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Mundane primer for the uninitiated

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Declarație

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Contents

Introduction

In Chapter 1, some basic concepts of QCD shall briefly be presented, with focus on gauge transformations and Yang-Mills equations.

Wilson loops on a lattice will be introduced in Chapter 2, as the appropriate degree of freedom for discretising the Yang-Mills action on a lattice without loss of inherent gauge invariance.

Chapter 1

QCD primer

Quantum chromodynamics is the theory of strong interactions[1]. It aims to describe the interactions between elementary constituents, namely the quarks[2], mediated by the carriers of the color force, the gluons.

The existence of color charge was proposed as an additional quantum number which would solve the violation of Pauli's exclusion principle for some particular baryons[3]. Since the quarks were never experimentally evidenced, it was proposed that the strong interaction constrains the free particles to only exist in color neutral states. This particularity of QCD is known as color confinement. Nevertheless, in the partonic picture[4], deep inelastic scattering[5] experiments between an electron and a proton confirmed the already predicted Bjorken scaling[6] of the electrons' differential cross-section.

In essence, QCD is an extension of the original $\mathsf{SU}(2)$ gauge theory of Yang and Mills [5] to local non-Abelian $\mathsf{SU}(3)$ gauge transformations.

1.1 Field content

Following textbook expositions [6–8], the QCD Lagrangian $\mathcal{L}$ is constructed from symmetry principles, namely $\mathsf{SO}(1,3)$ Lorentz invariance and local gauge invariance under $\mathsf{SU}(3)$. The fields should transform according to irreducible representations of these groups. The quark content of the Lagrangian is described by the quark and anti-quark fields $\psi_{\alpha,i,f}(x)$ and $\overline{\psi}_{\alpha,i,f}(x)$. They are Dirac spinors (spinorial index α), transform according to the fundamental representation of $\mathsf{SU}(3)$ (color index $i = 1, 2, 3$ or red, green, blue) and come in different flavours (flavour index $f = \overline{1, N_f}$ or up, down, strange, charm, bottom, up). The gluon fields $A^\mu_a(x)$ are Lorentz vectors and each correspond to a generator t^a ($a = \overline{1, 8}$) which, in the fundamental representation, is given by the Gell-Mann matrices $t^a = \lambda^a/2$.

[1] A brief historical review about the development of QCD may be found at [1].

[2] The quarks were firstly predicted in Gell-Mann's Eightfold Way [2].

[3] For example, Δ^{++}, which consists of three up quarks.

[4] Feynman proposed that high energy nuclei are made of elementary constituents, generically called partons [3].

[5] DIS is a process during which the structure of a hadron may be probed via interaction with, in general, a lepton.

[6] Bjorken deduced an expression for the cross-section of the electron by imagining that it interacts electromagnetically with each parton from the proton [4].

1.2 Gauge transformations

The quark and anti-quark fields must be invariant under local $\mathsf{SU}(3)$ gauge transformations[7]

$$\psi(x) \mapsto \mathsf{U}(x)\psi(x), \quad \overline{\psi} \mapsto \overline{\psi}(x)\mathsf{U}^\dagger(x)$$

with the group transformation expressible, via exponentiation, from the Lie algebra generators, with space-time dependent group parameters $\varepsilon^a(x)$, as

$$\boxed{\mathsf{U}(x) = \exp i \sum_a \varepsilon^a(x) t^a} \tag{1}$$

The gauge fields[8] $A_\mu(x) = \sum_a A_\mu^a(x) t^a$ must transform according to

$$\boxed{A_\mu(x) \mapsto \mathsf{U}(x) A_\mu(x) \mathsf{U}^\dagger(x) + \frac{i}{g}\mathsf{U}(x)\left[\partial_\mu \mathsf{U}^\dagger(x)\right]} \tag{2}$$

where g denotes the coupling constant. It is important to notice that one may generate gluon fields out of a null one, that is $A_\mu = 0$ by applying a local gauge transformation. Such field configurations take the form

$$A_\mu^{\text{pure}} = \frac{i}{g}\mathsf{U}\left(\partial_\mu \mathsf{U}^\dagger\right)$$

and are called pure gauge fields [9, 10]. The corresponding field strength tensor is null $F_{\mu\nu}^{\text{pure}} = 0$.

One may introduce the covariant derivative[9]

$$\mathsf{D}_\mu = \partial_\mu - igA_\mu.$$

Further, one may define the field strength tensor as the commutator between covariant derivatives[10]

$$F_{\mu\nu} = \frac{i}{g}\left[\mathsf{D}_\mu, \mathsf{D}_\nu\right],$$

[7] For simplicity, all the field indices will be dropped in the following computations.

[8] For each algebra element t^a, one may introduce a gauge field A_a^μ. These may be then used to construct a Lie-algebra valued gauge potential A^μ. This potential depends on the chosen representation.

[9] The covariant derivative has an elegant geometrical interpretation [11]: it represents the rate of change when fields from different space-time points are parallel transported along a given path. During this procedure, they are being aligned such that they may be properly compared. The corresponding connection is actually the gauge field.

[10] Since it arises as a commutator between covariant derivatives, which describe the parallel transport, the field strength tensor may be interpreted as a measure of the path dependence of parallel transport. For this reason, it is also referred to as the curvature [11].

which yields an expression in terms of gauge fields

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\big[A_\mu, A_\nu\big]$$

or equivalently, by color components $F_{\mu\nu} = F^a_{\mu\nu} t^a$, as

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\nu + g f^{abc} A^b_\mu A^c_\nu,$$

where f^{abc} are the structure constants of the Lie algebra $\mathfrak{su}(3)$. The last term from the above equation, when plugged in the Lagrangian, will give rise to gluonic self-interactions, a particular feature of QCD. The field strength tensor gauge transforms in the usual manner as

$$F_{\mu\nu}(x) \mapsto \mathsf{U}(x) F_{\mu\nu}(x) \mathsf{U}^\dagger(x)$$

1.3 QCD Lagrangian

One may now proceed to constructing the Lagrangian. The quark content is that of a free fermionic Lagrangian[11], but built with covariant derivatives, in order to satisfy gauge invariance

$$\mathcal{L}_{\mathsf{quarks}} = \overline{\psi}(x)\big(i\not{D} - \mathsf{M}\big)\psi(x),$$

where $\mathsf{M} = \mathsf{Diag}\left\{m_1, \dots, m_{N_f}\right\}$ is the diagonal quark mass matrix in flavour space[12].

The dynamics of the gluon fields is described by the following construction[13]

$$\mathcal{L}_{\mathsf{gluons}} = -\frac{1}{2}\mathsf{Tr}\left\{F_{\mu\nu}F^{\mu\nu}\right\},$$

where the color tracing over the contraction of field strength tensors assured gauge invariance. Equivalently, one may rewrite the above expression in terms of color components as

$$\mathcal{L}_{\mathsf{gluons}} = -\frac{1}{4}F^a_{\mu\nu}F^{a,\mu\nu},$$

valid in the fundamental representation, where $\mathsf{Tr}\left\{t^a t^b\right\} = \delta^{ab}/2$. Therefore, the QCD Lagrangian takes the form

$$\boxed{\mathcal{L}_{\mathsf{QCD}} = \overline{\psi}(x)\big(i\not{D} - \mathsf{M}\big)\psi(x) - \frac{1}{4}F^a_{\mu\nu}F^{a,\mu\nu}} \tag{3}$$

[11] After replacing the partial derivative with the covariant derivative, the Lagrangian will also contain an interaction term

$$\mathcal{L}_{\mathsf{int}} = g\overline{\psi}\gamma^\mu A^a_\mu t^a \psi.$$

[12] In the Standard Model, the quark mass matrix is no longer diagonal. After spontaneous symmetry breaking, the mixing between different flavoured quark masses is given by the CKM matrix [12].

[13] It is important to notice that such a construction contains not only standard kinetic terms but also interaction vertices with three gluons, which are proportional to g and four gluons, proportional to g^2.

The corresponding Yang-Mills action expressed in flat coordinates is given by

$$\mathsf{S} = \int \mathrm{d}^4x \left(-\frac{1}{2} \mathsf{Tr} \left\{ F_{\mu\nu} F^{\mu\nu} \right\} \right), \tag{4}$$

1.4 Field equations

The variational derivatives with respect to the color spinor fields give the colored Dirac equations

$$(i\not{\mathsf{D}} - \mathsf{M})\psi = 0,$$

and similarly for the anti-quark fields

$$\overline{\psi}(i \overleftarrow{\not{\mathsf{D}}} - \mathsf{M}) = 0.$$

The Euler-Lagrange equations corresponding to the gluon fields yield the Yang-Mills equations, or equivalently, colored Maxwell equations[14]

$$\boxed{\mathsf{D}_\nu F^{\nu\mu} = g J^\mu}$$

in which $J^\mu = \sum_a J^{a,\mu} t^a$ with $J^{a,\mu} = \overline{\psi}\gamma^\nu t^a \psi$ being the color current. The color current is covariantly conserved $\mathsf{D}_\mu J^\mu = 0$.

[14] There is an additional equation, called the Bianchi identity, which follows from the definition and properties of $F_{\mu\nu}$. It may be expressed as [13]

$$\mathsf{D}_\mu {}^*F^{\mu\nu} = 0,$$

where we introduced the dual field strength tensor as

$${}^*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}.$$

Chapter 2

Lattice gauge theory

2.1 Gauge invariance

Nevertheless, a naive discretization of the Yang-Mills action from Equation (4), in which one replaces all the partial derivatives appearing in the field strengths $F^{\mu\nu}$ with finite differences leads to the loss of gauge invariance [7, 14].

Remark
This is a consequence of imposing local gauge symmetry, with a $\mathsf{SU}(3)$ gauge transformation given by Equation (1), of the fields

$$\phi(x) \mapsto \mathsf{U}(x)\phi(x).$$

Fields at different space-time points cannot directly be compared. Because of this, the partial derivative, which contains the difference between fields at different points, is not well defined. By introducing a quantity which gauge transform as

$$\mathsf{W}(x, y) \mapsto \mathsf{U}(x)\mathsf{W}(x, y)\mathsf{U}^{\dagger}(y), \tag{1}$$

one may now properly define the covariant derivative as[1]

Definition 2.1 (Covariant derivative)

$$\mathsf{D}_{\mu}\phi(x) \triangleq \lim_{\delta\epsilon^{\mu}\to 0} \frac{\mathsf{W}(x, x+\epsilon)\phi(x+\epsilon) - \phi(x)}{\epsilon^{\mu}}. \tag{3}$$

We choose $\mathsf{W}(x, x) = \mathbb{1}$ and then write an expansion in terms of the gauge fields[2]

$$\mathsf{W}(x, x+\epsilon) = 1 + ig\epsilon^{\mu}A_{\mu} + \mathcal{O}(\epsilon^2).$$

This enables us to express the Wilson line as[3]

$$\boxed{\mathsf{W}(x, y) = \mathcal{P}\exp ig\int_y^x \mathrm{d}z^{\mu}A_{\mu}(z)} \tag{4}$$

[1] Notice that with this definition, the covariant derivative transforms as

$$\mathsf{D}_{\mu}\phi(x) \mapsto \mathsf{U}(x)\mathsf{D}_{\mu}\phi(x). \tag{2}$$

[2] If we plug in this relation back in Equation (3) and then in Equation (2), we deduce the gauge transformation of the gauge fields from Equation (2).

[3] Where the fields may further be written as $A_{\mu} = A_{\mu}^{a}T^{a}$ in the fundamental representation.

Remark
The path-ordering operator $\mathcal{P}\{\dots\}$ is necessary due to the non-Abelian nature of the fields. More clearly, after writing a Taylor expansion, the path-ordering operator acts as

$$\begin{aligned} W(x,y) =& 1 + ig\int_0^1 \frac{\mathrm{d}z^\mu_\lambda}{\mathrm{d}\lambda} A^a_\mu(z^\mu_\lambda) T^a \mathrm{d}\lambda - \frac{1}{2}g^2 \int_0^1 \mathrm{d}\lambda \int_0^1 \mathrm{d}\tau \frac{\mathrm{d}z^\mu_\lambda}{\mathrm{d}\lambda}\frac{\mathrm{d}z^\nu_\tau}{\mathrm{d}\tau}\times \\ &\times A^a_\mu(z^\mu_\lambda) A^b_\nu(z^\nu_\tau)\left[T^a T^b \theta(\lambda-\tau) + T^b T^a \theta(\tau-\lambda)\right] + \cdots . \end{aligned}$$

A Wilson line taken along a closed path γ is called a Wilson loop and it's given by

$$\boxed{\mathsf{W}_\gamma = \mathcal{P}\exp ig \oint_\gamma \mathrm{d}x^\mu A_\mu(x)} \tag{5}$$

It is important to emphasize that the trace of a Wilson loop is gauge invariant[4].

2.2 Real-time lattice gauge theory

Let us begin by discretizing the Minkowski space-time on a hypercubic lattice[5] whose points are given by

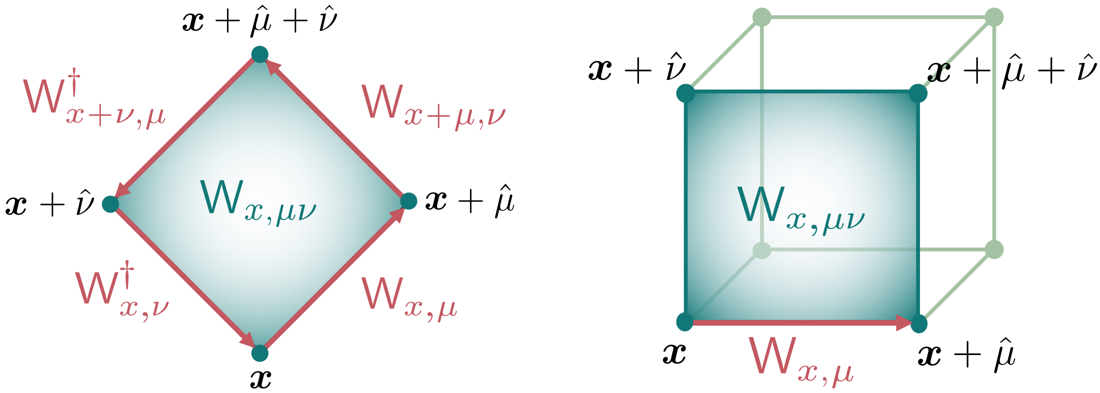
$$\mathsf{X}^4 = \left\{ x \mid x = \sum_{\mu=0}^{3} n_\mu \hat{a}^\mu, \quad n_\mu \in \mathbb{Z} \right\}.$$

A field $\phi(x)$ which resides on the lattice will be denoted as ϕ_x, with $x \in \mathsf{X}^4$. The gauge transformation $\mathsf{U}(x)$ will become, upon discretization, U_x. The discretized action and corresponding field equations written in terms of A_μ will not remain gauge invariant.

Nevertheless, a gauge invariant lattice action may be constructed. Instead of using the gauge fields A_μ, along with the corresponding conjugate momenta P_μ, as the main degrees of freedom, one may simply seek for a more suitable quantity which is already gauge invariant and preserves gauge invariance upon discretization. An action built from such a quantity will inherently be gauge invariant. The simplest choice is the trace of a Wilson line, as given in Equation (5). We already saw that such a construction is gauge invariant.

[4] This may easily be proven by inserting Equation (5) back in Equation (1) and using the invariance of the trace under cyclic permutations.

[5] The lattice spacing along direction μ, with the unit vector is $\hat{e}^\mu$, is a^μ. This enables us to write $\hat{a}^\mu = a^\mu \hat{e}^\mu$. More concisely, a^0 is the time step and a^i denote the spatial lattice spacings.



(a) 2d plaquette (b) 3d plaquette

Figure 2.1: Schematic representations of a plaquette, the shortest Wilson loop on a rectangular lattice.

A Wilson line taken between two neighboring lattice points, namely x and $x+\hat{a}^\mu$, is called a gauge link and it's given by[6]

$$\mathsf{W}_{x,\mu} = \overline{\mathcal{P}} \exp ig \int_x^{x+\hat{a}^\mu} dx^\mu A_{x,\mu}.$$

Remark
In a similar manner, we may also introduce a gauge link along the opposite direction, which would yield

$$W^\dagger_{x,\mu} = W_{x+\mu,-\mu}. \tag{6}$$

A gauge link transforms as

$$W_{x,\mu} \to U_x W_{x,\mu} U^\dagger_{x+\mu}.$$

One may express a gauge link and afterwards expand it, see Equation (4), as

$$W_{x,\mu} = \exp iga^\mu A_{x,\mu} \approx \mathbb{1} + iga^\mu A_{x,\mu(x)} - \frac{1}{2} g^2 a^\mu a^\nu A_{x,\mu} A_{x,\nu} + \mathcal{O}(a^3).$$

The simplest non-trivial[7] Wilson loop on the lattice may be constructed along the path connecting neighbouring points on a rectangular plaquette, see Figure 2.1,

[6] Here $\overline{\mathcal{P}}\{\dots\}$ denotes the anti-path-ordering. We introduce the notation

$$\mathsf{W}_{x,\mu} \triangleq \mathsf{W}(x, x+\hat{a}^\mu)$$

and similarly for $A_{x,\mu}$.

[7] The shortest Wilson line would just go back and forth between two neighbouring sites but such a combination would simply yield the trivial result

$$\mathsf{W}_{x,\mu}\mathsf{W}_{x+\mu,-\mu} \overset{(6)}{=\!=} \mathbb{1}.$$

as

$$\begin{aligned}\mathsf{W}_{x,\mu\nu} &= \mathsf{W}_{x,\mu}\mathsf{W}_{x+\mu,\nu}\mathsf{W}^{\dagger}_{x+\mu,\mu}\mathsf{W}^{\dagger}_{x,\nu}\\ &\overset{(6)}{=\!=} \mathsf{W}_{x,\mu}\mathsf{W}_{x+\mu,\nu}\mathsf{W}_{x+\mu+\nu,-\mu}\mathsf{W}_{x+\nu,-\nu},\end{aligned}$$

which may further be expressed as

$$\mathsf{W}_{x,\mu\nu} \approx \exp iga^{\mu}a^{\nu}F_{x,\mu\nu} + \mathcal{O}\left(a^3\right).$$

Proof

We may write the discretized Wilson loop as[8]

$$\begin{aligned}W_{x,\mu\nu} \approx \exp\Bigg\{ & ig(A_{x,\mu} + A_{x+\mu,\nu} - A_{x+\nu,\mu} - A_{x,\nu}) +\\ & + \frac{g^2}{2}\Big(\big[A_{x,\nu} + A_{x+\nu,\mu}, A_{x+\mu,\nu} + A_{x,\mu}\big] -\\ & - \big[A_{x,\nu}, A_{x+\nu,\mu}\big] - \big[A_{x+\mu,\nu}, A_{x,\mu}\big]\Big)\Bigg\}.\end{aligned}$$

By making use of the expansion

$$A_{x+\mu,\nu} \approx A_{x,\nu} + a^{\mu}\partial_{\mu}A_{x,\nu} + \mathcal{O}(a^2),$$

we may then derive

$$W_{x,\mu\nu} \approx \exp\Big\{iga^{\mu}a^{\nu}\underbrace{\left(\partial_{\mu}A_{x,\nu} - \partial_{\nu}A_{x,\mu} - ig[A_{x,\nu}, A_{x,\mu}]\right)}_{F_{x,\mu\nu}} + \mathcal{O}(a^3)\Big\}.$$

This may further be approximated as

$$W_{x,\mu\nu} = \mathbb{1} + iga^{\mu}a^{\nu}F_{x,\mu\nu} - \frac{1}{2}(ga^{\mu}a^{\nu})^2F^2_{x,\mu\nu} + \mathcal{O}(a^5).$$

in the limit of small lattice spacings. □

Therefore, we may construct a gauge invariant quantity, since in contains Wilson lines traced over, under the discretized gauge transformation U_x as

$$\mathsf{Tr}\left\{2 - \mathsf{W}_{x,\mu\nu} - \mathsf{W}^{\dagger}_{x,\mu\nu}\right\} \approx (ga^{\mu}a^{\nu})^2\,\mathsf{Tr}\left\{F^2_{x,\mu\nu}\right\} + \mathcal{O}(a^6). \tag{7}$$

[8] Using the Campbell-Baker-Hausdorff formula

$$\exp\mathsf{A}\exp\mathsf{B} \approx \exp\mathsf{A} + \mathsf{B} + \frac{1}{2}[\mathsf{A},\mathsf{B}] + \cdots.$$

The Yang-Mills action from Equation (4) may be split into an electric and a magnetic part

$$\mathsf{S} = \underbrace{\int \mathrm{d}^4x \sum_i \mathsf{Tr}\big\{F_{0i}^2(x)\big\}}_{\mathsf{S_E}} + \underbrace{\int \mathrm{d}^4x \sum_{i,j} \frac{1}{2}\mathsf{Tr}\big\{F_{ij}^2(x)\big\}}_{\mathsf{S_B}}.$$

Upon discretization, they become[9]

$$\mathsf{S_E} \approx V \sum_x \sum_i \frac{1}{\left(ga^0a^i\right)^2}\mathsf{Tr}\big\{2 - \mathsf{W}_{x,0i} - \mathsf{W}^\dagger_{x,0i}\big\},$$
$$\mathsf{S_B} \approx V \sum_x \sum_{i,j} \frac{1}{2\left(ga^ia^j\right)^2}\mathsf{Tr}\big\{2 - \mathsf{W}_{x,ij} - \mathsf{W}^\dagger_{x,ij}\big\}.$$

Thus, the Yang-Mills action on the lattice is given by

$$\mathsf{S} = V \sum_x \left(\sum_i \frac{1}{\left(ga^0a^i\right)^2}\mathsf{Tr}\big\{2 - \mathsf{W}_{x,0i} - \mathsf{W}^\dagger_{x,0i}\big\} - \sum_{i,j} \frac{1}{2\left(ga^ia^j\right)^2}\mathsf{Tr}\big\{2 - \mathsf{W}_{x,ij} - \mathsf{W}^\dagger_{x,ij}\big\}\right)$$

[9] By making the replacement

$$\int \mathrm{d}^4x(\ldots) \mapsto \underbrace{\prod_\mu a^\mu}_{\mathsf{V}} \sum_x (\ldots)$$

and using the result from Equation (7).

Conclusions

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