Examples

January 24, 2026

[25]:
```
include("notebook_init.jl");
```

This notebook generates the figures in the paper *Complexity of Projected Gradient Methods for Strongly Convex Optimization with Hölder Continuous Gradient Terms*

by X. Chen, C. T. Kelley, and L. Wang

1 Example 1

This problem is to solve the following two-dimensional PDE,

$$\mathcal{F}(u) = -\Delta u + \gamma u_+^\alpha = 0,$$

where $\alpha \in (0,1)$, $\gamma > 0$ is a constant and $u_+ = \max\{u, 0\}$. Discretizing this problem with the standard five point scheme leads to

$$\mathbf{F}(\mathbf{u}) = \mathbf{A}\mathbf{u} + \gamma \mathbf{u}_+^\alpha - \mathbf{b} = 0$$

$\mathbf{A} \in \mathbb{R}^{n\times n}$ is the discretization of $-\Delta$ with zero boundary conditions, $\mathbf{b} \in \mathbb{R}^n$ encodes the boundary conditions, and $\mathbf{u}_+^\alpha = \max\{\mathbf{u}, 0\}^\alpha$ is understood as a component-wise operation.

We now modify the above problem to enable direct computation of errors in the iterations. To this end, we take as the exact solution the function

$$u^*(x,y) = \left(\frac{3r-1}{2}\right)^2 \max\left\{0, r - \frac{1}{3}\right\}$$

where $r = \sqrt{x^2+y^2}$, and enforce the boundary conditions

$$u(x,1) = u^*(x,1),\ u(x,0) = u^*(x,0),\ u(1,y) = u^*(1,y),\ u(0,y) = u^*(0,y),$$

for $0 < x, y < 1$. Hence our modified equation is

$$\mathbf{F}(\mathbf{u}) - \mathbf{c}^* = 0,$$

where $\mathbf{c}^* = \mathbf{F}(\mathbf{u}^*)$.

The nonlinear system is first order optimality condition for the strongly convex optimization problem.

$$\min_{\mathbf{u}\in\mathbb{R}^n} \ f(\mathbf{u}) = \frac{1}{2}\mathbf{u}^\top \mathbf{A}\mathbf{u} + \frac{\gamma}{1+\alpha}\mathbf{e}^\top \mathbf{u}_+^{1+\alpha} - (\mathbf{b}+\mathbf{c}^*)^\top \mathbf{u},$$

where $\mathbf{e} \in \mathbb{R}^n$ is the vector of all ones.

1.1 Figures for Example 1

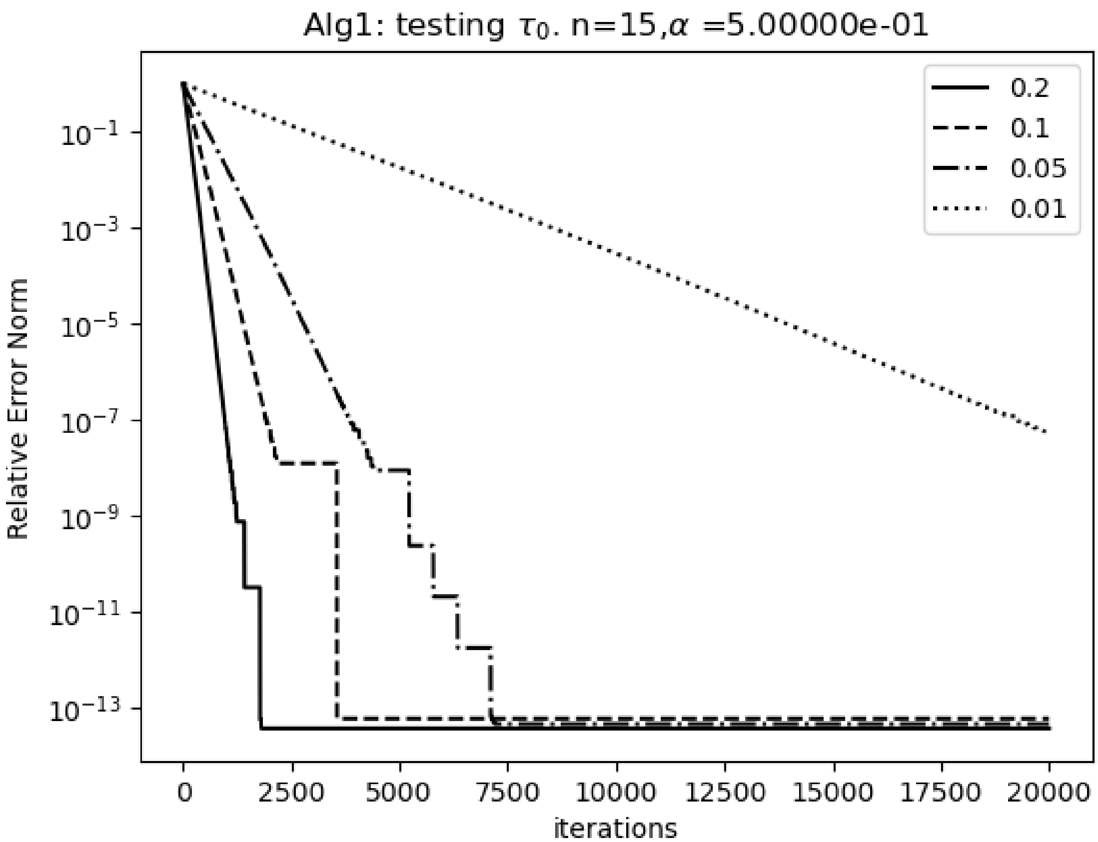
Generate Figures 1(a), 1(b), 2(a), and 2(b). Theses figures are for Algorithm 1. Figures 1(a) and 2(a) compare various stepsizes $\tau = \tau_0 h^2$ which are consistent with the CFL condition. Figures 1(b) and 2(b) examine values of the exponent p.

The files for building the figures are in /**src**/**Figures**. The code is **Figures_Alg1.jl** and the functions **Figure1_2a** and **Figure1_2b**. The functions take the dimension as an argument.

Figure 1a

[26]:
```
Figure1a(15; alpha=.5);
```

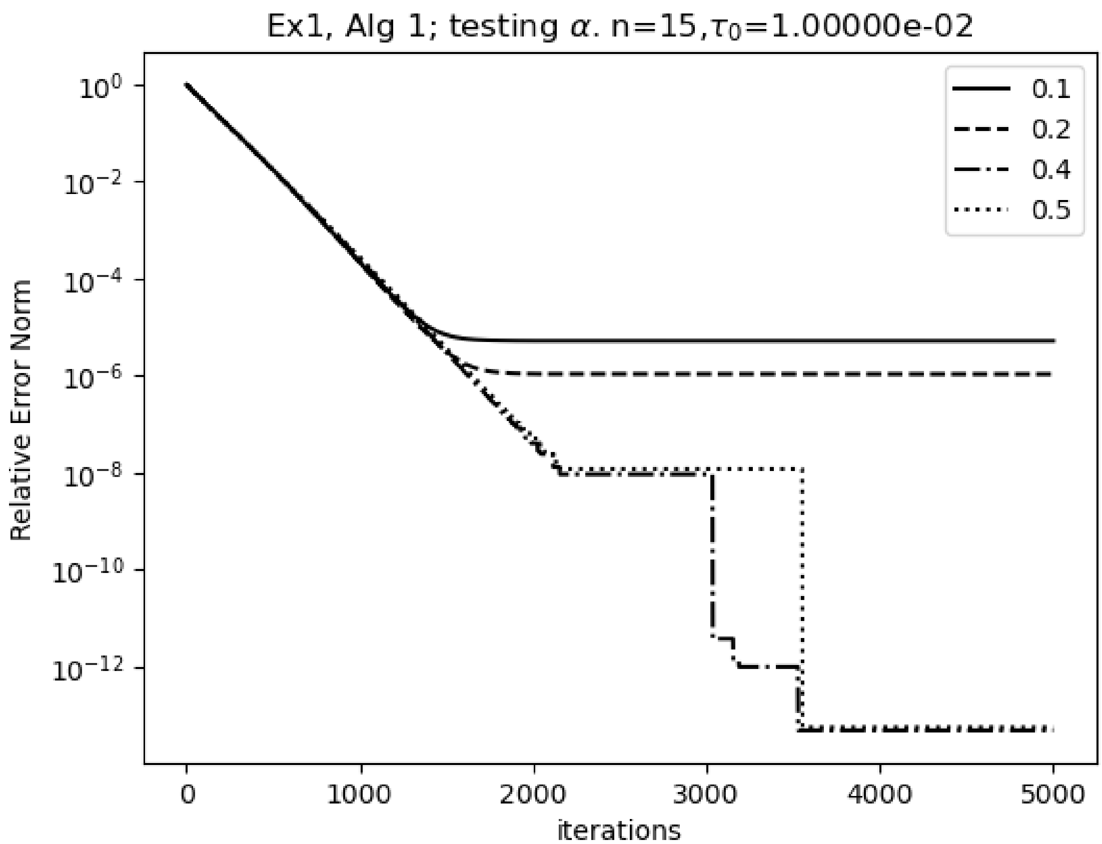
```
Alg1: testing $\tau_0$. n=15,$\alpha$ =5.00000e-01
```



Now we compare the effects of changing the exponent α. Here we can see the effects of the change I made in Alg 1 by letting the gradient norm increase without terminating the iteration.

Figure 1b

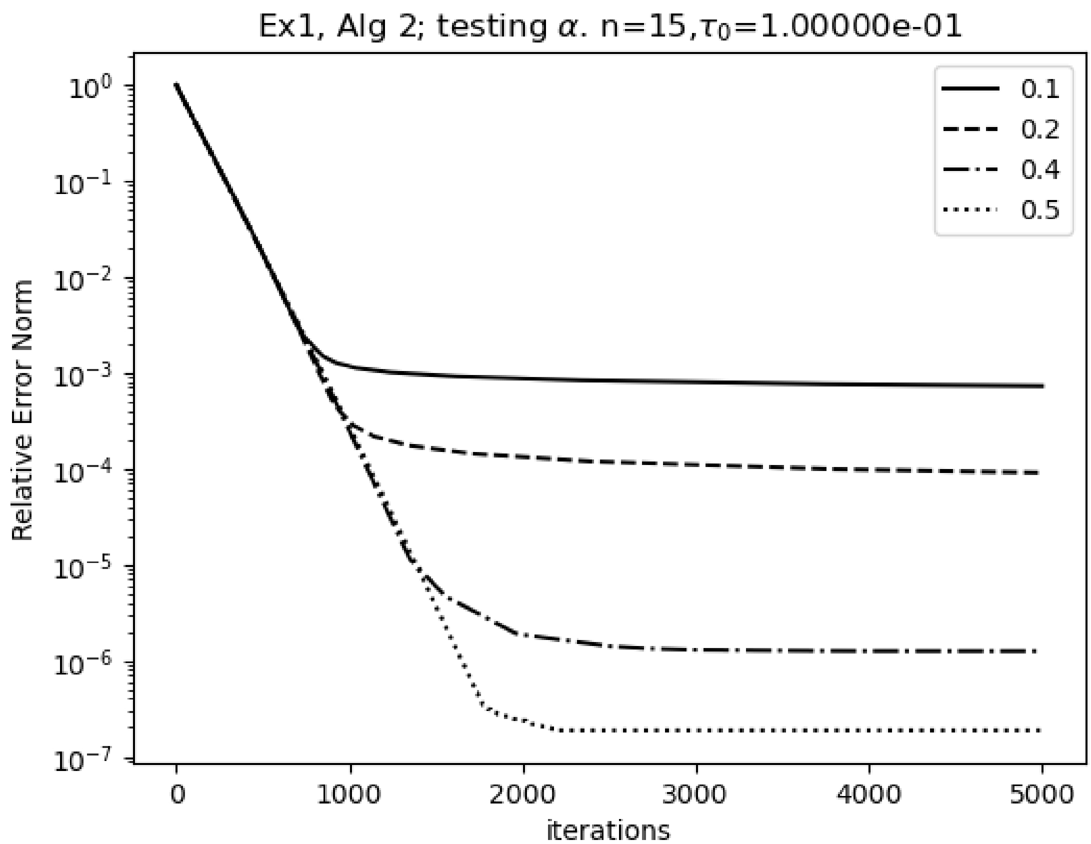
[27]:
```
Figure1b(15; maxit=5000);
```



And finally repeat the computation for a 31x31 grid. This will complete the computations for Example 1 + Algorithm 1.

Figure 2

[28]:
```
Figure2(15; maxit=5000);
```



The advantage of the line search is that one does not have to manually adjust τ_0.

The results for Algoirthm 3 are in Figures 4 and 5. We set $\nu = \tau_0 h^2$ in these examples and will need to modify that to use the estimate in Remark 4.2. In the first two figures 3(a) and 3(b) use use the values of τ_0 we used in Figure 1.

Figure 3a

[29]:
```
Figure3_4a(15);
```

```
Ex1, Alg3: testing $\tau_0$. n=15,$\alpha$ =5.00000e-01
```

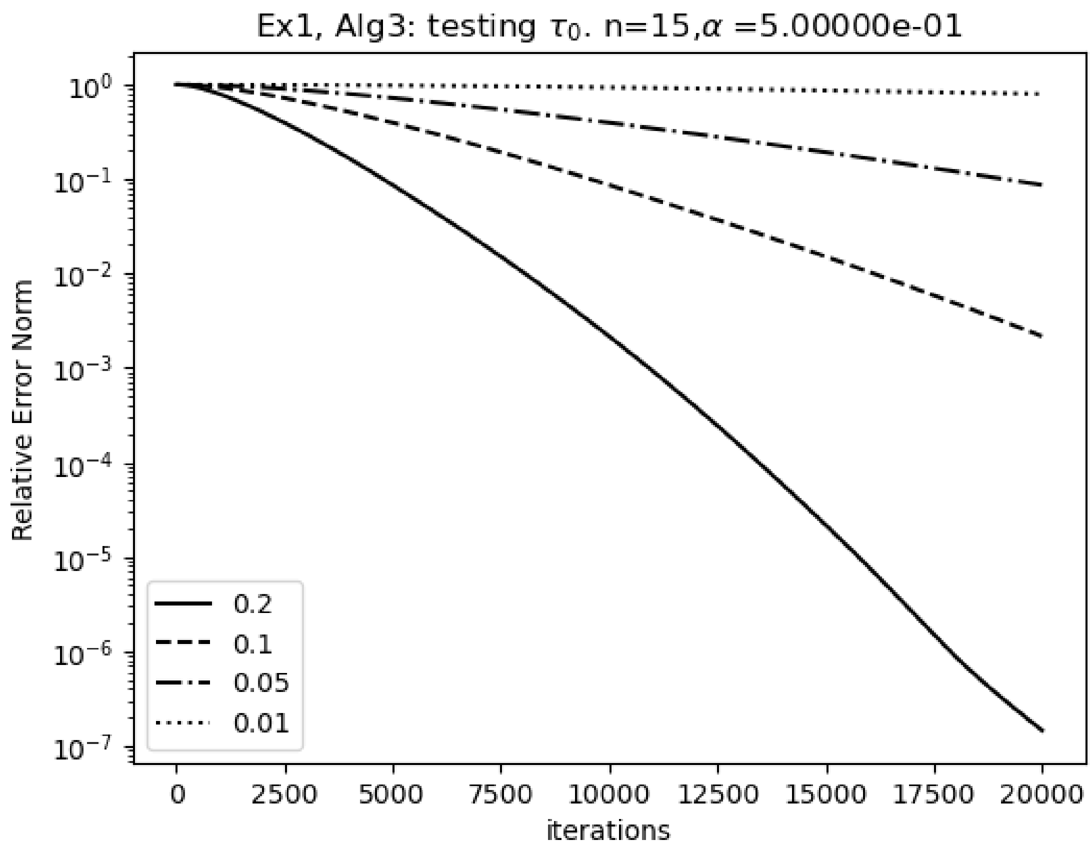
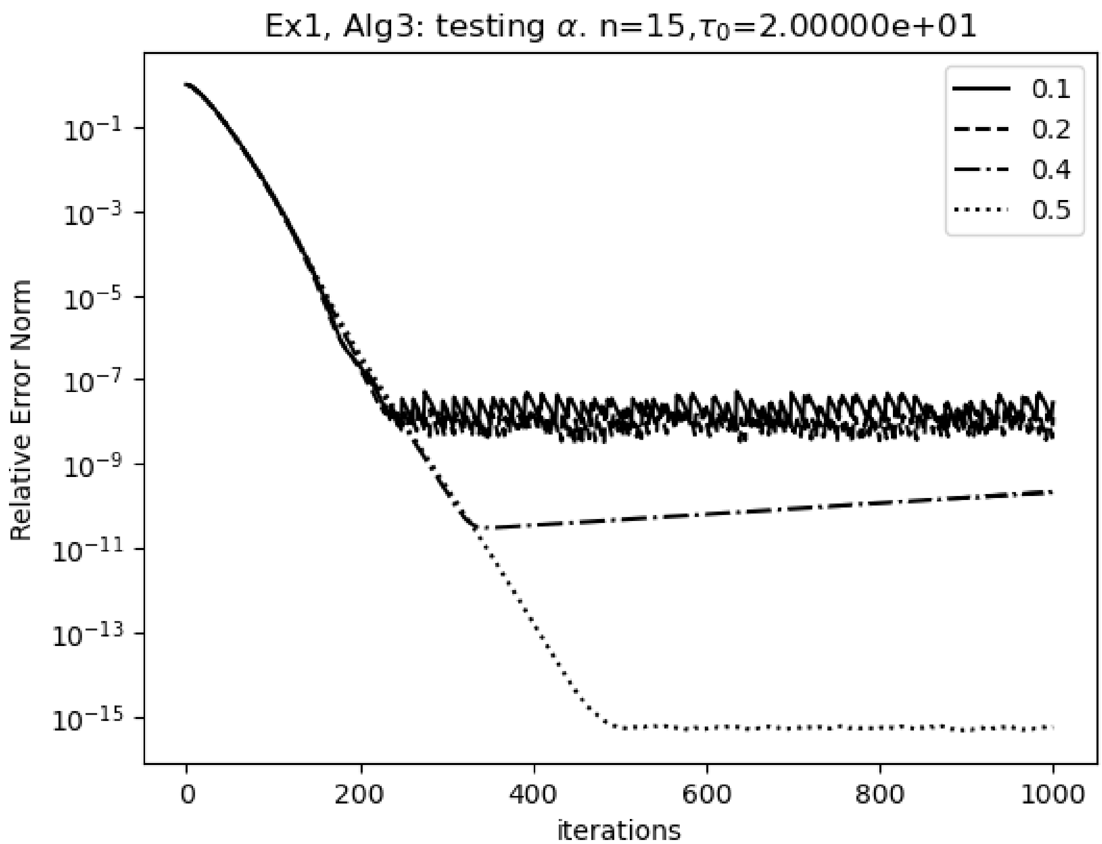


Figure 3b

[30]:
```
Figure3_4b(15; maxit=1000);
```



Now we use the larger values of τ_0.

Figure 4a

[31]:
```
Figure3_4a(15; maxit=1000, tauvec=[20.0, 10.0, 5.0, 1.0]);
```

```
Ex1, Alg3: testing $\tau_0$. n=15,$\alpha$ =5.00000e-01
```

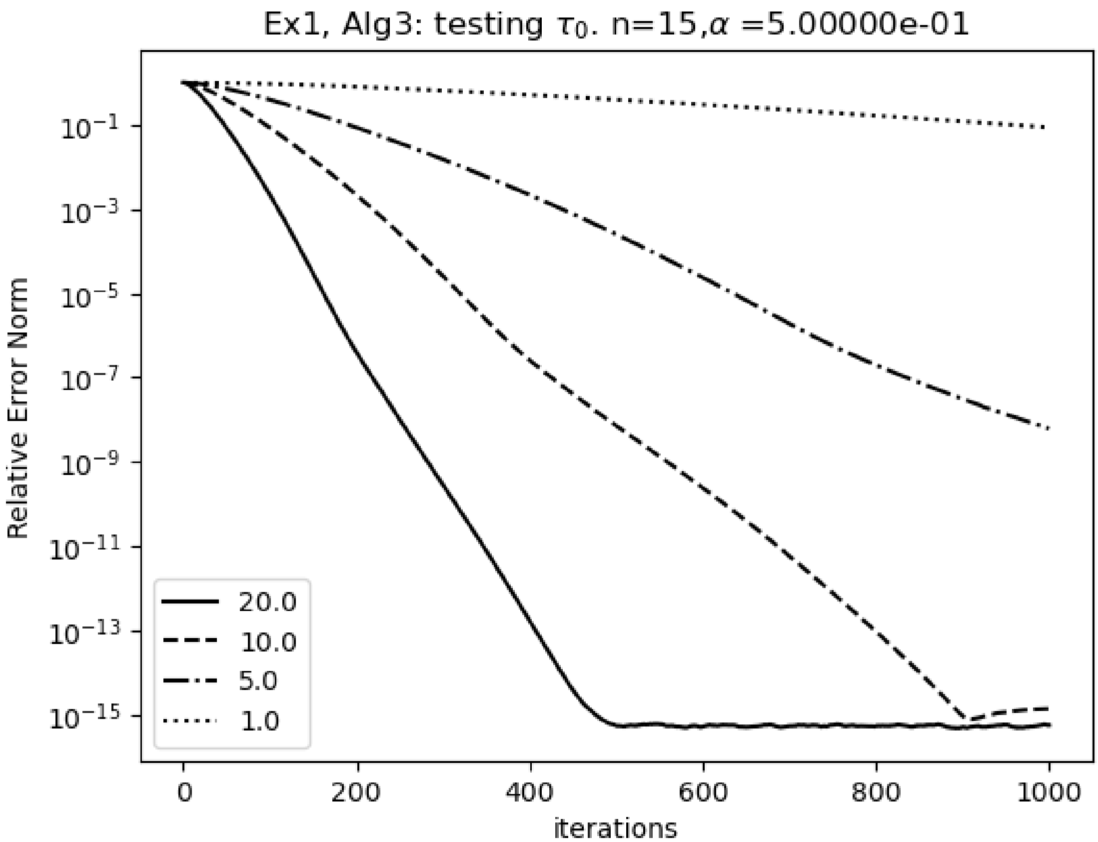
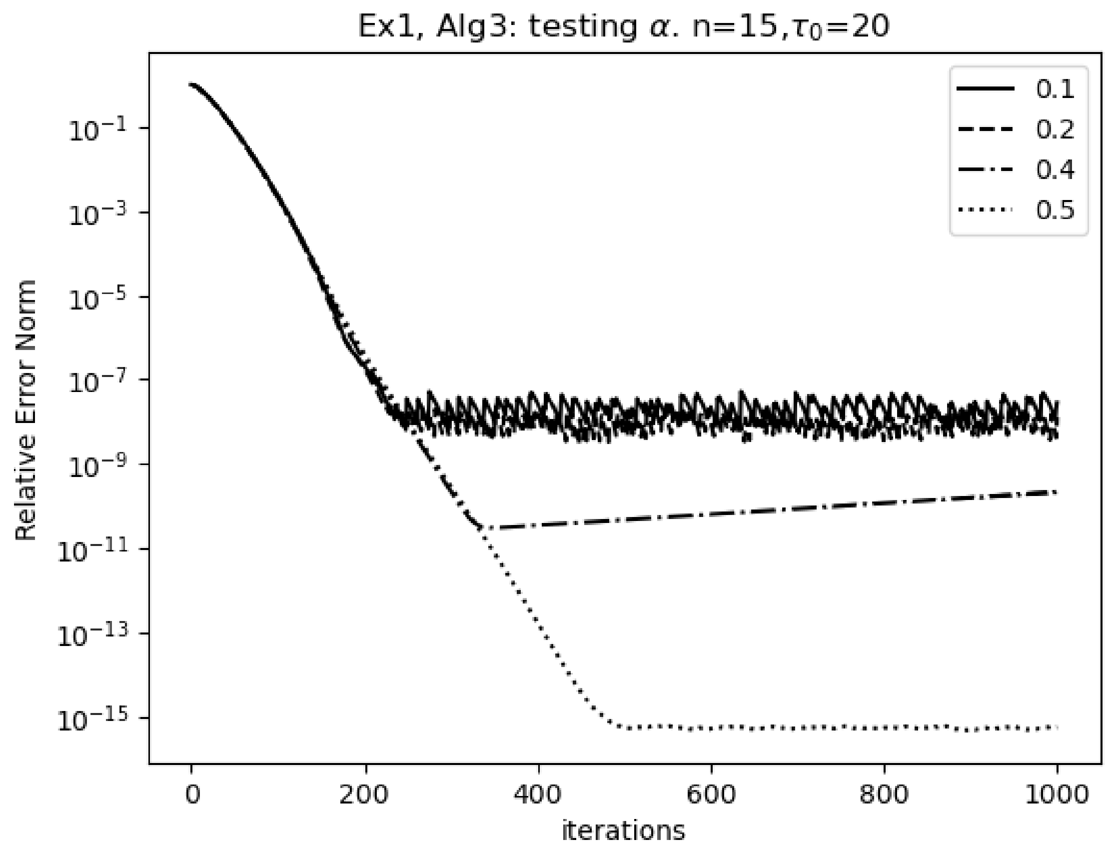


Figure 4b

[32]:
```
Figure3_4b(15; maxit=1000, tau0=20);
```



2 Example 2

This example is a constrained semi-linear elliptic boundary value problem. Define for sufficienlty smooth u

$$\mathcal{H}(u) = -\Delta u + \delta |u|^{\alpha} \mathsf{sign}(u) - |u|^{p-1} u$$

on $D = (0,1)^2$ with the boundary condition $u(x,y) = 0.5 - \sin(x)\sin(y)$ on ∂D. Here, $\alpha \in (0,1)$, $p > 1$, and $\delta > p/\alpha$ are three constants. We consider the variational inequality that is to find u^* with $u * (x,y) \in [-1,1]$ such that

$$\mathcal{H}(u^*)(u - u^*) \geq 0$$

for any u with values in $[-1,1]$.

This problem is equivalent to the following nonlinear equation,

$$0 = \mathcal{F}(u) := \begin{cases} \mathcal{H}(u), & \text{if } u - \mathcal{H}(u) \in [-1,1], \\ u - 1, & \text{if } u - \mathcal{H}(u) \geq 1, \\ u + 1, & \text{otherwise.} \end{cases}$$

After discretization we have the nonlinear system

$$0 = \mathbf{F}(\mathbf{u}) := \Big(\mathbf{u} - \Pi_{\mathbf{U}} \Big(\mathbf{u} - \tau \Big(\mathbf{A}\mathbf{u} + \delta \, |\mathbf{u}|^{\alpha} \, \mathsf{sign}(\mathbf{u}) - |\mathbf{u}|^{p-1} \, \mathbf{u} - \mathbf{b}\Big)\Big)\Big)$$

where $\mathbf{U} = [-1,1]^n$, $\tau > 0$ is the stepsize, $\mathbf{A} \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix, and $\mathbf{b} \in \mathbb{R}^n$ encodes the boundary conditions. The boundary conditions insure that the solution will change signs in the domain so the non-Lipschitz features are exercised. Hence the iteration for Algorithm 1 would look like

$$\mathbf{u}_{k+1} = \Pi_{\mathbf{U}} \left(\mathbf{u} - \tau \left(\mathbf{A}\mathbf{u}_k + \delta \left| \mathbf{u}_k \right|^{\alpha} \mathsf{sign}(\mathbf{u}_k) - \left| \mathbf{u}_k \right|^{p-1} \mathbf{u}_k - \mathbf{b} \right) \right)$$

In the example we use $\tau = \tau_0 h^{-2}$ as we did in Example 1.

The nonlinear equation is the necessary condition for the optimization problem

$$\min_{\mathbf{u}\in\Omega} f(\mathbf{u}) := \frac{1}{2} \left(f_1(\mathbf{u}) + f_2(\mathbf{u}) \right)$$

where

$$f_1(\mathbf{u}) = \mathbf{u}^\top \mathbf{A}\mathbf{u} - 2\mathbf{b}^\top \mathbf{u} - \frac{2}{1+p} \mathbf{e}^\top \left| \mathbf{u} \right|^{1+p}, \text{ and } f_2(\mathbf{u}) = \frac{2\delta}{1+\alpha} \mathbf{e}^\top \left| \mathbf{u} \right|^{1+\alpha}.$$

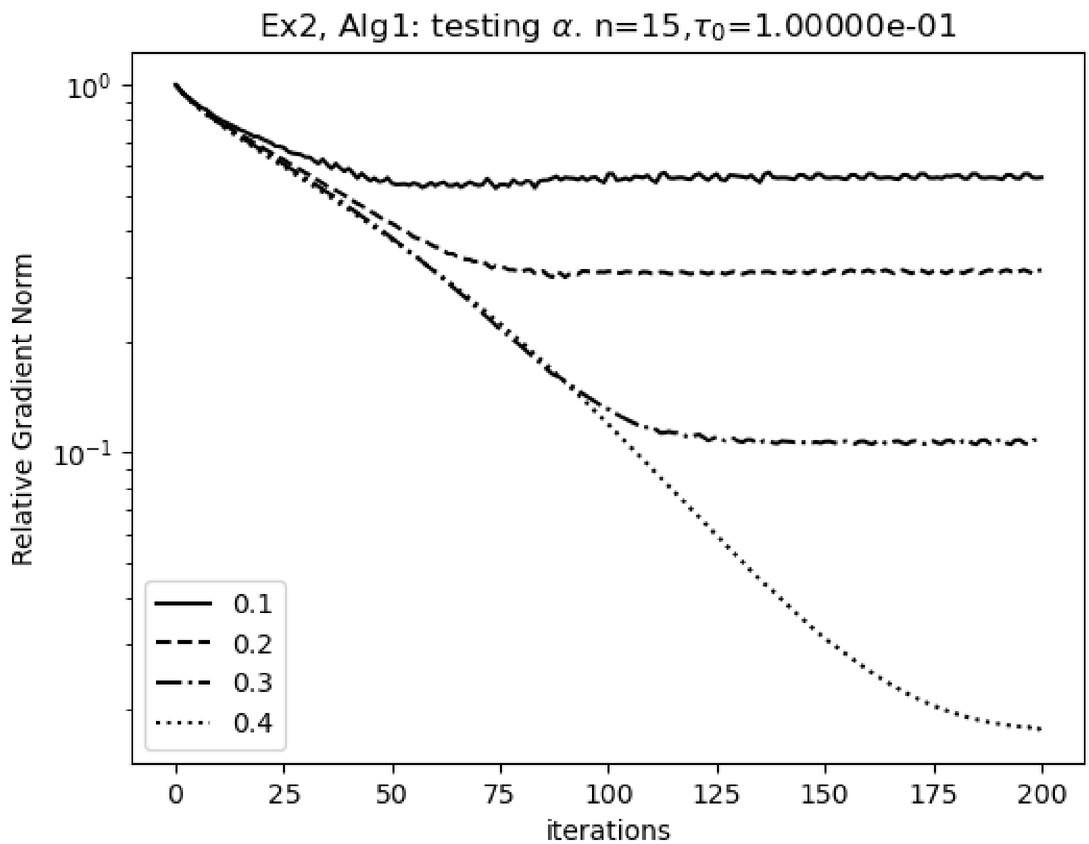
Our conditions on the problem struture hold with $\alpha_1 = 1$, $L_1 = 2\left\| \mathbf{A} \right\| + 2p$, $\alpha_2 = \alpha$, and $L_2 = 2\delta\alpha$. The condition that $\alpha \in (0,1)$, $p > 1$, and $\delta > p/\alpha$ implies strong convexity.

2.1 Figures for Example 2

In all the examples we use $\delta = 20$ and $p = 1.5$. The parameter α ranges from .1 to .8, so $\delta > \alpha/p$ in all cases.

Figure5a

[33]:
```
Figure5ac(15; tau0=.1, maxit=200, pvec=[.1, .2, .3, .4])
```



[33]: Python: Text(0.5, 1.0, 'Ex2, Alg1: testing $\\alpha$. n=15,$\\tau_0$=1.00000e-01')

Figure5b.

[34]:
```
Figure5bd(15; tau0=20.0, pvec=[.1, .2, .3, .4], maxit=50);
```

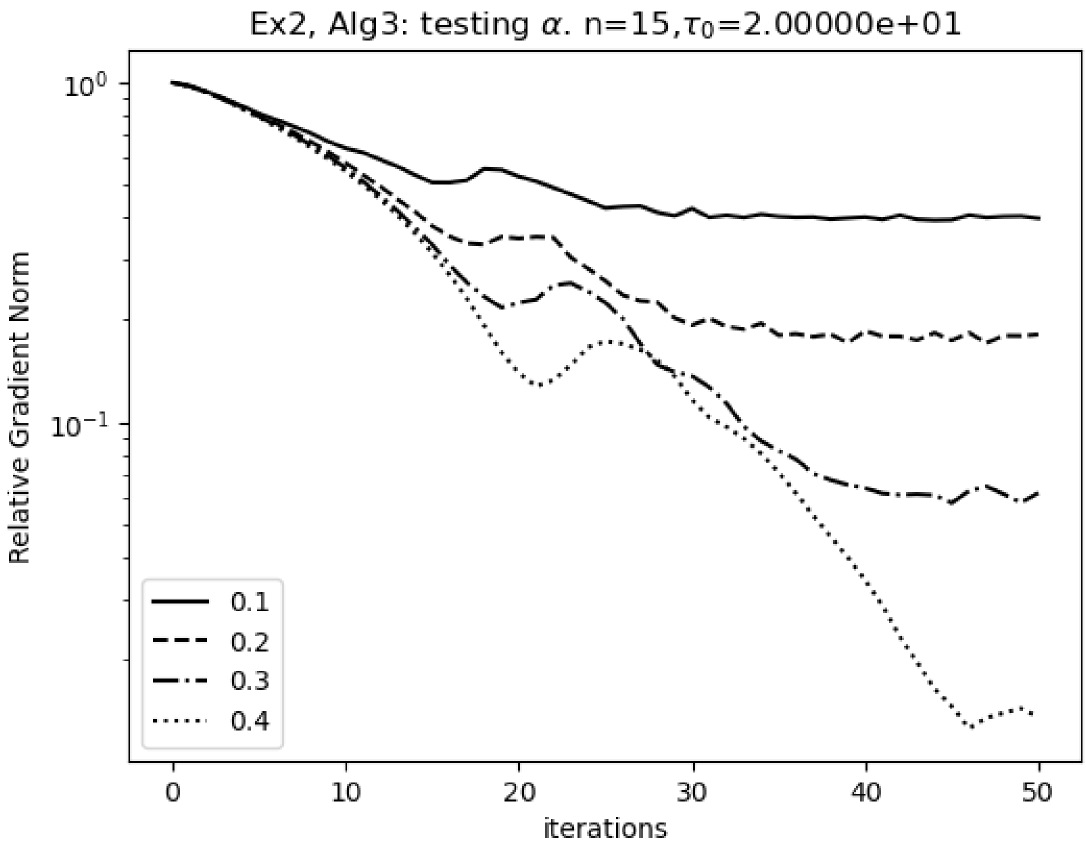


Figure 5c.

[35]:
```
Figure5ac(15; tau0=.1, pvec=[.5, .6, .7, .8], maxit=2000);
```

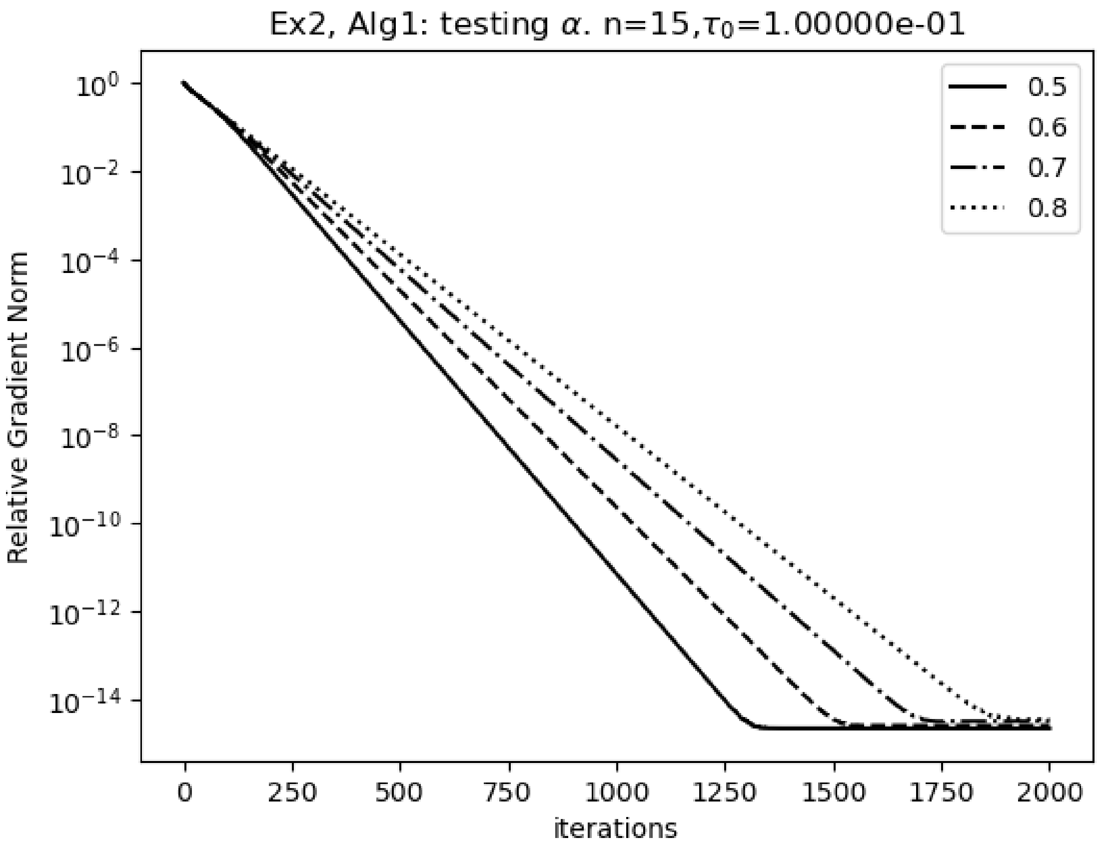
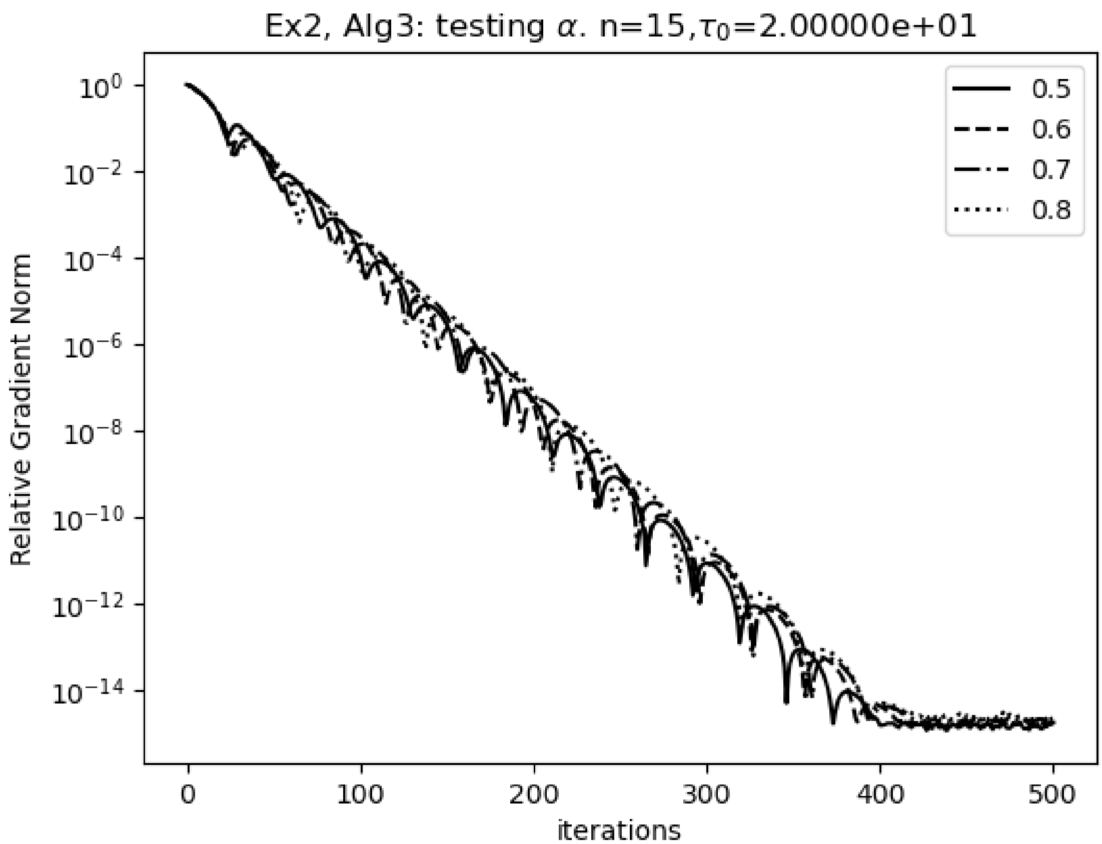


Figure 5d.

```
[36]: Figure5bd(15; tau0=20.0, pvec=[.5, .6, .7, .8], maxit=500);
```



Ex2, Alg3: testing α. n=15,τ_0=2.00000e+01
Relative Gradient Norm
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
0
100
200
300
400
500
iterations
0.5
0.6
0.7
0.8