MATH 33A Worksheet Week 3

TA: Emil Geisler and Caleb Partin

April 16, 2024

Exercise 1. Compute the following or state that it is not defined.

(a) $\begin{bmatrix} 4 & 2 & 0 \\ 1 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & 2 & 3 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & 2 & 3 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \end{bmatrix}$

Exercise 2. For each of the following linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$, find the corresponding matrix that represents T:

(a) Rotate any vector $\vec{v}$ counter-clockwise by an angle of $\frac{\pi}{2}$ radians

(b) Projection onto the x-axis

(c) Projection onto the y-axis

(d) First reflect a vector across the line $y = x$, then rotate it by $\frac{\pi}{2}$ radians. (We have matrices A and B that represent both steps of this linear transformation, and a single matrix C that represents the whole transformation. What is the relationship between A, B and C?)

Exercise 3. Let $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ be the standard basis vectors of $\mathbb{R}^n$. Show that if A is an $m \times n$ matrix such that $A\vec{e}_1 = A\vec{e}_2 = \cdots = A\vec{e}_n = 0$, then A is the zero matrix.

Exercise 4. Compute the following for all $\theta \in \mathbb{R}$:

$$\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

What linear transformation do each of these matrices represent? What is the geometric interpretation of the matrix you get as their product?

Exercise 5. (Challenge Problem): Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a function which satisfies $\underline{\mathbb{R}\text{-linearity}}$: $F(\vec{v} + a\vec{w}) = F(\vec{v}) + aF(\vec{w})$ for all $\vec{v}, \vec{w} \in \mathbb{R}^n$, $a \in \mathbb{R}$.
Prove that as functions $\mathbb{R}^n \to \mathbb{R}^m$, $F = A$ where A is the matrix with ith column vector equal to $F(e_i)$. (Notice that every $\mathbb{R}$-linear function $F : \mathbb{R}^n \to \mathbb{R}^m$ is also *linear*, by letting $\lambda = 1$.) This shows that every $\mathbb{R}$-linear function is a matrix.
