SVD :

1) $A = \begin{bmatrix} 6 & -7 \\ 2 & 6 \end{bmatrix}$, $A^T A = \begin{bmatrix} 6 & 2 \\ -7 & 6 \end{bmatrix} \begin{bmatrix} 6 & -7 \\ 2 & 6 \end{bmatrix}$

$\frac{125 \pm 75}{2}$ 25, 10 $= \begin{bmatrix} 40 & -30 \\ -30 & 85 \end{bmatrix}$

$\begin{bmatrix} 40-\lambda & -30 \\ -30 & 85-\lambda \end{bmatrix} \rightsquigarrow \lambda^2 - 125\lambda + 2500 = 0$

$(\lambda - 25)(\lambda - 100)$, $\lambda = 25, 100$

$$\left.\begin{aligned} \sigma_1 &= \sqrt{100} = 10 \\ \sigma_2 &= \sqrt{25} = 5 \end{aligned}\right\} \text{Singular values}$$

<u>eigenspaces of A^TA:</u>

<u>$\lambda = 100$:</u>

$$\begin{bmatrix} -60 & -30 \\ -30 & -15 \end{bmatrix} \begin{matrix} /-30 \\ /-15 \end{matrix} \sim \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \begin{matrix} 2x_1 + x_2 = 0 \\ x_2 \text{ free} \end{matrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}x_2 \\ x_2 \end{bmatrix} \quad \begin{matrix} \text{Choose} \\ x_2 = 2 \end{matrix} \quad \begin{bmatrix} -1 \\ 2 \end{bmatrix} \underset{\text{normalize}}{\sim} \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = V_1$$

$\underline{\lambda = 25}$:

$$\begin{bmatrix} 15 & -30 \\ -30 & 60 \end{bmatrix} \begin{matrix} /15 \\ /-30 \end{matrix} \to \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \begin{matrix} x_1 - 2x_2 = 0 \\ x_2 \text{ free} \end{matrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} \quad \text{Let } x_2 = 1 \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix} \xrightarrow{\text{normalize}} \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} = v_2$$

note v_2 is ortho. to v_1 as desired

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{10} \begin{bmatrix} 6 & -7 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \frac{1}{10} \begin{bmatrix} -20/\sqrt{5} \\ 10/\sqrt{5} \end{bmatrix} = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{5}\begin{bmatrix} 6 & -7 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 5/\sqrt{5} \\ 10/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

$$\begin{bmatrix} 6 & -7 \\ 2 & 6 \end{bmatrix} = \underbrace{\begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}}_{\Sigma}\underbrace{\begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}}_{V^T}$$

② $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$

$$A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

$$A = \overset{2\times 2}{U}\,\overset{2\times 3}{\Sigma}\,\overset{3\times 3}{V^T}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\lambda = 1, 4 \Rightarrow \sigma_1 = 2, \; \sigma_2 = 1$$

By observation, we see $v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = v_2$ are eigenvectors for 4 & 1 respectively

Then $u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Now, V is a 3×3 matrix & so we need to extend the $\{v_1, v_2\}$ vectors to an orthonormal basis for $\mathbb{R}^3$. We can see $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ will accomplish this.

So,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} = \underset{U}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \underset{\Sigma}{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}} \underset{V^T}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

3

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A^T A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \underset{3\times 3}{U} \; \underset{3\times 2}{\Sigma} \; \underset{2\times 2}{V^T} \qquad = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix} \rightsquigarrow \lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1)$$

$$\lambda = 1, 3 : \quad \sigma_1 = \sqrt{3}, \quad \sigma_2 = 1$$

$\underline{\lambda = 3}$:

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \rightsquigarrow \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{= v_1} \text{ eigenvector}$$

$$\xrightarrow{\text{normalize}} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$\underline{\lambda = 1}$:

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \rightsquigarrow \overset{\text{eigenvector}}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} \xrightarrow{\text{normalize}} \underbrace{\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}}_{= v_2}$$

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 2/\sqrt{2} \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{1} \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$\underset{\sim\sim\sim}{\text{Need } u_3}$: Since it must be orthogonal to u_1, u_2:

$$u_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 2/\sqrt{6} \\ -1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} = 0 \quad \Rightarrow \quad 2a - b + c = 0$$

$$u_3 \cdot \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = 0 \quad \Rightarrow \quad b + c = 0 \Rightarrow b = -c$$

$$\Rightarrow \quad 2a + 2c = 0 \Rightarrow a = -c$$

so $u_3 = \begin{bmatrix} -c \\ -c \\ c \end{bmatrix}$ for some c. Letting $c = 1/\sqrt{3}$ gives a normalized vector.

$$u_3 = \begin{bmatrix} -1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix},$$

So

$$A = \underbrace{\begin{bmatrix} 2/\sqrt{2} & 0 & -1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{2} & -1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{V^T}$$