

MPResults

June 25, 2020

1 Newton's Method in Multiple Precision: C. T. Kelley

This notebook documents the results in C. T. Kelley, *Newton's Method in Mixed Precision*, 2020.[8]

As an example we will solve the Chandrasekhar H-equation [2]. This equation, which we describe in detail in the Example section, has a fast $O(N \log(N))$ function evaluation, a Jacobian evaluation that is $O(N^2)$ work analytically and $O(N^2 \log(N))$ with a finite difference. This means that most of the work, if you do things right, is in the LU factorization of the Jacobian.

The difference between double, single, and half precision will be clear in the results from the examples. This notebook has no half-precision computations. Julia does half-precision in software and that is very slow.

1.1 Contents

- Section 1.2: Integral equations example
- Section 1.3: Install the application.
- Section 1.4: First solver test
- Section 1.5: Using nsold.jl, **version .01!!!**
- Section 5: Running the codes that generated the results

1.2 The Chandrasekhar H-Equation

The example is the mid-point rule discretization of the Chandrasekhar H-equation [2].

$$\mathcal{F}(H)(\mu) = H(\mu) - \left(1 - \frac{c}{2} \int_0^1 \frac{\mu H(\mu)}{\mu + \nu} d\nu\right)^{-1} = 0. \quad (1)$$

The nonlinear operator \mathcal{F} is defined on $C[0, 1]$, the space of continuous functions on $[0, 1]$.

The equation has a well-understood dependence on the parameter c [9], [3]. The equation has unique solutions at $c = 0$ and $c = 1$ and two solutions for $0 < c < 1$. There is a simple fold singularity [5] at $c = 1$. Only one [2], [1] of the two solutions for $0 < c < 1$ is of physical interest and that is the one easiest to find numerically. One must do a continuation computation to find the other one.

The structure of the singularity is preserved if one discretizes the integral with any rule that integrates constants exactly. For the purposes of this paper the composite midpoint rule will suffice.

The N -point composite midpoint rule is

$$\int_0^1 f(v) dv \approx \frac{1}{N} \sum_{j=1}^N f(v_j) \quad (2)$$

where $v_j = (j - 1/2)/N$ for $1 \leq j \leq N$. This rule is second-order accurate for sufficiently smooth functions f . The solution of the integral equation is, however, not smooth enough. $H'(\mu)$ has a logarithmic singularity at $\mu = 0$.

The discrete problem is

$$\mathbf{F}(\mathbf{u})_i \equiv u_i - \left(1 - \frac{c}{2N} \sum_{j=1}^N \frac{u_j \mu_i}{\mu_j + \mu_i} \right)^{-1} = 0. \quad (3)$$

One can simplify the approximate integral operator and expose some useful structure. Since

$$\frac{c}{2N} \sum_{j=1}^N \frac{u_j \mu_i}{\mu_j + \mu_i} = \frac{c(i - 1/2)}{2N} \sum_{j=1}^N \frac{u_j}{i + j - 1}. \quad (4)$$

hence the approximate integral operator is the product of a diagonal matrix and a Hankel matrix and one can use an FFT to evaluate that operator with $O(N \log(N))$ work [4].

We can express the approximation of the integral operator in matrix form

$$(\mathbf{L}\mathbf{u})_i = \frac{c(i - 1/2)}{2N} \sum_{j=1}^N \frac{u_j}{i + j - 1} \quad (5)$$

and compute the Jacobian analytically as

$$\mathbf{F}'(\mathbf{u}) = \mathbf{I} - \text{diag}(\mathbf{G}(\mathbf{u}))^2 \mathbf{L} \quad (6)$$

where

$$\mathbf{G}(\mathbf{u})_i = \left(1 - \frac{c}{2N} \sum_{j=1}^N \frac{u_j \mu_i}{\mu_j + \mu_i} \right)^{-1}. \quad (7)$$

Hence the data for the Jacobian is already available after one computes $\mathbf{F}(\mathbf{u}) = \mathbf{u} - \mathbf{G}(\mathbf{u})$ and the Jacobian can be computed with $O(N^2)$ work. We do that in this example and therefore the only $O(N^3)$ part of the solve is the matrix factorization.

One could also approximate the Jacobian with forward differences. In this case one approximates the j th column $\mathbf{F}'(\mathbf{u})_j$ of the Jacobian with

$$\frac{\mathbf{F}(\mathbf{u} + h\mathbf{\tilde{e}}_j) - \mathbf{F}(\mathbf{u})}{h} \quad (8)$$

where $\mathbf{\tilde{e}}_j$ is a unit vector in the j th coordinate direction and h is a suitable difference increment. If one computes \mathbf{F} in double precision with unit roundoff u_d , then $h = O(\|\mathbf{u}\| \sqrt{u_d})$ is a reasonable choice [6]. Then the error in the Jacobian is $O(\sqrt{u_d}) = O(u_s)$ where u_s is unit roundoff in single precision. The cost of a finite difference Jacobian in this example is $O(N^2 \log(N))$ work.

The analysis in [8] suggests that there is no significant difference in the nonlinear iteration from either the choice of analytic or finite difference Jacobians or the choice of single or double precision for the linear solver. This notebook has the data used in that paper to support that assertion. You will be able to duplicate the results and play with the codes.

Half precision is another story and we have those codes for you, too.

1.3 Setting up

You need to install these packages:

- PyPlot
- LinearAlgebra
- JLD
- Printf
- FFTW
- IJulia (You must have done this already or you would not be looking at this notebook.)
- AbstractFFTs

The directory is a Julia project. So all you should need to do to get going is to

1. Put the directory in your `LOAD_PATH`. The way to do this is to type

```
push!(LOAD_PATH, "/Users/yourid/whereyouputit/MPResults")
```

at the Julia prompt in the REPL or in a notebook code window.

2. Now load the modules with

```
using MPResults2019
```

3. Then you can do a simple solve and test that you did it right by typing

```
hout=heqtest()
```

which I will do now. Make sure **MPResults** is in your `LOAD_PATH`! If you forget to do the `push!` command, strange things may happen.

Make absolutely sure that you are in the MPResults directory. Then ...

```
[1]: MPDir=pwd()
      push!(LOAD_PATH,MPDir)
      using MPResults
```

1.4 Running the solver

The codes solve the H-equation and plot/tabulate the results in various ways. **heqtest** prints on column of the tables in Chandrasekhar's book. It calls **nsold.jl**. I've shown you the output from the solver but that is not important for now. You get the details on the solver in the next section.

```
[2]: heqtest()
```

```
0.00e+00      1.00000e+00
5.00e-02      1.04424e+00
```

1.00e-01	1.07236e+00
1.50e-01	1.09470e+00
2.00e-01	1.11346e+00
2.50e-01	1.12965e+00
3.00e-01	1.14389e+00
3.50e-01	1.15657e+00
4.00e-01	1.16797e+00
4.50e-01	1.17830e+00
5.00e-01	1.18773e+00
5.50e-01	1.19638e+00
6.00e-01	1.20435e+00
6.50e-01	1.21172e+00
7.00e-01	1.21856e+00
7.50e-01	1.22493e+00
8.00e-01	1.23088e+00
8.50e-01	1.23646e+00
9.00e-01	1.24169e+00
9.50e-01	1.24662e+00
1.00e+00	1.25126e+00

```
[1.54457e+00, 7.94167e-03, 1.55209e-07, 2.39149e-15, 1.61651e-15, 1.36877e-15,
1.47288e-15, 1.76242e-15, 2.20932e-15, 1.50598e-15, 1.06489e-15]
```

```
[2]: (exactout = (solution = [1.00707e+00, 1.01754e+00, 1.02622e+00, 1.03392e+00,
1.04094e+00, 1.04744e+00, 1.05352e+00, 1.05925e+00, 1.06469e+00, 1.06986e+00 ...
1.24220e+00, 1.24320e+00, 1.24419e+00, 1.24517e+00, 1.24614e+00, 1.24709e+00,
1.24804e+00, 1.24897e+00, 1.24989e+00, 1.25081e+00], ithist = [1.54457e+00,
7.94167e-03, 1.55209e-07, 2.39149e-15, 1.61651e-15, 1.36877e-15, 1.47288e-15,
1.76242e-15, 2.20932e-15, 1.50598e-15, 1.06489e-15]), fdout = (solution =
[1.00707e+00, 1.01754e+00, 1.02622e+00, 1.03392e+00, 1.04094e+00, 1.04744e+00,
1.05352e+00, 1.05925e+00, 1.06469e+00, 1.06986e+00 ... 1.24220e+00, 1.
↪24320e+00,
1.24419e+00, 1.24517e+00, 1.24614e+00, 1.24709e+00, 1.24804e+00, 1.24897e+00,
1.24989e+00, 1.25081e+00], ithist = [1.54457e+00, 7.94166e-03, 1.55234e-07,
2.02292e-15, 1.40433e-15, 1.29473e-15, 1.47288e-15, 1.52226e-15, 1.27555e-15,
1.43901e-15, 1.38667e-15]))
```

heqtest.jl, calls the solver and harvests some iteration statistics. The two columns of numbers are the results from [2] (page 125). The iteration statistics are from nsold.jl, the solver.

1.5 NSOLD

The solver is *nsold.jl* version .01. Keep in mind that nothing with a version number with a negative exponent field is likely to be very good. *nsold.jl* is included when you run the *MPResults2019.jl* module, which you do automatically when you type **using MPResults2019**

nsold.jl is a simple implementation of Newton's method (see [6] and [7]) using an LU factorization of the Jacobian to compute the Newton step with no line search or globalization. The code evaluates and factors the Jacobian at every nonlinear iteration.

1.6 Using nsold.jl

At the level of this notebook, it's pretty simple. Remember that Julia hates to allocate memory. So your function and Jacobian evaluation routines should expect the calling function to **preallocate** the storage for both the function and Jacobian. Your functions will then use `.=` to put the function and Jacobian where they are supposed to be.

It's worthwhile to look at the help screen.

```
[3]: ?nsold
```

```
search: nsold
plotnsold
```

```
[3]: nsold(x, FS, FPS, F!, J!=diffjac!; rtol=1.e-6, atol=1.e-12,
        maxit=20, dx=1.e-6, pdata=nothing)
```

This is Version .01. Nothing with a version number having a negative exponent field can be trusted.

Nonlinear solvers from my books in Julia. This version has no globalization, no quasi-Newton methods, and no Newton-Krylov. The mission here is to duplicate the mixed precision results in my SIREV-ED submission.

2 Inputs:

- `x`: initial iterate
- `FS`: Preallocated storage for function. It is an $N \times 1$ column vector
- `FPS`: preallocated storage for Jacobian. It is an $N \times N$ matrix
- `F!`: function evaluation, the `!` indicates that `F!` overwrites `FS`, your preallocated storage for the function.
- `J!`: Jacobian evaluation, the `!` indicates that `J!` overwrites `FPS`, your preallocated storage for the Jacobian. If you leave this out the default is a finite difference Jacobian.

3 keyword arguments (kwargs):

- `rtol` and `atol`: relative and absolute error tolerances
- `maxit`: limit on nonlinear iterations
- `dx`: difference increment in finite-difference derivatives $h=dx*\text{norm}(x)+1.e-8$

- `pdata`: precomputed data for the function/Jacobian. Things will go better if you use this rather than hide the data in global variables within the module for your function/Jacobian
-

4 Using `nsold.jl`

Here are the rules as of June 6, 2019

`F!` is the nonlinear residual. `J!` is the Jacobian evaluation.

I like to put everything in a module so that I can use global (within the module) variables.

1. You allocate storage for the function and Jacobian in advance -> in the calling program <- NOT in FS and FPS

`FV=F!(FV,x)` returns `FV=F(x)`

`FP=J!(FV,FP,x)` returns `FP=F'(x)`; `(FV,FP, x)` must be the argument list, even if `FP` does not need `FV`. One reason for this is that the finite-difference Jacobian does and that is the default in the solver.

In the future `J!` will also be a matrix-vector product and `FPS` will be the `PREALLOCATED (!)` storage for the `GMRES(m)` Krylov vectors.

Lemme tell ya 'bout precision. I designed this code for full precision functions and linear algebra in any precision you want. You can declare `FPS` as `Float64`, `Float32`, or `Float16` and `nsold` will do the right thing if YOU do not destroy the declaration in your `J!` function. I'm amazed that this works so easily.

If the Jacobian is reasonably well conditioned, I can see no reason to do linear algebra in double precision

Don't try to evaluate function and Jacobian all at once because that will cost you a extra function evaluation everytime the line search kicks in.

2. Any precomputed data for functions, Jacobians, matrix-vector products may live in global variables within a module containing `F!` and `J!`. Don't do that if you can avoid it. Use `pdata` instead.

4.1 How `nsold.jl` controls the precision of the Jacobian

You can control the precision of the Jacobian by simply allocating `FPS` in your favorite precision. So if I have a problem with `N=256` unknowns I will declare `FP` as `zeros(N,1)` and may declare `FPS` as `zeros(N,N)` or `Float32.(zeros(N,N))`.

Note the `.` between `Float32` and the paren. This, as is standard Julia practice, applies the conversion to `Float32` to every element in the array. If you forget the `.` Julia will complain.

5 The results in the paper

The paper has plots for double, single, and half precision computations for $c=.5$, $.99$, and 1.0 . The half precision results take a very long time to get. On my computer (2019 iMac; 8 cores, 64GB of memory) the half precision compute time was over two weeks. Kids, don't try this at home.

The data for the paper are in the cleverly named directory `Data_From_Paper`

`cd` to the directory `MPResults` and from that directory run

```
data_harvest("Data_From_Paper/MP_Data_")
```

at the julia prompt you will generate all the tables and plots in the paper.

If you have the time and patience you can also generate the data with `data_populate.jl`. This creates three directories named `Mixed_Precision_c=?` and you can see for yourself. Look at that file to see opportunities to edit the time-consuming jobs out. If you eliminate the half precision work and the larger dimensions, the code will run in a short time.

`data_populate(c; half='no', level=p)` does double and single precision solves for 1024×2^k point grids for $k=0, \dots, p-1$. Set `half='yes'` to make the computation take much longer.

Here is a simple example of using `data_populate` and the plotter code `plot_nsold.jl`. I'm only using the 1024, 2048 and 4096 point grids. The plot in the paper uses more levels. This is part of Figure 1 in the paper.

Look at the source to `data_populate.jl` and `plotnsold.jl` and you'll see how I did this. These codes only manage files, plots, and tables. There is nothing really exciting here. You don't need to know much Julia to understand this, but you do need to know something and I can't help with that.

To begin with, I will create a directory called `Data_Test` to put all this stuff. I will `cd` to that directory.

```
[4]: cd(MPRdir)
      Home4mp="Data_Test"
      try (mkdir(Home4mp))
      catch
      end
      cd(Home4mp)
      pwd()
```

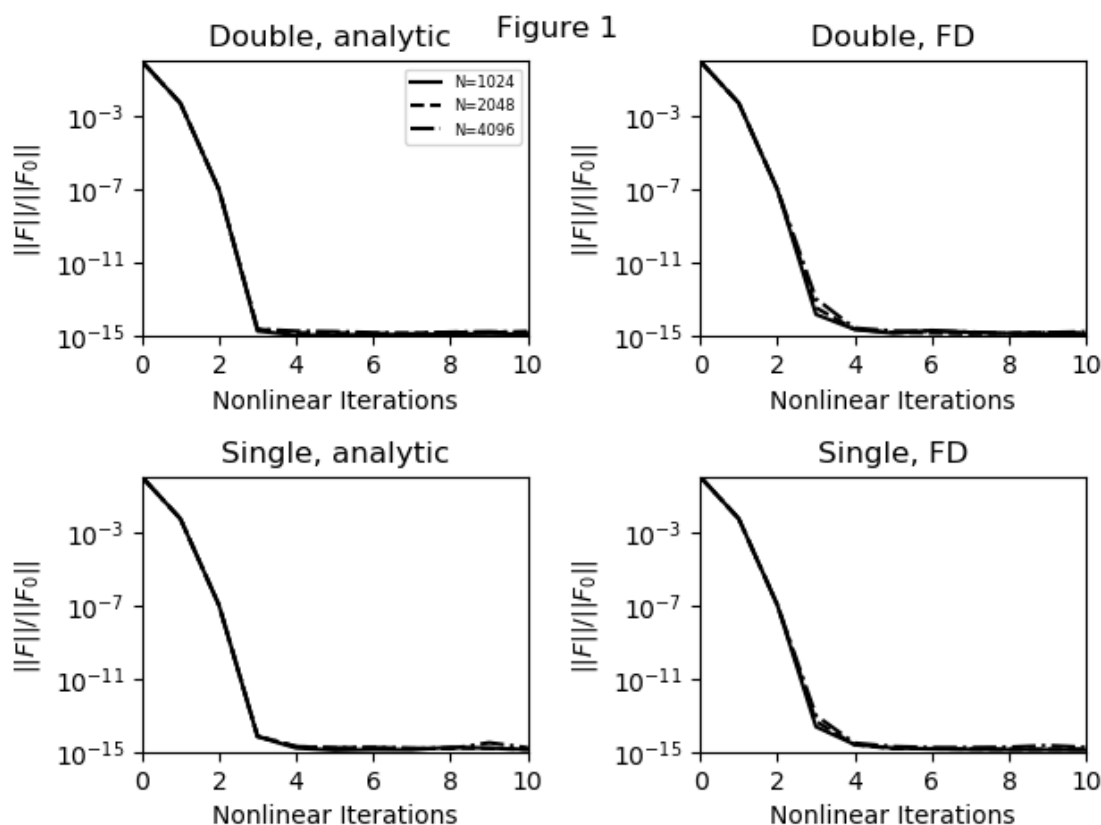
```
[4]: "/Users/ctk/Dropbox/Julia/dev/MPResults/Data_Test"
```

Now I'll run `data_populate` to create a subdirectory with the data. `cd` to that directory and run `plotnsold`. That's how I created the plots in the paper with all values of c and all the problem sizes. The half precision computation took two weeks.

plotnsold("no",c,10,3;title) makes the four-plot with only double and single (no half).

Even with this small problem, data_populate takes a while. Be patient. Once it's done the plots will appear pretty rapidly.

```
[5]: using PyPlot
data_populate(.5;half="no",level=3)
# after data_populate's done, the plots happen fast
cd("Mixed_Precision_c=5")
figtitle="Figure 1"
plotnsold("no",.5,10,3;bigtitle=figtitle)
```



Finally, I will duplicate the tables and plots in the paper with the precomputed data in the Data_From_Paper directory.

The three LaTeX tables are pretty vivid demonstrations that using a half-precision Jacobian is a poor idea. The columns are ratios of successive residual norms. Those ratios are supposed to go to zero if the convergence is q-superlinear. In the half precision case, it is not. You can also see that from the plots at the bottom.

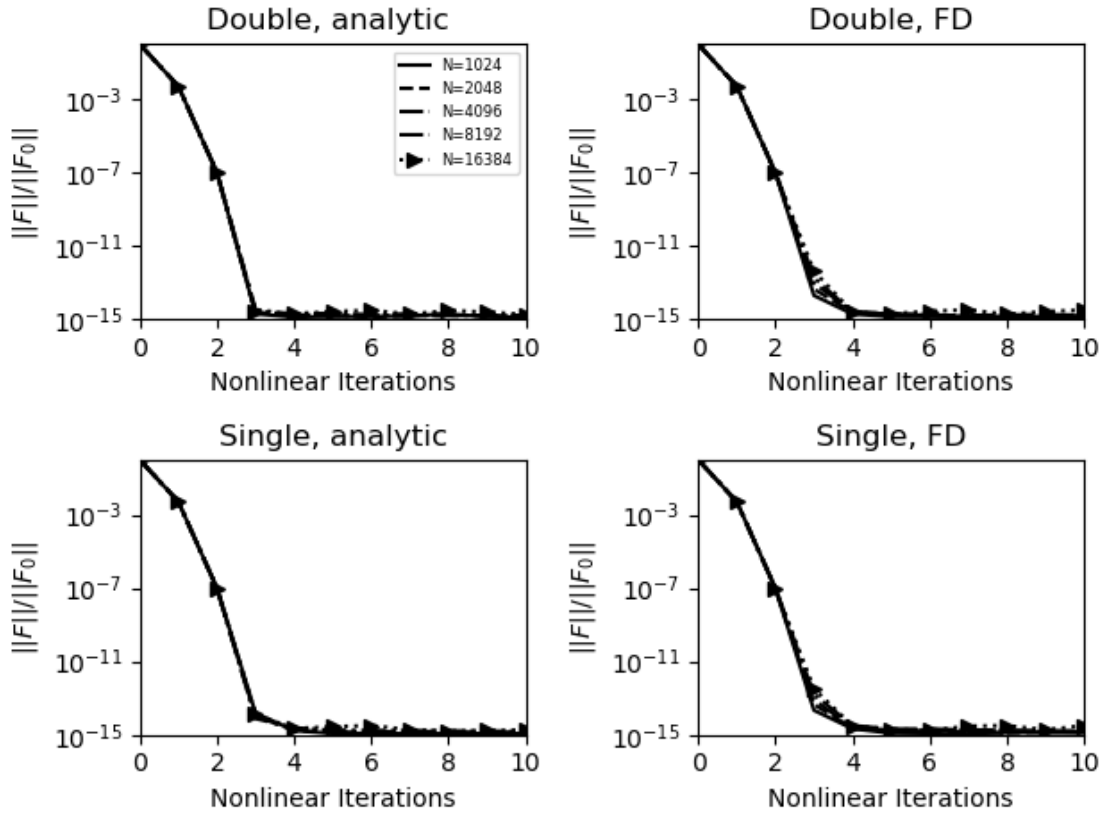

```
[6]: cd(MPRdir)
      data_harvest("Data_From_Paper/MP_Data_")
```

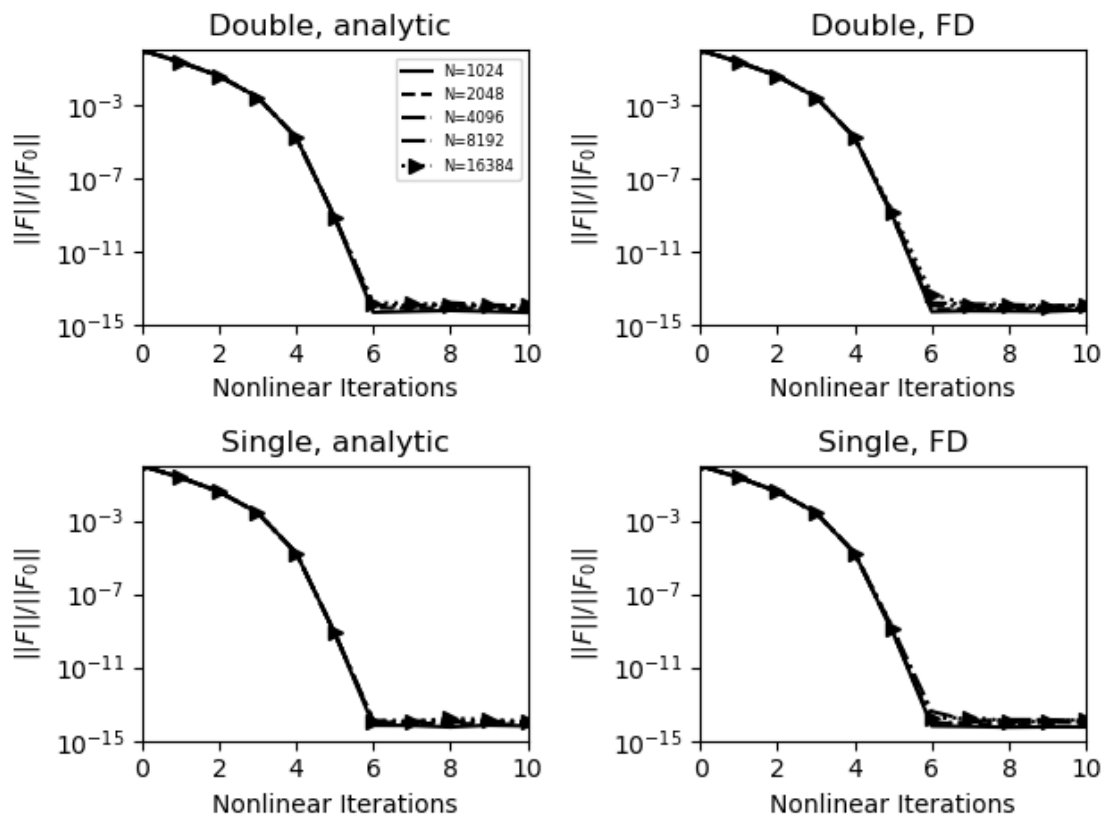
```
\begin{tabular}{l l l l l l l}
      n & 1024 & 2048 & 4096 & 8192 & 16384 & \\
\hline
1 & 1.51548e-02 & 1.50894e-02 & 1.50458e-02 & 1.50240e-02 & 1.50131e-02 & \\
2 & 1.07397e-02 & 1.06615e-02 & 1.06107e-02 & 1.05853e-02 & 1.05726e-02 & \\
3 & 1.04864e-02 & 1.04099e-02 & 1.03607e-02 & 1.03361e-02 & 1.03238e-02 & \\
4 & 1.04514e-02 & 1.03766e-02 & 1.03284e-02 & 1.03043e-02 & 1.02922e-02 & \\
5 & 1.04464e-02 & 1.03721e-02 & 1.03239e-02 & 1.03001e-02 & 1.02881e-02 & \\
6 & 1.04530e-02 & 1.03772e-02 & 1.03263e-02 & 1.02969e-02 & 1.02816e-02 & \\
7 & 1.01305e-02 & 9.79786e-03 & 1.05554e-02 & 1.05224e-02 & 1.06917e-02 & \\
8 & 8.73158e-02 & 9.79760e-02 & 1.09900e-01 & 1.31948e-01 & 1.24336e-01 & \\
9 & 8.46876e-01 & 7.99843e-01 & 9.21406e-01 & 7.13501e-01 & 1.60623e+00 & \\
10 & 8.73518e-01 & 9.65146e-01 & 8.40415e-01 & 9.29831e-01 & 7.88260e-01 & \\
\hline
\end{tabular}

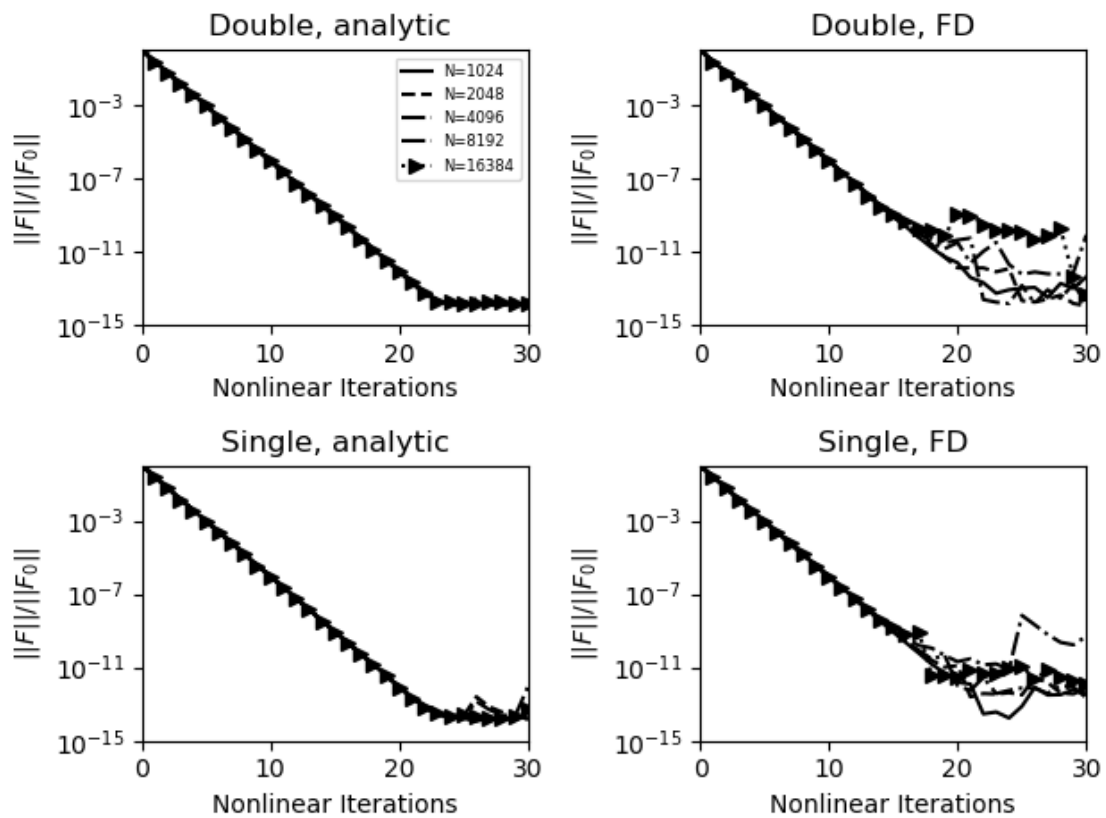
\begin{tabular}{l l l l l l l}
      n & 1024 & 2048 & 4096 & 8192 & 16384 & \\
\hline
1 & 2.47566e-01 & 2.93830e-01 & 2.94145e-01 & 2.94107e-01 & 2.94088e-01 & \\
2 & 2.05600e-01 & 3.48351e-01 & 3.77559e-01 & 3.77649e-01 & 3.77610e-01 & \\
3 & 1.40806e-01 & 3.16201e-01 & 4.38451e-01 & 4.38762e-01 & 4.38723e-01 & \\
4 & 9.31148e-02 & 3.28139e-01 & 4.73701e-01 & 4.73983e-01 & 4.73942e-01 & \\
5 & 8.59359e-02 & 3.38481e-01 & 4.92930e-01 & 4.93227e-01 & 4.93190e-01 & \\
6 & 8.65219e-02 & 3.40729e-01 & 5.03119e-01 & 5.03432e-01 & 5.03398e-01 & \\
7 & 8.66721e-02 & 3.41551e-01 & 5.08435e-01 & 5.08762e-01 & 5.08726e-01 & \\
8 & 8.66935e-02 & 3.41710e-01 & 5.11188e-01 & 5.11524e-01 & 5.11487e-01 & \\
9 & 8.66950e-02 & 3.41756e-01 & 5.12609e-01 & 5.12950e-01 & 5.12912e-01 & \\
10 & 8.66960e-02 & 3.41833e-01 & 5.13340e-01 & 5.13685e-01 & 5.13646e-01 & \\
\hline
\end{tabular}

\begin{tabular}{l l l l l l l}
      n & 1024 & 2048 & 4096 & 8192 & 16384 & \\
\hline
1 & 2.65171e-01 & 3.13548e-01 & 3.13867e-01 & 3.13831e-01 & 3.13812e-01 & \\
2 & 2.70831e-01 & 3.84700e-01 & 4.29466e-01 & 4.29666e-01 & 4.29633e-01 & \\
3 & 2.89581e-01 & 3.89214e-01 & 5.32231e-01 & 5.32478e-01 & 5.32453e-01 & \\
4 & 3.23427e-01 & 4.77575e-01 & 6.09958e-01 & 6.10101e-01 & 6.10123e-01 & \\
5 & 3.82508e-01 & 5.32789e-01 & 6.62135e-01 & 6.68348e-01 & 6.68369e-01 & \\
6 & 4.54440e-01 & 5.98778e-01 & 7.04067e-01 & 7.12635e-01 & 7.12645e-01 & \\
7 & 5.22841e-01 & 6.50593e-01 & 7.41734e-01 & 7.47025e-01 & 7.47031e-01 & \\
8 & 5.84470e-01 & 6.91184e-01 & 7.70896e-01 & 7.74339e-01 & 7.74343e-01 & \\
9 & 6.36332e-01 & 7.24333e-01 & 7.94007e-01 & 7.96482e-01 & 7.96485e-01 & \\
10 & 6.81255e-01 & 7.51888e-01 & 8.12841e-01 & 8.14760e-01 & 8.14762e-01 & \\
11 & 7.16247e-01 & 7.75225e-01 & 8.28523e-01 & 8.30083e-01 & 8.30085e-01 & \\
\hline
\end{tabular}
```

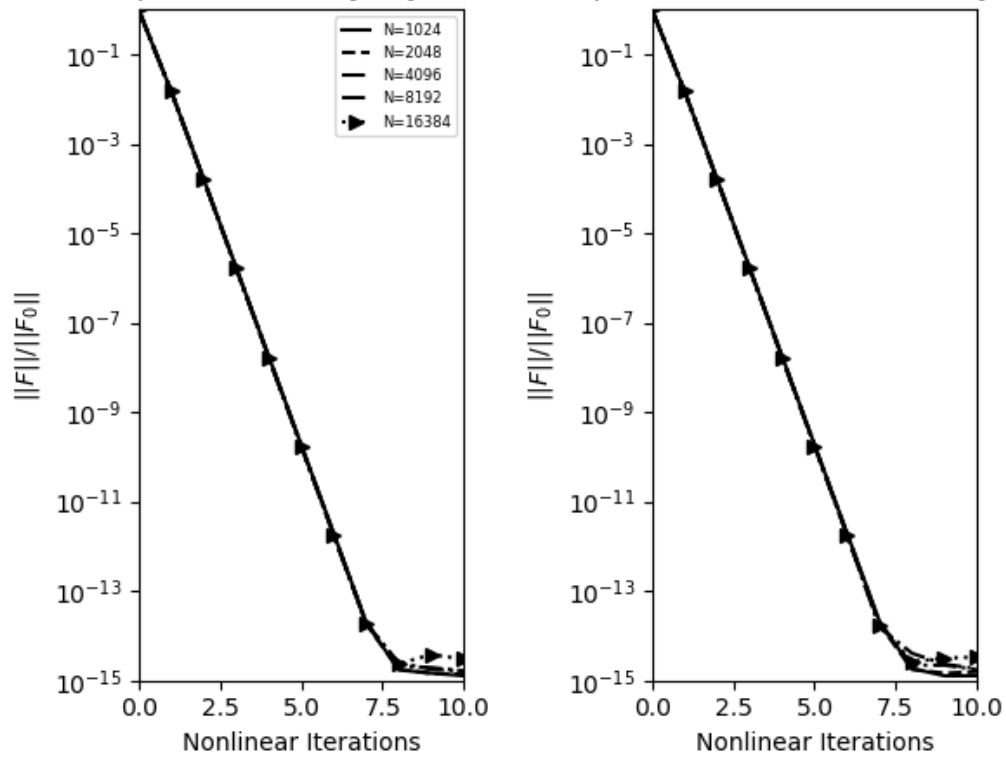
12 & 7.46058e-01 & 7.94854e-01 & 8.41796e-01 & 8.43103e-01 & 8.43105e-01 & \\
 13 & 7.69637e-01 & 8.11687e-01 & 8.53176e-01 & 8.54296e-01 & 8.54298e-01 & \\
 14 & 7.92049e-01 & 8.26179e-01 & 8.63042e-01 & 8.64018e-01 & 8.64019e-01 & \\
 15 & 8.07371e-01 & 8.38661e-01 & 8.71677e-01 & 8.72536e-01 & 8.72537e-01 & \\
 16 & 8.23999e-01 & 8.49675e-01 & 8.79296e-01 & 8.80061e-01 & 8.80062e-01 & \\
 17

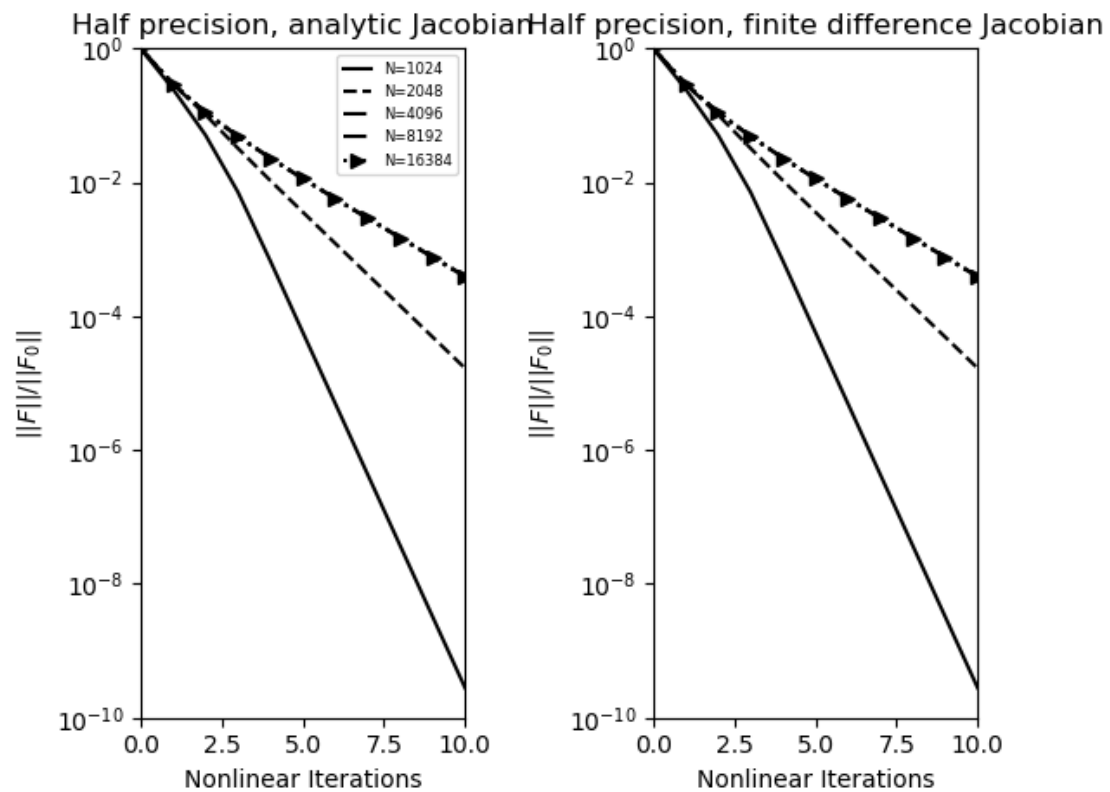


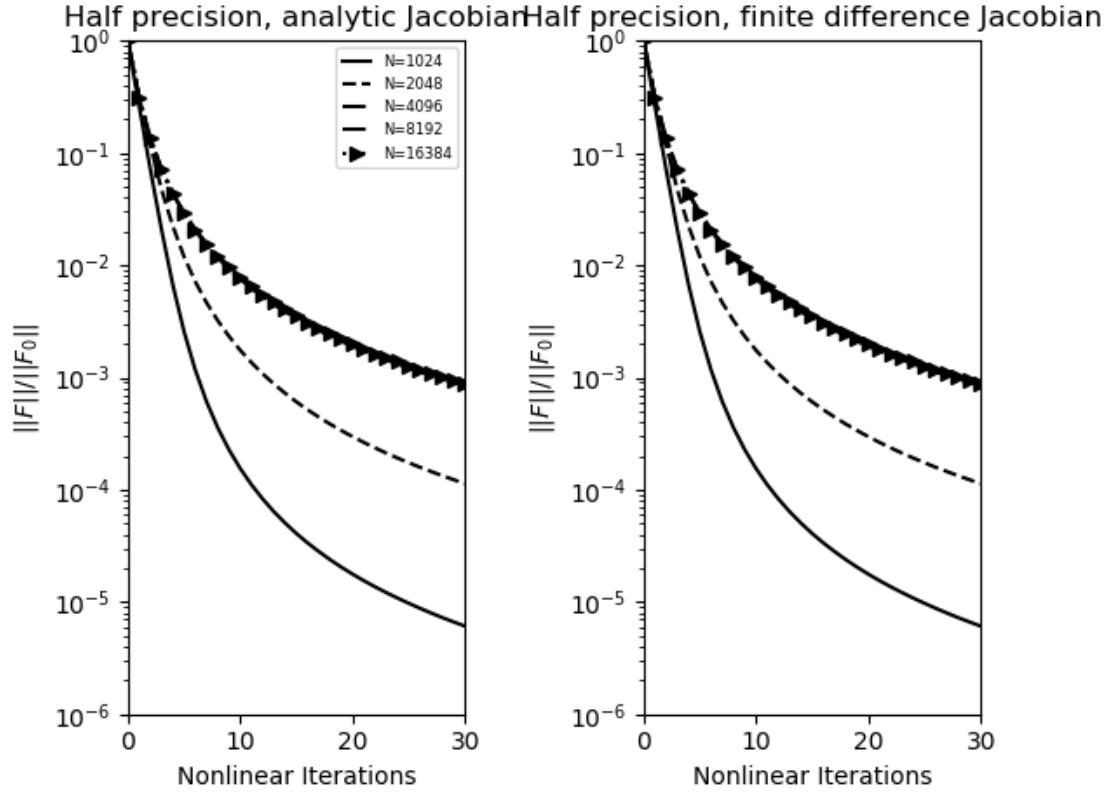




Half precision, analytic JacobianHalf precision, finite difference Jacobian







```

& 8.37356e-01 & 8.59263e-01 & 8.86068e-01 & 8.86754e-01 & 8.86755e-01 & \\\
18 & 8.47186e-01 & 8.67783e-01 & 8.92127e-01 & 8.92746e-01 & 8.92746e-01 & \\\
19 & 8.57336e-01 & 8.75404e-01 & 8.97578e-01 & 8.98140e-01 & 8.98141e-01 & \\\
20 & 8.66801e-01 & 8.82174e-01 & 9.02509e-01 & 9.03016e-01 & 9.03022e-01 & \\\
21 & 8.74347e-01 & 8.88296e-01 & 9.06990e-01 & 9.07453e-01 & 9.07460e-01 & \\\
22 & 8.81390e-01 & 8.93883e-01 & 9.11080e-01 & 9.11505e-01 & 9.11511e-01 & \\\
23 & 8.88034e-01 & 8.98900e-01 & 9.14828e-01 & 9.15219e-01 & 9.15225e-01 & \\\
24 & 8.93824e-01 & 9.03511e-01 & 9.18275e-01 & 9.18636e-01 & 9.18642e-01 & \\\
25 & 8.98283e-01 & 9.07706e-01 & 9.21455e-01 & 9.21789e-01 & 9.21795e-01 & \\\
26 & 9.02619e-01 & 9.11584e-01 & 9.24398e-01 & 9.24708e-01 & 9.24714e-01 & \\\
27 & 9.06563e-01 & 9.15125e-01 & 9.27130e-01 & 9.27418e-01 & 9.27424e-01 & \\\
28 & 9.10556e-01 & 9.18424e-01 & 9.29671e-01 & 9.29941e-01 & 9.29946e-01 & \\\
29 & 9.15203e-01 & 9.21451e-01 & 9.32043e-01 & 9.32295e-01 & 9.32300e-01 & \\\
30 & 9.18229e-01 & 9.24343e-01 & 9.34260e-01 & 9.34496e-01 & 9.34501e-01 & \\\
\hline
\end{tabular}

```

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References

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