The Journal of Open Source Software

MultiPrecisionArrays.jl: A Julia package for iterative refinement

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DOI: 10.xxxxxx/draft

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Editor: Open Journals

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Submitted: 01 January 1970
Published: unpublished

Summary

MultiPrecisionArrays.jl, (Kelley, 2024b, 2024c) provides data structures and solvers for several variations of iterative refinement (IR). IR can speed up an LU matrix factorization for solving linear systems of equations by factoring a low precision copy of the matrix and using that low precision factorization in a iteration loop to solve the system. For example, if high precision is double and low precision is single, then the factorization time is cut in half. The additional storage cost is the low precision copy, so IR is at time vs storage trade off. IR has a long history and a good account of the classical theory is in (Higham, 1996).

Statement of need

The solution of linear systems of equations is a ubiquitous task in computational science and engineering. A common method for dense systems is Gaussian elimination done via an LU factorization, (Higham, 1996). Iterative refinement is a way to reduce the factorization time at the cost of additional storage. MultiPrecisionArrays.jl enables IR with a simple interface in Julia (Bezanson et al., 2017) with an IR factorization object that one uses in the same way as the one for LU. The package offers several variants of IR, both classical (Higham, 1996; Wilkinson, 1948), and some from the recent literature (Amestoy et al., 2024; Carson & Higham, 2017).

Algorithm

This package will make solving dense systems of linear equations faster by using the LU factorization and IR. While other factorizations can be used in IR, the package is limited to LU for now. A very generic description of this for solving a linear system $Ax = b$ in a high (working) precision is

IR(A, b)

- $x = 0$
- $r = b$
- Factor $A = LU$ in a lower precision
- While $\|r\|$ is too large
  - $d = (LU)^{-1}r$
  - $x = x + d$
  - $r = b - Ax$
- end

- end

In Julia, a code to do this would solve the linear system $Ax = b$ in in the working precision, say double, by using a factorization in a lower (factorization) precision, say single, within a residual correction iteration. This means that one would need to allocate storage for a copy of A in the factorization precision and factor that copy.

The multiprecision factorization `mplu` makes the low precision copy of the matrix, factors that copy, and allocates some storage for the iteration. The original matrix and the low precision factorization are stored in a factorization object that you can use with `\`.

IR is a perfect example of a storage/time trade off. To solve a linear system $Ax = b$ in R^N with IR, one incurs the storage penalty of making a low precision copy of A and reaps the benefit of only having to factor the low precision copy.

Installation

The standard way to install a package is to type `import.Pkg; Pkg.add("MultiPrecisionArrays")` at the Julia prompt. One can run the unit tests with `Pkg.test("MultiPrecisionArrays")`. After installation, type `using MultiPrecisionArrays` when you want to use the functions in the package.

There are only two direct dependencies outside of the Julia standard libraries. The factorization in half precision (Float16) uses OhMyThreads.jl. The GMRES and Bi-CGSTAB solvers for Krylov-IR methods are taken from SIAMFANL.jl (Kelley, 2022c).

Example

Here is a simple example to show how `mplu` works. We will follow that with some benchmarking on the cost of factorizations. The computations were done with Julia 1.10.4 on an Apple Mac Mini with an M2 pro processor and 32GB RAM. We used OPENBLAS for LAPACK and the BLAS for this example. Other choices, such as the AppleAccelerate framework would work equally well.

In this example high (working) precision is double, `Float64`, and low (factorization) precision is single, `Float32`. The matrix is the sum of the identity and a constant multiple of the trapezoid rule discretization of the Greens operator for $-d^2/dx^2$ on $[0, 1]$

$$Gu(x) = \int_0^1 g(x, y)u(y)\, dy$$

where

$$g(x, y) = \min(x, y)(1 - \max(x, y)).$$

The discretization for an N-point discretization is the $N \times N$ maatrix

$$G_{ij} = g(x_i, x_j)/(N + 1)$$

where $x_i = i/(N + 1)$.

The code for construction G is in the **/src/Examples** directory. The file is **Gmat.jl**. You need to do

```
using MultiPrecisionArrays
using MultiPrecisionArrays.Examples
```

to use `mplu` and `Gmat`.

Now we will see how the results look. In this example we compare the result with iterative refinement with `A\b`, which is LAPACK's LU. As you can see the results are equally good. Note that a factorization object `MPF` is the output of `mplu`. This is analogous to `AF=lu(A)` in LAPACK.

```
julia> using MultiPrecisionArrays

julia> using MultiPrecisionArrays.Examples

julia> using BenchmarkTools

julia> N=4096;

julia> # Build G on an N x N grid

julia> G=Gmat(N);

julia> # The operator is I - G

julia> A = I - G;

julia> # Create a test problem where the solution is x = ones(N)

julia> x=ones(N); b=A*x;

julia> # Compute the mplu and lu factorizations of A

julia> MPF=mplu(A); AF=lu(A);

julia> # Solve A x = b with both factorizations

julia> z=MPF\b; w=AF\b;

julia> # Compute relative errors and residuals for mplu

julia> mpluerr=norm(z-x,Inf); mpluresid=norm(b-A*z,Inf)/norm(b,Inf);

julia> # Compute relative errors and residuals for lu

julia> luerr=norm(w-x,Inf); luresid=norm(b-A*w,Inf)/norm(b,Inf);

julia> # Print the results

julia> println("Errors: $mpluerr, $luerr. Residuals: $mpluresid, $luresid")
Errors: 4.44089e-16, 6.68354e-14. Residuals: 4.44089e-16, 6.68354e-14
```

So the results are equally good.

The compute time for `mplu` should be roughly half that of `lu`. A fair comparison is with `lu!`, which does not allocate new storage for the factorization. `mplu` factors a low precision array, so the factorization cost is cut in half. Memory is a different story because neither `mplu` nor `lu!` allocate storage for a new high precision array, but mplu allocates for a low precision copy, so the memory and allocation cost for mplu is 50% more than lu. One issue with smaller problems is that the triangular solve does not parallelize as well as the factorization, so does not exploit multi-core processor as well. We can see this in the IR solver times because each

iteration of IR needs a matrix-vector multiply and a triangular solve.

```
julia> using BenchmarkTools

julia> @belapsed mplu($A)
8.60945e-02

julia> @belapsed lu!(AC) setup=(AC=copy($A))
1.42840e-01

# And now for the solve times

julia> @belapsed ldiv!($AF,bb) setup=(bb = copy($b))
4.79117e-03

julia> @belapsed $MPF\$b
2.01195e-02
```

So the total solve time (factorization and solve) is less for IR by roughly 40%.

A Few Subtleties

Within the algorithm one has to determine what the line $d = (LU)^{-1}r$ means. Does one cast r into the lower precision before the solve or not? If one casts r into the lower precision, then the solve is done entirely in the factorization precision. If, however, r remains in the working precision, then the LU factors are promoted to the working precision on the fly. This makes little difference if TW is double and TF is single and there is a modest performance benefit to downcasting r into single. Therefore that is the default behavior in that case. If TF is half precision, `Float16`, then it is best to do the interprecision transfers on the fly and if one is using one of the Krylov-IR algorithms (Amestoy et al., 2024) then one must do the interprecision transfers on the fly and not downcast r.

There are two half precision (16 bit) formats. Julia has native support for IEEE 16 bit floats (Float16). A second format (BFloat16) has a larger exponent field and a smaller significand (mantissa), thereby trading precision for range. In fact, the exponent field in BFloat is the same size (8 bits) as that for single precision (Float32). The significand, however, is only 8 bits. Compare this to the size of the exponent fields for Float16 (11 bits) and single (24 bits). The size of the significand means that you can get in real trouble with half precision in either format and that IR is more likely to fail to converge. GMRES-IR can mitigate the convergence problems (Amestoy et al., 2024) by using the low-precision solve as a preconditioner. We support both GMRES (Saad & Schultz, 1986) and BiCGSTAB (Vorst, 1992) as solvers for Krylov-IR methods. One should also know that LAPACK and the BLAS do not yet support half precision arrays, so working in Float16 will be slower than using Float64.

The classic algorithm from (Wilkinson, 1948) and its recent extension (Carson & Higham, 2017) evaluate the residual in a higher precision that the working precision. This can give improved accuracy for ill-conditioned problems at a cost of the interprecision transfers in the residual computation. This needs to be implemented with some care and (Demmel et al., 2006) has an excellent account of the details.

MultiPrecisionArrays.jl provides infrastructure to manage these things and we refer the reader to (Kelley, 2024c) for the details.

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Projects using MultiPrecisionArrays.jl.

This package was motivated by the use of low-precision factorizations in Newton's method Kelley (2022c) and the interface between a preliminary version of this package and the solvers from (Kelley, 2022b) was reported in (Kelley, 2023). That paper used a three precision form of IR (TF=half, TW=single, nonlinear residual computed in double) and required direct use of multiprecision constructors that we do not export in **MultiPrecisionArrays.jl**. We will fully support the application to nonlinear solvers in a future version. We give a detailed account of interprecision transfers in (Kelley, 2024a) and use **MultiPrecisionArrays.jl** to generate the table in that paper.

Other Julia Packages for IR

The package IterativeRefinement.jl is an implementation of the IR method from (J.Dongarra et al., 1983). It has not been updated in four years.

The unregistered package Itref.jl implements IR and the GMRES-IR method from (Amestoy et al., 2024) and was used to obtain the numerical results in that paper. It does not provide the data structures for preallocation that we do and does not seem to have been updated lately.

Acknowledgements

This work was partially supported by Department of Energy grant DE-NA003967. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the United States Department of Energy.

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