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An approximation scheme for the optimal control of diffusion processes

AN APPROXIMATION SCHEME
FOR THE OPTIMAL CONTROL
OF DIFFUSION PROCESSES (*)

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Abstract — *We present a numerical approximation scheme for the infinite horizon problem related to diffusion processes The scheme is based on a discrete version of the dynamic programming principle and converges to the viscosity solution of the second order Hamilton-Jacobi-Bellman equation The diffusion can be degenerate The problem in $\mathbb{R}^n$ is solved in a bounded domain Ω using a truncation technique and without imposing invariance conditions on Ω We prove explicit estimates of the error due to the truncation technique*

Résumé — *Nous étudions un schema numérique d'approximation pour un problème de contrôle optimal a horizon infini pour un processus de diffusion Le schéma est construit à partir d'une version discrete du principe de la programmation dynamique et converge vers la solution de viscosité de l'équation d'Hamilton Jacobi-Bellman associée La diffusion peut dégénérer dans le domaine Le problème dans $\mathbb{R}^n$ est resolu sur un borné Ω en utilisant une technique de troncature et sans imposer des conditions d'invariance sur Ω On donne aussi des estimations de l'erreur due à la technique de troncature utilisée*

1. INTRODUCTION

We are interested in the approximation of the following discounted optimal control problem for a diffusion process. Let $(\Omega, F, \mathbb{P}, \{F_t\}_{t \geq 0})$ be a complete filtered probability space and suppose that given a progressively measurable processes $\alpha(t)$ there exists a unique (progressively measurable) process $X(t)$ satisfying the controlled stochastic differential equation :

$$\begin{cases} dX(t) = b(X(t), \alpha(t)) dt + \sigma(X(t), \alpha(t)) dW(t) \\ X(0) = x . \end{cases} \quad (1.1)$$

The process $X(t)$ represents the state of a system evolving in $\mathbb{R}^n$, the process $\alpha(t)$ is the control applied to the system at time t with values in the compact metric space A, $W(t)$ is a d-dimensional Wiener process

We consider the cost functional related to the infinite horizon problem

$$J_x(\alpha) \equiv \mathbb{E}\left\{\int_0^{+\infty} f(X(t), \alpha(t)) e^{-\lambda t} dt \,|\, X(0) = x\right\} \qquad (1\ 2)$$

where f $\mathbb{R}^n \times A \to \mathbb{R}$ is the *running cost* and λ is a positive parameter, the *discount factor* The set of admissible control laws will be given by the progressively measurable processes which take values in A, and will be denoted by $\mathscr{A}$

Under the above assumptions, the value function of the problem

$$v(x) = \inf\{J_\lambda(\alpha) \quad \alpha(t) \in \mathscr{A}\} \qquad (1\ 3)$$

is (Lions [L2]) the unique viscosity solution of the Hamilton-Jacobi-Bellman equation

$$(HJB) \quad \lambda u(x) = \inf_{a \in A}\{L(a) u(x) + f(x, a)\}, \quad x \in \mathbb{R}^n$$

where

$$L(a) = \frac{1}{2} \sum_{i\ j-1}^{n} \left(\sum_{m-1}^{d} \sigma_{im}(\,.\,,a)\,\sigma_{jm}(\,.\,,a)\right) \partial_{ij} + \sum_{i\ 1}^{n} b_i(\,.\,,a)\,\partial_i \qquad (1\ 4)$$

Several authors have studied approximation schemes for (HJB) using different techniques Kushner has proposed algorithms based on probabilistic methods   the key tool of these techniques is the approximation of the diffusion by proper Markov chains in finite dimensional spaces In this way (see [K], [KD] and references therein) one can obtain, *via* probabilistic arguments, convergence results for the value function of the optimal Markov problem to v but it is quite hard to establish a precise order of convergence Also Menaldi [M] has studied the problem mixing probability and analytic methods and obtaining a convergence result (to classical solutions) for a scheme derived by the dynamic programming principle More recently, Sun [Su] worked on an exit time problem using similar techniques coupled with a domain decomposition method

Numerical methods for (HJB) when the diffusion is non degenerate, i e in the case when it admits classical solutions, have been considered in Lions-Mercier [LM], Quadrat [Q] More recently, Akian [A] and Hoppe [H] worked on exit time problems using multigrid techniques in order to speed-up the convergence

In this paper we present a method for approximating the weak (viscosity) solution of (HJB). The scheme is based on a discretization of the equation both in the time and in the space variable and in this respect is similar to the one proposed in [M]. We should also mention that our method extends to the optimal control of diffusion processes the dynamic programming approach developed for deterministic problems in Capuzzo-Dolcetta [C], Capuzzo-Dolcetta and Ishii [CI], Capuzzo-Dolcetta and Falcone [CDF] and Falcone [F1]. The main feature of our method is the convergence to the weak solution (in the viscosity sense) also when the diffusion is degenerate, *i.e.* when $L(a)$ becomes a first order operator. This property guarantees the robustness of the scheme (see Test 3 in Section 4). As it is well known, a diffusion leaves any bounded domain with probability 1. Therefore any direct approximation of (HJB) is impossible since it would require the discretization of an unbounded domain. We propose here a truncation technique to restrict the problem to an arbitrary bounded domain Ω (where a numerical approximation becomes feasible) obtaining an error bound for the difference between v restricted to Ω and the solution of the truncated problem. We refer to [FiF] for a similar technique applied to the study of an economic model.

Finally, we point out that the method provide approximate feed-back controls at any point of the grid without extra computations.

The paper is organized as follows.

In Section 2 we introduce our basic assumptions, we build the time discretization and establish the main convergence theorem. Section 3 is devoted to the space discretization and to the truncation technique. Section 4 describes the numerical results on some test problems. In particular we present the experimental errors, the approximate feed-back controls and the approximate solutions when the diffusion tends to degenerate (vanishing viscosity).

2. DISCRETIZATION IN TIME AND BASIC ASSUMPTIONS

Let $b : \mathbb{R}^n \times A \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \times A \to \mathscr{L}(\mathbb{R}^d ; \mathbb{R}^n)$ and $f : \mathbb{R} \times A \to \mathbb{R}$. We shall assume in the sequel that b, σ and f are continuous and, for all $a \in A$, $x \in \mathbb{R}^n$

$$|g(x, a) - g(y, a)| \leqslant L_g |x - y|$$
$$|g(x, a)| \leqslant M_g \quad \text{for} \quad g = \sigma, b \tag{2.1}$$

$$|f(x, a) - f(y, a)| \leqslant L_f |x - y|^{\vartheta}$$
$$|f(x, a)| \leqslant M_f . \tag{2.2}$$

We also assume that

$$\lambda > 0 . \tag{2.3}$$

Assumptions (2 1), (2.2) and (2 3) guarantee that the strong solution of (1 1) is unique Let $h \in \left(0, \dfrac{1}{\lambda}\right]$ be a parameter and consider the following approximation of (HJB)

(HJB_h) $u_h(x) = \min_{a \in A} \{(1 - \lambda h) \, \Pi_h(a) \, u_h(x) + h f(x, a)\}$

where $\Pi_h(a)$ is the operator

$$\Pi_h(a) \, \phi(x) \equiv \sum_{m=1}^{d} \frac{1}{2d} [\phi(x + hb(x, a) + \sqrt{h} \, \sigma_m(x, a)) +$$

$$+ \phi(x + hb(x, a) - \sqrt{h} \, \sigma_m(x, a))] \qquad (2\ 4)$$

and σ_m is the m-th row of σ

Following the same ideas in [M] and [BeS] we can give a control interpretation of (HJB_h) thinking to it as the characterization of the value function of a discrete-time optimal control problem In fact, let us consider the Markov chain X_n

$$X_{n+1} = X_n + hb(X_n, a_n) + \sqrt{h} \sum_{m=1}^{d} \sigma_m(X_n, a_n) \, \xi_m^{n+1}, \quad X_0 = x$$

where ξ^{n+1} is a sequence of i i d random variables in $\mathbb{R}^d$ such that

$$\mathbb{P}(\xi_m^n = 1) = \mathbb{P}(\xi_m^n = -1) = \frac{1}{2d}$$

and

$$\mathbb{P}\left(\bigcup_{i,j=1}^{d} \{\xi_i^n \neq 0\} \cap \{\xi_j^n \neq 0\}\right) = 0$$

then

$$\Pi_h(a) \, \phi(x) = \mathbb{E}(X_1 \mid X_0 = x)$$

This implies that (HJB_h) is the dynamic programming equation related to the optimal control problem of $\{X_n\}$ with the infinite horizon cost functional

We recall that the viscosity solution of (HJB) belongs to $C^{0\,\gamma}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ (Lions [L1]) where γ is such that

  i) $\gamma = 1$ if $\lambda > \lambda_0$
  ii) $\gamma \in (0, 1)$ if $\lambda = \lambda_0$
  iii) $\gamma = \dfrac{\lambda}{\lambda_0}$ if $\lambda < \lambda_0$

and

$$\lambda_0 \equiv \sup_{x \neq y, \, a \in A} \left\{ \frac{1}{2} \frac{ \begin{array}{c} tr(\sigma(x, a) - \sigma(y, a), \sigma^t(x, a) - \sigma^t(y, a)) + \\ + 2(b(x, a) - b(y, a), x - y) \end{array} }{|x - y|^2} \right\} .$$

We will denote by B_ϑ the Banach space of bounded, Hölder continuous functions on $\mathbb{R}^n$, endowed with the norm

$$\|v\|_{0, \, \vartheta} = \|v\|_\infty + |v|_\vartheta$$

where

$$\|v\|_\infty \equiv \sup_{x \in \mathbb{R}^n} |v(x)|$$

$$\|v\|_\vartheta \equiv \sup_{x, \, y \in \mathbb{R}^n, \, x \neq y} \frac{|v(x) - v(y)|}{|x - y|^\vartheta} .$$

PROPOSITION 2.1 : *Let* $\lambda > \lambda_0 \, \vartheta$. *Then, for every* $h \in \left(0, \frac{1}{\lambda} \right]$, (HJB_h) *has a unique solution* $u_h \in B_\vartheta$ *and*

$$\|u_h\|_\infty \leq \frac{M_f}{\lambda} \tag{2.5}$$

$$\|u_h\|_\vartheta \leq \frac{L_f}{\lambda - \lambda_0 \, \vartheta} . \tag{2.6}$$

Proof :

i) Let us define the operator :

$$T_h v(x) \equiv \inf_{a \in A} \left\{ (1 - \lambda h) \, \Pi_h(a) \, v(x) + h f(x, a) \right\} .$$

We shall prove that T_h is contraction map on $L^\infty(\mathbb{R}^n)$. Let u, v be two bounded functions on $\mathbb{R}^n$ and $\bar{a} \in A$ be the control such that the minimum for $T_h v(x)$ is attained. Then

$$(T_h u - T_h v)(x) \leq (1 - \lambda h)$$

$$\left\{ \sum_{m=1}^d \frac{1}{2 d} \, [u(x + hb(x, \bar{a}) + \sqrt{h} \, \sigma_m(x, \bar{a})) - v(x + hb(x, \bar{a}) + \right.$$

$$+ \sqrt{h} \, \sigma_m(x, \bar{a}))] + \frac{1}{2 d} \, [u(x + hb(x, \bar{a}) - \sqrt{h} \, \sigma_m(x, \bar{a})) -$$

$$\left. - v(x + hb(x, \bar{a}) - \sqrt{h} \, \sigma_m(x, \bar{a}))] \right\} \leq$$

$$\leq (1 - h\lambda) \|u - v\|_\infty .$$

Reversing the role of u and v we get,

$$|T_h u(x) - T_h v(x)| \leq (1 - \lambda h) \|u - v\|_\infty, \quad \text{for all} \quad x \in \mathbb{R}^n.$$

Since $h \in \left(0, \dfrac{1}{\lambda}\right]$, T_h is a contraction map on $L^\infty(\mathbb{R}^n)$ and there exists a unique bounded function u_h such that $T_h u_h = u_h$.

ii) For every $h \in \left(0, \dfrac{1}{\lambda}\right]$,

$$|u_h(x)| = |T_h u_h(x)| \leq (1 - \lambda h)\|u_h\|_\infty + h M_f$$

then

$$\|u_h\|_\infty \leq (1 - \lambda h)\|u_h\|_\infty + h M_f$$

and this gives (2.5).

iii) Let us show now that u_h belongs to B_ϑ. Given $v \in B_\vartheta$ we prove that $T_h v$ belongs to B_ϑ. Let $\bar{a}$ be a control (not unique in general) where the minimum for $T_h v(y)$ is attained, then

$$
\begin{aligned}
T_h v(x) - T_h v(y) &\leq (1 - \lambda h)\left\{ \sum_{m=1}^{d} \frac{1}{2d} [v(x + hb(x, \bar{a}) + \sqrt{h}\,\sigma_m(x, \bar{a})) + \right. \\
&\quad + v(y + hb(y, \bar{a}) + \sqrt{h}\,\sigma_m(y, \bar{a}))] + \\
&\quad + \frac{1}{2d} [v(x + hb(x, \bar{a}) - \sqrt{h}\,\sigma_m(x, \bar{a})) + \\
&\quad + v(y + hb(y, \bar{a}) - \sqrt{h}\,\sigma_m(y, \bar{a}))] + h(f(x, \bar{a}) - f(y, \bar{a}))\right\} \leq \\
&\leq (1 - \lambda h)|v|_\vartheta \left\{ \sum_{m=1}^{d} \frac{1}{2d} [\,|x - y + h(b(x, \bar{a}) - b(y, \bar{a})) + \right. \\
&\quad + \sqrt{h}(\sigma_m(x, \bar{a}) - \sigma_m(y, \bar{a}))|^\vartheta \\
&\quad + |x - y + h(b(x, \bar{a}) - b(y, \bar{a})) - \sqrt{h}(\sigma_m(x, \bar{a}) - \sigma_m(y, \bar{a}))|^\vartheta]\right\} \\
&\quad + h L_f |x - y|^\vartheta.
\end{aligned}
$$

By applying the inequality $(\alpha + \beta) \leq 2^{(p-1)/p}(\alpha^p + \beta^p)^{1/p}$, for $p = \dfrac{2}{\vartheta} \geq 2$ and

$$\alpha = \frac{1}{2} \left| x - y + h(b(x, \bar{a}) - b(y, \bar{a})) + \sqrt{h}(\sigma_m(x, \bar{a}) - \sigma_m(y, \bar{a})) \right|^\vartheta$$

$$\beta = \frac{1}{2} \left| x - y + h(b(x, \bar{a}) - b(y, \bar{a})) - \sqrt{h}(\sigma_m(x, \bar{a}) - \sigma_m(y, \bar{a})) \right|^\vartheta$$

we get

$$T_h v(x) - T_h v(y) \leqslant$$

$$\leqslant (1 - \lambda h) \frac{1}{d} |v|_\vartheta \sum_{m=1}^d \left\{ |x - y|^2 \left[1 + 2 h \left(\frac{1}{2} L_\sigma^2 + L_b \right) + L_b^2 h^2 \right] \right\}^{\vartheta/2} +$$

$$+ hL_f |x - y|^\vartheta \leqslant (1 - \lambda h)(1 + \lambda_0 h)^\vartheta |v|_\vartheta |x - y|^\vartheta + hL_f |x - y|^\vartheta .$$

Hence, by symmetry,

$$|T_h v(x) - T_h v(y)| \leqslant (1 - \lambda h)(1 + \lambda_0 h)^\vartheta |v|_\vartheta |x - y|^\vartheta + hL_f |x - y|^\vartheta$$

for all x and y in $\mathbb{R}^n$. Therefore

$$|T_h v|_\vartheta \leqslant (1 - \lambda h)(1 + \lambda_0 h)^\vartheta |v|_\vartheta + hL_f . \qquad (2.8)$$

Let us define,

$$C_h \equiv hL_f / [1 - (1 - \lambda h)(1 + \lambda_0 h)^\vartheta] .$$

Since $\lambda > \lambda_0 \vartheta$, C_h is strictly positive. By (2.8) $|v|_\vartheta \leqslant C_h$ implies $|T_h v|_\vartheta \leqslant C_h$. Then, for any $h \in \left(0, \frac{1}{\lambda} \right]$, the solution u_h of the equation (HJB_h) belongs to B_ϑ and verifies $|u_h|_\vartheta \leqslant C_h$. Since C_h is a decreasing function of $h > 0$, we have

$$|u_h|_\vartheta \leqslant \lim_{h \to 0^+} C_h = L_f / (\lambda - \vartheta \lambda_0)$$

and this proves (2.6). □

The next theorem shows that the sequence u_h converges to the viscosity solution of (HJB).

THEOREM 2.2 : *Let u be the viscosity solution of (HJB). Then, for $h \to 0^+$, $u_h \to u$ locally uniformly in $\mathbb{R}^n$.*

Proof : From (2.5), (2.6) and the Ascoli-Arzelà theorem, there exist a subsequence $h_p \to 0^+$ and a function $u \in B_\vartheta$ such that $u_{h_p} \to u$ locally uniformly in $\mathbb{R}^n$. We shall prove that u is the viscosity solution of (HJB). Let $\phi \in C^2(\mathbb{R}^n)$ and assume that x_0 is a local maximum point for $u - \phi$. There exists then a closed ball $B(x_0, R)$ such that

$$(u - \phi)(x_0) \geqslant (u - \phi)(x) \quad \text{for any} \quad x \in B(x_0, R) .$$

Let x_p be a local maximum point for u_{h_p} in $B(x_0, R)$. Since $\{u_{h_p}\}_p$ converges locally uniformly to u, we have $x_p \to x_0$ as $p \to \infty$. Since b and σ are

bounded, the points $x_p + h_p b(x_p, a) \pm \sqrt{h_p} \sigma_m(x_p, a)$, $m = 1, ..., d$, belong to $B(x_0, R)$ for p large enough, so that

$$(u_{h_p} - \phi)(x_p) \geq (u_{h_p} - \phi)(x_p + h_p b(x_p, a) \pm \sqrt{h_p} \sigma_m(x_p, a))$$
$$m = 1, ..., d.$$

Since u_{h_p} is the solution of (HJB_{h_p}), we have :

$$0 = \max_{a \in A} \left\{ -(1 - \lambda h_p) \Pi_{h_p}(a) u_{h_p}(x_p) - h_p f(x_p, a) \right\} + u_{h_p}(x_p) =$$

$$= \max_{a \in A} \left\{ \sum_{m=1}^{d} \frac{1}{2d} [u_{h_p}(x_p) - u_{h_p}(x_p + h_p b(x_p, a) + \sqrt{h_p} \sigma_m(x_p, a))] + \right.$$

$$+ \sum_{m=1}^{d} \frac{1}{2d} [u_{h_p}(x_p) - u_{h_p}(x_p + h_p b(x_0, a) - \sqrt{h_p} \sigma_m(x_p, a))] +$$

$$\left. + \lambda h_p \Pi_{h_p}(a) u_{h_p}(x_p) - h_p f(x_p, a) \right\} \geq$$

$$\geq \max_{a \in A} \left\{ \sum_{m=1}^{d} \frac{1}{2d} [\phi(x_p) - \phi(x_p + h_p b(x_p, a) + \sqrt{h_p} \sigma_m(x_p, a))] + \right.$$

$$+ \sum_{m=1}^{d} \frac{1}{2d} [\phi(x_p) - \phi(x_p + h_p b(x_p, a) - \sqrt{h_p} \sigma_m(x_p, a))] +$$

$$\left. + \lambda h_p \Pi_{h_p}(a) u_{h_p}(x_p) - h_p f(x_p, a) \right\}.$$

Since $\phi \in C^2(\mathbb{R}^n)$, the above inequality gives

$$0 \geq \max_{a \in A} \left\{ - \sum_{m=1}^{d} \frac{1}{2d} \sum_{\iota=1}^{n} \partial_\iota \phi(x_p + h_p b_\iota(x_p, a) + \right.$$

$$+ \sqrt{h_p} \sigma_{\iota m}(x_p, a))[h_p b_\iota(x_p, a) +$$

$$+ h_p b_\iota(x_p, a)] - \sum_{m=1}^{d} \frac{1}{2d} \sum_{\iota, \jmath=1}^{n} \frac{1}{2} \partial_{\iota \jmath} \phi(x_o + \rho_{h_p}^{m,+}(a)(h_p b(x_p, a) +$$

$$+ \sqrt{h_p} \sigma_m(x_p, a)))[h_p b_\iota(x_p, a) + \sqrt{h_p} \sigma_{\iota m}(x_p, a)]$$

$$[h_p b_\jmath(x_p, a) + \sqrt{h_p} \sigma_{\jmath m}(x_p, a)] -$$

$$- \sum_{m=1}^{d} \frac{1}{2d} \sum_{\iota \jmath=1}^{n} \frac{1}{2} \partial_{\iota \jmath} \phi(x_p + \rho_{h_p}^{m,-}(a)(h_p b(x_p, a)) + \sqrt{h_p} \sigma_m(x_p, a))$$

$$[h_p b_\iota(x_p, a) - \sqrt{h_p} \sigma_{\iota m}(x_p, a)][h_p b_\jmath(x_p, a) - \sqrt{h_p} \sigma_{\jmath m}(x_p, a)] +$$

$$\left. + \lambda h_p \Pi_{h_p}(a) u_{h_p}(x_p) - h_p f(x_p, a) \right\}$$

for some $\rho_{h_p}^{m,+}(a)$, $\rho_{h_p}^{m,-}(a) \in (0, 1)$. Dividing the above relation by

h_p and passing to the limit as $p \to \infty$, we get

$$0 \geqslant \max_{a \in A} \left\{ -\frac{1}{2} \sum_{m=1}^{d} \sum_{i,j=1}^{n} \sigma_{im}(x_0, a)\, \sigma_{jm}(x_0, a)\, \partial_{ij} \phi(x_0) + \right.$$

$$\left. + \sum_{i=1}^{n} b_i(x_0, a)\, \partial_i \phi(x_0) + \lambda u(x_0) - f(x_0, a) \right\} =$$

$$= \max_{a \in A} \left\{ - L(a)\, \phi(x_0) - f(x_0, a) + \lambda u(x_0) \right\}.$$

Repeating the same argument for the minimum of $u - \phi$ one can easily complete the proof. The convergence of u_{h_p} to u follows from the uniqueness of the viscosity solution of (HJB). $\square$

Remark 2.1 : The proof of the above theorem can also be obtained by applying the general convergence result (Theorem 2.1) in Barles-Souganidis [BS] which does not require the estimate (2.6) (see also [FS]). Since (2.6) will be used to prove the estimate (3.10), we preferred to give a direct proof.

Remark 2.2 : It is rather easy to use the above discretization to construct an approximation scheme for the evolutive problem related to a non autonomous dynamics and the finite horizon cost functional

$$J_{(x, y)}(\alpha) \equiv \mathbb{E}_x \left\{ \int_t^T f(y_x(t), \alpha(t), t) + \psi(y(T)) \right\}.$$

The time-discrete scheme is

$$\left\{ \begin{array}{l} \dfrac{\partial u}{\partial t}(x, t) + \sup_{a \in A} \left\{ -\dfrac{1}{2} Tr[\sigma(x, a, t)\, \sigma(x, a, t)^T D^2 u(x, t)] - \right. \\[2mm] \left. - b(x, a, t)\, Du(x, t) - f(x, a, t) \right\} = 0 \quad (x, t) \in \mathbb{R}^n \times [0, T) \\[2mm] u(x, T) = \psi(x) \quad x \in \mathbb{R}^n. \end{array} \right.$$

Under assumptions on the coefficients analogous to (2.1) and (2.2), for any $\psi \in C_b^{0, \vartheta}(\mathbb{R}^n)$ there exists a unique viscosity solution u of $(HJBE)$ and $u \in C_b^{0, \vartheta}(\mathbb{R}^n \times [0, T])$ (see [IL]). By a simple adaptation of the approximation technique described above, we obtain the following explicit scheme

$$\left\{ \begin{array}{l} w_h(x, n) = \inf_{a \in A} \left\{ -\dfrac{1}{2d} \sum_{m-1}^{d} [w_h(x + hb(x, a, nh) + \right. \\[2mm] + \sqrt{h}\, \sigma_m(x, a, nh), (n + 1) h)] \\[2mm] - w_h(x + hb(x, a, nh) - \sqrt{h}\, \sigma_m(x, a, nh), (n + 1) h) \\[2mm] \left. + hf(x, a, nh) \right\} \quad x \in \mathbb{R}^n, \quad n = 0, ..., N - 1 \\[2mm] w_h(x, N) = \psi(x), \quad \text{for} \quad x \in \mathbb{R}^n \end{array} \right.$$

where $Nh - T$ The function w_h satisfies

$$\| w_h(. , n) \|_{L^\infty(\mathbb{R})} \leq \| \psi \|_{L^\infty(\mathbb{R})} + (N - n) hM_f \tag{2 9}$$

$$| w_h(. , n) |_\vartheta \leq (1 + \lambda_0 h)^{N - n} \left(| \psi |_\vartheta + \frac{L_f}{\lambda_0} \right) \tag{2 10}$$

with ϑ defined as in Proposition 2 1 The convergence of this scheme is guaranteed since the discrete scheme, defined by

$$u_h(x, t) = w_h(x, n) \quad \text{if} \quad t \in [nh, (n + 1) h), \quad n = 0, 1, \quad , N - 1$$
$$u_h(x, T) = \psi(x)$$

satisfies the basic assumptions of the general convergence theorem in [BS] In fact, stability follows from (2 9), whereas the proof of the monotonicity and consistency properties is straightforward Since a comparison principle holds true for the continuous problem, $u_h(x, t) \to u(x, t)$ locally uniformly in $\mathbb{R}^n \times [0, T]$

3 DISCRETIZATION IN THE SPACE VARIABLE

As we have seen in Section 2, the discretization in the time variable gives the approximating equation (HJB_h) where the state variable x is still continuous We will make the discretization in the state variable by means of piecewise affine finite elements In order to obtain a finite dimensional approximation of the Hamilton-Jacobi-Bellman equation we must restrict our problem to a bounded subset of $\mathbb{R}^n$

One can of course assume that the system (1 1) verifies an invariance condition, f e that there exists a polyhedron Ω such that

$$(IC_1) \quad \begin{cases} \sigma(x, a) = 0 \\ \\ b(x, a) . n(x) \leq c_0 < 0 \end{cases} \quad \text{for all} \quad x \in \partial\Omega, \quad a \in A$$

This condition is very restrictive since it corresponds to the degeneracy of the diffusion σ on $\partial\Omega$ (see also Remark 3 3 for the construction of an approximation scheme in this situation and [F] for the results related to the corresponding deterministic problem) Even more general invariance conditions such as

$$(IC_2) \quad \begin{cases} [D\varphi(x)]' \, \sigma(x, a) = 0 \\ \\ \frac{1}{2} Tr(\sigma(x, a) \sigma'(x, a) D^2 \varphi(x)) + b(x, a) D\varphi(x) \geq c > 0 \end{cases} \quad \text{for all} \quad x \in \partial\Omega, \quad a \in A$$

are unsatisfactory (see the Appendix for a proof of (IC_2))

We will focus our attention on a truncation technique which leads to a new approximation. This technique is more adequate to obtain approximate solutions since does not require supplementary assumptions on the diffusion and on the dynamics (1.1).

We introduce a cut-off function, restricting the problem to an arbitrary bounded set. Let us fix a parameter μ, $\mu > 0$, and define $I_\mu \equiv \{x \in \mathbb{R}^n : |x| < \frac{1}{\mu}\}$. Let

$$\xi_\mu : \mathbb{R}^n \to [0, 1], \quad \xi_\mu = 1 \quad \text{for} \quad x \in I_\mu \quad \text{and} \quad \xi_\mu \in C_0^\infty(\mathbb{R}^n) \quad (3.1)$$

we consider a new truncated control problem setting

$$\sigma_\mu \equiv \xi_\mu(x) \sigma(x, a), \quad b_\mu \equiv \xi_\mu^2(x) b(x, a). \quad (3.2)$$

Since b_μ, σ_μ are bounded and Lipschitz continuous, there exists a unique viscosity solution u_μ of the equation

$$(HJB)_\mu \qquad \lambda u(x) = \min_{a \in A} \{L_\mu(a) u(x) + f(x, a)\}, \quad x \in \mathbb{R}^n,$$

where

$$L_\mu(a) \equiv$$

$$\equiv \frac{1}{2} \sum_{i, j = 1}^n \left(\sum_{m = 1}^d \sigma_{im}(., a) \sigma_{jm}(., a)\right) \xi_\mu^2(.) \partial_{ij} + \sum_{i = 1}^n b_i(., a) \xi_\mu^2(.) \partial_i.$$

Notice that, by definition, the solution u_μ of the truncated problem satisfies

$$u_\mu = \frac{1}{\lambda} \min_{a \in A} f(x, a), \quad \forall x \notin \text{supp}(\xi_\mu). \quad (3.3)$$

We look for an estimate of $|u - u_\mu|$.

PROPOSITION 3.1 : *Assume (2.1)-(2.3) and (3.1), (3.2). Then there exists a positive constant C such that*

$$|u(x) - u_\mu(x)| \leq C \mu^2 (1 + |x|^2), \quad x \in I_\mu. \quad (3.4)$$

Proof : Let us denote by $\tilde{X}_\mu(t)$ the solution of the stochastic differential equation

$$\begin{cases} dX(t) = b_\mu(X(t), \alpha(t)) dt + \sigma_\mu(X(t), \alpha(t)) dW(t) \\ X(0) = x \end{cases}$$

and define $\tilde{\tau}_\mu \equiv \inf\{t > 0 : \tilde{X}_\mu(t) \notin I_\mu\}$, $\tau_\mu \equiv \inf\{t > 0 : X(t) \notin I_\mu\}$

(where $X(t)$ still denotes the solution of (1.1)). By the results in [L2] we have

$$u(x) = \inf_{\mathscr{A}} \mathbb{E} \left\{ \int_0^{\tau_\mu} f(X(t), \alpha(t)) e^{-\lambda t} dt + u(X(\tau_\mu)) e^{-\lambda \tau_\mu} \right\}$$

$$u_\mu(x) = \inf_{\mathscr{A}} \mathbb{E} \left\{ \int_0^{\tilde{\tau}_\mu} f(\tilde{X}_\mu(t), \alpha(t)) e^{-\lambda t} dt + u_\mu(\tilde{X}_\mu(\tilde{\tau}_\mu)) e^{-\lambda \tilde{\tau}_\mu} \right\} .$$

Since $\tau_\mu = \tilde{\tau}_\mu$ and $X(t) = \tilde{X}_\mu(t)$ for $t \leqslant \tau$ ($\mathbb{P} = 1$), we get

$$\left| u(x) - u_\mu(x) \right| \leqslant \left\| u(x) - u_\mu(x) \right\|_\infty \mathbb{E} \left[e^{-\lambda \tau_\mu} \right] \leqslant \frac{2 M_f}{\lambda} \mathbb{E} \left[e^{-\lambda \tau_\mu} \right] .$$

In order to obtain an estimate for $\mathbb{E}(e^{-\lambda \tau_\mu})$ in terms of μ, we observe that, for any $x \in I_\mu$

$$\mathbb{P} \left[\tau_\mu \leqslant t \right] \leqslant \mathbb{P} \left[\sup_{[0, t]} |X(s)| \geqslant 1/\mu \right] \leqslant C_1 \mu^2 (1 + |x|^2) \max [t, t^2]$$

where C_1 is a constant depending on M_σ, M_b. Then,

$$\mathbb{E} \left[e^{-\lambda \tau_\mu} \right] = \int_0^\infty \lambda \, e^{-\lambda t} \, \mathbb{P} \left[\tau_\mu \leqslant t \right] dt \leqslant C_2 (1 + |x|^2) \mu^2$$

which gives

$$\left| u(x) - u_\mu(x) \right| \leqslant C \mu^2 (1 + |x|^2) \quad \text{for} \quad x \in I_\mu$$

where C is a constant depending on M_σ, M_b, λ and M_f. $\square$

We remark that the term $(1 + |x|^2)$ in the estimate gives the dependence from $d(x, \partial \Omega)$, whereas the term μ^2 says how the estimate depends from the radius of the truncation domain. Obviously we can center I_μ at any point $x \in \mathbb{R}^n$.

The approximation scheme described in Section 2 can be applied to the truncated problem. We get the following equation

$(HJB_{h\mu})$

$$u_{h, \mu}(x) = \inf_{a \in A} \left\{ (1 - \lambda h) \, \Pi_{h, \mu}(a) u_{h, \mu}(x) + h f(x, a) \right\} , \quad x \in \mathbb{R}^n$$

where the operator $\Pi_{h, \mu}$ is obtained replacing in (2.4) σ and b by σ_μ and b_μ. As we have shown in Section 2, the above equation has a unique bounded and Hölder continuous solution, $u_{h\mu}$, and the sequence $\{u_{h_\mu}\}_h$ converges locally uniformly to u_μ, as $h \to 0^+$. It is important for the sequel to

notice that, for any positive h,

$$u_{h\mu}(x) = u_\mu(x), \quad x \notin \text{supp } (\xi_\mu).$$

Remark 3.1 : The truncation technique has a control interpretation. In fact, it is known that one can approximate the problem over $\mathbb{R}^n$ by a sequence of stopping time problems defined over $\Omega_n \subset \mathbb{R}^n$ with diam $(\Omega_n) \to \infty$. We point out that the definition of ξ_μ guarantees the regularity of u_μ in $\mathbb{R}^n$ and the fact that it is the solution of an (HJB) type problem in $\mathbb{R}^n$. This choice corresponds to a stopping time problem on $\Omega = \text{supp } \xi_\mu$ with a stopping cost on $\partial\Omega$ given by (3.3).

We want to construct the discretization of $(HJB_{h\mu})$ in the space variable in the domain $\Omega \equiv \text{int supp } (\xi_\mu)$.

Let $\{S_j\}$ be a family of simplices which set up a regular triangulation of $\mathbb{R}^n$ such that

$$\text{diam } (S_j) \leqslant k, \quad \forall j. \tag{3.5}$$

Let $\{x_i : i \in I\}$ be the set of vertices of the triangulation and $\{x_i : i \in I^0\}$ be the finite set of the vertices belonging to Ω.

Let W^k be the family of the functions which are continuous in $\mathbb{R}^n$ and affine on the simplices of the triangulation. We look for a solution in W^k of the following problem,

$(HJB^k_{h\mu})$

$$\begin{cases} w(x_i) = \min_{a \in A} \left\{ (1 - \lambda h)\, \Pi_{h\mu}(a)\, w(x_i) + hf(x_i, a) \right\} & \text{for } i \in I^0 \\[2mm] w(x_i) = \dfrac{1}{\lambda} \min_{a \in A} f(x_i, a) & \text{for } i \in I \backslash I^0. \end{cases}$$

THEOREM 3.2 : *Let $h \in \left(0, \dfrac{1}{\lambda} \right]$. Then, for every $k \in R_+$, there exists a unique solution $w \in W^k$ of $(HJB^k_{h\mu})$.*

Proof : Let us remark that $(HJB^k_{h\mu})$ is equivalent to a finite dimensional non linear problem. In fact, let us define

$$y^\pm_{i,m}(a) = x_i + hb_\mu(x_i, a) \pm \sqrt{h}\, \sigma_{\mu m}(x_i, a) \quad i \in I, \quad m = 1, ..., d$$

($\sigma_{\mu m}$ is the m-th row of σ_μ).

Let $M^+(m, a), M^-(m, a), m = 1, ..., d$, be the matrices such that

$$\sum_{j \in I} M^\pm_{ij}(m, a) = 1, \quad 0 \leqslant M^\pm_{ij}(m, a) \leqslant 1 \text{ for } m = 1, ..., d \text{ and } i \in I \tag{3.6}$$

and

$$y_{i,\,m}^{\pm}(a) = \sum_{j\,\in\,I} M_{ij}^{\pm}(m,\,a)\,x_j\ .$$

The matrices $M_{ij}^{\pm}$ are the baricentric coordinates of $y_{i,\,m}^{\pm}(a)$ with respect to the vertices of triangulation. It is a simple check to show that for $i \in I \backslash I^0$, $M_{ij}^{\pm}(m,\,a) = \delta_j^i$ and that $M_{ij}^{\pm}(m,\,a) \neq 0$ for at most $n + 1$ indices. Let us observe first that for $x \in \mathbb{R}^n$ such that $x = \sum_{i\,\in\,I} \mu_j\,x_j$ and $w \in W^k$, we can write $w(x) = \sum_{j\,\in\,I} \mu_j\,w(x_j)$. This allows to reduce $(HJB_{h\mu}^k)$ to the search for a vector $U \in \mathbb{R}^I$ such that

$$\begin{cases} U_i = \min_{a\,\in\,A} \left\{ (1 - \lambda h) \sum_{m\,=\,1}^d \frac{1}{2\,d}\,[M_i^+(m,\,a) + M_i^-(m,\,a)]\,U + hF_i(a) \right\} \\[2mm] \qquad \text{for}\quad i \in I^0 \\[2mm] U_i = \frac{1}{\lambda} \min_{a\,\in\,A} f(x_i,\,a) \quad \text{for}\quad i \in I \backslash I^0 \end{cases} \tag{3.7}$$

where $F_i(a) \equiv f(x_i,\,a)$.

Let us define

$$\mathscr{U}_{\mu} = \left\{ U \in \mathbb{R}^I : U_i = \frac{1}{\lambda} \min_{a\,\in\,A} f(x_i,\,a) \quad \text{for}\quad i \in I \backslash I^0 \right\} \tag{3.8}$$

and the operator $T_h^k : \mathscr{U}_{\mu} \to \mathscr{U}_{\mu}$,

$$[T_h^k\,W]_i \equiv$$

$$\equiv \min_{a\,\in\,A} \left\{ (1 - \lambda h) \sum_{m\,=\,1}^d \frac{1}{2\,d}\,[M_i^+(m,\,a) + M_i^-(m,\,a)]\,U + hF_i(a) \right\}$$

$$\text{for}\quad i \in I^0 \quad (3.9a)$$

$$[T_h^k\,U]_i \equiv U_i,\quad \text{for}\quad i \in I \backslash I^0\ . \tag{3.9b}$$

In order to prove that T_h^k is a contraction map in $\mathbb{R}^I$ it clearly suffices to show that it is a contraction in $\mathbb{R}^{I^0}$.

In fact, let $U, W \in \mathscr{U}_{\mu}$ and assume that $\bar{a} \in A$ is a control such that the minimum in $(3.9a)$ is attained. We have,

$$[T_h^k\,U]_i - [T_h^k\,W]_i \leqslant$$

$$\leqslant (1 - \lambda h) \left\{ \sum_1^d m\,\frac{1}{2\,d} \sum_{j\,\in\,I} [M_{ij}^+(m,\,\bar{a}) + M_{ij}^-(m,\,\bar{a})] \right\} |U_j - W_j|\ .$$

Since $\sum_{j \in I} M_{ij}^{\pm}(m, a) = 1$ for any a, by symmetry :

$$\left\| T_k^h U - T_k^h W \right\|_{\infty} \leq (1 - \lambda h) \left\| U - W \right\|_{\infty}$$

where $\|U - W\|_{\infty} = \max_{i \in I} |U_i - W_i|$. $\square$

The following theorem shows that w converges to $u_{h\mu}$ as k tends to zero.

THEOREM 3.3 : *Let* $\lambda > \vartheta \lambda_0$. *Then, for every* $h \in \left(0, \dfrac{1}{\lambda}\right]$,

$$\max_{x \in \mathbb{R}^n} |u_{h\mu}(x) - w(x)| \leq \frac{L_f}{\lambda (\lambda - \vartheta \lambda_0)} \frac{k^{\vartheta}}{h}. \qquad (3.10)$$

Proof : Let $x \in \mathbb{R}^n$ and $x = \sum_{j \in I} \mu_j(x) x_j$ where $0 \leq \mu_j(x) \leq 1$ and $\sum_{j \in I} \mu_j(x_j) = 1$. Then,

$$|u_{h\mu}(x) - w(x)| \leq$$

$$\leq \sum_{j \in I} \mu_j(x) |u_{h\mu}(x) - u_{h\mu}(x_j)| + \sum_{j \in I} \mu_j(x) |u_{h\mu}(x_j) - w(x_j)|. \qquad (3.11)$$

Since $\mu_j(x) \neq 0$ if and only if x_j is a vertex of the simplex containing x and $u_{h\mu} \in B_{\vartheta}$, we have

$$\sum_{j \in I} \mu_j(x) |u_{h\mu}(x) - u_{h\mu}(x_j)| \leq [L_f/(\lambda - \lambda_0 \vartheta)] k^{\vartheta}. \qquad (3.12)$$

Since $u_{h\mu}$ and w are solutions of the equations $(HJB_{h\mu})$ and $(HJB_{h\mu}^k)$, it is easy to check that

$$\sum_{j \in I} \mu_j(x) |u_{h\mu}(x_j) - w(x_j)| \leq (1 - \lambda h) \|u_{h\mu} - w\|_{L^{\infty}(\mathbb{R}^n)}. \qquad (3.13)$$

Substituting (3.12) and (3.13) in (3.11), we get

$$|u_{h\mu}(x) - w(x)| \leq (1 - \lambda h) \|u_{h\mu} - w\|_{\infty} + [L_f/(\lambda - \vartheta \lambda_0)] k^{\vartheta}$$

and this proves (3.10). $\square$

Remark 3.2 : Let the invariance conditions (IC1) or (IC2) be verified. The above approximation scheme applies without truncating the coefficients since we can build directly the triangulation of the invariant set. Equation $(HJB_{h\mu}^k)$ is now valid for all the vertices of triangulation and the proofs of Theorem 3.2 and Theorem 3.3 remain valid without changes.

Remark 3.3 : The Markov chain interpretation.

We shall give now a control interpretation of (3.7).
Define the matrix M as

$$M(a) \equiv \sum_{m=1}^{d} \frac{1}{2d} [M^+(m, a) + M^-(m, a)], \quad a \in A.$$

Let us consider as state space the set $\mathcal{V}$ of the vertices x_i of the triangulation of $\mathbb{R}^n$. We can define a Markov chain $\{Y^n\}$ belonging to $\mathcal{V}$ and such that

$$\mathbb{P}(Y_{n+1} = x_j \mid Y_n = x_i) = m_{ij}.$$

In fact, (3.6) guarantees that $M(a)$ is a stochastic matrix so that its elements m_{ij} can be interpreted as the transition probabilities of Y_n. We define the following cost functional on $\{Y_n\}$,

$$J^{h,k}_{x_i}(\{a_n\}) \equiv \mathbb{E}\left\{\sum_{0}^{\infty} {}_n h(1 - \lambda h)^n f(Y_n, a_n) \mid Y_0 = x_i\right\}, \quad i \in I,$$

where the control law $\{a_n\}$ is a sequence of random variables, $a_n :$ $\Omega \to A$ and a_n is measurable with respect to the σ-algebra generated by $Y_0, \ldots, Y_n$. The value function of the problem will be

$$U_i = \inf_{\{a_n\}} J^{h,k}_{x_i}(\{a_n\}), \quad i \in I.$$

and it is the unique solution of (3.7) (Bertsekas-Shreve [BeS]) Numerical methods for the solution of (3.7) can be found in [CDF], [K].

Remark 3.3 : We briefly describe how to obtain a fully discrete scheme for the evolutive problem described in Remark 2.2. Fix a triangulation of $\mathbb{R}^n$ of size k (i.e. k is the maximum diameter of the simplices of the triangulation). The fully discrete scheme is

$(HJBE^k_h)$

$$\begin{cases} V^n_i = \inf_{a \in A} \left\{ -\frac{1}{2d} \sum_{m=1}^{d} [M^+_i(a, n) + M^-_i(a, n)] V^{n+1} + hF_i(a, n) \right\} \\ V^N_i = \Psi_i \end{cases}$$

where $\Psi_i = \psi(\dot{x}_i)$, $F_i(a, n) = f(x_i, a, nh)$ and $M^+(a, n)$, $M^-(a, n)$ are defined as usual.

Let $w(x, n)$ be the linear interpolate function of the vector V^n on the simplices of the triangulation. The convergence of the fully discrete scheme

follows by the a priori estimate (see [CF])

$$\sup_{n = 0, \ N} \left\| w_h(\,.\,, n) - w(\,.\,, n) \right\|_{L^\infty(\mathbb{R}^n)} \leqslant C_0 \, k^\vartheta + C_1 \frac{k^\vartheta}{h}$$

where $C_0 = |\psi|_\vartheta$ and $C_1 = \dfrac{e^{\lambda_0 T}}{\lambda_0} \left[|\psi|_\vartheta + \dfrac{L_f}{\lambda_0} \right]$.

4. NUMERICAL EXPERIMENTS

In this section we present some tests based on the results of the preceeding sections In all the examples below we can compute the exact solution of (HJB) so that it is possible to obtain explicit numerical errors.

Test 1

Let $\chi(x, y)$ be the characteristic function of $B \equiv \{(x, y) \in R^2 : x^2 + y^2 \leqslant 1\}$. We set

$b(x, y, a) \equiv$

$$\equiv \left(-\frac{a}{2} x (\sqrt{x^2 + y^2} - 1)^2 \, \chi(x, y), \ -\frac{a}{2} y (\sqrt{x^2 + y^2} - 1)^2 \, \chi(x, y) \right)$$

$$\sigma(x, y, a) \equiv \begin{pmatrix} \dfrac{ax}{\sqrt{2}} (\sqrt{x^2 + y^2} - 1) \, \chi(x, y) & \dfrac{ax}{\sqrt{2}} (\sqrt{x^2 + y^2} - 1) \, \chi(x, y) \\ \dfrac{ay}{\sqrt{2}} (\sqrt{x^2 + y^2} - 1) \, \chi(x, y) & \dfrac{ay}{\sqrt{2}} (\sqrt{x^2 + y^2} - 1) \, \chi(x, y) \end{pmatrix}$$

$$f(x, y, a) \equiv a (x^2 + y^2 - 1) \, \chi(x, y)$$

$$\lambda \equiv 2 \quad \text{and} \quad A \equiv [0, 1]$$

The unique viscosity solution of (HJB) is

$$u(x, y) = \frac{1}{\lambda} (x^2 + y^2 - 1) \, \chi(x, y).$$

Notice that the second derivatives of $u(x, y)$ are discontinuous on the boundary of B. The optimal feed-back control is $a^*(x, y) = \chi(x, y)$, i.e is a bang-bang control.

Since the problem vanishes out of B, it naturally satisfies the invariance condition so that there is no truncation error.

The solution of the non linear problem (HJB_h^k) is computed by an acceleration method which speeds up the search for the fixed point (see [F2]

and [CDF]) Figure 1 compares the exact and the approximate solution (respectively indicated by the continuous line and the dotted line) along the x axis, this is meaningful since the solution is rotationally symmetric Table 1a shows the errors in L^∞ corresponding to different choices of h and k We remark that the estimate (3 10) is confirmed by the numerical result In fact, for a fixed k, the error increases as h goes to 0 and for a constant ratio of k/h, the error decreases as h goes to 0

Test 2

Let

$$b(x, y, a) \equiv \left(x + \frac{ax}{\sqrt{x^2 + y^2}}, y + \frac{ay}{\sqrt{x^2 + y^2}} \right)$$

$$\sigma(x, y, a) \equiv \begin{pmatrix} \dfrac{sx}{\sqrt{2}} & \dfrac{sx}{\sqrt{2}} \\ \dfrac{sy}{\sqrt{2}} & \dfrac{sy}{\sqrt{2}} \end{pmatrix}$$

$$f(x, y, a) \equiv x^2 + y^2 + ca^2$$

where s and c are positive parameters Let p be the positive constant

$$p \equiv \frac{-\lambda + 2 + s^2 + \sqrt{(-\lambda + 2 + s^2)^2 + 4/c}}{1/c}$$

The classical solution of (HJB) is

$$u(x, y) = \frac{p}{2} (x^2 + y^2)$$

and the optimal control law is

$$a^*(x, y) = -\frac{p}{2c} (x^2 + y^2)^{\frac{1}{2}}$$

For the numerical experiments, we have fixed

$$s = 0 1, \quad c = 0 5 \quad \text{and} \quad \lambda = 4$$

The problem is defined in $\mathbb{R}^2$ so we need to apply the truncation technique described in Section 3 In this problem the controls vary in $\mathbb{R}_+$ Since supp $\xi_\mu = [-5, 5] \times [-5, 5]$ and $\xi = 1$ in the square $[-4 5, 4 5] \times [-4 5, 4 5]$, we can fix $A \equiv [0, 4]$ It is useful to compute the solution for a discrete set of controls, replacing A with $\hat{A} \equiv \{jq \quad j = 0, \quad , N\}$ This adds
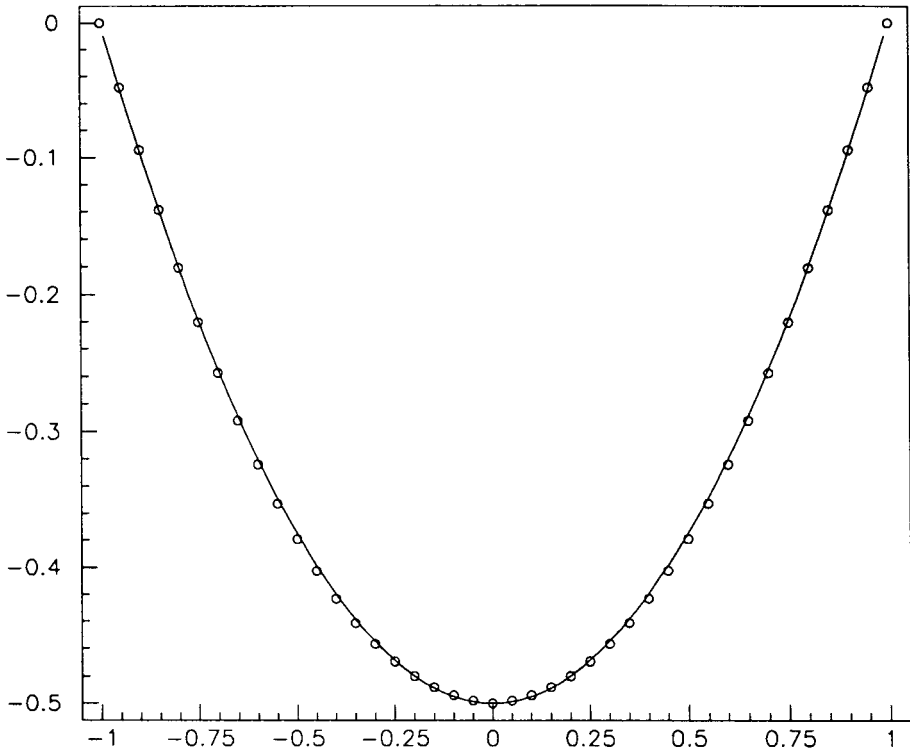
Figure 1. — Test 1 : Comparison of exact and approximate solutions ($h = 0.05$, $k = 0.05$).

Table 1. — Errors for the approximation scheme (1a) Test 1, (1b) Test 2.

	$k = 0.2$	$k = 0.1$	$k = 0.05$	$k = 0.025$
$h = 0.1$	0.0156	0.0073	0.0052	0.0050
$h = 0.05$	0.0268	0.0096	0.0065	0.0059
$h = 0.025$	0.0397	0.0184	0.0079	0.0062
$h = 0.0125$	0.0541	0.0290	0.0103	0.0070

Table 1a

h	k	$x = 0.0$	$x = 0.5$	$x = 1.0$	$x = 2.0$	$x = 3.0$	$x = 4.0$
0.2	1.0	0.0000	******	0.0750	0.3334	0.7304	0.9664
0.1	0.5	0.0000	0.0013	0.0217	0.1148	0.2348	0.0709
0.05	0.25	0.0000	0.0007	0.0041	0.0472	0.0969	0.9050

Table 1b

to the approximation error a term of the form Cq where $q = \sup \{d(z, z')$: $z \in A, \; z' \in \hat{A}\}$ and C is a constant depending from the coefficients of the continuous problem (in fact an estimate of this type has been established in [FF] for the deterministic problem and can be easily adapted to our case). We have chosen $\hat{A} \equiv \{jq : q = 0.4$ and $j = 0, ..., 10\}$.

Since the solution is rotationally symmetric it suffices to compare exact and approximate solutions along the x-axis (see *fig.* 2). Figure 3 shows the optimal control for the continuous problem and the computed feed-back control along the x-axis. Finally in Table 1b we present the approximation errors for some sample points (in this table h and k are such that their ratio is constant).

Test 3.

Finally we study the numerical stability of our scheme when the diffusion vanishes. Let us consider the deterministic optimal control problem in R^1 with the following coefficients :

$$b(x, a) \equiv a(x - 2)(x + 2), \quad \sigma(x, a) \equiv 0$$

$$f(x, a) \equiv 2\, ax(x - 2)(x + 2)\, sgn(x - 1)\, e^{-x^2} - (a^2 - 2)\left| \frac{1}{e} - e^{-x^2} \right|$$

$$A = [0, 1] \quad \text{and} \quad \lambda = 1 .$$

The viscosity solution of (HJB) is :

$$u(x) = \left| \frac{1}{e} - e^{-x^2} \right|$$

(notice that $u(x)$ is a continuous function with jumps in the derivative at $x = -1$ and $x = 1$).

In this case our scheme coincides with the one studied in [C], [CI], [F] and we have $\|u - u_h\|_\infty \leqslant h^{\frac{1}{2}}$. By (3.10) and (3.4) we can get an estimate of $\|u - w\|_\infty$.
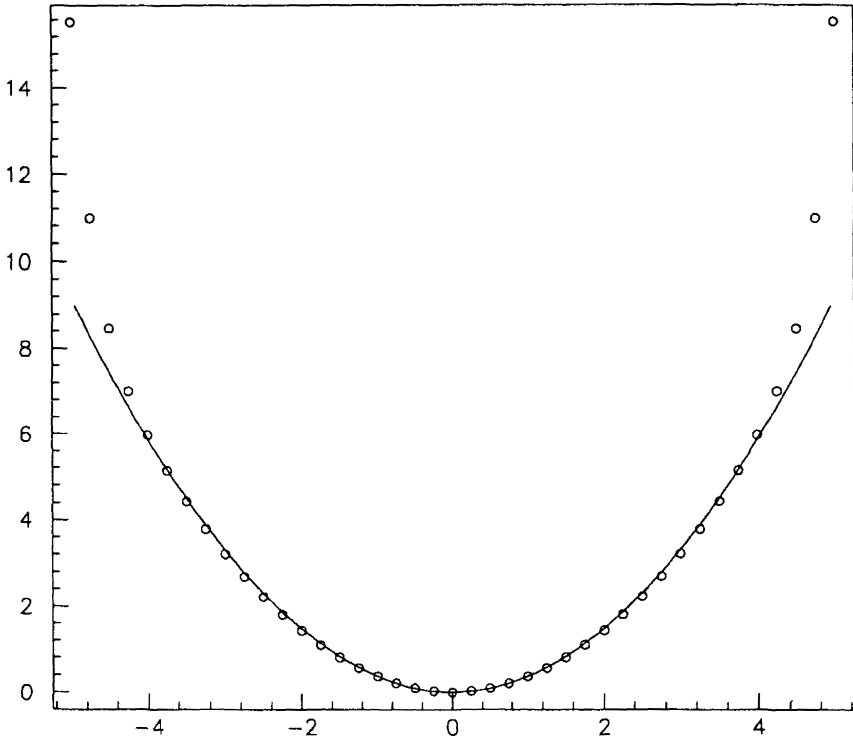
Figure 2. — Test 2 : Comparison of exact and approximate solutions ($h = 0.05$, $k = 0.05$).

Figure 4 compares u with the approximate solutions corresponding to $\sigma(x, a) \equiv s$ for $s = 1$ and $s = 0$. We choose $I_\mu = [-3.5, 3.5]$ and supp $\{\xi_\mu\} = [-4, 4]$.

Notice the smoothing effect on the solution due to the presence of a non-zero diffusion coefficient and that the scheme is stable for $s = 0$.

APPENDIX : an invariant condition for diffusion processes

For completeness we give a proof of the invariance condition (IC_2). The proof refers for simplicity to a stochastic differential equation without control, the extension to (1.1) is straightforward.

THEOREM : *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set such that $\Omega = \{x \in \mathbb{R}^n : \varphi(x) > 0\}$ and $\partial\Omega = \{x \in \mathbb{R}^n : \varphi(x) = 0\}$ with $\varphi \in C^2(\mathbb{R}^n)$ and $\inf\{|D\varphi(x)| : x \in \partial\Omega\} > 0$. Let $X(t)$ be the solution of the stochastic differential equation :*

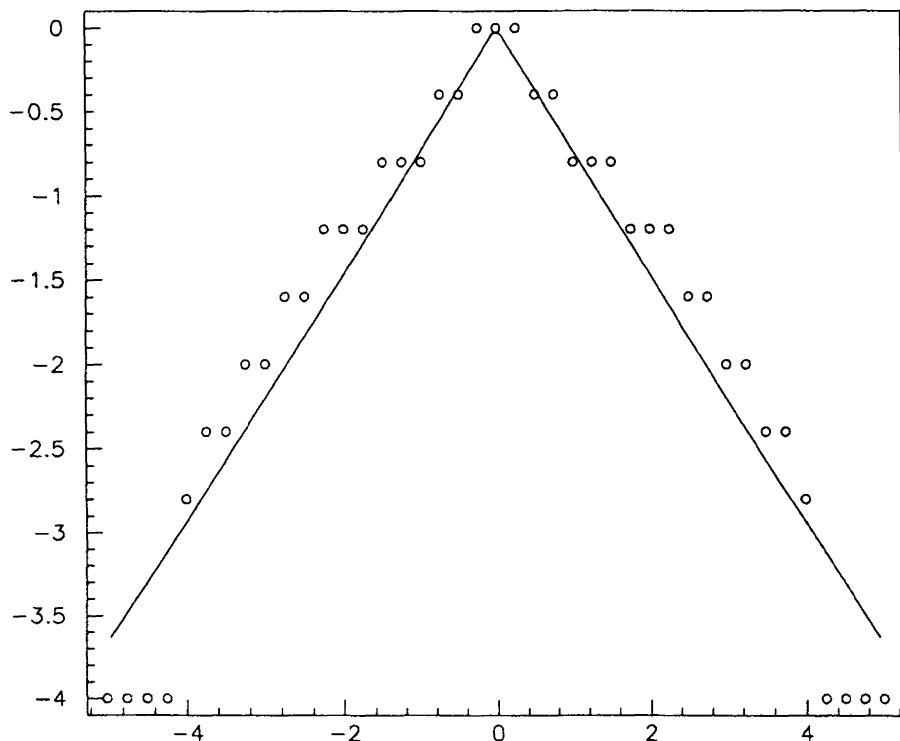$$dX(t) = b(X(t)) \, dt + \sigma(X(t)) \, dW(t)$$

Figure 3. — Test 2 : Continuous and approximate feedback controls.

with $b(x)$, $\sigma(x)$ satisfying (2.1) and (2.2). Then

$$(IC_2) \quad \begin{cases} [D\varphi(x)]^t \, \sigma(x) = 0 & \text{for all } x \in \partial\Omega \\ \dfrac{1}{2} Tr(\sigma(x) \, \sigma^t(x) \, D^2 \, \varphi(x)) + b(x) \, D\varphi(x) \geq c > 0 \end{cases}$$

implies that if $X(0) = x \in \bar{\Omega}$, $X(t) \in \bar{\Omega}$ for all $t > 0$ ($\mathbb{P} = 1$).

Proof : We will define $L = \dfrac{1}{2} Tr(\sigma(x) \, \sigma^t(x) \, D^2) + b(x) \, D$. If $\psi \in C^2(\mathbb{R})$,
then :

$$L(\psi \circ \varphi)(x) = \psi'(\varphi(x)) \, L\varphi + \frac{1}{2} \psi''(\varphi(x))|[D\varphi(x)]^t \, \sigma(x)|^2 \,.$$

Let $c_n = \sup \left\{ |[D\varphi(x)]^t \, \sigma(x)| \, ; x \in \Omega \text{ such that } \varphi(x) \leq \dfrac{3}{n} \right\}$. From the
hypotheses on σ and φ, we have :

$$c_n^2 \leq C \, (1/n)^2 \,.$$

---- exact sol. with s=0.
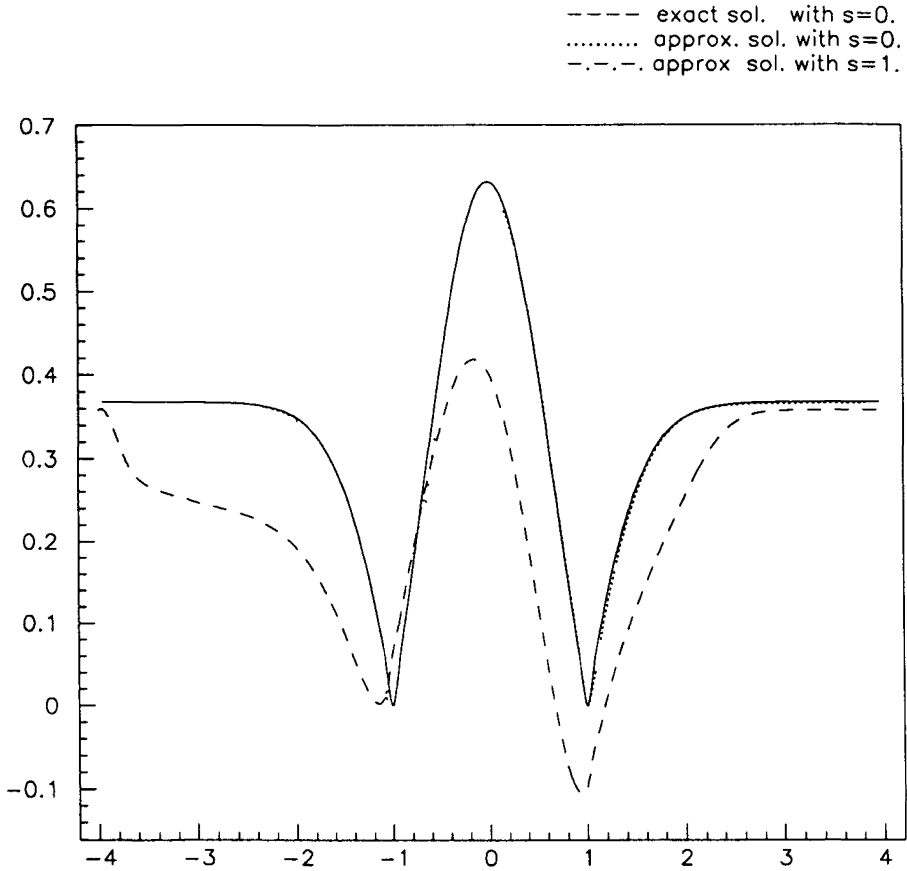.......... approx. sol. with s=0.
-.-.-. approx sol. with s=1.

Figure 4. — Test 3 : Vanishing viscosity. Steps sizes $h = 0.05$, $k = 0.0625$. Viscosity coefficients :
$\varsigma = 1$, $s = 0$.

Define $K_n = \dfrac{1}{c_n^2}$ and a sequence of functions ψ_n such that .

$$\psi_n(t) = \begin{cases} (K_n/n)(1/n - t) + K_n/n^2 & \text{if } \quad t < 1/n \\ (nK_n)(t - 1/n)^3 - (K_n/n^2)\, t + 2\, K_n^2/n & \text{if } \quad 1/n \leqslant t \leqslant 2/n \\ nK_n(3/n - t)^3/6 & \text{if } \quad 2/n \leqslant t < 3/n \\ 0 & \text{if } \quad 3/n < t \, . \end{cases}$$

The function ψ_n belong to C^2 ; if n is large enough, $\varepsilon > 0$ small (fixed), we have

$$\inf \{ L\varphi(x) : x \in \varOmega \text{ such that } \varphi(x) \in (-\varepsilon, 3/n) \} \geqslant c_0 > 0 \, .$$

Therefore,

$$L(\psi_n \circ \varphi)(x) \leqslant c_0 \, \psi_n'(\varphi(x)) + (c_n^2/2) \, \psi_n''(\varphi(x)) \leqslant 1/2 \ .$$

Moreover, if x is such that $\varphi(x) = -\varepsilon$,

$$(\psi_n \circ \varphi)(x) = \psi_n(-\varepsilon) = \varepsilon(K_n/n) > C \, \varepsilon n^{-1} \ .$$

Define $\tau_\varepsilon = \inf \{t \geqslant 0 : \varphi(X(t)) \leqslant -\varepsilon\}$. By Dynkin's formula, we have for all $T > 0$ and $x \in \Omega$

$$\mathbb{E}_x[(\psi_n \circ \varphi)(X(\tau_\varepsilon \wedge T))] = (\psi_n \circ \varphi)(x) +$$

$$+ \mathbb{E}_x\left[\int_0^{\tau_\varepsilon \wedge T} L(\psi_n \circ \varphi)(X(s)) \, ds\right] \leqslant (\psi_n \circ \varphi)(x) + T/2$$

and $(\psi_n \circ \varphi)(x)$ is zero for n large enough ($\varphi(x)$ is greater than $\frac{3}{n}$ for n large enough, if $x \in \Omega$). Moreover

$$\mathbb{E}_x[(\psi_n \circ \varphi)(X(\tau_\varepsilon \wedge T))] \geqslant \mathbb{P}(\tau_\varepsilon \leqslant T) \, \psi_n(-\varepsilon) \ .$$

By the above inequalities, we get

$$\mathbb{P}(\tau_\varepsilon \leqslant T) \leqslant T/(2 \, \varepsilon n) \to 0 \quad \text{for} \quad n \to +\infty , \quad \text{for all} \quad \varepsilon > 0 \ .$$

Then $X(t) \in \{x : \varphi(x) > -\varepsilon\}$ for all $\varepsilon > 0$ and for all $t \leqslant T$ with probability 1, that is $X(t) \in \Omega$ for all $t \leqslant T$ with probability 1. By the arbitrariness of T we complete the proof.   □

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