

COMPLEXITY OF PROJECTED GRADIENT METHODS FOR STRONGLY CONVEX OPTIMIZATION WITH HÖLDER CONTINUOUS GRADIENT TERMS*

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Abstract. This paper studies the complexity of projected gradient descent methods for a class of strongly convex constrained optimization problems where the objective function is expressed as a summation of m component functions, each possessing a gradient that is Hölder continuous with an exponent $\alpha_i \in (0, 1]$. Under this formulation, the gradient of the objective function may fail to be globally Hölder continuous, thereby existing complexity results inapplicable to this class of problems. Our theoretical analysis reveals that, in this setting, the complexity of projected gradient methods is determined by $\hat{\alpha} = \min_{i \in \{1, \dots, m\}} \alpha_i$. We first prove that, with an appropriately fixed stepsize, the complexity bound for finding an approximate minimizer with a distance to the true minimizer less than ε is $O(\log(\varepsilon^{-1})\varepsilon^{2(\hat{\alpha}-1)/(1+\hat{\alpha})})$, which extends the well-known complexity result for $\hat{\alpha} = 1$. Next we show that the complexity bound can be improved to $O(\log(\varepsilon^{-1})\varepsilon^{2(\hat{\alpha}-1)/(1+3\hat{\alpha})})$ if the stepsize is updated by the universal scheme. We illustrate our complexity results by numerical examples arising from elliptic equations with a non-Lipschitz term.

Key words. projected gradient descent, complexity, Hölder continuity

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1. Introduction. Given a closed and convex set $\Omega \subseteq \mathbb{R}^n$, this paper considers the following optimization problem,

$$21 \quad (1.1) \qquad \min_{\mathbf{u} \in \Omega} \quad f(\mathbf{u}) := \frac{1}{m} \sum_{i=1}^m f_i(\mathbf{u}),$$

22 where the objective function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies the following assumption.

ASSUMPTION 1.1.

24 (i) The function f is μ -strongly convex with a parameter $\mu > 0$ on Ω , that is,

$$f(\mathbf{u}) \geq f(\mathbf{v}) + \langle \nabla f(\mathbf{v}), \mathbf{u} - \mathbf{v} \rangle + \frac{\mu}{2} \|\mathbf{u} - \mathbf{v}\|^2,$$

26 for all $\mathbf{u}, \mathbf{v} \in \Omega$.

(ii) For each $i \in [m] := \{1, 2, \dots, m\}$, the function $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuously differentiable and the gradient ∇f_i is (globally) Hölder continuous with an exponent $\alpha_i \in (0, 1]$ on Ω , namely, there exists a constant $L_i > 0$ such that

$$30 \quad (1.2) \quad \|\nabla f_i(\mathbf{u}) - \nabla f_i(\mathbf{v})\| \leq L_i \|\mathbf{u} - \mathbf{v}\|^{\alpha_i},$$

31 *for all* $\mathbf{u}, \mathbf{v} \in \Omega$.

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32 Here, $\|\cdot\|$ is the ℓ_2 norm and $\langle \cdot, \cdot \rangle$ is the inner product on \mathbb{R}^n . We also denote by
 33 $\mathbf{u}^* \in \Omega$ and $f^* = f(\mathbf{u}^*)$ the global minimizer and the optimal value of problem (1.1),
 34 respectively.

35 Suppose that each ∇f_i is Lipschitz continuous, which corresponds to condition
 36 (1.2) with $\alpha_i = 1$ for all $\mathbf{u}, \mathbf{v} \in \Omega$. Then ∇f is also Lipschitz continuous and
 37 the associated Lipschitz constant is $L = \sum_{i=1}^m L_i/m$. Let $\Pi_\Omega(\cdot)$ be the projection
 38 operator onto the set Ω . It is well known that the classical projected gradient descent
 39 method

40 (1.3)
$$\mathbf{u}_{k+1} = \Pi_\Omega(\mathbf{u}_k - \tau \nabla f(\mathbf{u}_k)),$$

41 with any initial point $\mathbf{u}_0 \in \mathbb{R}^n$ and the stepsize $\tau \in (0, 2/(\mu + L)]$, achieves a linear
 42 rate of convergence [11, Theorem 2.2.14] as follows,

43
$$\|\mathbf{u}_k - \mathbf{u}^*\| \leq (1 - \mu\tau)^k \|\mathbf{u}_0 - \mathbf{u}^*\|.$$

44 Therefore, for a given $\varepsilon > 0$, method (1.3) is guaranteed to find a point $\mathbf{u}_k \in \Omega$
 45 satisfying $\|\mathbf{u}_k - \mathbf{u}^*\| \leq \varepsilon$ after at most $O(\log(\varepsilon^{-1}))$ iterations. Unfortunately, this
 46 analysis fails if there exists at least one index $i \in [m]$ such that $\alpha_i < 1$. We explain
 47 the failure of the convergence of method (1.3) to \mathbf{u}^* by the following example.

48 *Example 1.2.* [6, Example 1] Consider the following univariate optimization prob-
 49 lem on $\Omega = \mathbb{R}$,

50 (1.4)
$$\min_{x \in \mathbb{R}} f(x) = \frac{1}{2}x^2 + \frac{2}{3}|x|^{3/2},$$

51 which is a special instance of problem (1.1) with $f_1(x) = x^2/2$ and $f_2(x) = 2|x|^{3/2}/3$.
 52 It is easy to see that the global minimizer is $x^* = 0$. Method (1.3) with the fixed
 53 stepsize $\tau > 0$ starting from $x_0 \neq 0$ proceeds as follows,

54
$$x_{k+1} = x_k - \tau \nabla f(x_k) = (1 - \tau)x_k - \tau \text{sign}(x_k)|x_k|^{1/2},$$

55 where $\text{sign}(x) = 1$ if $x > 0$, 0 if $x = 0$, and -1 otherwise. A straightforward verification
 56 reveals that

57
$$|x_{k+1}|^2 - |x_k|^2 = -\tau(2 - \tau)|x_k|^2 - 2\tau(1 - \tau)|x_k|^{3/2} + \tau^2|x_k|.$$

58 It is evident that, when $|x_k|$ is sufficiently small, the last term in the right-hand side
 59 becomes dominant, resulting in that $|x_{k+1}|^2 - |x_k|^2 \geq 0$. Therefore, the distance to
 60 the global minimizer ceases to decrease once it achieves a certain level.

61 Moreover, in [6] we show that ∇f is locally, but not globally, Hölder continuous.
 62 In fact, from

63
$$|\nabla f(|h|) - \nabla f(0)| = |h| + |h|^{1/2} = \left(|h|^{1-\alpha} + |h|^{1/2-\alpha}\right)|h|^\alpha,$$

64 we can obtain that, $|h|^{1-\alpha} \rightarrow \infty$ when $\alpha \in (0, 1)$ and $|h| \rightarrow \infty$, while $|h|^{1/2-\alpha} \rightarrow \infty$
 65 when $\alpha = 1$ and $|h| \rightarrow 0$. Therefore, ∇f cannot be globally Hölder continuous for all
 66 $\alpha \in (0, 1]$.

67 On the other hand, problem (1.4) satisfies all the conditions in Assumption 1.1.
 68 It is clear that f is strongly convex. In addition, we have

69
$$|\nabla f_1(x) - \nabla f_1(y)| = |x - y|,$$

70 and

$$71 \quad |\nabla f_2(x) - \nabla f_2(y)| = \left| \text{sign}(x) |x|^{1/2} - \text{sign}(y) |y|^{1/2} \right| \leq \sqrt{2} |x - y|^{1/2},$$

72 for all $x, y \in \mathbb{R}$.

73 This simple example demonstrates that, in problem (1.1), a function f expressed
 74 as a sum of component functions f_i , each endowed with a Hölder continuous gradient,
 75 may itself fail to possess a Hölder continuous gradient. This phenomenon, initially
 76 observed in our previous work [6], was later revisited and further highlighted by
 77 Nesterov (see [12, Example 1]).

78 Since ∇f may not be globally Hölder continuous, most existing complexity results
 79 are inapplicable to problem (1.1). For the special case where $m = 1$, namely, ∇f is
 80 globally Hölder continuous with an exponent $\alpha \in (0, 1]$, Devolder et al. [7] presented
 81 the following bound for method (1.3),

$$82 \quad f(\hat{\mathbf{u}}_N) - f(\mathbf{u}^*) \leq K(N) := \frac{L_\alpha \|\mathbf{u}_0 - \mathbf{u}^*\|^{1+\alpha}}{1 + \alpha} \left(\frac{2}{N} \right)^{\frac{1+\alpha}{2}},$$

83 where L_α is the Hölder constant and $\hat{\mathbf{u}}_N = \sum_{k=1}^N \mathbf{u}_k / N$. In the strongly convex case,
 84 (51) in [7] comes to

$$85 \quad \|\hat{\mathbf{u}}_N - \mathbf{u}^*\|^2 \leq \frac{2}{\mu} K(N),$$

86 which implies that finding an N average of iterations $\hat{\mathbf{u}}_N$ satisfying $\|\hat{\mathbf{u}}_N - \mathbf{u}^*\| \leq \varepsilon$
 87 requires $O(\varepsilon^{-4/(1+\alpha)})$ iterations.

88 The contribution of this paper is to provide new complexity results of the pro-
 89 jected gradient descent methods for problem (1.1), which are dictated by the parame-
 90 ter $\hat{\alpha} = \min_{i \in [m]} \alpha_i \in (0, 1]$. We first show that, with an appropriately fixed stepsize,
 91 the complexity bound for finding an iterate with a distance to the global minimizer
 92 less than ε is $O(\log(\varepsilon^{-1})\varepsilon^{2(\hat{\alpha}-1)/(1+\hat{\alpha})})$, which extends the well-known complexity re-
 93 sult for $\hat{\alpha} = 1$. Next, we demonstrate that this complexity bound can be improved
 94 to $O(\log(\varepsilon^{-1})\varepsilon^{2(\hat{\alpha}-1)/(1+3\hat{\alpha})})$ if the stepsize is updated at each iteration using the
 95 universal scheme. Even in the special case where $m = 1$, our complexity bound is
 96 at least $O(\varepsilon^{-1})$ lower than (51) in [7]. For example, when $\hat{\alpha} = 1/2$, our bound is
 97 $O(\log(\varepsilon^{-1})\varepsilon^{-2/5})$ but (51) in [7] is $O(\varepsilon^{-8/3})$.

98 Our study is motivated by elliptic equations with a non-Lipschitz term [3, 14],
 99 complementarity problems [1, 13], and optimization problems with an ℓ_p -norm ($1 <$
 100 $p < 2$) regularization term [2, 5]. We illustrate our complexity results by two numerical
 101 examples arising from elliptic equations with a non-Lipschitz term in section 5, after
 102 we present complexity of projected gradient methods with fixed stepsizes and updated
 103 stepsizes in sections 2 to 4, respectively.

104 2. Vanilla Projected Gradient Descent Method with a Fixed Stepsize.

105 In this section, we attempt to employ the vanilla projected gradient descent method
 106 (1.3) with a fixed stepsize to solve problem (1.1), whose complexity bound is also
 107 provided. Example 1.2 illustrates that the projected gradient descent method (1.3)
 108 with a fixed stepsize will experience stagnation before reaching the global minimizer.

109 To obtain an approximate solution to problem (1.1), it is necessary to choose
 110 a sufficiently small stepsize τ in the projected gradient descent method (1.3), the

111 magnitude of which depends on the desired level of accuracy. Let $M > 0$ be a
 112 constant defined as

$$113 \quad (2.1) \quad M = \max_{i \in [m]} \left\{ \left[\frac{2(1 - \alpha_i)}{\mu(1 + \alpha_i)} \right]^{(1 - \alpha_i)/(1 + \alpha_i)} L_i^{2/(1 + \alpha_i)} \right\},$$

114 with the convention $0^0 = 1$. We select a specific stepsize $\tau = \varepsilon^{2(1 - \hat{\alpha})/(1 + \hat{\alpha})}/M$ in
 115 the projected gradient descent method, whose complete framework is presented in
 116 Algorithm 1. Two sequences $\{\mathbf{v}_k\}$ and $\{\mathbf{u}_k\}$ are maintained in Algorithm 1, where
 117 \mathbf{v}_k is generated by the projected gradient descent method and \mathbf{u}_k corresponds to the
 118 iterate achieving the smallest objective function value among the first k iterations.

Algorithm 1: Projected Gradient Descent Method (PGDM).

Input: $\varepsilon > 0$.

Initialize $\mathbf{u}_0 = \mathbf{v}_0 \in \Omega$.

Choose the stepsize $\tau = \varepsilon^{2(1 - \hat{\alpha})/(1 + \hat{\alpha})}/M$.

for $k = 0, 1, 2, \dots$ **do**

Compute

$$\mathbf{v}_{k+1} = \Pi_\Omega (\mathbf{v}_k - \tau \nabla f(\mathbf{v}_k)).$$

Set

$$\mathbf{u}_{k+1} = \begin{cases} \mathbf{v}_{k+1}, & \text{if } f(\mathbf{v}_{k+1}) \leq f(\mathbf{u}_k), \\ \mathbf{u}_k, & \text{otherwise.} \end{cases}$$

Output: \mathbf{u}_{k+1} .

119 Our subsequent analysis is based on the inexact oracle [7] derived from the Hölder
 120 continuity condition of gradients, which is generalized to problem (1.1) and demon-
 121 strated in the following proposition.

122 **PROPOSITION 2.1.** *Suppose that Assumption 1.1 holds. Let $\delta > 0$ and*

$$123 \quad \rho \geq \max_{i \in [m]} \left\{ \left[\frac{1 - \alpha_i}{(1 + \alpha_i)\delta} \right]^{(1 - \alpha_i)/(1 + \alpha_i)} L_i^{2/(1 + \alpha_i)} \right\}.$$

124 Then for all $\mathbf{u}, \mathbf{v} \in \Omega$, we have

$$125 \quad f(\mathbf{v}) \leq f(\mathbf{u}) + \langle \nabla f(\mathbf{u}), \mathbf{v} - \mathbf{u} \rangle + \frac{\rho}{2} \|\mathbf{v} - \mathbf{u}\|^2 + \frac{\delta}{2}.$$

126 *Proof.* Since ∇f_i is Hölder continuous with an exponent α_i , we can obtain from
 127 [15, Lemma 1] that

$$128 \quad f_i(\mathbf{v}) \leq f_i(\mathbf{u}) + \langle \nabla f_i(\mathbf{u}), \mathbf{v} - \mathbf{u} \rangle + \frac{L_i}{1 + \alpha_i} \|\mathbf{v} - \mathbf{u}\|^{1 + \alpha_i},$$

129 for all $\mathbf{u}, \mathbf{v} \in \Omega$. Then, for each i , it follows from [10, Lemma 2] that

$$130 \quad f_i(\mathbf{v}) \leq f_i(\mathbf{u}) + \langle \nabla f_i(\mathbf{u}), \mathbf{v} - \mathbf{u} \rangle + \frac{\rho}{2} \|\mathbf{v} - \mathbf{u}\|^2 + \frac{\delta}{2}.$$

131 Summing the above relationship over $i \in [m]$, we immediately arrive at the assertion
 132 of this proposition. The proof is completed. \square

133 Now, we are able to derive the complexity bound of Algorithm 1 in the following
 134 theorem.

135 THEOREM 2.2. *Let $\varepsilon \in (0, 1)$ be a sufficiently small constant. Then after at most*

$$136 \quad O\left(\log\left(\frac{M^{(1+\hat{\alpha})/4}}{\varepsilon}\right) \frac{M}{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}}\right)$$

137 iterations, Algorithm 1 will find an iterate $\mathbf{u}_k \in \Omega$ satisfying $\|\mathbf{u}_k - \mathbf{u}^*\| \leq \varepsilon$.

138 *Proof.* In view of Proposition 2.1, we take

$$139 \quad \rho = \frac{1}{\tau} = \frac{M}{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}} \geq \max_{i \in [m]} \left\{ \left[\frac{2(1-\alpha_i)}{\mu(1+\alpha_i)\varepsilon^2} \right]^{(1-\alpha_i)/(1+\alpha_i)} L_i^{2/(1+\alpha_i)} \right\}.$$

140 Then it holds that

$$141 \quad f(\mathbf{v}_{k+1}) \leq f(\mathbf{v}_k) + \langle \nabla f(\mathbf{v}_k), \mathbf{v}_{k+1} - \mathbf{v}_k \rangle + \frac{1}{2\tau} \|\mathbf{v}_{k+1} - \mathbf{v}_k\|^2 + \frac{\mu\varepsilon^2}{4},$$

142 which, after a suitable rearrangement, can be equivalently written as

$$143 \quad (2.2) \quad \langle \nabla f(\mathbf{v}_k), \mathbf{v}_k - \mathbf{v}_{k+1} \rangle \leq f(\mathbf{v}_k) - f(\mathbf{v}_{k+1}) + \frac{\mu\varepsilon^2}{4} + \frac{1}{2\tau} \|\mathbf{v}_{k+1} - \mathbf{v}_k\|^2.$$

144 Recall that $f^* = f(\mathbf{u}^*)$. By virtue of the strong convexity of f , we can obtain that

$$145 \quad (2.3) \quad \langle \nabla f(\mathbf{v}_k), \mathbf{u}^* - \mathbf{v}_k \rangle \leq f^* - f(\mathbf{v}_k) - \frac{\mu}{2} \|\mathbf{v}_k - \mathbf{u}^*\|^2.$$

146 The optimality condition of the projection problem defining \mathbf{v}_{k+1} yields that

$$147 \quad \langle \mathbf{v}_{k+1} - \mathbf{v}_k + \tau \nabla f(\mathbf{v}_k), \mathbf{u} - \mathbf{v}_{k+1} \rangle \geq 0,$$

148 for all $\mathbf{u} \in \Omega$. Upon taking $\mathbf{u} = \mathbf{u}^*$, we have

$$149 \quad \begin{aligned} \langle \mathbf{v}_{k+1} - \mathbf{v}_k, \mathbf{v}_{k+1} - \mathbf{u}^* \rangle &\leq \tau \langle \nabla f(\mathbf{v}_k), \mathbf{u}^* - \mathbf{v}_{k+1} \rangle \\ &= \tau \langle \nabla f(\mathbf{v}_k), \mathbf{u}^* - \mathbf{v}_k \rangle + \tau \langle \nabla f(\mathbf{v}_k), \mathbf{v}_k - \mathbf{v}_{k+1} \rangle, \end{aligned}$$

150 which together with (2.2) and (2.3) implies that

$$151 \quad \begin{aligned} \langle \mathbf{v}_{k+1} - \mathbf{v}_k, \mathbf{v}_{k+1} - \mathbf{u}^* \rangle &\leq \tau \left(f^* - f(\mathbf{v}_{k+1}) + \frac{\mu\varepsilon^2}{4} \right) - \frac{\mu\tau}{2} \|\mathbf{v}_k - \mathbf{u}^*\|^2 \\ &\quad + \frac{1}{2} \|\mathbf{v}_{k+1} - \mathbf{v}_k\|^2. \end{aligned}$$

152 Moreover, it can be readily verified that

$$153 \quad (2.4) \quad \begin{aligned} \|\mathbf{v}_{k+1} - \mathbf{u}^*\|^2 &= \|\mathbf{v}_{k+1} - \mathbf{v}_k + \mathbf{v}_k - \mathbf{u}^*\|^2 \\ &= \|\mathbf{v}_k - \mathbf{u}^*\|^2 + 2 \langle \mathbf{v}_{k+1} - \mathbf{v}_k, \mathbf{v}_k - \mathbf{u}^* \rangle + \|\mathbf{v}_{k+1} - \mathbf{v}_k\|^2 \\ &= \|\mathbf{v}_k - \mathbf{u}^*\|^2 + 2 \langle \mathbf{v}_{k+1} - \mathbf{v}_k, \mathbf{v}_{k+1} - \mathbf{u}^* \rangle - \|\mathbf{v}_{k+1} - \mathbf{v}_k\|^2. \end{aligned}$$

154 Collecting the above two relationships together, we arrive at

$$155 \quad \|\mathbf{v}_{k+1} - \mathbf{u}^*\|^2 \leq (1 - \mu\tau) \|\mathbf{v}_k - \mathbf{u}^*\|^2 + 2\tau \left(f^* - f(\mathbf{v}_{k+1}) + \frac{\mu\varepsilon^2}{4} \right).$$

156 From the construction of \mathbf{u}_k in Algorithm 1, it then follows that $f(\mathbf{v}_l) \geq f(\mathbf{u}_k)$ for
157 all $l \in \{1, 2, \dots, k\}$. Let $C_k = \sum_{l=1}^k (1 - \mu\tau)^{l-1}$ be a constant. Applying the above
158 relationship recursively for k times leads to that

$$159 \quad \begin{aligned} \|\mathbf{v}_k - \mathbf{u}^*\|^2 &\leq (1 - \mu\tau)^k \|\mathbf{u}_0 - \mathbf{u}^*\|^2 + 2\tau \sum_{l=1}^k (1 - \mu\tau)^{l-1} \left(f^* - f(\mathbf{v}_l) + \frac{\mu\varepsilon^2}{4} \right) \\ &\leq (1 - \mu\tau)^k \|\mathbf{u}_0 - \mathbf{u}^*\|^2 + 2\tau \left(f^* - f(\mathbf{u}_k) + \frac{\mu\varepsilon^2}{4} \right) C_k, \end{aligned}$$

160 which together with $\|\mathbf{v}_k - \mathbf{u}^*\| \geq 0$ and $C_k \geq 1$ implies that

$$161 \quad f(\mathbf{u}_k) - f^* \leq \frac{(1 - \mu\tau)^k}{2\tau C_k} \|\mathbf{u}_0 - \mathbf{u}^*\|^2 + \frac{\mu\varepsilon^2}{4} \leq \frac{(1 - \mu\tau)^k}{2\tau} \|\mathbf{u}_0 - \mathbf{u}^*\|^2 + \frac{\mu\varepsilon^2}{4}.$$

162 According to the strong convexity of f and the optimality condition of problem (1.1),
163 we have

$$164 \quad (2.5) \quad f(\mathbf{u}_k) - f^* \geq \langle \nabla f(\mathbf{u}^*), \mathbf{u}_k - \mathbf{u}^* \rangle + \frac{\mu}{2} \|\mathbf{u}_k - \mathbf{u}^*\|^2 \geq \frac{\mu}{2} \|\mathbf{u}_k - \mathbf{u}^*\|^2.$$

165 Hence, it holds that

$$166 \quad \begin{aligned} \|\mathbf{u}_k - \mathbf{u}^*\|^2 &\leq \frac{2}{\mu} (f(\mathbf{u}_k) - f^*) \leq \frac{(1 - \mu\tau)^k}{\mu\tau} \|\mathbf{u}_0 - \mathbf{u}^*\|^2 + \frac{\varepsilon^2}{2} \\ &\leq \frac{M \|\mathbf{u}_0 - \mathbf{u}^*\|^2}{\mu\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}} \left(1 - \frac{\mu}{M} \varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})} \right)^k + \frac{\varepsilon^2}{2}. \end{aligned}$$

167 We denote by K_ε^* the smallest iteration number k such that $\|\mathbf{u}_k - \mathbf{u}^*\| \leq \varepsilon$. Then
168 solving the inequality $M \|\mathbf{u}_0 - \mathbf{u}^*\|^2 \varepsilon^{-2(1-\hat{\alpha})/(1+\hat{\alpha})} (1 - \mu\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}/M)^k / \mu \leq \varepsilon^2/2$
169 indicates that

$$170 \quad \begin{aligned} K_\varepsilon^* &\leq \frac{4 \log((2M \|\mathbf{u}_0 - \mathbf{u}^*\|^2 / \mu)^{(1+\hat{\alpha})/4} / \varepsilon)}{-\log(1 - \mu\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}/M)(1 + \hat{\alpha})} \\ &\leq \frac{4M \log((2M \|\mathbf{u}_0 - \mathbf{u}^*\|^2 / \mu)^{(1+\hat{\alpha})/4} / \varepsilon)}{\mu(1 + \hat{\alpha})\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}}. \end{aligned}$$

171 The proof is completed. \square

172 Theorem 2.2 demonstrates that the iteration complexity of Algorithm 1 with a
173 fixed stepsize is $O(\log(\varepsilon^{-1})\varepsilon^{2(\hat{\alpha}-1)/(1+\hat{\alpha})})$ for problem (1.1). This complexity result
174 generalizes the classical linear convergence when $\hat{\alpha} = 1$, which highlights the perfor-
175 mance degradation incurred by non-Lipschitz gradients.

176 **3. Universal Primal Gradient Method.** The fixed stepsize τ chosen in Algo-
177 rithm 1 depends on the parameters α_i and L_i for all $i \in [m]$, which are often unknown
178 and hard to estimate in practice. To address this issue, we adopt the universal primal
179 gradient method (UPGM) proposed by Nesterov [10] to solve problem (1.1). This

Algorithm 2: Universal Primal Gradient Method (UPGM).**Input:** $\varepsilon > 0$.Initialize $\mathbf{u}_0 = \mathbf{v}_0 \in \Omega$ and $\rho_0 > 0$.**for** $k = 0, 1, 2, \dots$ **do** **for** $j_k = 0, 1, 2, \dots$ **do**

Compute

$$\mathbf{v}_{k+1} = \Pi_{\Omega} \left(\mathbf{v}_k - \frac{1}{2^{j_k} \rho_k} \nabla f(\mathbf{v}_k) \right).$$

If \mathbf{v}_{k+1} satisfies the following line-search condition,

$$(3.1) \quad \begin{aligned} f(\mathbf{v}_{k+1}) &\leq f(\mathbf{v}_k) + \langle \nabla f(\mathbf{v}_k), \mathbf{v}_{k+1} - \mathbf{v}_k \rangle \\ &+ \frac{2^{j_k} \rho_k}{2} \|\mathbf{v}_{k+1} - \mathbf{v}_k\|^2 + \frac{\mu \varepsilon^2}{4}, \end{aligned}$$

then break. Update $\rho_{k+1} = 2^{j_k} \rho_k$.

Set

$$\mathbf{u}_{k+1} = \begin{cases} \mathbf{v}_{k+1}, & \text{if } f(\mathbf{v}_{k+1}) \leq f(\mathbf{u}_k), \\ \mathbf{u}_k, & \text{otherwise.} \end{cases}$$

Output: \mathbf{u}_{k+1} .

180 method incorporates a line-search procedure to adaptively determine the stepsize at
 181 each iteration, and its overall framework is outlined in Algorithm 2.

182 Next, we establish the iteration complexity of Algorithm 2, which remains on the
 183 same order as that of the projected gradient descent method with a fixed stepsize.

184 **THEOREM 3.1.** *Let $\varepsilon \in (0, 1)$ be a sufficiently small constant. Then after at most*

$$185 \quad O \left(\log \left(\frac{M^{(1+\hat{\alpha})/4}}{\varepsilon} \right) \frac{M}{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}} \right)$$

186 iterations, Algorithm 2 will attain an iterate $\mathbf{u}_k \in \Omega$ satisfying that $\|\mathbf{u}_k - \mathbf{u}^*\| \leq \varepsilon$.

187 *Proof.* Obviously, there exists $j_k \in \mathbb{N}$ such that

$$188 \quad 2^{j_k} \rho_k \geq \max_{i \in [m]} \left\{ \left[\frac{2(1-\alpha_i)}{\mu(1+\alpha_i)\varepsilon^2} \right]^{(1-\alpha_i)/(1+\alpha_i)} L_i^{2/(1+\alpha_i)} \right\}.$$

189 By invoking the results of Proposition 2.1, we know that condition (3.1) is satisfied.
 190 Hence, the line-search step in Algorithm 2 can be terminated after a finite number of
 191 trials and the required number of trials j_k satisfies

$$192 \quad (3.2) \quad 2^{j_k} \rho_k \leq 2 \max_{i \in [m]} \left\{ \left[\frac{2(1-\alpha_i)}{\mu(1+\alpha_i)\varepsilon^2} \right]^{(1-\alpha_i)/(1+\alpha_i)} L_i^{2/(1+\alpha_i)} \right\} \leq \frac{2M}{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}},$$

193 where $M > 0$ is a constant defined in (2.1). Moreover, the line-search condition (3.1)

194 directly yields that

195 (3.3)
$$\langle \nabla f(\mathbf{v}_k), \mathbf{v}_k - \mathbf{v}_{k+1} \rangle \leq f(\mathbf{v}_k) - f(\mathbf{v}_{k+1}) + \frac{2^{j_k} \rho_k}{2} \|\mathbf{v}_{k+1} - \mathbf{v}_k\|^2 + \frac{\mu \varepsilon^2}{4}.$$

196 According to the optimality condition of the projection problem defining \mathbf{v}_{k+1} , we
197 have

198
$$\left\langle \mathbf{v}_{k+1} - \mathbf{v}_k + \frac{1}{2^{j_k} \rho_k} \nabla f(\mathbf{v}_k), \mathbf{u}^* - \mathbf{v}_{k+1} \right\rangle \geq 0,$$

199 which further implies that

200
$$\begin{aligned} \langle \mathbf{v}_{k+1} - \mathbf{v}_k, \mathbf{v}_{k+1} - \mathbf{u}^* \rangle &\leq \frac{1}{2^{j_k} \rho_k} \langle \nabla f(\mathbf{v}_k), \mathbf{u}^* - \mathbf{v}_{k+1} \rangle \\ &\leq \frac{1}{2^{j_k} \rho_k} \langle \nabla f(\mathbf{v}_k), \mathbf{u}^* - \mathbf{v}_k \rangle + \frac{1}{2^{j_k} \rho_k} \langle \nabla f(\mathbf{v}_k), \mathbf{v}_k - \mathbf{v}_{k+1} \rangle. \end{aligned}$$

201 Substituting (2.3) and (3.3) into the above relationship leads to that

202
$$\begin{aligned} \langle \mathbf{v}_{k+1} - \mathbf{v}_k, \mathbf{v}_{k+1} - \mathbf{u}^* \rangle &\leq \frac{1}{2^{j_k} \rho_k} \left(f^* - f(\mathbf{v}_{k+1}) + \frac{\mu \varepsilon^2}{4} \right) \\ &\quad + \frac{1}{2} \|\mathbf{v}_{k+1} - \mathbf{v}_k\|^2 - \frac{\mu}{2^{j_k+1} \rho_k} \|\mathbf{v}_k - \mathbf{u}^*\|^2, \end{aligned}$$

203 Thus, it follows from relationship (2.4) that

204
$$\begin{aligned} \|\mathbf{v}_{k+1} - \mathbf{u}^*\|^2 &\leq \left(1 - \frac{\mu}{2^{j_k} \rho_k} \right) \|\mathbf{v}_k - \mathbf{u}^*\|^2 + \frac{2}{2^{j_k} \rho_k} \left(f^* - f(\mathbf{v}_{k+1}) + \frac{\mu \varepsilon^2}{4} \right) \\ &\leq \left(1 - \frac{\mu \varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}}{2M} \right) \|\mathbf{v}_k - \mathbf{u}^*\|^2 + \frac{2}{\rho_0} \left(f^* - f(\mathbf{v}_{k+1}) + \frac{\mu \varepsilon^2}{4} \right), \end{aligned}$$

205 where the last inequality comes from (3.2) and $2^{j_k} \rho_k \geq \rho_0$. The remaining part of
206 the proof follows the same line of reasoning as that of Theorem 2.2 and is therefore
207 omitted here for the sake of brevity. \square

208 We end this section by estimating the total number of line-search steps required
209 by Algorithm 2.

210 COROLLARY 3.2. *Let $\varepsilon \in (0, 1)$ be a sufficiently small constant. Then Algorithm 2
211 requires at most*

212
$$O \left(\log \left(\frac{M^{(1+\hat{\alpha})/4}}{\varepsilon} \right) \frac{M}{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}} \right)$$

213 line-search steps for the generated sequence $\{\mathbf{u}_k\}$ to satisfy $\|\mathbf{u}_k - \mathbf{u}^*\| \leq \varepsilon$.

214 *Proof.* Let N_k be the total number of line-search steps after k iterations in Algo-
215 rithm 2. From the update rule $\rho_{k+1} = 2^{j_k} \rho_k$, we can obtain that $j_k = \log \rho_{k+1} - \log \rho_k$.
216 Then a straightforward verification reveals that

217 (3.4)
$$N_k = \sum_{l=0}^k (j_l + 1) = k + 1 + \log \rho_{k+1} - \log \rho_0,$$

218 which together with relationship (3.2) implies that

$$\begin{aligned} N_k &\leq k + \log\left(\frac{2M}{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}}\right) - \log\rho_0 \\ 219 \quad &\leq k + \frac{2(1-\hat{\alpha})}{1+\hat{\alpha}} \log\left(\frac{1}{\varepsilon}\right) + \log\left(\frac{2M}{\rho_0}\right) + 1. \end{aligned}$$

220 By invoking the results of Theorem 3.1, we conclude that Algorithm 2 requires at
221 most $O(\log(\varepsilon^{-1})\varepsilon^{2(\hat{\alpha}-1)/(1+\hat{\alpha})})$ line-search steps, which completes the proof. \square

222 At each iteration of Algorithm 2, we evaluate both the function value and the
223 gradient at \mathbf{v}_k . In addition, an extra function evaluation at \mathbf{v}_{k+1,j_k} is involved during
224 each line-search step. Therefore, Theorem 3.1 and Corollary 3.2 together reveal that
225 the total number of function and gradient evaluations required by Algorithm 2 is
226 $O(\log(\varepsilon^{-1})\varepsilon^{2(\hat{\alpha}-1)/(1+\hat{\alpha})})$.

227 **4. Universal Fast Gradient Method.** To obtain a sharper complexity bound,
228 we devise in this section a universal fast gradient method (UFGM) tailored to prob-
229 lem (1.1). The proposed scheme, summarized in Algorithm 3, exhibits slight but
230 essential differences from the algorithm introduced by Nesterov [10] to exploit the
231 strong convexity of the objective function.

232 The following lemma illustrates that the line-search process in (4.4) is well-defined,
233 which is guaranteed to terminate in a finite number of trials.

234 **LEMMA 4.1.** *There exists an integer $j_k \in \mathbb{N}$ such that the line-search condition
235 (4.4) is satisfied in Algorithm 3.*

236 *Proof.* It follows from the definition of η_k and $\nu_k \leq 1$ that

$$237 \quad \eta_k = \frac{\nu_k}{1+\nu_k} \geq \frac{\nu_k}{2}, \quad \text{and} \quad \frac{\mu}{\nu_k^2} = 2^{j_k} \rho_k.$$

238 Recall that $\hat{\alpha} = \min_{i \in [m]} \alpha_i \in (0, 1]$. Then we have

$$\begin{aligned} 239 \quad \frac{\mu}{\nu_k^2} \eta_k^{(1-\hat{\alpha})/(1+\hat{\alpha})} &\geq \frac{2^{j_k} \rho_k}{2^{(1-\hat{\alpha})/(1+\hat{\alpha})}} \nu_k^{(1-\hat{\alpha})/(1+\hat{\alpha})} \\ &= \frac{2^{j_k} \rho_k}{2^{(1-\hat{\alpha})/(1+\hat{\alpha})}} \left[\frac{\mu}{2^{j_k} \rho_k} \right]^{(1-\hat{\alpha})/(2(1+\hat{\alpha}))} \\ &= \frac{\mu^{(1-\hat{\alpha})/(2(1+\hat{\alpha}))}}{2^{(1-\hat{\alpha})/(1+\hat{\alpha})}} [2^{j_k} \rho_k]^{(1+3\hat{\alpha})/(2(1+\hat{\alpha}))}, \end{aligned}$$

240 where the first equality comes from the definition of ν_k . Now it is clear that

$$241 \quad \frac{\mu}{\nu_k^2} \eta_k^{(1-\hat{\alpha})/(1+\hat{\alpha})} \rightarrow \infty,$$

242 as $j_k \rightarrow \infty$. Thus, there exists $j_k \in \mathbb{N}$ such that

$$243 \quad (4.6) \quad \frac{\mu}{\nu_k^2} \eta_k^{(1-\hat{\alpha})/(1+\hat{\alpha})} \geq \max_{i \in [m]} \left\{ \left[\frac{2(1-\alpha_i)}{\mu(1+\alpha_i)\varepsilon^2} \right]^{(1-\alpha_i)/(1+\alpha_i)} L_i^{2/(1+\alpha_i)} \right\},$$

Algorithm 3: Universal Fast Gradient Method (UFGM).**Input:** $\varepsilon > 0$.Initialize $\mathbf{u}_0 = \mathbf{w}_0 \in \Omega$ and $\rho_0 \geq \mu$.**for** $k = 0, 1, 2, \dots$ **do** **for** $j_k = 0, 1, 2, \dots$ **do** Set $\nu_k = \sqrt{\mu/(2^{j_k} \rho_k)}$ and $\eta_k = \nu_k/(1 + \nu_k)$.

Compute

(4.1)
$$\mathbf{v}_k = (1 - \eta_k)\mathbf{u}_k + \eta_k \Pi_{\Omega}(\mathbf{w}_k),$$

and

(4.2)
$$\mathbf{z}_k = \Pi_{\Omega} \left(\Pi_{\Omega}(\mathbf{w}_k) - \frac{\nu_k}{\mu} \nabla f(\mathbf{v}_k) \right).$$

Set

(4.3)
$$\mathbf{u}_{k+1} = (1 - \eta_k)\mathbf{u}_k + \eta_k \mathbf{z}_k.$$

If \mathbf{u}_{k+1} satisfies the following line-search condition,

(4.4)
$$\begin{aligned} f(\mathbf{u}_{k+1}) &\leq f(\mathbf{v}_k) + \langle \nabla f(\mathbf{v}_k), \mathbf{u}_{k+1} - \mathbf{v}_k \rangle \\ &+ \frac{\mu}{2\nu_k^2} \|\mathbf{u}_{k+1} - \mathbf{v}_k\|^2 + \frac{\eta_k \mu \varepsilon^2}{4}, \end{aligned}$$

then break.Set $\rho_{k+1} = 2^{j_k} \rho_k$ and update \mathbf{w}_{k+1} by

(4.5)
$$\mathbf{w}_{k+1} = (1 - \eta_k)\mathbf{w}_k + \eta_k \mathbf{v}_k - \frac{\eta_k}{\mu} \nabla f(\mathbf{v}_k).$$

Output: \mathbf{u}_{k+1} .

244 which further implies that

$$\begin{aligned} \frac{\mu}{\nu_k^2} &\geq \frac{1}{\eta_k^{(1-\hat{\alpha})/(1+\hat{\alpha})}} \max_{i \in [m]} \left\{ \left[\frac{2(1-\alpha_i)}{\mu(1+\alpha_i)\varepsilon^2} \right]^{(1-\alpha_i)/(1+\alpha_i)} L_i^{2/(1+\alpha_i)} \right\} \\ 245 \quad &\geq \max_{i \in [m]} \left\{ \left[\frac{2(1-\alpha_i)}{\eta_k \mu(1+\alpha_i)\varepsilon^2} \right]^{(1-\alpha_i)/(1+\alpha_i)} L_i^{2/(1+\alpha_i)} \right\}. \end{aligned}$$

246 As a direct consequence of Proposition 2.1, we can proceed to show that the line-search
247 condition (4.4) is satisfied, which completes the proof. \square 248 *Remark 4.2.* When the parameters of problem (1.1) are fully specified, Algo-
249 rithm 3 may alternatively be implemented with a fixed stepsize. Recall that $M > 0$
250 is a constant defined in (2.1). By invoking the result of Lemma 4.1, we can fix

251
$$\nu_k = 2 \left[\frac{\mu}{4M} \right]^{(1+\hat{\alpha})/(1+3\hat{\alpha})} \varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})},$$

252 and dispense with the parameter ρ_k and the line-search procedure in (4.4). Under

253 this choice, Algorithm 3 continues to enjoy the same iteration complexity established
 254 later.

255 We now introduce the estimating sequences associated with Algorithm 3, which
 256 play a crucial role in our subsequent analysis.

257 LEMMA 4.3. *Let $\{\sigma_k\}$ be a sequence of positive constants defined recursively by*

258 (4.7)
$$\sigma_{k+1} = (1 + \nu_k)\sigma_k,$$

259 with $\sigma_0 = 1$. And let $\{\phi_k\}$ be a sequence of functions defined recursively by

260 (4.8)
$$\begin{aligned} \phi_{k+1}(\mathbf{u}) &= \phi_k(\mathbf{u}) - \nu_k\sigma_k f^* + \nu_k\sigma_k f(\mathbf{v}_k) + \nu_k\sigma_k \langle \nabla f(\mathbf{v}_k), \mathbf{u} - \mathbf{v}_k \rangle \\ &\quad + \frac{\nu_k\sigma_k\mu}{2} \|\mathbf{u} - \mathbf{v}_k\|^2, \end{aligned}$$

261 with $\phi_0(\mathbf{u}) = c_0 + \sigma_0\mu \|\mathbf{u} - \mathbf{w}_0\|^2 / 2$ for $c_0 = f(\mathbf{u}_0) - f^* - \mu\varepsilon^2/4$ and $\mathbf{w}_0 \in \Omega$. Then,
 262 for all $k \in \mathbb{N}$, the function ϕ_k preserves the following canonical form,

263 (4.9)
$$\phi_k(\mathbf{u}) = c_k + \frac{\sigma_k\mu}{2} \|\mathbf{u} - \mathbf{w}_k\|^2,$$

264 where $\{c_k\}$ is a sequence of real numbers and $\{\mathbf{w}_k\}$ is defined recursively by (4.5).

265 Proof. We first prove that $\nabla^2\phi_k = \sigma_k\mu I$ for all $k \in \mathbb{N}$ by induction. It is evident
 266 that $\nabla^2\phi_0 = \sigma_0\mu I$. Now we assume that $\nabla^2\phi_k = \sigma_k\mu I$ for some k . Then relationships
 267 (4.7) and (4.8) imply that

268
$$\nabla^2\phi_{k+1} = \nabla^2\phi_k + \nu_k\sigma_k\mu I = \sigma_k\mu I + \nu_k\sigma_k\mu I = \sigma_{k+1}\mu I.$$

269 Thus, we know that $\nabla^2\phi_k = \sigma_k\mu I$ for all $k \in \mathbb{N}$, which, in turn, justifies the canonical
 270 form of ϕ_k in (4.9).

271 Next, by combining two relationships (4.8) and (4.9) together, we can obtain that

272
$$\begin{aligned} \phi_{k+1}(\mathbf{u}) &= c_k + \frac{\sigma_k\mu}{2} \|\mathbf{u} - \mathbf{w}_k\|^2 - \nu_k\sigma_k f^* + \nu_k\sigma_k f(\mathbf{v}_k) \\ &\quad + \nu_k\sigma_k \langle \nabla f(\mathbf{v}_k), \mathbf{u} - \mathbf{v}_k \rangle + \frac{\nu_k\sigma_k\mu}{2} \|\mathbf{u} - \mathbf{v}_k\|^2. \end{aligned}$$

273 Since \mathbf{w}_{k+1} is a global minimizer of ϕ_{k+1} over \mathbb{R}^n , the first-order optimality condition
 274 yields that

275
$$\begin{aligned} 0 &= \nabla\phi_{k+1}(\mathbf{w}_{k+1}) = \sigma_k\mu(\mathbf{w}_{k+1} - \mathbf{w}_k) + \nu_k\sigma_k\nabla f(\mathbf{v}_k) + \nu_k\sigma_k\mu(\mathbf{w}_{k+1} - \mathbf{v}_k) \\ &= (1 + \nu_k)\sigma_k\mu\mathbf{w}_{k+1} - \sigma_k\mu\mathbf{w}_k - \nu_k\sigma_k\mu\mathbf{v}_k + \nu_k\sigma_k\nabla f(\mathbf{v}_k), \end{aligned}$$

276 from which the closed-form expression of \mathbf{w}_{k+1} in (4.5) can be derived. The proof is
 277 completed. \square

278 The following lemma characterizes the relationship between the objective function
 279 of problem (1.1) and the estimating sequences.

280 LEMMA 4.4. *Let σ_k and $\{\phi_k\}$ be the sequences defined in Lemma 4.3. Then we
 281 have*

282 (4.10)
$$\phi_k(\mathbf{u}) \leq \sigma_k(f(\mathbf{u}) - f^*) + \phi_0(\mathbf{u}),$$

283 for all $\mathbf{u} \in \Omega$ and $k \in \mathbb{N}$.

284 *Proof.* We prove that $\{\phi_k\}$ and $\{\sigma_k\}$ satisfy relationship (4.10) by induction. It
 285 is obvious that (4.10) holds for $k = 0$ since $f(\mathbf{u}) \geq f^*$ for any $\mathbf{u} \in \Omega$. Now we assume
 286 that (4.10) holds for some $k \in \mathbb{N}$. It follows from the strong convexity of f that

$$287 \quad f(\mathbf{u}) \geq f(\mathbf{v}_k) + \langle \nabla f(\mathbf{v}_k), \mathbf{u} - \mathbf{v}_k \rangle + \frac{\mu}{2} \|\mathbf{u} - \mathbf{v}_k\|^2,$$

288 for all $\mathbf{u} \in \Omega$. Then substituting the above relationship into (4.8) leads to that

$$\begin{aligned} 289 \quad \phi_{k+1}(\mathbf{u}) &\leq \phi_k(\mathbf{u}) - \nu_k \sigma_k f^* + \nu_k \sigma_k f(\mathbf{u}) \\ &\leq \sigma_k(f(\mathbf{u}) - f^*) + \phi_0(\mathbf{u}) + \nu_k \sigma_k(f(\mathbf{u}) - f^*) \\ &= \sigma_{k+1}(f(\mathbf{u}) - f^*) + \phi_0(\mathbf{u}), \end{aligned}$$

290 which indicates that (4.10) also holds for $k + 1$. We complete the proof. \square

291 Next, we proceed to show that the function value error of Algorithm 3 is controlled
 292 by the estimating sequences.

293 PROPOSITION 4.5. *Let $\{\sigma_k\}$ and $\{\phi_k\}$ be the sequences defined in Lemma 4.3.
 294 Then the sequence $\{\mathbf{u}_k\}$ generated by Algorithm 3 satisfies*

$$295 \quad (4.11) \quad f(\mathbf{u}_k) - f^* \leq \frac{1}{\sigma_k} \phi_0(\mathbf{u}^*) + \frac{\mu \varepsilon^2}{4},$$

296 for all $k \in \mathbb{N}$.

297 *Proof.* Let $\phi_k^* := \min_{\mathbf{u} \in \Omega} \phi_k(\mathbf{u})$. We first prove by induction that

$$298 \quad (4.12) \quad \sigma_k \left(f(\mathbf{u}_k) - f^* - \frac{\mu \varepsilon^2}{4} \right) \leq \phi_k^*,$$

299 for any $k \in \mathbb{N}$. It is clear that (4.12) holds for $k = 0$ since $\sigma_0 = 1$ and $\phi_0^* = \phi_0(\mathbf{w}_0) =$
 300 $f(\mathbf{u}_0) - f^* - \mu \varepsilon^2 / 4$. Now we assume that (4.12) holds for some $k \in \mathbb{N}$ and investigate
 301 the situation for $k + 1$.

302 From the canonical form (4.9), it follows that ϕ_k is a strongly convex function
 303 and $\Pi_\Omega(\mathbf{w}_k) = \arg \min_{\mathbf{u} \in \Omega} \phi_k(\mathbf{u})$. By invoking the result of [11, Corollary 2.2.1], we
 304 have

$$\begin{aligned} 305 \quad \phi_k(\mathbf{u}) &\geq \phi_k^* + \frac{\sigma_k \mu}{2} \|\mathbf{u} - \Pi_\Omega(\mathbf{w}_k)\|^2 \\ &\geq \sigma_k \left(f(\mathbf{u}_k) - f^* - \frac{\mu \varepsilon^2}{4} \right) + \frac{\sigma_k \mu}{2} \|\mathbf{u} - \Pi_\Omega(\mathbf{w}_k)\|^2, \end{aligned}$$

306 for all $\mathbf{u} \in \Omega$. Then relationship (4.8) yields that

$$\begin{aligned} 307 \quad \phi_{k+1}(\mathbf{u}) &\geq \sigma_k \left(f(\mathbf{u}_k) - f^* - \frac{\mu \varepsilon^2}{4} \right) + \frac{\sigma_k \mu}{2} \|\mathbf{u} - \Pi_\Omega(\mathbf{w}_k)\|^2 - \nu_k \sigma_k f^* \\ &\quad + \nu_k \sigma_k f(\mathbf{v}_k) + \nu_k \sigma_k \langle \nabla f(\mathbf{v}_k), \mathbf{u} - \mathbf{v}_k \rangle + \frac{\nu_k \sigma_k \mu}{2} \|\mathbf{u} - \mathbf{v}_k\|^2 \\ &\geq \sigma_{k+1} (f(\mathbf{v}_k) - f^*) - \frac{\sigma_k \mu \varepsilon^2}{4} + \langle \nabla f(\mathbf{v}_k), \sigma_k \mathbf{u}_k - \sigma_{k+1} \mathbf{v}_k \rangle \\ &\quad + \nu_k \sigma_k \langle \nabla f(\mathbf{v}_k), \mathbf{u} \rangle + \frac{\sigma_k \mu}{2} \|\mathbf{u} - \Pi_\Omega(\mathbf{w}_k)\|^2 \\ &= \sigma_{k+1} (f(\mathbf{v}_k) - f^*) - \frac{\sigma_k \mu \varepsilon^2}{4} + \nu_k \sigma_k \langle \nabla f(\mathbf{v}_k), \mathbf{u} - \Pi_\Omega(\mathbf{w}_k) \rangle \\ &\quad + \frac{\sigma_k \mu}{2} \|\mathbf{u} - \Pi_\Omega(\mathbf{w}_k)\|^2, \end{aligned}$$

308 where the second inequality comes from the strong convexity of f and (4.7), and the
 309 last equality holds due to the definition of \mathbf{v}_k in (4.1). According to the definition of
 310 \mathbf{z}_k in (4.2), we can obtain that

$$\begin{aligned} & \nu_k \sigma_k \langle \nabla f(\mathbf{v}_k), \mathbf{u} - \Pi_\Omega(\mathbf{w}_k) \rangle + \frac{\sigma_k \mu}{2} \|\mathbf{u} - \Pi_\Omega(\mathbf{w}_k)\|^2 \\ &= \frac{\sigma_k \mu}{2} \left\| \mathbf{u} - \left(\Pi_\Omega(\mathbf{w}_k) - \frac{\nu_k}{\mu} \nabla f(\mathbf{v}_k) \right) \right\|^2 - \frac{\nu_k^2 \sigma_k}{2\mu} \|\nabla f(\mathbf{v}_k)\|^2 \\ &\geq \frac{\sigma_k \mu}{2} \left\| \mathbf{z}_k - \left(\Pi_\Omega(\mathbf{w}_k) - \frac{\nu_k}{\mu} \nabla f(\mathbf{v}_k) \right) \right\|^2 - \frac{\nu_k^2 \sigma_k}{2\mu} \|\nabla f(\mathbf{v}_k)\|^2 \\ &= \nu_k \sigma_k \langle \nabla f(\mathbf{v}_k), \mathbf{z}_k - \Pi_\Omega(\mathbf{w}_k) \rangle + \frac{\sigma_k \mu}{2} \|\mathbf{z}_k - \Pi_\Omega(\mathbf{w}_k)\|^2. \end{aligned}$$

312 As a result, it holds that

$$\begin{aligned} & \phi_{k+1}(\mathbf{u}) \geq \sigma_{k+1} (f(\mathbf{v}_k) - f^*) - \frac{\sigma_k \mu \varepsilon^2}{4} + \nu_k \sigma_k \langle \nabla f(\mathbf{v}_k), \mathbf{z}_k - \Pi_\Omega(\mathbf{w}_k) \rangle \\ & \quad + \frac{\sigma_k \mu}{2} \|\mathbf{z}_k - \Pi_\Omega(\mathbf{w}_k)\|^2, \end{aligned} \tag{4.13}$$

314 for all $\mathbf{u} \in \Omega$. From the definitions of \mathbf{v}_k and \mathbf{u}_{k+1} in (4.1) and (4.3), it can be derived
 315 that $\mathbf{z}_k - \Pi_\Omega(\mathbf{w}_k) = (\mathbf{u}_{k+1} - \mathbf{v}_k)/\eta_k$. Substituting this relationship into (4.13) and
 316 taking $\mathbf{u} = \Pi_\Omega(\mathbf{w}_{k+1})$, we arrive at

$$\frac{\phi_{k+1}^*}{\sigma_{k+1}} \geq f(\mathbf{v}_k) - f^* + \langle \nabla f(\mathbf{v}_k), \mathbf{u}_{k+1} - \mathbf{v}_k \rangle + \frac{\mu}{2\nu_k^2} \|\mathbf{u}_{k+1} - \mathbf{v}_k\|^2 - \frac{(1 - \eta_k)\mu \varepsilon^2}{4},$$

318 which together with the line-search condition (4.4) implies that

$$\frac{\phi_{k+1}^*}{\sigma_{k+1}} \geq f(\mathbf{u}_{k+1}) - f^* - \frac{\eta_k \mu \varepsilon^2}{4} - \frac{(1 - \eta_k)\mu \varepsilon^2}{4} = f(\mathbf{u}_{k+1}) - f^* - \frac{\mu \varepsilon^2}{4}.$$

320 Therefore, relationship (4.12) also holds for $k + 1$.

321 Finally, by collecting two relationships (4.10) and (4.12) together, we can obtain
 322 that

$$\begin{aligned} & \sigma_k \left(f(\mathbf{u}_k) - f^* - \frac{\mu \varepsilon^2}{4} \right) \leq \min_{\mathbf{u} \in \Omega} \phi_k(\mathbf{u}) \leq \min_{\mathbf{u} \in \Omega} \{ \sigma_k(f(\mathbf{u}) - f^*) + \phi_0(\mathbf{u}) \} \\ & \leq \sigma_k(f(\mathbf{u}^*) - f^*) + \phi_0(\mathbf{u}^*) \\ & = \phi_0(\mathbf{u}^*), \end{aligned}$$

324 which completes the proof. \square

325 With the above preparatory results in place, we are now in a position to establish
 326 the iteration complexity of Algorithm 3, as articulated in the theorem below.

327 **THEOREM 4.6.** *Let $\varepsilon \in (0, 1)$ be a sufficiently small constant. Then after at most*

$$O \left(\log \left(\frac{1}{\varepsilon} \right) \frac{M^{(1+\hat{\alpha})/(1+3\hat{\alpha})}}{\varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})}} \right)$$

329 iterations, Algorithm 3 will reach an iterate \mathbf{u}_k satisfying $\|\mathbf{u}_k - \mathbf{u}^*\| \leq \varepsilon$.

330 *Proof.* In view of relationship (4.6), the number of line-search steps j_k in (4.4)
 331 satisfies

$$332 \quad \frac{\mu}{\nu_k^2} \eta_k^{(1-\hat{\alpha})/(1+\hat{\alpha})} \leq 2 \max_{i \in [m]} \left\{ \left[\frac{2(1-\alpha_i)}{\mu(1+\alpha_i)\varepsilon^2} \right]^{(1-\alpha_i)/(1+\alpha_i)} L_i^{2/(1+\alpha_i)} \right\} \leq \frac{2M}{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}},$$

333 where $M > 0$ is a constant defined in (2.1). Since $\eta_k = \nu_k/(1+\nu_k) \geq \nu_k/2$, we arrive
 334 at

$$335 \quad (4.14) \quad \frac{\nu_k^2}{\mu} \geq \frac{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}}{2M} \eta_k^{(1-\hat{\alpha})/(1+\hat{\alpha})} \geq \frac{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}}{2^{2/(1+\hat{\alpha})} M} \nu_k^{(1-\hat{\alpha})/(1+\hat{\alpha})}.$$

336 Let $\omega > 0$ be a constant defined as

$$337 \quad \omega = \frac{1}{2^{2/(1+3\hat{\alpha})}} \left[\frac{\mu}{M} \right]^{(1+\hat{\alpha})/(1+3\hat{\alpha})}.$$

338 Then it follows from relationship (4.14) that

$$339 \quad (4.15) \quad \nu_k \geq \omega \varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})},$$

340 which further infers that

$$341 \quad \sigma_{k+1} = (1 + \nu_k) \sigma_k \geq \left(1 + \omega \varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})} \right) \sigma_k.$$

342 Applying the above inequality for k times recursively yields that

$$343 \quad \sigma_k \geq \left(1 + \omega \varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})} \right)^k.$$

344 As a direct consequence of (2.5) and (4.11), we can show that

$$345 \quad \begin{aligned} \|\mathbf{u}_k - \mathbf{u}^*\|^2 &\leq \frac{2}{\mu} (f(\mathbf{u}_k) - f^*) \leq \frac{2}{\mu} \left(\frac{1}{\sigma_k} \phi_0(\mathbf{u}^*) + \frac{\mu \varepsilon^2}{4} \right) \\ &\leq \chi \left(1 + \omega \varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})} \right)^{-k} + \frac{\varepsilon^2}{2}, \end{aligned}$$

346 where $\chi = 2(f(\mathbf{u}_0) - f^*)/\mu + \|\mathbf{u}_0 - \mathbf{u}^*\|^2 > 0$ is a constant. Let K_ε^* be the small-
 347 est iteration number k such that $\|\mathbf{u}_k - \mathbf{u}^*\| \leq \varepsilon$. By solving the inequality $\chi(1 + \omega \varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})})^{-k} \leq \varepsilon^2/2$, we have

$$349 \quad K_\varepsilon^* \leq \log \left(\frac{\sqrt{2\chi}}{\varepsilon} \right) \frac{2}{\log(1 + \omega \varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})})} \leq \log \left(\frac{\sqrt{2\chi}}{\varepsilon} \right) \frac{4}{\omega \varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})}}.$$

350 The proof is completed. \square

351 The complexity bound established in Theorem 4.6 is markedly lower than those
 352 presented in Theorems 2.2 and 3.1, thereby highlighting the acceleration effect at-
 353 tained by Algorithm 3. Finally, we demonstrate that the number of line-search steps
 354 required by Algorithm 3 is also $O(\log(\varepsilon^{-1}) \varepsilon^{2(\hat{\alpha}-1)/(1+3\hat{\alpha})})$.

355 COROLLARY 4.7. *Let $\varepsilon \in (0, 1)$ be a sufficiently small constant. Then, to achieve
 356 an iterate \mathbf{u}_k satisfying $\|\mathbf{u}_k - \mathbf{u}^*\| \leq \varepsilon$, Algorithm 3 requires at most*

$$357 \quad O \left(\log \left(\frac{1}{\varepsilon} \right) \frac{M^{(1+\hat{\alpha})/(1+3\hat{\alpha})}}{\varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})}} \right)$$

358 line-search steps.

359 *Proof.* It follows from relationship (4.14) that

$$360 \quad \rho_{k+1} = 2^{j_k} \rho_k = \frac{\mu}{\nu_k^2} \leq \frac{2^{2/(1+\hat{\alpha})} M}{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}} \left[\frac{1}{\nu_k} \right]^{(1-\hat{\alpha})/(1+\hat{\alpha})},$$

361 which together with (4.15) implies that

$$362 \quad \rho_{k+1} \leq \frac{2^{2/(1+\hat{\alpha})} M}{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}} \left[\frac{1}{\omega \varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})}} \right]^{(1-\hat{\alpha})/(1+\hat{\alpha})} = \frac{2^{2/(1+\hat{\alpha})} M}{\omega^{(1-\hat{\alpha})/(1+\hat{\alpha})} \varepsilon^{4(1-\hat{\alpha})/(1+3\hat{\alpha})}}.$$

363 Let N_k be the total number of line-search steps after k iterations in Algorithm 3. In
364 view of (3.4), we have

$$365 \quad N_k \leq k + 1 + \log \left(\frac{2^{2/(1+\hat{\alpha})} M}{\omega^{(1-\hat{\alpha})/(1+\hat{\alpha})} \varepsilon^{4(1-\hat{\alpha})/(1+3\hat{\alpha})}} \right) - \log \rho_0 \\ \leq k + \frac{4(1-\hat{\alpha})}{1+3\hat{\alpha}} \log \left(\frac{1}{\varepsilon} \right) + \log \left(\frac{2^{2/(1+\hat{\alpha})} M}{\omega^{(1-\hat{\alpha})/(1+\hat{\alpha})} \rho_0} \right) + 1.$$

366 Consequently, Theorem 4.6 indicates that the total number of line-search steps in
367 Algorithm 3 is at most $O(\log(\varepsilon^{-1})\varepsilon^{2(\hat{\alpha}-1)/(1+3\hat{\alpha})})$, which completes the proof. \square

368 *Remark 4.8.* By an analogous argument, we can also prove that Algorithm 3
369 requires at most $O(\log(\varepsilon^{-1})\varepsilon^{(\hat{\alpha}-1)/(1+3\hat{\alpha})})$ iterations to generate an iterate \mathbf{u}_k such
370 that $f(\mathbf{u}_k) - f^* \leq \varepsilon$ for problem (1.1). Very recently, Doikov [8] has shown that,
371 in the case $m = 2$, where f_1 is a convex function with a Hölder continuous gradient
372 and $f_2(\mathbf{u}) = \|\mathbf{u}\|^2$, the lower complexity bound for first-order methods is precisely
373 $O(\log(\varepsilon^{-1})\varepsilon^{(\hat{\alpha}-1)/(1+3\hat{\alpha})})$ in terms of function value accuracy. This finding confirms
374 that Algorithm 3 achieves the optimal iteration complexity.

375 **5. Numerical Experiments.** Preliminary numerical results are presented in
376 this section to provide additional insights into the performance guarantees of the al-
377 gorithms proposed in this paper. We aim to elucidate that the final error attained
378 by the algorithm is influenced by both the stepsize and the Hölder exponent. The
379 numerical experiments are conducted using Julia [4] (version 1.12) on an Apple Mac-
380 intosh Mini with an M2 processor, 8 performance cores, and 32GB of memory. We
381 have placed the Julia codes in the GitHub repository (https://github.com/ctkelley/Grad_Des_CKW.jl) with instructions for reproducing the figures.

383 **5.1. Two-dimensional PDE with a non-Lipschitz term.** Hölder continu-
384 ous gradients arise naturally in partial differential equations (PDEs) involving non-
385 Lipschitz nonlinearity [3, 14]. In this subsection, we introduce a numerical example
386 from [3]. This problem is to solve the following two-dimensional PDE,

$$387 \quad (5.1) \quad \mathcal{F}(u) = -\Delta u + \gamma u_+^\alpha = 0,$$

388 where $\alpha \in (0, 1)$, $\gamma > 0$ is a constant and $u_+ = \max\{u, 0\}$. Discretizing (5.1) with the
389 standard five point difference scheme [9] leads to the following nonlinear system,

$$390 \quad (5.2) \quad \mathbf{F}(\mathbf{u}) = \mathbf{A}\mathbf{u} + \gamma \mathbf{u}_+^\alpha - \mathbf{b} = 0,$$

391 where $\mathbf{A} \in \mathbb{R}^{n \times n}$ is the discretization of $-\Delta$ with zero boundary conditions, $\mathbf{b} \in$
392 \mathbb{R}^n encodes the boundary conditions, and $\mathbf{u}_+^\alpha = \max\{\mathbf{u}, 0\}^\alpha$ is understood as a
393 component-wise operation.

394 We now modify the above problem to enable direct computation of errors in the
 395 iterations. To this end, we follow [13, Example 4.4] and take as the exact solution the
 396 function

$$397 \quad u^*(x, y) = \left(\frac{3r - 1}{2} \right)^2 \max \left\{ 0, r - \frac{1}{3} \right\},$$

398 where $r = \sqrt{x^2 + y^2}$. We enforce the following boundary conditions,

$$399 \quad u(x, 1) = u^*(x, 1), \quad u(x, 0) = u^*(x, 0), \quad u(1, y) = u^*(1, y), \quad u(0, y) = u^*(0, y),$$

400 for $0 < x, y < 1$. And these conditions are encoded into \mathbf{b} . Then our modified
 401 equation is

$$402 \quad (5.3) \quad \mathbf{F}(\mathbf{u}) - \mathbf{c}^* = 0,$$

403 where $\mathbf{c}^* = \mathbf{F}(\mathbf{u}^*)$. The nonlinear system (5.3) corresponds to the optimality condition
 404 of the following problem,

$$405 \quad (5.4) \quad \min_{\mathbf{u} \in \mathbb{R}^n} f(\mathbf{u}) = \frac{1}{2} \mathbf{u}^\top \mathbf{A} \mathbf{u} + \frac{\gamma}{1 + \alpha} \mathbf{e}^\top \mathbf{u}_+^{1+\alpha} - (\mathbf{b} + \mathbf{c}^*)^\top \mathbf{u},$$

406 where $\mathbf{e} \in \mathbb{R}^n$ is the vector of all ones.

407 The optimization model (5.4) is a special instance of problem (1.1) with $\Omega = \mathbb{R}^n$,
 408 $m = 2$,

$$409 \quad f_1(\mathbf{u}) = \mathbf{u}^\top \mathbf{A} \mathbf{u} - 2(\mathbf{b} + \mathbf{c}^*)^\top \mathbf{u}, \quad \text{and} \quad f_2(\mathbf{u}) = \frac{2\gamma}{1 + \alpha} \mathbf{e}^\top \mathbf{u}_+^{1+\alpha}.$$

410 It is clear that, ∇f_1 is Lipschitz continuous with the corresponding Lipschitz constant
 411 $L_1 = 2 \|\mathbf{A}\|$, and ∇f_2 is Hölder continuous with the Hölder exponent α and $L_2 = 2\gamma$
 412 from

$$413 \quad \|\nabla f_2(\mathbf{u}) - \nabla f_2(\mathbf{v})\| = 2\gamma \|\mathbf{u}_+^\alpha - \mathbf{v}_+^\alpha\| \leq 2\gamma \|\mathbf{u} - \mathbf{v}\|^\alpha,$$

414 for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Moreover, the function $f = (f_1 + f_2)/2$ is $\lambda(\mathbf{A})$ -strongly convex,
 415 where $\lambda(\mathbf{A})$ is the smallest eigenvalue of the symmetric positive definite matrix \mathbf{A} .
 416 Let \mathbf{u}^* be the vector obtained by evaluating u^* at the interior grid points. Then \mathbf{u}^*
 417 serves as the unique global minimizer of problem (5.4).

418 In the subsequent experiments, we use the solution of $\mathbf{A}\mathbf{u}_0 = -\mathbf{b}$ as the initial
 419 iterate. This is the discretization of Laplace's equation with the boundary conditions.
 420 In this way, we ensure that the entire iteration satisfies the boundary conditions.
 421 Unless otherwise specified, we set the spatial mesh width as $h = 2^{-4}$ in this subsection.
 422 The dimension of the discretized problem is $n = (h^{-1} - 1)^2$.

423 **5.1.1. Numerical results of Algorithm 1.** In the first experiment, we scrutinize
 424 the performance of Algorithm 1 under different stepsizes for problem (5.4) with
 425 $\alpha = 0.5$ and $\gamma = 0.5$. Specifically, Algorithm 1 is tested for stepsizes of the form
 426 $\tau = \tau_0 h^2$, where τ_0 is taken from the set $\{0.2, 0.1, 0.05, 0.01\}$. The corresponding nu-
 427 mercial results, presented in Figure 1(a), illustrate the decay of the distance between
 428 the iterates and the global minimizer over iterations. It can be observed that, a larger
 429 stepsize facilitates a more rapid descent in the early stage of iterations, albeit at the

430 expense of a greater asymptotic error. This phenomenon corroborates our theoretical
431 predictions.

432 In the second experiment, we vary the Hölder exponent α over the values in
433 $\{0.1, 0.2, 0.5, 0.8\}$, while fixing $\tau_0 = 0.01$. Figure 1(b) similarly tracks the decay of
434 the distance to the global minimizer over iterations. It is evident that, as the value
435 of α decreases, the final error attained by Algorithm 1 increases under the same
436 stepsize. Therefore, the associated optimization problems become increasingly ill-
437 conditioned and thus more challenging to solve for smaller values of α . These findings
438 offer empirical support for our theoretical analysis.

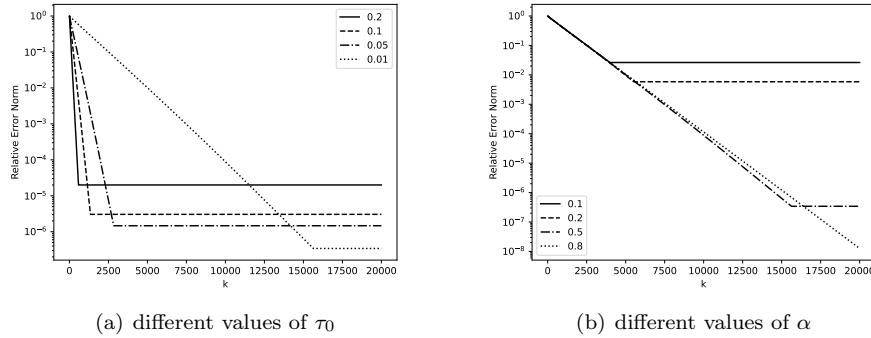


FIG. 1. Numerical performance of Algorithm 1 for problem (5.4) with $h = 2^{-4}$.

439 We now repeat the experiment with $h = 2^{-5}$, so we reduce the mesh width by a
440 factor of two and increase the norm of \mathbf{A} by a factor of four. As one would expect
441 the stepsize must decrease by a factor of four for stability.

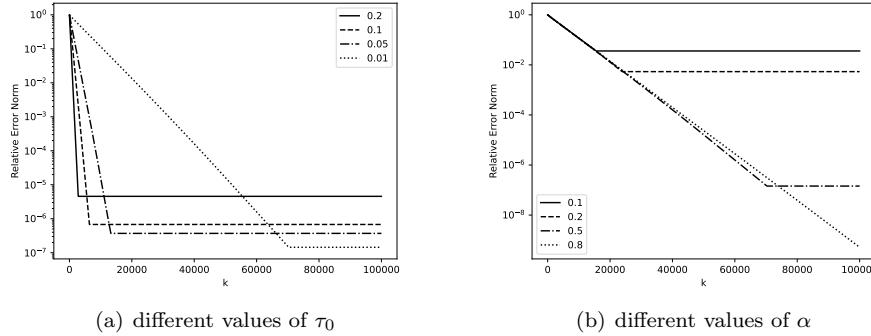


FIG. 2. Numerical performance of Algorithm 1 for problem (5.4) with $h = 2^{-5}$.

442 **5.1.2. Numerical results of Algorithm 2.** We repeat the study in subsec-
443 tion 5.1.1 for Algorithm 2 by varying the values of the Hölder exponent α . We set
444 $\varepsilon = 10^{-6}$ and $\mu = 2\pi^2$ in Algorithm 2, which is a lower estimate for the smallest
445 eigenvalue of \mathbf{A} . The stepsize is initialized to $0.1h^2$ in the line-search procedure. The
446 corresponding numerical results are depicted in Figure 3. Comparing Figure 3 to Fig-

ure 2(b) shows the benefits of the line-search procedure in Algorithm 2, which does not need to manually adjust the value of τ_0 to converge for a given value of ε .

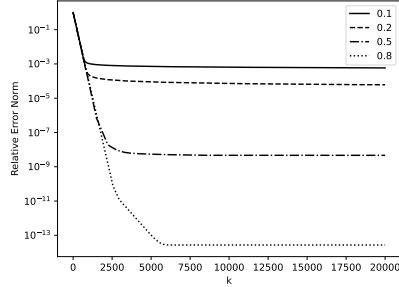


FIG. 3. Numerical performance of Algorithm 2 for problem (5.4) with different values of α .

5.1.3. Numerical results of Algorithm 3. We report the numerical performance of Algorithm 3 on two experiments. Guided by the observation in Remark 4.2, we test Algorithm 3 with a fixed stepsize $\nu = \tau_0 h^2$. In the first example, we use the values for τ_0 from Figure 1. In this way we can directly compare the performance of Algorithm 3 with that of Algorithm 1. The corresponding results, shown in Figure 4, are poor. The reason for this is that we are not exploiting the ability of Algorithm 3 to use larger stepsizes. In the second example, we consider larger values for τ_0 in Figure 5(a) and set $\tau_0 = 20$ in Figure 5(b). The convergence is much better in all cases. The hardest case ($\alpha = 0.1$) has very irregular convergence in the terminal phase of iterations.

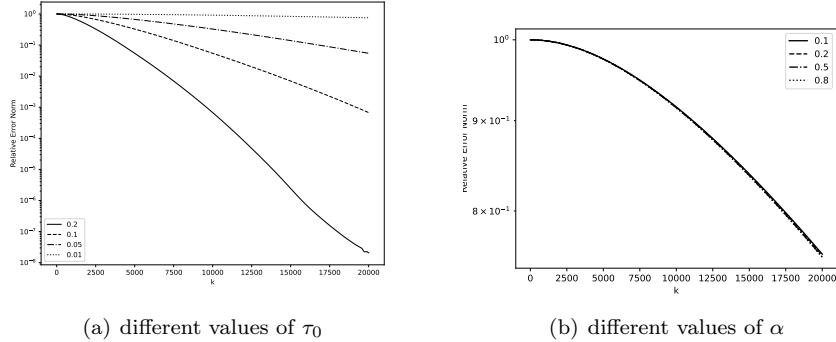


FIG. 4. Numerical performance of Algorithm 3 for problem (5.4) with smaller stepsizes.

5.1.4. Stepsize and termination. It is useful to look at the values of stepsizes from Remark 4.2. We note that for problem (5.4), $M = O(h^{-2})$. We are using $\hat{\alpha} = \alpha$ and neglecting constants in the estimate. We tabulate in Table 1 the value of

$$(5.5) \quad \nu = h^{2p_1} \varepsilon^{p_2}$$

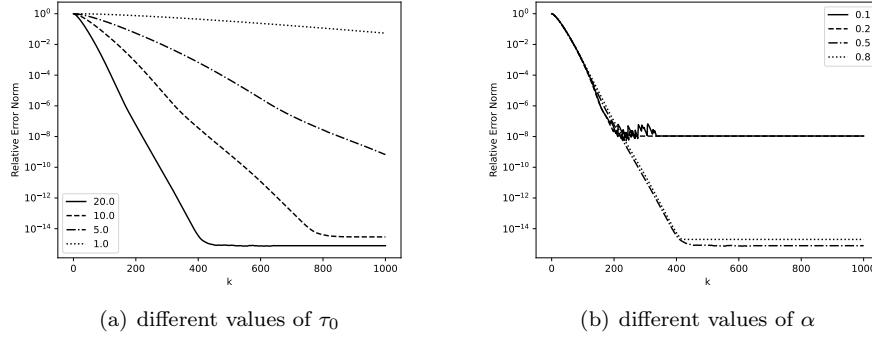


FIG. 5. Numerical performance of Algorithm 3 for problem (5.4) with larger stepsizes.

463 where

464
$$p_1 = (1 + \alpha)/(1 + 3\alpha), \text{ and } p_2 = 2(1 - \alpha)/(1 + 3\alpha).$$

465 Contrasting the values of ν in Table 1 to the value of $20h^2 \approx 0.08$, we can see that the
466 stepsize estimate from (5.5) is very pessimistic. For smaller values of α , the predicted
467 stepsize is too small to be useful in practice.TABLE 1
Representative values of ν .

$\alpha \setminus \varepsilon$	1.00e-02	1.00e-03	1.00e-05	1.00e-08
0.1	1.56e-05	6.43e-07	1.09e-09	7.68e-14
0.2	1.56e-04	1.56e-05	1.56e-07	1.56e-10
0.5	5.69e-03	2.26e-03	3.59e-04	2.26e-05
0.8	3.09e-02	2.36e-02	1.37e-02	6.08e-03

468 Next, we consider the complexity bound

469
$$O\left(\log\left(\frac{1}{\varepsilon}\right) M^{p_1} \varepsilon^{-p_2}\right).$$

470 In Table 2 we present the predicted number of iterations. The estimates are pessimistic
471 except for the larger values of α when compared to the findings we report in Figure 5.TABLE 2
Representative iteration numbers.

$\alpha \setminus \varepsilon$	1.00e-02	1.00e-03	1.00e-05	1.00e-08
0.1	4.26e+05	1.55e+07	1.52e+10	3.46e+14
0.2	4.25e+04	6.38e+05	1.06e+08	1.70e+11
0.5	1.17e+03	4.40e+03	4.63e+04	1.17e+06
0.8	2.15e+02	4.23e+02	1.21e+03	4.37e+03

472 Finally, we consider termination of the iteration. In problem (5.4), we know the
473 exact solution and can evaluate the algorithms in terms of the error. In practice we

cannot do that and must use the gradient norm as a surrogate for the error. While this is standard for smooth optimization, it could be a problem when the gradient is not Lipschitz continuous. We illustrate this in Figure 6, where we compare the gradient norm with the error for the case $\tau_0 = 20$ using Algorithm 3. The numerical results in Figure 6 indicate that, when the gradient norm stops decreasing, the error has also stopped decreasing. However, the gradient norm is larger than the error norm, especially when the error is small, which is consistent with Hölder continuity.

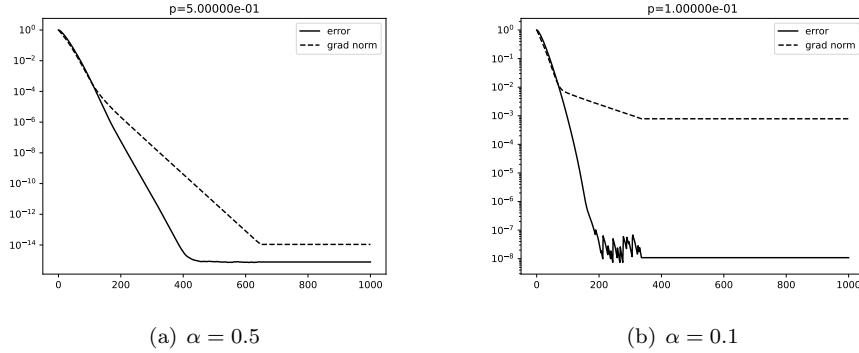


FIG. 6. Gradient and error norms for problem (5.4).

5.2. Semi-linear elliptic problem with a constraint. We consider a second numerical example motivated by a semi-linear elliptic problem with a constraint on the solution in a certain set [14]. Let

$$(5.6) \quad \mathcal{H}(u) = -\Delta u + \delta|u|^\alpha \text{sign}(u) - |u|^{p-1}u,$$

on $D = (0, 1)^2$ with the boundary condition $u(x, y) = 0.5 - \sin(x)\sin(y)$ on ∂D . Here, $\alpha \in (0, 1)$, $p > 1$, and $\delta > p/\alpha$ are three constants. We consider the variational inequality that is to find $u^* \in [-1, 1]$ such that

$$\mathcal{H}(u^*)(u - u^*) \geq 0,$$

for any $u \in [-1, 1]$. This problem is equivalent to the following nonlinear equation,

$$(5.7) \quad 0 = \mathcal{F}(u) := \begin{cases} \mathcal{H}(u), & \text{if } u - \mathcal{H}(u) \in [-1, 1], \\ u - 1, & \text{if } u - \mathcal{H}(u) \geq 1, \\ u + 1, & \text{otherwise.} \end{cases}$$

By discretizing (5.6) with the standard five point difference scheme [9], problem (5.7) leads to the following system of nonlinear equations,

$$(5.8) \quad 0 = \mathbf{F}(\mathbf{u}) := \mathbf{u} - \Pi_{\mathbf{U}} \left(\mathbf{u} - \theta \left(\mathbf{A}\mathbf{u} + \delta |\mathbf{u}|^\alpha \text{sign}(\mathbf{u}) - |\mathbf{u}|^{p-1} \mathbf{u} - \mathbf{b} \right) \right),$$

where $\mathbf{U} = [-1, 1]^n$, $\theta > 0$ is a constant, $\mathbf{A} \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix, and $\mathbf{b} \in \mathbb{R}^n$ encodes the boundary conditions. Note that (5.8) is the optimality condition of the following problem,

$$(5.9) \quad \min_{\mathbf{u} \in \mathbf{U}} f(\mathbf{u}) := \frac{1}{2} \mathbf{u}^\top \mathbf{A} \mathbf{u} + \frac{\delta}{1+\alpha} \mathbf{e}^\top |\mathbf{u}|^{1+\alpha} - \frac{1}{1+p} \mathbf{e}^\top |\mathbf{u}|^{1+p} - \mathbf{b}^\top \mathbf{u}.$$

498 The Hessian matrix of f at \mathbf{u} with $\mathbf{u}_i \neq 0$ ($i = 1, \dots, n$) has the form

$$499 \quad \nabla^2 f(\mathbf{u}) = \mathbf{A} + \delta\alpha \text{Diag}(|\mathbf{u}|^{\alpha-1}) - p \text{Diag}(|\mathbf{u}|^{p-1}),$$

500 Since $\delta > p/\alpha$, $\nabla^2 f(\mathbf{u})$ is symmetric positive definite for any $\mathbf{u} \in \mathbf{U}$ with $\mathbf{u}_i \neq 0$ ($i = 1, \dots, n$). Hence, the function f is μ -strongly convex in \mathbf{U} with $\mu = \lambda(\mathbf{A})$ and the system (5.8) has a unique solution in \mathbf{U} . The optimization model (5.9) is a special instance of problem (1.1) with $\Omega = \mathbf{U}$, $m = 2$,

$$504 \quad f_1(\mathbf{u}) = \mathbf{u}^\top \mathbf{A} \mathbf{u} - 2\mathbf{b}^\top \mathbf{u} - \frac{2}{1+p} \mathbf{e}^\top |\mathbf{u}|^{1+p}, \text{ and } f_2(\mathbf{u}) = \frac{2\delta}{1+\alpha} \mathbf{e}^\top |\mathbf{u}|^{1+\alpha}.$$

505 It is clear that Assumption 1.1 (ii) holds with $\alpha_1 = 1$, $L_1 = 2\|\mathbf{A}\| + 2p$, $\alpha_2 = \alpha$, and
506 $L_2 = 2\delta\alpha$.

507 In this example we do not have an analytic solution, so we only plot the residual
508 norms $\|\mathbf{F}(\mathbf{u})\|$. We compare the performance of Algorithm 1 and Algorithm 3 on
509 problem (5.9). The stepsizes of Algorithm 1 and Algorithm 3 are set to $0.1h^2$ and
510 $20h^2$, respectively. In our examples, we vary α over the values in $\{0.1, 0.2, 0.5, 0.8\}$,
511 while fixing $p = 1.5$ and $\delta = 20$. The numerical results are provided in Figure 7. It
512 can be observed that, Algorithm 3 exhibits a faster convergence rate, benefiting from
513 the use of a larger stepsize.

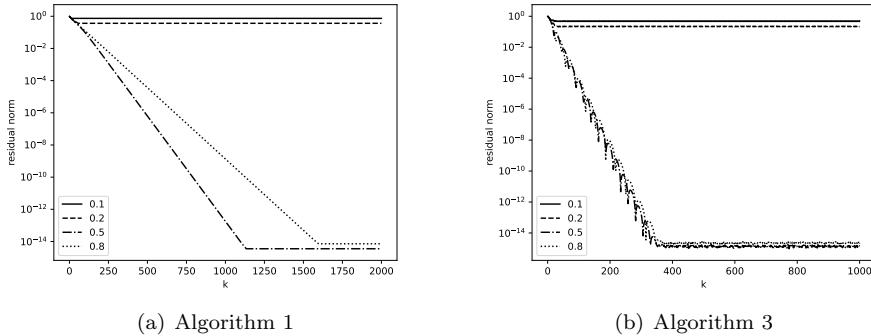


FIG. 7. Numerical performance of Algorithm 1 and Algorithm 3 for problem (5.9) with different values of α .

514 **6. Conclusion.** In this paper, we consider a class of strongly convex constrained
515 optimization problems of the form (1.1). Example 1.2 shows that although each com-
516 ponent function f_i of the objective function f admits a Hölder continuous gradient
517 with an component $\alpha_i \in (0, 1]$, the gradient of f is not necessarily Hölder continuous.
518 To establish the iteration complexity of the projected gradient descent methods for
519 this class of problems, we use the parameter $\hat{\alpha} = \min_{i \in [m]} \alpha_i$ to determine the com-
520 plexity bound. Algorithm 1 is a new version of projected gradient method for prob-
521 lem (1.1) with an appropriately fixed stepsize. Theorem 2.2 shows that Algorithm 1
522 can find an iterate in the feasible set Ω with a distance to the global minimizer less
523 than ε at most $O(\log(\varepsilon^{-1})\varepsilon^{2(\hat{\alpha}-1)/(1+\hat{\alpha})})$ iterations. This recovers the classical com-
524 plexity result when $\hat{\alpha} = 1$ and reveals the additional difficulty imposed by the weaker
525 smoothness of the objective function for $\hat{\alpha} < 1$. Algorithm 2 is a modification of Algo-
526 rithm 1 for problems where the parameters α_i and L_i are difficult to estimate for the

527 stepsize. In Algorithm 3, the stepsize is updated by the universal scheme at each iteration,
 528 which improves the complexity bound to $O(\log(\varepsilon^{-1})\varepsilon^{2(\hat{\alpha}-1)/(1+3\hat{\alpha})})$. Numerical
 529 experiments are conducted to validate our theoretical findings, demonstrating the ex-
 530 pected behavior of projected gradient descent methods under different stepsizes and
 531 Hölder exponents. These results offer new insights into the performance guarantees
 532 of the classic projected gradient descent methods for a broader class of optimization
 533 problems with non-Lipschitz gradients.

534

REFERENCES

- 535 [1] G. ALEFELD AND X. CHEN, *A regularized projection method for complementarity problems with*
 536 *non-Lipschitzian functions*, Math. Comput., 77 (2008), pp. 379–395.
- 537 [2] J.-C. BARITAUX, K. HASSLER, AND M. UNSER, *An efficient numerical method for general L_p*
 538 *regularization in fluorescence molecular tomography*, IEEE Trans. Med. Imaging, 29 (2010),
 539 pp. 1075–1087.
- 540 [3] J. W. BARRETT AND R. M. SHANAHAN, *Finite element approximation of a model reaction-*
 541 *diffusion problem with a non-Lipschitz nonlinearity*, Numer. Math., 59 (1991), pp. 217–
 542 242.
- 543 [4] J. BEZANSON, A. EDELMAN, S. KARPINSKI, AND V. B. SHAH, *Julia: A fresh approach to nu-*
 544 *merical computing*, SIAM Rev., 59 (2017), pp. 65–98.
- 545 [5] L. S. BORGES, F. S. V. BAZÁN, AND L. BEDIN, *A projection-based algorithm for ℓ_2 - ℓ_p Tikhonov*
 546 *regularization*, Math. Methods Appl. Sci., 41 (2018), pp. 5919–5938.
- 547 [6] X. CHEN, C. T. KELLEY, AND L. WANG, *A new complexity result for strongly convex optimiza-*
 548 *tion with locally α -Hölder continuous gradients*, arXiv:2505.03506v1, (2025).
- 549 [7] O. DEVOLDER, F. GLINEUR, AND Y. NESTEROV, *First-order methods of smooth convex opti-*
 550 *mization with inexact oracle*, Math. Program., 146 (2014), pp. 37–75.
- 551 [8] N. DOIKOV, *Lower complexity bounds for minimizing regularized functions*, Optim. Lett.,
 552 (2025), pp. 1–20.
- 553 [9] R. J. LEVEQUE, *Finite Difference Methods for Ordinary and Partial Differential Equations:*
 554 *Steady-State and Time-Dependent Problems*, Society for Industrial and Applied Mathe-
 555 *matics*, 2007.
- 556 [10] Y. NESTEROV, *Universal gradient methods for convex optimization problems*, Math. Program.,
 557 152 (2015), pp. 381–404.
- 558 [11] Y. NESTEROV, *Lectures on Convex Optimization*, Springer, 2018.
- 559 [12] Y. NESTEROV, *Universal complexity bounds for universal gradient methods in nonlinear opti-*
 560 *mization*, arXiv:2509.20902, (2025).
- 561 [13] X. QU, W. BIAN, AND X. CHEN, *An extra gradient Anderson-accelerated algorithm for pseu-*
 562 *domonotone variational inequalities*, Math. Comput., (2025).
- 563 [14] M. TANG, *Uniqueness of bound states to $\Delta u - u + |u|^{p-1}u = 0$ in \mathbb{R}^n , $n \geq 3$* , Invent. Math.,
 564 (2025), pp. 1–47.
- 565 [15] M. YASHTINI, *On the global convergence rate of the gradient descent method for functions with*
 566 *Hölder continuous gradients*, Optim. Lett., 10 (2016), pp. 1361–1370.