

# MultiPrecisionArrays.jl: A Julia package for iterative refinement

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## Summary

[MultiPrecisionArrays.jl](#), ([Kelley, 2024a, 2024b](#)) provides data structures and solvers for several variations of iterative refinement (IR). IR can speed up an LU matrix factorization for solving linear systems of equations by factoring a low precision copy of the matrix and using that low precision factorization in a iteration loop to solve the system. For example, if high precision is double and low precision is single, then the factorization time is cut in half. The additional storage cost is the low precision copy, so IR is at time vs storage trade off. IR has a long history and a good account of the classical theory is in ([Higham, 1996](#)).

## Statement of need

The solution of linear systems of equations is a ubiquitous task in computational science and engineering. A common method for dense systems is Gaussian elimination done via an LU factorization, ([Higham, 1996](#)). Iterative refinement is a way to reduce the factorization time at the cost of additional storage. [MultiPrecisionArrays.jl](#) enables IR with a simple interface in Julia ([Bezanson et al., 2017](#)) with an IR factorization object that one uses in the same way as the one for LU. The package offers several variants of IR, both classical ([Higham, 1996](#); [Wilkinson, 1948](#)), and some from the recent literature ([Amestoy et al., 2024](#); [Carson & Higham, 2017](#)).

## Algorithm

This package will make solving dense systems of linear equations faster by using the LU factorization and IR. While other factorizations can be used in IR, the package is limited to LU for now. A very generic description of this for solving a linear system  $Ax = b$  in a high (working) precision is

### IR(A, b)

- $x = 0$
- $r = b$
- Factor  $A = LU$  in a lower precision
- While  $\|r\|$  is too large
  - $d = (LU)^{-1}r$
  - $x = x + d$
  - $r = b - Ax$
- end

36     ▪ end

37 In Julia, a code to do this would solve the linear system  $Ax = b$  in the working precision,  
38 say double, by using a factorization in a lower (factorization) precision, say single, within a  
39 residual correction iteration. This means that one would need to allocate storage for a copy of  
40  $A$  in the factorization precision and factor that copy.

41 The multiprecision factorization `mplu` makes the low precision copy of the matrix, factors that  
42 copy, and allocates some storage for the iteration. The original matrix and the low precision  
43 factorization are stored in a factorization object that you can use with `\`.

44 IR is a perfect example of a storage/time trade off. To solve a linear system  $Ax = b$  in  $R^N$   
45 with IR, one incurs the storage penalty of making a low precision copy of  $A$  and reaps the  
46 benefit of only having to factor the low precision copy.

## 47 Installation

48 The standard way to install a package is to type `import Pkg; Pkg.add("MultiPrecisionArrays")`  
49 at the Julia prompt. One can run the unit tests with `Pkg.test("MultiPrecisionArrays")`.  
50 After installation, type using `MultiPrecisionArrays` when you want to use the functions in  
51 the package.

52 There are only two direct dependencies outside of the Julia standard libraries. The factorization  
53 in half precision (`Float16`) uses [OhMyThreads.jl](#). The GMRES and Bi-CGSTAB solvers for  
54 Krylov-IR methods are taken from [SIAMFANL.jl](#) ([Kelley, 2022c](#)).

## 55 Example

56 Here is a simple example to show how `mplu` works. We will follow that with some benchmarking  
57 on the cost of factorizations. The computations were done with Julia 1.10.2 on an Apple Mac  
58 Mini with an M2 pro processor and 32GB RAM. We used OPENBLAS for LAPACK and the  
59 BLAS for this example. Other choices, such as the [AppleAccelerate](#) framework would work  
60 equally well.

61 In this example high (working) precision is double, `Float64`, and low (factorization) precision is  
62 single, `Float32`. The matrix is the sum of the identity and a constant multiple of the trapezoid  
63 rule discretization of the Greens operator for  $-d^2/dx^2$  on  $[0, 1]$

$$Gu(x) = \int_0^1 g(x, y)u(y) dy$$

64 where

$$g(x, y) = \min(x, y)(1 - \max(x, y)).$$

65 The discretization for an  $N$ -point discretization is the  $N \times N$  maatrix

$$G_{ij} = g(x_i, x_j)/(N + 1)$$

66 where  $x_i = i/(N + 1)$ .

67 The code for construction  $G$  is in the `/src/Examples` directory. The file is `Gmat.jl`. You need  
68 to do

```
69 using MultiPrecisionArrays
70 using MultiPrecisionArrays.Examples
```

```

71  to use mpla and Gmat.

72  Now we will see how the results look. In this example we compare the result with iterative
73  refinement with A\b, which is LAPACK's LU. As you can see the results are equally good.
74  Note that a factorization object MPF is the output of mpla. This is analogous to AF=lu(A) in
75  LAPACK.

76  julia> using MultiPrecisionArrays
77
78  julia> using MultiPrecisionArrays.Examples
79
80  julia> using BenchmarkTools
81
82  julia> N=4096;
83
84  julia> # Build G on an N x N grid
85
86  julia> G=Gmat(N);
87
88  julia> # The operator is I - G
89
90  julia> A = I - G;
91
92  julia> # Create a test problem where the solution is x = ones(N)
93
94  julia> x=ones(N); b=A*x;
95
96  julia> # Compute the mpla and lu factorizations of A
97
98  julia> MPF=mpla(A); AF=lu(A);
99
100 julia> # Solve A x = b with both factorizations
101
102 julia> z=MPF\b; w=AF\b;
103
104 julia> # Compute relative errors and residuals for mpla
105
106 julia> mplaerr=norm(z-x,Inf); mplaresid=norm(b-A*z,Inf)/norm(b,Inf);
107
108 julia> # Compute relative errors and residuals for lu
109
110 julia> luerr=norm(w-x,Inf); luresid=norm(b-A*w,Inf)/norm(b,Inf);
111
112 julia> # Print the results
113
114 julia> println("Errors: $mplaerr, $luerr. Residuals: $mplaresid, $luresid")
115 Errors: 4.44089e-16, 6.68354e-14. Residuals: 4.44089e-16, 6.68354e-14

116 So the results are equally good.

117 The compute time for mpla should be roughly half that of lu. A fair comparison is with lu!,
118 which does not allocate new storage for the factorization. mpla factors a low precision array,
119 so the factorization cost is cut in half. Memory is a different story because neither mpla nor
120 lu! allocate storage for a new high precision array, but mpla allocates for a low precision copy,
121 so the memory and allocation cost for mpla is 50% more than lu.

122 julia> using BenchmarkTools

```

```

123
124 julia> @belapsed mpla($A)
125 8.60945e-02
126
127 julia> @belapsed lu!(AC) setup=(AC=copy($A))
128 1.42840e-01

```

## A Few Subtleties

Within the algorithm one has to determine what the line  $d = (LU)^{-1}r$  means. Does one cast  $r$  into the lower precision before the solve or not? If one casts  $r$  into the lower precision, then the solve is done entirely in the factorization precision. If, however,  $r$  remains in the working precision, then the LU factors are promoted to the working precision on the fly. This makes little difference if TW is double and TF is single and there is a modest performance benefit to downcasting  $r$  into single. Therefore that is the default behavior in that case. If TF is half precision, Float16, then it is best to do the interprecision transfers on the fly and if one is using one of the Krylov-IR algorithms (Amestoy et al., 2024) then one must do the interprecision transfers on the fly and not downcast  $r$ .

There are two half precision (16 bit) formats. Julia has native support for IEEE 16 bit floats (Float16). A second format (BFloat16) has a larger exponent field and a smaller significand (mantissa), thereby trading precision for range. In fact, the exponent field in BFloat is the same size (8 bits) as that for single precision (Float32). The significand, however, is only 8 bits. Compare this to the size of the exponent fields for Float16 (11 bits) and single (24 bits). The size of the significand means that you can get in real trouble with half precision in either format and that IR is more likely to fail to converge. GMRES-IR can mitigate the convergence problems (Amestoy et al., 2024) by using the low-precision solve as a preconditioner. We support both GMRES (Saad & Schultz, 1986) and BiCGSTAB (van der Vorst, 1992) as solvers for Krylov-IR methods. One should also know that LAPACK and the BLAS do not yet support half precision arrays, so working in Float16 will be slower than using Float64.

The classic algorithm from (Wilkinson, 1948) and its recent extension (Carson & Higham, 2017) evaluate the residual in a higher precision than the working precision. This can give improved accuracy for ill-conditioned problems at a cost of the interprecision transfers in the residual computation. This needs to be implemented with some care and (Demmel et al., 2006) has an excellent account of the details.

**MultiPrecisionArrays.jl** provides infrastructure to manage these things and we refer the reader to (Kelley, 2024b) for the details.

## Projects using MultiPrecisionArrays.jl.

This package was motivated by the use of low-precision factorizations in Newton's method Kelley (2022c) and the interface between a preliminary version of this package and the solvers from (Kelley, 2022b) was reported in (Kelley, 2023). That paper used a three precision form of IR (TF=half, TW=single, nonlinear residual computed in double) and required direct use of multiprecision constructors that we do not export in **MultiPrecisionArrays.jl**. We will fully support the application to nonlinear solvers in a future version.

## Other Julia Packages for IR

The package **IterativeRefinement.jl** is an implementation of the IR method from (J.Dongarra et al., 1983). It has not been updated in four years.

The unregistered package [ltref.jl](#) implements IR and the GMRES-IR method from (Amestoy et al., 2024) and was used to obtain the numerical results in that paper. It does not provide the data structures for preallocation that we do and does not seem to have been updated lately.

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