

# COMPLEXITY OF PROJECTED GRADIENT METHODS FOR STRONGLY CONVEX OPTIMIZATION WITH HÖLDER CONTINUOUS GRADIENT TERMS\*

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**Abstract.** This paper studies the complexity of projected gradient descent methods for a class of strongly convex constrained optimization problems where the objective function is expressed as a summation of  $m$  component functions, each possessing a gradient that is Hölder continuous with an exponent  $\alpha_i \in (0, 1]$ . Under this formulation, the gradient of the objective function may fail to be globally Hölder continuous, thereby rendering existing complexity results inapplicable to this class of problems. Our theoretical analysis reveals that, in this setting, the complexity of projected gradient methods is determined by  $\hat{\alpha} = \min_{i \in \{1, \dots, m\}} \alpha_i$ . We first prove that, with an appropriately fixed stepsize, the complexity bound for finding an approximate minimizer with a distance to the true minimizer less than  $\varepsilon$  is  $O(\log(\varepsilon^{-1})\varepsilon^{2(\hat{\alpha}-1)/(1+\hat{\alpha})})$ , which extends the well-known complexity result for  $\hat{\alpha} = 1$ . Next we show that the complexity bound can be improved to  $O(\log(\varepsilon^{-1})\varepsilon^{2(\hat{\alpha}-1)/(1+3\hat{\alpha})})$  if the stepsize is updated by the universal scheme. We illustrate our complexity results by numerical examples arising from elliptic equations with a non-Lipschitz term.

**Key words.** projected gradient descent, complexity, Hölder continuity

**MSC codes.** 90C25, 65L05, 65Y20

**1. Introduction.** Given a closed and convex set  $\Omega \subseteq \mathbb{R}^n$ , this paper considers the following optimization problem,

$$(1.1) \quad \min_{\mathbf{u} \in \Omega} f(\mathbf{u}) := \frac{1}{m} \sum_{i=1}^m f_i(\mathbf{u}),$$

where the objective function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  satisfies the following blanket assumption.

ASSUMPTION 1.1.

(i) The function  $f$  is  $\mu$ -strongly convex with a parameter  $\mu > 0$  on  $\Omega$ , that is,

$$f(\mathbf{u}) \geq f(\mathbf{v}) + \langle \nabla f(\mathbf{v}), \mathbf{u} - \mathbf{v} \rangle + \frac{\mu}{2} \|\mathbf{u} - \mathbf{v}\|^2,$$

for all  $\mathbf{u}, \mathbf{v} \in \Omega$ .

(ii) For each  $i \in [m] := \{1, 2, \dots, m\}$ , the function  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$  is continuously differentiable and the gradient  $\nabla f_i$  is (globally) Hölder continuous with an exponent  $\alpha_i \in (0, 1]$  on  $\Omega$ , namely, there exists a constant  $L_i > 0$  such that

$$(1.2) \quad \|\nabla f_i(\mathbf{u}) - \nabla f_i(\mathbf{v})\| \leq L_i \|\mathbf{u} - \mathbf{v}\|^{\alpha_i},$$

for all  $\mathbf{u}, \mathbf{v} \in \Omega$ .

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Here,  $\|\cdot\|$  is the  $\ell_2$  norm and  $\langle \cdot, \cdot \rangle$  is the inner product on  $\mathbb{R}^n$ . We also denote by  $\mathbf{u}^* \in \Omega$  and  $f^* = f(\mathbf{u}^*)$  the global minimizer and the optimal value of problem (1.1), respectively.

Suppose that each  $\nabla f_i$  is Lipschitz continuous, which corresponds to condition (1.2) with  $\alpha_i = 1$  for all  $\mathbf{u}, \mathbf{v} \in \Omega$ . Then  $\nabla f$  is also Lipschitz continuous and the associated Lipschitz constant is  $L = \sum_{i=1}^m L_i/m$ . Let  $\Pi_\Omega(\cdot)$  be the projection operator onto the set  $\Omega$ . It is well known that the classical projected gradient descent method

$$(1.3) \quad \mathbf{u}_{k+1} = \Pi_\Omega(\mathbf{u}_k - \tau \nabla f(\mathbf{u}_k)),$$

with any initial point  $\mathbf{u}_0 \in \mathbb{R}^n$  and the stepsize  $\tau \in (0, 2/(\mu + L)]$ , achieves a linear rate of convergence [11, Theorem 2.2.14] as follows,

$$\|\mathbf{u}_k - \mathbf{u}^*\| \leq (1 - \mu\tau)^k \|\mathbf{u}_0 - \mathbf{u}^*\|.$$

Therefore, for a given  $\varepsilon > 0$ , method (1.3) is guaranteed to find a point  $\mathbf{u}_k \in \Omega$  satisfying  $\|\mathbf{u}_k - \mathbf{u}^*\| \leq \varepsilon$  after at most  $O(\log(\varepsilon^{-1}))$  iterations. Unfortunately, this analysis fails if there exists at least one index  $i \in [m]$  such that  $\alpha_i < 1$ . We explain the failure of the convergence of method (1.3) to  $\mathbf{u}^*$  by the following example.

*Example 1.2.* [6, Example 1] Consider the following univariate optimization problem on  $\Omega = \mathbb{R}$ ,

$$(1.4) \quad \min_{x \in \mathbb{R}} f(x) = \frac{1}{2}x^2 + \frac{2}{3}|x|^{3/2},$$

which is a special instance of problem (1.1) with  $f_1(x) = x^2/2$  and  $f_2(x) = 2|x|^{3/2}/3$ . It is easy to see that the global minimizer is  $x^* = 0$ . Method (1.3) with the fixed stepsize  $\tau > 0$  starting from  $x_0 \neq 0$  proceeds as follows,

$$x_{k+1} = x_k - \tau \nabla f(x_k) = (1 - \tau)x_k - \tau \text{sign}(x_k) |x_k|^{1/2},$$

where  $\text{sign}(x) = 1$  if  $x > 0$ , 0 if  $x = 0$ , and  $-1$  otherwise. A straightforward verification reveals that

$$|x_{k+1}|^2 - |x_k|^2 = -\tau(2 - \tau)|x_k|^2 - 2\tau(1 - \tau)|x_k|^{3/2} + \tau^2|x_k|.$$

It is evident that, when  $|x_k|$  is sufficiently small, the last term in the right-hand side becomes dominant, resulting in that  $|x_{k+1}|^2 - |x_k|^2 \geq 0$ . Therefore, the distance to the global minimizer ceases to decrease once it achieves a certain level.

Moreover, in [6] we show that  $\nabla f$  is locally, but not globally, Hölder continuous. In fact, from

$$|\nabla f(|h|) - \nabla f(0)| = |h| + |h|^{1/2} = \left(|h|^{1-\alpha} + |h|^{1/2-\alpha}\right) |h|^\alpha,$$

we can obtain that,  $|h|^{1-\alpha} \rightarrow \infty$  when  $\alpha \in (0, 1)$  and  $|h| \rightarrow \infty$ , while  $|h|^{1/2-\alpha} \rightarrow \infty$  when  $\alpha = 1$  and  $|h| \rightarrow 0$ . Therefore,  $\nabla f$  cannot be globally Hölder continuous for all  $\alpha \in (0, 1]$ .

On the other hand, problem (1.4) satisfies all the conditions in Assumption 1.1. It is clear that  $f$  is strongly convex. In addition, we have

$$|\nabla f_1(x) - \nabla f_1(y)| = |x - y|,$$

70 and

$$71 \quad |\nabla f_2(x) - \nabla f_2(y)| = |\text{sign}(x)|x|^{1/2} - \text{sign}(y)|y|^{1/2}| \leq \sqrt{2}|x - y|^{1/2},$$

72 for all  $x, y \in \mathbb{R}$ .

73 This simple example demonstrates that, in problem (1.1), a function  $f$  expressed  
74 as a sum of component functions  $f_i$ , each endowed with a Hölder continuous gradient,  
75 may itself fail to possess a Hölder continuous gradient. This phenomenon, initially  
76 observed in our previous work [6], was later revisited and further highlighted by  
77 Nesterov (see [12, Example 1]).

78 Since  $\nabla f$  may not be globally Hölder continuous, most existing complexity results  
79 are inapplicable to problem (1.1). For the special case where  $m = 1$ , namely,  $\nabla f$  is  
80 globally Hölder continuous with an exponent  $\alpha \in (0, 1]$ , Devolder et al. [7] presented  
81 the following bound for method (1.3),

$$82 \quad f(\hat{\mathbf{u}}_N) - f(\mathbf{u}^*) \leq K(N) := \frac{L_\alpha \|\mathbf{u}_0 - \mathbf{u}^*\|^{1+\alpha}}{1+\alpha} \left( \frac{2}{N} \right)^{\frac{1+\alpha}{2}},$$

83 where  $L_\alpha$  is the Hölder constant and  $\hat{\mathbf{u}}_N = \sum_{k=1}^N \mathbf{u}_k / N$ . In the strongly convex case,  
84 (51) in [7] comes to

$$85 \quad \|\hat{\mathbf{u}}_N - \mathbf{u}^*\|^2 \leq \frac{2}{\mu} K(N),$$

86 which implies that finding an  $N$  average of iterations  $\hat{\mathbf{u}}_N$  satisfying  $\|\hat{\mathbf{u}}_N - \mathbf{u}^*\| \leq \varepsilon$   
87 requires  $O(\varepsilon^{-4/(1+\alpha)})$  iterations.

88 The contribution of this paper is to provide new complexity results for the pro-  
89 jected gradient descent methods to solve problem (1.1), which are dictated by the  
90 parameter  $\hat{\alpha} = \min_{i \in [m]} \alpha_i \in (0, 1]$ . We first show that, with an appropriately fixed  
91 stepsize, the complexity bound for finding an iterate with a distance to the global  
92 minimizer less than  $\varepsilon$  is  $O(\log(\varepsilon^{-1})\varepsilon^{2(\hat{\alpha}-1)/(1+\hat{\alpha})})$ , which extends the well-known com-  
93 plexity result for  $\hat{\alpha} = 1$ . Next, we demonstrate that this complexity bound can be  
94 improved to  $O(\log(\varepsilon^{-1})\varepsilon^{2(\hat{\alpha}-1)/(1+3\hat{\alpha})})$  if the stepsize is updated at each iteration us-  
95 ing the universal scheme. Even in the special case where  $m = 1$ , our complexity bound  
96 is at least  $O(\varepsilon^{-1})$  lower than (51) in [7]. For example, when  $\hat{\alpha} = 1/2$ , our bound is  
97  $O(\log(\varepsilon^{-1})\varepsilon^{-2/5})$  but (51) in [7] is  $O(\varepsilon^{-8/3})$ .

98 Our study is motivated by elliptic equations with a non-Lipschitz term [3, 14],  
99 complementarity problems [1, 13], and optimization problems with an  $\ell_p$ -norm ( $1 <$   
100  $p < 2$ ) regularization term [2, 5]. We illustrate our complexity results by two numerical  
101 examples arising from elliptic equations with a non-Lipschitz term in section 5, after  
102 we present complexity of projected gradient methods with fixed stepsizes and updated  
103 stepsizes in sections 2 to 4, respectively.

## 104 2. Vanilla Projected Gradient Descent Method with a Fixed Stepsize.

105 In this section, we attempt to employ the vanilla projected gradient descent method  
106 (1.3) with a fixed stepsize to solve problem (1.1), whose complexity bound is also  
107 provided. Example 1.2 illustrates that the projected gradient descent method (1.3)  
108 with a fixed stepsize will experience stagnation before reaching the global minimizer.

109 To obtain an approximate solution to problem (1.1), it is necessary to choose  
110 a sufficiently small stepsize  $\tau$  in the projected gradient descent method (1.3), the

111 magnitude of which depends on the desired level of accuracy. Let  $M > 0$  be a  
 112 constant defined as

$$113 \quad (2.1) \quad M = \max_{i \in [m]} \left\{ \left[ \frac{2(1 - \alpha_i)}{\mu(1 + \alpha_i)} \right]^{(1 - \alpha_i)/(1 + \alpha_i)} L_i^{2/(1 + \alpha_i)} \right\},$$

114 with the convention  $0^0 = 1$ . We select a specific stepsize  $\tau = \varepsilon^{2(1 - \hat{\alpha})/(1 + \hat{\alpha})}/M$  in  
 115 the projected gradient descent method, whose complete framework is presented in  
 116 Algorithm 1. Two sequences  $\{\mathbf{v}_k\}$  and  $\{\mathbf{u}_k\}$  are maintained in Algorithm 1, where  
 117  $\mathbf{v}_k$  is generated by the projected gradient descent method and  $\mathbf{u}_k$  corresponds to the  
 118 iterate achieving the smallest objective function value among the first  $k$  iterations.

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**Algorithm 1:** Projected Gradient Descent Method (PGDM).

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**Input:**  $\varepsilon > 0$ .

Initialize  $\mathbf{u}_0 = \mathbf{v}_0 \in \Omega$ .

Choose the stepsize  $\tau = \varepsilon^{2(1 - \hat{\alpha})/(1 + \hat{\alpha})}/M$ .

**for**  $k = 0, 1, 2, \dots$  **do**

    Compute

$$\mathbf{v}_{k+1} = \Pi_{\Omega}(\mathbf{v}_k - \tau \nabla f(\mathbf{v}_k)).$$

    Set

$$\mathbf{u}_{k+1} = \begin{cases} \mathbf{v}_{k+1}, & \text{if } f(\mathbf{v}_{k+1}) \leq f(\mathbf{u}_k), \\ \mathbf{u}_k, & \text{otherwise.} \end{cases}$$

**Output:**  $\mathbf{u}_{k+1}$ .

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119 Our subsequent analysis is based on the inexact oracle [7] derived from the Hölder  
 120 continuity condition of gradients, which is generalized to problem (1.1) and demon-  
 121 strated in the following proposition.

122 **PROPOSITION 2.1.** *Let  $\delta > 0$  and*

$$123 \quad \rho \geq \max_{i \in [m]} \left\{ \left[ \frac{1 - \alpha_i}{(1 + \alpha_i)\delta} \right]^{(1 - \alpha_i)/(1 + \alpha_i)} L_i^{2/(1 + \alpha_i)} \right\}.$$

124 *Then for all  $\mathbf{u}, \mathbf{v} \in \Omega$ , we have*

$$125 \quad f(\mathbf{v}) \leq f(\mathbf{u}) + \langle \nabla f(\mathbf{u}), \mathbf{v} - \mathbf{u} \rangle + \frac{\rho}{2} \|\mathbf{v} - \mathbf{u}\|^2 + \frac{\delta}{2}.$$

126 *Proof.* Since  $\nabla f_i$  is Hölder continuous with an exponent  $\alpha_i$ , we can obtain from  
 127 [15, Lemma 1] that

$$128 \quad f_i(\mathbf{v}) \leq f_i(\mathbf{u}) + \langle \nabla f_i(\mathbf{u}), \mathbf{v} - \mathbf{u} \rangle + \frac{L_i}{1 + \alpha_i} \|\mathbf{v} - \mathbf{u}\|^{1 + \alpha_i},$$

129 for all  $\mathbf{u}, \mathbf{v} \in \Omega$ . Then, for each  $i$ , it follows from [10, Lemma 2] that

$$130 \quad f_i(\mathbf{v}) \leq f_i(\mathbf{u}) + \langle \nabla f_i(\mathbf{u}), \mathbf{v} - \mathbf{u} \rangle + \frac{\rho}{2} \|\mathbf{v} - \mathbf{u}\|^2 + \frac{\delta}{2}.$$

Summing the above relationship over  $i \in [m]$ , we immediately arrive at the assertion of this proposition. The proof is completed.  $\square$

Now, we are able to derive the complexity bound of Algorithm 1 in the following theorem.

**THEOREM 2.2.** *Let  $\varepsilon \in (0, 1)$  be a sufficiently small constant. Then after at most*

$$O\left(\log\left(\frac{M^{(1+\hat{\alpha})/4}}{\varepsilon}\right)\frac{M}{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}}\right)$$

*iterations, Algorithm 1 will find an iterate  $\mathbf{u}_k \in \Omega$  satisfying  $\|\mathbf{u}_k - \mathbf{u}^*\| \leq \varepsilon$ .*

*Proof.* In view of Proposition 2.1, we take

$$\rho = \frac{1}{\tau} = \frac{M}{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}} \geq \max_{i \in [m]} \left\{ \left[ \frac{2(1-\alpha_i)}{\mu(1+\alpha_i)\varepsilon^2} \right]^{(1-\alpha_i)/(1+\alpha_i)} L_i^{2/(1+\alpha_i)} \right\}.$$

Then it holds that

$$f(\mathbf{v}_{k+1}) \leq f(\mathbf{v}_k) + \langle \nabla f(\mathbf{v}_k), \mathbf{v}_{k+1} - \mathbf{v}_k \rangle + \frac{1}{2\tau} \|\mathbf{v}_{k+1} - \mathbf{v}_k\|^2 + \frac{\mu\varepsilon^2}{4},$$

which, after a suitable rearrangement, can be equivalently written as

$$(2.2) \quad \langle \nabla f(\mathbf{v}_k), \mathbf{v}_k - \mathbf{v}_{k+1} \rangle \leq f(\mathbf{v}_k) - f(\mathbf{v}_{k+1}) + \frac{\mu\varepsilon^2}{4} + \frac{1}{2\tau} \|\mathbf{v}_{k+1} - \mathbf{v}_k\|^2.$$

Recall that  $f^* = f(\mathbf{u}^*)$ . By virtue of the strong convexity of  $f$ , we can obtain that

$$(2.3) \quad \langle \nabla f(\mathbf{v}_k), \mathbf{u}^* - \mathbf{v}_k \rangle \leq f^* - f(\mathbf{v}_k) - \frac{\mu}{2} \|\mathbf{v}_k - \mathbf{u}^*\|^2.$$

The optimality condition of the projection problem defining  $\mathbf{v}_{k+1}$  yields that

$$\langle \mathbf{v}_{k+1} - \mathbf{v}_k + \tau \nabla f(\mathbf{v}_k), \mathbf{u} - \mathbf{v}_{k+1} \rangle \geq 0,$$

for all  $\mathbf{u} \in \Omega$ . Upon taking  $\mathbf{u} = \mathbf{u}^*$ , we have

$$\begin{aligned} \langle \mathbf{v}_{k+1} - \mathbf{v}_k, \mathbf{v}_{k+1} - \mathbf{u}^* \rangle &\leq \tau \langle \nabla f(\mathbf{v}_k), \mathbf{u}^* - \mathbf{v}_{k+1} \rangle \\ &= \tau \langle \nabla f(\mathbf{v}_k), \mathbf{u}^* - \mathbf{v}_k \rangle + \tau \langle \nabla f(\mathbf{v}_k), \mathbf{v}_k - \mathbf{v}_{k+1} \rangle, \end{aligned}$$

which together with (2.2) and (2.3) implies that

$$\begin{aligned} \langle \mathbf{v}_{k+1} - \mathbf{v}_k, \mathbf{v}_{k+1} - \mathbf{u}^* \rangle &\leq \tau \left( f^* - f(\mathbf{v}_{k+1}) + \frac{\mu\varepsilon^2}{4} \right) - \frac{\mu\tau}{2} \|\mathbf{v}_k - \mathbf{u}^*\|^2 \\ &\quad + \frac{1}{2} \|\mathbf{v}_{k+1} - \mathbf{v}_k\|^2. \end{aligned}$$

Moreover, it can be readily verified that

$$\begin{aligned} \|\mathbf{v}_{k+1} - \mathbf{u}^*\|^2 &= \|\mathbf{v}_{k+1} - \mathbf{v}_k + \mathbf{v}_k - \mathbf{u}^*\|^2 \\ &= \|\mathbf{v}_k - \mathbf{u}^*\|^2 + 2 \langle \mathbf{v}_{k+1} - \mathbf{v}_k, \mathbf{v}_k - \mathbf{u}^* \rangle + \|\mathbf{v}_{k+1} - \mathbf{v}_k\|^2 \\ &= \|\mathbf{v}_k - \mathbf{u}^*\|^2 + 2 \langle \mathbf{v}_{k+1} - \mathbf{v}_k, \mathbf{v}_{k+1} - \mathbf{u}^* \rangle - \|\mathbf{v}_{k+1} - \mathbf{v}_k\|^2. \end{aligned}$$

Collecting the above two relationships together, we arrive at

$$\|\mathbf{v}_{k+1} - \mathbf{u}^*\|^2 \leq (1 - \mu\tau) \|\mathbf{v}_k - \mathbf{u}^*\|^2 + 2\tau \left( f^* - f(\mathbf{v}_{k+1}) + \frac{\mu\varepsilon^2}{4} \right).$$

From the construction of  $\mathbf{u}_k$  in Algorithm 1, it then follows that  $f(\mathbf{v}_l) \geq f(\mathbf{u}_k)$  for all  $l \in \{1, 2, \dots, k\}$ . Let  $C_k = \sum_{l=1}^k (1 - \mu\tau)^{l-1}$  be a constant. Applying the above relationship recursively for  $k$  times leads to that

$$\begin{aligned} \|\mathbf{v}_k - \mathbf{u}^*\|^2 &\leq (1 - \mu\tau)^k \|\mathbf{u}_0 - \mathbf{u}^*\|^2 + 2\tau \sum_{l=1}^k (1 - \mu\tau)^{l-1} \left( f^* - f(\mathbf{v}_l) + \frac{\mu\varepsilon^2}{4} \right) \\ &\leq (1 - \mu\tau)^k \|\mathbf{u}_0 - \mathbf{u}^*\|^2 + 2\tau \left( f^* - f(\mathbf{u}_k) + \frac{\mu\varepsilon^2}{4} \right) C_k, \end{aligned}$$

which together with  $\|\mathbf{v}_k - \mathbf{u}^*\| \geq 0$  and  $C_k \geq 1$  implies that

$$f(\mathbf{u}_k) - f^* \leq \frac{(1 - \mu\tau)^k}{2\tau C_k} \|\mathbf{u}_0 - \mathbf{u}^*\|^2 + \frac{\mu\varepsilon^2}{4} \leq \frac{(1 - \mu\tau)^k}{2\tau} \|\mathbf{u}_0 - \mathbf{u}^*\|^2 + \frac{\mu\varepsilon^2}{4}.$$

According to the strong convexity of  $f$  and the optimality condition of problem (1.1), we have

$$(2.5) \quad f(\mathbf{u}_k) - f^* \geq \langle \nabla f(\mathbf{u}^*), \mathbf{u}_k - \mathbf{u}^* \rangle + \frac{\mu}{2} \|\mathbf{u}_k - \mathbf{u}^*\|^2 \geq \frac{\mu}{2} \|\mathbf{u}_k - \mathbf{u}^*\|^2.$$

Hence, it holds that

$$\begin{aligned} \|\mathbf{u}_k - \mathbf{u}^*\|^2 &\leq \frac{2}{\mu} (f(\mathbf{u}_k) - f^*) \leq \frac{(1 - \mu\tau)^k}{\mu\tau} \|\mathbf{u}_0 - \mathbf{u}^*\|^2 + \frac{\varepsilon^2}{2} \\ &\leq \frac{M \|\mathbf{u}_0 - \mathbf{u}^*\|^2}{\mu\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}} \left( 1 - \frac{\mu}{M} \varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})} \right)^k + \frac{\varepsilon^2}{2}. \end{aligned}$$

We denote by  $K_\varepsilon^*$  the smallest iteration number  $k$  such that  $\|\mathbf{u}_k - \mathbf{u}^*\| \leq \varepsilon$ . Then solving the inequality  $M \|\mathbf{u}_0 - \mathbf{u}^*\|^2 \varepsilon^{-2(1-\hat{\alpha})/(1+\hat{\alpha})} (1 - \mu\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}/M)^k / \mu \leq \varepsilon^2/2$  indicates that

$$\begin{aligned} K_\varepsilon^* &\leq \frac{4 \log((2M \|\mathbf{u}_0 - \mathbf{u}^*\|^2 / \mu)^{(1+\hat{\alpha})/4} / \varepsilon)}{-\log(1 - \mu\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}/M)(1 + \hat{\alpha})} \\ &\leq \frac{4M \log((2M \|\mathbf{u}_0 - \mathbf{u}^*\|^2 / \mu)^{(1+\hat{\alpha})/4} / \varepsilon)}{\mu(1 + \hat{\alpha})\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}}. \end{aligned}$$

The proof is completed.  $\square$

Theorem 2.2 demonstrates that the iteration complexity of Algorithm 1 with a fixed stepsize is  $O(\log(\varepsilon^{-1})\varepsilon^{2(\hat{\alpha}-1)/(1+\hat{\alpha})})$  for problem (1.1). This complexity result generalizes the classical linear convergence when  $\hat{\alpha} = 1$ , which highlights the performance degradation incurred by non-Lipschitz gradients.

**3. Universal Primal Gradient Method.** The fixed stepsize  $\tau$  chosen in Algorithm 1 depends on the parameters  $\alpha_i$  and  $L_i$  for all  $i \in [m]$ , which are often unknown and hard to estimate in practice. To address this issue, we adopt the universal primal gradient method (UPGM) proposed by Nesterov [10] to solve problem (1.1). This

**Algorithm 2:** Universal Primal Gradient Method (UPGM).**Input:**  $\varepsilon > 0$ .Initialize  $\mathbf{u}_0 = \mathbf{v}_0 \in \Omega$  and  $\rho_0 > 0$ .**for**  $k = 0, 1, 2, \dots$  **do**    **for**  $j_k = 0, 1, 2, \dots$  **do**

Compute

$$\mathbf{v}_{k+1} = \Pi_{\Omega} \left( \mathbf{v}_k - \frac{1}{2^{j_k} \rho_k} \nabla f(\mathbf{v}_k) \right).$$

**If**  $\mathbf{v}_{k+1}$  satisfies the following line-search condition,

$$(3.1) \quad \begin{aligned} f(\mathbf{v}_{k+1}) &\leq f(\mathbf{v}_k) + \langle \nabla f(\mathbf{v}_k), \mathbf{v}_{k+1} - \mathbf{v}_k \rangle \\ &\quad + \frac{2^{j_k} \rho_k}{2} \|\mathbf{v}_{k+1} - \mathbf{v}_k\|^2 + \frac{\mu \varepsilon^2}{4}, \end{aligned}$$

**then** break.Update  $\rho_{k+1} = 2^{j_k} \rho_k$ .

Set

$$\mathbf{u}_{k+1} = \begin{cases} \mathbf{v}_{k+1}, & \text{if } f(\mathbf{v}_{k+1}) \leq f(\mathbf{u}_k), \\ \mathbf{u}_k, & \text{otherwise.} \end{cases}$$

**Output:**  $\mathbf{u}_{k+1}$ .

180 method incorporates a line-search procedure to adaptively determine the stepsize at  
 181 each iteration, and its overall framework is outlined in Algorithm 2.

182 Next, we establish the iteration complexity of Algorithm 2, which remains on the  
 183 same order as that of the projected gradient descent method with a fixed stepsize.

184 **THEOREM 3.1.** *Let  $\varepsilon \in (0, 1)$  be a sufficiently small constant. Then after at most*

$$185 \quad O \left( \log \left( \frac{M^{(1+\hat{\alpha})/4}}{\varepsilon} \right) \frac{M}{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}} \right)$$

186 *iterations, Algorithm 2 will attain an iterate  $\mathbf{u}_k \in \Omega$  satisfying that  $\|\mathbf{u}_k - \mathbf{u}^*\| \leq \varepsilon$ .*

187 *Proof.* Obviously, there exists  $j_k \in \mathbb{N}$  such that

$$188 \quad 2^{j_k} \rho_k \geq \max_{i \in [m]} \left\{ \left[ \frac{2(1-\alpha_i)}{\mu(1+\alpha_i)\varepsilon^2} \right]^{(1-\alpha_i)/(1+\alpha_i)} L_i^{2/(1+\alpha_i)} \right\}.$$

189 By invoking the results of Proposition 2.1, we know that condition (3.1) is satisfied.

190 Hence, the line-search step in Algorithm 2 can be terminated after a finite number of  
 191 trials and the required number of trials  $j_k$  satisfies

$$192 \quad (3.2) \quad 2^{j_k} \rho_k \leq 2 \max_{i \in [m]} \left\{ \left[ \frac{2(1-\alpha_i)}{\mu(1+\alpha_i)\varepsilon^2} \right]^{(1-\alpha_i)/(1+\alpha_i)} L_i^{2/(1+\alpha_i)} \right\} \leq \frac{2M}{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}},$$

193 where  $M > 0$  is a constant defined in (2.1). Moreover, the line-search condition (3.1)

directly yields that

$$(3.3) \quad \langle \nabla f(\mathbf{v}_k), \mathbf{v}_k - \mathbf{v}_{k+1} \rangle \leq f(\mathbf{v}_k) - f(\mathbf{v}_{k+1}) + \frac{2^{j_k} \rho_k}{2} \|\mathbf{v}_{k+1} - \mathbf{v}_k\|^2 + \frac{\mu \varepsilon^2}{4}.$$

According to the optimality condition of the projection problem defining  $\mathbf{v}_{k+1}$ , we have

$$\left\langle \mathbf{v}_{k+1} - \mathbf{v}_k + \frac{1}{2^{j_k} \rho_k} \nabla f(\mathbf{v}_k), \mathbf{u}^* - \mathbf{v}_{k+1} \right\rangle \geq 0,$$

which further implies that

$$\begin{aligned} \langle \mathbf{v}_{k+1} - \mathbf{v}_k, \mathbf{v}_{k+1} - \mathbf{u}^* \rangle &\leq \frac{1}{2^{j_k} \rho_k} \langle \nabla f(\mathbf{v}_k), \mathbf{u}^* - \mathbf{v}_{k+1} \rangle \\ &\leq \frac{1}{2^{j_k} \rho_k} \langle \nabla f(\mathbf{v}_k), \mathbf{u}^* - \mathbf{v}_k \rangle + \frac{1}{2^{j_k} \rho_k} \langle \nabla f(\mathbf{v}_k), \mathbf{v}_k - \mathbf{v}_{k+1} \rangle. \end{aligned}$$

Substituting (2.3) and (3.3) into the above relationship leads to that

$$\begin{aligned} \langle \mathbf{v}_{k+1} - \mathbf{v}_k, \mathbf{v}_{k+1} - \mathbf{u}^* \rangle &\leq \frac{1}{2^{j_k} \rho_k} \left( f^* - f(\mathbf{v}_{k+1}) + \frac{\mu \varepsilon^2}{4} \right) \\ &\quad + \frac{1}{2} \|\mathbf{v}_{k+1} - \mathbf{v}_k\|^2 - \frac{\mu}{2^{j_k+1} \rho_k} \|\mathbf{v}_k - \mathbf{u}^*\|^2, \end{aligned}$$

Thus, it follows from relationship (2.4) that

$$\begin{aligned} \|\mathbf{v}_{k+1} - \mathbf{u}^*\|^2 &\leq \left( 1 - \frac{\mu}{2^{j_k} \rho_k} \right) \|\mathbf{v}_k - \mathbf{u}^*\|^2 + \frac{2}{2^{j_k} \rho_k} \left( f^* - f(\mathbf{v}_{k+1}) + \frac{\mu \varepsilon^2}{4} \right) \\ &\leq \left( 1 - \frac{\mu \varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}}{2M} \right) \|\mathbf{v}_k - \mathbf{u}^*\|^2 + \frac{2}{\rho_0} \left( f^* - f(\mathbf{v}_{k+1}) + \frac{\mu \varepsilon^2}{4} \right), \end{aligned}$$

where the last inequality comes from (3.2) and  $2^{j_k} \rho_k \geq \rho_0$ . The remaining part of the proof follows the same line of reasoning as that of Theorem 2.2 and is therefore omitted here for the sake of brevity.  $\square$

We end this section by estimating the total number of line-search steps required by Algorithm 2.

**COROLLARY 3.2.** *Let  $\varepsilon \in (0, 1)$  be a sufficiently small constant. Then Algorithm 2 requires at most*

$$O \left( \log \left( \frac{M^{(1+\hat{\alpha})/4}}{\varepsilon} \right) \frac{M}{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}} \right)$$

*line-search steps for the generated sequence  $\{\mathbf{u}_k\}$  to satisfy  $\|\mathbf{u}_k - \mathbf{u}^*\| \leq \varepsilon$ .*

*Proof.* Let  $N_k$  be the total number of line-search steps after  $k$  iterations in Algorithm 2. From the update rule  $\rho_{k+1} = 2^{j_k} \rho_k$ , we can obtain that  $j_k = \log \rho_{k+1} - \log \rho_k$ . Then a straightforward verification reveals that

$$(3.4) \quad N_k = \sum_{l=0}^k (j_l + 1) = k + 1 + \log \rho_{k+1} - \log \rho_0,$$



218 which together with relationship (3.2) implies that

$$\begin{aligned}
 219 \quad N_k &\leq k + 1 + \log \left( \frac{2M}{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}} \right) - \log \rho_0 \\
 &\leq k + \frac{2(1-\hat{\alpha})}{1+\hat{\alpha}} \log \left( \frac{1}{\varepsilon} \right) + \log \left( \frac{2M}{\rho_0} \right) + 1.
 \end{aligned}$$

220 By invoking the results of Theorem 3.1, we conclude that Algorithm 2 requires at  
 221 most  $O(\log(\varepsilon^{-1})\varepsilon^{2(\hat{\alpha}-1)/(1+\hat{\alpha})})$  line-search steps, which completes the proof.  $\square$

222 At each iteration of Algorithm 2, we evaluate both the function value and the  
 223 gradient at  $\mathbf{v}_k$ . In addition, an extra function evaluation at  $\mathbf{v}_{k+1,j_k}$  is involved during  
 224 each line-search step. Therefore, Theorem 3.1 and Corollary 3.2 together reveal that  
 225 the total number of function and gradient evaluations required by Algorithm 2 is  
 226  $O(\log(\varepsilon^{-1})\varepsilon^{2(\hat{\alpha}-1)/(1+\hat{\alpha})})$ .

227 **4. Universal Fast Gradient Method.** To obtain a sharper complexity bound,  
 228 we devise, in this section, a universal fast gradient method (UFGM) tailored to prob-  
 229 lem (1.1). The proposed scheme, summarized in Algorithm 3, exhibits slight but  
 230 essential differences from the algorithm introduced by Nesterov [10] to exploit the  
 231 strong convexity of the objective function.

232 The following lemma illustrates that the line-search process in (4.4) is well-defined,  
 233 which is guaranteed to terminate in a finite number of trials.

234 **LEMMA 4.1.** *There exists an integer  $j_k \in \mathbb{N}$  such that the line-search condition*  
 235 *(4.4) is satisfied in Algorithm 3.*

236 *Proof.* It follows from the definition of  $\eta_k$  and  $\nu_k \leq 1$  that

$$237 \quad \eta_k = \frac{\nu_k}{1 + \nu_k} \geq \frac{\nu_k}{2}, \quad \text{and} \quad \frac{\mu}{\nu_k^2} = 2^{j_k} \rho_k.$$

238 Recall that  $\hat{\alpha} = \min_{i \in [m]} \alpha_i \in (0, 1]$ . Then we have

$$\begin{aligned}
 \frac{\mu}{\nu_k^2} \eta_k^{(1-\hat{\alpha})/(1+\hat{\alpha})} &\geq \frac{2^{j_k} \rho_k}{2^{(1-\hat{\alpha})/(1+\hat{\alpha})}} \nu_k^{(1-\hat{\alpha})/(1+\hat{\alpha})} \\
 239 \quad &= \frac{2^{j_k} \rho_k}{2^{(1-\hat{\alpha})/(1+\hat{\alpha})}} \left[ \frac{\mu}{2^{j_k} \rho_k} \right]^{(1-\hat{\alpha})/(2(1+\hat{\alpha}))} \\
 &= \frac{\mu^{(1-\hat{\alpha})/(2(1+\hat{\alpha}))}}{2^{(1-\hat{\alpha})/(1+\hat{\alpha})}} [2^{j_k} \rho_k]^{(1+3\hat{\alpha})/(2(1+\hat{\alpha}))},
 \end{aligned}$$

240 where the first equality comes from the definition of  $\nu_k$ . Now it is clear that

$$241 \quad \frac{\mu}{\nu_k^2} \eta_k^{(1-\hat{\alpha})/(1+\hat{\alpha})} \rightarrow \infty,$$

242 as  $j_k \rightarrow \infty$ . Thus, there exists  $j_k \in \mathbb{N}$  such that

$$243 \quad (4.6) \quad \frac{\mu}{\nu_k^2} \eta_k^{(1-\hat{\alpha})/(1+\hat{\alpha})} \geq \max_{i \in [m]} \left\{ \left[ \frac{2(1-\alpha_i)}{\mu(1+\alpha_i)\varepsilon^2} \right]^{(1-\alpha_i)/(1+\alpha_i)} L_i^{2/(1+\alpha_i)} \right\},$$

**Algorithm 3:** Universal Fast Gradient Method (UFGM).**Input:**  $\varepsilon > 0$ .Initialize  $\mathbf{u}_0 = \mathbf{w}_0 \in \Omega$  and  $\rho_0 \geq \mu$ .**for**  $k = 0, 1, 2, \dots$  **do**    **for**  $j_k = 0, 1, 2, \dots$  **do**        Set  $\nu_k = \sqrt{\mu/(2^{j_k} \rho_k)}$  and  $\eta_k = \nu_k/(1 + \nu_k)$ .

Compute

$$(4.1) \quad \mathbf{v}_k = (1 - \eta_k)\mathbf{u}_k + \eta_k \Pi_\Omega(\mathbf{w}_k),$$

and

$$(4.2) \quad \mathbf{z}_k = \Pi_\Omega \left( \Pi_\Omega(\mathbf{w}_k) - \frac{\nu_k}{\mu} \nabla f(\mathbf{v}_k) \right).$$

Set

$$(4.3) \quad \mathbf{u}_{k+1} = (1 - \eta_k)\mathbf{u}_k + \eta_k \mathbf{z}_k.$$

**If**  $\mathbf{u}_{k+1}$  satisfies the following line-search condition,

$$(4.4) \quad \begin{aligned} f(\mathbf{u}_{k+1}) &\leq f(\mathbf{v}_k) + \langle \nabla f(\mathbf{v}_k), \mathbf{u}_{k+1} - \mathbf{v}_k \rangle \\ &\quad + \frac{\mu}{2\nu_k^2} \|\mathbf{u}_{k+1} - \mathbf{v}_k\|^2 + \frac{\eta_k \mu \varepsilon^2}{4}, \end{aligned}$$

**then break.**    Set  $\rho_{k+1} = 2^{j_k} \rho_k$  and update  $\mathbf{w}_{k+1}$  by

$$(4.5) \quad \mathbf{w}_{k+1} = (1 - \eta_k)\mathbf{w}_k + \eta_k \mathbf{v}_k - \frac{\eta_k}{\mu} \nabla f(\mathbf{v}_k).$$

**Output:**  $\mathbf{u}_{k+1}$ .

244 which further implies that

$$\begin{aligned} \frac{\mu}{\nu_k^2} &\geq \frac{1}{\eta_k^{(1-\hat{\alpha})/(1+\hat{\alpha})}} \max_{i \in [m]} \left\{ \left[ \frac{2(1 - \alpha_i)}{\mu(1 + \alpha_i)\varepsilon^2} \right]^{(1-\alpha_i)/(1+\alpha_i)} L_i^{2/(1+\alpha_i)} \right\} \\ &\geq \max_{i \in [m]} \left\{ \left[ \frac{2(1 - \alpha_i)}{\eta_k \mu(1 + \alpha_i)\varepsilon^2} \right]^{(1-\alpha_i)/(1+\alpha_i)} L_i^{2/(1+\alpha_i)} \right\}. \end{aligned}$$

246 As a direct consequence of Proposition 2.1, we can proceed to show that the line-search  
 247 condition (4.4) is satisfied, which completes the proof.  $\square$

248 *Remark 4.2.* When the parameters of problem (1.1) are fully specified, Algo-  
 249 rithm 3 may alternatively be implemented with a fixed stepsize. Recall that  $M > 0$   
 250 is a constant defined in (2.1). By invoking the result of Lemma 4.1, we can fix

$$251 \quad \nu_k = 2 \left[ \frac{\mu}{4M} \right]^{(1+\hat{\alpha})/(1+3\hat{\alpha})} \varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})},$$

252 and dispense with the parameter  $\rho_k$  and the line-search procedure in (4.4). Under

this choice, Algorithm 3 continues to enjoy the same iteration complexity established later.

We now introduce the estimating sequences associated with Algorithm 3, which play a crucial role in our subsequent analysis.

LEMMA 4.3. *Let  $\{\sigma_k\}$  be a sequence of positive constants defined recursively by*

$$(4.7) \quad \sigma_{k+1} = (1 + \nu_k)\sigma_k,$$

with  $\sigma_0 = 1$ . And let  $\{\phi_k\}$  be a sequence of functions defined recursively by

$$(4.8) \quad \begin{aligned} \phi_{k+1}(\mathbf{u}) = & \phi_k(\mathbf{u}) - \nu_k \sigma_k f^* + \nu_k \sigma_k f(\mathbf{v}_k) + \nu_k \sigma_k \langle \nabla f(\mathbf{v}_k), \mathbf{u} - \mathbf{v}_k \rangle \\ & + \frac{\nu_k \sigma_k \mu}{2} \|\mathbf{u} - \mathbf{v}_k\|^2, \end{aligned}$$

with  $\phi_0(\mathbf{u}) = c_0 + \sigma_0 \mu \|\mathbf{u} - \mathbf{w}_0\|^2 / 2$  for  $c_0 = f(\mathbf{u}_0) - f^* - \mu \varepsilon^2 / 4$  and  $\mathbf{w}_0 \in \Omega$ . Then, for all  $k \in \mathbb{N}$ , the function  $\phi_k$  preserves the following canonical form,

$$(4.9) \quad \phi_k(\mathbf{u}) = c_k + \frac{\sigma_k \mu}{2} \|\mathbf{u} - \mathbf{w}_k\|^2,$$

where  $\{c_k\}$  is a sequence of real numbers and  $\{\mathbf{w}_k\}$  is defined recursively by (4.5).

*Proof.* We first prove that  $\nabla^2 \phi_k = \sigma_k \mu I$  for all  $k \in \mathbb{N}$  by induction. It is evident that  $\nabla^2 \phi_0 = \sigma_0 \mu I$ . Now we assume that  $\nabla^2 \phi_k = \sigma_k \mu I$  for some  $k$ . Then relationships (4.7) and (4.8) imply that

$$\nabla^2 \phi_{k+1} = \nabla^2 \phi_k + \nu_k \sigma_k \mu I = \sigma_k \mu I + \nu_k \sigma_k \mu I = \sigma_{k+1} \mu I.$$

Thus, we know that  $\nabla^2 \phi_k = \sigma_k \mu I$  for all  $k \in \mathbb{N}$ , which, in turn, justifies the canonical form of  $\phi_k$  in (4.9).

Next, by combining two relationships (4.8) and (4.9) together, we can obtain that

$$\begin{aligned} \phi_{k+1}(\mathbf{u}) = & c_k + \frac{\sigma_k \mu}{2} \|\mathbf{u} - \mathbf{w}_k\|^2 - \nu_k \sigma_k f^* + \nu_k \sigma_k f(\mathbf{v}_k) \\ & + \nu_k \sigma_k \langle \nabla f(\mathbf{v}_k), \mathbf{u} - \mathbf{v}_k \rangle + \frac{\nu_k \sigma_k \mu}{2} \|\mathbf{u} - \mathbf{v}_k\|^2. \end{aligned}$$

Since  $\mathbf{w}_{k+1}$  is a global minimizer of  $\phi_{k+1}$  over  $\mathbb{R}^n$ , the first-order optimality condition yields that

$$\begin{aligned} 0 = \nabla \phi_{k+1}(\mathbf{w}_{k+1}) = & \sigma_k \mu (\mathbf{w}_{k+1} - \mathbf{w}_k) + \nu_k \sigma_k \nabla f(\mathbf{v}_k) + \nu_k \sigma_k \mu (\mathbf{w}_{k+1} - \mathbf{v}_k) \\ = & (1 + \nu_k) \sigma_k \mu \mathbf{w}_{k+1} - \sigma_k \mu \mathbf{w}_k - \nu_k \sigma_k \mu \mathbf{v}_k + \nu_k \sigma_k \nabla f(\mathbf{v}_k), \end{aligned}$$

from which the closed-form expression of  $\mathbf{w}_{k+1}$  in (4.5) can be derived. The proof is completed.  $\square$

The following lemma characterizes the relationship between the objective function of problem (1.1) and the estimating sequences.

LEMMA 4.4. *Let  $\sigma_k$  and  $\{\phi_k\}$  be the sequences defined in Lemma 4.3. Then we have*

$$(4.10) \quad \phi_k(\mathbf{u}) \leq \sigma_k (f(\mathbf{u}) - f^*) + \phi_0(\mathbf{u}),$$

for all  $\mathbf{u} \in \Omega$  and  $k \in \mathbb{N}$ .

284 *Proof.* We prove that  $\{\phi_k\}$  and  $\{\sigma_k\}$  satisfy relationship (4.10) by induction. It  
 285 is obvious that (4.10) holds for  $k = 0$  since  $f(\mathbf{u}) \geq f^*$  for any  $\mathbf{u} \in \Omega$ . Now we assume  
 286 that (4.10) holds for some  $k \in \mathbb{N}$ . It follows from the strong convexity of  $f$  that

$$287 \quad f(\mathbf{u}) \geq f(\mathbf{v}_k) + \langle \nabla f(\mathbf{v}_k), \mathbf{u} - \mathbf{v}_k \rangle + \frac{\mu}{2} \|\mathbf{u} - \mathbf{v}_k\|^2,$$

288 for all  $\mathbf{u} \in \Omega$ . Then substituting the above relationship into (4.8) leads to that

$$\begin{aligned} \phi_{k+1}(\mathbf{u}) &\leq \phi_k(\mathbf{u}) - \nu_k \sigma_k f^* + \nu_k \sigma_k f(\mathbf{u}) \\ 289 \quad &\leq \sigma_k (f(\mathbf{u}) - f^*) + \phi_0(\mathbf{u}) + \nu_k \sigma_k (f(\mathbf{u}) - f^*) \\ &= \sigma_{k+1} (f(\mathbf{u}) - f^*) + \phi_0(\mathbf{u}), \end{aligned}$$

290 which indicates that (4.10) also holds for  $k + 1$ . We complete the proof.  $\square$

291 Next, we proceed to show that the function value error of Algorithm 3 is controlled  
 292 by the estimating sequences.

293 **PROPOSITION 4.5.** *Let  $\{\sigma_k\}$  and  $\{\phi_k\}$  be the sequences defined in Lemma 4.3.*  
 294 *Then the sequence  $\{\mathbf{u}_k\}$  generated by Algorithm 3 satisfies*

$$295 \quad (4.11) \quad f(\mathbf{u}_k) - f^* \leq \frac{1}{\sigma_k} \phi_0(\mathbf{u}^*) + \frac{\mu \varepsilon^2}{4},$$

296 for all  $k \in \mathbb{N}$ .

297 *Proof.* Let  $\phi_k^* := \min_{\mathbf{u} \in \Omega} \phi_k(\mathbf{u})$ . We first prove by induction that

$$298 \quad (4.12) \quad \sigma_k \left( f(\mathbf{u}_k) - f^* - \frac{\mu \varepsilon^2}{4} \right) \leq \phi_k^*,$$

299 for any  $k \in \mathbb{N}$ . It is clear that (4.12) holds for  $k = 0$  since  $\sigma_0 = 1$  and  $\phi_0^* = \phi_0(\mathbf{w}_0) =$   
 300  $f(\mathbf{u}_0) - f^* - \mu \varepsilon^2/4$ . Now we assume that (4.12) holds for some  $k \in \mathbb{N}$  and investigate  
 301 the situation for  $k + 1$ .

302 From the canonical form (4.9), it follows that  $\phi_k$  is a strongly convex function  
 303 and  $\Pi_\Omega(\mathbf{w}_k) = \arg \min_{\mathbf{u} \in \Omega} \phi_k(\mathbf{u})$ . By invoking the result of [11, Corollary 2.2.1], we  
 304 have

$$\begin{aligned} \phi_k(\mathbf{u}) &\geq \phi_k^* + \frac{\sigma_k \mu}{2} \|\mathbf{u} - \Pi_\Omega(\mathbf{w}_k)\|^2 \\ 305 \quad &\geq \sigma_k \left( f(\mathbf{u}_k) - f^* - \frac{\mu \varepsilon^2}{4} \right) + \frac{\sigma_k \mu}{2} \|\mathbf{u} - \Pi_\Omega(\mathbf{w}_k)\|^2, \end{aligned}$$

306 for all  $\mathbf{u} \in \Omega$ . Then relationship (4.8) yields that

$$\begin{aligned} \phi_{k+1}(\mathbf{u}) &\geq \sigma_k \left( f(\mathbf{u}_k) - f^* - \frac{\mu \varepsilon^2}{4} \right) + \frac{\sigma_k \mu}{2} \|\mathbf{u} - \Pi_\Omega(\mathbf{w}_k)\|^2 - \nu_k \sigma_k f^* \\ &\quad + \nu_k \sigma_k f(\mathbf{v}_k) + \nu_k \sigma_k \langle \nabla f(\mathbf{v}_k), \mathbf{u} - \mathbf{v}_k \rangle + \frac{\nu_k \sigma_k \mu}{2} \|\mathbf{u} - \mathbf{v}_k\|^2 \\ &\geq \sigma_{k+1} (f(\mathbf{v}_k) - f^*) - \frac{\sigma_k \mu \varepsilon^2}{4} + \langle \nabla f(\mathbf{v}_k), \sigma_k \mathbf{u}_k - \sigma_{k+1} \mathbf{v}_k \rangle \\ 307 \quad &\quad + \nu_k \sigma_k \langle \nabla f(\mathbf{v}_k), \mathbf{u} \rangle + \frac{\sigma_k \mu}{2} \|\mathbf{u} - \Pi_\Omega(\mathbf{w}_k)\|^2 \\ &= \sigma_{k+1} (f(\mathbf{v}_k) - f^*) - \frac{\sigma_k \mu \varepsilon^2}{4} + \nu_k \sigma_k \langle \nabla f(\mathbf{v}_k), \mathbf{u} - \Pi_\Omega(\mathbf{w}_k) \rangle \\ &\quad + \frac{\sigma_k \mu}{2} \|\mathbf{u} - \Pi_\Omega(\mathbf{w}_k)\|^2, \end{aligned}$$

where the second inequality comes from the strong convexity of  $f$  and (4.7), and the last equality holds due to the definition of  $\mathbf{v}_k$  in (4.1). According to the definition of  $\mathbf{z}_k$  in (4.2), we can obtain that

$$\begin{aligned}
& \nu_k \sigma_k \langle \nabla f(\mathbf{v}_k), \mathbf{u} - \Pi_\Omega(\mathbf{w}_k) \rangle + \frac{\sigma_k \mu}{2} \|\mathbf{u} - \Pi_\Omega(\mathbf{w}_k)\|^2 \\
&= \frac{\sigma_k \mu}{2} \left\| \mathbf{u} - \left( \Pi_\Omega(\mathbf{w}_k) - \frac{\nu_k}{\mu} \nabla f(\mathbf{v}_k) \right) \right\|^2 - \frac{\nu_k^2 \sigma_k}{2\mu} \|\nabla f(\mathbf{v}_k)\|^2 \\
&\geq \frac{\sigma_k \mu}{2} \left\| \mathbf{z}_k - \left( \Pi_\Omega(\mathbf{w}_k) - \frac{\nu_k}{\mu} \nabla f(\mathbf{v}_k) \right) \right\|^2 - \frac{\nu_k^2 \sigma_k}{2\mu} \|\nabla f(\mathbf{v}_k)\|^2 \\
&= \nu_k \sigma_k \langle \nabla f(\mathbf{v}_k), \mathbf{z}_k - \Pi_\Omega(\mathbf{w}_k) \rangle + \frac{\sigma_k \mu}{2} \|\mathbf{z}_k - \Pi_\Omega(\mathbf{w}_k)\|^2.
\end{aligned}$$

As a result, it holds that

$$\begin{aligned}
(4.13) \quad \phi_{k+1}(\mathbf{u}) &\geq \sigma_{k+1} (f(\mathbf{v}_k) - f^*) - \frac{\sigma_k \mu \varepsilon^2}{4} + \nu_k \sigma_k \langle \nabla f(\mathbf{v}_k), \mathbf{z}_k - \Pi_\Omega(\mathbf{w}_k) \rangle \\
&\quad + \frac{\sigma_k \mu}{2} \|\mathbf{z}_k - \Pi_\Omega(\mathbf{w}_k)\|^2,
\end{aligned}$$

for all  $\mathbf{u} \in \Omega$ . From the definitions of  $\mathbf{v}_k$  and  $\mathbf{u}_{k+1}$  in (4.1) and (4.3), it can be derived that  $\mathbf{z}_k - \Pi_\Omega(\mathbf{w}_k) = (\mathbf{u}_{k+1} - \mathbf{v}_k)/\eta_k$ . Substituting this relationship into (4.13) and taking  $\mathbf{u} = \Pi_\Omega(\mathbf{w}_{k+1})$ , we arrive at

$$\frac{\phi_{k+1}^*}{\sigma_{k+1}} \geq f(\mathbf{v}_k) - f^* + \langle \nabla f(\mathbf{v}_k), \mathbf{u}_{k+1} - \mathbf{v}_k \rangle + \frac{\mu}{2\nu_k^2} \|\mathbf{u}_{k+1} - \mathbf{v}_k\|^2 - \frac{(1 - \eta_k)\mu\varepsilon^2}{4},$$

which together with the line-search condition (4.4) implies that

$$\frac{\phi_{k+1}^*}{\sigma_{k+1}} \geq f(\mathbf{u}_{k+1}) - f^* - \frac{\eta_k \mu \varepsilon^2}{4} - \frac{(1 - \eta_k)\mu\varepsilon^2}{4} = f(\mathbf{u}_{k+1}) - f^* - \frac{\mu\varepsilon^2}{4}.$$

Therefore, relationship (4.12) also holds for  $k + 1$ .

Finally, by collecting two relationships (4.10) and (4.12) together, we can obtain that

$$\begin{aligned}
\sigma_k \left( f(\mathbf{u}_k) - f^* - \frac{\mu\varepsilon^2}{4} \right) &\leq \min_{\mathbf{u} \in \Omega} \phi_k(\mathbf{u}) \leq \min_{\mathbf{u} \in \Omega} \{ \sigma_k (f(\mathbf{u}) - f^*) + \phi_0(\mathbf{u}) \} \\
&\leq \sigma_k (f(\mathbf{u}^*) - f^*) + \phi_0(\mathbf{u}^*) \\
&= \phi_0(\mathbf{u}^*),
\end{aligned}$$

which completes the proof.  $\square$

With the above preparatory results in place, we are now in a position to establish the iteration complexity of Algorithm 3, as articulated in the theorem below.

**THEOREM 4.6.** *Let  $\varepsilon \in (0, 1)$  be a sufficiently small constant. Then after at most*

$$O \left( \log \left( \frac{1}{\varepsilon} \right) \frac{M^{(1+\hat{\alpha})/(1+3\hat{\alpha})}}{\varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})}} \right)$$

*iterations, Algorithm 3 will reach an iterate  $\mathbf{u}_k$  satisfying  $\|\mathbf{u}_k - \mathbf{u}^*\| \leq \varepsilon$ .*

*Proof.* In view of relationship (4.6), the number of line-search steps  $j_k$  in (4.4) satisfies

$$\frac{\mu}{\nu_k^2} \eta_k^{(1-\hat{\alpha})/(1+\hat{\alpha})} \leq 2 \max_{i \in [m]} \left\{ \left[ \frac{2(1-\alpha_i)}{\mu(1+\alpha_i)\varepsilon^2} \right]^{(1-\alpha_i)/(1+\alpha_i)} L_i^{2/(1+\alpha_i)} \right\} \leq \frac{2M}{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}},$$

where  $M > 0$  is a constant defined in (2.1). Since  $\eta_k = \nu_k/(1+\nu_k) \geq \nu_k/2$ , we arrive at

$$(4.14) \quad \frac{\nu_k^2}{\mu} \geq \frac{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}}{2M} \eta_k^{(1-\hat{\alpha})/(1+\hat{\alpha})} \geq \frac{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}}{2^{2/(1+\hat{\alpha})}M} \nu_k^{(1-\hat{\alpha})/(1+\hat{\alpha})}.$$

Let  $\omega > 0$  be a constant defined as

$$\omega = \frac{1}{2^{2/(1+3\hat{\alpha})}} \left[ \frac{\mu}{M} \right]^{(1+\hat{\alpha})/(1+3\hat{\alpha})}.$$

Then it follows from relationship (4.14) that

$$(4.15) \quad \nu_k \geq \omega \varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})},$$

which further infers that

$$\sigma_{k+1} = (1 + \nu_k) \sigma_k \geq \left( 1 + \omega \varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})} \right) \sigma_k.$$

Applying the above inequality for  $k$  times recursively yields that

$$\sigma_k \geq \left( 1 + \omega \varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})} \right)^k.$$

As a direct consequence of (2.5) and (4.11), we can show that

$$\begin{aligned} \|\mathbf{u}_k - \mathbf{u}^*\|^2 &\leq \frac{2}{\mu} (f(\mathbf{u}_k) - f^*) \leq \frac{2}{\mu} \left( \frac{1}{\sigma_k} \phi_0(\mathbf{u}^*) + \frac{\mu \varepsilon^2}{4} \right) \\ &\leq \chi \left( 1 + \omega \varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})} \right)^{-k} + \frac{\varepsilon^2}{2}, \end{aligned}$$

where  $\chi = 2(f(\mathbf{u}_0) - f^*)/\mu + \|\mathbf{u}_0 - \mathbf{u}^*\|^2 > 0$  is a constant. Let  $K_\varepsilon^*$  be the smallest iteration number  $k$  such that  $\|\mathbf{u}_k - \mathbf{u}^*\| \leq \varepsilon$ . By solving the inequality  $\chi(1 + \omega \varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})})^{-k} \leq \varepsilon^2/2$ , we have

$$K_\varepsilon^* \leq \log \left( \frac{\sqrt{2\chi}}{\varepsilon} \right) \frac{2}{\log(1 + \omega \varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})})} \leq \log \left( \frac{\sqrt{2\chi}}{\varepsilon} \right) \frac{4}{\omega \varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})}}.$$

The proof is completed.  $\square$

The complexity bound established in Theorem 4.6 is markedly lower than those presented in Theorems 2.2 and 3.1, thereby highlighting the acceleration effect attained by Algorithm 3. Finally, we demonstrate that the number of line-search steps required by Algorithm 3 is also  $O(\log(\varepsilon^{-1})\varepsilon^{2(\hat{\alpha}-1)/(1+3\hat{\alpha})})$ .

**COROLLARY 4.7.** *Let  $\varepsilon \in (0, 1)$  be a sufficiently small constant. Then, to achieve an iterate  $\mathbf{u}_k$  satisfying  $\|\mathbf{u}_k - \mathbf{u}^*\| \leq \varepsilon$ , Algorithm 3 requires at most*

$$O \left( \log \left( \frac{1}{\varepsilon} \right) \frac{M^{(1+\hat{\alpha})/(1+3\hat{\alpha})}}{\varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})}} \right)$$

*line-search steps.*

*Proof.* It follows from relationship (4.14) that

$$\rho_{k+1} = 2^{j_k} \rho_k = \frac{\mu}{\nu_k^2} \leq \frac{2^{2/(1+\hat{\alpha})} M}{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}} \left[ \frac{1}{\nu_k} \right]^{(1-\hat{\alpha})/(1+\hat{\alpha})},$$

which together with (4.15) implies that

$$\rho_{k+1} \leq \frac{2^{2/(1+\hat{\alpha})} M}{\varepsilon^{2(1-\hat{\alpha})/(1+\hat{\alpha})}} \left[ \frac{1}{\omega \varepsilon^{2(1-\hat{\alpha})/(1+3\hat{\alpha})}} \right]^{(1-\hat{\alpha})/(1+\hat{\alpha})} = \frac{2^{2/(1+\hat{\alpha})} M}{\omega^{(1-\hat{\alpha})/(1+\hat{\alpha})} \varepsilon^{4(1-\hat{\alpha})/(1+3\hat{\alpha})}}.$$

Let  $N_k$  be the total number of line-search steps after  $k$  iterations in Algorithm 3. In view of (3.4), we have

$$\begin{aligned} N_k &\leq k + 1 + \log \left( \frac{2^{2/(1+\hat{\alpha})} M}{\omega^{(1-\hat{\alpha})/(1+\hat{\alpha})} \varepsilon^{4(1-\hat{\alpha})/(1+3\hat{\alpha})}} \right) - \log \rho_0 \\ &\leq k + \frac{4(1-\hat{\alpha})}{1+3\hat{\alpha}} \log \left( \frac{1}{\varepsilon} \right) + \log \left( \frac{2^{2/(1+\hat{\alpha})} M}{\omega^{(1-\hat{\alpha})/(1+\hat{\alpha})} \rho_0} \right) + 1. \end{aligned}$$

Consequently, Theorem 4.6 indicates that the total number of line-search steps in Algorithm 3 is at most  $O(\log(\varepsilon^{-1}) \varepsilon^{2(\hat{\alpha}-1)/(1+3\hat{\alpha})})$ , which completes the proof.  $\square$

*Remark 4.8.* By an analogous argument, we can also prove that Algorithm 3 requires at most  $O(\log(\varepsilon^{-1}) \varepsilon^{(\hat{\alpha}-1)/(1+3\hat{\alpha})})$  iterations to generate an iterate  $\mathbf{u}_k$  such that  $f(\mathbf{u}_k) - f^* \leq \varepsilon$  for problem (1.1). Very recently, Doikov [8] has shown that, in the case  $m = 2$ , where  $f_1$  is a convex function with a Hölder continuous gradient and  $f_2(\mathbf{u}) = \|\mathbf{u}\|^2$ , the lower complexity bound for first-order methods is precisely  $O(\log(\varepsilon^{-1}) \varepsilon^{(\hat{\alpha}-1)/(1+3\hat{\alpha})})$  in terms of function value accuracy. This finding confirms that Algorithm 3 achieves the optimal iteration complexity.

**5. Numerical Experiments.** Preliminary numerical results are presented in this section to provide additional insights into the performance guarantees of the algorithms proposed in this paper. We aim to elucidate that the final error attained by the algorithm is influenced by both the stepsize and the Hölder exponent. The numerical experiments are conducted using Julia [4] (version 1.12) on an Apple Macintosh Mini with an M2 processor, 8 performance cores, and 32GB of memory. We have placed the Julia codes in the GitHub repository ([https://github.com/ctkelly/Grad\\_Des\\_CKW.jl](https://github.com/ctkelly/Grad_Des_CKW.jl)) with instructions for reproducing the figures. In this section, we set the spatial mesh width as  $h = 2^{-4}$  for the discretization of partial differential equations (PDEs). Then the dimension of the discretized problem is  $n = (h^{-1} - 1)^2$ .

**5.1. Two-dimensional PDE with a non-Lipschitz term.** Hölder continuous gradients arise naturally in PDEs involving non-Lipschitz nonlinearity [3, 14]. In this subsection, we introduce a numerical example from [3]. This problem is to solve the following two-dimensional PDE,

$$(5.1) \quad \mathcal{F}(u) = -\Delta u + \gamma u_+^\alpha = 0,$$

where  $\alpha \in (0, 1)$ ,  $\gamma > 0$  is a constant and  $u_+ = \max\{u, 0\}$ . Discretizing (5.1) with the standard five point difference scheme [9] leads to the following nonlinear system,

$$(5.2) \quad \mathbf{F}(\mathbf{u}) = \mathbf{A}\mathbf{u} + \gamma \mathbf{u}_+^\alpha - \mathbf{b} = 0,$$

where  $\mathbf{A} \in \mathbb{R}^{n \times n}$  is the discretization of  $-\Delta$  with zero boundary conditions,  $\mathbf{b} \in \mathbb{R}^n$  encodes the boundary conditions, and  $\mathbf{u}_+^\alpha = \max\{\mathbf{u}, 0\}^\alpha$  is understood as a component-wise operation.

We now modify the above problem to enable direct computation of errors in the iterations. To this end, we follow [13, Example 4.4] and take as the exact solution the function

$$u^*(x, y) = \left( \frac{3r-1}{2} \right)^2 \max \left\{ 0, r - \frac{1}{3} \right\},$$

where  $r = \sqrt{x^2 + y^2}$ . We enforce the following boundary conditions,

$$u(x, 1) = u^*(x, 1), u(x, 0) = u^*(x, 0), u(1, y) = u^*(1, y), u(0, y) = u^*(0, y),$$

for  $0 < x, y < 1$ . And these conditions are encoded into  $\mathbf{b}$ . Then our modified equation is

$$(5.3) \quad \mathbf{F}(\mathbf{u}) - \mathbf{c}^* = 0,$$

where  $\mathbf{c}^* = \mathbf{F}(\mathbf{u}^*)$ . The nonlinear system (5.3) corresponds to the optimality condition of the following problem,

$$(5.4) \quad \min_{\mathbf{u} \in \mathbb{R}^n} f(\mathbf{u}) = \frac{1}{2} \mathbf{u}^\top \mathbf{A} \mathbf{u} + \frac{\gamma}{1+\alpha} \mathbf{e}^\top \mathbf{u}_+^{1+\alpha} - (\mathbf{b} + \mathbf{c}^*)^\top \mathbf{u},$$

where  $\mathbf{e} \in \mathbb{R}^n$  is the vector of all ones.

The optimization model (5.4) is a special instance of problem (1.1) with  $\Omega = \mathbb{R}^n$ ,  $m = 2$ ,

$$f_1(\mathbf{u}) = \mathbf{u}^\top \mathbf{A} \mathbf{u} - 2(\mathbf{b} + \mathbf{c}^*)^\top \mathbf{u}, \quad \text{and} \quad f_2(\mathbf{u}) = \frac{2\gamma}{1+\alpha} \mathbf{e}^\top \mathbf{u}_+^{1+\alpha}.$$

It is clear that,  $\nabla f_1$  is Lipschitz continuous with the corresponding Lipschitz constant  $L_1 = 2 \|\mathbf{A}\|$ , and  $\nabla f_2$  is Hölder continuous with the Hölder exponent  $\alpha$  and  $L_2 = 2\gamma$  from

$$\|\nabla f_2(\mathbf{u}) - \nabla f_2(\mathbf{v})\| = 2\gamma \|\mathbf{u}_+^\alpha - \mathbf{v}_+^\alpha\| \leq 2\gamma \|\mathbf{u} - \mathbf{v}\|^\alpha,$$

for all  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ . Moreover, the function  $f = (f_1 + f_2)/2$  is  $\lambda(\mathbf{A})$ -strongly convex, where  $\lambda(\mathbf{A})$  is the smallest eigenvalue of the symmetric positive definite matrix  $\mathbf{A}$ . Let  $\mathbf{u}^*$  be the vector obtained by evaluating  $u^*$  at the interior grid points. Then  $\mathbf{u}^*$  serves as the unique global minimizer of problem (5.4).

In the subsequent experiments, we use the solution of  $\mathbf{A} \mathbf{u}_0 = \mathbf{b}$  as the initial iterate. This is the discretization of Laplace's equation with the boundary conditions. In this way, we ensure that the entire iteration satisfies the boundary conditions.

**5.1.1. Numerical results for Algorithm 1.** In the first experiment, we scrutinize the performance of Algorithm 1 under different stepsizes for problem (5.4) with  $\alpha = 0.5$  and  $\gamma = 0.5$ . Specifically, Algorithm 1 is tested for stepsizes of the form  $\tau = \tau_0 h^2$ , where  $\tau_0$  is taken from the set  $\{0.2, 0.1, 0.05, 0.01\}$ . The corresponding numerical results, presented in Figure 1(a), illustrate the decay of the distance between the iterates and the global minimizer over iterations. It can be observed that a larger stepsize facilitates a more rapid descent in the iterations.



In the second experiment, we vary the Hölder exponent  $\alpha$  over the values in  $\{0.1, 0.2, 0.5, 0.8\}$ , while fixing  $\tau_0 = 0.1$ . Figure 1(b) similarly tracks the decay of the distance to the global minimizer over iterations. It is evident that, as the value of  $\alpha$  decreases, the final error attained by Algorithm 1 increases under the same stepsize. Therefore, the associated optimization problems become increasingly ill-conditioned and thus more challenging to solve for smaller values of  $\alpha$ . These findings offer empirical support for our theoretical analysis. In Figure 1(b), and in several others, the thick lines are artifacts of small rapid oscillations in the error norm which reflect stagnation in the iteration.

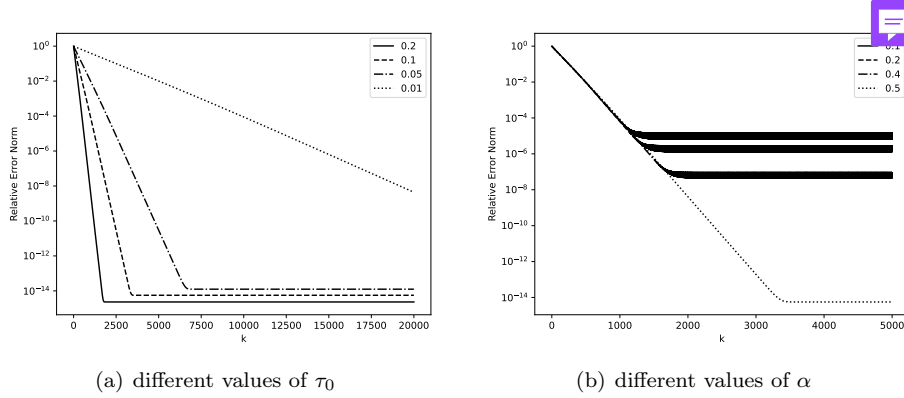


FIG. 1. Numerical performance of Algorithm 1 for problem (5.4).

**5.1.2. Numerical results for Algorithm 2.** We repeat the study in subsection 5.1.1 for Algorithm 2 by varying the values of the Hölder exponent  $\alpha$ . We set  $\varepsilon = 10^{-6}$  and  $\mu = 2\pi^2$  in Algorithm 2, which is a lower estimate for the smallest eigenvalue of  $\mathbf{A}$ . The stepsize is initialized to  $20h^2$  in the line-search procedure. The corresponding numerical results are depicted in Figure 2. Comparing Figure 2 to Figure 1(b) shows the benefits of the line-search procedure in Algorithm 2, which does not need to manually adjust the value of  $\tau_0$  to converge for a given value of  $\varepsilon$ .

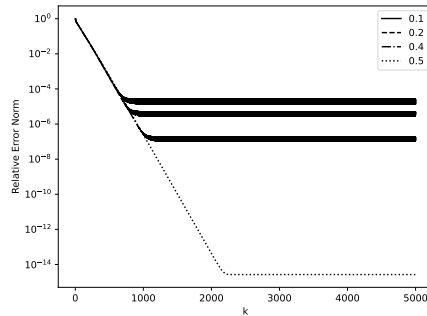


FIG. 2. Numerical performance of Algorithm 2 for problem (5.4) with different values of  $\alpha$ .

**5.1.3. Numerical results for Algorithm 3.** We report the numerical performance of Algorithm 3 on two experiments. Guided by the observation in Remark 4.2, we test Algorithm 3 with a fixed stepsize  $\nu = \tau_0 h^2$ . We first use the values for  $\tau_0$

from Figure 1. In this way we can directly compare the performance of Algorithm 3 with that of Algorithm 1. The corresponding results, shown in Figure 3, are poor. The reason for this is that we are not exploiting the ability of Algorithm 3 to use larger stepsizes. Consequently, we consider larger values for  $\tau_0$  in Figure 4(a) and set  $\tau_0 = 20$  in Figure 4(b). The convergence is much better in all cases.

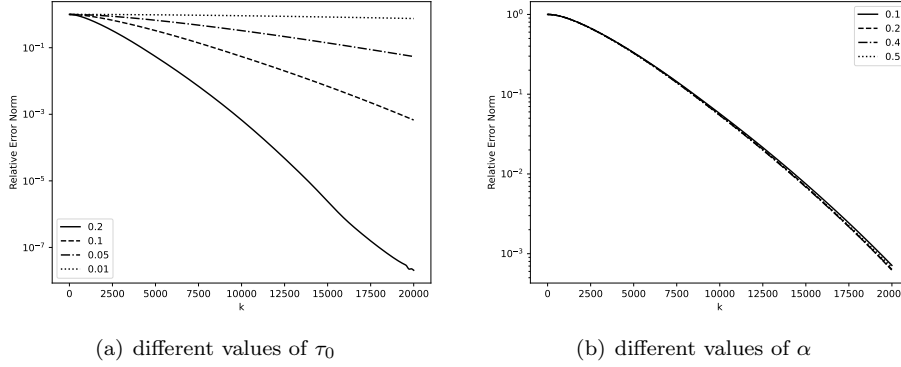


FIG. 3. Numerical performance of Algorithm 3 for problem (5.4) with smaller stepsizes.

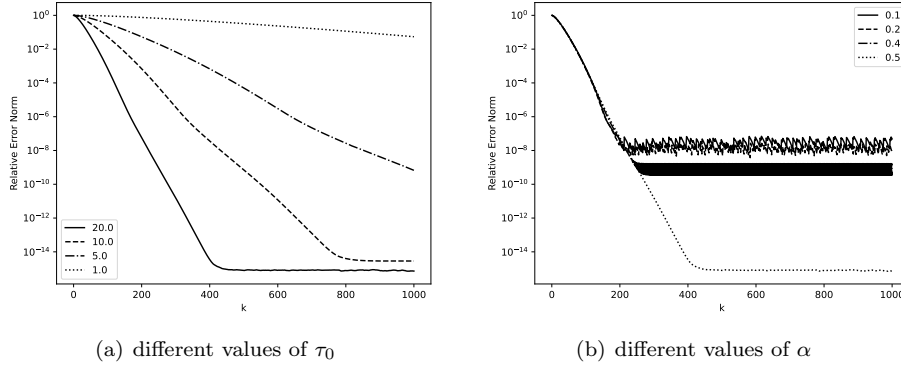


FIG. 4. Numerical performance of Algorithm 3 for problem (5.4) with larger stepsizes.

**5.2. Semi-linear elliptic problem with a constraint.** We consider a second numerical example motivated by a semi-linear elliptic problem with a constraint on the solution in a certain set [14]. Let

$$(5.5) \quad \mathcal{H}(u) = -\Delta u + \delta |u|^\alpha \text{sign}(u) - |u|^{p-1} u,$$

on  $D = (0, 1)^2$  with the boundary condition  $u(x, y) = 0.5 - \sin(x) \sin(y)$  on  $\partial D$ . Here,  $\alpha \in (0, 1)$ ,  $p > 1$ , and  $\delta > p/\alpha$  are three constants. We consider the variational inequality that is to find  $u^* \in [-1, 1]$  such that

$$\mathcal{H}(u^*)(u - u^*) \geq 0,$$

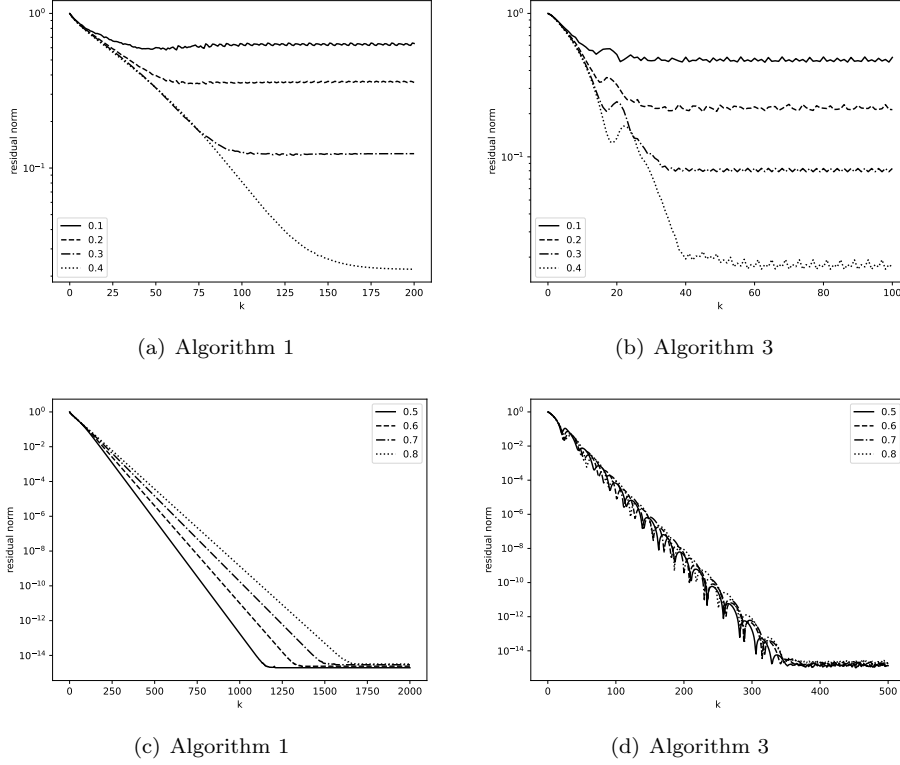


FIG. 5. Numerical performance of Algorithm 1 and Algorithm 3 for problem (5.8) with different values of  $\alpha$ .

for any  $u \in [-1, 1]$ . This problem is equivalent to the following nonlinear equation,

$$0 = \mathcal{F}(u) := \begin{cases} \mathcal{H}(u), & \text{if } u - \mathcal{H}(u) \in [-1, 1], \\ u - 1, & \text{if } u - \mathcal{H}(u) \geq 1, \\ u + 1, & \text{otherwise.} \end{cases} \quad (5.6)$$

By discretizing (5.5) with the standard five point difference scheme [9], problem (5.6) leads to the following system of nonlinear equations,

$$0 = \mathbf{F}(\mathbf{u}) := \mathbf{u} - \Pi_{\mathbf{U}} \left( \mathbf{u} - \tau \left( \mathbf{A}\mathbf{u} + \delta |\mathbf{u}|^\alpha \text{sign}(\mathbf{u}) - |\mathbf{u}|^{p-1} \mathbf{u} - \mathbf{b} \right) \right), \quad (5.7)$$

where  $\mathbf{U} = [-1, 1]^n$ ,  $\tau > 0$  is a constant,  $\mathbf{A} \in \mathbb{R}^{n \times n}$  is a symmetric positive definite matrix, and  $\mathbf{b} \in \mathbb{R}^n$  encodes the boundary conditions. Note that (5.7) is the optimality condition of the following problem,

$$\min_{\mathbf{u} \in \mathbf{U}} f(\mathbf{u}) := \frac{1}{2} \mathbf{u}^\top \mathbf{A} \mathbf{u} + \frac{\delta}{1+\alpha} \mathbf{e}^\top |\mathbf{u}|^{1+\alpha} - \frac{1}{1+p} \mathbf{e}^\top |\mathbf{u}|^{1+p} - \mathbf{b}^\top \mathbf{u}. \quad (5.8)$$

The Hessian matrix of  $f$  at  $\mathbf{u}$  with  $\mathbf{u}_i \neq 0$  ( $i = 1, \dots, n$ ) has the form

$$\nabla^2 f(\mathbf{u}) = \mathbf{A} + \delta \alpha \text{Diag} \left( |\mathbf{u}|^{\alpha-1} \right) - p \text{Diag} \left( |\mathbf{u}|^{p-1} \right),$$

Since  $\delta > p/\alpha$ ,  $\nabla^2 f(\mathbf{u})$  is symmetric positive definite for any  $\mathbf{u} \in \mathbf{U}$  with  $\mathbf{u}_i \neq 0$  ( $i = 1, \dots, n$ ). Hence, the function  $f$  is  $\mu$ -strongly convex in  $\mathbf{U}$  with  $\mu = \lambda(\mathbf{A})$  and the system (5.7) has a unique solution in  $\mathbf{U}$ . The optimization model (5.8) is a special instance of problem (1.1) with  $\Omega = \mathbf{U}$ ,  $m = 2$ ,

$$f_1(\mathbf{u}) = \mathbf{u}^\top \mathbf{A} \mathbf{u} - 2\mathbf{b}^\top \mathbf{u} - \frac{2}{1+p} \mathbf{e}^\top |\mathbf{u}|^{1+p}, \text{ and } f_2(\mathbf{u}) = \frac{2\delta}{1+\alpha} \mathbf{e}^\top |\mathbf{u}|^{1+\alpha}.$$

It is clear that Assumption 1.1 (ii) holds with  $\alpha_1 = 1$ ,  $L_1 = 2\|\mathbf{A}\| + 2p$ ,  $\alpha_2 = \alpha$ , and  $L_2 = 2\delta\alpha$ .

In this example, we do not have an analytic solution and we only plot the residual norm  $\|\mathbf{F}(\mathbf{u})\|$  with  $\tau$  being the stepsize. We compare the performance of Algorithm 1 and Algorithm 3 on problem (5.8) with  $p = 1.5$  and  $\delta = 20$ . The stepsizes of Algorithm 1 and Algorithm 3 are set to  $\tau = 0.1h^2$  and  $\tau = 20h^2$ , respectively. Figure 5(a) and Figure 5(c) present the performance of Algorithm 1 for  $\alpha \in \{0.1, 0.2, 0.3, 0.4\}$  and  $\alpha \in \{0.5, 0.6, 0.7, 0.8\}$ , respectively. In a similar vein, Figure 5(b) and Figure 5(d) illustrate the behavior of Algorithm 3 across the same ranges of  $\alpha$ . Similar as the case in subsection 5.1, problems where the exponent  $\alpha$  for the non-Lipschitz term in the gradients is small are difficult. In particular, one cannot drive the residual to a small value. For larger values of  $\alpha$ , both algorithms demonstrate strong performance. Furthermore, Algorithm 3 exhibits a faster convergence rate, benefiting from the use of a larger stepsize.

**6. Conclusion.** In this paper, we consider a class of strongly convex constrained optimization problems of the form (1.1). Example 1.2 shows that although each component function  $f_i$  of the objective function  $f$  admits a Hölder continuous gradient with an component  $\alpha_i \in (0, 1]$ , the gradient of  $f$  is not necessarily Hölder continuous. To establish the iteration complexity of the projected gradient descent methods for this class of problems, we use the parameter  $\hat{\alpha} = \min_{i \in [m]} \alpha_i$  to determine the complexity bound. Algorithm 1 is a new version of projected gradient method for problem (1.1) with an appropriately fixed stepsize. Theorem 2.2 shows that Algorithm 1 can find an iterate in the feasible set  $\Omega$  with a distance to the global minimizer less than  $\varepsilon$  at most  $O(\log(\varepsilon^{-1})\varepsilon^{2(\hat{\alpha}-1)/(1+\hat{\alpha})})$  iterations. This recovers the classical complexity result when  $\hat{\alpha} = 1$  and reveals the additional difficulty imposed by the weaker smoothness of the objective function for  $\hat{\alpha} < 1$ . Algorithm 2 is a modification of Algorithm 1 for problems where the parameters  $\alpha_i$  and  $L_i$  are difficult to estimate for the stepsize. In Algorithm 3, the stepsize is updated by the universal scheme at each iteration, which improves the complexity bound to  $O(\log(\varepsilon^{-1})\varepsilon^{2(\hat{\alpha}-1)/(1+3\hat{\alpha})})$ . Numerical experiments are conducted to validate our theoretical findings, demonstrating the expected behavior of projected gradient descent methods under different stepsizes and Hölder exponents. These results offer new insights into the performance guarantees of the classic projected gradient descent methods for a broader class of optimization problems with non-Lipschitz gradients.

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