

COMPLEXITY OF PROJECTED GRADIENT METHODS FOR STRONGLY CONVEX OPTIMIZATION WITH HÖLDER CONTINUOUS GRADIENT TERMS*

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Abstract. This paper studies the complexity of projected gradient descent methods for a class of strongly convex constrained optimization problems where the objective function is expressed as a summation of m component functions, each possessing a gradient that is Hölder continuous with an exponent $\alpha_i \in (0, 1]$. Under this formulation, the gradient of the objective function may fail to be globally Hölder continuous, thereby rendering existing complexity results inapplicable to this class of problems. Our theoretical analysis reveals that, in this setting, the complexity of projected gradient methods is determined by $\hat{\alpha} = \min_{i \in \{1, \dots, m\}} \alpha_i$. We first prove that, with an appropriately fixed stepsize, the complexity bound for finding an approximate minimizer with a distance to the true minimizer less than ε is $O(\log(\varepsilon^{-1})\varepsilon^{2(\hat{\alpha}-1)/(1+\hat{\alpha})})$, which extends the well-known complexity result for $\hat{\alpha} = 1$. Next we show that the complexity bound can be improved to $O(\log(\varepsilon^{-1})\varepsilon^{2(\hat{\alpha}-1)/(1+3\hat{\alpha})})$ if the stepsize is updated by the universal scheme. We illustrate our complexity results by numerical examples arising from elliptic equations with a non-Lipschitz term.

Key words. projected gradient descent, complexity, Hölder continuity

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1. Introduction. Given a closed and convex set $\Omega \subseteq \mathbb{R}^n$, this paper considers the following optimization problem,

$$21 \quad (1.1) \qquad \min_{\mathbf{u} \in \Omega} \ f(\mathbf{u}) := \frac{1}{m} \sum_{i=1}^m f_i(\mathbf{u}),$$

where the objective function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies the following blanket assumption.

ASSUMPTION 1.1.

24 (i) The function f is μ -strongly convex with a parameter $\mu > 0$ on Ω , that is,

$$f(\mathbf{u}) \geq f(\mathbf{v}) + \langle \nabla f(\mathbf{v}), \mathbf{u} - \mathbf{v} \rangle + \frac{\mu}{2} \|\mathbf{u} - \mathbf{v}\|^2,$$

26 for all $\mathbf{u}, \mathbf{v} \in \Omega$.

(ii) For each $i \in [m] := \{1, 2, \dots, m\}$, the function $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuously differentiable and the gradient ∇f_i is (globally) Hölder continuous with an exponent $\alpha_i \in (0, 1]$ on Ω , namely, there exists a constant $L_i > 0$ such that

$$30 \quad (1.2) \quad \|\nabla f_i(\mathbf{u}) - \nabla f_i(\mathbf{v})\| \leq L_i \|\mathbf{u} - \mathbf{v}\|^{\alpha_i},$$

31 for all $\mathbf{u}, \mathbf{v} \in \Omega$.

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32 Here, $\|\cdot\|$ is the ℓ_2 norm and $\langle \cdot, \cdot \rangle$ is the inner product on \mathbb{R}^n . We also denote by
 33 $\mathbf{u}^* \in \Omega$ and $f^* = f(\mathbf{u}^*)$ the global minimizer and the optimal value of problem (1.1),
 34 respectively.

35 Suppose that each ∇f_i is Lipschitz continuous, which corresponds to condition
 36 (1.2) with $\alpha_i = 1$ for all $\mathbf{u}, \mathbf{v} \in \Omega$. Then ∇f is also Lipschitz continuous and
 37 the associated Lipschitz constant is $L = \sum_{i=1}^m L_i/m$. Let $\Pi_\Omega(\cdot)$ be the projection
 38 operator onto the set Ω . It is well known that the classical projected gradient descent
 39 method

40 (1.3)
$$\mathbf{u}_{k+1} = \Pi_\Omega(\mathbf{u}_k - \tau \nabla f(\mathbf{u}_k)),$$

41 with any initial point $\mathbf{u}_0 \in \mathbb{R}^n$ and the stepsize $\tau \in (0, 2/(\mu + L)]$, achieves a linear
 42 rate of convergence [11, Theorem 2.2.14] as follows,

43
$$\|\mathbf{u}_k - \mathbf{u}^*\| \leq (1 - \mu\tau)^k \|\mathbf{u}_0 - \mathbf{u}^*\|.$$

44 Therefore, for a given $\varepsilon > 0$, method (1.3) is guaranteed to find a point $\mathbf{u}_k \in \Omega$
 45 satisfying $\|\mathbf{u}_k - \mathbf{u}^*\| \leq \varepsilon$ after at most $O(\log(\varepsilon^{-1}))$ iterations. Unfortunately, this
 46 analysis fails if there exists at least one index $i \in [m]$ such that $\alpha_i < 1$. We explain
 47 the failure of the convergence of method (1.3) to \mathbf{u}^* by the following example.

48 *Example 1.2.* [6, Example 1] Consider the following univariate optimization prob-
 49 lem on $\Omega = \mathbb{R}$,

50 (1.4)
$$\min_{x \in \mathbb{R}} f(x) = \frac{1}{2}x^2 + \frac{2}{3}|x|^{3/2},$$

51 which is a special instance of problem (1.1) with $f_1(x) = x^2/2$ and $f_2(x) = 2|x|^{3/2}/3$.
 52 It is easy to see that the global minimizer is $x^* = 0$. Method (1.3) with the fixed
 53 stepsize $\tau > 0$ starting from $x_0 \neq 0$ proceeds as follows,

54
$$x_{k+1} = x_k - \tau \nabla f(x_k) = (1 - \tau)x_k - \tau \text{sign}(x_k)|x_k|^{1/2},$$

55 where $\text{sign}(x) = 1$ if $x > 0$, 0 if $x = 0$, and -1 otherwise. A straightforward verification
 56 reveals that

57
$$|x_{k+1}|^2 - |x_k|^2 = -\tau(2 - \tau)|x_k|^2 - 2\tau(1 - \tau)|x_k|^{3/2} + \tau^2|x_k|.$$

58 It is evident that, when $|x_k|$ is sufficiently small, the last term in the right-hand side
 59 becomes dominant, resulting in that $|x_{k+1}|^2 - |x_k|^2 \geq 0$. Therefore, the distance to
 60 the global minimizer ceases to decrease once it achieves a certain level.

61 Moreover, in [6] we show that ∇f is locally, but not globally, Hölder continuous.
 62 In fact, from

63
$$|\nabla f(|h|) - \nabla f(0)| = |h| + |h|^{1/2} = \left(|h|^{1-\alpha} + |h|^{1/2-\alpha}\right)|h|^\alpha,$$

64 we can obtain that, $|h|^{1-\alpha} \rightarrow \infty$ when $\alpha \in (0, 1)$ and $|h| \rightarrow \infty$, while $|h|^{1/2-\alpha} \rightarrow \infty$
 65 when $\alpha = 1$ and $|h| \rightarrow 0$. Therefore, ∇f cannot be globally Hölder continuous for all
 66 $\alpha \in (0, 1]$.

67 On the other hand, problem (1.4) satisfies all the conditions in Assumption 1.1.
 68 It is clear that f is strongly convex. In addition, we have

69
$$|\nabla f_1(x) - \nabla f_1(y)| = |x - y|,$$

70 and

$$71 \quad |\nabla f_2(x) - \nabla f_2(y)| = \left| \text{sign}(x) |x|^{1/2} - \text{sign}(y) |y|^{1/2} \right| \leq \sqrt{2} |x - y|^{1/2},$$

72 for all $x, y \in \mathbb{R}$.

73 This simple example demonstrates that, in problem (1.1), a function f expressed
 74 as a sum of component functions f_i , each endowed with a Hölder continuous gradient,
 75 may itself fail to possess a Hölder continuous gradient. This phenomenon, initially
 76 observed in our previous work [6], was later revisited and further highlighted by
 77 Nesterov (see [12, Example 1]).

78 Since ∇f may not be globally Hölder continuous, most existing complexity results
 79 are inapplicable to problem (1.1). For the special case where $m = 1$, namely, ∇f is
 80 globally Hölder continuous with an exponent $\alpha \in (0, 1]$, Devolder et al. [7] presented
 81 the following bound for method (1.3),

$$82 \quad f(\hat{\mathbf{u}}_N) - f(\mathbf{u}^*) \leq K(N) := \frac{L_\alpha \|\mathbf{u}_0 - \mathbf{u}^*\|^{1+\alpha}}{1 + \alpha} \left(\frac{2}{N} \right)^{\frac{1+\alpha}{2}},$$

83 where L_α is the Hölder constant and $\hat{\mathbf{u}}_N = \sum_{k=1}^N \mathbf{u}_k / N$. In the strongly convex case,
 84 (51) in [7] comes to

$$85 \quad \|\hat{\mathbf{u}}_N - \mathbf{u}^*\|^2 \leq \frac{2}{\mu} K(N),$$

86 which implies that finding an N average of iterations $\hat{\mathbf{u}}_N$ satisfying $\|\hat{\mathbf{u}}_N - \mathbf{u}^*\| \leq \varepsilon$
 87 requires $O(\varepsilon^{-4/(1+\alpha)})$ iterations.

88 The contribution of this paper is to provide new complexity results of the pro-
 89 jected gradient descent methods for problem (1.1), which are dictated by the parame-
 90 ter $\hat{\alpha} = \min_{i \in [m]} \alpha_i \in (0, 1]$. We first show that, with an appropriately fixed stepsize,
 91 the complexity bound for finding an iterate with a distance to the global minimizer
 92 less than ε is $O(\log(\varepsilon^{-1})\varepsilon^{2(\hat{\alpha}-1)/(1+\hat{\alpha})})$, which extends the well-known complexity re-
 93 sult for $\hat{\alpha} = 1$. Next, we demonstrate that this complexity bound can be improved
 94 to $O(\log(\varepsilon^{-1})\varepsilon^{2(\hat{\alpha}-1)/(1+3\hat{\alpha})})$ if the stepsize is updated at each iteration using the
 95 universal scheme. Even in the special case where $m = 1$, our complexity bound is
 96 at least $O(\varepsilon^{-1})$ lower than (51) in [7]. For example, when $\hat{\alpha} = 1/2$, our bound is
 97 $O(\log(\varepsilon^{-1})\varepsilon^{-2/5})$ but (51) in [7] is $O(\varepsilon^{-8/3})$.

98 Our study is motivated by elliptic equations with a non-Lipschitz term [3, 14],
 99 complementarity problems [1, 13], and optimization problems with an ℓ_p -norm ($1 <$
 100 $p < 2$) regularization term [2, 5]. We illustrate our complexity results by two numerical
 101 examples arising from elliptic equations with a non-Lipschitz term in section 5, after
 102 we present complexity of projected gradient methods with fixed stepsizes and updated
 103 stepsizes in sections 2 to 4, respectively.

104 2. Vanilla Projected Gradient Descent Method with a Fixed Stepsize.

105 In this section, we attempt to employ the vanilla projected gradient descent method
 106 (1.3) with a fixed stepsize to solve problem (1.1), whose complexity bound is also
 107 provided. Example 1.2 illustrates that the projected gradient descent method (1.3)
 108 with a fixed stepsize will experience stagnation before reaching the global minimizer.

109 To obtain an approximate solution to problem (1.1), it is necessary to choose
 110 a sufficiently small stepsize τ in the projected gradient descent method (1.3), the

111 magnitude of which depends on the desired level of accuracy. Let $M > 0$ be a
 112 constant defined as

$$113 \quad (2.1) \quad M = \max_{i \in [m]} \left\{ \left[\frac{2(1 - \alpha_i)}{\mu(1 + \alpha_i)} \right]^{(1 - \alpha_i)/(1 + \alpha_i)} L_i^{2/(1 + \alpha_i)} \right\},$$

114 with the convention $0^0 = 1$. We select a specific stepsize $\tau = \varepsilon^{2(1 - \hat{\alpha})/(1 + \hat{\alpha})}/M$ in
 115 the projected gradient descent method, whose complete framework is presented in
 116 Algorithm 1. Two sequences $\{\mathbf{v}_k\}$ and $\{\mathbf{u}_k\}$ are maintained in Algorithm 1, where
 117 \mathbf{v}_k is generated by the projected gradient descent method and \mathbf{u}_k corresponds to the
 118 iterate achieving the smallest objective function value among the first k iterations.

Algorithm 1: Projected Gradient Descent Method (PGDM).

Input: $\varepsilon > 0$.

Initialize $\mathbf{u}_0 = \mathbf{v}_0 \in \Omega$.

Choose the stepsize $\tau = \varepsilon^{2(1 - \hat{\alpha})/(1 + \hat{\alpha})}/M$.

for $k = 0, 1, 2, \dots$ **do**

Compute

$$\mathbf{v}_{k+1} = \Pi_{\Omega} (\mathbf{v}_k - \tau \nabla f(\mathbf{v}_k)).$$

Set

$$\mathbf{u}_{k+1} = \begin{cases} \mathbf{v}_{k+1}, & \text{if } f(\mathbf{v}_{k+1}) \leq f(\mathbf{u}_k), \\ \mathbf{u}_k, & \text{otherwise.} \end{cases}$$

Output: \mathbf{u}_{k+1} .

119 Our subsequent analysis is based on the inexact oracle [7] derived from the Hölder
 120 continuity condition of gradients, which is generalized to problem (1.1) and demon-
 121 strated in the following proposition.

122 PROPOSITION 2.1. *Let $\delta > 0$ and*

$$123 \quad \rho \geq \max_{i \in [m]} \left\{ \left[\frac{1 - \alpha_i}{(1 + \alpha_i)\delta} \right]^{(1 - \alpha_i)/(1 + \alpha_i)} L_i^{2/(1 + \alpha_i)} \right\}.$$

124 Then for all $\mathbf{u}, \mathbf{v} \in \Omega$, we have

$$125 \quad f(\mathbf{v}) \leq f(\mathbf{u}) + \langle \nabla f(\mathbf{u}), \mathbf{v} - \mathbf{u} \rangle + \frac{\rho}{2} \|\mathbf{v} - \mathbf{u}\|^2 + \frac{\delta}{2}.$$

126 *Proof.* Since ∇f_i is Hölder continuous with an exponent α_i , we can obtain from
 127 [15, Lemma 1] that

$$128 \quad f_i(\mathbf{v}) \leq f_i(\mathbf{u}) + \langle \nabla f_i(\mathbf{u}), \mathbf{v} - \mathbf{u} \rangle + \frac{L_i}{1 + \alpha_i} \|\mathbf{v} - \mathbf{u}\|^{1 + \alpha_i},$$

129 for all $\mathbf{u}, \mathbf{v} \in \Omega$. Then, for each i , it follows from [10, Lemma 2] that

$$130 \quad f_i(\mathbf{v}) \leq f_i(\mathbf{u}) + \langle \nabla f_i(\mathbf{u}), \mathbf{v} - \mathbf{u} \rangle + \frac{\rho}{2} \|\mathbf{v} - \mathbf{u}\|^2 + \frac{\delta}{2}.$$

131 Summing the above relationship over $i \in [m]$, we immediately arrive at the assertion
 132 of this proposition. The proof is completed. \square

133 Now, we are able to derive the complexity bound of Algorithm 1 in the following
 134 theorem.

135 THEOREM 2.2. *Let $\varepsilon \in (0, 1)$ be a sufficiently small constant. Then after at most*

$$136 \quad O\left(\log\left(\frac{M^{(1+\hat{\alpha})/4}}{\varepsilon}\right) \frac{M}{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}}\right)$$

137 iterations, Algorithm 1 will find an iterate $\mathbf{u}_k \in \Omega$ satisfying $\|\mathbf{u}_k - \mathbf{u}^*\| \leq \varepsilon$.

138 *Proof.* In view of Proposition 2.1, we take

$$139 \quad \rho = \frac{1}{\tau} = \frac{M}{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}} \geq \max_{i \in [m]} \left\{ \left[\frac{2(1-\alpha_i)}{\mu(1+\alpha_i)\varepsilon^2} \right]^{(1-\alpha_i)/(1+\alpha_i)} L_i^{2/(1+\alpha_i)} \right\}.$$

140 Then it holds that

$$141 \quad f(\mathbf{v}_{k+1}) \leq f(\mathbf{v}_k) + \langle \nabla f(\mathbf{v}_k), \mathbf{v}_{k+1} - \mathbf{v}_k \rangle + \frac{1}{2\tau} \|\mathbf{v}_{k+1} - \mathbf{v}_k\|^2 + \frac{\mu\varepsilon^2}{4},$$

142 which, after a suitable rearrangement, can be equivalently written as

$$143 \quad (2.2) \quad \langle \nabla f(\mathbf{v}_k), \mathbf{v}_k - \mathbf{v}_{k+1} \rangle \leq f(\mathbf{v}_k) - f(\mathbf{v}_{k+1}) + \frac{\mu\varepsilon^2}{4} + \frac{1}{2\tau} \|\mathbf{v}_{k+1} - \mathbf{v}_k\|^2.$$

144 Recall that $f^* = f(\mathbf{u}^*)$. By virtue of the strong convexity of f , we can obtain that

$$145 \quad (2.3) \quad \langle \nabla f(\mathbf{v}_k), \mathbf{u}^* - \mathbf{v}_k \rangle \leq f^* - f(\mathbf{v}_k) - \frac{\mu}{2} \|\mathbf{v}_k - \mathbf{u}^*\|^2.$$

146 The optimality condition of the projection problem defining \mathbf{v}_{k+1} yields that

$$147 \quad \langle \mathbf{v}_{k+1} - \mathbf{v}_k + \tau \nabla f(\mathbf{v}_k), \mathbf{u} - \mathbf{v}_{k+1} \rangle \geq 0,$$

148 for all $\mathbf{u} \in \Omega$. Upon taking $\mathbf{u} = \mathbf{u}^*$, we have

$$149 \quad \begin{aligned} \langle \mathbf{v}_{k+1} - \mathbf{v}_k, \mathbf{v}_{k+1} - \mathbf{u}^* \rangle &\leq \tau \langle \nabla f(\mathbf{v}_k), \mathbf{u}^* - \mathbf{v}_{k+1} \rangle \\ &= \tau \langle \nabla f(\mathbf{v}_k), \mathbf{u}^* - \mathbf{v}_k \rangle + \tau \langle \nabla f(\mathbf{v}_k), \mathbf{v}_k - \mathbf{v}_{k+1} \rangle, \end{aligned}$$

150 which together with (2.2) and (2.3) implies that

$$151 \quad \begin{aligned} \langle \mathbf{v}_{k+1} - \mathbf{v}_k, \mathbf{v}_{k+1} - \mathbf{u}^* \rangle &\leq \tau \left(f^* - f(\mathbf{v}_{k+1}) + \frac{\mu\varepsilon^2}{4} \right) - \frac{\mu\tau}{2} \|\mathbf{v}_k - \mathbf{u}^*\|^2 \\ &\quad + \frac{1}{2} \|\mathbf{v}_{k+1} - \mathbf{v}_k\|^2. \end{aligned}$$

152 Moreover, it can be readily verified that

$$153 \quad (2.4) \quad \begin{aligned} \|\mathbf{v}_{k+1} - \mathbf{u}^*\|^2 &= \|\mathbf{v}_{k+1} - \mathbf{v}_k + \mathbf{v}_k - \mathbf{u}^*\|^2 \\ &= \|\mathbf{v}_k - \mathbf{u}^*\|^2 + 2 \langle \mathbf{v}_{k+1} - \mathbf{v}_k, \mathbf{v}_k - \mathbf{u}^* \rangle + \|\mathbf{v}_{k+1} - \mathbf{v}_k\|^2 \\ &= \|\mathbf{v}_k - \mathbf{u}^*\|^2 + 2 \langle \mathbf{v}_{k+1} - \mathbf{v}_k, \mathbf{v}_{k+1} - \mathbf{u}^* \rangle - \|\mathbf{v}_{k+1} - \mathbf{v}_k\|^2. \end{aligned}$$

154 Collecting the above two relationships together, we arrive at

$$155 \quad \|\mathbf{v}_{k+1} - \mathbf{u}^*\|^2 \leq (1 - \mu\tau) \|\mathbf{v}_k - \mathbf{u}^*\|^2 + 2\tau \left(f^* - f(\mathbf{v}_{k+1}) + \frac{\mu\varepsilon^2}{4} \right).$$

156 From the construction of \mathbf{u}_k in Algorithm 1, it then follows that $f(\mathbf{v}_l) \geq f(\mathbf{u}_k)$ for
157 all $l \in \{1, 2, \dots, k\}$. Let $C_k = \sum_{l=1}^k (1 - \mu\tau)^{l-1}$ be a constant. Applying the above
158 relationship recursively for k times leads to that

$$159 \quad \begin{aligned} \|\mathbf{v}_k - \mathbf{u}^*\|^2 &\leq (1 - \mu\tau)^k \|\mathbf{u}_0 - \mathbf{u}^*\|^2 + 2\tau \sum_{l=1}^k (1 - \mu\tau)^{l-1} \left(f^* - f(\mathbf{v}_l) + \frac{\mu\varepsilon^2}{4} \right) \\ &\leq (1 - \mu\tau)^k \|\mathbf{u}_0 - \mathbf{u}^*\|^2 + 2\tau \left(f^* - f(\mathbf{u}_k) + \frac{\mu\varepsilon^2}{4} \right) C_k, \end{aligned}$$

160 which together with $\|\mathbf{v}_k - \mathbf{u}^*\| \geq 0$ and $C_k \geq 1$ implies that

$$161 \quad f(\mathbf{u}_k) - f^* \leq \frac{(1 - \mu\tau)^k}{2\tau C_k} \|\mathbf{u}_0 - \mathbf{u}^*\|^2 + \frac{\mu\varepsilon^2}{4} \leq \frac{(1 - \mu\tau)^k}{2\tau} \|\mathbf{u}_0 - \mathbf{u}^*\|^2 + \frac{\mu\varepsilon^2}{4}.$$

162 According to the strong convexity of f and the optimality condition of problem (1.1),
163 we have

$$164 \quad (2.5) \quad f(\mathbf{u}_k) - f^* \geq \langle \nabla f(\mathbf{u}^*), \mathbf{u}_k - \mathbf{u}^* \rangle + \frac{\mu}{2} \|\mathbf{u}_k - \mathbf{u}^*\|^2 \geq \frac{\mu}{2} \|\mathbf{u}_k - \mathbf{u}^*\|^2.$$

165 Hence, it holds that

$$166 \quad \begin{aligned} \|\mathbf{u}_k - \mathbf{u}^*\|^2 &\leq \frac{2}{\mu} (f(\mathbf{u}_k) - f^*) \leq \frac{(1 - \mu\tau)^k}{\mu\tau} \|\mathbf{u}_0 - \mathbf{u}^*\|^2 + \frac{\varepsilon^2}{2} \\ &\leq \frac{M \|\mathbf{u}_0 - \mathbf{u}^*\|^2}{\mu\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}} \left(1 - \frac{\mu}{M} \varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})} \right)^k + \frac{\varepsilon^2}{2}. \end{aligned}$$

167 We denote by K_ε^* the smallest iteration number k such that $\|\mathbf{u}_k - \mathbf{u}^*\| \leq \varepsilon$. Then
168 solving the inequality $M \|\mathbf{u}_0 - \mathbf{u}^*\|^2 \varepsilon^{-2(1-\hat{\alpha})/(1+\hat{\alpha})} (1 - \mu\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}/M)^k / \mu \leq \varepsilon^2/2$
169 indicates that

$$170 \quad \begin{aligned} K_\varepsilon^* &\leq \frac{4 \log((2M \|\mathbf{u}_0 - \mathbf{u}^*\|^2 / \mu)^{(1+\hat{\alpha})/4} / \varepsilon)}{-\log(1 - \mu\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}/M)(1 + \hat{\alpha})} \\ &\leq \frac{4M \log((2M \|\mathbf{u}_0 - \mathbf{u}^*\|^2 / \mu)^{(1+\hat{\alpha})/4} / \varepsilon)}{\mu(1 + \hat{\alpha})\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}}. \end{aligned}$$

171 The proof is completed. \square

172 Theorem 2.2 demonstrates that the iteration complexity of Algorithm 1 with a
173 fixed stepsize is $O(\log(\varepsilon^{-1})\varepsilon^{2(\hat{\alpha}-1)/(1+\hat{\alpha})})$ for problem (1.1). This complexity result
174 generalizes the classical linear convergence when $\hat{\alpha} = 1$, which highlights the perfor-
175 mance degradation incurred by non-Lipschitz gradients.

176 **3. Universal Primal Gradient Method.** The fixed stepsize τ chosen in Algo-
177 rithm 1 depends on the parameters α_i and L_i for all $i \in [m]$, which are often unknown
178 and hard to estimate in practice. To address this issue, we adopt the universal primal
179 gradient method (UPGM) proposed by Nesterov [10] to solve problem (1.1). This

Algorithm 2: Universal Primal Gradient Method (UPGM).**Input:** $\varepsilon > 0$.Initialize $\mathbf{u}_0 = \mathbf{v}_0 \in \Omega$ and $\rho_0 > 0$.**for** $k = 0, 1, 2, \dots$ **do** **for** $j_k = 0, 1, 2, \dots$ **do**
 Compute

$$\mathbf{v}_{k+1} = \Pi_{\Omega} \left(\mathbf{v}_k - \frac{1}{2^{j_k} \rho_k} \nabla f(\mathbf{v}_k) \right).$$

If \mathbf{v}_{k+1} satisfies the following line-search condition,

$$(3.1) \quad \begin{aligned} f(\mathbf{v}_{k+1}) &\leq f(\mathbf{v}_k) + \langle \nabla f(\mathbf{v}_k), \mathbf{v}_{k+1} - \mathbf{v}_k \rangle \\ &+ \frac{2^{j_k} \rho_k}{2} \|\mathbf{v}_{k+1} - \mathbf{v}_k\|^2 + \frac{\mu \varepsilon^2}{4}, \end{aligned}$$

then break. Update $\rho_{k+1} = 2^{j_k} \rho_k$.

Set

$$\mathbf{u}_{k+1} = \begin{cases} \mathbf{v}_{k+1}, & \text{if } f(\mathbf{v}_{k+1}) \leq f(\mathbf{u}_k), \\ \mathbf{u}_k, & \text{otherwise.} \end{cases}$$

Output: \mathbf{u}_{k+1} .

180 method incorporates a line-search procedure to adaptively determine the stepsize at
 181 each iteration, and its overall framework is outlined in Algorithm 2.

182 Next, we establish the iteration complexity of Algorithm 2, which remains on the
 183 same order as that of the projected gradient descent method with a fixed stepsize.

184 **THEOREM 3.1.** *Let $\varepsilon \in (0, 1)$ be a sufficiently small constant. Then after at most*

$$185 \quad O \left(\log \left(\frac{M^{(1+\hat{\alpha})/4}}{\varepsilon} \right) \frac{M}{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}} \right)$$

186 iterations, Algorithm 2 will attain an iterate $\mathbf{u}_k \in \Omega$ satisfying that $\|\mathbf{u}_k - \mathbf{u}^*\| \leq \varepsilon$.

187 *Proof.* Obviously, there exists $j_k \in \mathbb{N}$ such that

$$188 \quad 2^{j_k} \rho_k \geq \max_{i \in [m]} \left\{ \left[\frac{2(1-\alpha_i)}{\mu(1+\alpha_i)\varepsilon^2} \right]^{(1-\alpha_i)/(1+\alpha_i)} L_i^{2/(1+\alpha_i)} \right\}.$$

189 By invoking the results of Proposition 2.1, we know that condition (3.1) is satisfied.
 190 Hence, the line-search step in Algorithm 2 can be terminated after a finite number of
 191 trials and the required number of trials j_k satisfies

$$192 \quad (3.2) \quad 2^{j_k} \rho_k \leq 2 \max_{i \in [m]} \left\{ \left[\frac{2(1-\alpha_i)}{\mu(1+\alpha_i)\varepsilon^2} \right]^{(1-\alpha_i)/(1+\alpha_i)} L_i^{2/(1+\alpha_i)} \right\} \leq \frac{2M}{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}},$$

193 where $M > 0$ is a constant defined in (2.1). Moreover, the line-search condition (3.1)

194 directly yields that

$$195 \quad (3.3) \quad \langle \nabla f(\mathbf{v}_k), \mathbf{v}_k - \mathbf{v}_{k+1} \rangle \leq f(\mathbf{v}_k) - f(\mathbf{v}_{k+1}) + \frac{2^{j_k} \rho_k}{2} \|\mathbf{v}_{k+1} - \mathbf{v}_k\|^2 + \frac{\mu \varepsilon^2}{4}.$$

196 According to the optimality condition of the projection problem defining \mathbf{v}_{k+1} , we
197 have

$$198 \quad \left\langle \mathbf{v}_{k+1} - \mathbf{v}_k + \frac{1}{2^{j_k} \rho_k} \nabla f(\mathbf{v}_k), \mathbf{u}^* - \mathbf{v}_{k+1} \right\rangle \geq 0,$$

199 which further implies that

$$200 \quad \begin{aligned} \langle \mathbf{v}_{k+1} - \mathbf{v}_k, \mathbf{v}_{k+1} - \mathbf{u}^* \rangle &\leq \frac{1}{2^{j_k} \rho_k} \langle \nabla f(\mathbf{v}_k), \mathbf{u}^* - \mathbf{v}_{k+1} \rangle \\ &\leq \frac{1}{2^{j_k} \rho_k} \langle \nabla f(\mathbf{v}_k), \mathbf{u}^* - \mathbf{v}_k \rangle + \frac{1}{2^{j_k} \rho_k} \langle \nabla f(\mathbf{v}_k), \mathbf{v}_k - \mathbf{v}_{k+1} \rangle. \end{aligned}$$

201 Substituting (2.3) and (3.3) into the above relationship leads to that

$$202 \quad \begin{aligned} \langle \mathbf{v}_{k+1} - \mathbf{v}_k, \mathbf{v}_{k+1} - \mathbf{u}^* \rangle &\leq \frac{1}{2^{j_k} \rho_k} \left(f^* - f(\mathbf{v}_{k+1}) + \frac{\mu \varepsilon^2}{4} \right) \\ &\quad + \frac{1}{2} \|\mathbf{v}_{k+1} - \mathbf{v}_k\|^2 - \frac{\mu}{2^{j_k+1} \rho_k} \|\mathbf{v}_k - \mathbf{u}^*\|^2, \end{aligned}$$

203 Thus, it follows from relationship (2.4) that

$$204 \quad \begin{aligned} \|\mathbf{v}_{k+1} - \mathbf{u}^*\|^2 &\leq \left(1 - \frac{\mu}{2^{j_k} \rho_k} \right) \|\mathbf{v}_k - \mathbf{u}^*\|^2 + \frac{2}{2^{j_k} \rho_k} \left(f^* - f(\mathbf{v}_{k+1}) + \frac{\mu \varepsilon^2}{4} \right) \\ &\leq \left(1 - \frac{\mu \varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}}{2M} \right) \|\mathbf{v}_k - \mathbf{u}^*\|^2 + \frac{2}{\rho_0} \left(f^* - f(\mathbf{v}_{k+1}) + \frac{\mu \varepsilon^2}{4} \right), \end{aligned}$$

205 where the last inequality comes from (3.2) and $2^{j_k} \rho_k \geq \rho_0$. The remaining part of
206 the proof follows the same line of reasoning as that of Theorem 2.2 and is therefore
207 omitted here for the sake of brevity. \square

208 We end this section by estimating the total number of line-search steps required
209 by Algorithm 2.

210 COROLLARY 3.2. *Let $\varepsilon \in (0, 1)$ be a sufficiently small constant. Then Algorithm 2
211 requires at most*

$$212 \quad O \left(\log \left(\frac{M^{(1+\hat{\alpha})/4}}{\varepsilon} \right) \frac{M}{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}} \right)$$

213 line-search steps for the generated sequence $\{\mathbf{u}_k\}$ to satisfy $\|\mathbf{u}_k - \mathbf{u}^*\| \leq \varepsilon$.

214 *Proof.* Let N_k be the total number of line-search steps after k iterations in Algo-
215 rithm 2. From the update rule $\rho_{k+1} = 2^{j_k} \rho_k$, we can obtain that $j_k = \log \rho_{k+1} - \log \rho_k$.
216 Then a straightforward verification reveals that

$$217 \quad (3.4) \quad N_k = \sum_{l=0}^k (j_l + 1) = k + 1 + \log \rho_{k+1} - \log \rho_0,$$

218 which together with relationship (3.2) implies that

$$\begin{aligned} N_k &\leq k + \log\left(\frac{2M}{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}}\right) - \log\rho_0 \\ 219 \quad &\leq k + \frac{2(1-\hat{\alpha})}{1+\hat{\alpha}} \log\left(\frac{1}{\varepsilon}\right) + \log\left(\frac{2M}{\rho_0}\right) + 1. \end{aligned}$$

220 By invoking the results of Theorem 3.1, we conclude that Algorithm 2 requires at
221 most $O(\log(\varepsilon^{-1})\varepsilon^{2(\hat{\alpha}-1)/(1+\hat{\alpha})})$ line-search steps, which completes the proof. \square

222 At each iteration of Algorithm 2, we evaluate both the function value and the
223 gradient at \mathbf{v}_k . In addition, an extra function evaluation at \mathbf{v}_{k+1,j_k} is involved during
224 each line-search step. Therefore, Theorem 3.1 and Corollary 3.2 together reveal that
225 the total number of function and gradient evaluations required by Algorithm 2 is
226 $O(\log(\varepsilon^{-1})\varepsilon^{2(\hat{\alpha}-1)/(1+\hat{\alpha})})$.

227 **4. Universal Fast Gradient Method.** To obtain a sharper complexity bound,
228 we devise, in this section, a universal fast gradient method (UFGM) tailored to prob-
229 lem (1.1). The proposed scheme, summarized in Algorithm 3, exhibits slight but
230 essential differences from the algorithm introduced by Nesterov [10] to exploit the
231 strong convexity of the objective function.

232 The following lemma illustrates that the line-search process in (4.4) is well-defined,
233 which is guaranteed to terminate in a finite number of trials.

234 **LEMMA 4.1.** *There exists an integer $j_k \in \mathbb{N}$ such that the line-search condition
235 (4.4) is satisfied in Algorithm 3.*

236 *Proof.* It follows from the definition of η_k and $\nu_k \leq 1$ that

$$237 \quad \eta_k = \frac{\nu_k}{1+\nu_k} \geq \frac{\nu_k}{2}, \quad \text{and} \quad \frac{\mu}{\nu_k^2} = 2^{j_k} \rho_k.$$

238 Recall that $\hat{\alpha} = \min_{i \in [m]} \alpha_i \in (0, 1]$. Then we have

$$\begin{aligned} 239 \quad \frac{\mu}{\nu_k^2} \eta_k^{(1-\hat{\alpha})/(1+\hat{\alpha})} &\geq \frac{2^{j_k} \rho_k}{2^{(1-\hat{\alpha})/(1+\hat{\alpha})}} \nu_k^{(1-\hat{\alpha})/(1+\hat{\alpha})} \\ &= \frac{2^{j_k} \rho_k}{2^{(1-\hat{\alpha})/(1+\hat{\alpha})}} \left[\frac{\mu}{2^{j_k} \rho_k} \right]^{(1-\hat{\alpha})/(2(1+\hat{\alpha}))} \\ &= \frac{\mu^{(1-\hat{\alpha})/(2(1+\hat{\alpha}))}}{2^{(1-\hat{\alpha})/(1+\hat{\alpha})}} [2^{j_k} \rho_k]^{(1+3\hat{\alpha})/(2(1+\hat{\alpha}))}, \end{aligned}$$

240 where the first equality comes from the definition of ν_k . Now it is clear that

$$241 \quad \frac{\mu}{\nu_k^2} \eta_k^{(1-\hat{\alpha})/(1+\hat{\alpha})} \rightarrow \infty,$$

242 as $j_k \rightarrow \infty$. Thus, there exists $j_k \in \mathbb{N}$ such that

$$243 \quad (4.6) \quad \frac{\mu}{\nu_k^2} \eta_k^{(1-\hat{\alpha})/(1+\hat{\alpha})} \geq \max_{i \in [m]} \left\{ \left[\frac{2(1-\alpha_i)}{\mu(1+\alpha_i)\varepsilon^2} \right]^{(1-\alpha_i)/(1+\alpha_i)} L_i^{2/(1+\alpha_i)} \right\},$$

Algorithm 3: Universal Fast Gradient Method (UFGM).**Input:** $\varepsilon > 0$.Initialize $\mathbf{u}_0 = \mathbf{w}_0 \in \Omega$ and $\rho_0 \geq \mu$.**for** $k = 0, 1, 2, \dots$ **do** **for** $j_k = 0, 1, 2, \dots$ **do** Set $\nu_k = \sqrt{\mu/(2^{j_k} \rho_k)}$ and $\eta_k = \nu_k/(1 + \nu_k)$.

Compute

(4.1)
$$\mathbf{v}_k = (1 - \eta_k)\mathbf{u}_k + \eta_k \Pi_{\Omega}(\mathbf{w}_k),$$

and

(4.2)
$$\mathbf{z}_k = \Pi_{\Omega} \left(\Pi_{\Omega}(\mathbf{w}_k) - \frac{\nu_k}{\mu} \nabla f(\mathbf{v}_k) \right).$$

Set

(4.3)
$$\mathbf{u}_{k+1} = (1 - \eta_k)\mathbf{u}_k + \eta_k \mathbf{z}_k.$$

If \mathbf{u}_{k+1} satisfies the following line-search condition,

(4.4)
$$\begin{aligned} f(\mathbf{u}_{k+1}) &\leq f(\mathbf{v}_k) + \langle \nabla f(\mathbf{v}_k), \mathbf{u}_{k+1} - \mathbf{v}_k \rangle \\ &+ \frac{\mu}{2\nu_k^2} \|\mathbf{u}_{k+1} - \mathbf{v}_k\|^2 + \frac{\eta_k \mu \varepsilon^2}{4}, \end{aligned}$$

then break.Set $\rho_{k+1} = 2^{j_k} \rho_k$ and update \mathbf{w}_{k+1} by

(4.5)
$$\mathbf{w}_{k+1} = (1 - \eta_k)\mathbf{w}_k + \eta_k \mathbf{v}_k - \frac{\eta_k}{\mu} \nabla f(\mathbf{v}_k).$$

Output: \mathbf{u}_{k+1} .

244 which further implies that

$$\begin{aligned} \frac{\mu}{\nu_k^2} &\geq \frac{1}{\eta_k^{(1-\hat{\alpha})/(1+\hat{\alpha})}} \max_{i \in [m]} \left\{ \left[\frac{2(1-\alpha_i)}{\mu(1+\alpha_i)\varepsilon^2} \right]^{(1-\alpha_i)/(1+\alpha_i)} L_i^{2/(1+\alpha_i)} \right\} \\ 245 \quad &\geq \max_{i \in [m]} \left\{ \left[\frac{2(1-\alpha_i)}{\eta_k \mu(1+\alpha_i)\varepsilon^2} \right]^{(1-\alpha_i)/(1+\alpha_i)} L_i^{2/(1+\alpha_i)} \right\}. \end{aligned}$$

246 As a direct consequence of Proposition 2.1, we can proceed to show that the line-search
247 condition (4.4) is satisfied, which completes the proof. \square 248 *Remark 4.2.* When the parameters of problem (1.1) are fully specified, Algo-
249 rithm 3 may alternatively be implemented with a fixed stepsize. Recall that $M > 0$
250 is a constant defined in (2.1). By invoking the result of Lemma 4.1, we can fix

251
$$\nu_k = 2 \left[\frac{\mu}{4M} \right]^{(1+\hat{\alpha})/(1+3\hat{\alpha})} \varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})},$$

252 and dispense with the parameter ρ_k and the line-search procedure in (4.4). Under

253 this choice, Algorithm 3 continues to enjoy the same iteration complexity established
 254 later.

255 We now introduce the estimating sequences associated with Algorithm 3, which
 256 play a crucial role in our subsequent analysis.

257 LEMMA 4.3. *Let $\{\sigma_k\}$ be a sequence of positive constants defined recursively by*

258 (4.7)
$$\sigma_{k+1} = (1 + \nu_k)\sigma_k,$$

259 with $\sigma_0 = 1$. And let $\{\phi_k\}$ be a sequence of functions defined recursively by

260 (4.8)
$$\begin{aligned} \phi_{k+1}(\mathbf{u}) &= \phi_k(\mathbf{u}) - \nu_k\sigma_k f^* + \nu_k\sigma_k f(\mathbf{v}_k) + \nu_k\sigma_k \langle \nabla f(\mathbf{v}_k), \mathbf{u} - \mathbf{v}_k \rangle \\ &\quad + \frac{\nu_k\sigma_k\mu}{2} \|\mathbf{u} - \mathbf{v}_k\|^2, \end{aligned}$$

261 with $\phi_0(\mathbf{u}) = c_0 + \sigma_0\mu \|\mathbf{u} - \mathbf{w}_0\|^2 / 2$ for $c_0 = f(\mathbf{u}_0) - f^* - \mu\varepsilon^2/4$ and $\mathbf{w}_0 \in \Omega$. Then,
 262 for all $k \in \mathbb{N}$, the function ϕ_k preserves the following canonical form,

263 (4.9)
$$\phi_k(\mathbf{u}) = c_k + \frac{\sigma_k\mu}{2} \|\mathbf{u} - \mathbf{w}_k\|^2,$$

264 where $\{c_k\}$ is a sequence of real numbers and $\{\mathbf{w}_k\}$ is defined recursively by (4.5).

265 Proof. We first prove that $\nabla^2\phi_k = \sigma_k\mu I$ for all $k \in \mathbb{N}$ by induction. It is evident
 266 that $\nabla^2\phi_0 = \sigma_0\mu I$. Now we assume that $\nabla^2\phi_k = \sigma_k\mu I$ for some k . Then relationships
 267 (4.7) and (4.8) imply that

268
$$\nabla^2\phi_{k+1} = \nabla^2\phi_k + \nu_k\sigma_k\mu I = \sigma_k\mu I + \nu_k\sigma_k\mu I = \sigma_{k+1}\mu I.$$

269 Thus, we know that $\nabla^2\phi_k = \sigma_k\mu I$ for all $k \in \mathbb{N}$, which, in turn, justifies the canonical
 270 form of ϕ_k in (4.9).

271 Next, by combining two relationships (4.8) and (4.9) together, we can obtain that

272
$$\begin{aligned} \phi_{k+1}(\mathbf{u}) &= c_k + \frac{\sigma_k\mu}{2} \|\mathbf{u} - \mathbf{w}_k\|^2 - \nu_k\sigma_k f^* + \nu_k\sigma_k f(\mathbf{v}_k) \\ &\quad + \nu_k\sigma_k \langle \nabla f(\mathbf{v}_k), \mathbf{u} - \mathbf{v}_k \rangle + \frac{\nu_k\sigma_k\mu}{2} \|\mathbf{u} - \mathbf{v}_k\|^2. \end{aligned}$$

273 Since \mathbf{w}_{k+1} is a global minimizer of ϕ_{k+1} over \mathbb{R}^n , the first-order optimality condition
 274 yields that

275
$$\begin{aligned} 0 &= \nabla\phi_{k+1}(\mathbf{w}_{k+1}) = \sigma_k\mu(\mathbf{w}_{k+1} - \mathbf{w}_k) + \nu_k\sigma_k\nabla f(\mathbf{v}_k) + \nu_k\sigma_k\mu(\mathbf{w}_{k+1} - \mathbf{v}_k) \\ &= (1 + \nu_k)\sigma_k\mu\mathbf{w}_{k+1} - \sigma_k\mu\mathbf{w}_k - \nu_k\sigma_k\mu\mathbf{v}_k + \nu_k\sigma_k\nabla f(\mathbf{v}_k), \end{aligned}$$

276 from which the closed-form expression of \mathbf{w}_{k+1} in (4.5) can be derived. The proof is
 277 completed. \square

278 The following lemma characterizes the relationship between the objective function
 279 of problem (1.1) and the estimating sequences.

280 LEMMA 4.4. *Let σ_k and $\{\phi_k\}$ be the sequences defined in Lemma 4.3. Then we
 281 have*

282 (4.10)
$$\phi_k(\mathbf{u}) \leq \sigma_k(f(\mathbf{u}) - f^*) + \phi_0(\mathbf{u}),$$

283 for all $\mathbf{u} \in \Omega$ and $k \in \mathbb{N}$.

284 *Proof.* We prove that $\{\phi_k\}$ and $\{\sigma_k\}$ satisfy relationship (4.10) by induction. It
 285 is obvious that (4.10) holds for $k = 0$ since $f(\mathbf{u}) \geq f^*$ for any $\mathbf{u} \in \Omega$. Now we assume
 286 that (4.10) holds for some $k \in \mathbb{N}$. It follows from the strong convexity of f that

$$287 \quad f(\mathbf{u}) \geq f(\mathbf{v}_k) + \langle \nabla f(\mathbf{v}_k), \mathbf{u} - \mathbf{v}_k \rangle + \frac{\mu}{2} \|\mathbf{u} - \mathbf{v}_k\|^2,$$

288 for all $\mathbf{u} \in \Omega$. Then substituting the above relationship into (4.8) leads to that

$$\begin{aligned} 289 \quad \phi_{k+1}(\mathbf{u}) &\leq \phi_k(\mathbf{u}) - \nu_k \sigma_k f^* + \nu_k \sigma_k f(\mathbf{u}) \\ &\leq \sigma_k(f(\mathbf{u}) - f^*) + \phi_0(\mathbf{u}) + \nu_k \sigma_k(f(\mathbf{u}) - f^*) \\ &= \sigma_{k+1}(f(\mathbf{u}) - f^*) + \phi_0(\mathbf{u}), \end{aligned}$$

290 which indicates that (4.10) also holds for $k + 1$. We complete the proof. \square

291 Next, we proceed to show that the function value error of Algorithm 3 is controlled
 292 by the estimating sequences.

293 PROPOSITION 4.5. *Let $\{\sigma_k\}$ and $\{\phi_k\}$ be the sequences defined in Lemma 4.3.
 294 Then the sequence $\{\mathbf{u}_k\}$ generated by Algorithm 3 satisfies*

$$295 \quad (4.11) \quad f(\mathbf{u}_k) - f^* \leq \frac{1}{\sigma_k} \phi_0(\mathbf{u}^*) + \frac{\mu \varepsilon^2}{4},$$

296 for all $k \in \mathbb{N}$.

297 *Proof.* Let $\phi_k^* := \min_{\mathbf{u} \in \Omega} \phi_k(\mathbf{u})$. We first prove by induction that

$$298 \quad (4.12) \quad \sigma_k \left(f(\mathbf{u}_k) - f^* - \frac{\mu \varepsilon^2}{4} \right) \leq \phi_k^*,$$

299 for any $k \in \mathbb{N}$. It is clear that (4.12) holds for $k = 0$ since $\sigma_0 = 1$ and $\phi_0^* = \phi_0(\mathbf{w}_0) =$
 300 $f(\mathbf{u}_0) - f^* - \mu \varepsilon^2 / 4$. Now we assume that (4.12) holds for some $k \in \mathbb{N}$ and investigate
 301 the situation for $k + 1$.

302 From the canonical form (4.9), it follows that ϕ_k is a strongly convex function
 303 and $\Pi_\Omega(\mathbf{w}_k) = \arg \min_{\mathbf{u} \in \Omega} \phi_k(\mathbf{u})$. By invoking the result of [11, Corollary 2.2.1], we
 304 have

$$\begin{aligned} 305 \quad \phi_k(\mathbf{u}) &\geq \phi_k^* + \frac{\sigma_k \mu}{2} \|\mathbf{u} - \Pi_\Omega(\mathbf{w}_k)\|^2 \\ &\geq \sigma_k \left(f(\mathbf{u}_k) - f^* - \frac{\mu \varepsilon^2}{4} \right) + \frac{\sigma_k \mu}{2} \|\mathbf{u} - \Pi_\Omega(\mathbf{w}_k)\|^2, \end{aligned}$$

306 for all $\mathbf{u} \in \Omega$. Then relationship (4.8) yields that

$$\begin{aligned} 307 \quad \phi_{k+1}(\mathbf{u}) &\geq \sigma_k \left(f(\mathbf{u}_k) - f^* - \frac{\mu \varepsilon^2}{4} \right) + \frac{\sigma_k \mu}{2} \|\mathbf{u} - \Pi_\Omega(\mathbf{w}_k)\|^2 - \nu_k \sigma_k f^* \\ &\quad + \nu_k \sigma_k f(\mathbf{v}_k) + \nu_k \sigma_k \langle \nabla f(\mathbf{v}_k), \mathbf{u} - \mathbf{v}_k \rangle + \frac{\nu_k \sigma_k \mu}{2} \|\mathbf{u} - \mathbf{v}_k\|^2 \\ &\geq \sigma_{k+1} (f(\mathbf{v}_k) - f^*) - \frac{\sigma_k \mu \varepsilon^2}{4} + \langle \nabla f(\mathbf{v}_k), \sigma_k \mathbf{u}_k - \sigma_{k+1} \mathbf{v}_k \rangle \\ &\quad + \nu_k \sigma_k \langle \nabla f(\mathbf{v}_k), \mathbf{u} \rangle + \frac{\sigma_k \mu}{2} \|\mathbf{u} - \Pi_\Omega(\mathbf{w}_k)\|^2 \\ &= \sigma_{k+1} (f(\mathbf{v}_k) - f^*) - \frac{\sigma_k \mu \varepsilon^2}{4} + \nu_k \sigma_k \langle \nabla f(\mathbf{v}_k), \mathbf{u} - \Pi_\Omega(\mathbf{w}_k) \rangle \\ &\quad + \frac{\sigma_k \mu}{2} \|\mathbf{u} - \Pi_\Omega(\mathbf{w}_k)\|^2, \end{aligned}$$

308 where the second inequality comes from the strong convexity of f and (4.7), and the
 309 last equality holds due to the definition of \mathbf{v}_k in (4.1). According to the definition of
 310 \mathbf{z}_k in (4.2), we can obtain that

$$\begin{aligned} & \nu_k \sigma_k \langle \nabla f(\mathbf{v}_k), \mathbf{u} - \Pi_\Omega(\mathbf{w}_k) \rangle + \frac{\sigma_k \mu}{2} \|\mathbf{u} - \Pi_\Omega(\mathbf{w}_k)\|^2 \\ &= \frac{\sigma_k \mu}{2} \left\| \mathbf{u} - \left(\Pi_\Omega(\mathbf{w}_k) - \frac{\nu_k}{\mu} \nabla f(\mathbf{v}_k) \right) \right\|^2 - \frac{\nu_k^2 \sigma_k}{2\mu} \|\nabla f(\mathbf{v}_k)\|^2 \\ &\geq \frac{\sigma_k \mu}{2} \left\| \mathbf{z}_k - \left(\Pi_\Omega(\mathbf{w}_k) - \frac{\nu_k}{\mu} \nabla f(\mathbf{v}_k) \right) \right\|^2 - \frac{\nu_k^2 \sigma_k}{2\mu} \|\nabla f(\mathbf{v}_k)\|^2 \\ &= \nu_k \sigma_k \langle \nabla f(\mathbf{v}_k), \mathbf{z}_k - \Pi_\Omega(\mathbf{w}_k) \rangle + \frac{\sigma_k \mu}{2} \|\mathbf{z}_k - \Pi_\Omega(\mathbf{w}_k)\|^2. \end{aligned}$$

312 As a result, it holds that

$$\begin{aligned} & \phi_{k+1}(\mathbf{u}) \geq \sigma_{k+1} (f(\mathbf{v}_k) - f^*) - \frac{\sigma_k \mu \varepsilon^2}{4} + \nu_k \sigma_k \langle \nabla f(\mathbf{v}_k), \mathbf{z}_k - \Pi_\Omega(\mathbf{w}_k) \rangle \\ & \quad + \frac{\sigma_k \mu}{2} \|\mathbf{z}_k - \Pi_\Omega(\mathbf{w}_k)\|^2, \end{aligned} \tag{4.13}$$

314 for all $\mathbf{u} \in \Omega$. From the definitions of \mathbf{v}_k and \mathbf{u}_{k+1} in (4.1) and (4.3), it can be derived
 315 that $\mathbf{z}_k - \Pi_\Omega(\mathbf{w}_k) = (\mathbf{u}_{k+1} - \mathbf{v}_k)/\eta_k$. Substituting this relationship into (4.13) and
 316 taking $\mathbf{u} = \Pi_\Omega(\mathbf{w}_{k+1})$, we arrive at

$$\frac{\phi_{k+1}^*}{\sigma_{k+1}} \geq f(\mathbf{v}_k) - f^* + \langle \nabla f(\mathbf{v}_k), \mathbf{u}_{k+1} - \mathbf{v}_k \rangle + \frac{\mu}{2\nu_k^2} \|\mathbf{u}_{k+1} - \mathbf{v}_k\|^2 - \frac{(1 - \eta_k)\mu \varepsilon^2}{4},$$

318 which together with the line-search condition (4.4) implies that

$$\frac{\phi_{k+1}^*}{\sigma_{k+1}} \geq f(\mathbf{u}_{k+1}) - f^* - \frac{\eta_k \mu \varepsilon^2}{4} - \frac{(1 - \eta_k)\mu \varepsilon^2}{4} = f(\mathbf{u}_{k+1}) - f^* - \frac{\mu \varepsilon^2}{4}.$$

320 Therefore, relationship (4.12) also holds for $k + 1$.

321 Finally, by collecting two relationships (4.10) and (4.12) together, we can obtain
 322 that

$$\begin{aligned} & \sigma_k \left(f(\mathbf{u}_k) - f^* - \frac{\mu \varepsilon^2}{4} \right) \leq \min_{\mathbf{u} \in \Omega} \phi_k(\mathbf{u}) \leq \min_{\mathbf{u} \in \Omega} \{ \sigma_k(f(\mathbf{u}) - f^*) + \phi_0(\mathbf{u}) \} \\ & \leq \sigma_k(f(\mathbf{u}^*) - f^*) + \phi_0(\mathbf{u}^*) \\ & = \phi_0(\mathbf{u}^*), \end{aligned}$$

324 which completes the proof. \square

325 With the above preparatory results in place, we are now in a position to establish
 326 the iteration complexity of Algorithm 3, as articulated in the theorem below.

327 **THEOREM 4.6.** *Let $\varepsilon \in (0, 1)$ be a sufficiently small constant. Then after at most*

$$O \left(\log \left(\frac{1}{\varepsilon} \right) \frac{M^{(1+\hat{\alpha})/(1+3\hat{\alpha})}}{\varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})}} \right)$$

329 iterations, Algorithm 3 will reach an iterate \mathbf{u}_k satisfying $\|\mathbf{u}_k - \mathbf{u}^*\| \leq \varepsilon$.

330 *Proof.* In view of relationship (4.6), the number of line-search steps j_k in (4.4)
 331 satisfies

$$332 \quad \frac{\mu}{\nu_k^2} \eta_k^{(1-\hat{\alpha})/(1+\hat{\alpha})} \leq 2 \max_{i \in [m]} \left\{ \left[\frac{2(1-\alpha_i)}{\mu(1+\alpha_i)\varepsilon^2} \right]^{(1-\alpha_i)/(1+\alpha_i)} L_i^{2/(1+\alpha_i)} \right\} \leq \frac{2M}{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}},$$

333 where $M > 0$ is a constant defined in (2.1). Since $\eta_k = \nu_k/(1+\nu_k) \geq \nu_k/2$, we arrive
 334 at

$$335 \quad (4.14) \quad \frac{\nu_k^2}{\mu} \geq \frac{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}}{2M} \eta_k^{(1-\hat{\alpha})/(1+\hat{\alpha})} \geq \frac{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}}{2^{2/(1+\hat{\alpha})} M} \nu_k^{(1-\hat{\alpha})/(1+\hat{\alpha})}.$$

336 Let $\omega > 0$ be a constant defined as

$$337 \quad \omega = \frac{1}{2^{2/(1+3\hat{\alpha})}} \left[\frac{\mu}{M} \right]^{(1+\hat{\alpha})/(1+3\hat{\alpha})}.$$

338 Then it follows from relationship (4.14) that

$$339 \quad (4.15) \quad \nu_k \geq \omega \varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})},$$

340 which further infers that

$$341 \quad \sigma_{k+1} = (1 + \nu_k) \sigma_k \geq \left(1 + \omega \varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})} \right) \sigma_k.$$

342 Applying the above inequality for k times recursively yields that

$$343 \quad \sigma_k \geq \left(1 + \omega \varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})} \right)^k.$$

344 As a direct consequence of (2.5) and (4.11), we can show that

$$345 \quad \begin{aligned} \|\mathbf{u}_k - \mathbf{u}^*\|^2 &\leq \frac{2}{\mu} (f(\mathbf{u}_k) - f^*) \leq \frac{2}{\mu} \left(\frac{1}{\sigma_k} \phi_0(\mathbf{u}^*) + \frac{\mu \varepsilon^2}{4} \right) \\ &\leq \chi \left(1 + \omega \varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})} \right)^{-k} + \frac{\varepsilon^2}{2}, \end{aligned}$$

346 where $\chi = 2(f(\mathbf{u}_0) - f^*)/\mu + \|\mathbf{u}_0 - \mathbf{u}^*\|^2 > 0$ is a constant. Let K_ε^* be the small-
 347 est iteration number k such that $\|\mathbf{u}_k - \mathbf{u}^*\| \leq \varepsilon$. By solving the inequality $\chi(1 + \omega \varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})})^{-k} \leq \varepsilon^2/2$, we have

$$349 \quad K_\varepsilon^* \leq \log \left(\frac{\sqrt{2\chi}}{\varepsilon} \right) \frac{2}{\log(1 + \omega \varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})})} \leq \log \left(\frac{\sqrt{2\chi}}{\varepsilon} \right) \frac{4}{\omega \varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})}}.$$

350 The proof is completed. \square

351 The complexity bound established in Theorem 4.6 is markedly lower than those
 352 presented in Theorems 2.2 and 3.1, thereby highlighting the acceleration effect at-
 353 tained by Algorithm 3. Finally, we demonstrate that the number of line-search steps
 354 required by Algorithm 3 is also $O(\log(\varepsilon^{-1}) \varepsilon^{2(\hat{\alpha}-1)/(1+3\hat{\alpha})})$.

355 COROLLARY 4.7. *Let $\varepsilon \in (0, 1)$ be a sufficiently small constant. Then, to achieve
 356 an iterate \mathbf{u}_k satisfying $\|\mathbf{u}_k - \mathbf{u}^*\| \leq \varepsilon$, Algorithm 3 requires at most*

$$357 \quad O \left(\log \left(\frac{1}{\varepsilon} \right) \frac{M^{(1+\hat{\alpha})/(1+3\hat{\alpha})}}{\varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})}} \right)$$

358 line-search steps.

359 *Proof.* It follows from relationship (4.14) that

$$360 \quad \rho_{k+1} = 2^{j_k} \rho_k = \frac{\mu}{\nu_k^2} \leq \frac{2^{2/(1+\hat{\alpha})} M}{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}} \left[\frac{1}{\nu_k} \right]^{(1-\hat{\alpha})/(1+\hat{\alpha})},$$

361 which together with (4.15) implies that

$$362 \quad \rho_{k+1} \leq \frac{2^{2/(1+\hat{\alpha})} M}{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}} \left[\frac{1}{\omega \varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})}} \right]^{(1-\hat{\alpha})/(1+\hat{\alpha})} = \frac{2^{2/(1+\hat{\alpha})} M}{\omega^{(1-\hat{\alpha})/(1+\hat{\alpha})} \varepsilon^{4(1-\hat{\alpha})/(1+3\hat{\alpha})}}.$$

363 Let N_k be the total number of line-search steps after k iterations in Algorithm 3. In
364 view of (3.4), we have

$$365 \quad \begin{aligned} N_k &\leq k + 1 + \log \left(\frac{2^{2/(1+\hat{\alpha})} M}{\omega^{(1-\hat{\alpha})/(1+\hat{\alpha})} \varepsilon^{4(1-\hat{\alpha})/(1+3\hat{\alpha})}} \right) - \log \rho_0 \\ &\leq k + \frac{4(1-\hat{\alpha})}{1+3\hat{\alpha}} \log \left(\frac{1}{\varepsilon} \right) + \log \left(\frac{2^{2/(1+\hat{\alpha})} M}{\omega^{(1-\hat{\alpha})/(1+\hat{\alpha})} \rho_0} \right) + 1. \end{aligned}$$

366 Consequently, Theorem 4.6 indicates that the total number of line-search steps in
367 Algorithm 3 is at most $O(\log(\varepsilon^{-1})\varepsilon^{2(\hat{\alpha}-1)/(1+3\hat{\alpha})})$, which completes the proof. \square

368 *Remark 4.8.* By an analogous argument, we can also prove that Algorithm 3
369 requires at most $O(\log(\varepsilon^{-1})\varepsilon^{(\hat{\alpha}-1)/(1+3\hat{\alpha})})$ iterations to generate an iterate \mathbf{u}_k such
370 that $f(\mathbf{u}_k) - f^* \leq \varepsilon$ for problem (1.1). Very recently, Doikov [8] has shown that,
371 in the case $m = 2$, where f_1 is a convex function with a Hölder continuous gradient
372 and $f_2(\mathbf{u}) = \|\mathbf{u}\|^2$, the lower complexity bound for first-order methods is precisely
373 $O(\log(\varepsilon^{-1})\varepsilon^{(\hat{\alpha}-1)/(1+3\hat{\alpha})})$ in terms of function value accuracy. This finding confirms
374 that Algorithm 3 achieves the optimal iteration complexity.

375 **5. Numerical Experiments.** Preliminary numerical results are presented in
376 this section to provide additional insights into the performance guarantees of the al-
377 gorithms proposed in this paper. We aim to elucidate that the final error attained
378 by the algorithm is influenced by both the stepsize and the Hölder exponent. The
379 numerical experiments are conducted using Julia [4] (version 1.12) on an Apple Mac-
380 intosh Mini with an M2 processor, 8 performance cores, and 32GB of memory. We
381 have placed the Julia codes in the GitHub repository (https://github.com/ctkelle/y/Grad_Des_CKW.jl) with instructions for reproducing the figures. In this section,
382 we set the spatial mesh width as $h = 2^{-4}$ for the discretization of partial differential
383 equations (PDEs). Then the dimension of the discretized problem is $n = (h^{-1} - 1)^2$.

385 **5.1. Two-dimensional PDE with a non-Lipschitz term.** Hölder continuous
386 gradients arise naturally in PDEs involving non-Lipschitz nonlinearity [3, 14]. In this
387 subsection, we introduce a numerical example from [3]. This problem is to solve the
388 following two-dimensional PDE,

$$389 \quad (5.1) \quad \mathcal{F}(u) = -\Delta u + \gamma u_+^\alpha = 0,$$

390 where $\alpha \in (0, 1)$, $\gamma > 0$ is a constant and $u_+ = \max\{u, 0\}$. Discretizing (5.1) with the
391 standard five point difference scheme [9] leads to the following nonlinear system,

$$392 \quad (5.2) \quad \mathbf{F}(\mathbf{u}) = \mathbf{A}\mathbf{u} + \gamma \mathbf{u}_+^\alpha - \mathbf{b} = 0,$$

393 where $\mathbf{A} \in \mathbb{R}^{n \times n}$ is the discretization of $-\Delta$ with zero boundary conditions, $\mathbf{b} \in$
 394 \mathbb{R}^n encodes the boundary conditions, and $\mathbf{u}_+^\alpha = \max\{\mathbf{u}, 0\}^\alpha$ is understood as a
 395 component-wise operation.

396 We now modify the above problem to enable direct computation of errors in the
 397 iterations. To this end, we follow [13, Example 4.4] and take as the exact solution the
 398 function

$$399 \quad u^*(x, y) = \left(\frac{3r-1}{2}\right)^2 \max\left\{0, r - \frac{1}{3}\right\},$$

400 where $r = \sqrt{x^2 + y^2}$. We enforce the following boundary conditions,

$$401 \quad u(x, 1) = u^*(x, 1), u(x, 0) = u^*(x, 0), u(1, y) = u^*(1, y), u(0, y) = u^*(0, y),$$

402 for $0 < x, y < 1$. And these conditions are encoded into \mathbf{b} . Then our modified
 403 equation is

$$404 \quad (5.3) \quad \mathbf{F}(\mathbf{u}) - \mathbf{c}^* = 0,$$

405 where $\mathbf{c}^* = \mathbf{F}(\mathbf{u}^*)$. The nonlinear system (5.3) corresponds to the optimality condition
 406 of the following problem,

$$407 \quad (5.4) \quad \min_{\mathbf{u} \in \mathbb{R}^n} f(\mathbf{u}) = \frac{1}{2} \mathbf{u}^\top \mathbf{A} \mathbf{u} + \frac{\gamma}{1+\alpha} \mathbf{e}^\top \mathbf{u}_+^{1+\alpha} - (\mathbf{b} + \mathbf{c}^*)^\top \mathbf{u},$$

408 where $\mathbf{e} \in \mathbb{R}^n$ is the vector of all ones.

409 The optimization model (5.4) is a special instance of problem (1.1) with $\Omega = \mathbb{R}^n$,
 410 $m = 2$,

$$411 \quad f_1(\mathbf{u}) = \mathbf{u}^\top \mathbf{A} \mathbf{u} - 2(\mathbf{b} + \mathbf{c}^*)^\top \mathbf{u}, \text{ and } f_2(\mathbf{u}) = \frac{2\gamma}{1+\alpha} \mathbf{e}^\top \mathbf{u}_+^{1+\alpha}.$$

412 It is clear that, ∇f_1 is Lipschitz continuous with the corresponding Lipschitz constant
 413 $L_1 = 2 \|\mathbf{A}\|$, and ∇f_2 is Hölder continuous with the Hölder exponent α and $L_2 = 2\gamma$
 414 from

$$415 \quad \|\nabla f_2(\mathbf{u}) - \nabla f_2(\mathbf{v})\| = 2\gamma \|\mathbf{u}_+^\alpha - \mathbf{v}_+^\alpha\| \leq 2\gamma \|\mathbf{u} - \mathbf{v}\|^\alpha,$$

416 for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Moreover, the function $f = (f_1 + f_2)/2$ is $\lambda(\mathbf{A})$ -strongly convex,
 417 where $\lambda(\mathbf{A})$ is the smallest eigenvalue of the symmetric positive definite matrix \mathbf{A} .
 418 Let \mathbf{u}^* be the vector obtained by evaluating u^* at the interior grid points. Then \mathbf{u}^*
 419 serves as the unique global minimizer of problem (5.4).

420 In the subsequent experiments, we use the solution of $\mathbf{A}\mathbf{u}_0 = \mathbf{b}$ as the initial
 421 iterate. This is the discretization of Laplace's equation with the boundary conditions.
 422 In this way, we ensure that the entire iteration satisfies the boundary conditions.

423 **5.1.1. Numerical results of Algorithm 1.** In the first experiment, we scrutinize
 424 the performance of Algorithm 1 under different stepsizes for problem (5.4) with
 425 $\alpha = 0.5$ and $\gamma = 0.5$. Specifically, Algorithm 1 is tested for stepsizes of the form
 426 $\tau = \tau_0 h^2$, where τ_0 is taken from the set $\{0.2, 0.1, 0.05, 0.01\}$. The corresponding nu-
 427 mercial results, presented in Figure 1(a), illustrate the decay of the distance between
 428 the iterates and the global minimizer over iterations. It can be observed that a larger
 429 stepsize facilitates a more rapid descent in the iterations.

430 In the second experiment, we vary the Hölder exponent α over the values in
 431 $\{0.1, 0.2, 0.5, 0.8\}$, while fixing $\tau_0 = 0.01$. Figure 1(b) similarly tracks the decay of
 432 the distance to the global minimizer over iterations. It is evident that, as the value
 433 of α decreases, the final error attained by Algorithm 1 increases under the same
 434 stepsize. Therefore, the associated optimization problems become increasingly ill-
 435 conditioned and thus more challenging to solve for smaller values of α . These findings
 436 offer empirical support for our theoretical analysis.

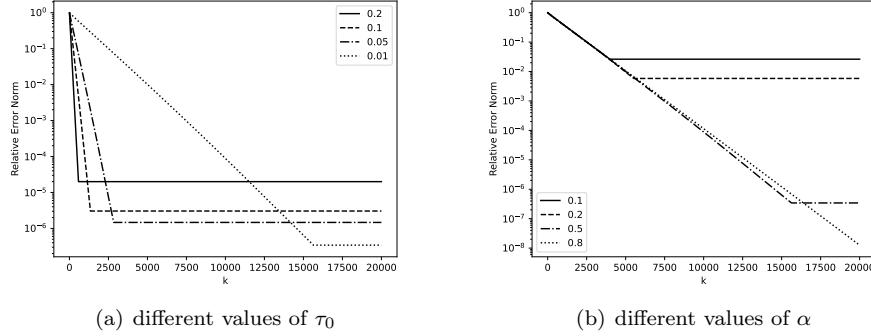


FIG. 1. Numerical performance of Algorithm 1 for problem (5.4).

437 **5.1.2. Numerical results of Algorithm 2.** We repeat the study in subsec-
 438 tion 5.1.1 for Algorithm 2 by varying the values of the Hölder exponent α . We set
 439 $\varepsilon = 10^{-6}$ and $\mu = 2\pi^2$ in Algorithm 2, which is a lower estimate for the smallest
 440 eigenvalue of \mathbf{A} . The stepsize is initialized to $0.1h^2$ in the line-search procedure. The
 441 corresponding numerical results are depicted in Figure 2. Comparing Figure 2 to ??
 442 shows the benefits of the line-search procedure in Algorithm 2, which does not need
 443 to manually adjust the value of τ_0 to converge for a given value of ε .

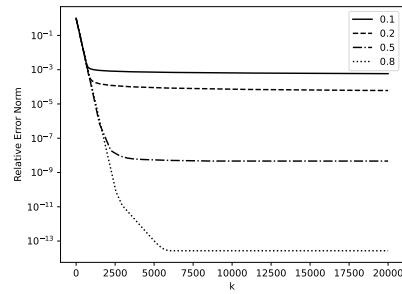


FIG. 2. Numerical performance of Algorithm 2 for problem (5.4) with different values of α .

444 **5.1.3. Numerical results of Algorithm 3.** We report the numerical perfor-
 445 mance of Algorithm 3 on two experiments. Guided by the observation in Remark 4.2,
 446 we test Algorithm 3 with a fixed stepsize $\nu = \tau_0 h^2$. In the first example, we use the
 447 values for τ_0 from Figure 1. In this way we can directly compare the performance of
 448 Algorithm 3 with that of Algorithm 1. The corresponding results, shown in Figure 3,

449 are poor. The reason for this is that we are not exploiting the ability of Algorithm 3
 450 to use larger stepsizes. In the second example, we consider larger values for τ_0 in Fig-
 451 ure 4(a) and set $\tau_0 = 20$ in Figure 4(b). The convergence is much better in all cases.
 452 The hardest case ($\alpha = 0.1$) has very irregular convergence in the terminal phase of
 453 iterations.

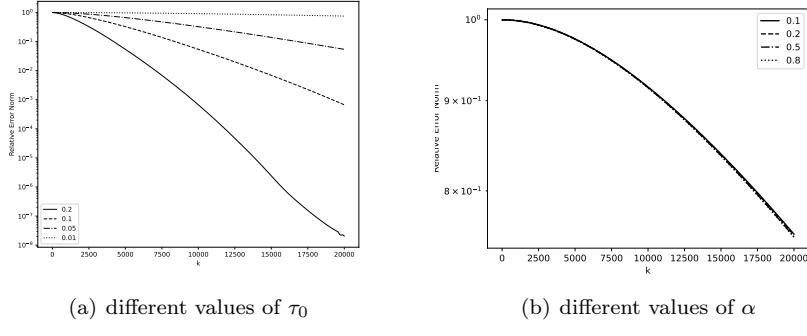


FIG. 3. Numerical performance of Algorithm 3 for problem (5.4) with smaller stepsizes.

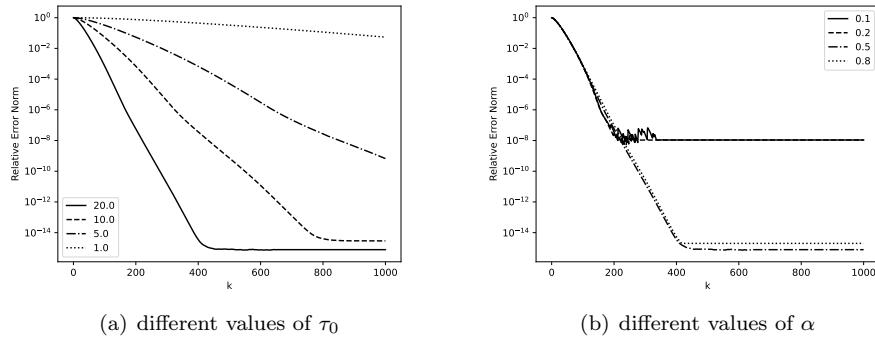


FIG. 4. Numerical performance of Algorithm 3 for problem (5.4) with larger stepsizes.

454 **5.2. Semi-linear elliptic problem with a constraint.** We consider a second
 455 numerical example motivated by a semi-linear elliptic problem with a constraint on
 456 the solution in a certain set [14]. Let

457 (5.5)
$$\mathcal{H}(u) = -\Delta u + \delta|u|^\alpha \text{sign}(u) - |u|^{p-1}u,$$

458 on $D = (0, 1)^2$ with the boundary condition $u(x, y) = 0.5 - \sin(x)\sin(y)$ on ∂D . Here,
 459 $\alpha \in (0, 1)$, $p > 1$, and $\delta > p/\alpha$ are three constants. We consider the variational
 460 inequality that is to find $u^* \in [-1, 1]$ such that

461
$$\mathcal{H}(u^*)(u - u^*) \geq 0,$$

462 for any $u \in [-1, 1]$. This problem is equivalent to the following nonlinear equation,

$$463 \quad (5.6) \quad 0 = \mathcal{F}(u) := \begin{cases} \mathcal{H}(u), & \text{if } u - \mathcal{H}(u) \in [-1, 1], \\ u - 1, & \text{if } u - \mathcal{H}(u) \geq 1, \\ u + 1, & \text{otherwise.} \end{cases}$$

464 By discretizing (5.5) with the standard five point difference scheme [9], problem (5.6)
465 leads to the following system of nonlinear equations,

$$466 \quad (5.7) \quad 0 = \mathbf{F}(\mathbf{u}) := \frac{1}{\theta} \left(\mathbf{u} - \Pi_{\mathbf{U}} \left(\mathbf{u} - \theta \left(\mathbf{A}\mathbf{u} + \delta |\mathbf{u}|^\alpha \operatorname{sign}(\mathbf{u}) - |\mathbf{u}|^{p-1} \mathbf{u} - \mathbf{b} \right) \right) \right),$$

467 where $\mathbf{U} = [-1, 1]^n$, $\theta > 0$ is a constant, $\mathbf{A} \in \mathbb{R}^{n \times n}$ is a symmetric positive definite
468 matrix, and $\mathbf{b} \in \mathbb{R}^n$ encodes the boundary conditions. Note that (5.7) is the optimality
469 condition of the following problem,

$$470 \quad (5.8) \quad \min_{\mathbf{u} \in \mathbf{U}} f(\mathbf{u}) := \frac{1}{2} \mathbf{u}^\top \mathbf{A} \mathbf{u} + \frac{\delta}{1+\alpha} \mathbf{e}^\top |\mathbf{u}|^{1+\alpha} - \frac{1}{1+p} \mathbf{e}^\top |\mathbf{u}|^{1+p} - \mathbf{b}^\top \mathbf{u}.$$

471 The Hessian matrix of f at \mathbf{u} with $\mathbf{u}_i \neq 0$ ($i = 1, \dots, n$) has the form

$$472 \quad \nabla^2 f(\mathbf{u}) = \mathbf{A} + \delta \alpha \operatorname{Diag}(|\mathbf{u}|^{\alpha-1}) - p \operatorname{Diag}(|\mathbf{u}|^{p-1}),$$

473 Since $\delta > p/\alpha$, $\nabla^2 f(\mathbf{u})$ is symmetric positive definite for any $\mathbf{u} \in \mathbf{U}$ with $\mathbf{u}_i \neq 0$ ($i = 1, \dots, n$). Hence, the function f is μ -strongly convex in \mathbf{U} with $\mu = \lambda(\mathbf{A})$ and the
474 system (5.7) has a unique solution in \mathbf{U} . The optimization model (5.8) is a special
475 instance of problem (1.1) with $\Omega = \mathbf{U}$, $m = 2$,

$$477 \quad f_1(\mathbf{u}) = \mathbf{u}^\top \mathbf{A} \mathbf{u} - 2\mathbf{b}^\top \mathbf{u} - \frac{2}{1+p} \mathbf{e}^\top |\mathbf{u}|^{1+p}, \text{ and } f_2(\mathbf{u}) = \frac{2\delta}{1+\alpha} \mathbf{e}^\top |\mathbf{u}|^{1+\alpha}.$$

478 It is clear that Assumption 1.1 (ii) holds with $\alpha_1 = 1$, $L_1 = 2 \|\mathbf{A}\| + 2p$, $\alpha_2 = \alpha$, and
479 $L_2 = 2\delta\alpha$.

480 In this example, we do not have an analytic solution and we only plot the residual
481 norm $\|\mathbf{F}(\mathbf{u})\|$ with θ being the stepsize. We compare the performance of Algorithm 1
482 and Algorithm 3 on problem (5.8) with $p = 1.5$ and $\delta = 20$. The stepsizes of Algo-
483 rithm 1 and Algorithm 3 are set to $0.1h^2$ and $20h^2$, respectively. Figure 5 illustrates
484 the performance of two algorithms for different values of α . Similar as the case in
485 subsection 5.1, problems where the exponent α for the non-Lipschitz term in the
486 gradients is small are difficult. In particular, one cannot drive the residual to a small
487 value. For larger values of α , both algorithms demonstrate strong performance. Fur-
488 thermore, Algorithm 3 exhibits a faster convergence rate, benefiting from the use of
489 a larger stepsize.

490 **6. Conclusion.** In this paper, we consider a class of strongly convex constrained
491 optimization problems of the form (1.1). Example 1.2 shows that although each com-
492 ponent function f_i of the objective function f admits a Hölder continuous gradient
493 with an component $\alpha_i \in (0, 1]$, the gradient of f is not necessarily Hölder continuous.
494 To establish the iteration complexity of the projected gradient descent methods for
495 this class of problems, we use the parameter $\hat{\alpha} = \min_{i \in [m]} \alpha_i$ to determine the com-
496 plexity bound. Algorithm 1 is a new version of projected gradient method for prob-
497 lem (1.1) with an appropriately fixed stepsize. Theorem 2.2 shows that Algorithm 1

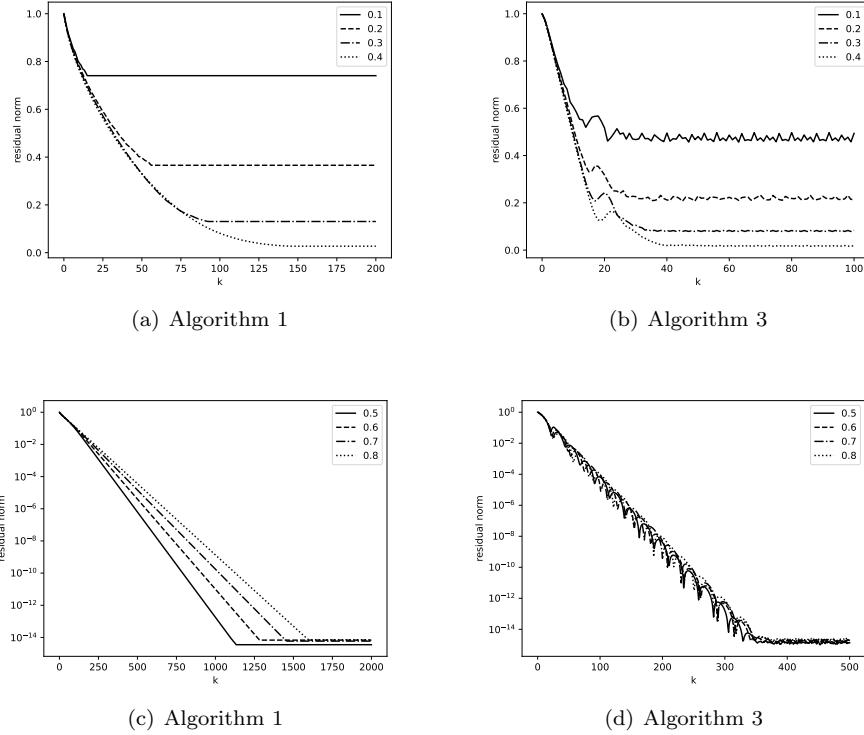


FIG. 5. Numerical performance of Algorithm 1 and Algorithm 3 for problem (5.8) with different values of α .

498 can find an iterate in the feasible set Ω with a distance to the global minimizer less
499 than ε at most $O(\log(\varepsilon^{-1})\varepsilon^{2(\hat{\alpha}-1)/(1+\hat{\alpha})})$ iterations. This recovers the classical com-
500 plexity result when $\hat{\alpha} = 1$ and reveals the additional difficulty imposed by the weaker
501 smoothness of the objective function for $\hat{\alpha} < 1$. Algorithm 2 is a modification of Algo-
502 rithm 1 for problems where the parameters α_i and L_i are difficult to estimate for the
503 stepsize. In Algorithm 3, the stepsize is updated by the universal scheme at each iter-
504 ation, which improves the complexity bound to $O(\log(\varepsilon^{-1})\varepsilon^{2(\hat{\alpha}-1)/(1+3\hat{\alpha})})$. Numerical
505 experiments are conducted to validate our theoretical findings, demonstrating the ex-
506 pected behavior of projected gradient descent methods under different stepsizes and
507 Hölder exponents. These results offer new insights into the performance guarantees
508 of the classic projected gradient descent methods for a broader class of optimization
509 problems with non-Lipschitz gradients.

510

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