

# Examples

January 24, 2026

```
[13]: include("notebook_init.jl");
```

This notebook generates the figures in the paper *Complexity of Projected Gradient Methods for Strongly Convex Optimization with Hölder Continuous Gradient Terms*

by X. Chen, C. T. Kelley, and L. Wang

## 1 Example 1

This problem is to solve the following two-dimensional PDE,

$$\mathcal{F}(u) = -\Delta u + \gamma u_+^\alpha = 0,$$

where  $\alpha \in (0, 1)$ ,  $\gamma > 0$  is a constant and  $u_+ = \max\{u, 0\}$ . Discretizing this problem with the standard five point scheme leads to

$$\mathbf{F}(\mathbf{u}) = \mathbf{A}\mathbf{u} + \gamma \mathbf{u}_+^\alpha - \mathbf{b} = 0$$

$\mathbf{A} \in \mathbb{R}^{n \times n}$  is the discretization of  $-\Delta$  with zero boundary conditions,  $\mathbf{b} \in \mathbb{R}^n$  encodes the boundary conditions, and  $\mathbf{u}_+^\alpha = \max\{\mathbf{u}, 0\}^\alpha$  is understood as a component-wise operation.

We now modify the above problem to enable direct computation of errors in the iterations. To this end, we take as the exact solution the function

$$u^*(x, y) = \left(\frac{3r-1}{2}\right)^2 \max\left\{0, r - \frac{1}{3}\right\}$$

where  $r = \sqrt{x^2 + y^2}$ , and enforce the boundary conditions

$$u(x, 1) = u^*(x, 1), u(x, 0) = u^*(x, 0), u(1, y) = u^*(1, y), u(0, y) = u^*(0, y),$$

for  $0 < x, y < 1$ . Hence our modified equation is

$$\mathbf{F}(\mathbf{u}) - \mathbf{c}^* = 0,$$

where  $\mathbf{c}^* = \mathbf{F}(\mathbf{u}^*)$ .

The nonlinear system is first order optimality condition for the strongly convex optimization problem.

$$\min_{\mathbf{u} \in \mathbb{R}^n} f(\mathbf{u}) = \frac{1}{2} \mathbf{u}^\top \mathbf{A} \mathbf{u} + \frac{\gamma}{1+\alpha} \mathbf{e}^\top \mathbf{u}_+^{1+\alpha} - (\mathbf{b} + \mathbf{c}^*)^\top \mathbf{u},$$

where  $\mathbf{e} \in \mathbb{R}^n$  is the vector of all ones.

## 1.1 Figures for Example 1

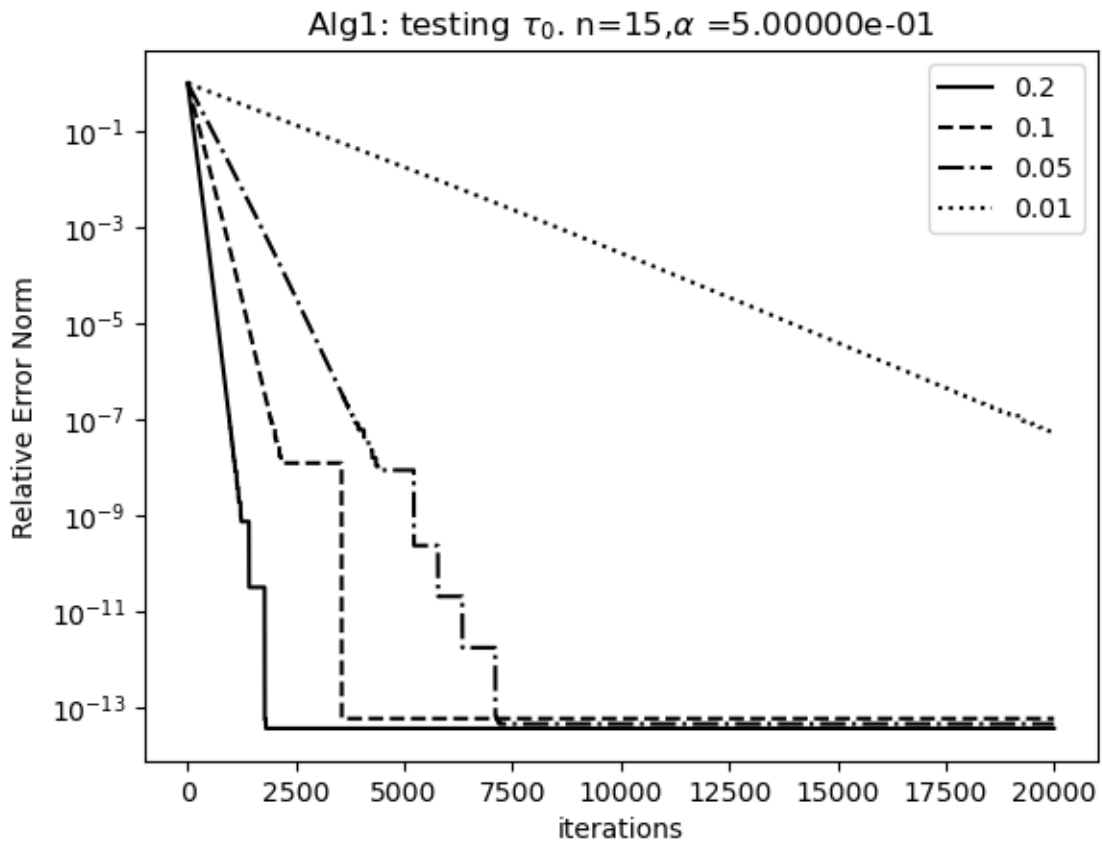
Generate Figures 1(a), 1(b), 2(a), and 2(b). These figures are for Algorithm 1. Figures 1(a) and 2(a) compare various stepsizes  $\tau = \tau_0 h^2$  which are consistent with the CFL condition. Figures 1(b) and 2(b) examine values of the exponent  $p$ .

The files for building the figures are in `/src/Figures`. The code is `Figures_Alg1.jl` and the functions `Figure1_2a` and `Figure1_2b`. The functions take the dimension as an argument.

Figure 1a

```
[14]: Figure1a(15; alpha=.5);
```

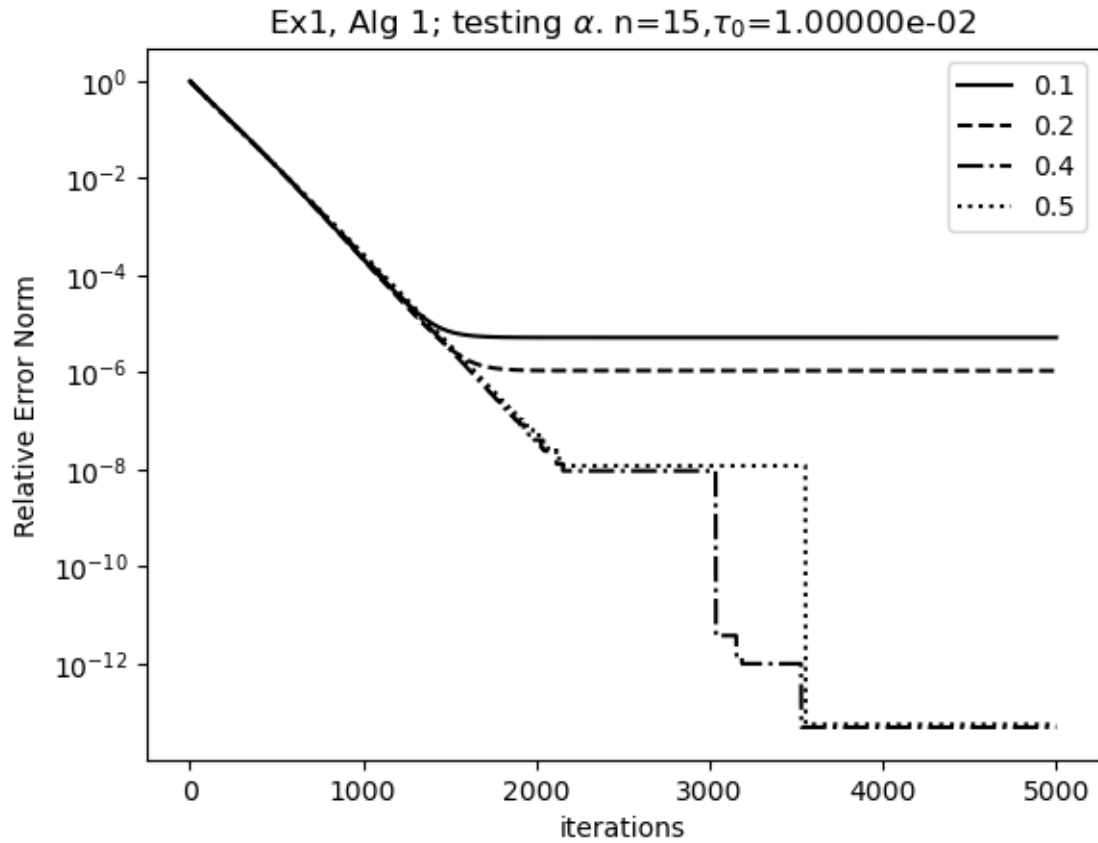
Alg1: testing  $\tau_0$ . n=15,  $\alpha = 5.00000e-01$



Now we compare the effects of changing the exponent  $\alpha$ . Here we can see the effects of the change I made in Alg 1 by letting the gradient norm increase without terminating the iteration.

Figure 1b

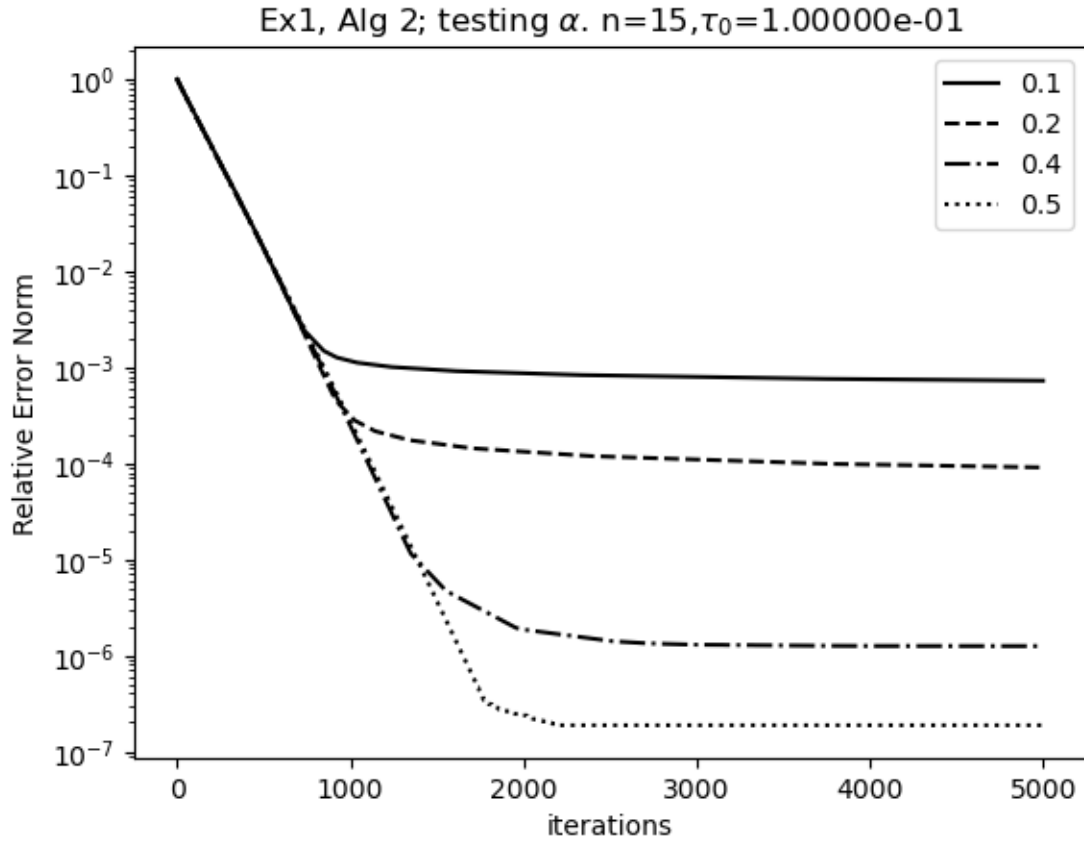
```
[15]: Figure1b(15; maxit=5000);
```



And finally repeat the computation for a 31x31 grid. This will complete the computations for Example 1 + Algorithm 1.

Figure 2

[16]: `Figure2(15; maxit=5000);`



The advantage of the line search is that one does not have to manually adjust  $\tau_0$ .

The results for Algorithm 3 are in Figures 4 and 5. We set  $\nu = \tau_0 h^2$  in these examples and will need to modify that to use the estimate in Remark 4.2. In the first two figures 3(a) and 3(b) use the values of  $\tau_0$  we used in Figure 1.

Figure 3a

[17]: `Figure3_4a(15);`

Ex1, Alg3: testing  $\tau_0$ . n=15,  $\alpha = 5.00000e-01$

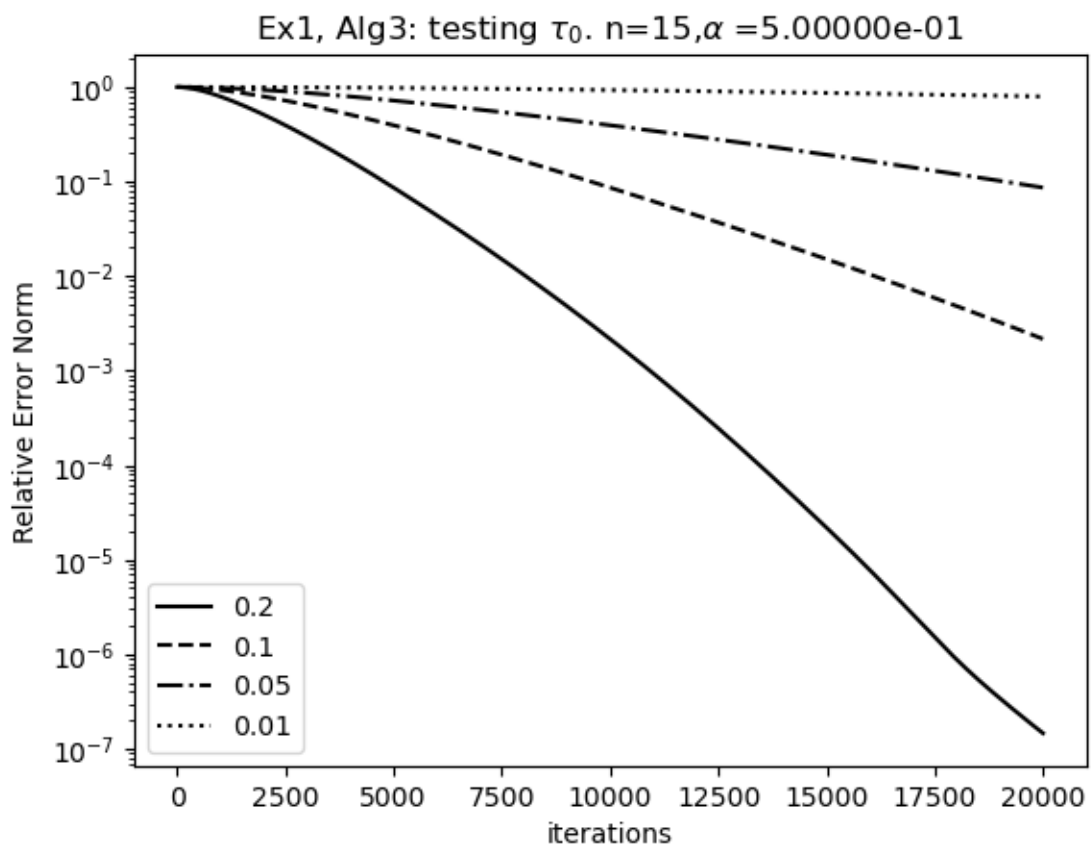
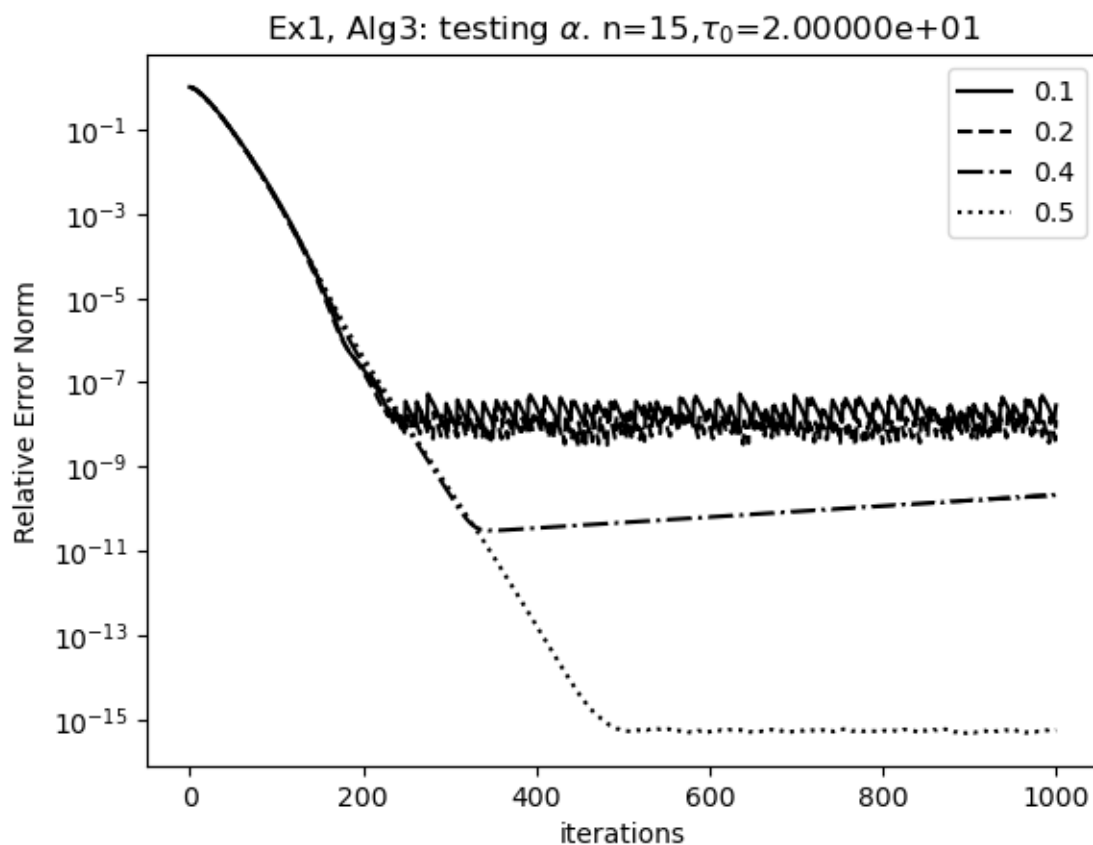


Figure 3b

[18]: `Figure3_4b(15; maxit=1000);`



Now we use the larger values of  $\tau_0$ .

Figure 4a

```
[19]: Figure3_4a(15; maxit=1000, tauvec=[20.0, 10.0, 5.0, 1.0]);
```

Ex1, Alg3: testing  $\tau_0$ .  $n=15, \alpha=5.00000e-01$

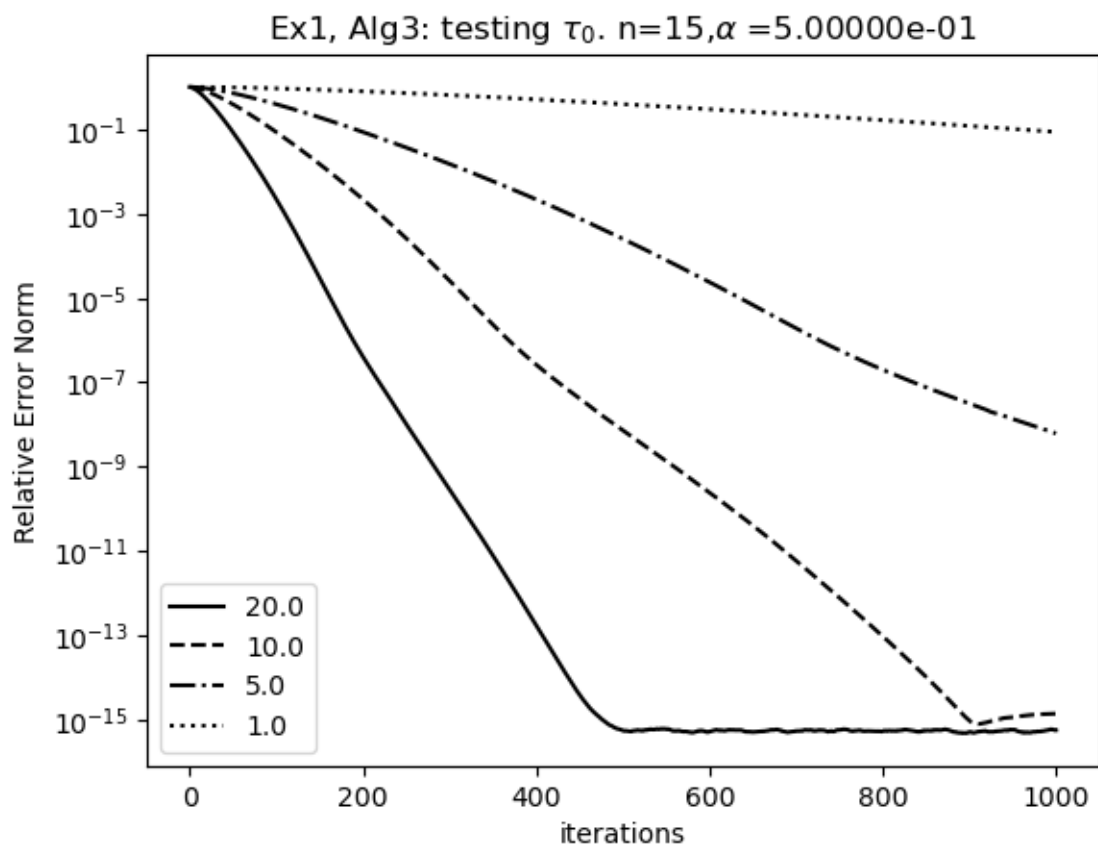
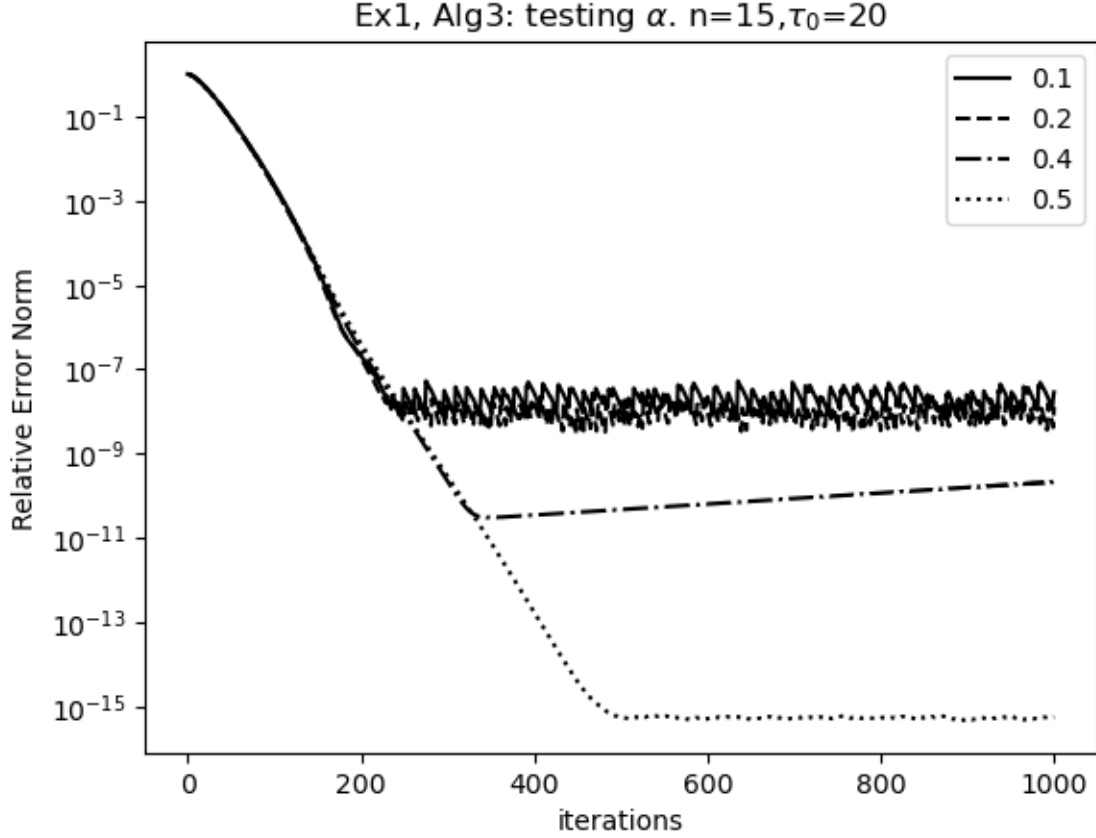


Figure 4b

[20]: `Figure3_4b(15; maxit=1000, tau0=20);`



## 2 Example 2

This example is a constrained semi-linear elliptic boundary value problem. Define for sufficiently smooth  $u$

$$\mathcal{H}(u) = -\Delta u + \delta |u|^\alpha \text{sign}(u) - |u|^{p-1}u$$

on  $D = (0, 1)^2$  with the boundary condition  $u(x, y) = 0.5 - \sin(x) \sin(y)$  on  $\partial D$ . Here,  $\alpha \in (0, 1)$ ,  $p > 1$ , and  $\delta > p/\alpha$  are three constants. We consider the variational inequality that is to find  $u^*$  with  $u^*(x, y) \in [-1, 1]$  such that

$$\mathcal{H}(u^*)(u - u^*) \geq 0$$

for any  $u$  with values in  $[-1, 1]$ .

This problem is equivalent to the following nonlinear equation,

$$0 = \mathcal{F}(u) := \begin{cases} \mathcal{H}(u), & \text{if } u - \mathcal{H}(u) \in [-1, 1], \\ u - 1, & \text{if } u - \mathcal{H}(u) \geq 1, \\ u + 1, & \text{otherwise.} \end{cases}$$

After discretization we have the nonlinear system

$$0 = \mathbf{F}(\mathbf{u}) := \left( \mathbf{u} - \Pi_U \left( \mathbf{u} - \tau \left( \mathbf{A}\mathbf{u} + \delta |\mathbf{u}|^\alpha \text{sign}(\mathbf{u}) - |\mathbf{u}|^{p-1} \mathbf{u} - \mathbf{b} \right) \right) \right)$$



where  $\mathbf{U} = [-1, 1]^n$ ,  $\tau > 0$  is the stepsize,  $\mathbf{A} \in \mathbb{R}^{n \times n}$  is a symmetric positive definite matrix, and  $\mathbf{b} \in \mathbb{R}^n$  encodes the boundary conditions. The boundary conditions insure that the solution will change signs in the domain so the non-Lipschitz features are exercised. Hence the iteration for Algorithm 1 would look like

$$\mathbf{u}_{k+1} = \Pi_{\mathbf{U}} \left( \mathbf{u} - \tau \left( \mathbf{A}\mathbf{u}_k + \delta |\mathbf{u}_k|^\alpha \text{sign}(\mathbf{u}_k) - |\mathbf{u}_k|^{p-1} \mathbf{u}_k - \mathbf{b} \right) \right)$$

In the example we use  $\tau = \tau_0 h^{-2}$  as we did in Example 1.

The nonlinear equation is the necessary condition for the optimization problem

$$\min_{\mathbf{u} \in \Omega} f(\mathbf{u}) := \frac{1}{2} (f_1(\mathbf{u}) + f_2(\mathbf{u}))$$

where

$$f_1(\mathbf{u}) = \mathbf{u}^\top \mathbf{A} \mathbf{u} - 2\mathbf{b}^\top \mathbf{u} - \frac{2}{1+p} \mathbf{e}^\top |\mathbf{u}|^{1+p}, \text{ and } f_2(\mathbf{u}) = \frac{2\delta}{1+\alpha} \mathbf{e}^\top |\mathbf{u}|^{1+\alpha}.$$

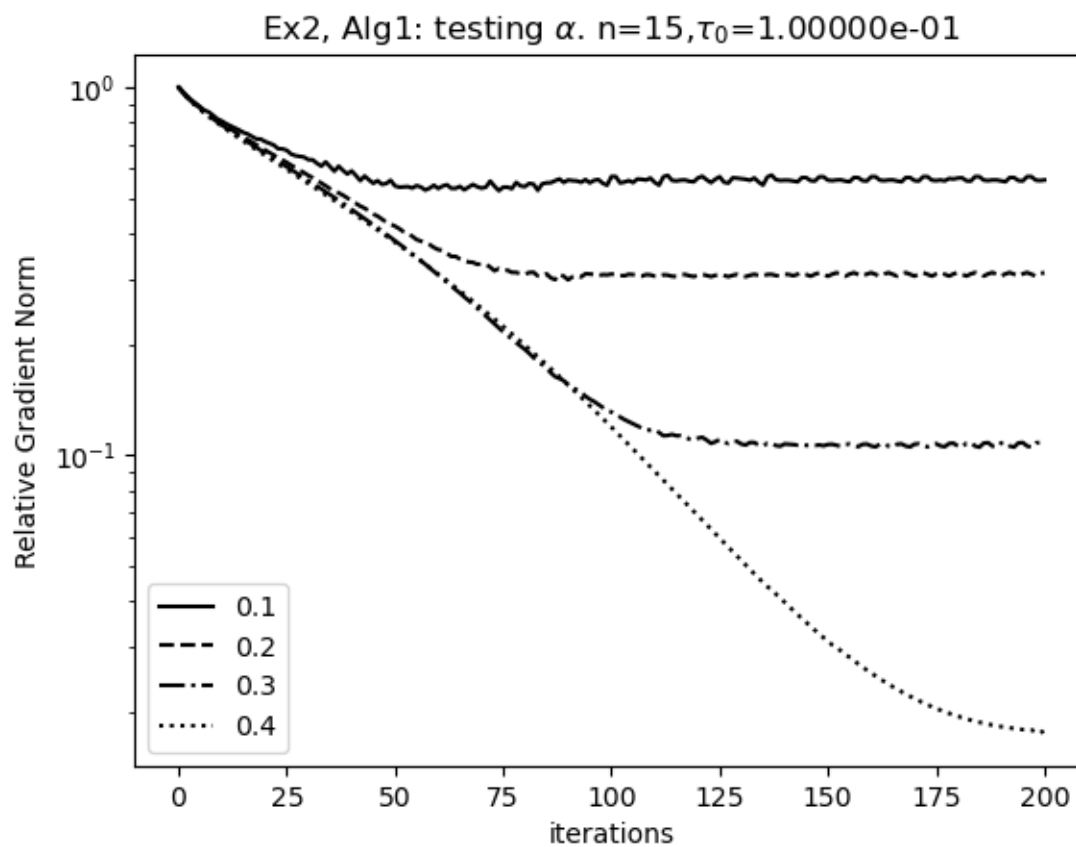
Our conditions on the problem struture hold with  $\alpha_1 = 1$ ,  $L_1 = 2 \|\mathbf{A}\| + 2p$ ,  $\alpha_2 = \alpha$ , and  $L_2 = 2\delta\alpha$ . The condition that  $\alpha \in (0, 1)$ ,  $p > 1$ , and  $\delta > p/\alpha$  implies strong convexity.

## 2.1 Figures for Example 2

In all the examples we use  $\delta = 20$  and  $p = 1.5$ . The parameter  $\alpha$  ranges from .1 to .8, so  $\delta > \alpha/p$  in all cases.

Figure5a

[21]: `Figure5ac(15; tau0=.1, maxit=200, pvec=[.1, .2, .3, .4])`



[21]: Python: Text(0.5, 1.0, 'Ex2, Alg1: testing  $\alpha$ .  
 $n=15, \tau_0=1.00000e-01$ ')

Figure5b.

[22]: Figure5bd(15; tau0=20.0, pvec=[.1, .2, .3, .4], maxit=50);

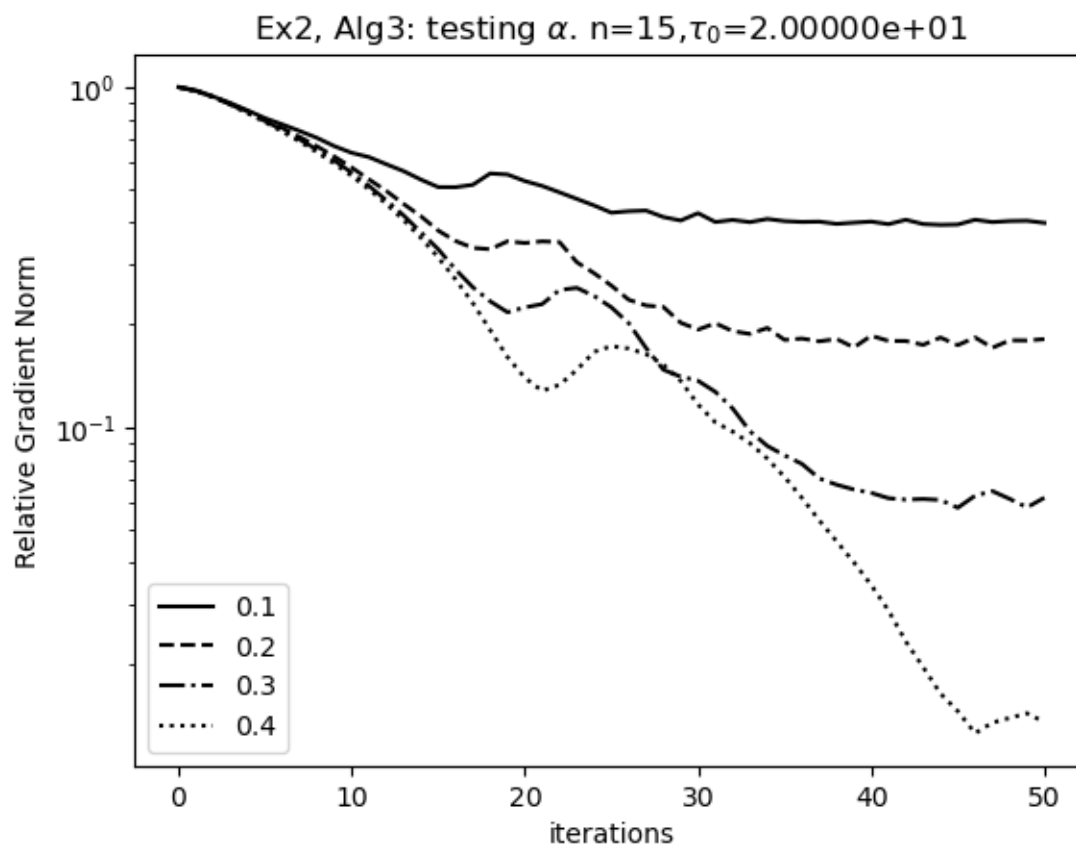


Figure 5c.

[23]: `Figure5ac(15; tau0=.1, pvec=[.5, .6, .7, .8], maxit=2000);`

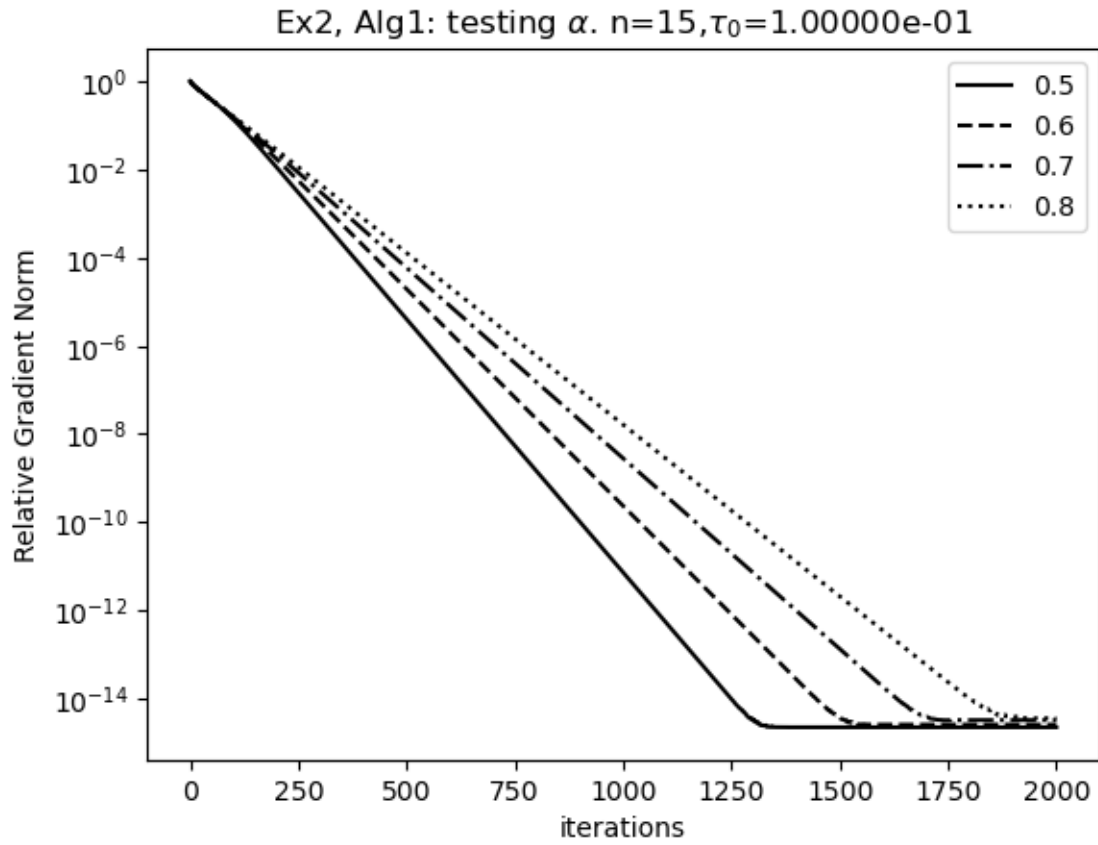
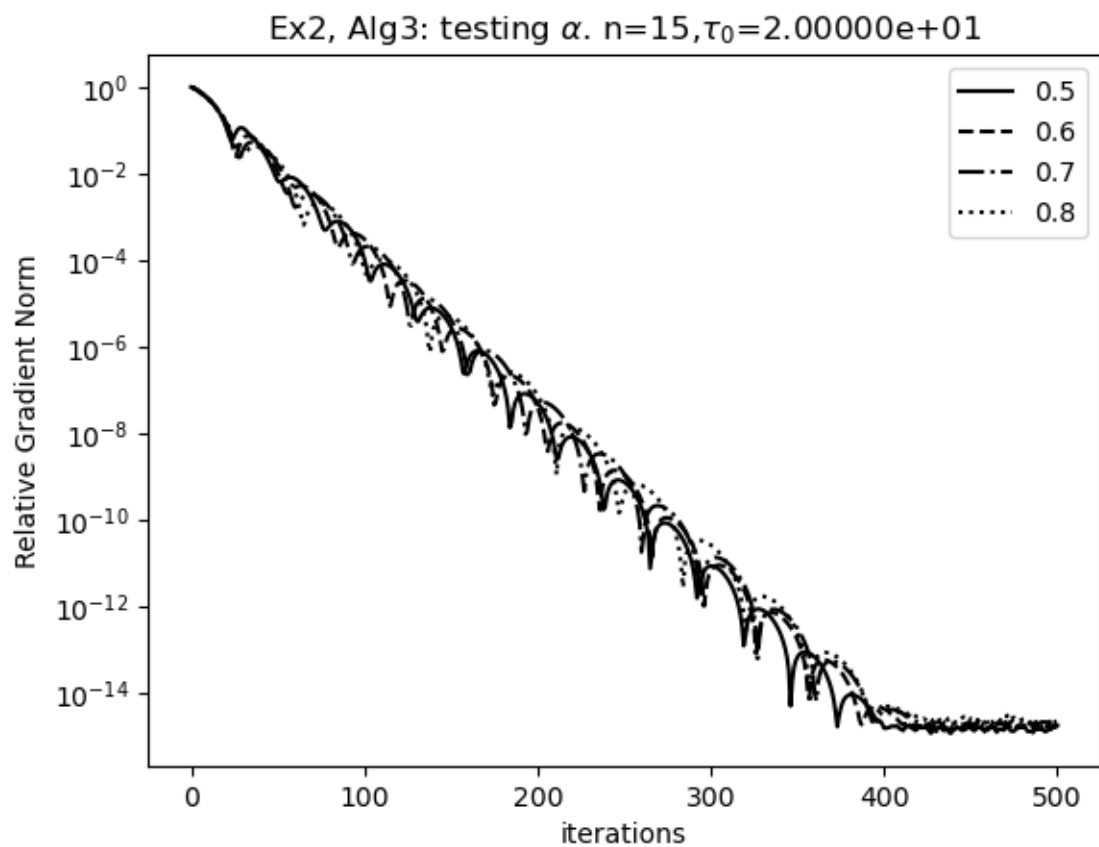


Figure 5d.

[24]: `Figure5bd(15; tau0=20.0, pvec=[.5, .6, .7, .8], maxit=500);`



[ ]:

[ ]: