Checking the sum and product rules, and their consequences

Goal: Check using a very simple example that the Bayesian rules are consistent with standard probabilities based on frequencies. Also check notation and vocabulary.

Bayesian rules of probability as principles of logic

Notation: $p(x \mid I)$ is the probability (or pdf) of x being true given information I

1. **Sum rule:** If set $\{x_i\}$ is exhaustive and exclusive,

$$\sum_i p(x_i \mid I) = 1 \quad \longrightarrow \quad \int dx\, p(x \mid I) = 1$$

   - cf. complete and orthonormal
   - implies *marginalization* (cf. inserting complete set of states or integrating out variables - but be careful!)

$$p(x \mid I) = \sum_j p(x, y_j \mid I) \quad \longrightarrow \quad p(x \mid I) = \int dy\, p(x, y \mid I)$$

2. **Product rule:** expanding a joint probability of x and y

$$p(x, y \mid I) = p(x \mid y, I)\, p(y \mid I) = p(y \mid x, I)\, p(x \mid I)$$

   - If x and y are *mutually independent*: $p(x \mid y, I) = p(x \mid I)$, then

$$p(x, y \mid I) \longrightarrow p(x \mid I)\, p(y \mid I)$$

   - Rearranging the second equality yields *Bayes' Rule (or Theorem)*

$$p(x \mid y, I) = \frac{p(y \mid x, I)\, p(x \mid I)}{p(y \mid I)}$$

See Cox for the proof.

Answer the questions in *italics*.

Check answers with your neighbors. Ask for help if you get stuck or are unsure.

In [1]:
```
%%html
<style>
 table { width:70% !important; }
 table td, th { border: 1px solid black !important;
         text-align:center !important;
         font-size: 16px }
</style>
```

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1
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TABLE 1	Blue	Brown	Total
Tall	1	17	18
Short	37	20	57
Total	38	37	75

TABLE 2	Blue	Brown	Total
Tall			
Short			
Total			

1. Table 1 shows the number of blue- or brown-eyed and tall or short individuals in a population of 75.

   Fill in the blanks in Table 2 with probabilities (in decimals with three places, not fractions) based on the usual "frequentist" interpretations of probability (which would say that the probability of randomly drawing an ace from a deck of cards is 4/52 = 1/13). *Add x's in the row and/or column that illustrates the sum rule.*

2. *What is $p(short, blue)$? Is this a joint or conditional probability? What is $p(blue)$?*
   From the product rule, what is $p(short|blue)$? Can you read this result directly from the table?

3. *Apply Bayes' theorem to find $p(blue|short)$ from your answers to the last part.*

4. *What rule does the second row (the one starting with "Short") illustrate? Write it out in $p(\cdot)$ notation.*

5. *Are the probabilities of being tall and having brown eyes mutually independent? Why or why not?*