General ARMA(p,q) models

Shumway and Stoffer: 3.1-3.2

Introduction to autoregressive models

- *Autoregression* - our earlier example of $x_t = x_{t-1} - 0.9x_{t-2} + w_t$
- Intuitively, makes forecasting possible - an exciting perpective
- More formally, a stationary process

$$x_t = \phi_1 x_{t-1} + \ldots + \phi_p x_{t-p} + \omega_t$$

  is an autoregressive process of order p with $\phi_p \neq 0$.

- For simplicity, ω_t is Gaussian with mean zero and variance σ^2

Some remarks about an AR(p) model

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- Let $\Phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \phi_p B^p$ be an **autoregressive operator**. It can be viewed as a polynomial in B of order p. Then the autoregressive model of order p can be written concisely as

$$\Phi(B)x_t = w_t$$

- If the mean μ of x_t is not zero, it is useful to center the series and consider the process in terms of $x_t - \mu$, e.g.

$$x_t = \alpha + \phi_1 x_{t-1} + \ldots + \phi_p x_{t-p} + \omega_t$$

where $\alpha = \mu(1 - \phi_1 - \ldots - \phi_p)$

AR(1) process as easy example

- Assume $p = 1$ and consider AR(1) process first for simplicity

$$x_t = \phi x_{t-1} + w_t \tag{1}$$

- If $|\phi| < 1$, (1) allows the infinite representation $x_t = \sum_{j=0}^{\infty} \phi^j w_{t-j}$
- Clearly, the mean of the process is $E\, x_t = \sum_{j=0}^{\infty} \phi^j E\, w_{t-j} = 0$.
- Its autocovariance function is

$$\gamma(h) = cov(x_{t+h}, x_t) = \sigma_w^2 \sum_{j=0}^{\infty} \phi^j \phi^{j+h} = \frac{\sigma_w^2 \phi^h}{1 - \phi^2}$$

- As a corollary, the autocorrelation function of the AR(1) process is

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \phi^h$$

for all $h \geq 0$

- Note also that $\rho(h)$ satisfies the recursive relationship $\rho(h) = \phi\rho(h-1)$, $h = 1, 2, \ldots$

Causality of an autoregressive process

- Representation of the AR(1) process as linear process $\sum_{j=0}^{\infty} \psi_j w_{t-j}$ is only possible when $|\phi| < 1$
- If $|\phi| > 1$ we have the process that depends on the **future** and is thus non-causal; more specifically, the solution is

$$x_t = -\sum_{j=1}^{\infty} \phi^{-j} \omega_{t+j}$$

and its autocovariance is $\gamma_x(h) = \sigma_\omega^2 \phi^{-2} \phi^{-h}/(1 - \phi^{-2})$

- The above makes it clear that we can define a process $y_t = \phi^{-1} y_{t-1} + v_t$ with v_t iid $N(0, \phi^{-2}\sigma_\omega^2)$ that is stochastically equal to the process x_t; this suggests there's no need to consider non-causal autoregressive stationary processes

General stationary solution of the AR(p) model

- Represent x_t as a linear process with unknown coefficients:

$$x_t = \sum_{j=0}^{\infty} \psi_j w_{t-j} = \Psi(B) w_t$$

where the polynomial $\Psi(B) = \sum_{j=0}^{\infty} \psi_j w_{t-j}$

- From the identity

$$\Phi(B)\Psi(B) w_t = w_t$$

determine the coefficients ψ_j recursively

- As an example, for AR(1) the sequence of equations is

$$\begin{aligned} \psi_1 - \phi_1 &= 0 \\ \psi_2 - \psi_1 \phi &= 0 \\ &\dots \\ \psi_j - \psi_{j-1} \phi &= 0 \end{aligned}$$

- Clearly, $\psi_j = \psi_{j-1}\phi$ and, thus, $\psi_j = \phi^j$

General moving average models MA(q) (moving average of order q)

- The general MA(q) process is defined as

$$x_t = w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2} + \ldots + \theta_q w_{t-q} \tag{2}$$

  where $\theta_q \neq 0$ and w_t is a white noise

- Again, in general $w_t \sim N(0, \sigma_w^2)$
- Introduce the **moving average operator** $\Theta(B) = 1+\theta_1 B+\theta_2 B^2+\ldots+\theta_q B^q$
- Then, the concise form of (2) is

$$x_t = \Theta(B) w_t$$

An example: MA(1)

- Consider

$$x_t = w_t + \theta w_{t-1}$$

- Clearly, $E\,x_t = 0$; moreover, it is easy to check that the autocovariance function is

$$\gamma(h) = \begin{cases} (1+\theta^2)\sigma_w^2, h = 0 \\ \theta\sigma_w^2, h = 1 \\ 0, |h| \geq 1 \end{cases}$$

- Note that the autocovariance function of MA(1) cuts off at the lag 1

Non-uniqueness and invertibility of MA models

- It is easy to find out that the MA(1) processes with $\sigma_w^2 = 1$ and $\theta = 5$, on one hand, and $\sigma_w^2 = 25$ and $\theta = \frac{1}{5}$ possess the same autocovariance function
- Let us assume these processes are driven by the Gaussian white noise; then, they are $x_t = w_t + \frac{1}{5}w_{t-1}$, $w_t \sim iid\, N(0, 25)$ and $x_t = v_t + 5v_{t-1}$, $v_t \sim iid\, N(0, 1)$.
- All of their finite-dimensional distributions are identical$\rightarrow$ the processes themselves are identical!!!
- To avoid this, it is necessary to assume that $|\theta| < 1$. If this is the case, there is a unique representation of the MA(1) process as

$$w_t = \sum_{j=0}^{\infty} (-\theta)^j x_{t-j}$$

- Note that $MA(1)$ process is **Always** stationary.

General invertibility condition

- For general $MA(q)$ processes, it is not easy to state the invertibility condition in terms of the moving average polynomial coefficients
- However, it can be easily done in terms of the *roots* of the polynomial $\Theta(z)$ viewed as a complex variable function
- The MA(q) process is causal if and only if $\Theta(z) \neq 0$ for all $|z| \leq 1.$

Example: a general MA(q) model

- Let $x_t = \Theta(B)\omega_t$ where $\Theta(B) = 1 + \theta_1 B + \ldots + \theta_q B^q$. Clearly, $E\,x_t = 0$
- Moreover,

$$\gamma(h) = cov(x_{t+h}, x_t) = \sigma_w^2 \sum_{j=0}^{q-h} \theta_j \theta_{j+h}$$

  for any $0 \le h \le q$ while it is equal to zero for any $h > q$

- The autocorrelation is

$$\rho(h) = \frac{\sigma_w^2 \sum_{j=0}^{q-h} \theta_j \theta_{j+h}}{1 + \theta_1^2 + \ldots + \theta_q^2}$$

  for any $0 \le h \le q$

- The autocovariance(autocorrelation) that cuts off after q lags is the signature of an MA(q) model

General ARMA (autoregressive moving average) models

- A general ARMA(p,q) model is defined as

$$x_t = \phi_1 x_{t-1} + \ldots + \phi_p x_{t-p} + w_t + \theta_1 w_{t-1} + \ldots + \theta_q w_{t-q}$$

  where $\phi_p \neq 0$ and $\theta_q \neq 0$; w_t is a white noise with $\sigma_w^2 > 0$.

- In concise form it is

$$\Phi(B)x_t = \Theta(B)w_t$$

Parameter redundancy in ARMA(p,q) models

- It is **always** assumed that the polynomials $\Phi(B)$ and $\Theta(B)$ do not have common factors
- To ensure that the model is *causal* we require that the autoregressive polynomial $\Phi(B)$ does not have any roots inside the unit circle
- To ensure that the model is *invertible* we require, likewise, that the moving average polynomial $\Theta(B)$ does not have any roots inside the unit circle

Causality of ARMA(p,q) models

- The ARMA (p,q) process x_t that satisfies

$$\Phi(B)x_t = \Theta(B)\omega_t$$

is said to be **causal** when it can be represented as

$$x_t = \sum_{j=0}^{\infty} \psi_j \omega_{t-j} = \Psi(B)\omega_t$$

where $\sum_{j=0}^{\infty} |\psi_j| < \infty$ and $\Psi(B) = \sum_{j=0}^{\infty} \psi_j B^j$

- **Property** The ARMA(p,q) process x_t is causal if and only if all of the roots of its autoregressive polynomial $\phi(z)$ are outside the unit circle: $\phi(z) = 0$ only if $|z| > 1$. In that case, $\psi(z) = \frac{\theta(z)}{\phi(z)}$ for any $|z| \leq 1$

Invertibility of ARMA(p,q) processes

- The ARMA (p,q) x_t process that satisfies

$$\Phi(B)x_t = \Theta(B)\omega_t$$

is said to be **invertible** when there exists it can be represented as

$$\pi(B)x_t = \omega_t$$

where $\pi(B) = \sum_{j=0}^{\infty} \pi_j B^j$ and $\sum_{j=0}^{\infty} |\pi_j| < \infty$ and

- **Property** The ARMA(p,q) process x_t is invertible if and only if all of the roots of its moving average polynomial $\theta(z)$ are outside the unit circle: $\theta(z) = 0$ only if $|z| > 1$. In that case, $\pi(z) = \frac{\phi(z)}{\pi(z)}$ for any $|z| \leq 1$