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Disturbance attenuation using proportional integral observers

KRISHNA K. BUSAWON†* and POUSGA KABORE‡

In this paper, we first propose a proportional integral observer design for single-output uncertain linear systems which permit us to attenuate either measurement noise or modelling errors. We show that, when only sensor noise is present in the system, an integral observer alone suffices to achieve good convergence and filtering properties. On the other hand, when modelling errors and sensor noise are present, we show that, for some classes of linear systems, the proportional integral observer allows us to decouple completely the modelling uncertainties while keeping satisfactory convergence properties. A comparison of the classical proportional observer to the proposed observers are given via academic simulation examples. We next extend the design to the class of single-output uniformly observable non-linear systems. We show through a practical simulation example, dealing with a flexible joint robot, that the non-linear proportional integral has very satisfactory disturbance attenuating properties.

1. Introduction

Proportional integral observers (PIO) are observers in which an additional term, which is proportional to the integral of the output estimation error, is added in order to achieve some desired robustness performance (see, e.g. Beale and Shafai 1987, Saif 1992, Shafai and Caroll 1985, Weinmann 1991). In fact, the addition of an integral term permits us to obtain additional degrees of freedom for control or observer design purposes. In Beale and Shafai (1987), in particular, this additional degree of freedom was used for recovering the traditional stability margins of linear quadratic regulators. Such an additional degree may be used, for instance, to decouple modelling uncertainties or measurement of noise. This last property may be interesting when the attention is focused on the detection and isolation of faults in a system. In fact, the detection and isolation of faults relies on the accuracy of the sensors being used. The PIO presented in this paper can allow us to minimize the effect of sensor noise or errors while increasing the accuracy of detection.

In this context, the PIO appears to be an adequate alternative to deal with uncertain systems. On the other hand, the design of observers for uncertain systems or equivalently the unknown input observer (UIO) design has been an important topic of research during the last decade because of their particular importance in the field of model-based fault detection (see Frank and Wunnenberg 1989). Most of the existing UIO designs are based on the combined notion of disturbance decoupling together with the classical proportional observer design. While PIOs have been used in connection to full state feedback robustness recovery problems and loop transfer recovery problems (see Beale and Shafai 1987, Shafai and Caroll 1985, Niemann *et al.* 1995), this method has not gained much attention in the area of observer design for uncertain systems or UIO design, except a few works related to disturbance estimation (see, e.g. Saif 1992). In the classical approach to PIO design, one would generally use the proportional gain to stabilize the disturbance free error dynamics and would seek for an integral gain to decouple or attentuate the disturbances. However, such an approach is not particularly suitable for handling measurement noise since the latter is amplified by the proportional gain as in the case of the conventional proportional observer. This suggest that, in a certain way, the PIO design has to be reformulated.

In this paper, we first give a new formulation to the PIO design for single-output linear uncertain systems. The reformulation basically lies in that, unlike the classical approach to PIO design, the integral gain is used to stabilize the noise free error dynamics while the proportional gain is used to decouple the disturbance. In using such a formulation, it is shown that a PIO can be designed for single output uncertain linear systems. We show that, when modelling errors and sensor noise are present, the PIO allows us to decouple completely the modelling uncertainties. In the particular case when only sensor noise is present, an integral observer (IO) is sufficient. Furthermore, both the PIO and the IO permits us to achieve good filtering and convergence properties. A comparison of the classical proportional observer to the proposed observers is made via simulation examples. A generalization of the proposed PIO was made to the non-linear case, namely for the class of single-output uniformly observable systems. A simulation example, dealing with a flexible joint robot, is carried out to show the disturbance attenuating proper-

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ties of the non-linear PIO observer. Finally, some conclusions are given.

2. Classical PIO design and problem statement

Consider the following single output linear system

$$\left.\begin{aligned} \dot{x} &= Fx + Gu \\ y &= Hx \end{aligned}\right\} \qquad (1)$$

where $x = (x_1, \dots, x_n)^{\mathrm{T}} \in R^n$, $u \in R^m$, $y \in R$. The matrices F, G and H are of appropriate dimensions and the pair (F, H) is supposed to be rank observable.

The classical approach to PIO design is established as

$$\left.\begin{aligned} \dot{\hat{x}} &= F\hat{x} + Gu + K_p(y - H\hat{x}) + K_I w \\ \dot{w} &= y - H\hat{x} \end{aligned}\right\} \qquad (2)$$

Note that the variable $w(t)$ is defined as the integral of the difference $y - H\hat{x}$. The vectors K_p and K_I stands for the proportional and integral gains respectively.

Setting $e = x - \hat{x}$, we obtain the estimation error dynamics as

$$\begin{bmatrix} \dot{e} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} F - K_p H & -K_I \\ H & 0 \end{bmatrix} \begin{bmatrix} e \\ w \end{bmatrix} \qquad (3)$$

The gains K_p and K_I are chosen such that the error dynamics is stable.

Now assume that the above system is affected by some measurement noise $d(t)$ as

$$\left.\begin{aligned} \dot{x} &= Fx + Gu \\ y &= Hx + d \end{aligned}\right\} \qquad (4)$$

We assume, for convenience, that the disturbance $d(t)$ does not affect the rank observability of the system. In this case, the error dynamics would have the expression

$$\begin{bmatrix} \dot{e} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} F - K_p H & -K_I \\ H & 0 \end{bmatrix} \begin{bmatrix} e \\ w \end{bmatrix} - \begin{bmatrix} K_p d \\ -d \end{bmatrix} \qquad (5)$$

From this we can see that the disturbance, $d(t)$, is multiplied by the proportional gain K_p. Therefore, if K_p is a high gain then necessarily the disturbance will be amplified. In fact, the conventional proportional observer also has the same characteristics as shown in the next section. Consequently, regarding this approach to design, it implies that the above PIO will not bring satisfactory performance while dealing with sensor noise. This is somewhat surprising because, in its very essence, the PIO was designed in an attempt to recover the well-known robustness properties of its dual the PI controller. As a result, there is a real incentive to reformulate the PIO design.

In the next section, we give a new formulation of the PIO. The reformulation basically lies in that, unlike the above approach, the integral gain is used to stabilize the error dynamics while the proportional gain is used to deal with the disturbance. Some special classes of linear systems are treated first in order to simplify the presentation of the paper.

3. A new PIO design approach for a class of linear systems

Consider the special class of single output linear systems described by

$$\left.\begin{aligned} \dot{x} &= A_0 x + G_0 u \\ y_1 &= C_0 x + d \end{aligned}\right\} \qquad (6)$$

where $x = (x_1, \dots, x_n)^{\mathrm{T}} \in R^n$, $u \in R^m$, $y_1 \in R$ and $d \in R$ is an unknown bounded disturbance. The $(n \times n)$ matrix A_0 and the $(1 \times n)$ matrix C_0 are of the observable companion form

$$A_0 = \begin{pmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ & & & 1 \\ a_1 & & \dots & a_n \end{pmatrix}, \qquad C_0 = [1, 0, \dots, 0] \qquad (7)$$

G_0 is a constant matrix of appropriate dimension. Notice that single-output observable systems can be brought into the above form by a change of coordinates.

Regarding system (6) it is important to note that if $d(t) = -x_1(t)$ (this condition means that there is a total failure of the sensor) then the above system is not observable. On account of this, we set the hypothesis

(A1) We assume that $|d(t)| < |x_1(t)|$ for all $t \geq 0$.

Assumption (A1) is not restrictive since it implies that the amplitude of the noise does not exceed the amplitude of the noise free measurement. Indeed, we cannot expect to extract useful information from the output if the amplitude of the noise $|d(t)|$ is several times greater than $|x_1(t)|$.

3.1. *Proportional observer*

If we had to design a proportional observer for system (6) we would usually design an observer of the classical Luenberger form

$$\begin{aligned} \dot{\hat{x}} &= A_0\hat{x} + G_0 u + K_p(y_1 - C_0\hat{x}) \\ &= A_0\hat{x} + G_0 u + K_p(C_0 x - C_0\hat{x}) + K_p d \end{aligned}$$

Setting $e = x - \hat{x}$, the estimation error dynamics is

$$\dot{e} = (A_0 - K_p C_0)e - K_p d \qquad (8)$$

From this we can see that the disturbance $d(t)$ is multiplied by the factor K_p. If K_p is a high gain then necessarily the disturbance will be amplified. For this reason, as is well-known, one of the shortcomings of proportional high gain observers is that they have a tendency to amplify measurement noise. On the other hand, if we choose a small gain K_p in order to attenuate the effect of the disturbance, then the convergence properties of the observer may be affected. This implies that the proportional observer is definitely not adequate for handling measurement noise.

3.2. *Intergral observer*

Consider again system (6). Set $x_0(t) = \int_0^t y_1(\tau)\,\mathrm{d}\tau$ so that $\dot{x}_0(t) = y_1(t) = C_0 x(t) + d(t)$. We then have the augmented system

$$\left.\begin{aligned} \dot{x}_0 &= C_0 x + d \\ \dot{x} &= A_0 x + G_0 u \\ y_0 &= x_0 \end{aligned}\right\} \tag{9}$$

Setting

$$z = \begin{bmatrix} x_0 \\ x \end{bmatrix}, \qquad G = \begin{bmatrix} 0 \\ G_0 \end{bmatrix}, \qquad E = \begin{bmatrix} 1 \\ \mathbf{0}_{n\times 1} \end{bmatrix}$$

and

$$A = \begin{bmatrix} 0 & C_0 \\ \mathbf{0}_{n\times 1} & A_0 \end{bmatrix}$$

we can rewrite system (9) as

$$\left.\begin{aligned} \dot{z} &= Az + Gu + Ed \\ y_0 &= C_1 z \end{aligned}\right\} \tag{10}$$

where $C_1 = (1 \;\; 0 \;\; \cdots \;\; 0)$. Notice that the pair (A, C_1) is observable and that the augmented system remains observable on the basis of Assumption (A1).

Now, consider the system

$$\dot{\hat{z}} = A\hat{z} + Gu + K_I(y_0 - C_1\hat{z}) \tag{11}$$

where K_I is chosen such that the matrix $A - K_I C_1$ is Hurwitz. We shall show that system (11) is an asymptotic observer for system (10). Indeed, let $e = z - \hat{z}$, then error dynamics is now

$$\dot{e} = (A - K_I C_1)e + C_1^{\mathrm{T}} d \tag{12}$$

since $E = C_1^{\mathrm{T}}$. Comparing (8) and (12) we can see that the disturbance $d(t)$ in equation (12) is not affected by the observer gain. Therefore, we can choose a reasonably high gain K_I such that the stabilizing term prevails over the perturbation term $C_1^{\mathrm{T}} d$. In other words, system (11) is an asymptotic observer for system (10).

Example 1: Consider the following single-input–single-output system

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \\ y_1 &= x_1 + d \end{aligned}\right\} \tag{13}$$

Proportional observer: A proportional observer for (13) is given by

$$\left.\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + l_1(x_1 + d - \hat{x}_1) \\ \dot{\hat{x}}_2 &= u + l_2(x_1 + d - \hat{x}_1) \end{aligned}\right\} \tag{14}$$

where l_1 and l_2 are chosen such that the disturbance free error dynamics are stable.

A simulation of the above proportional observer (14) was carried out when the two poles of the observer was chosen to be equal to -4; that is $l_1 = 8$ and $l_2 = 16$. The input was set as $u(t) = \sin t$. Figure 1 shows the profile of the disturbance $d(t)$. Figure 2 shows the behaviour of the observer. For the sake of simplicity, we have shown only the x_2 state variable and its estimate. The estimate $\hat{x}_2$ is represented in solid lines. It can easily be seen that the convergence is not satisfactory.

Integral observer: Now, let us design an integral observer for system (13). By setting $x_0(t) = \int_0^t y_1(\tau)\,\mathrm{d}\tau$ and proceeding as above, we get the augmented system

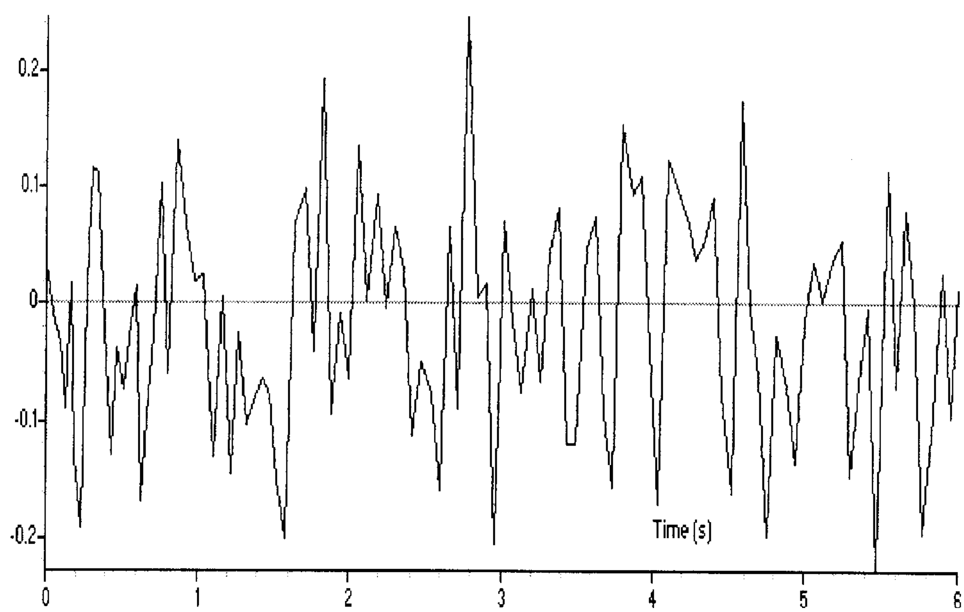

Figure 1. The distrubance $d(t)$.

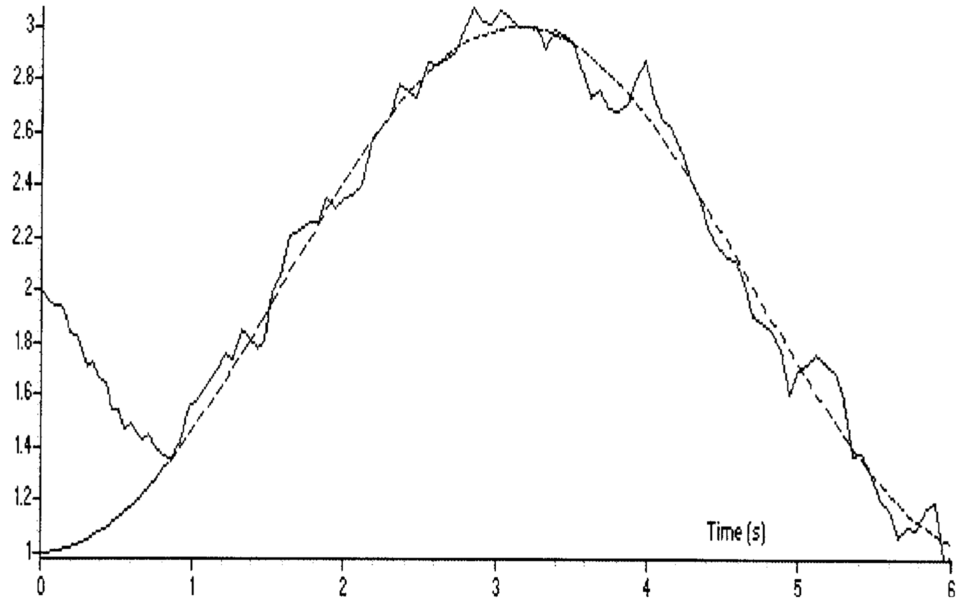

Figure 2. Estimation of x_2 via the proportional observer.

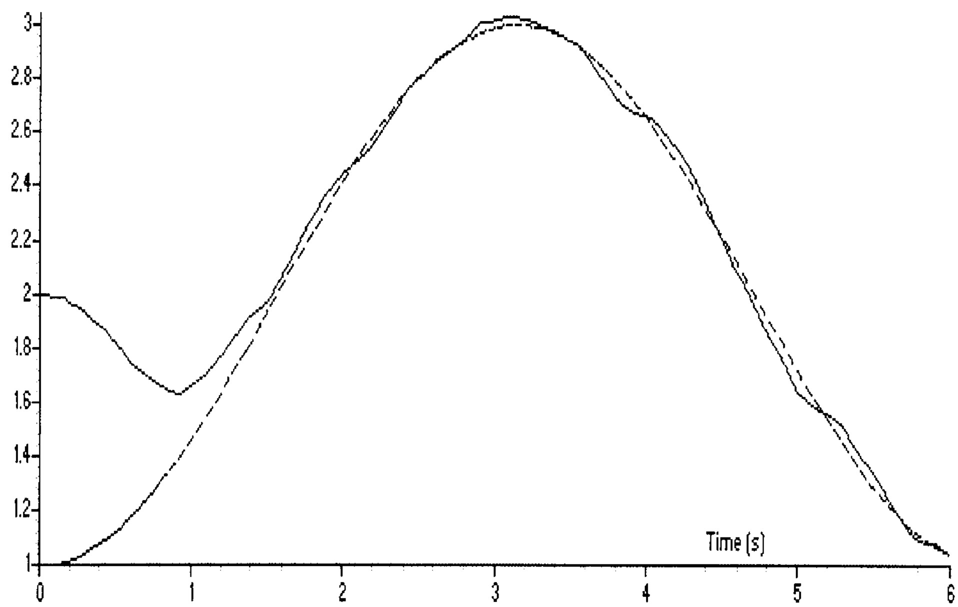

Figure 3. Estimation of x_2 via the integral observer.

$$\left.\begin{aligned} \dot{x}_0 &= x_1 + d \\ \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \\ y_0 &= x_0 \end{aligned}\right\} \tag{15}$$

The integral observer is then given by

$$\left.\begin{aligned} \dot{\hat{x}}_0 &= \hat{x}_1 + k_1(x_0 - \hat{x}_0) \\ \dot{\hat{x}}_1 &= \hat{x}_2 + k_2(x_0 - \hat{x}_0) \\ \dot{\hat{x}}_2 &= u + k_3(x_0 - \hat{x}_0) \end{aligned}\right\} \tag{16}$$

where the gains were chosen as $k_1 = 12$, $k_2 = 48$ and $k_3 = 64$ so that the poles of the observer are all located at -4. Figure 3 show the performance of the observer (16). It can be seen that the convergence of the estimate $\hat{x}_2$ (in solid lines) is smoother than in the case where the proportional observer (14) is used. Therefore, integral observer (16) has good filtering properties.

4. Main result

In this section we shall extend the previous PIO design approach to a more general class of linear uncertain systems. More specifically, we shall consider linear systems of the form (6) which are affected both by modelling and measurement errors as

$$\left.\begin{aligned} \dot{x} &= A_0x + G_0u + E_0d \\ y_1 &= C_0x + d \end{aligned}\right\} \qquad (17)$$

where E_0 is a known $(n \times 1)$ matrix. We shall show that the addition of a proportional term to the above integral observer permits us to decouple the effect of modelling errors.

Setting again $x_0(t) = \int_0^t y_1(\tau)\,d\tau$, we get the augmented system

$$\left.\begin{aligned} \dot{x}_0 &= C_0x + d \\ \dot{x} &= A_0x + G_0u + E_0d \\ y_0 &= x_0 \\ y_1 &= C_0x + d \end{aligned}\right\}$$

which can be written in a compact form as

$$\left.\begin{aligned} \dot{z} &= Az + Gu + C_1^{\mathrm{T}}d + Ed \\ y_0 &= C_1z \\ y_1 &= C_2z + d \end{aligned}\right\} \qquad (18)$$

where z, A, G and C_1 have the same meanings as above,

$$E = \begin{bmatrix} 0 \\ E_0 \end{bmatrix} \quad \text{and} \quad C_2 = (0 \;\; C_0)$$

Consider the system

$$\dot{\hat{z}} = A\hat{z} + Gu + K_I(y_0 - C_1\hat{z}) + E(y_1 - C_2\hat{z}) \qquad (19)$$

We can now state the following.

Theorem 1: *Assume that $|d(t)| < |x_1(t)|$ for all $t \geq 0$. Then, the system (19) is an asymptotic observer for system (18) which allows us to decouple the effect of the system's uncertainty from the state estimate.*

Proof: First note that

$$\dot{\hat{z}} = A\hat{z} + Gu + K_I(C_1z - C_1\hat{z}) + E(C_2z - C_2\hat{z}) + Ed$$

Set $e = z - \hat{z}$, then

$$\begin{aligned} \dot{e} &= (A - K_IC_1)e + C_1^{\mathrm{T}}d - EC_2e \\ &= (A - EC_2 - K_IC_1)e + C_1^{\mathrm{T}}d \end{aligned}$$

Now, since the pair $(A - EC_2, C_1)$ is observable we can choose K_I such that the matrix $A - EC_2 - K_IC_1$ is stable. Therefore, system (19) is an asymptotic observer for system (18). □

Remarks:

(1) We have assumed that the matrix E_0 is known. This is not generally true in practice; except in some special situations such as actuator noise or disturbance. Indeed, if we assume that we have an additive actuator failure or disturbance which is modelled as $u(t) = u_0(t) + d(t)$, then system (17) would take the form

$$\left.\begin{aligned} \dot{x} &= A_0x + G_0u_0 + G_0d \\ y_1 &= C_0x + d \end{aligned}\right\}$$

In this case, $E_0 = G_0$.

(2) Finally, we have assumed that the disturbance $d(t)$ appearing at the output and in the system are the same. This may not be generally true either. However, such situations exist in many practical and theoretical problems. For example, such situations will occur when we have an uncertain parameter which appears in both the measurement and in the model. Another case where such situations appear is in the problem of observer design by output injection (see remark in § 6) or in the geometric approach to fault detection by output injection (see Massoumnia *et al.* 1989). Indeed, if the output has an additive disturbance $d(t)$ and is injected in the system, then the same disturbance $d(t)$ will appear in the model as well.

Example 2: Consider the system given in Example 1 but which is now affected by modelling uncertainty as

$$\left.\begin{aligned} \dot{x} &= x_2 + d \\ \dot{x}_2 &= u - 2d \\ y_1 &= x_1 + d \end{aligned}\right\} \qquad (20)$$

Proportional observer: A proportional observer for system (20) is given by

$$\left.\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + l_1(x_1 + d - \hat{x}_1) \\ \dot{\hat{x}}_2 &= u + l_2(x_1 + d - \hat{x}_1) \end{aligned}\right\} \qquad (21)$$

As before a simulation of the above proportional observer (21) was carried out, with the same disturbance as in figure 1. The two poles of the observer was again set at -4 and the input was chosen as previously. Figure 4 shows the behaviour of the state estimate $\hat{x}_2$ (in solid lines) with respect to the real state variable x_2 (in dotted lines). It can again be seen that the convergence is not satisfactory.

Proportional integral observer: We now design a proportional integral observer for (20). By setting $x_0(t) = \int_0^t y_1(\tau)\,d\tau$ we get the augmented system

$$\left.\begin{aligned} \dot{x}_0 &= x_1 + d \\ \dot{x}_1 &= x_2 + d \\ \dot{x}_2 &= u - 2d \\ y_0 &= x_0 \\ y_1 &= x_1 + d \end{aligned}\right\} \qquad (22)$$

Consequently, the PIO for system (22) is given by

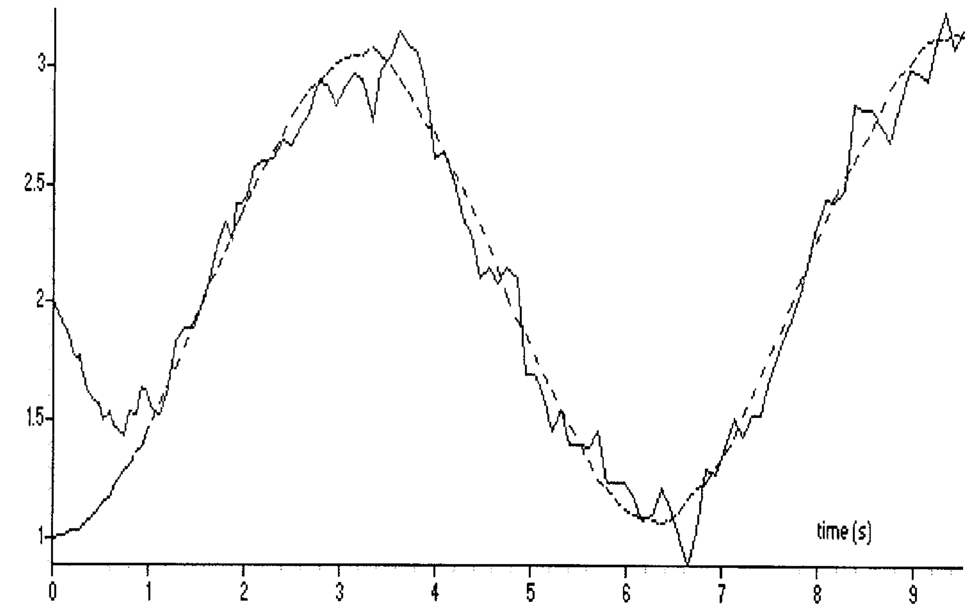

Figure 4. Estimation of x_2 via the proportional observer.

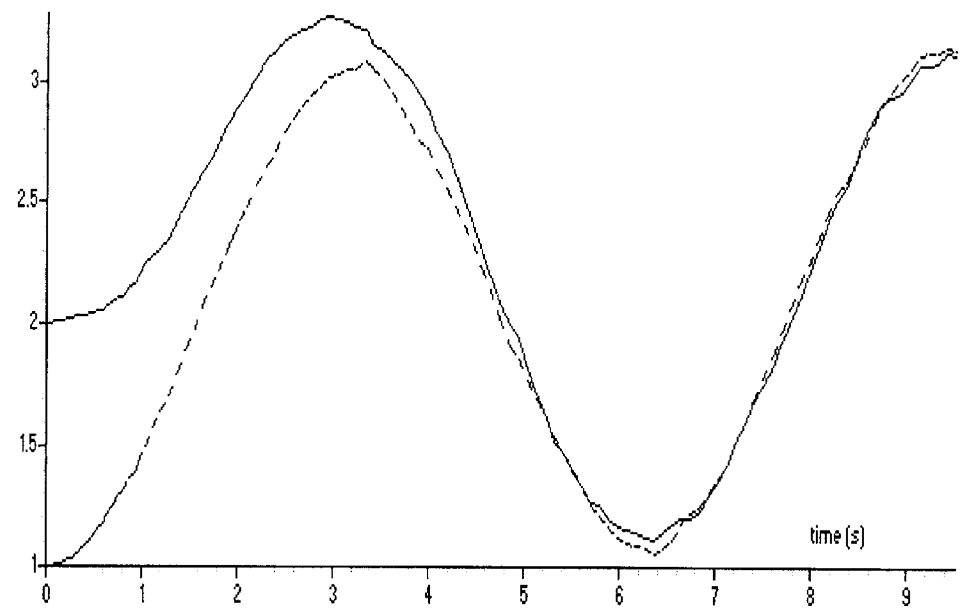

Figure 5. Estimation of x_2 via PIO.

$$\left.\begin{aligned}\dot{\hat{x}}_0 &= \hat{x}_1 + k_1(x_0 - \hat{x}_0)\\ \dot{\hat{x}}_1 &= \hat{x}_2 + k_2(x_0 - \hat{x}_0) + (y_1 - \hat{x}_1)\\ \dot{\hat{x}}_2 &= u + k_3(x_0 - \hat{x}_0) - 2(y_1 - \hat{x}_1)\end{aligned}\right\} \tag{23}$$

where the gains of the observer is chosen as $k_1 = 11$, $k_2 = 39$ and $k_3 = 42$ so that the poles of the observer are all located at -4. Figure 5 show the performance of the PIO (23). It can be seen that the convergence of the estimate $\hat{x}_2$ (in solid lines) is smoother than in the case where the proportional observer (21) is used. Furthermore, the PIO (23) has satisfactory filtering properties.

5. Extension to uniformly observable non-linear systems

In this section, we are going to extend the previous design to the class of single-output uniformly observable systems

$$\left.\begin{aligned}\dot{x} &= A_0 x + g(x, u) + E_0 d\\ y_1 &= C_0 x + d\end{aligned}\right\} \tag{24}$$

where $x \in R^n$, $u \in R^m$, $y \in R$ and the unknown bounded disturbance $d \in R$. The matrix E_0 is known and the matrices A_0 and C_0 are as in (7). The non-linearity $g(x, u)$ has a triangular structure, i.e.

$$g(x,u) = \begin{pmatrix} g_1(x_1,u) \\ g_2(x_1,x_2,u) \\ \vdots \\ g_{n-1}(x_1,\dots,x_{n-1},u) \\ g_n(x,u) \end{pmatrix}$$

We assume that

(H1) g is globally Lipschitzian with respect to x uniformly in u.
(H2) the disturbance $d(t)$ is such that $|d(t)| < |x_1(t)|$ for all $t \geq 0$.

Setting $x_0(t) = \int_0^t y_1(\tau)\,\mathrm{d}\tau$, we obtain the augmented system

$$\left.\begin{aligned} \dot{x}_0 &= C_0 x + d \\ \dot{x} &= A_0 x + g(x,u) + E_0 d \\ y_0 &= x_0 \\ y_1 &= C_0 x + d \end{aligned}\right\} \qquad (25)$$

which can be written in a compact form as

$$\left.\begin{aligned} \dot{z} &= Az + \bar{g}(z,u) + Ed + C_1^{\mathrm{T}} d \\ y_0 &= C_1 z \\ y_1 &= C_2 z + d \end{aligned}\right\} \qquad (26)$$

where z, A, E, C_1 and C_2 is defined as previously and $\bar{g} = \begin{bmatrix} 0 \\ g \end{bmatrix}$.

Note that if we set $d(t) = 0$ in (26) and neglect the output y_1, we would obtain a $(n+1)$-dimensional system which is still uniformly observable. Therefore, to update the integral gain we can use the high gain observer design technique given by Gauthier *et al.* (1992).

Indeed, consider the system

$$\dot{\hat{z}} = A\hat{z} + \bar{g}(\hat{z},u) + S_\theta^{-1} C_1^{\mathrm{T}} (y_0 - C_1\hat{z}) + E(y_1 - C_2\hat{z}) \quad (27)$$

where S_θ is the unique symmetric positive definite matrix satisfying

$$\theta S_\theta + A^{\mathrm{T}} S_\theta + S_\theta A = C_1^{\mathrm{T}} C_1$$

with $\theta > 0$.

Theorem 2: *Assume that system* (26) *satisfies assumptions* (H1) *and* (H2). *Then, there exists* $\theta_0 > 0$ *such that, for all* $\theta \geq \theta_0$, *system* (27) *is an asymptotic observer for system* (26) *which allows us to decouple the effect of the system's uncertainty from the state estimate.*

Proof: The proof of the theorem uses similar ideas as in Gauthier *et al.* (1992). Setting, $e = z - \hat{z}$, then error dynamics is

$$\dot{e} = (A - S_\theta^{-1} C_1^{\mathrm{T}} C_1)e + (\bar{g}(z,u) - \bar{g}(\hat{z},u)) - EC_2 e + C_1^{\mathrm{T}} d$$

Choosing $V = e^{\mathrm{T}} S_\theta e$ as a candidate Lyapunov function, we get

$$\begin{aligned} \dot{V} &= 2e^{\mathrm{T}} S_\theta \dot{e} \\ &= 2e^{\mathrm{T}} S_\theta (A - S_\theta^{-1} C_1^{\mathrm{T}} C_1)e + 2e^{\mathrm{T}} S_\theta (\bar{g}(z,u) - \bar{g}(\hat{z},u)) \\ &\quad - 2e^{\mathrm{T}} S_\theta E C_2 e + 2e^{\mathrm{T}} S_\theta C_1^{\mathrm{T}} d \\ &= -\theta V - \|Ce\|^2 + 2e^{\mathrm{T}} S_\theta (\bar{g}(z,u) - \bar{g}(\hat{z},u)) \\ &\quad - 2e^{T} S_\theta E C_2 e + 2e^{\mathrm{T}} S_\theta C_1^{\mathrm{T}} d \\ &\leq -\theta V + 2e^{\mathrm{T}} S_\theta (\bar{g}(z,u) - \bar{g}(\hat{z},u)) \\ &\quad - 2e^{\mathrm{T}} S_\theta E C_2 e + 2e^{\mathrm{T}} S_\theta C_1^{\mathrm{T}} d \end{aligned}$$

Since $\bar{g}(z,u)$ is lower triangular and globally Lipschitzian with respect to x uniformly in u, using a similar argument as in Gauthier *et al.* (1992), we have

$$2\|e^{\mathrm{T}} S_\theta (\bar{g}(z,u) - \bar{g}(\hat{z},u))\| \leq k_1 V$$

for some positive constant k_1 depending on the Lipschitz constant of g and the upper bound of u but which is independent on θ. Similarly, since the matrix EC_2 is lower triangular, there exist a constant k_2 which is independent on θ such that

$$2\|e^{\mathrm{T}} S_\theta E C_2 e\| \leq k_1 V$$

Recall that $S_\theta = (1/\theta)\Delta_\theta S \Delta_\theta$ (see Busawon *et al.* 1997) where

$$\Delta_\theta = \mathrm{diag}\left(\frac{1}{\theta}, \frac{1}{\theta^2}, \dots, \frac{1}{\theta^{n+1}}\right)$$

and S is a symmetric positive definite matrix which is independent on θ. Consequently, assuming $\theta \geq 1$, we have

$$\frac{\lambda_0}{\theta}\|\Delta_\theta e\|^2 \leq e^{\mathrm{T}} S_\theta e \leq \frac{\lambda_1}{\theta}\|\Delta_\theta e\|^2$$

where λ_0 and λ_1 is the smallest and largest eigenvalue of S respectively. As a result

$$\begin{aligned} 2\|e^{\mathrm{T}} S_\theta C_1^{\mathrm{T}} d\| &= \frac{2}{\theta}\|e^{\mathrm{T}} \Delta_\theta S \Delta_\theta C_1^{\mathrm{T}} d\| \\ &\leq \frac{2\lambda_1}{\theta}\|e^{\mathrm{T}} \Delta_0\| \|\Delta_0 C_1^{\mathrm{T}} d\| \end{aligned}$$

Now, $\|\Delta_\theta C_1^{\mathrm{T}} d\| \leq (1/\theta)|d|$ owing to the special form of C_1. Hence

$$2\|e^{\mathrm{T}} S_\theta C_1^{\mathrm{T}} d\| \leq \frac{2\lambda_1}{\theta^2}\|\Delta_\theta e\| |d| \leq \frac{2\lambda_1}{\theta\sqrt{\lambda_0}}\sqrt{V}|d|$$

Therefore

$$\begin{aligned}\dot V &\le -\theta V + 2\|e^{\mathrm T} S_\theta(\bar g(z,u) - \bar g(\hat z,u))\| \\ &\quad + 2\|e^{\mathrm T} S_\theta E C_2 e\| + 2\|e^{\mathrm T} S_\theta C_1^{\mathrm T} d\| \\ &\le -\theta V + kV + \frac{c_1}{\theta}\sqrt{V}|d|\end{aligned}$$

where $c_0 = k_1 + k_2$ and $c_1 = 2\lambda_1/\sqrt{\lambda_0}$. Now, choosing $\theta > k$, we obtain

$$\dot V \le -\mu V + \frac{c_1}{\theta}\sqrt{V}|d|$$

for some positive constant μ. Since, $\dot V = 2\sqrt{V}\dot{\sqrt{V}}$, we get

$$\dot{\sqrt{V}} \le -\mu\sqrt{V} + \frac{c_1}{\theta}|d|$$

From this last inequality we see that if $d = 0$, we get the classical exponential convergence properties of the noise-free high gain observer. On the other hand, if $d \ne 0$, we obtain an asymptotic observer instead, and, the perturbation term can be made sufficiently small for large values of θ. □

6. Application to a flexible joint mechanism

The dynamics of a single link robot arm with a revolute elastic joint rotating in a vertical plane are given by (see Lewis *et al.* 1993 and Marino and Tomei 1995)

$$J_1\ddot q_1 + k(q_1 - q_2) + mgl\sin q_1 = 0 \tag{28}$$

$$J_2\ddot q_2 - k(q_1 - q_2) = u \tag{29}$$

in which q_1 and q_2 are the link displacement and the rotor displacement, respectively. The link inertia J_1, the motor rotor inertia J_2, the elastic constant k, the link mass m, the gravity constant g and the centre of mass l are all positive.

In what follows, we shall use the following numerical values: $J_2 = 0.2\,\mathrm{kg\,m^2}$, $k = 10\,\mathrm{N}$, $m = 1\,\mathrm{kg}$, $g = 10\,\mathrm{m/s^2}$, $J_1 = 1\,\mathrm{kg\,m^2}$, $l = 1\,\mathrm{m}$. We assume that the position q_1 is measured.

We set $x_1 = q_1$, $x_2 = \dot q_1$, $x_3 = 10q_2$, $x_4 = 10\dot q_2$, and we assume that the position sensor is affected by a measurement noise $d(t)$. Then, the above system can be written in a state-space representation as

$$\left.\begin{aligned}\dot x_1 &= x_2 \\ \dot x_2 &= x_3 - 10\sin x_1 - 10x_1 \\ \dot x_3 &= x_4 \\ \dot x_4 &= 50x_1 - 5x_3 + 5u \\ y &= x_1 + d\end{aligned}\right\} \tag{30}$$

which is of the form (24) with $E_0 = 0$. A proportional high gain observer as described in Gauthier *et al.* (1992) for (30) is given by

$$\left.\begin{aligned}\dot{\hat x}_1 &= \hat x_2 + 4\theta(x_1 + d - \hat x_1) \\ \dot{\hat x}_2 &= \hat x_3 - 10\sin\hat x_1 - 10\hat x_1 + 6\theta^2(x_1 + d - \hat x_1) \\ \dot{\hat x}_3 &= \hat x_4 + 4\theta^3(x_1 + d - \hat x_1) \\ \dot{\hat x}_4 &= 50\hat x_1 - 5\hat x_3 + 5u + \theta^4(x_1 + d - \hat x_1)\end{aligned}\right\} \tag{31}$$

A proportional integral observer as described by (27) is given by

$$\left.\begin{aligned}\dot{\hat x}_0 &= \hat x_1 + 5\theta(x_0 - \hat x_0) \\ \dot{\hat x}_1 &= \hat x_2 + 10\theta^2(x_0 - \hat x_0) \\ \dot{\hat x}_2 &= \hat x_3 - 10\sin\hat x_1 - 10\hat x_1 + 10\theta^3(x_0 - \hat x_0) \\ \dot{\hat x}_3 &= \hat x_4 + 5\theta^4(x_0 - \hat x_0) \\ \dot{\hat x}_4 &= 50\hat x_1 - 5\hat x_3 + 5u + \theta^5(x_0 - \hat x_0)\end{aligned}\right\} \tag{32}$$

Remark: It is clear that the above PIO design can be readily extended to systems of the form

$$\left.\begin{aligned}\dot x &= A_0x + g(x,u) + \varphi(y) + E_0d \\ y_1 &= C_0x + d\end{aligned}\right\} \tag{33}$$

since the non-linearity $\varphi(y)$ will be cancelled in the error dynamics. As an example, consider the above flexible robot model. If an output injection in the linear terms is performed, then we would have the system

$$\left.\begin{aligned}\dot x_1 &= x_2 \\ \dot x_2 &= x_3 - 10\sin x_1 - 10y + 10d \\ \dot x_3 &= x_4 \\ \dot x_4 &= 50y - 5x_3 + 5u - 50d \\ y &= x_1 + d\end{aligned}\right\} \tag{34}$$

which is of the form (33) with $g(x,u) = \mathrm{col}\,(0, -10\sin x_1, 0, -5x_3 + 5u)$ and $E_0 = \mathrm{col}\,(0, 10, 0, -50)$.

6.1. *Simulation results*

A simulation was carried out to compare the performance of the high gain observer (31) and the proportional integral high gain observer (32). The same initial condition and the value of $\theta = 4$ was used.

The disturbance $d(t)$ is a measurement noise of 5% of the maximum value of the output with zero mean and standard deviation 1. Figure 6 shows the plot of the noisy measurements $y(t)$. In order to economize space only the estimation of x_3 is shown. Figure 7 shows the estimation of x_3 when the high gain proportional observer (31) was used. Figure 8

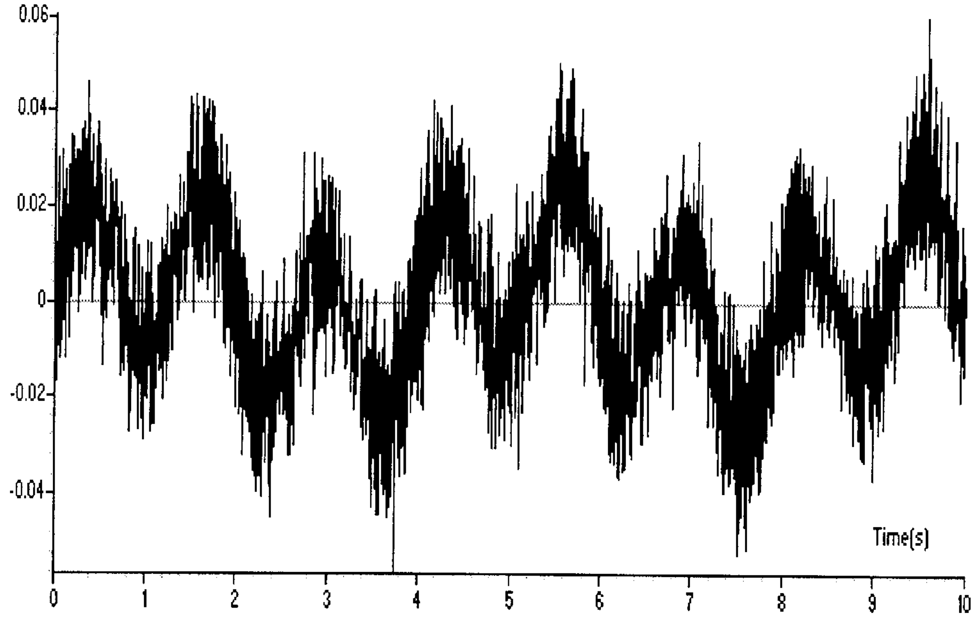

Figure 6. The noisy measurements $y(t)$.

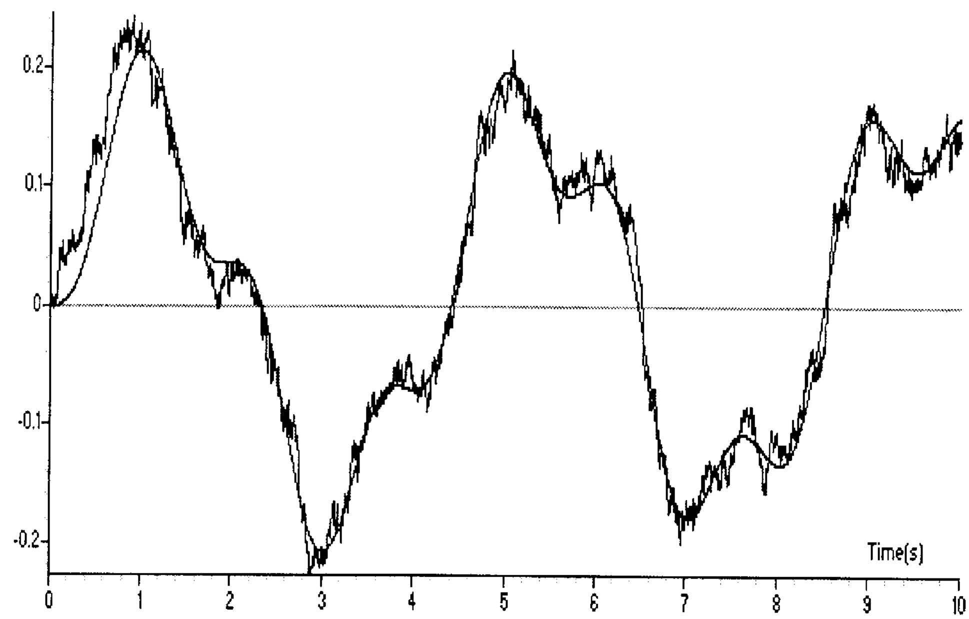

Figure 7. Estimation of x_3 with proportional high gain observer.

shows the estimation of x_3 when the PIO (32) was used. The disturbance attenuating properties of the PIO (32) can clearly be observed.

7. Conclusions

In this paper, we have, first of all, given a design approach of proportional integral observer for linear uncertain systems. It is shown that, when only sensor noise is present in the system, integral observers permit us to achieve good filtering properties. On the other hand, when modelling errors and sensor noise are present, it is shown that the proportional integral observer allows us to decouple completely the modelling uncertainties. A comparison of the classical proportional observer to the proposed observers are given via a simulation example. An extension of the proposed design to a special class of non-linear systems is also given. The extention of the design to a more general class of non-linear systems is an open problem. A simulation example dealing with a flexible joint robot have shown the very satisfactory disturbance attenuating properties of the proposed observer. We have treated only the case of single-output systems, the case of multi-output systems is a topic of current research.

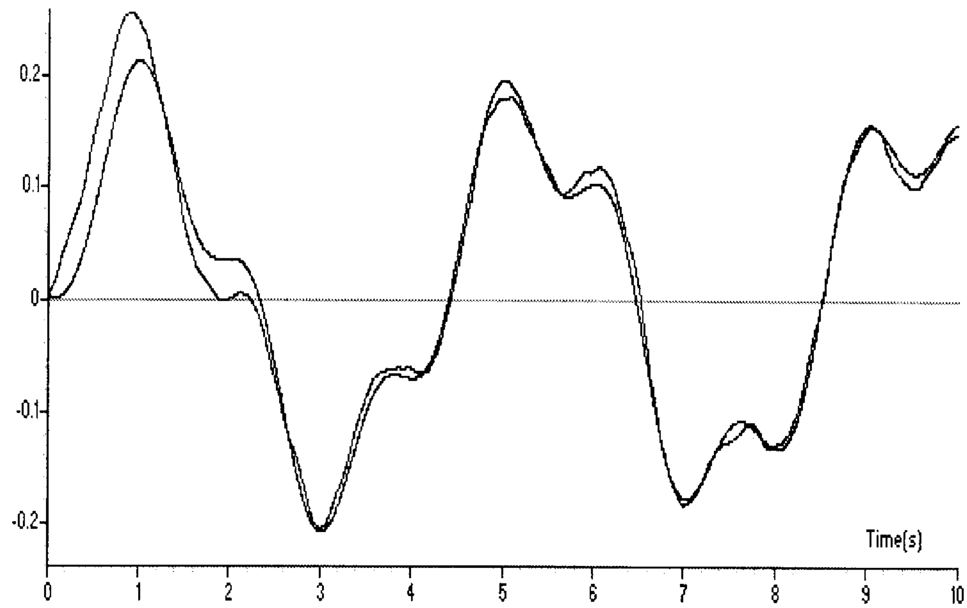

Figure 8. Estimation of x_3 with high gain PIO.

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