

# COMPLEXITY OF PROJECTED GRADIENT METHODS FOR STRONGLY CONVEX OPTIMIZATION WITH HÖLDER CONTINUOUS GRADIENT TERMS\*

XIAOJUN CHEN<sup>†</sup>, C. T. KELLEY<sup>‡</sup>, AND LEI WANG<sup>§</sup>

December 5, 2025

**Abstract.** This paper studies the complexity of projected gradient descent methods for a class of strongly convex constrained optimization problems where the objective function is expressed as a summation of  $m$  component functions, each possessing a gradient that is Hölder continuous with an exponent  $\alpha_i \in (0, 1]$ . Under this formulation, the gradient of the objective function may fail to be globally Hölder continuous, thereby existing complexity results inapplicable to this class of problems. Our theoretical analysis reveals that, in this setting, the complexity of projected gradient methods is determined by  $\hat{\alpha} = \min_{i \in \{1, \dots, m\}} \alpha_i$ . We first prove that, with an appropriately fixed stepsize, the complexity bound for finding an approximate minimizer with a distance to the true minimizer less than  $\varepsilon$  is  $O(\log(\varepsilon^{-1})\varepsilon^{2(\hat{\alpha}-1)/(1+\hat{\alpha})})$ , which extends the well-known complexity result for  $\hat{\alpha} = 1$ . Next we show that the complexity bound can be improved to  $O(\log(\varepsilon^{-1})\varepsilon^{2(\hat{\alpha}-1)/(1+3\hat{\alpha})})$  if the stepsize is updated by the universal scheme. We illustrate our complexity results by numerical examples arising from elliptic equations with a non-Lipschitz term.

**Key words.** projected gradient descent, complexity, Hölder continuity

**MSC codes.** 90C25, 65L05, 65Y20

**1. Introduction.** Given a closed and convex set  $\Omega \subseteq \mathbb{R}^n$ , this paper considers the following optimization problem,

$$(1.1) \quad \min_{\mathbf{u} \in \Omega} f(\mathbf{u}) := \frac{1}{m} \sum_{i=1}^m f_i(\mathbf{u}),$$

where the objective function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  satisfies the following assumption.

ASSUMPTION 1.1.

1. The function  $f$  is  $\mu$ -strongly convex with a parameter  $\mu > 0$  on  $\Omega$ , that is,

$$f(\mathbf{u}) \geq f(\mathbf{v}) + \langle \nabla f(\mathbf{v}), \mathbf{u} - \mathbf{v} \rangle + \frac{\mu}{2} \|\mathbf{u} - \mathbf{v}\|^2,$$

for all  $\mathbf{u}, \mathbf{v} \in \Omega$ .

2. For each  $i \in [m] := \{1, 2, \dots, m\}$ , the function  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$  is continuously differentiable and the gradient  $\nabla f_i$  is (globally) Hölder continuous with an exponent  $\alpha_i \in (0, 1]$  on  $\Omega$ , namely, there exists a constant  $L_i > 0$  such that

$$(1.2) \quad \|\nabla f_i(\mathbf{u}) - \nabla f_i(\mathbf{v})\| \leq L_i \|\mathbf{u} - \mathbf{v}\|^{\alpha_i},$$

for all  $\mathbf{u}, \mathbf{v} \in \Omega$ .

---

\*Submitted to the editors DATE.

**Funding:** We would like to acknowledge support for this project from RGC grant JLFS/P-501/24 for the CAS AMSS-PolyU Joint Laboratory in Applied Mathematics and Hong Kong Research Grant Council project PolyU15300024.

<sup>†</sup>Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong, China (maxjchen@polyu.edu.hk).

<sup>‡</sup>Department of Mathematics, Box 8205, North Carolina State University, Raleigh, NC 27695-8205, USA (Tim.Kelley@ncsu.edu).

<sup>§</sup>Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong, China (lei2wang@polyu.edu.hk).

Here,  $\|\cdot\|$  is the  $\ell_2$  norm and  $\langle \cdot, \cdot \rangle$  is the inner product on  $\mathbb{R}^n$ . We also denote by  $\mathbf{u}^* \in \Omega$  and  $f^* = f(\mathbf{u}^*)$  the global minimizer and the optimal value of problem (1.1), respectively.

Suppose that each  $\nabla f_i$  is Lipschitz continuous, which corresponds to condition (1.2) with  $\alpha_i = 1$  for all  $\mathbf{u}, \mathbf{v} \in \Omega$ . Then  $\nabla f$  is also Lipschitz continuous and the associated Lipschitz constant is  $L = \sum_{i=1}^m L_i/m$ . Let  $\Pi_\Omega(\cdot)$  be the projection operator onto the set  $\Omega$ . It is well known that the classical projected gradient descent method

$$(1.3) \quad \mathbf{u}_{k+1} = \Pi_\Omega(\mathbf{u}_k - \tau \nabla f(\mathbf{u}_k)),$$

with any initial point  $\mathbf{u}_0 \in \mathbb{R}^n$  and the stepsize  $\tau \in (0, 2/(\mu + L)]$ , achieves a linear rate of convergence [9, Theorem 2.2.14] as follows,

$$\|\mathbf{u}_k - \mathbf{u}^*\| \leq (1 - \mu\tau)^k \|\mathbf{u}_0 - \mathbf{u}^*\|.$$

Therefore, for a given  $\varepsilon > 0$ , method (1.3) is guaranteed to find a point  $\mathbf{u}_k \in \Omega$  satisfying  $\|\mathbf{u}_k - \mathbf{u}^*\| \leq \varepsilon$  after at most  $O(\log(\varepsilon^{-1}))$  iterations. Unfortunately, this analysis fails if there exists at least one index  $i \in [m]$  such that  $\alpha_i < 1$ . We explain the failure of the convergence of method (1.3) to  $\mathbf{u}^*$  by the following example.

*Example 1.1.* [5, Example 1] Consider the following univariate optimization problem,

$$\min_{x \in \mathbb{R}} f(x) = \frac{1}{2}x^2 + \frac{2}{3}|x|^{3/2},$$

which is a special instance of problem (1.1) with  $f_1(x) = x^2/2$ ,  $f_2(x) = 2|x|^{3/2}/3$ , and  $\Omega = \mathbb{R}$ . It is easy to see that the global minimizer is  $x^* = 0$ . Method (1.3) with the fixed stepsize  $\tau > 0$  starting from  $x_0 \neq 0$  reads as follows,

$$x_{k+1} = x_k - \tau \nabla f(x_k) = (1 - \tau)x_k - \tau \text{sign}(x_k) |x_k|^{1/2},$$

where  $\text{sign}(x) = 1$  if  $x > 0$ , 0 if  $x = 0$ , and  $-1$  otherwise. A straightforward verification reveals that

$$|x_{k+1}|^2 - |x_k|^2 = -\tau(2 - \tau)|x_k|^2 - 2\tau(1 - \tau)|x_k|^{3/2} + \tau^2|x_k|.$$

It is evident that, when  $|x_k|$  is sufficiently small, the last term in the right-hand side becomes dominant, resulting in that  $|x_{k+1}|^2 - |x_k|^2 \geq 0$ . Therefore, the distance to the global minimizer ceases to decrease once it achieves a certain level.

Moreover, in [5] we show that  $\nabla f$  is locally Hölder continuous, but not globally Hölder continuous. In fact, from

$$\nabla f(x^* + |h|) - \nabla f(x^*) = |h| + |h|^{\frac{1}{2}} = (|h|^{1-\alpha} + |h|^{\frac{1}{2}-\alpha})|h|^\alpha =: (\hat{L}_1(h) + \hat{L}_2(h))|h|^\alpha,$$

we have  $\hat{L}_1(h) \rightarrow \infty$  when  $\alpha = (0, 1)$  and  $|h| \rightarrow \infty$ , while  $\hat{L}_2(h) \rightarrow \infty$  when  $\alpha = 1$  and  $|h| \rightarrow 0$ .

This example demonstrates that a function  $f$  expressed as a sum of component functions  $f_i$ , each endowed with a Hölder continuous gradient, may itself fail to possess a Hölder continuous gradient. This phenomenon was revisited by Nesterov (see [10, Example 1]).

On the other hand, this example satisfies Assumption 1.1 (ii) as

$$|\nabla f_1(x) - \nabla f_1(y)| \leq L_1|x - y| \quad \text{and} \quad |\nabla f_2(x) - \nabla f_2(y)| \leq L_2|x - y|^{1/2}, \quad \forall x, y \in \mathbb{R}$$

with  $L_1 = L_2 = 1$ .

Since  $\nabla f$  may not be globally Hölder continuous, most existing complexity results are inapplicable to problem (1.1). For the special case where  $m = 1$ , namely,  $\nabla f$  is globally Hölder continuous with an exponent  $\alpha \in (0, 1]$ , Devolder et al. [6] presented the following bound for method (1.3),

$$f(\hat{\mathbf{u}}_N) - f(\mathbf{u}^*) \leq K(N) := \frac{L_\alpha \|\mathbf{u}_0 - \mathbf{u}^*\|^{1+\alpha}}{1+\alpha} \left( \frac{2}{N} \right)^{\frac{1+\alpha}{2}},$$

where  $L_\alpha$  is the Hölder constant and  $\hat{\mathbf{u}}_N = \sum_{k=1}^N \mathbf{u}_k / N$ . In the strongly convex case, (51) in [6] comes to

$$\|\hat{\mathbf{u}}_N - \mathbf{u}^*\|^2 \leq \frac{2}{\mu} K(N),$$

which implies that finding an  $N$  average of iterations  $\hat{\mathbf{u}}_N$  satisfying  $\|\hat{\mathbf{u}}_N - \mathbf{u}^*\| \leq \varepsilon$  requires  $O(\varepsilon^{-4/(1+\alpha)})$  iterations.

The contribution of this paper is to provide new complexity results of the projected gradient descent methods for problem (1.1), which are dictated by the parameter  $\hat{\alpha} = \min_{i \in [m]} \alpha_i \in (0, 1]$ . We first show that, with an appropriately fixed stepsize, the complexity bound for finding an iterate with a distance to the global minimizer less than  $\varepsilon$  is  $O(\log(\varepsilon^{-1}) \varepsilon^{2(\hat{\alpha}-1)/(1+\hat{\alpha})})$ , which extends the well-known complexity result for  $\hat{\alpha} = 1$ . Next, we demonstrate that this complexity bound can be improved to  $O(\log(\varepsilon^{-1}) \varepsilon^{2(\hat{\alpha}-1)/(1+3\hat{\alpha})})$  if the stepsize is updated at each iteration using the universal scheme. Even in the special case where  $m = 1$ , our complexity bound is at least  $O(\varepsilon^{-1})$  lower than (51) in [6]. For example, when  $\hat{\alpha} = 1$ , our bound is  $O(\log(\varepsilon^{-1}))$  but (51) in [6] is  $O(\varepsilon^{-2})$ .

Our study is motivated by elliptic equations with a non-Lipschitz term [2, 12], as well as optimization problems with an  $\ell_p$ -norm ( $1 < p < 2$ ) regularization term [1, 4]. We illustrate our complexity results by two numerical examples arising from elliptic equations with a non-Lipschitz term in section 5, after we present complexity of projected gradient methods with fixed stepsizes and updated stepsizes in sections 2 to 4, respectively.

## 2. Vanilla Projected Gradient Descent Method with a Fixed Stepsize.

In this section, we attempt to employ the vanilla projected gradient descent method (1.3) with a fixed stepsize to solve problem (1.1), whose complexity bound is also provided. Example 1.1 illustrates that the projected gradient descent method (1.3) with a fixed stepsize will experience stagnation before reaching the global minimizer.

To obtain an approximate solution to problem (1.1), it is necessary to choose a sufficiently small stepsize  $\tau$  in the projected gradient descent method (1.3), the magnitude of which depends on the desired level of accuracy. Let  $M > 0$  be a constant defined as

$$(2.1) \quad M = \max_{i \in [m]} \left\{ \left[ \frac{2(1 - \alpha_i)}{\mu(1 + \alpha_i)} \right]^{(1 - \alpha_i)/(1 + \alpha_i)} L_i^{2/(1 + \alpha_i)} \right\}.$$

We select a specific stepsize  $\tau = \varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}/M$  in the projected gradient descent method, whose complete framework is presented in Algorithm 1. Two sequences  $\{\mathbf{v}_k\}$  and  $\{\mathbf{u}_k\}$  are maintained in Algorithm 1, where  $\mathbf{v}_k$  is generated by the projected gradient descent method and  $\mathbf{u}_k$  corresponds to the iterate achieving the smallest objective function value among the first  $k$  iterations.

---

**Algorithm 1:** Projected Gradient Descent Method (PGDM).

---

**Input:**  $\varepsilon > 0$ .

Initialize  $\mathbf{u}_0 = \mathbf{v}_0 \in \Omega$ .

Choose the stepsize  $\tau = \varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}/M$ .

**for**  $k = 0, 1, 2, \dots$  **do**

    Compute

$$\mathbf{v}_{k+1} = \Pi_{\Omega}(\mathbf{v}_k - \tau \nabla f(\mathbf{v}_k)).$$

    Set

$$\mathbf{u}_{k+1} = \begin{cases} \mathbf{v}_{k+1}, & \text{if } f(\mathbf{v}_{k+1}) \leq f(\mathbf{u}_k), \\ \mathbf{u}_k, & \text{otherwise.} \end{cases}$$

**Output:**  $\mathbf{u}_{k+1}$ .

---

Our subsequent analysis is based on the inexact oracle [6] derived from the Hölder continuity condition of gradients, which is generalized to problem (1.1) and demonstrated in the following proposition.

**PROPOSITION 2.1.** *Suppose that Assumption 1.1 holds. Let  $\delta > 0$  and*

$$\rho \geq \max_{i \in [m]} \left\{ \left[ \frac{1 - \alpha_i}{(1 + \alpha_i)\delta} \right]^{(1-\alpha_i)/(1+\alpha_i)} L_i^{2/(1+\alpha_i)} \right\}.$$

*Then for all  $\mathbf{u}, \mathbf{v} \in \Omega$ , we have*

$$f(\mathbf{v}) \leq f(\mathbf{u}) + \langle \nabla f(\mathbf{u}), \mathbf{v} - \mathbf{u} \rangle + \frac{\rho}{2} \|\mathbf{v} - \mathbf{u}\|^2 + \frac{\delta}{2}.$$

*Proof.* Since  $\nabla f_i$  is Hölder continuous with an exponent  $\alpha_i$ , we can obtain that

$$f_i(\mathbf{v}) \leq f_i(\mathbf{u}) + \langle \nabla f_i(\mathbf{u}), \mathbf{v} - \mathbf{u} \rangle + \frac{L_i}{1 + \alpha_i} \|\mathbf{v} - \mathbf{u}\|^{1+\alpha_i},$$

for all  $\mathbf{u}, \mathbf{v} \in \Omega$ . Then, for each  $i$ , it follows from [8, Lemma 2] that

$$f_i(\mathbf{v}) \leq f_i(\mathbf{u}) + \langle \nabla f_i(\mathbf{u}), \mathbf{v} - \mathbf{u} \rangle + \frac{\rho}{2} \|\mathbf{v} - \mathbf{u}\|^2 + \frac{\delta}{2}.$$

Summing the above relationship over  $i \in [m]$ , we immediately arrive at the assertion of this proposition. The proof is completed.  $\square$

Now, we are in the position to derive the complexity bound of Algorithm 1 in the following theorem.

THEOREM 2.2. Let  $\varepsilon \in (0, 1)$  be a sufficiently small constant. Then after at most

$$O\left(\log\left(\frac{1}{\varepsilon}\right) \frac{1}{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}}\right)$$

iterations, Algorithm 1 will find an iterate  $\mathbf{u}_k \in \Omega$  satisfying  $\|\mathbf{u}_k - \mathbf{u}^*\| \leq \varepsilon$ .

*Proof.* In view of Proposition 2.1, we take

$$\rho = \frac{1}{\tau} = \frac{M}{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}} \geq \max_{i \in [m]} \left\{ \left[ \frac{2(1-\alpha_i)}{\mu(1+\alpha_i)\varepsilon^2} \right]^{(1-\alpha_i)/(1+\alpha_i)} L_i^{2/(1+\alpha_i)} \right\}.$$

Then it holds that

$$f(\mathbf{v}_{k+1}) \leq f(\mathbf{v}_k) + \langle \nabla f(\mathbf{v}_k), \mathbf{v}_{k+1} - \mathbf{v}_k \rangle + \frac{1}{2\tau} \|\mathbf{v}_{k+1} - \mathbf{v}_k\|^2 + \frac{\mu\varepsilon^2}{4},$$

which, after a suitable rearrangement, can be equivalently written as

$$(2.2) \quad \langle \nabla f(\mathbf{v}_k), \mathbf{v}_k - \mathbf{v}_{k+1} \rangle \leq f(\mathbf{v}_k) - f(\mathbf{v}_{k+1}) + \frac{\mu\varepsilon^2}{4} + \frac{1}{2\tau} \|\mathbf{v}_{k+1} - \mathbf{v}_k\|^2.$$

Recall that  $f^* = f(\mathbf{u}^*)$ . By virtue of the strong convexity of  $f$ , we can obtain that

$$(2.3) \quad \langle \nabla f(\mathbf{v}_k), \mathbf{u}^* - \mathbf{v}_k \rangle \leq f^* - f(\mathbf{v}_k) - \frac{\mu}{2} \|\mathbf{v}_k - \mathbf{u}^*\|^2.$$

The optimality condition of the projection problem defining  $\mathbf{v}_{k+1}$  yields that

$$\langle \mathbf{v}_{k+1} - \mathbf{v}_k + \tau \nabla f(\mathbf{v}_k), \mathbf{u} - \mathbf{v}_{k+1} \rangle \geq 0,$$

for all  $\mathbf{u} \in \Omega$ . Upon taking  $\mathbf{u} = \mathbf{u}^*$ , we have

$$\begin{aligned} \langle \mathbf{v}_{k+1} - \mathbf{v}_k, \mathbf{v}_{k+1} - \mathbf{u}^* \rangle &\leq \tau \langle \nabla f(\mathbf{v}_k), \mathbf{u}^* - \mathbf{v}_{k+1} \rangle \\ &= \tau \langle \nabla f(\mathbf{v}_k), \mathbf{u}^* - \mathbf{v}_k \rangle + \tau \langle \nabla f(\mathbf{v}_k), \mathbf{v}_k - \mathbf{v}_{k+1} \rangle, \end{aligned}$$

which together with (2.2) and (2.3) implies that

$$\begin{aligned} \langle \mathbf{v}_{k+1} - \mathbf{v}_k, \mathbf{v}_{k+1} - \mathbf{u}^* \rangle &\leq \tau \left( f^* - f(\mathbf{v}_{k+1}) + \frac{\mu\varepsilon^2}{4} \right) - \frac{\mu\tau}{2} \|\mathbf{v}_k - \mathbf{u}^*\|^2 \\ &\quad + \frac{1}{2} \|\mathbf{v}_{k+1} - \mathbf{v}_k\|^2. \end{aligned}$$

Moreover, it can be readily verified that

$$\begin{aligned} \|\mathbf{v}_{k+1} - \mathbf{u}^*\|^2 &= \|\mathbf{v}_{k+1} - \mathbf{v}_k + \mathbf{v}_k - \mathbf{u}^*\|^2 \\ &= \|\mathbf{v}_k - \mathbf{u}^*\|^2 + 2 \langle \mathbf{v}_{k+1} - \mathbf{v}_k, \mathbf{v}_k - \mathbf{u}^* \rangle + \|\mathbf{v}_{k+1} - \mathbf{v}_k\|^2 \\ &= \|\mathbf{v}_k - \mathbf{u}^*\|^2 + 2 \langle \mathbf{v}_{k+1} - \mathbf{v}_k, \mathbf{v}_{k+1} - \mathbf{u}^* \rangle - \|\mathbf{v}_{k+1} - \mathbf{v}_k\|^2. \end{aligned} \quad (2.4)$$

Collecting the above two relationships together, we arrive at

$$\|\mathbf{v}_{k+1} - \mathbf{u}^*\|^2 \leq (1 - \mu\tau) \|\mathbf{v}_k - \mathbf{u}^*\|^2 + 2\tau \left( f^* - f(\mathbf{v}_{k+1}) + \frac{\mu\varepsilon^2}{4} \right).$$

From the construction of  $\mathbf{u}_k$  in Algorithm 1, it then follows that  $f(\mathbf{v}_l) \geq f(\mathbf{u}_k)$  for all  $l \in \{1, 2, \dots, k\}$ . Let  $C_k = \sum_{l=1}^k (1 - \mu\tau)^{l-1}$  be a constant. Applying the above relationship recursively for  $k$  times leads to that

$$\begin{aligned} \|\mathbf{v}_k - \mathbf{u}^*\|^2 &\leq (1 - \mu\tau)^k \|\mathbf{u}_0 - \mathbf{u}^*\|^2 + 2\tau \sum_{l=1}^k (1 - \mu\tau)^{l-1} \left( f^* - f(\mathbf{v}_l) + \frac{\mu\varepsilon^2}{4} \right) \\ &\leq (1 - \mu\tau)^k \|\mathbf{u}_0 - \mathbf{u}^*\|^2 + 2\tau \left( f^* - f(\mathbf{u}_k) + \frac{\mu\varepsilon^2}{4} \right) C_k, \end{aligned}$$

which together with  $\|\mathbf{v}_k - \mathbf{u}^*\| \geq 0$  and  $C_k \geq 1$  implies that

$$f(\mathbf{u}_k) - f^* \leq \frac{(1 - \mu\tau)^k}{2\tau C_k} \|\mathbf{u}_0 - \mathbf{u}^*\|^2 + \frac{\mu\varepsilon^2}{4} \leq \frac{(1 - \mu\tau)^k}{2\tau} \|\mathbf{u}_0 - \mathbf{u}^*\|^2 + \frac{\mu\varepsilon^2}{4}.$$

According to the strong convexity of  $f$  and the optimality condition of problem (1.1), we have

$$(2.5) \quad f(\mathbf{u}_k) - f^* \geq \langle \nabla f(\mathbf{u}^*), \mathbf{u}_k - \mathbf{u}^* \rangle + \frac{\mu}{2} \|\mathbf{u}_k - \mathbf{u}^*\|^2 \geq \frac{\mu}{2} \|\mathbf{u}_k - \mathbf{u}^*\|^2.$$

Hence, it holds that

$$\begin{aligned} \|\mathbf{u}_k - \mathbf{u}^*\|^2 &\leq \frac{2}{\mu} (f(\mathbf{u}_k) - f^*) \leq \frac{(1 - \mu\tau)^k}{\mu\tau} \|\mathbf{u}_0 - \mathbf{u}^*\|^2 + \frac{\varepsilon^2}{2} \\ &\leq \frac{M \|\mathbf{u}_0 - \mathbf{u}^*\|^2}{\mu\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}} \left( 1 - \frac{\mu}{M} \varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})} \right)^k + \frac{\varepsilon^2}{2}. \end{aligned}$$

We denote by  $K_\varepsilon^*$  the smallest iteration number  $k$  such that  $\|\mathbf{u}_k - \mathbf{u}^*\| \leq \varepsilon$ . Then solving the inequality  $M \|\mathbf{u}_0 - \mathbf{u}^*\|^2 \varepsilon^{-2(1-\hat{\alpha})/(1+\hat{\alpha})} (1 - \mu\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}/M)^k / \mu \leq \varepsilon^2/2$  indicates that

$$\begin{aligned} K_\varepsilon^* &\leq \frac{4 \log((2M \|\mathbf{u}_0 - \mathbf{u}^*\|^2 / \mu)^{(1+\hat{\alpha})/4} / \varepsilon)}{-\log(1 - \mu\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}/M)(1 + \hat{\alpha})} \\ &\leq \frac{4M \log((2M \|\mathbf{u}_0 - \mathbf{u}^*\|^2 / \mu)^{(1+\hat{\alpha})/4} / \varepsilon)}{\mu(1 + \hat{\alpha})\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}}. \end{aligned}$$

The proof is completed.  $\square$

Theorem 2.2 demonstrates that the iteration complexity of Algorithm 1 with a fixed stepsize is  $O(\log(\varepsilon^{-1})\varepsilon^{2(\hat{\alpha}-1)/(1+\hat{\alpha})})$  for problem (1.1). This complexity result generalizes the classical linear convergence when  $\hat{\alpha} = 1$ , which highlights the performance degradation incurred by non-Lipschitz gradients.

**3. Universal Primal Gradient Method.** The fixed stepsize  $\tau$  chosen in Algorithm 1 depends on the parameters  $\alpha_i$  and  $L_i$  for all  $i \in [m]$ , which are often unknown and hard to estimate in practice. To address this issue, we adopt the universal primal gradient method (UPGM) proposed by Nesterov [8] to solve problem (1.1). This method incorporates a line-search procedure to adaptively determine the stepsize at each iteration, and its overall framework is outlined in Algorithm 2.

Next, we establish the iteration complexity of Algorithm 2, which remains on the same order as that of the projected gradient descent method with a fixed stepsize.

**Algorithm 2:** Universal Primal Gradient Method (UPGM).**Input:**  $\varepsilon > 0$ .Initialize  $\mathbf{u}_0 = \mathbf{v}_0 \in \Omega$  and  $\rho_0 > 0$ .**for**  $k = 0, 1, 2, \dots$  **do**    **for**  $j_k = 0, 1, 2, \dots$  **do**

Compute

$$\mathbf{v}_{k+1} = \Pi_{\Omega} \left( \mathbf{v}_k - \frac{1}{2^{j_k} \rho_k} \nabla f(\mathbf{v}_k) \right).$$

**If**  $\mathbf{v}_{k+1}$  satisfies the following line-search condition,

$$(3.1) \quad \begin{aligned} f(\mathbf{v}_{k+1}) &\leq f(\mathbf{v}_k) + \langle \nabla f(\mathbf{v}_k), \mathbf{v}_{k+1} - \mathbf{v}_k \rangle \\ &\quad + \frac{2^{j_k} \rho_k}{2} \|\mathbf{v}_{k+1} - \mathbf{v}_k\|^2 + \frac{\mu \varepsilon^2}{4}, \end{aligned}$$

**then break.**Update  $\rho_{k+1} = 2^{j_k} \rho_k$ .

Set

$$\mathbf{u}_{k+1} = \begin{cases} \mathbf{v}_{k+1}, & \text{if } f(\mathbf{v}_{k+1}) \leq f(\mathbf{u}_k), \\ \mathbf{u}_k, & \text{otherwise.} \end{cases}$$

**Output:**  $\mathbf{u}_{k+1}$ .

174      **THEOREM 3.1.** *Let  $\varepsilon \in (0, 1)$  be a sufficiently small constant. Then after at most*

$$175 \quad O \left( \log \left( \frac{1}{\varepsilon} \right) \frac{1}{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}} \right)$$

176 *iterations, Algorithm 2 will attain an iterate  $\mathbf{u}_k \in \Omega$  satisfying that  $\|\mathbf{u}_k - \mathbf{u}^*\| \leq \varepsilon$ .*

177 *Proof.* Obviously, there exists  $j_k \in \mathbb{N}$  such that

$$178 \quad 2^{j_k} \rho_k \geq \max_{i \in [m]} \left\{ \left[ \frac{2(1-\alpha_i)}{\mu(1+\alpha_i)\varepsilon^2} \right]^{(1-\alpha_i)/(1+\alpha_i)} L_i^{2/(1+\alpha_i)} \right\}.$$

179 By invoking the results of Proposition 2.1, we know that condition (3.1) is satisfied.

180 Hence, the line-search step in Algorithm 2 can be terminated after a finite number of

181 trials and the required number of trials  $j_k$  satisfies

$$182 \quad (3.2) \quad 2^{j_k} \rho_k \leq 2 \max_{i \in [m]} \left\{ \left[ \frac{2(1-\alpha_i)}{\mu(1+\alpha_i)\varepsilon^2} \right]^{(1-\alpha_i)/(1+\alpha_i)} L_i^{2/(1+\alpha_i)} \right\} \leq \frac{2M}{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}},$$

183 where  $M > 0$  is a constant defined in (2.1). Moreover, the line-search condition (3.1)

184 directly yields that

$$185 \quad (3.3) \quad \langle \nabla f(\mathbf{v}_k), \mathbf{v}_k - \mathbf{v}_{k+1} \rangle \leq f(\mathbf{v}_k) - f(\mathbf{v}_{k+1}) + \frac{2^{j_k} \rho_k}{2} \|\mathbf{v}_{k+1} - \mathbf{v}_k\|^2 + \frac{\mu \varepsilon^2}{4}.$$

186 According to the optimality condition of the projection problem defining  $\mathbf{v}_{k+1}$ , we  
 187 have

$$188 \quad \left\langle \mathbf{v}_{k+1} - \mathbf{v}_k + \frac{1}{2^{j_k} \rho_k} \nabla f(\mathbf{v}_k), \mathbf{u}^* - \mathbf{v}_{k+1} \right\rangle \geq 0,$$

189 which further implies that

$$\begin{aligned} 190 \quad \langle \mathbf{v}_{k+1} - \mathbf{v}_k, \mathbf{v}_{k+1} - \mathbf{u}^* \rangle &\leq \frac{1}{2^{j_k} \rho_k} \langle \nabla f(\mathbf{v}_k), \mathbf{u}^* - \mathbf{v}_{k+1} \rangle \\ &\leq \frac{1}{2^{j_k} \rho_k} \langle \nabla f(\mathbf{v}_k), \mathbf{u}^* - \mathbf{v}_k \rangle + \frac{1}{2^{j_k} \rho_k} \langle \nabla f(\mathbf{v}_k), \mathbf{v}_k - \mathbf{v}_{k+1} \rangle. \end{aligned}$$

191 Substituting (2.3) and (3.3) into the above relationship leads to that

$$\begin{aligned} 192 \quad \langle \mathbf{v}_{k+1} - \mathbf{v}_k, \mathbf{v}_{k+1} - \mathbf{u}^* \rangle &\leq \frac{1}{2^{j_k} \rho_k} \left( f^* - f(\mathbf{v}_{k+1}) + \frac{\mu \varepsilon^2}{4} \right) \\ &\quad + \frac{1}{2} \|\mathbf{v}_{k+1} - \mathbf{v}_k\|^2 - \frac{\mu}{2^{j_k+1} \rho_k} \|\mathbf{v}_k - \mathbf{u}^*\|^2, \end{aligned}$$

193 Thus, it follows from relationship (2.4) that

$$\begin{aligned} 194 \quad \|\mathbf{v}_{k+1} - \mathbf{u}^*\|^2 &\leq \left( 1 - \frac{\mu}{2^{j_k} \rho_k} \right) \|\mathbf{v}_k - \mathbf{u}^*\|^2 + \frac{2}{2^{j_k} \rho_k} \left( f^* - f(\mathbf{v}_{k+1}) + \frac{\mu \varepsilon^2}{4} \right) \\ &\leq \left( 1 - \frac{\mu \varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}}{2M} \right) \|\mathbf{v}_k - \mathbf{u}^*\|^2 + \frac{2}{\rho_0} \left( f^* - f(\mathbf{v}_{k+1}) + \frac{\mu \varepsilon^2}{4} \right), \end{aligned}$$

195 where the last inequality comes from (3.2) and  $2^{j_k} \rho_k \geq \rho_0$ . The remaining part of  
 196 the proof follows the same line of reasoning as that of Theorem 2.2 and is therefore  
 197 omitted here for the sake of brevity.  $\square$

198 We end this section by estimating the total number of line-search steps required  
 199 by Algorithm 2.

200 **COROLLARY 3.2.** *Let  $\varepsilon \in (0, 1)$  be a sufficiently small constant. Then Algorithm 2*  
 201 *requires at most*

$$202 \quad O \left( \log \left( \frac{1}{\varepsilon} \right) \frac{1}{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}} \right)$$

203 *line-search steps for the generated sequence  $\{\mathbf{u}_k\}$  to satisfy  $\|\mathbf{u}_k - \mathbf{u}^*\| \leq \varepsilon$ .*

204 *Proof.* Let  $N_k$  be the total number of line-search steps after  $k$  iterations in Algo-  
 205 rithm 2. From the update rule  $\rho_{k+1} = 2^{j_k} \rho_k$ , we can obtain that  $j_k = \log \rho_{k+1} - \log \rho_k$ .  
 206 Then a straightforward verification reveals that

$$207 \quad (3.4) \quad N_k = \sum_{l=0}^k (j_l + 1) = k + 1 + \log \rho_{k+1} - \log \rho_0,$$

208 which together with relationship (3.2) implies that

$$\begin{aligned} 209 \quad N_k &\leq k + 1 + \log \left( \frac{2M}{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}} \right) - \log \rho_0 \\ &\leq k + \frac{2(1-\hat{\alpha})}{1+\hat{\alpha}} \log \left( \frac{1}{\varepsilon} \right) + \log \left( \frac{2M}{\rho_0} \right) + 1. \end{aligned}$$



210 By invoking the results of Theorem 3.1, we conclude that Algorithm 2 requires at  
 211 most  $O(\log(\varepsilon^{-1})\varepsilon^{2(\hat{\alpha}-1)/(1+\hat{\alpha})})$  line-search steps, which completes the proof.  $\square$

212 At each iteration of Algorithm 2, we evaluate both the function value and the  
 213 gradient at  $\mathbf{v}_k$ . In addition, an extra function evaluation at  $\mathbf{v}_{k+1,j_k}$  is involved during  
 214 each line-search step. Therefore, Theorem 3.1 and Corollary 3.2 together reveal that  
 215 the total number of function and gradient evaluations required by Algorithm 2 is  
 216  $O(\log(\varepsilon^{-1})\varepsilon^{2(\hat{\alpha}-1)/(1+\hat{\alpha})})$ .

217 **4. Universal Fast Gradient Method.** To obtain a sharper complexity bound,  
 218 we devise in this section a universal fast gradient method (UFGM) tailored to prob-  
 219 lem (1.1). The proposed scheme, summarized in Algorithm 3, exhibits slight but  
 220 essential differences from the algorithm introduced by Nesterov [8] to exploit the  
 221 strong convexity of the objective function.

---

**Algorithm 3:** Universal Fast Gradient Method (UFGM).

---

**Input:**  $\varepsilon > 0$ .

Initialize  $\mathbf{u}_0 = \mathbf{w}_0 \in \Omega$ ,  $\rho_0 \geq \mu$ , and  $\sigma_0 = 1$ .

**for**  $k = 0, 1, 2, \dots$  **do**

**for**  $j_k = 0, 1, 2, \dots$  **do**

        Set  $\nu_k = \sqrt{\mu/(2^{j_k}\rho_k)}$  and  $\eta_k = \nu_k/(1 + \nu_k)$ .

        Compute

$$(4.1) \quad \mathbf{v}_k = (1 - \eta_k)\mathbf{u}_k + \eta_k \Pi_{\Omega}(\mathbf{w}_k),$$

        and

$$(4.2) \quad \mathbf{z}_k = \Pi_{\Omega} \left( \Pi_{\Omega}(\mathbf{w}_k) - \frac{\nu_k}{\mu} \nabla f(\mathbf{v}_k) \right).$$

        Set

$$(4.3) \quad \mathbf{u}_{k+1} = (1 - \eta_k)\mathbf{u}_k + \eta_k \mathbf{z}_k.$$

**If**  $\mathbf{u}_{k+1}$  satisfies the following line-search condition,

$$(4.4) \quad \begin{aligned} f(\mathbf{u}_{k+1}) &\leq f(\mathbf{v}_k) + \langle \nabla f(\mathbf{v}_k), \mathbf{u}_{k+1} - \mathbf{v}_k \rangle \\ &\quad + \frac{\mu}{2\nu_k^2} \|\mathbf{u}_{k+1} - \mathbf{v}_k\|^2 + \frac{\eta_k \mu \varepsilon^2}{4}, \end{aligned}$$

**then** break.

    Set  $\rho_{k+1} = 2^{j_k}\rho_k$  and update  $\mathbf{w}_{k+1}$  by

$$(4.5) \quad \mathbf{w}_{k+1} = (1 - \eta_k)\mathbf{w}_k + \eta_k \mathbf{v}_k - \frac{\eta_k}{\mu} \nabla f(\mathbf{v}_k).$$

**Output:**  $\mathbf{u}_{k+1}$ .

---

222 The following lemma illustrates that the line-search process in (4.4) is well-defined,  
 223 which is guaranteed to terminate in a finite number of trials.

224 **LEMMA 4.1.** *There exists an integer  $j_k \in \mathbb{N}$  such that the line-search condition*  
 225 *(4.4) is satisfied in Algorithm 3.*

*Proof.* It follows from the definition of  $\eta_k$  and  $\nu_k \leq 1$  that

$$\eta_k = \frac{\nu_k}{1 + \nu_k} \geq \frac{\nu_k}{2}, \quad \text{and} \quad \frac{\mu}{\nu_k^2} = 2^{j_k} \rho_k.$$

Recall that  $\hat{\alpha} = \min_{i \in [m]} \alpha_i \in (0, 1]$ . Then we have

$$\begin{aligned} \frac{\mu}{\nu_k^2} \eta_k^{(1-\hat{\alpha})/(1+\hat{\alpha})} &\geq \frac{2^{j_k} \rho_k}{2^{(1-\hat{\alpha})/(1+\hat{\alpha})}} \nu_k^{(1-\hat{\alpha})/(1+\hat{\alpha})} \\ &= \frac{2^{j_k} \rho_k}{2^{(1-\hat{\alpha})/(1+\hat{\alpha})}} \left[ \frac{\mu}{2^{j_k} \rho_k} \right]^{(1-\hat{\alpha})/(2(1+\hat{\alpha}))} \\ &= \frac{\mu^{(1-\hat{\alpha})/(2(1+\hat{\alpha}))}}{2^{(1-\hat{\alpha})/(1+\hat{\alpha})}} [2^{j_k} \rho_k]^{(1+3\hat{\alpha})/(2(1+\hat{\alpha}))}, \end{aligned}$$

where the first equality comes from the definition of  $\nu_k$ . Now it is clear that

$$\frac{\mu}{\nu_k^2} \eta_k^{(1-\hat{\alpha})/(1+\hat{\alpha})} \rightarrow \infty,$$

as  $j_k \rightarrow \infty$ . Thus, there exists  $j_k \in \mathbb{N}$  such that

$$(4.6) \quad \frac{\mu}{\nu_k^2} \eta_k^{(1-\hat{\alpha})/(1+\hat{\alpha})} \geq \max_{i \in [m]} \left\{ \left[ \frac{2(1-\alpha_i)}{\mu(1+\alpha_i)\varepsilon^2} \right]^{(1-\alpha_i)/(1+\alpha_i)} L_i^{2/(1+\alpha_i)} \right\},$$

which further implies that

$$\begin{aligned} \frac{\mu}{\nu_k^2} &\geq \frac{1}{\eta_k^{(1-\hat{\alpha})/(1+\hat{\alpha})}} \max_{i \in [m]} \left\{ \left[ \frac{2(1-\alpha_i)}{\mu(1+\alpha_i)\varepsilon^2} \right]^{(1-\alpha_i)/(1+\alpha_i)} L_i^{2/(1+\alpha_i)} \right\} \\ &\geq \max_{i \in [m]} \left\{ \left[ \frac{2(1-\alpha_i)}{\eta_k \mu(1+\alpha_i)\varepsilon^2} \right]^{(1-\alpha_i)/(1+\alpha_i)} L_i^{2/(1+\alpha_i)} \right\}. \end{aligned}$$

As a direct consequence of Proposition 2.1, we can proceed to show that the line-search condition (4.4) is satisfied, which completes the proof.  $\square$

LEMMA 4.2. Let  $\{\sigma_k\}$  be a sequence of positive constants defined recursively by

$$(4.7) \quad \sigma_{k+1} = (1 + \nu_k) \sigma_k,$$

with  $\sigma_0 = 1$ . And let  $\{\phi_k\}$  be a sequence of functions defined recursively by

$$(4.8) \quad \begin{aligned} \phi_{k+1}(\mathbf{u}) &= \phi_k(\mathbf{u}) - \nu_k \sigma_k f^* + \nu_k \sigma_k f(\mathbf{v}_k) + \nu_k \sigma_k \langle \nabla f(\mathbf{v}_k), \mathbf{u} - \mathbf{v}_k \rangle \\ &\quad + \frac{\nu_k \sigma_k \mu}{2} \|\mathbf{u} - \mathbf{v}_k\|^2, \end{aligned}$$

with  $\phi_0(\mathbf{u}) = c_0 + \sigma_0 \mu \|\mathbf{u} - \mathbf{w}_0\|^2 / 2$  for  $c_0 = f(\mathbf{u}_0) - f^* - \mu \varepsilon^2 / 4$  and  $\mathbf{w}_0 \in \Omega$ . Then, for all  $k \in \mathbb{N}$ , the function  $\phi_k$  preserves the following canonical form,

$$(4.9) \quad \phi_k(\mathbf{u}) = c_k + \frac{\sigma_k \mu}{2} \|\mathbf{u} - \mathbf{w}_k\|^2,$$

where  $\{c_k\}$  is a sequence of real numbers and  $\{\mathbf{w}_k\}$  is defined recursively by (4.5).

246 *Proof.* We first prove that  $\nabla^2\phi_k = \sigma_k\mu I$  for all  $k \in \mathbb{N}$  by induction. It is evident  
 247 that  $\nabla^2\phi_0 = \sigma_0\mu I$ . Now we assume that  $\nabla^2\phi_k = \sigma_k\mu I$  for some  $k$ . Then relationships  
 248 (4.7) and (4.8) imply that

$$249 \quad \nabla^2\phi_{k+1} = \nabla^2\phi_k + \nu_k\sigma_k\mu I = \sigma_k\mu I + \nu_k\sigma_k\mu I = \sigma_{k+1}\mu I.$$

250 Thus, we know that  $\nabla^2\phi_k = \sigma_k\mu I$  for all  $k \in \mathbb{N}$ , which, in turn, justifies the canonical  
 251 form of  $\phi_k$  in (4.9).

252 Next, by combining two relationships (4.8) and (4.9) together, we can obtain that

$$253 \quad \begin{aligned} \phi_{k+1}(\mathbf{u}) &= c_k + \frac{\sigma_k\mu}{2} \|\mathbf{u} - \mathbf{w}_k\|^2 - \nu_k\sigma_k f^* + \nu_k\sigma_k f(\mathbf{v}_k) \\ &\quad + \nu_k\sigma_k \langle \nabla f(\mathbf{v}_k), \mathbf{u} - \mathbf{v}_k \rangle + \frac{\nu_k\sigma_k\mu}{2} \|\mathbf{u} - \mathbf{v}_k\|^2. \end{aligned}$$

254 Since  $\mathbf{w}_{k+1}$  is a global minimizer of  $\phi_{k+1}$  over  $\mathbb{R}^n$ , the first-order optimality condition  
 255 yields that

$$256 \quad \begin{aligned} 0 &= \nabla\phi_{k+1}(\mathbf{w}_{k+1}) = \sigma_k\mu(\mathbf{w}_{k+1} - \mathbf{w}_k) + \nu_k\sigma_k\nabla f(\mathbf{v}_k) + \nu_k\sigma_k\mu(\mathbf{w}_{k+1} - \mathbf{v}_k) \\ &= (1 + \nu_k)\sigma_k\mu\mathbf{w}_{k+1} - \sigma_k\mu\mathbf{w}_k - \nu_k\sigma_k\mu\mathbf{v}_k + \nu_k\sigma_k\nabla f(\mathbf{v}_k), \end{aligned}$$

257 from which the closed-form expression of  $\mathbf{w}_{k+1}$  in (4.5) can be derived. The proof is  
 258 completed.  $\square$

259 **LEMMA 4.3.** *Let  $\sigma_k$  and  $\{\phi_k\}$  be the sequences defined in Lemma 4.2. Then we*  
 260 *have*

$$261 \quad (4.10) \quad \phi_k(\mathbf{u}) \leq \sigma_k(f(\mathbf{u}) - f^*) + \phi_0(\mathbf{u}),$$

262 *for all  $\mathbf{u} \in \Omega$  and  $k \in \mathbb{N}$ .*

263 *Proof.* We prove that  $\{\phi_k\}$  and  $\{\sigma_k\}$  satisfy relationship (4.10) by induction. It  
 264 is obvious that (4.10) holds for  $k = 0$  since  $f(\mathbf{u}) \geq f^*$  for any  $\mathbf{u} \in \Omega$ . Now we assume  
 265 that (4.10) holds for some  $k \in \mathbb{N}$ . It follows from the strong convexity of  $f$  that

$$266 \quad f(\mathbf{u}) \geq f(\mathbf{v}_k) + \langle \nabla f(\mathbf{v}_k), \mathbf{u} - \mathbf{v}_k \rangle + \frac{\mu}{2} \|\mathbf{u} - \mathbf{v}_k\|^2,$$

267 for all  $\mathbf{u} \in \Omega$ . Then substituting the above relationship into (4.8) leads to that

$$268 \quad \begin{aligned} \phi_{k+1}(\mathbf{u}) &\leq \phi_k(\mathbf{u}) - \nu_k\sigma_k f^* + \nu_k\sigma_k f(\mathbf{u}) \\ &\leq \sigma_k(f(\mathbf{u}) - f^*) + \phi_0(\mathbf{u}) + \nu_k\sigma_k(f(\mathbf{u}) - f^*) \\ &= \sigma_{k+1}(f(\mathbf{u}) - f^*) + \phi_0(\mathbf{u}), \end{aligned}$$

269 which indicates that (4.10) also holds for  $k + 1$ . We complete the proof.  $\square$

270 **LEMMA 4.4.** *Let  $\{\sigma_k\}$  and  $\{\phi_k\}$  be the sequences defined in Lemma 4.2. Then,*  
 271 *for the sequence  $\{\mathbf{u}_k\}$  generated by Algorithm 3, it holds that*

$$272 \quad (4.11) \quad \sigma_k \left( f(\mathbf{u}_k) - f^* - \frac{\mu\varepsilon^2}{4} \right) \leq \phi_k^* := \min_{\mathbf{u} \in \Omega} \phi_k(\mathbf{u}),$$

273 *for all  $k \in \mathbb{N}$ .*

274 *Proof.* We aim to prove the assertion of this lemma by induction. It is clear that  
 275 (4.11) holds for  $k = 0$  since  $\sigma_0 = 1$  and  $\phi_0^* = \phi_0(\mathbf{w}_0) = f(\mathbf{u}_0) - f^* - \mu\varepsilon^2/4$ . Now we  
 276 assume that (4.11) holds for some  $k \in \mathbb{N}$  and investigate the situation for  $k + 1$ .

From the canonical form (4.9), it follows that  $\phi_k$  is a strongly convex function and  $\Pi_\Omega(\mathbf{w}_k) = \arg \min_{\mathbf{u} \in \Omega} \phi_k(\mathbf{u})$ . By invoking the result of [9, Corollary 2.2.1], we have

$$\begin{aligned} \phi_k(\mathbf{u}) &\geq \phi_k^* + \frac{\sigma_k \mu}{2} \|\mathbf{u} - \Pi_\Omega(\mathbf{w}_k)\|^2 \\ &\geq \sigma_k \left( f(\mathbf{u}_k) - f^* - \frac{\mu \varepsilon^2}{4} \right) + \frac{\sigma_k \mu}{2} \|\mathbf{u} - \Pi_\Omega(\mathbf{w}_k)\|^2, \end{aligned}$$

for all  $\mathbf{u} \in \Omega$ . Then relationship (4.8) yields that

$$\begin{aligned} \phi_{k+1}(\mathbf{u}) &\geq \sigma_k \left( f(\mathbf{u}_k) - f^* - \frac{\mu \varepsilon^2}{4} \right) + \frac{\sigma_k \mu}{2} \|\mathbf{u} - \Pi_\Omega(\mathbf{w}_k)\|^2 - \nu_k \sigma_k f^* \\ &\quad + \nu_k \sigma_k f(\mathbf{v}_k) + \nu_k \sigma_k \langle \nabla f(\mathbf{v}_k), \mathbf{u} - \mathbf{v}_k \rangle + \frac{\nu_k \sigma_k \mu}{2} \|\mathbf{u} - \mathbf{v}_k\|^2 \\ &\geq \sigma_{k+1} (f(\mathbf{v}_k) - f^*) - \frac{\sigma_k \mu \varepsilon^2}{4} + \langle \nabla f(\mathbf{v}_k), \sigma_k \mathbf{u}_k - \sigma_{k+1} \mathbf{v}_k \rangle \\ &\quad + \nu_k \sigma_k \langle \nabla f(\mathbf{v}_k), \mathbf{u} \rangle + \frac{\sigma_k \mu}{2} \|\mathbf{u} - \Pi_\Omega(\mathbf{w}_k)\|^2 \\ &= \sigma_{k+1} (f(\mathbf{v}_k) - f^*) - \frac{\sigma_k \mu \varepsilon^2}{4} + \nu_k \sigma_k \langle \nabla f(\mathbf{v}_k), \mathbf{u} - \Pi_\Omega(\mathbf{w}_k) \rangle \\ &\quad + \frac{\sigma_k \mu}{2} \|\mathbf{u} - \Pi_\Omega(\mathbf{w}_k)\|^2, \end{aligned}$$

where the second inequality comes from the strong convexity of  $f$  and (4.7), and the last equality holds due to the definition of  $\mathbf{v}_k$  in (4.1). According to the definition of  $\mathbf{z}_k$  in (4.2), we can obtain that

$$\begin{aligned} &\nu_k \sigma_k \langle \nabla f(\mathbf{v}_k), \mathbf{u} - \Pi_\Omega(\mathbf{w}_k) \rangle + \frac{\sigma_k \mu}{2} \|\mathbf{u} - \Pi_\Omega(\mathbf{w}_k)\|^2 \\ &= \frac{\sigma_k \mu}{2} \left\| \mathbf{u} - \left( \Pi_\Omega(\mathbf{w}_k) - \frac{\nu_k}{\mu} \nabla f(\mathbf{v}_k) \right) \right\|^2 - \frac{\nu_k^2 \sigma_k}{2\mu} \|\nabla f(\mathbf{v}_k)\|^2 \\ &\geq \frac{\sigma_k \mu}{2} \left\| \mathbf{z}_k - \left( \Pi_\Omega(\mathbf{w}_k) - \frac{\nu_k}{\mu} \nabla f(\mathbf{v}_k) \right) \right\|^2 - \frac{\nu_k^2 \sigma_k}{2\mu} \|\nabla f(\mathbf{v}_k)\|^2 \\ &= \nu_k \sigma_k \langle \nabla f(\mathbf{v}_k), \mathbf{z}_k - \Pi_\Omega(\mathbf{w}_k) \rangle + \frac{\sigma_k \mu}{2} \|\mathbf{z}_k - \Pi_\Omega(\mathbf{w}_k)\|^2. \end{aligned}$$

As a result, it holds that

$$\begin{aligned} \phi_{k+1}(\mathbf{u}) &\geq \sigma_{k+1} (f(\mathbf{v}_k) - f^*) - \frac{\sigma_k \mu \varepsilon^2}{4} + \nu_k \sigma_k \langle \nabla f(\mathbf{v}_k), \mathbf{z}_k - \Pi_\Omega(\mathbf{w}_k) \rangle \\ &\quad + \frac{\sigma_k \mu}{2} \|\mathbf{z}_k - \Pi_\Omega(\mathbf{w}_k)\|^2, \end{aligned} \tag{4.12}$$

for all  $\mathbf{u} \in \Omega$ . From the definitions of  $\mathbf{v}_k$  and  $\mathbf{u}_{k+1}$  in (4.1) and (4.3), it can be derived that  $\mathbf{z}_k - \Pi_\Omega(\mathbf{w}_k) = (\mathbf{u}_{k+1} - \mathbf{v}_k)/\eta_k$ . Substituting this relationship into (4.12) and taking  $\mathbf{u} = \Pi_\Omega(\mathbf{w}_{k+1})$ , we arrive at

$$\frac{\phi_{k+1}^*}{\sigma_{k+1}} \geq f(\mathbf{v}_k) - f^* + \langle \nabla f(\mathbf{v}_k), \mathbf{u}_{k+1} - \mathbf{v}_k \rangle + \frac{\mu}{2\nu_k^2} \|\mathbf{u}_{k+1} - \mathbf{v}_k\|^2 - \frac{(1 - \eta_k)\mu \varepsilon^2}{4},$$

which together with the line-search condition (4.4) implies that

$$\frac{\phi_{k+1}^*}{\sigma_{k+1}} \geq f(\mathbf{u}_{k+1}) - f^* - \frac{\eta_k \mu \varepsilon^2}{4} - \frac{(1 - \eta_k)\mu \varepsilon^2}{4} = f(\mathbf{u}_{k+1}) - f^* - \frac{\mu \varepsilon^2}{4}.$$

Therefore, relationship (4.11) also holds for  $k + 1$ . The proof is completed.  $\square$

COROLLARY 4.5. *Let  $\{\sigma_k\}$  and  $\{\phi_k\}$  be the sequences defined in Lemma 4.2. Then the sequence  $\{\mathbf{u}_k\}$  generated by Algorithm 3 satisfies*

$$(4.13) \quad f(\mathbf{u}_k) - f^* \leq \frac{1}{\sigma_k} \phi_0(\mathbf{u}^*) + \frac{\mu \varepsilon^2}{4},$$

for any  $k \in \mathbb{N}$ .

*Proof.* Collecting two relationships (4.10) and (4.11) together, we can obtain that

$$\begin{aligned} \sigma_k \left( f(\mathbf{u}_k) - f^* - \frac{\mu \varepsilon^2}{4} \right) &\leq \min_{\mathbf{u} \in \Omega} \phi_k(\mathbf{u}) \leq \min_{\mathbf{u} \in \Omega} \{ \sigma_k (f(\mathbf{u}) - f^*) + \phi_0(\mathbf{u}) \} \\ &\leq \sigma_k (f(\mathbf{u}^*) - f^*) + \phi_0(\mathbf{u}^*) \\ &= \phi_0(\mathbf{u}^*), \end{aligned}$$

which completes the proof.  $\square$

We proceed to establish the iteration complexity of Algorithm 3, as articulated in the theorem below.

THEOREM 4.6. *Let  $\varepsilon \in (0, 1)$  be a sufficiently small constant. Then after at most*

$$O \left( \log \left( \frac{1}{\varepsilon} \right) \frac{1}{\varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})}} \right)$$

*iterations, Algorithm 3 will reach an iterate  $\mathbf{u}_k$  satisfying  $\|\mathbf{u}_k - \mathbf{u}^*\| \leq \varepsilon$ .*

*Proof.* In view of relationship (4.6), the number of line-search steps  $j_k$  in (4.4) satisfies

$$\frac{\mu}{\nu_k^2} \eta_k^{(1-\hat{\alpha})/(1+\hat{\alpha})} \leq 2 \max_{i \in [m]} \left\{ \left[ \frac{2(1-\alpha_i)}{\mu(1+\alpha_i)\varepsilon^2} \right]^{(1-\alpha_i)/(1+\alpha_i)} L_i^{2/(1+\alpha_i)} \right\} \leq \frac{2M}{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}},$$

where  $M > 0$  is a constant defined in (2.1). Since  $\eta_k = \nu_k/(1+\nu_k) \geq \nu_k/2$ , we arrive at

$$(4.14) \quad \frac{\nu_k^2}{\mu} \geq \frac{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}}{2M} \eta_k^{(1-\hat{\alpha})/(1+\hat{\alpha})} \geq \frac{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}}{2^{2/(1+\hat{\alpha})} M} \nu_k^{(1-\hat{\alpha})/(1+\hat{\alpha})}.$$

Let  $\omega > 0$  be a constant defined as

$$\omega = \frac{1}{2^{2/(1+3\hat{\alpha})}} \left[ \frac{\mu}{M} \right]^{(1+\hat{\alpha})/(1+3\hat{\alpha})}.$$

Then it follows from relationship (4.14) that

$$(4.15) \quad \nu_k \geq \omega \varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})},$$

which further infers that

$$\sigma_{k+1} = (1 + \nu_k) \sigma_k \geq \left( 1 + \omega \varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})} \right) \sigma_k.$$

Applying the above inequality for  $k$  times recursively yields that

$$\sigma_k \geq \left( 1 + \omega \varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})} \right)^k.$$

As a direct consequence of (2.5) and (4.13), we can show that

$$\begin{aligned} \|\mathbf{u}_k - \mathbf{u}^*\|^2 &\leq \frac{2}{\mu} (f(\mathbf{u}_k) - f^*) \leq \frac{2}{\mu} \left( \frac{1}{\sigma_k} \phi_0(\mathbf{u}^*) + \frac{\mu \varepsilon^2}{4} \right) \\ &\leq \chi \left( 1 + \omega \varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})} \right)^{-k} + \frac{\varepsilon^2}{2}, \end{aligned}$$

where  $\chi = 2(f(\mathbf{u}_0) - f^*)/\mu + \|\mathbf{u}_0 - \mathbf{u}^*\|^2 > 0$  is a constant. Let  $K_\varepsilon^*$  be the smallest iteration number  $k$  such that  $\|\mathbf{u}_k - \mathbf{u}^*\| \leq \varepsilon$ . By solving the inequality  $\chi(1 + \omega \varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})})^{-k} \leq \varepsilon^2/2$ , we have

$$K_\varepsilon^* \leq \log \left( \frac{\sqrt{2\chi}}{\varepsilon} \right) \frac{2}{\log(1 + \omega \varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})})} \leq \log \left( \frac{\sqrt{2\chi}}{\varepsilon} \right) \frac{4}{\omega \varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})}}.$$

The proof is completed.  $\square$

The complexity bound established in Theorem 4.6 is markedly lower than those presented in Theorems 2.2 and 3.1, thereby highlighting the acceleration effect attained by Algorithm 3. Finally, we demonstrate that the number of line-search steps required by Algorithm 3 is also  $O(\log(\varepsilon^{-1})\varepsilon^{2(\hat{\alpha}-1)/(1+3\hat{\alpha})})$ .

**COROLLARY 4.7.** *Let  $\varepsilon \in (0, 1)$  be a sufficiently small constant. Then, to achieve an iterate  $\mathbf{u}_k$  satisfying  $\|\mathbf{u}_k - \mathbf{u}^*\| \leq \varepsilon$ , Algorithm 3 requires at most*

$$O \left( \log \left( \frac{1}{\varepsilon} \right) \frac{1}{\varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})}} \right)$$

*line-search steps.*

*Proof.* It follows from relationship (4.14) that

$$\rho_{k+1} = 2^{j_k} \rho_k = \frac{\mu}{\nu_k^2} \leq \frac{2^{2/(1+\hat{\alpha})} M}{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}} \left[ \frac{1}{\nu_k} \right]^{(1-\hat{\alpha})/(1+\hat{\alpha})},$$

which together with (4.15) implies that

$$\rho_{k+1} \leq \frac{2^{2/(1+\hat{\alpha})} M}{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}} \left[ \frac{1}{\omega \varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})}} \right]^{(1-\hat{\alpha})/(1+\hat{\alpha})} = \frac{2^{2/(1+\hat{\alpha})} M}{\omega^{(1-\hat{\alpha})/(1+\hat{\alpha})} \varepsilon^{4(1-\hat{\alpha})/(1+3\hat{\alpha})}}.$$

Let  $N_k$  be the total number of line-search steps after  $k$  iterations in Algorithm 3. In view of (3.4), we have

$$\begin{aligned} N_k &\leq k + 1 + \log \left( \frac{2^{2/(1+\hat{\alpha})} M}{\omega^{(1-\hat{\alpha})/(1+\hat{\alpha})} \varepsilon^{4(1-\hat{\alpha})/(1+3\hat{\alpha})}} \right) - \log \rho_0 \\ &\leq k + \frac{4(1-\hat{\alpha})}{1+3\hat{\alpha}} \log \left( \frac{1}{\varepsilon} \right) + \log \left( \frac{2^{2/(1+\hat{\alpha})} M}{\omega^{(1-\hat{\alpha})/(1+\hat{\alpha})} \rho_0} \right) + 1. \end{aligned}$$

Consequently, Theorem 4.6 indicates that the total number of line-search steps in Algorithm 3 is at most  $O(\log(\varepsilon^{-1})\varepsilon^{2(\hat{\alpha}-1)/(1+3\hat{\alpha})})$ , which completes the proof.  $\square$

**5. Numerical Experiments.** Preliminary numerical results are presented in this section to provide additional insights into the performance guarantees of the gradient descent method (1.3). We aim to elucidate that the final error attained by

the gradient descent method (1.3) is influenced by both the stepsize  $\tau$  and the Hölder exponent  $p$ .

We generated the results using Julia [3] version 1.12 on an Apple Macintosh Mini with a M2 processor, 8 performance cores, and 32GB of memory.

We have placed the Julia codes for the results in the GitHub repository [https://github.com/ctkelley/Grad\\_Des\\_CKW.jl](https://github.com/ctkelley/Grad_Des_CKW.jl) with instructions for reproducing the figures.

**5.1. Two-dimensional PDE with a non-Lipschitz term.** Hölder continuous gradients arise naturally in partial differential equations (PDEs) involving non-Lipschitz nonlinearity [2, 12]. In this subsection, we introduce a numerical example from [2]. This problem is to solve the following two-dimensional PDE,

$$(5.1) \quad \mathcal{F}(u) = -\Delta u + \nu u_+^p = 0,$$

where  $p \in (0, 1)$ ,  $\nu > 0$  is a constant and  $u_+ = \max\{u, 0\}$ . It should be noted that  $\mathcal{F}$  is the gradient of the following energy functional,

$$\hat{f}(u) = \frac{1}{2} \|\nabla u\|^2 + \frac{\nu}{p+1} \int_D u_+^{p+1}(y) \, dy.$$

Discretizing (5.1) with the standard five point difference scheme [7] leads to the following nonlinear system,

$$(5.2) \quad \mathbf{F}(\mathbf{u}) = \mathbf{A}\mathbf{u} + \nu \mathbf{u}_+^{1/2} - \mathbf{b} = 0,$$

where  $\mathbf{A} \in \mathbb{R}^{n \times n}$  is the discretization of  $-\Delta$  with zero boundary conditions,  $\mathbf{b} \in \mathbb{R}^n$  encodes the boundary conditions, and  $\mathbf{u}_+^{1/2} = \max\{\mathbf{u}, 0\}^{1/2}$  is understood as a component-wise operation. Problem (5.2) is equivalent to optimization problem (1.1) with  $\Omega = \mathbb{R}^n$ , and

$$f(\mathbf{u}) = \frac{1}{2}(f_1(\mathbf{u}) + f_2(\mathbf{u})) \quad \text{with} \quad f_1(\mathbf{u}) = \mathbf{u}^\top \mathbf{A} \mathbf{u} - 2\mathbf{b}^\top \mathbf{u}, \quad f_2(\mathbf{u}) = \frac{\nu}{p+1} \mathbf{e}^\top \mathbf{u}_+^{1+p},$$

where  $\mathbf{e} \in \mathbb{R}^n$  is the vector of all ones.

It is clear that  $\nabla f_1$  is Lipschitz continuous with the Lipschitz constant  $L_1 = \|\mathbf{A}\|$ , and  $\nabla f_2$  is locally Hölder continuous with  $\alpha = 1/2$  and  $L_2 = \nu n^{1/4}$  from

$$\|\nabla f_2(\mathbf{u}) - \nabla f_2(\mathbf{v})\| = \nu \left\| \mathbf{u}_+^{1/2} - \mathbf{v}_+^{1/2} \right\| \leq \nu n^{1/4} \|\mathbf{u} - \mathbf{v}\|^{1/2},$$

for all  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ . The function  $f$  is  $\lambda(\mathbf{A})$ -strongly convex, where  $\lambda(\mathbf{A})$  is the smallest eigenvalue of the symmetric positive definite matrix  $\mathbf{A}$ .

We now modify the problem to enable direct computation of the errors in the iteration. To this end we follow Example 4.4 in [11] and take as the exact solution the function

$$u^*(x, y) = \left( \frac{3r-1}{2} \right)^2 \max(0, r-1/3)$$

where  $r = \sqrt{x^2 + y^2}$ , and let  $\mathbf{u}^*$  be  $u^*$  evaluated at the interior grid points. We enforce the boundary conditions

$$u(x, 1) = u^*(x, 1), u(x, 0) = u^*(x, 0), u(1, y) = u^*(1, y), u(0, y) = u^*(0, y)$$

for  $0 < x, y < 1$  and encode this into  $\mathbf{b}$  Letting  $\mathbf{c}^* = \mathbf{F}(\mathbf{u}^*)$  our modified equation is

$$(5.3) \quad \mathbf{F}(\mathbf{u}) - \mathbf{c}^* = 0.$$

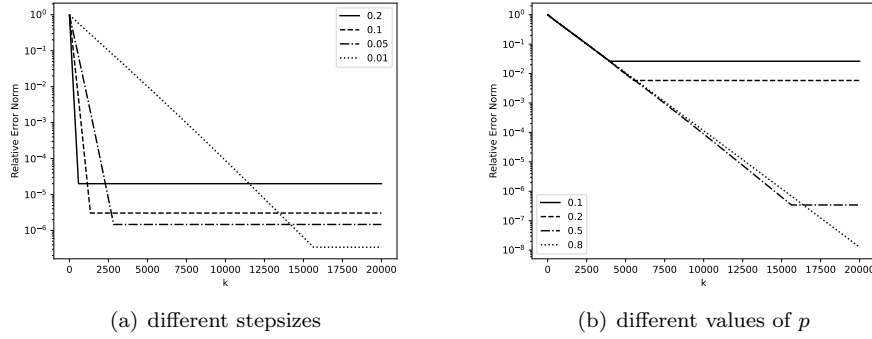


FIG. 1. Numerical performance of Algorithm 1 for problem (5.4).

Equation 5.3 is the necessary condition for the optimization problem

$$(5.4) \quad \min_{\mathbf{u} \in \mathbb{R}^n} f(\mathbf{u}) = \frac{1}{2} \mathbf{u}^\top \mathbf{A} \mathbf{u} + \frac{1}{1+p} \mathbf{e}^\top \mathbf{u}_+^{1+p} - (\mathbf{c}^*)^\top \mathbf{u}.$$

In the iteration we use the solution of  $\mathbf{A} \mathbf{u}_0 = -\mathbf{b}$  as the initial iterate. This is the discretization of Laplace's equation with the problem boundary conditions. In this way we ensure that the entire iteration satisfies the boundary conditions. We use a  $n \times n$  grid with  $n = 15$  for the examples in this section and examine the effects of grid refinement in a later section.

In the first experiment, we scrutinize the performance of the gradient descent method (1.3) under different stepsizes. Specifically, with the parameters  $p$  and  $\nu$  fixed at 0.5.

We test the algorithm is tested for stepsizes of the form  $\tau = \tau_0 h^2$ , where  $h = 1/(n+1)$  is the spatial meshwidth and  $\tau_0$  is taken from the set  $\{.2, .1, .05, .01\}$ .

The corresponding numerical results, presented in Figure 1(a), illustrate the decay of the distance between the iterates and the global minimizer over iterations. It can be observed that a larger stepsize facilitates a more rapid descent in the early stage of iterations, albeit at the expense of a greater asymptotic error. This phenomenon corroborates our theoretical predictions.

In the second experiment, we fix  $\tau_0$  is fixed at 0.01, while the parameter  $p$  is varied over the values  $\{0.2, 0.4, 0.6, 0.8\}$ . Figure 1(b) similarly tracks the decay of the distance to the global minimizer over iterations. It is evident that, as the value of  $p$  decreases, the final error attained by the algorithm increases under the same stepsize. Therefore, the associated optimization problems become increasingly ill-conditioned and thus more challenging to solve for smaller values of  $p$ . These findings offer empirical support for our theoretical analysis.

**5.2. Example 2.** We consider a second numerical example motivated by a semi-linear elliptic problem with a constraint on the solution in a certain set [12]. Let  $D = (0, 1)^3$  and

$$(5.5) \quad \mathcal{H}(u) = -\Delta u + \lambda |u|^\nu - |u|^{p-1} u$$

on  $D$  with the boundary condition  $u = 1$  on the boundary  $\partial D$ , where  $p > 1$ ,  $\nu \in (0, 1)$  and  $\lambda > p/\nu$  are constants. We consider the variational inequality that is to find



416  $u^* \in [-1, 1]$  such that for any  $u \in [-1, 1]$ ,

417 
$$\mathcal{H}(u^*)(u - u^*) \geq 0.$$

418 This problem is equivalent to the nonlinear equation

419 (5.6) 
$$0 = \mathcal{F}(u) := \begin{cases} \mathcal{H}(u) & \text{if } u - \mathcal{H}(u) \in [-1, 1], \\ u - 1 & \text{if } u - \mathcal{H}(u) \geq 1, \\ u + 1 & \text{otherwise.} \end{cases}$$

420 Discretizing (5.5) with the standard five point difference scheme [7], problem (5.6)  
421 leads to the following system of nonlinear equations

422 (5.7) 
$$\mathbf{F}(\mathbf{u}) = \mathbf{u} - \Pi_{\mathbf{U}}\left(\mathbf{u} - \tau(\mathbf{A}\mathbf{u} + \lambda|\mathbf{u}|^\nu - |\mathbf{u}|^{p-1}\mathbf{u} - \mathbf{b})\right) = 0,$$

423 where  $\mathbf{U} = [-1, 1]^n$ ,  $\tau > 0$  is a constant,  $\mathbf{A} \in \mathbb{R}^{n \times n}$  is a symmetric positive defi-  
424 nite matrix and  $\mathbf{b} \in \mathbb{R}^n$ . Note that (5.7) is the first-order optimal condition of the  
425 minimization problem

426 (5.8) 
$$\min_{\mathbf{u} \in [-1, 1]^n} f(\mathbf{u}) := \frac{1}{2}\mathbf{u}^\top \mathbf{A}\mathbf{u} + \frac{\lambda}{1+\nu}\mathbf{e}^\top |\mathbf{u}|^{\nu+1} - \frac{1}{1+p}\mathbf{e}^\top \max(\mathbf{u}, -\mathbf{u})^{p+1} + \mathbf{b}^\top \mathbf{u}.$$

The Hessian matrix of  $f$  at  $\mathbf{u}$  with  $\mathbf{u}_i \neq 0$ ,  $i = 1, \dots, n$  has the form

$$\nabla^2 f(\mathbf{u}) = \mathbf{A} + \lambda\nu|\mathbf{u}|^{\nu-1} - p\text{diag}\left(\max(-\mathbf{u}, \mathbf{u})^{p-1}\right),$$

427 Since  $\lambda\nu > p$ ,  $\nabla^2 f(\mathbf{u})$  is symmetric positive definite for any  $\mathbf{u} \in [-1, 1]^n$  with  $\mathbf{u}_i \neq 0$ ,  
428  $i = 1, \dots, n$ . Hence  $f$  is  $\mu$ -strongly convex in  $[-1, 1]^n$  with  $\mu = \lambda_{\min}(\mathbf{A})$  and the  
429 system (5.7) has a unique solution in  $[-1, 1]^n$ . However,  $\nabla f$  is not Lipschitz continuous  
430 in  $[-1, 1]^n$ .

Let

$$f_1(\mathbf{u}) = \frac{1}{2}\mathbf{u}^\top \mathbf{A}\mathbf{u} + \mathbf{b}^\top \mathbf{u}, f_2(\mathbf{u}) = \frac{\lambda}{1+\nu}\mathbf{e}^\top |\mathbf{u}|^{\nu+1}, f_3(\mathbf{u}) = -\frac{1}{1+p}\mathbf{e}^\top \max(\mathbf{u}, -\mathbf{u})^{p+1}$$

431 This example satisfies Assumption 1.1 (ii) with  $L_1 = \lambda_{\max}(\mathbf{A})$ ,  $L_2 = \lambda\nu$ ,  $L_3 =$   
432  $pn^{\frac{1}{2}}$ ,  $\alpha_1 = \alpha_3 = 1$ ,  $\alpha_2 = 1 - \nu$ .

433 **6. Conclusion.** In this paper, we consider a class of strongly convex constrained  
434 optimization problems of the form (1.1).

435 This example satisfies Assumption 1.1 (ii) with  $L_1 = \lambda_{\max}(\mathbf{A})$ ,  $L_2 = \lambda\nu$ ,  $L_3 =$   
436  $pn^{\frac{1}{2}}$ ,  $\alpha_1 = \alpha_3 = 1$ ,  $\alpha_2 = 1 - \nu$ . Example 1.1 shows that although each component  
437 function  $f_i$  of the objective function  $f$  admits a Hölder continuous gradient with an  
438 component  $\alpha_i \in (0, 1]$ , the gradient of  $f$  is not necessarily Hölder continuous. To  
439 establish the iteration complexity of the projected gradient descent methods for this  
440 class of problems, we use the parameter  $\hat{\alpha} = \min_{i \in [m]} \alpha_i$  to determine the complex-  
441 ity bound. Algorithm 1 is a new version of projected gradient method for problem  
442 (1.1) with an appropriately fixed stepsize. Theorem 2.2 shows that Algorithm 1 can  
443 find an iterate in the feasible set  $\Omega$  with a distance to the global minimizer less than  
444  $\varepsilon$  at most  $O(\log(\varepsilon^{-1})\varepsilon^{2(\hat{\alpha}-1)/(1+\hat{\alpha})})$  iterations. This recovers the classical complex-  
445 ity result when  $\hat{\alpha} = 1$  and reveals the additional difficulty imposed by the weaker  
446 smoothness of the objective function for  $\hat{\alpha} < 1$ . Algorithm 2 is a modification of

Algorithm 1 for problems where the parameters  $\alpha_i$  and  $L_i$  are difficult to estimate for the stepsize. In Algorithm 3, the stepsize is updated by the universal scheme at each iteration, which improves the complexity bound to  $O(\log(\varepsilon^{-1})\varepsilon^{2(\hat{\alpha}-1)/(1+3\hat{\alpha})})$ . Numerical experiments are conducted to validate our theoretical findings, demonstrating the expected behavior of projected gradient descent methods under different stepsizes and Hölder exponents. These results offer new insights into the performance guarantees of the classic projected gradient descent methods for a broader class of optimization problems with non-Lipschitz gradients.

## REFERENCES

- [1] J.-C. BARITAUX, K. HASSLER, AND M. UNSER, *An efficient numerical method for general  $L_p$  regularization in fluorescence molecular tomography*, IEEE Trans. Med. Imaging, 29 (2010), pp. 1075–1087.
- [2] J. W. BARRETT AND R. M. SHANAHAN, *Finite element approximation of a model reaction-diffusion problem with a non-lipschitz nonlinearity*, Numer. Math., 59 (1991), pp. 217–242.
- [3] J. BEZANSON, A. EDELMAN, S. KARPINSKI, AND V. B. SHAH, *Julia: A fresh approach to numerical computing*, SIAM Review, 59 (2017), pp. 65–98.
- [4] L. S. BORGES, F. S. V. BAZÁN, AND L. BEDIN, *A projection-based algorithm for  $\ell_2$ - $\ell_p$  Tikhonov regularization*, Math. Methods Appl. Sci., 41 (2018), pp. 5919–5938.
- [5] X. CHEN, C. T. KELLEY, AND L. WANG, *A new complexity result for strongly convex optimization with locally  $\alpha$ -hölder continuous gradients*, arXiv:2505.03506v1, (2025).
- [6] O. DEVOLDER, F. GLINEUR, AND Y. NESTEROV, *First-order methods of smooth convex optimization with inexact oracle*, Math. Program., 146 (2014), pp. 37–75.
- [7] R. J. LEVEQUE, *Finite Difference Methods for Ordinary and Partial Differential Equations: Steady-State and Time-Dependent Problems*, Society for Industrial and Applied Mathematics, 2007.
- [8] Y. NESTEROV, *Universal gradient methods for convex optimization problems*, Math. Program., 152 (2015), pp. 381–404.
- [9] Y. NESTEROV, *Lectures on Convex Optimization*, Springer, 2018.
- [10] Y. NESTEROV, *Universal complexity bounds for universal gradient methods in nonlinear optimization*, arXiv:2509.20902, (2025).
- [11] X. QU, W. BIAN, AND X. CHEN, *An extra gradient Anderson-accelerated algorithm for pseudomonotone variational inequalities*, 2024, <https://arxiv.org/abs/2408.06606>, <https://arxiv.org/abs/2408.06606>.
- [12] M. TANG, *Uniqueness of bound states to  $\Delta u - u + |u|^{p-1}u = 0$  in  $\mathbb{R}^n$ ,  $n \geq 3$* , Invent. Math., (2025), pp. 1–47.