MATH 33A Worksheet Week 2 Solutions

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Topic 1: Invertibility

Exercise 1.1. Determine whether the following matrices are invertible, and if they are, find the inverse.

(a) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$

(a) Not invertible, rank of the matrix is 1

(b) Invertible, with inverse: $\begin{bmatrix} -\frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix}$

(c) Invertible, with inverse: $\begin{bmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{bmatrix}$

Exercise 1.2. Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$.

(a) Compute A^{-1}.

(b) Use the inverse to find all solutions to $A\vec{x} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, and all solutions to $A\vec{x} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$.

(a) $A^{-1} = \begin{bmatrix} 2 & -2 & 3 \\ -3 & 4 & -6 \\ 1 & -1 & 2 \end{bmatrix}$.

(b) $A\vec{x} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ implies $\vec{x} = A^{-1} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$,

$$\vec{x} = \begin{bmatrix} 2 & -2 & 3 \\ -3 & 4 & -6 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 15 \\ -29 \\ 9 \end{bmatrix}$$

Thus the only solution is $\vec{x} = \begin{bmatrix} 15 \\ -29 \\ 9 \end{bmatrix}$.

Similarly for the second equation, we find that

$$\vec{x} = \begin{bmatrix} 2 & -2 & 3 \\ -3 & 4 & -6 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

is the only solution.

Exercise 1.3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation of projection onto the line $y = x$. Is T invertible? Argue both (1) geometrically, and (2) by finding the matrix representation for T and computing its determinant.

Geometrically: Since T is projection onto the line $y = x$, for every point (a, b) on the line $y = -x$ perpendicular to $y = x$, $T \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Therefore, T cannot have an inverse since it is not one-to-one/injective.

Algebraically: Since T is projection onto $y = x$, for every point (a, b) on the line $y = x$, $T \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$. Similarly for every point (a, b) on the line $y = -x$ perpendicular to $y = x$, $T \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Therefore, $T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $T \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Using linearity of T, we have:

$$T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = T\left(\frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \frac{1}{2} T \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2} T \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

Similarly,

$$T\begin{bmatrix}0\\1\end{bmatrix} = T\left(\frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}-1\\1\end{bmatrix}\right) = \frac{1}{2}T\begin{bmatrix}1\\1\end{bmatrix} + \frac{1}{2}T\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}1/2\\1/2\end{bmatrix}$$

Therefore, T is represented by the matrix $A = \begin{bmatrix}1/2 & 1/2\\1/2 & 1/2\end{bmatrix}$. Since A is a rank 1 matrix, T is not invertible.

Exercise 1.4. Find the inverse of the following matrix (in terms of $c \in \mathbb{R}$. Verify your answer with matrix multiplication: $A = \begin{bmatrix}1 & c & c^3\\0 & 1 & c\\0 & 0 & 1\end{bmatrix}$.

$$A^{-1} = \begin{bmatrix}1 & -c & c^2 - c^3\\0 & 1 & -c\\0 & 0 & 1\end{bmatrix}$$

Exercise 1.5. True or False?

(a) There exists an invertible $n \times n$ matrix with two identical rows.

(b) There exists an invertible 2×2 matrix A such that $A^{-1} = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$

(a) False, a square matrix with two identical rows will not have rank n, since we will get a row of all 0's in the process of row reduction.

(b) False, A^{-1} must be an invertible matrix, since $(A^{-1})^{-1} = A$, but $\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$ is not invertible.

Topic 2: Subspaces and Linear Independence

Exercise 2.1. Show that the following subsets are *not* subspaces of $\mathbb{R}^2$:

(a) $V = \{\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}\}$

(b) $V = \{\begin{bmatrix}3s+1\\2-s\end{bmatrix} \mid s \in \mathbb{R}\}$

Show that the following subsets *are* subspaces of $\mathbb{R}^2$:

(c) $V = \{\begin{bmatrix}t\\3s\end{bmatrix} \mid s, t \in \mathbb{R}\}$.

(d) $V = \{\begin{bmatrix}0\\0\end{bmatrix}\}$

To show a subset is *not* a subspace, we need to find $u, v \in V$ such that $u + v \notin V$ (i.e., V is not *closed under addition*), or some $u \in V$ and a scalar $c \in \mathbb{R}$ such that $c\dot{u} \notin V$ (i.e., V is not *closed under scalar multiplication)*.

(a) Let $u = \begin{bmatrix}1\\0\end{bmatrix}, v = \begin{bmatrix}2\\0\end{bmatrix}$, which are both in V. Then $u + v = \begin{bmatrix}3\\0\end{bmatrix}$ is not an element of V, so V is not a subspace.

(b) Let $u = \begin{bmatrix}3+1\\2-1\end{bmatrix} = \begin{bmatrix}4\\1\end{bmatrix}, v = \begin{bmatrix}7\\0\end{bmatrix} \in V$, so $u + v = \begin{bmatrix}11\\1\end{bmatrix}$. Let us show that $u + v$ is not in V, so there does not exist $s \in \mathbb{R}$ such that $\begin{bmatrix}3s+1\\2-s\end{bmatrix} = \begin{bmatrix}11\\1\end{bmatrix}$. In other words, we aim to show that there are no solutions to the linear system

$$\begin{bmatrix}3\\-1\end{bmatrix}\begin{bmatrix}s\end{bmatrix} = \begin{bmatrix}10\\-1\end{bmatrix}$$

Thus performing augmented row reduction, we have

$$\begin{bmatrix}13 & 10\\-1 & -1\end{bmatrix} \Rightarrow^{\text{swap and multiply by } -1} \begin{bmatrix}11 & 1\\3 & 10\end{bmatrix} \Rightarrow \begin{bmatrix}11 & 1\\0 & 7\end{bmatrix}$$

Thus, there are no solutions to $\begin{bmatrix}3\\-1\end{bmatrix}\begin{bmatrix}s\end{bmatrix} = \begin{bmatrix}10\\-1\end{bmatrix}$, so $\begin{bmatrix}11\\1\end{bmatrix}$ is not in V, so V is not a subspace.

To show subsets are subspaces of $\mathbb{R}^2$, we have to show that for **all** $u, v \in V$ that $u + v$ is in V (V is *closed under addition*), and that for all $v \in V, c \in \mathbb{R}$, that $c \cdot v \in V$ (V is *closed under scalar multiplication*).

(c) Let u, v be two elements of V, so $u = \begin{bmatrix} t_1 \\ 3s_1 \end{bmatrix}$ for some $t_1, s_1 \in \mathbb{R}$ and $v = \begin{bmatrix} t_2 \\ 3s_2 \end{bmatrix}$ for some $t_2, s_2 \in \mathbb{R}$. Then, $u + v = \begin{bmatrix} t_1 + t_2 \\ 3s_1 + 3s_2 \end{bmatrix}$. In particular, $u + v = \begin{bmatrix} t \\ 3s \end{bmatrix}$ for $t = t_1 + t_2, s = s_1 + s_2$, so $u + v \in V$. Now let u be an element of V, so $u = \begin{bmatrix} t_1 \\ 3s_1 \end{bmatrix}$ for some $t_1, s_1 \in \mathbb{R}$, and let $c \in \mathbb{R}$ be arbitrary. Then $c \cdot u = \begin{bmatrix} ct_1 \\ 3cs_1 \end{bmatrix} = \begin{bmatrix} t \\ 3s \end{bmatrix}$ for $t = ct_1, s = 3s_1$, so $c \cdot u \in V$.

(d) Let u, v be two elements of V. Since V only has a single vector, we must have $u = v = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Then $u + v = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in V$, so V is closed under addition. Now let $u \in V$, so $u = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and let $c \in \mathbb{R}$ arbitrary. Then $c \cdot u = \begin{bmatrix} c \cdot 0 \\ c \cdot 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in V$, so V is closed under scalar multiplication.

Exercise 2.2. Determine whether the following sets of vectors are linearly independent or linearly dependent:

(a) $\begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

(b) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 5 \end{bmatrix}$

(a) Any 3 vectors in a two dimensional space must be linearly dependent, so they are linearly dependent. In particular, we can write:

$$\frac{2}{3}\begin{bmatrix} 3 \\ -1 \end{bmatrix} + \frac{8}{3}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

(b) This is linearly dependent since $1 \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$. Notice that any set of vectors including zero must be linearly dependent.

(c) The matrix with columns corresponding to these 4 vectors is upper triangular and has determinant $1 * 3 * 4 * 5 \neq 0$, and so these 4 vectors are linearly independent.

Topic 3: Image and Kernel

Exercise 3.1. Find a linear transformation $A : \mathbb{R}^3 \to \mathbb{R}^3$ which satisfies each of the following conditions, or explain why such a linear transformation doesn't exist:

(a) $\ker A = \{\vec{0}\}$, $\operatorname{im} A = \{\vec{0}\}$.

(b) $\ker A = \operatorname{span}\{\begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}\}$ and A is not invertible.

(c) $\ker A = \operatorname{span}\{\begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}\}$ and A is invertible.

(d) $\ker A = \operatorname{span}\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\}$, $\operatorname{im} A = \operatorname{span}\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\}$.

(a) If $\operatorname{im} A = \{\vec{0}\}$, then every column of A must be the zero vector, so $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. However, this matrix has $v \in \ker A$ for every single vector $v \in \mathbb{R}^3$, thus $\ker A \neq \{\vec{0}\}$. So no such matrix A can exist.

(b) Since $\begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix} \in \ker A$, we have $x_1 = -x_3/3$, $x_2 = -x_3$, x_3 free as a solution to the equation $Ax = 0$. Therefore, the matrix

$$A = \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

has the desired kernel. A also is not invertible since it contains a row of 0's and thus has rank less than 3.

(c) If A is invertible, then the kernel of A must just consist of the zero vector, so such an A is impossible to construct.

(d) Recall that $A\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the first column of A. Since $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in \ker A$, $A\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$, so the first column of A must be zero. Similarly, the second column of A must be zero. In order for the image of A to be the span of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, its columns must must span $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, so the following works:

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Exercise 3.2. Find the kernel of the following matrices:

(a) $\begin{bmatrix} 3 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & -1 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}$

(c) A for $A : \mathbb{R}^n \to \mathbb{R}^n$ invertible.

(d) A with row reduced echelon form $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$.

(a) RREF:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

so

$$\ker = \left\{ \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} \mid x_3 \in \mathbb{R} \right\}$$

(b) RREF has a leading one in both columns, so $\ker = \{\vec{0}\}$.

(c) A linear invertible transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ has RREF equal to the identity matrix, so every column has a leading one, so $\ker A = \{\vec{0}\}$. Alternatively, since A is invertible, $Ax = 0$ has the unique solution $\vec{0}$.

(d) Refer to previous question part (b).

Exercise 3.3.
Let A be any matrix, and B be the RREF for A.

(a) Is $\ker A = \ker B$?

(b) Is $\operatorname{im} A = \operatorname{im} B$?

(a) The kernel of A is equal to the kernel of B. Note that the kernel of A is the same as the set of solutions to the matrix equation $Ax = 0$, which we set up as the augmented matrix equation $(A|0)$. We row reduce this to RREF and get the augmented matrix $(B|0)$, and so we are now solving the equation $Bx = 0$! Since row reduction doesn't change our solution set, the kernel of B will be the same as the kernel of A.

(b) No! The image of A and image of B will not necessarily be the same. Here's an example:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \mathrm{rref}(A) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

The matrix A has as its image the line spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, i.e. the line $y = x$. However, its rref has the x-axis as its image.