High-dimensional Takens embeddings

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Abstract

J. Perea and C. Tralie introduced a method of detecting quasiperiodic behavior in time-dependent data [**TrPe**] appearling to Takens delay embedding theorem: generically, one can reconstruct a dynamical system from a uniform finite sampling of an observation function along trajectories. We characterize precisely quasiperiodic functions with delay embedding yielding Tori and Klein bottles via an infinitesimal analysis which generalizes to arbitrary compact manifolds.

1 Sliding window

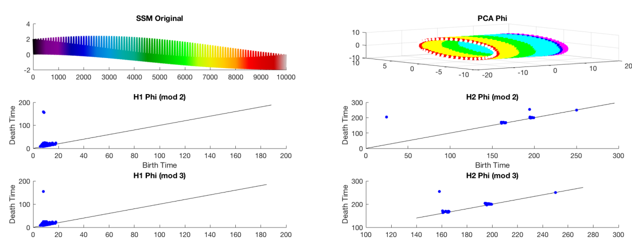
The quasiperiodic function

$$g(t) = \cos(\sqrt{2}t) + \cos(t)$$

is a sum of two periodic functions with incommensurate periods. A point cloud densely sampled from *sliding window*

$$\big(g(t), g(t+1), g(t+2), g(t+3), g(t+4), g(t+5)\big)$$

curve in $\mathbb{R}^6$ with coordinates given by uniform shifts of g produces persistence diagrams exhibiting two classes in first degree and one in second degree, as follows:



The computation suggests that the sliding window curve is a dense winding on a torus. This is essentially a consequence of Takens' delay embedding theorem [**Ta**], as g is obtained by applying the *observation function*

$$G(x, y) = \cos(x) + \cos(y)$$

to the irrational flow $(\sqrt{2}t, t)$ on the planar torus $\mathbb{T} = \mathbb{R}^2/2\pi\mathbb{Z}^2$.

More generally, a sliding window

$$\mathrm{SW}_\tau^N\, g := \big(g(t), g(t+\tau), g(t+2\tau), \ldots, g(t+N\tau)\big)$$

in $\mathbb{R}^N$ of $G(x, y)$ on any irrational flow in $\mathbb{T}$ will trace out an embedded torus for sufficiently large positive integer N and small τ, as $G(x, y)$ is a "sufficiently injective" observation. Theorem 3.1 below gives a classification of all observations with toroidal sliding window embedding.

2 Takens map

Let M be a compact manifold of dimension m, $G : M \to \mathbb{R}$ a smooth function, and X a vector field with flow ψ_t. Applying G to an integral curve

$$\gamma_p(t) := \psi_t(p)$$

collapses it to a real-valued function

$$g_p := G \circ \gamma_p$$

in t, the *observation curve* of p. For sufficiently nice G and X, one can recover the point p from a finite uniform sampling of g_p. More precisely, the *Takens map* $\Psi_\tau^N : M \to \mathbb{R}^{N+1}$ defined by

$$\Psi_\tau^N(p) = \big(g_p(0), g_p(\tau), g_p(2\tau), \ldots, g_p(N\tau)\big)$$

is an embedding for some dimension $N > 0$ and flow time $\tau > 0$. For such G and X we say G is a *good observation* for X.

As a simple example, take $M = S^1 = \mathbb{R}/2\pi\mathbb{Z}$, $\psi_t(x) = x + t$, and $G(x) = \cos(x)$. x is uniquely determined by sampling the two values $g_x(0) = \cos(x)$ and $g_x(\pi/2) = -\sin(x)$ and the Takens map

$$\Psi^1_{\pi/2}(x) = (\cos(x), -\sin(x))$$

is an embedding, so G is a good observation.

On the other hand, the doubly periodic function $G(x) = \cos(2x)$ is not a good observation function. Indeed, any integral curve $g_x(t)$ is invariant under a π-shift of x, as G cannot distinguish between any flow of x and $x + \pi$. In fact, the good observation functions on S^1 for the rotational dynamic are precisely ones with minimum period 2π

In higher dimensions the task of recovering p from g_p becomes less clear. Consider the torus $\mathbb{T} = S^1 \times S^1$ and $G : \mathbb{T} \to \mathbb{R}$ given by

$$G(x, y) = \cos(x) + \cos(y)$$

and ψ_t an irrational flow

$$\psi_t(x, y) = (x + \alpha t, y + \beta t)$$

and thus for $p = (x, y) \in \mathbb{T}$ we have the observation curve

$$g_p(t) = \cos(x + \alpha t) + \cos(y + \beta t)$$

For G to be good, there must be a τ such that each p is uniquely determined by sampling G along the integral curve γ_p at finitely many τ-steps. Since we are free to shrink τ and increase N, it is natural to examine infinitesimal changes of G along the flow ψ_t. The derivatives

$$\begin{aligned}
g_p(0) &= \cos(x) + \cos(y) \\
g_p'(0) &= -\alpha \sin(x) - \beta \sin(y) \\
g_p^{(2)}(0) &= -\alpha^2 \cos(x) - \beta^2 \cos(y) \\
g_p^{(3)}(0) &= \alpha^3 \sin(x) + \beta^3 \sin(y)
\end{aligned}$$

up to 3$^{\text{rd}}$ order yield the linear equation

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ -\alpha^2 & -\beta^2 & 0 & 0 \\ 0 & 0 & -\alpha & -\beta \\ 0 & 0 & \alpha^3 & \beta^3 \end{pmatrix} \begin{pmatrix} \cos(x) \\ \cos(y) \\ \sin(x) \\ \sin(y) \end{pmatrix} = \begin{pmatrix} g_p(0) \\ g_p'(0) \\ g_p^{(2)}(0) \\ g_p^{(3)}(0) \end{pmatrix}$$

Equivalently, over $\mathbb{C}$, the linear system

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ i\alpha & -i\alpha & i\beta & -i\beta \\ -\alpha^2 & -\alpha^2 & -\beta^2 & -\beta^2 \\ -i\alpha^3 & i\alpha^3 & -i\beta^3 & i\beta^3 \end{pmatrix} \begin{pmatrix} e^{ix} \\ e^{-ix} \\ e^{iy} \\ e^{-iy} \end{pmatrix} = \begin{pmatrix} g_p(0) \\ g_p'(0) \\ g_p^{(2)}(0) \\ g_p^{(3)}(0) \end{pmatrix}$$

has invertible Vandermonde matrix and one can solve for e^{ix} and e^{iy}. Therefore (x, y) is uniquely determined by $g_p^{(k)}(0)$'s. Choosing τ small enough so that $g_p(\tau)$ is close to the 3rd order Taylor polynomial of g_p about 0, we see that p is uniquely determined (modulo 2π) by a τ-uniform finite sampling of g_p.

The above calculation illustrates our approach to determining whether G is good: by studying the Taylor coefficients $g_p^{(k)}$. Note that $g_p^{(k)}(t)$ is the k-fold derivation of X applied to G at $\psi_t(p)$, i.e., in Lie derivative notation,

$$g_p^{(k)}(t) = \mathcal{L}_X^k G(\psi_t(p)).$$

The main result of this paper generalizes this infinitesimal analysis to arbitrary G and X on a compact manifold M, yielding two conditions as follows:

Theorem 2.1. *The Takens map Ψ_τ^N is an embedding for some $N > 0$ and flow time $\tau > 0$ if the following conditions hold:*

1. *For any point of $p \in M$ there is an m-tuple $J \in \mathbb{Z}_{\geq 0}^m$ of nonnegative integers such that the m-form*
$$\mathcal{L}_X^{\wedge J} dG := \bigwedge_{j \in J} \mathcal{L}_X^j dG$$
is nonzero at some point on the integral curve $\gamma_p(s)$.

2. *For any pair of distinct points $p, q \in M$ the observation curves $g_p(s)$ and $g_q(s)$ are not identical.*

Proof. For Ψ_τ^N to be an immersion, the cotangent vectors

$$dG|_p,\ d(G \circ \psi_\tau)|_p,\ d(G \circ \psi_{2\tau})|_p,\ \ldots,\ d(G \circ \psi_{N\tau})|_p$$

must span an m-dimensional space. Equivalently, for any point $p \in M$ there must be a strictly increasing m-tuple $I = (i_1, i_2, \ldots, i_m) \in \mathbb{Z}_{\geq 0}^m$ of indices such that the determinant m-form $\bigwedge d(G \circ \psi_{i_k \tau})$ does not vanish at p, i.e.

$$\omega_p^I(\tau) := \bigwedge_{i_k \in I} d(G \circ \psi_{i_k \tau})|_p \neq 0$$

The idea is to convolute the Taylor series of the cotangent curves $d(G \circ \psi_{i_k t})$ and use condition 1 above to choose sufficiently small τ so that $\omega_p^I(\tau) \neq 0$ for some I. By compactness one makes a uniform choice of small τ so that Ψ_τ is immersive and each observation curve is distinguished on some integer multiple of τ, thereby making Ψ_τ^N injective.

See section 5 below for details. □

3 Irrational flow on a torus

As a consequence of Theorem 2.1, we characterize functions on the torus

$$\mathbb{T} = \mathbb{R}^2/2\pi\mathbb{Z}^2$$

which are good for an irrational flow, i.e. a vector field X with flow

$$\psi_t(x, y) = (x + \alpha t, y + \beta t)$$

where $\alpha/\beta \notin \mathbb{Q}$. A smooth function $G : \mathbb{T} \to \mathbb{R}$ has a Fourier expansion

$$G(x, y) = \sum_{(n,m)\in\mathbb{Z}^2} \hat{G}(n, m) \cdot \exp(i(nx + my))$$

where $\hat{G}(n, m) \in \mathbb{C}$ is the $(n, m)^{\text{th}}$ Fourier coefficient of G. Set

$$\operatorname{Supp} \hat{G} = \{(n, m) \in \mathbb{Z}^2 \mid \hat{G}(n, m) \neq 0\}$$

the *support* of $\hat{G}$. The following theorem shows that whether or not G is good for an irrational flow depends only on the support of $\hat{G}$:

Theorem 3.1. *A smooth function $G : \mathbb{T} \to \mathbb{R}$ is a good observation for an irrational winding if and only if the support* $\operatorname{Supp} \hat{G}$ *of the Fourier coefficients generates $\mathbb{Z}^2$ as an abelian group.*

Proof. Write $e_{n,m} = \exp(i(nx + my))$ for the $(n, m)^{\text{th}}$ Fourier basis element. The k-fold Lie derivative $\mathcal{L}_X^k G$ has Fourier coefficient

$$\widehat{\mathcal{L}_X^k G}(n, m) = i^k(n\alpha + m\beta)^k \cdot \hat{G}(n, m)$$

and thus Fourier expansion

$$\mathcal{L}_X^k G = \sum_{(n,m)\in\mathbb{Z}^2} i^k(n\alpha + m\beta)^k \hat{G}(n, m) \cdot e_{n,m}$$

Since α/β is irrational, the coefficient

$$c_{n,m} = i \cdot (n\alpha + m\beta)$$

is always nonzero. The Vandermonde matrix with $(n, m) \times j^{\text{th}}$ entry

$$(c_{n,m}^k)$$

is nonsingular because the $c_{n,m}$ are distinct. Therefore the projection

$$G * e_{n,m} = \hat{G}(n, m) \cdot \exp(i(nx + my))$$

can be written as an infinite sum

$$\hat{G}(n, m) \cdot \exp(i(nx + my)) = \sum_{j=0}^{\infty} b_j \mathcal{L}_X^j G \tag{1}$$

Hence the values of $\mathcal{L}_X^k G$ on a point $(u,v) \in \mathbb{T}$ uniquely determine

$$\hat{G}(n,m) \cdot e^{i(nu+mv)}$$

If $\operatorname{Supp} \hat{G}$ generates $\mathbb{Z}^2$, then there is some finite product

$$\prod_{(n_j,m_j) \in \operatorname{Supp} \hat{G}} e^{i(n_j u + m_j v)} = e^{iu}$$

and thus u, and similarly v, are uniquely determined modulo 2π by the observation curve $G \circ \gamma_{u,v}$ and condition 2 of Theorem 2.1 above is satisfied.

If $d\mathcal{L}_X^j G \wedge d\mathcal{L}_X^k G$ vanishes at p for all $j, k \geq 0$, then by equation 1 above, the 2-form

$$d(G * e_{n,m}) \wedge d(G * e_{n',m'}) = \det \begin{pmatrix} n & m \\ n' & m' \end{pmatrix} \hat{G}(n,m)\hat{G}(n',m') \cdot e_{n,m} e_{n',m'}$$

also vanishes at p for all pairs $(n,m), (n',m') \in \mathbb{Z}^2$. Thus

$$\det \begin{pmatrix} n & m \\ n' & m' \end{pmatrix} = 0 \text{ for all } (n,m), (n',m') \in \operatorname{Supp} \hat{G}$$

and $\operatorname{Supp} \hat{G}$ cannot generate $\mathbb{Z}^2$. So condition 1 of Theorem 2.1 is satisfied if $\operatorname{Supp} \hat{G}$ generates $\mathbb{Z}^2$.

Conversely, suppose $\operatorname{Supp} \hat{G}$ does not generate $\mathbb{Z}^2$. By the classification of finitely generated abelian groups, there is a $\mathbb{Z}$-basis

$$(n_1, m_1), (n_2, m_2)$$

for $\mathbb{Z}^2$ such that $\operatorname{Supp} \hat{G}$ is generated by

$$a \cdot (n_1, m_1), b \cdot (n_2, m_2)$$

where a and b are integers not both ± 1. Then there is some $(u,v) \notin 2\pi\mathbb{Z}^2$ such that

$$\begin{pmatrix} an_1 & am_1 \\ bn_2 & bm_2 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix}$$

takes values in $2\pi\mathbb{Z}$, so that $\exp(i(nu+mv)) = 1$ for all $(n,m) \in \operatorname{Supp} \hat{G}$. So for any point $(x,y) \in \mathbb{T}$, $(x+u, y+v) \in \mathbb{T}$ is a distinct point with the same observation curve, and no Takens map can distinguish between (x,y) and $(x+u, y+v)$.

□

4 Klein bottle

Let $\mathbb{K}$ be the Klein bottle obtained as the quotient of the torus $\mathbb{T}$ by the automorphism $\kappa : (x,y) \mapsto (x+\pi, -y)$. We wish to characterize good observations on $\mathbb{K}$. However, the irrational flow on the $\mathbb{T}$ is not κ-invariant since κ is orientation reversing in the y coordinate. We therefore consider the closest approximation to the irrational winding by restricting it to the fundamental domain $[0, 2\pi] \times [0, \pi]$ and smoothing it out on the boundary circles $y = 0, \pi$. For $\alpha, \beta \in \mathbb{R}$ with $0 < \alpha/\beta << 1$ irrational, let X_ϵ be a vector field on the rectangle given by

$$X_\epsilon(x,y) = \begin{cases} (\alpha, \rho(y)) & 0 \leq y \leq \epsilon \\ (\alpha, \beta) & \epsilon < y \leq \pi - \epsilon \\ (\alpha, \rho(\pi + \epsilon - y)) & \pi - \epsilon < y \leq \pi \end{cases}$$

where ρ is a smooth function on a neighborhood of $[0, \epsilon]$ with $\rho(0) = 0$, $\rho(\epsilon) = \beta$ making X_ϵ smooth. For example, $\rho = \beta \exp(1/(y/\epsilon - 1)^2 - 1)$. Then X_ϵ extends uniquely to a κ-invariant vector field on on $\mathbb{T}$, and therefore induces a vector field on $\mathbb{K}$.

Theorem 4.1. *Let $G : \mathbb{T} \to \mathbb{R}$ be a κ-invariant function on $\mathbb{T}$. For fixed $N\tau$, the Takens map*

$$\Psi_\tau^N : \mathbb{K} \to \mathbb{R}$$

induced by G and X_ϵ for arbitrarily small ϵ and slope $\alpha/\beta << 1$ is an embedding if and only if the following conditions hold:

1. *$G(x, \pi)$ and $G(x, 0)$ have period π in x do not differ by a shift*
2. Supp $\hat{G}$ *generates* $\mathbb{Z}^2$

Proof. Suppose G is good for X_ϵ. Since X_ϵ flows horizontally at $y = 0, \pi$, condition 1) must hold so that each point is uniquely determined by its observation curve. Condition 2) must hold as well, since X_ϵ is given by an irrational winding away from the ϵ-neighborhood of $y = 0, \pi$ and the same argument as in Theorem 3.1 above applies for sufficiently shallow slope α/β because $N\tau$ is fixed.

Conversely, suppose conditions 1) and 2) hold. X_ϵ is given by an irrational flow away from the ϵ-neighborhood of $y = 0, \pi$. Furthermore, any point in the ϵ-strip with $y \neq 0, \pi$ may be flowed to a point where X_ϵ has irrational slope. The same argument as in Theorem 3.1 shows that the Takens map restricts to an embedding on $y \neq 0, \pi$.

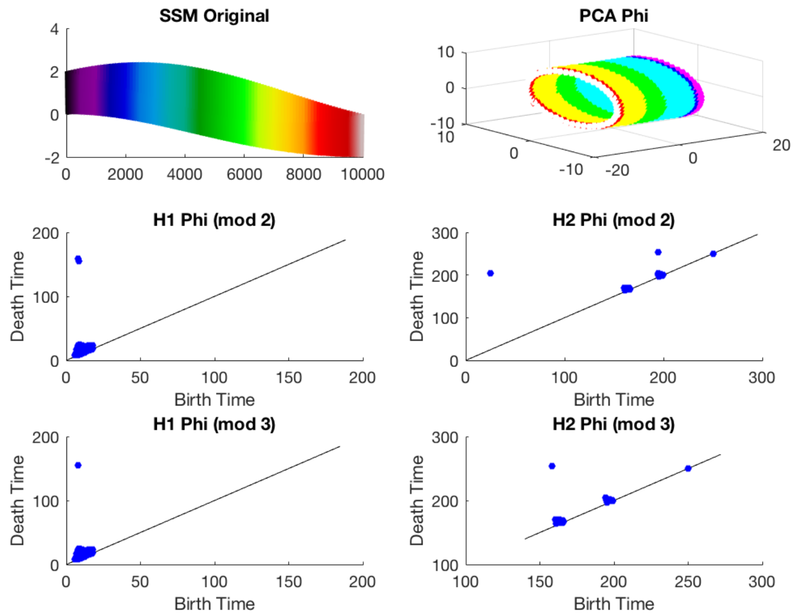
By condition 1, the observation curve of a point (x, y) where $y = 0, \pi$ uniquely determines x modulo π, and is periodic and therefore distinct from any observation curve for $y \neq 0, \pi$. So each point is uniquely determined by its observation curve as per condition 2) of Theorem 2.1.

It remains to show that the Takens map is immersive at $y = 0, \pi$. If not, then $\frac{\partial G}{\partial y}$ vanishes on the circles $y = 0, \pi$, a neighorhood about which Ψ_τ^N would fail to immerse, a contradiction. □

4.1 Example

Let $G(x,y) = \cos(2x) + \cos(x)\sin(y) + \cos(y)$. The Fourier coefficients of G are supported at $(\pm 2, 0), (\pm 1, \pm 1), (0, \pm 1)$, which generates $\mathbb{Z}^2$. Also we have $G(x,0) = \cos(2x) + 1$ and $G(x,0) = \cos(2x) - 1$, which are distinct and doubly periodic. So G is good for X_ϵ.

Accordingly, a sliding window persistence of $G(x,y)$ and X_ϵ shows Klein bottle homology—an



integral homology class in first degree, and a binary class in degree 1 and 2.

5 Proof of Theorem 2.1

Let $s \geq 0$ be a time parameter for p such that

$$\mathcal{L}_X^{\wedge J} dG$$

is nonzero at $\gamma_p(s)$. Write $\tilde{p} = \gamma_p(s)$ and $\mathcal{J}_k$ for the set of all strictly increasing m-tuples $J = (j_1, j_2, \ldots, j_m)$ with degree

$$j_1 + j_2 + \ldots + j_m = k$$

satisfying

$$\mathcal{L}_X^{\wedge J} dG|_{\tilde{p}} \neq 0$$

Fix $n > 0$ to be the minimal integer for which $\mathcal{J}_n$ is nonempty (possible by condition 1 above).

Let $A(t)$ be the m by n matrix with $(k,j)^{\text{th}}$ entry

$$A_{k,j}(t) = \frac{i_k^j (t-s)^j}{j!}$$

and $L : T_{\tilde{p}}M \to \mathbb{R}^n$ the linear map given by $\mathcal{L}_X^j dG|_{\tilde{p}}$ in the j^{th} coordinate. So the k^{th} component of the composition $A(t) \circ L$ yields the n^{th} order Taylor polynomial about $t = s$ of the cotangent curve $d(G \circ \psi_{i_k t})|_p$.

By Cauchy-Binet formula applied to $A(t)$ and L, the top exterior product

$$\omega_p^I(t) = \bigwedge_{i_k \in I} d(G \circ \psi_{i_k t})|_p$$

has n^{th} order Taylor series expansion about $t = s$ with n^{th} coefficient

$$C_n = \frac{\det(V)}{a_n} \sum_{J \in \mathcal{J}_n} |I^J| \cdot \mathcal{L}_X^{\wedge J} dG|_{\tilde{p}}$$

where

- a_n is a nonzero constant depending only on n
- $|I^J| = \prod i_k^{j_k - k + 1}$

and

$$\det(V) = \prod_{k<k'} (i_{k'} - i_k) \neq 0$$

is the nonzero determinant of the $m \times m$ Vandermonde matrix V with $(k, j)^{\text{th}}$ entry

$$V_{k,j} = {i_k}^{j-1}$$

where we take $0^0 = 1$.

$$C_n \neq 0$$

is nonzero. Hence we may choose a time $\eta > s$ sufficiently close to s so that the Taylor error $R_p^n(\eta)$ is small and the inequality

$$\omega_q^I(\eta) \neq 0$$

holds for all q in a neighborhood of p, and this property remains invariant under shrinking η closer to s. By compactness of M there is a finite collection of triples (I_r, η_r, s_r) such that the collection of m-forms

$$\{\omega^{I_r}(\eta_r)\}$$

do not all vanish at any given point of M and the cotangent vectors

$$\{d(G \circ \psi_{i_k \eta_r})|_q\}_{i_k \in I_r}$$

specified by I_r are linearly independent. Choose $\tau > 0$ small enough so that there is an integer multiple of τ lying in the interval (s_r, η_r) for each r. Then the Takens map Ψ_τ^N is an immersion for all $N > 0$ bounding I_r and η_r/τ.

So Ψ_τ^N is locally injective and the difference map

$$\Psi_\tau^N(p) - \Psi_\tau^N(q)$$

does not vanish for all $p \neq q$ in an open neighborhood U of the diagonal in $M \times M$, and this property is invariant under scaling $N \mapsto Nd$ and $\tau \mapsto \tau/d$ for an integer $d > 0$ (with U fixed).

For distinct $(p, q) \in M \times M \setminus U$, we may shrink τ so that g_p and g_q are distinguished on some integer multiple of τ and $\Psi_\tau(p) \neq \Psi_\tau(q)$. By compactness of $M \times M \setminus U$, there is a uniform choice of τ and N making Ψ_τ^N injective, hence an embedding. $\square$

References

[1] Christopher J. Tralie; Jose A. Perea, *(Quasi)Periodicity Quantification in Video Data, Using Topology*

[2] Floris Takens; *Detecting strange attractors in turbulence*