COMP 330B 2022, Assignment 4
Due Tuesday, April 12th 2022 23:59

[8%] **4.4** Let $A\varepsilon_{\mathsf{CFG}} = \{\langle G\rangle |\ G \text{ is a CFG that generates } \varepsilon\}$. Show that $A\varepsilon_{\mathsf{CFG}}$ is decidable.

[10%] **4.13** Let $A = \{\langle R, S\rangle |\ R \text{ and } S \text{ are regular expressions and } L(R) \subseteq L(S)\}$. Show that A is decidable.

[10%] **4.28** Let $C = \{\langle G, x\rangle |\ G \text{ is a CFG } x \text{ is a substring of some } y \in L(G)\}$. Show that C is decidable. (Hint: An elegant solution to this problem uses the decider for E_{CFG}.)

[8%] **5.16** Let $\Gamma = \{0, 1, \sqcup\}$ be the tape alphabet for all TMs in this problem. Define the ***busy beaver function*** $BB: \mathcal{N} \longrightarrow \mathcal{N}$ as follows. For each value of k, consider all k-state TMs that halt when started with a blank tape. Let $BB(k)$ be the maximum number of 1s that remain on the tape among all of these machines. Show that BB is not a computable function.

[+5%] + Prove that $BB(k)$ grows faster than any computable function $f(k)$, i.e. $\lim_{k\to\infty} f(k)/BB(k)=0$.

[10%] **5.19** In the *silly Post Correspondence Problem*, *SPCP*, in each pair the top string has the same length as the bottom string. Show that the *SPCP* is decidable.

[10%] **5.20** Prove that there exists an undecidable subset of $\{1\}^*$.

[10%] **5.21** Let $AMBIG_{\mathsf{CFG}} = \{\langle G\rangle |\ G \text{ is an ambiguous CFG}\}$. Show that $AMBIG_{\mathsf{CFG}}$ is undecidable. (Hint: Use a reduction from *PCP*. Given an instance

$$P = \left\{ \left[\frac{t_1}{b_1}\right], \left[\frac{t_2}{b_2}\right], \ldots, \left[\frac{t_k}{b_k}\right] \right\},$$

of the Post Correspondence Problem, construct a CFG G with the rules

$$\begin{aligned} S &\to T \mid B \\ T &\to t_1 T \mathtt{a}_1 \mid \cdots \mid t_k T \mathtt{a}_k \mid t_1 \mathtt{a}_1 \mid \cdots \mid t_k \mathtt{a}_k \\ B &\to b_1 B \mathtt{a}_1 \mid \cdots \mid b_k B \mathtt{a}_k \mid b_1 \mathtt{a}_1 \mid \cdots \mid b_k \mathtt{a}_k\ , \end{aligned}$$

where $\mathtt{a}_1, \ldots, \mathtt{a}_k$ are new terminal symbols. Prove that this reduction works.)

[8%] **5.31** Let

$$f(x) = \begin{cases} 3x+1 & \text{for odd } x \\ x/2 & \text{for even } x \end{cases}$$

for any natural number x. If you start with an integer x and iterate f, you obtain a sequence, $x, f(x), f(f(x)), \ldots$ Stop if you ever hit 1. For example, if $x = 17$, you get the sequence 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1. Extensive computer tests have shown that every starting point between 1 and a large positive integer gives a sequence that ends in 1. But, the question of whether all positive starting points end up at 1 is unsolved; it is called the $3x + 1$ problem.

Suppose that A_{TM} were decidable by a TM H. Use H to describe a TM that is guaranteed to state the answer to the $3x + 1$ problem.

[8%] **6.19** Recall the Post correspondence problem that we defined in Section 5.2 and its associated language *PCP*. Show that *PCP* is decidable relative to A_{TM}.

[8%] **7.21** Let $DOUBLE\text{-}SAT = \{\langle\phi\rangle \mid \phi$ has at least two satisfying assignments$\}$. Show that $DOUBLE\text{-}SAT$ is NP-complete.

[10%] **7.27** A ***coloring*** of a graph is an assignment of colors to its nodes so that no two adjacent nodes are assigned the same color. Let

$$3COLOR = \{\langle G\rangle \mid \text{the nodes of } G \text{ can be colored with three colors such that no two nodes joined by an edge have the same color}\}.$$

Show that *3COLOR* is NP-complete. (Hint: Use the following three subgraphs.)