$\theta(t) = \left[\frac{b}{p_1 p_2} + \frac{b}{p_1(p_1 - p_2)}\, e^{p_1 t} + \frac{b}{p_2(p_2 - p_1)}\, e^{p_2 t}\right] u(t).$$

(a) Telescope system

$$\lim_{t \to \infty} \theta(t) = \frac{1}{K}\, .$$

If the magnitude of K is on the order of inverse seconds, the final value of $\theta(t)$ is approached very quickly. This final value is equal to the total angular rotation required to move the telescope from its original direction to the designated new direction.

Simple Inverted Pendulum on a Cart

- Input $x(t)$ is the back-and-forth position of the cart.
- Output $\theta(t)$ is the angle made with the vertical.
- Pendulum has length L and mass m, all at its end. The pendulum thus has moment of inertia mL^2.
- The pendulum has a frictionless hinge at its base.

There are three torques around the pendulum hinge acting on the mass m at the end of the pendulum.

(a) The torque $mLg\sin(\theta)$ due to gravity on the mass.

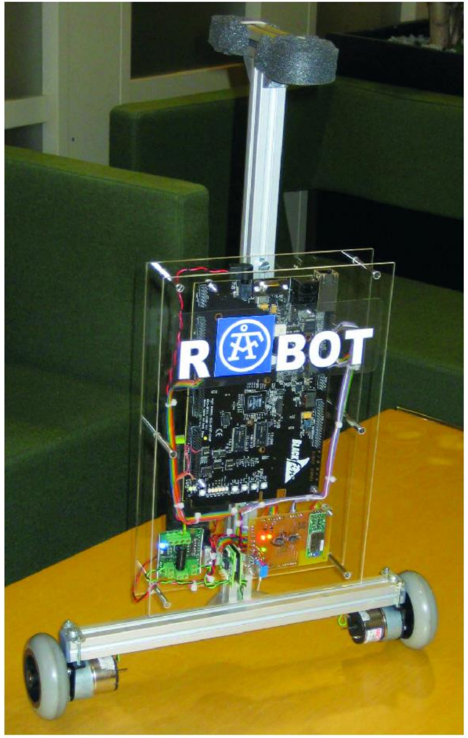
(b) The torque $-mL\cos(\theta)\,(d^2x/dt^2)$ due to the motion of the cart. Note that forward motion of the cart makes the pendulum tend to swing backwards.

(c) The torque $mL^2(d^2\theta/dt^2)$ due to angular acceleration of the pendulum.

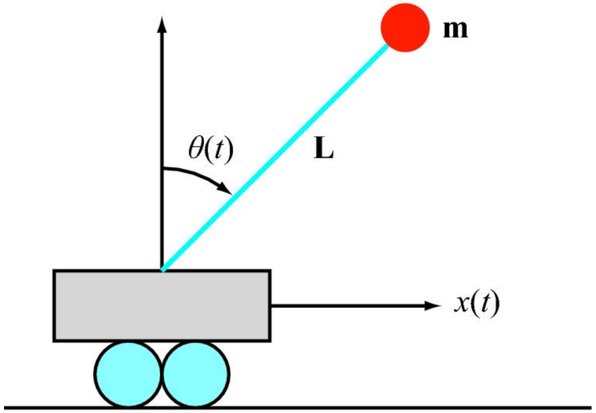
Equating these torques around the pendulum hinge gives

$$mL^2\frac{d^2\theta}{dt^2} = mLg\sin(\theta) - mL\cos(\theta)\frac{d^2x}{dt^2}, \qquad (4.159)$$

where $g = 9.8\ \mathrm{m\cdot s^{-2}}$ is the acceleration of gravity, and the angle θ has units of radians.



(a) Inverted pendulum on wheels



(b) Diagram

Linearization

$$mL^2 \frac{d^2\theta}{dt^2} = mLg\sin(\theta) - mL\cos(\theta)\,\frac{d^2x}{dt^2}$$

$$\sin(\theta) = \theta - \frac{\theta^3}{3!} + \cdots \approx \theta \qquad \text{if } \theta \ll 1,$$

$$\cos(\theta) = 1 - \frac{\theta^2}{2!} + \cdots \approx 1 \qquad \text{if } \theta \ll 1$$

$$mL^2 \frac{d^2\theta}{dt^2} = mLg\,\theta(t) - mL(1)\,\frac{d^2x}{dt^2}. \tag{4.161}$$

Finally, dividing by mL^2 leads to the linearized equation

$$\frac{d^2\theta}{dt^2} = \frac{g}{L}\,\theta(t) - \frac{1}{L}\,\frac{d^2x}{dt^2}. \tag{4.162}$$

This system model is LTI; in fact, it is an LCCDE. To compute its transfer function, we transform the differential equation to the **s**-domain:

$$\mathbf{s}^2\,\boldsymbol{\theta}(\mathbf{s}) = \frac{g}{L}\,\boldsymbol{\theta}(\mathbf{s}) - \frac{1}{L}\,\mathbf{s}^2\,\mathbf{X}(\mathbf{s}). \tag{4.163}$$

Solving for $\boldsymbol{\theta}(\mathbf{s})$ and then dividing by $\mathbf{X}(\mathbf{s})$ gives

$$\mathbf{H}(\mathbf{s}) = \frac{\boldsymbol{\theta}(\mathbf{s})}{\mathbf{X}(\mathbf{s})} = -\frac{\mathbf{s}^2/L}{\mathbf{s}^2 - g/L}. \tag{4.164}$$

Example 4-14: Open-Loop Inverted Pendulum

A cart carrying an inverted pendulum of length 61.25 cm is suddenly moved 1 mm. Compute its response.

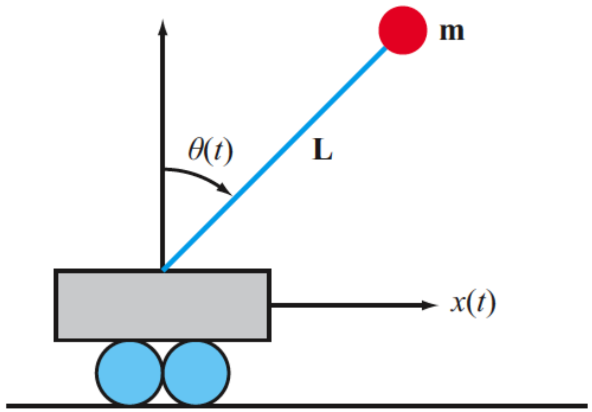
Solution:

Inserting $L = 61.25$ cm and $g = 9.8 \text{ m} \cdot \text{s}^{-2}$ into Eq. (4.164) gives

$$\mathbf{H}(\mathbf{s}) = \frac{-\mathbf{s}^2/0.6125}{\mathbf{s}^2 - 9.8/0.6125} = \frac{-1.633\mathbf{s}^2}{\mathbf{s}^2 - 16}. \tag{4.165}$$

A sudden movement of length 1 mm means that the input $x(t) = 0.001u(t)$. Its Laplace transform is $\mathbf{X}(\mathbf{s}) = 0.001/\mathbf{s}$, and the corresponding response is

$$\boldsymbol{\theta}(\mathbf{s}) = \mathbf{H}(\mathbf{s})\ \mathbf{X}(\mathbf{s}) = \frac{-1.633\mathbf{s}^2}{\mathbf{s}^2 - 16}\,\frac{0.001}{\mathbf{s}} = \frac{-0.001633\mathbf{s}}{\mathbf{s}^2 - 16}.$$



Partial fraction expansion gives

$$\boldsymbol{\theta}(\mathbf{s}) = \frac{-0.001633\mathbf{s}}{(\mathbf{s}-4)(\mathbf{s}+4)} = \frac{-0.000817}{\mathbf{s}-4} + \frac{-0.000817}{\mathbf{s}+4}. \tag{4.167}$$

The inverse Laplace transform of $\boldsymbol{\theta}(\mathbf{s})$ is

$$\theta(t) = -0.000817(e^{4t} + e^{-4t})\ u(t). \tag{4.168}$$

The pendulum topples over in less than two seconds, since $-0.000817e^{8} = -2.435 \text{ rad} = -139.5° < -90°$.

Note that forward motion of the cart makes the pendulum fall backwards. That is why the angle $\theta(t) < 0$.

4-12.5 Control Using PI Feedback

Since neither proportional nor PD control can stabilize the inverted pendulum, let us now try something new: ***proportional-plus-integral (PI)*** feedback, wherein

$$\mathbf{G}(\mathbf{s}) = K_1 + \frac{K_2}{\mathbf{s}}. \tag{4.176}$$

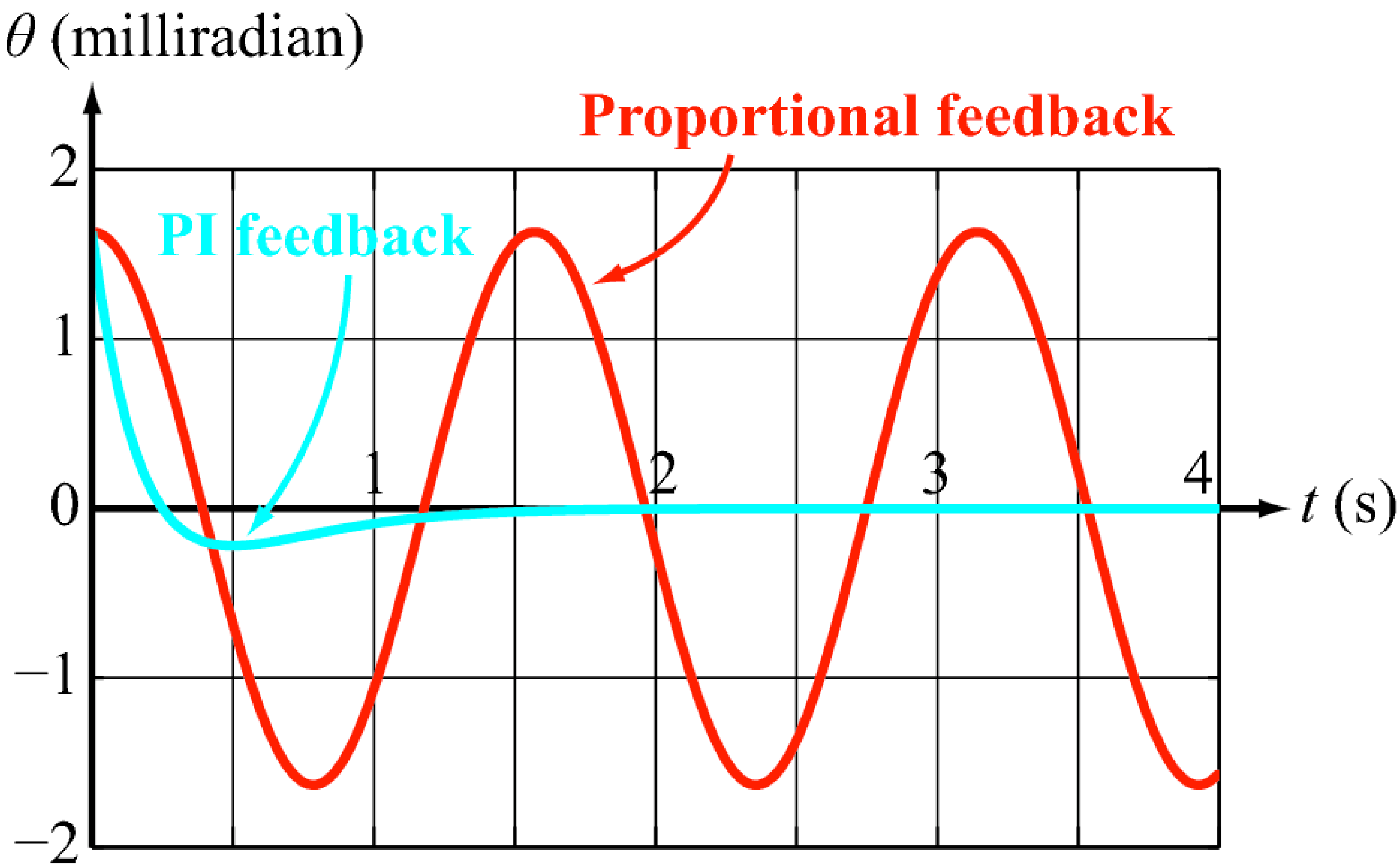
Now the feedback depends on both the angle $\theta(t)$ and its integral $\int_0^t \theta(t')\,dt'$. The integral can be computed using an op-amp integrator circuit. The closed-loop transfer function $\mathbf{Q}(\mathbf{s})$ is then

$$\begin{aligned}
\mathbf{Q}(\mathbf{s}) &= \frac{\mathbf{H}(\mathbf{s})}{1 + \mathbf{G}(\mathbf{s})\,\mathbf{H}(\mathbf{s})} \\
&= -\frac{\mathbf{s}^2/L}{\mathbf{s}^2 - g/L} \Bigg/ \left(1 - (K_1 + K_2/\mathbf{s})\,\frac{\mathbf{s}^2/L}{\mathbf{s}^2 - g/L}\right) \\
&= -\frac{\mathbf{s}^2}{(L - K_1)\mathbf{s}^2 - K_2\mathbf{s} - g}.
\end{aligned} \tag{4.177}$$

The denominator of $\mathbf{Q}(\mathbf{s})$ is now a quadratic polynomial, so it has only two poles. With two arbitrary constants, K_1 and K_2, to determine two poles, we can not only stabilize the closed-loop system, but we can also place its poles anywhere we want!

Comparison of Proportional Feedback to Proportional-Plus-Integral Feedback for the inverted Pendulum

- Response to a sudden 1-mm movement



Module 4.2 Inverted Pendulum Response The input can be a step, ramp, or sinusoid. The period of the sinusoidal input and the gain of the feedback (K for proportional control, and K_1 and K_2 for PD control), are selectable parameters. Waveforms of the input and angular response are displayed.