Krylov Linear Solvers and Quasi Monte Carlo Methods for Transport Simulations

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Abstract

QMC + Krylov

Keywords — Quasi Monte Carlo Methods, Krylov Linear Solvers

I. INTRODUCTION

II. COMPUTATIONAL RESULTS

In this section we consider an example from [1]. The formulation of the transport problem is taken from [2]. The equation for the angular flux ψ is

$$\mu \frac{\partial \psi}{\partial x}(x, \mu) + \Sigma_t(x)\psi(x, \mu) = \frac{1}{2}\left[\Sigma_s(x) \int_{-1}^{1} \psi(x, \mu')\, d\mu' + q(x)\right] \text{ for } 0 \leq x \leq \tau \qquad (1)$$

The boundary conditions are

$$\psi(0, \mu) = \psi_l(\mu), \mu > 0; \psi(\tau, \mu) = \psi_r(\mu), \mu < 0.$$

In the computaions we use

$$\tau = 5, \Sigma_s(x) = \omega_0 e^{-x/s}, \Sigma_t(x) = 1, q(x) = 0, \psi_l(\mu) = 1, \psi_r(\mu) = 0,$$

and consider two cases $s = 1$ and $s = \infty$

II.A. Using Krylov Linear Solvers

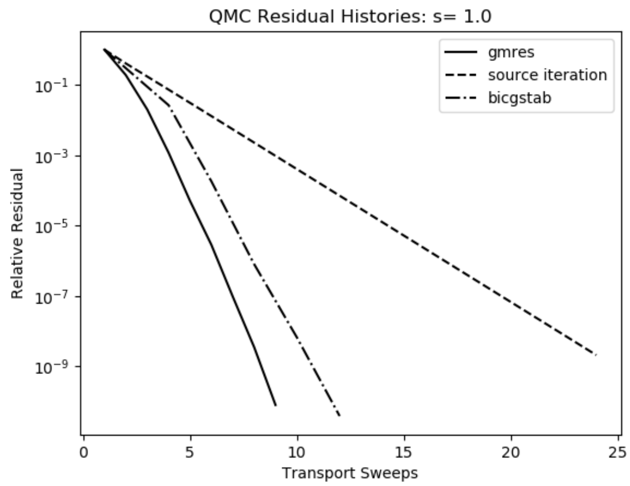
We use two krylov methods [3], GMRES [4] and Bi-CGSTAB [5].

The linear and nonlinear solvers come from the Julia package SIAMFANLEQ.jl [6]. The documentation for these codes is in the Juila notebooks [7] and the book [8] that accompany the package.

II.B. Source Iteration and Krylov Methods

II.C. QMC and Krylov Linear Solvers

I can solve the QMC linear problem with both Krylov methods now and, just like the classical case, I'm seeing fewer than half of the number of transport sweeps. Herewith the results for N=1000 and Nx= 100.



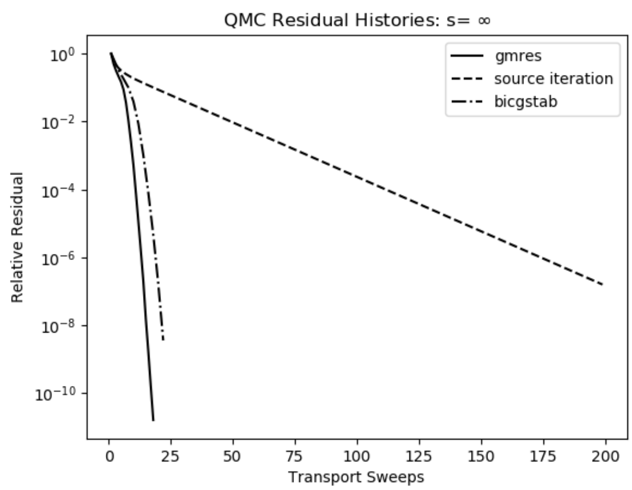


Fig. 2. $s = \infty$

II.D. Validation and calibration study

We conclude this section with a validation study. We compare the QMC results with the results from [1]. The results in [1] are exit distributions and are accurate to six figures. We have duplicated those results with an Sn computation on a fine angular and spatial mesh.

Sam, Ryan, should we use more or different values of N and Nx?

For $N = 1000$ and $Nx = 100$ we obtain the cell-average fluxes from the QMC approximation. We then use a single Sn transport sweep to recover the exit distributions from the QMC cell-average fluxes. We report the results and the corresponding results from [1] in Tables I and II.

The exit distributions, as is clear from Table I can vary by five orders of magnitude. Even so, the results from QMC agree with the benchmarks to roughly two figues.

TABLE I
Exit Distributiions: $s = 1$

	Garcia/Siewert		QMC	
μ	$\psi(0, -\mu)$	$\psi(\tau, -\mu)$	$\psi(0, -\mu)$	$\psi(\tau, -\mu)$
5.00e-02	5.89664e-01	6.07488e-06	5.71197e-01	5.85487e-06
1.00e-01	5.31120e-01	6.92516e-06	5.22137e-01	6.66741e-06
2.00e-01	4.43280e-01	9.64232e-06	4.41567e-01	9.25261e-06
3.00e-01	3.80306e-01	1.62339e-05	3.81029e-01	1.54416e-05
4.00e-01	3.32964e-01	4.38580e-05	3.34673e-01	4.09691e-05
5.00e-01	2.96090e-01	1.69372e-04	2.98224e-01	1.57373e-04
6.00e-01	2.66563e-01	5.73465e-04	2.68871e-01	5.35989e-04
7.00e-01	2.42390e-01	1.51282e-03	2.44749e-01	1.42448e-03
8.00e-01	2.22235e-01	3.24369e-03	2.24583e-01	3.07431e-03
9.00e-01	2.05174e-01	5.96036e-03	2.07478e-01	5.67991e-03
1.00e+00	1.90546e-01	9.77123e-03	1.92789e-01	9.35351e-03

TABLE II
Exit Distributiions: $s = \infty$

	Garcia/Siewert		QMC	
μ	$\psi(0, -\mu)$	$\psi(\tau, -\mu)$	$\psi(0, -\mu)$	$\psi(\tau, -\mu)$
5.00e-02	8.97798e-01	1.02202e-01	8.47454e-01	1.00663e-01
1.00e-01	8.87836e-01	1.12164e-01	8.52822e-01	1.10325e-01
2.00e-01	8.69581e-01	1.30419e-01	8.47710e-01	1.29064e-01
3.00e-01	8.52299e-01	1.47701e-01	8.35879e-01	1.46849e-01
4.00e-01	8.35503e-01	1.64497e-01	8.22291e-01	1.64034e-01
5.00e-01	8.18996e-01	1.81004e-01	8.08044e-01	1.80827e-01
6.00e-01	8.02676e-01	1.97324e-01	7.93459e-01	1.97336e-01
7.00e-01	7.86493e-01	2.13507e-01	7.78672e-01	2.13625e-01
8.00e-01	7.70429e-01	2.29571e-01	7.63768e-01	2.29725e-01
9.00e-01	7.54496e-01	2.45504e-01	7.48818e-01	2.45642e-01
1.00e+00	7.38721e-01	2.61279e-01	7.33889e-01	2.61361e-01

III. CONCLUSION

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