Math 32A Midterm 2 Review : 11/19/23

General Agenda :

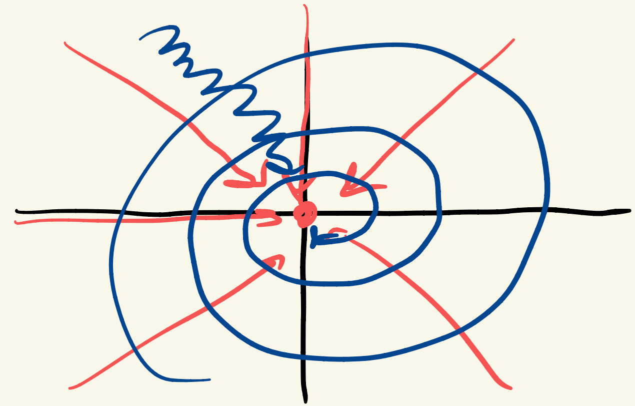
11 am - 11:45 am : Topic Review

~ 10 min break

12 pm - 1 pm : Answering Questions

Follow link in chat to submit midterm related questions you'd like me to answer. Even if you can't stay the whole time, submit your questions!

Limits & Continuity : 15.2



$f(x,y)$ function

$$\lim_{(x,y)\to(a,b)} f(x,y) = ?$$

DO NOT NEED ε-δ (epsilon-delta) for midterm

Ex: $f(x,y) = x^2 + y^2$

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0^2 + 0^2 = 0$$

Ex: $f(x,y) = \left(e^{x^2}\right) \cdot \sin(y)$

$$\lim_{(x,y)\to(0,0)} f(x,y) = e^0 \cdot \sin(0)$$

$$= 1 \cdot 0 = 0$$

Polar Coordinates:

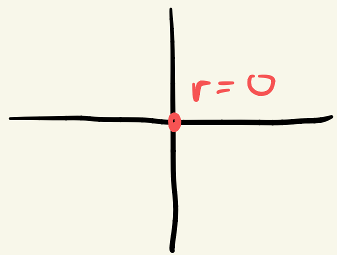
$x \longmapsto r\cos\theta$

$y \longmapsto r\sin\theta$

$(x,y) = (0,0)$

$\parallel$

$(r,\theta) = (0,0)$ $\boxed{r=0}$



Limit laws:

$$\lim_{(x,y)\to(a,b)} f(x,y) \cdot g(x,y)$$

$$= \underbrace{\lim f(x,y)}_{L_1} \cdot \underbrace{\lim g(x,y)}_{L_2}$$

if L_1, L_2 both exist

$$\lim_{(x,y)\to(a,b)} \left(f(x,y) + g(x,y)\right)$$

$$= \lim\left(f(x,y)\right) + \lim\left(g(x,y)\right)$$

Ex:

$$f(x,y) = \frac{xy}{\sqrt{x^2+y^2}}$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{0}{0} \quad \times$$

$$f(r,\theta) = \frac{(r\cos\theta)(r\sin\theta)}{\sqrt{(r\cos\theta)^2+(r\sin\theta)^2}}$$

$$= \frac{r^2 \cos\theta \cdot \sin\theta}{\sqrt{r^2}} = \frac{r^{\cancel{2}} \cos\theta \sin\theta}{\cancel{r}}$$

$$= r\cos\theta\sin\theta$$

$$\lim_{r\to 0} f(r,\theta) = 0 \cdot \cos\theta\sin\theta = \boxed{0}$$

When to use polar:

1. Substitution fails
2. If you see $\boxed{x^2+y^2}$ $= r^2$

Squeeze Thm

$$\underline{-1} \le \underline{\cos\theta\sin\theta} \le \underline{1}$$

$$f(r,\theta) = r\cos\theta\sin\theta$$

$$-r \le f(r,\theta) \le r$$

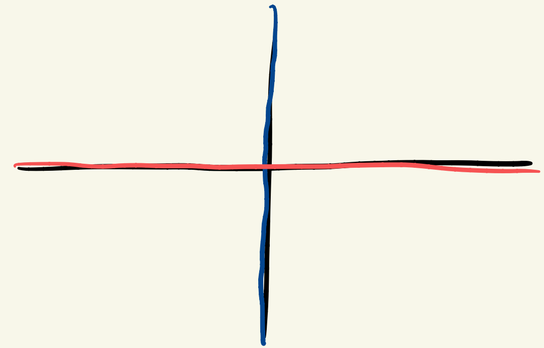
$r \to 0$

$$0 \le f(r,\theta) \le 0$$

$$= \boxed{0}$$

Limits DNE :

$$f(x,y) = \frac{x^2}{x^2+y^2}$$



$$\lim_{(x,y)\to(0,0)} \quad \frac{0}{0} \quad \times$$

[$x = 0$] $\quad f(0,y) = \dfrac{0}{0+y^2} = 0$

$$\lim_{(x,y)\to(0,0)} 0 = 0$$

$y = 0$: $\quad f(x,0) = \dfrac{x^2}{x^2+0} = \dfrac{x^2}{x^2} = 1$

$$\lim_{(x,y)\to(0,0)} 1 = 1$$

Two different values along two different lines

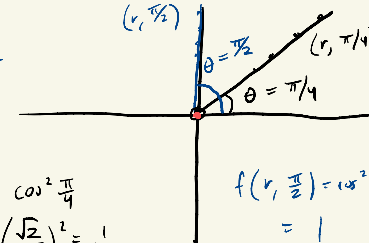
$\Rightarrow$ Limit DNE

Plug in polar:

$$\frac{(r\cos\theta)^2}{r^2} = \cos^2\theta$$

Thinking about this as a function of r & θ

$= f(r,\theta)$



$f(r, \pi/4) = \cos^2\frac{\pi}{4}$

$= \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}$

$f(r, \frac{\pi}{2}) = \cos^2\frac{\pi}{2}$

$= 1$

y-axis: $\theta = \pi/2 \Rightarrow 0$

x-axis: $\theta = 0 \Rightarrow 1$

Taking limit as $(x,y) \to (0,0)$ is same as $r \to 0$, so need r in function

Looking at all straight lines $y = mx$:

$$f(x,y) = \frac{x^2}{x^2+y^2} \qquad y = mx$$

$$f(x, mx) = \frac{\cancel{x^2}}{\cancel{x^2}+m^2\cancel{x^2}} = \frac{1}{1+m^2}$$

$$\lim_{(x,y)\to(0,0)} \frac{1}{1+m^2} = \frac{1}{1+m^2}$$

$\Rightarrow$ limit DNE, b/c we have dependence on slope m

This method <u>only</u> works for showing the limit <u>DNE</u>.

<u>Q</u>: Find the limit of $f(x,y)$ or show it DNE:

① Try substitution

② Look if a polar substitution would help

③ Then try to show limit DNE by checking different lines

<u>Tip</u>: Be as familiar as you can with trig limits

Continuity: A function $f(x,y)$ is <u>continuous</u> at (a,b) if:

$$f(a,b) = \lim_{(x,y)\to(a,b)} f(x,y)$$

Ex:
$$f(x,y) = \begin{cases} \dfrac{\sin(x^2+y^2)}{x^2+y^2} & (x,y) \neq (0,0) \\ 1 & (x,y) = (0,0) \end{cases}$$

Is $f(x,y)$ continuous at $(0,0)$?

$$f(0,0) = 1 \overset{?}{=} \lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} \qquad \frac{\sin(0)}{0} = \frac{0}{0} \times$$

polar substitution: $x = r\cos\theta$, $y = r\sin\theta$ $\qquad r\to 0 \quad \dfrac{\sin(r^2)}{r^2}$

$\left[\sin(x) \sim x \text{ when } x \text{ is close to } 0\right]$

Could use L'Hopital since one variable now

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} \frac{\sin(r^2)}{r^2} = 1$$

<u>Continuous everywhere</u> : $f(x,y)$ cont. at <u>every</u> pt. (x,y)

Arc length : $r(t)$ parameterization for a curve C $a \le t \le b$

$v(t) = r'(t)$

$a(t) = r''(t)$

$$s(t) = \int_a^t \|r'(u)\| \, du$$ arc length function

$$s = g(t) = \int_a^t \|r'(u)\| \, du$$ => Find inverse of g $t = g^{-1}(s)$

$r(g^{-1}(s))$: arc length parameterization $\leftarrow$ This always has $\|r'(s)\| = 1$

Tangent, Normal, Binormal:

$$T(t) = \frac{r'(t)}{\|r'(t)\|} \qquad N(t) = \frac{T'(t)}{\|T'(t)\|} \qquad B(t) = T(t) \times N(t)$$

All have unit length and are mutually orthogonal

$$s(t) = \|v(t)\|$$

$$a(t) = a_T T + \boxed{a_N N}$$

$$a_T = a \cdot T = a \cdot \frac{v}{\|v\|} = s'(t)$$

Important side note:

We also know T & N are unit vectors

$$a_N = a \cdot N = \sqrt{\|a\|^2 - |a_T|^2} = \underline{K(t)\,\|v(t)\|^2}$$

a_T : acceleration due to change in speed

a_N : acceleration due to change in direction

Curvature:

$$K(s) = \left\| \frac{dT}{ds} \right\|$$ s: w/ respect to arc length parametrization

$$= \left\| \frac{dT}{dt} \right\| \frac{1}{\|r'(t)\|}$$

$$K(t) = \frac{\| r'(t) \times r''(t) \|}{\|r'(t)\|^3}$$

$a(t) = \langle 1, 2, 3 \rangle$

$a_T T = \langle 0, 1, 2 \rangle$

Ex: $a(t)$, $a_T T$

$$a_N N = a(t) - a_T T = \langle 1, 2, 3 \rangle - \langle 0, 1, 2 \rangle$$
$$= \langle 1, 1, 1 \rangle$$

Want: N

$$a_N N = \langle 1, 1, 1 \rangle \quad , \quad N = \frac{a_N N}{\|a_N N\|} = \frac{\langle 1, 1, 1 \rangle}{\sqrt{3}} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$$

Problems :

Quiz Q3 : A particle is traveling. At $t = t_0$, $\vec{v} = \langle 1, 2, 3 \rangle$ and its acceleration $\vec{a} = \langle 3, 2, 1 \rangle$. Compute the unit normal vector $\vec{N}$ at $t = t_0$.

$$\| \langle 1, 2, 3 \rangle \| = \sqrt{1+4+9} = \sqrt{14}$$

$$a(t) = a_T T + a_N N$$

$$a(t_0) = a_T T(t_0) + a_N N(t_0)$$

$$a_T = a \cdot T$$

$$= \langle 3, 2, 1 \rangle \cdot \frac{1}{\sqrt{14}} \langle 1, 2, 3 \rangle$$

$$= \frac{1}{\sqrt{14}} (3+4+3) = \frac{10}{\sqrt{14}}$$

$$T(t_0) = \frac{\langle 1, 2, 3 \rangle}{\| \langle 1, 2, 3 \rangle \|} = \frac{1}{\sqrt{14}} \langle 1, 2, 3 \rangle$$

$$a_n N(t_0) = a(t_0) - a_T T(t_0)$$

$$= \langle 3, 2, 1 \rangle - \frac{10}{\sqrt{14}} \cdot \frac{1}{\sqrt{14}} \langle 1, 2, 3 \rangle = \langle 3, 2, 1 \rangle - \frac{10}{14} \langle 1, 2, 3 \rangle$$

$$= \underbrace{\langle 3, 2, 1 \rangle}_{\left\langle \frac{42}{14}, \frac{28}{14}, \frac{14}{14} \right\rangle} - \left\langle \frac{10}{14}, \frac{20}{14}, \frac{30}{14} \right\rangle$$

$$= \left\langle \frac{32}{14}, \frac{8}{14}, -\frac{16}{14} \right\rangle = \frac{1}{7} \langle 16, 4, -8 \rangle$$

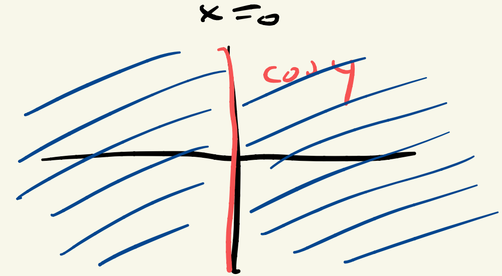
$$= \frac{4}{7} \langle 4, 1, -2 \rangle$$

$$\| a_N N \| = \frac{4}{7} \sqrt{16 + 1 + 4} = \frac{4}{7} \sqrt{21}$$

$$N = \frac{a_N N}{\| a_N N \|} = \frac{7}{4\sqrt{21}} \cdot \frac{4}{7} \langle 4, 1, -2 \rangle = \boxed{\frac{1}{\sqrt{21}} \langle 4, 1, -2 \rangle}$$

Enakoutsa S 23

(3) $f(x,y) = \begin{cases} \boxed{\dfrac{\cos y \sin x}{x}} & x \neq 0 \\ \cos y & x = 0 \end{cases}$



Is $f(x,y)$ cont. at $(0,0)$? Is it cont. everywhere?

$$f(0,0) = \lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \begin{cases} \dfrac{\cos y \sin x}{x} & x \neq 0 \\ \cos y & x = 0 \end{cases}$$

$$f(0,0) = \cos(0) = 1$$

To see $f(x,y)$ cont., for any value of y

$$\lim_{x\to 0} \frac{\cos y \sin x}{x} = \cos y \qquad \lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x\to 0} \frac{\cos y \sin x}{x} = \cos y \underbrace{\lim_{x\to 0} \frac{\sin x}{x}}_{1} = \cos y \cdot 1 = \cos y$$

$$\lim_{(x,y)\to(0,0)} \frac{\cos y \sin x}{x} = \lim_{(x,y)\to(0,0)} \cos y \cdot \lim_{(x,y)\to(0,0)} \frac{\sin x}{x}$$

$$= \cos(0) \cdot 1 = 1 \checkmark$$

$$\lim_{(x,y)\to(0,b)} \frac{\cos y \cdot \sin x}{x} = \lim_{(x,y)\to(0,b)} \cos y \cdot \lim_{(x,y)\to(0,b)} \frac{\sin x}{x}$$

$y = b$

$$= \cos(b) \cdot 1 = \cos(b) \checkmark$$

Sideroute: Squeeze Thm

$$f(r,\theta) = r\cos\theta\sin\theta$$

$$\lim_{r\to 0} f(r,\theta) = r\cos\theta\sin\theta$$

$$0 \le |f(r,\theta)| = |r\cos\theta\sin\theta| = r|\cos\theta\sin\theta| \le r\cdot 1 = r$$

$$r\to 0, \quad \underbrace{0}_{} \le |f(r,\theta)| \le \underbrace{r}_{} \implies |f(r,\theta)| \longrightarrow 0$$

$$\implies \lim_{r\to 0} f(r,\theta) = 0$$

By Squeeze Thm

Emakoutsa F21 :

2 (i) $f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 \quad ; & (x,y) = (0,0) \end{cases}$

At which pts. is $f(x,y)$ continous?

If $x^2+y^2=0$, only happens when $(x,y)=(0,0)$.

If $(x,y) \neq 0 \Rightarrow x^2+y^2 \neq 0 \Rightarrow \dfrac{x^3+y^3}{x^2+y^2}$ cont. at that pt. since its a rational function with non-zero denominator

To check : $(x,y) = (0,0)$

$$x^3+y^3 = (\quad)(x^2+y^2)$$

Polar : $x = r\cos\theta$

$y = r\sin\theta$

$$\frac{r^3\cos^3\theta + r^3\sin^3\theta}{r^2}$$

$$= r\cos^3\theta + r\sin^3\theta = r(\cos^3\theta + \sin^3\theta)$$

$$\underbrace{|\cos^3\theta + \sin^3\theta|}\ \overset{*}{\le}\ |\cos^3\theta| + |\sin^3\theta| \le 1 + 1 = \underbrace{2}$$

$$0 \le |r(\cos^3\theta + \sin^3\theta)| \le r\underbrace{|\cos^3\theta + \sin^3\theta|} \le r\cdot 2$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} f(r,\theta) = 0$$

$* \ |a+b| \le |a| + |b|$

$$\begin{array}{ccccc} 0 \le & |f(r,\theta)| & \le 2r \\ \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 \end{array}$$

by squeeze thm.

Quiz Q1:

What is parallel to $N(t)$?

$$N(t) = \frac{T'(t)}{\|T'(t)\|}$$

a) ~~$\frac{d}{dt}\left(f(t)\,T(t)\right)$~~

b) $\frac{d}{dt}\left(T(t)\right)$ ✓

$$N(t) = \frac{\frac{d}{dt}\left(T(t)\right)}{\left\|\frac{d}{dt}\left(T(t)\right)\right\|}$$

c) ~~$a(t)$~~

$$a(t) = a_T T + a_N N$$

$$\frac{d}{dt}\left(f(t)\,T(t)\right) = \underbrace{f'(t)\,T(t)}_{\text{\underline{not} parallel to } N} + \underbrace{f(t)\,T'(t)}_{\text{parallel to } N}$$

Quiz Q2 :

If a particle is traveling in a circle, $N(t)$ is constant

$$r(t) = \langle \cos t, \sin t \rangle$$

$$T(t) = \langle -\sin t, \cos t \rangle$$

$$N(t) = \frac{T'(t)}{\|T'(t)\|} = \frac{\langle -\cos t, \sin t \rangle}{1} = \underline{\langle -\cos t, \sin t \rangle}$$