Practice Midterm

Please note that this is just a practice midterm, and the ACTUAL MIDTERM WILL BE DIFFERENT IN TERMS OF QUESTIONS THAT WILL BE ASKED. The purpose of this practice midterm is to give you an idea of the format of the actual midterm, and to test your knowledge as you prepare for the midterm.

Problem 1. *Fix some (unknown) number a and consider the system of linear equations*

$$\begin{cases} x + y + z & = 5 \\ x + 2y + z & = 9 \\ x + y + (a^2 - 24)z & = a \end{cases}$$

For which value of a does the system have infinitely many solutions? To receive full credit, you must show all the steps you have taken to find the solution.

$$\begin{array}{ccc} x & y & z \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & 2 & 1 & 9 \\ 1 & 1 & a^2-24 & a \end{array}\right] - (I) \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & 4 \\ 1 & 1 & a^2-24 & a \end{array}\right] -(I)$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & 4 \\ \underbrace{0 \quad 0 \quad a^2-25}_{\text{free variable?}} & & & a-5 \end{array}\right]$$

$$\boxed{a^2 - 25 = 0}$$
$$\boxed{a - 5 = 0}$$
$$\Rightarrow \quad a = 5$$

A row of all 0's $\Rightarrow$ ∞ many solutions

$[0 \; \cdots \; 0 \mid \text{non-zero}] \Rightarrow$ no solutions

Problem 2. *If $T : \mathbb{R}^3 \to \mathbb{R}^2$ is the linear function defined by the formula*

$$T\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} x_1 + 2x_2 \\ x_2 - x_1 \end{pmatrix}$$

Find the associated matrix of the linear transformation. To receive full credit, you must show all the steps you have taken to find the matrix.

A : 2×3 matrix

$m \times n$

$A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}$

$A \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{a}_1$

$A \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \vec{a}_i$

i^{th} compnt is 1

$$A = \begin{bmatrix} \underline{T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}} & \underline{T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} & \underline{T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 2\cdot 0 & 0 + 2\cdot 1 & 0 + 2\cdot 0 \\ 0 - 1 & 1 - 0 & 0 - 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 0 \end{bmatrix}$$

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ x_2 - x_1 \\ x_3 \end{pmatrix} \qquad \underbrace{\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right\}}$$

Find A that represents T in the basis $\mathcal{B}$.

$$A = \left[\left[T\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right]_{\mathcal{B}} \left[T\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right]_{\mathcal{B}} \left[T\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right]_{\mathcal{B}} \right]$$

$$\left[T\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right]_{\mathcal{B}} = \left[\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \right]_{\mathcal{B}}$$

$$\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -1 & 1 & 0 \\ 1 & 0 & -2 & 1 \end{array} \right] \qquad \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix}^{-1} ?$$

$$\text{sol} = \left[T\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right]_{\mathcal{B}}$$

Problem 3. *Assume* $\begin{pmatrix} p \\ q \end{pmatrix}$ *is a vector in* $\mathbb{R}^2$ *such that its orthogonal projection onto the line* $y = x$ *is* $\begin{pmatrix} 10 \\ 10 \end{pmatrix}$ *and such that its orthogonal projection onto the line* $y = 2x$ *is* $\begin{pmatrix} 5 \\ 10 \end{pmatrix}$*. Compute* p *and* q*.*

orthogonal projection onto $y = x$: same thing as ortho. proj. onto a non-zero vector that lies on $y = x$.

$$\text{proj}_{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \begin{pmatrix} p \\ q \end{pmatrix} = \frac{\begin{pmatrix} p \\ q \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{p+q}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 10 \end{pmatrix}$$

$$\frac{p+q}{2} = 10 \Rightarrow p + q = 20$$

orthogonal projection onto $y = 2x$:

$$\text{proj}_{\begin{pmatrix} 1 \\ 2 \end{pmatrix}} \begin{pmatrix} p \\ q \end{pmatrix} = \frac{\begin{pmatrix} p \\ q \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}}{\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{p+2q}{5} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$$

$$= 5 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\frac{p+2q}{5} = 5 \Rightarrow p + 2q = 25$$

$$-(p + q = 20)$$

$$\boxed{q = 5, \ p = 15}$$

$$\boxed{\begin{pmatrix} 15 \\ 5 \end{pmatrix}}$$

Problem 4. *Compute the inverse of the matrix*

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

To receive full credit, you must show all the steps you have taken to find the inverse.

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 & 1 \end{array}\right)\begin{array}{l} \\ \\ \\ /5 \end{array}$$

$$\rightsquigarrow$$

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1/5 \end{array}\right)\begin{array}{l} -(\mathrm{II}) \\ \\ \\ \\ \end{array}$$

$$\rightsquigarrow$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1/5 \end{array}\right)\begin{array}{l} \\ -(\mathrm{III}) \\ \\ \\ \end{array}$$

$$\rightsquigarrow$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1/5 \end{array}\right)\begin{array}{l} \\ \\ -(\mathrm{IV}) \\ \\ \end{array}$$

$$\rightsquigarrow$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1/5 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1/5 \end{array}\right)$$

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1/5 \\ 0 & 0 & 0 & 1/5 \end{bmatrix}$$

sq. matrix invertible $\iff$ rank $= n$ $\iff$ $\ker A = \{0\}$
$n \times n$ (n leading 1's)

Problem 5. *Find the orthogonal projection of the vector*

$$\begin{pmatrix} 2 \\ -2 \\ 0 \\ 2 \end{pmatrix} \qquad P = \left[\mathrm{proj}_V \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \ \mathrm{proj}_V \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \ \mathrm{proj}_V \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \ \mathrm{proj} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right]$$

onto the $\underset{V}{\text{subspace}}$ *of* $\mathbb{R}^4$ *spanned by the vectors*

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 9 \\ 9 \\ 1 \end{pmatrix}$$

To receive full credit, you must show all the steps you have taken to find the solution.

Gram - Schmidt :

$$v_1^\perp = v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad \|v_1^\perp\| = \sqrt{1+1+1+1} = \sqrt{4} = 2$$

$$v_2^\perp = v_2 - \mathrm{proj}_{v_1^\perp} v_2 = \begin{pmatrix} 1 \\ 9 \\ 9 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} 1 \\ 9 \\ 9 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 9 \\ 9 \\ 1 \end{pmatrix} - \frac{1+9+9+1}{1+1+1+1} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 9 \\ 9 \\ 1 \end{pmatrix} - \overset{5}{\cancel{\frac{20}{4}}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -4 \\ 4 \\ 4 \\ -4 \end{pmatrix} \qquad \|v_2^\perp\| = \sqrt{16+16+16+16} = \sqrt{64} = 8$$

$$u_1 = \frac{v_1^\perp}{\|v_1^\perp\|} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad u_2 = \frac{v_2^\perp}{\|v_2^\perp\|} = \frac{1}{8} \begin{pmatrix} -4 \\ 4 \\ 4 \\ -4 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$$

$$\operatorname{proj}_V \begin{pmatrix} 2 \\ -2 \\ 0 \\ 2 \end{pmatrix} = \operatorname{proj}_{\begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}} \begin{pmatrix} 2 \\ -2 \\ 0 \\ 2 \end{pmatrix} + \operatorname{proj}_{\begin{pmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{pmatrix}} \begin{pmatrix} 2 \\ -2 \\ 0 \\ 2 \end{pmatrix}$$

$$= \left(\begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ 0 \\ 2 \end{pmatrix} \right) \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} + \left(\begin{pmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ 0 \\ 2 \end{pmatrix} \right) \begin{pmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{pmatrix}$$

$$= \overset{1}{\cancel{(1 - 1 + 1)}} \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} + \overset{-3}{\cancel{(-1 - 1 - 1)}} \begin{pmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 + 3/2 \\ 1/2 - 3/2 \\ 1/2 - 3/2 \\ 1/2 + 3/2 \end{pmatrix} = \boxed{\begin{pmatrix} 2 \\ -1 \\ -1 \\ 2 \end{pmatrix}}$$

$$v_1^{\perp} = v_1$$

$$v_2^{\perp} = v_2 - \operatorname{proj}_{v_1^{\perp}} v_2$$

$$v_3^{\perp} = v_3 - \operatorname{proj}_{v_1^{\perp}} v_3 - \operatorname{proj}_{v_2^{\perp}} v_3$$