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Galilean Transformations of Kinetic Energy, Work, and Potential Energy

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Work, kinetic energy (KE), and potential energy (PE) are key physics concepts, taught in essentially every introductory physics course along with the physical laws such as the work-kinetic energy theorem and the conservation of mechanical energy, which describe how the values of those quantities change during various processes. Quite appropriately for introductory courses, little attention is paid to how those concepts play out when implemented via observations in reference frames moving with respect to each other.

For most introductory course work, it is sufficient to argue that the laws of physics are the same (have the same form) in all inertial reference frames, and hence we may choose any convenient reference frame to describe a situation, carry out the calculations, and interpret the results. There are almost no practical situations in which viewing a process from another reference frame is required. However, as we shall see, understanding how these properties change when transforming from one reference frame to another illuminates some of the subtleties associated with the concepts. Furthermore, the formalism developed in what follows could serve as a prelude to Lorentz transformations and special relativity.

Within the context of Newtonian mechanics, transforming observations from one reference frame to another is carried out formally by using the Galilean transformation equations to find out how work, KE, and PE change (or might change) when described in reference frames that are moving with respect to each other with constant velocities.[1] We are not surprised to recognize that the KE $(\frac{1}{2}mv^2)$ associated with an object depends on the reference frame since an object's velocity depends on the reference frame used to describe the situation. It is less obvious, though easily verified, that the *change* in KE also depends on the reference frame. And it may be very surprising to find that the work done on an object is also frame dependent. For PE, there is a bit of a muddle in the literature. Several recent papers[2-4] assert, incorrectly as we shall see, that changes in potential energy are frame dependent.[1] Another paper[5] claims that conservation of energy can apparently be violated in a moving reference frame. In this paper, we examine frame dependence or independence of KE, work, and PE in a simple context and then discuss why it is easy to arrive at erroneous conclusions.

In a stimulating recent article,[4] Ginsberg notes many of the subtleties associated with the interpretation of KE, work, and PE as viewed from reference frames moving relative to one another. The basic argument is the following: the changes in the object's KE are different as viewed in different frames. Hence, to have the work-KE theorem $\Delta K = W$ be the same in all inertial frames, the work done on the object (by an external force, of course) must be frame dependent.[1,6] (In what follows, for the sake of typographical simplicity we will use the symbol K in equations to represent KE.)

So far, the logic is impeccable. But then Ginsberg[4] and others[2,3] claim that if there is a change in PE (the object is acted on by a conservative force), that change of PE (equal, as usual, to the negative of the work done on the object by the conservative force) must also be frame dependent. That claim should raise a red flag: We all know that PE (sometimes called configuration energy) is energy associated with the relative positions of the objects that make up a system. While the displacements of each of the objects over time is frame dependent (and that

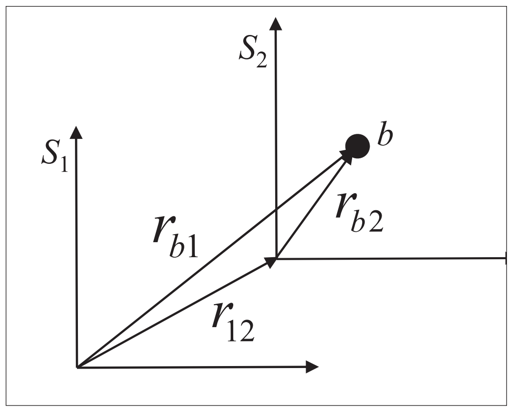

Fig. 1. The position vectors of an object *b* relative to two reference frames S_1 and S_2. r_{21} is the position vector of the origin of S_2 relative to the origin of S_1.

is why the work done on the object is frame dependent), the relative positions of the objects at any one time are independent of the reference frame as shown explicitly below. Since PE depends only on those relative positions, it should be independent of the reference frame. How do we reconcile those two claims?

These kinds of issues are rightly, we believe, ignored in introductory physics courses and in the textbooks written for those courses. As one counterexample, we note that Chapter 6 of Ref. 7 provides a thorough discussion of how KE and changes in KE depend on the reference frame but nevertheless conservation of energy has the same form, independent of the reference frame.

In what follows, we will show in a very general way that Ginsberg's claims are correct if we focus on the motion of just a single object. But when our attention turns to a system of objects, it turns out that potential energy and changes in potential energy are in fact frame independent. All our results are in agreement with those of Ref. 1, but we work in a simpler context, which, we hope, will make the basic issues and their resolution more transparent and accessible to both physics instructors and their students.

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General formulation

Let's start from the basics by introducing two frames of reference S_1 and S_2 with position vectors for an object b as shown in Fig. 1. The position vectors are related by

$$\boldsymbol{r}_{b1}(t) = \boldsymbol{r}_{21}(t) + \boldsymbol{r}_{b2}(t). \tag{1}$$

Throughout this paper we will use the subscript notation introduced in Eq. (1) and Fig. 1: $\boldsymbol{r}_{b1}$ is the position vector of object b relative to the origin of frame of reference 1, and so on. If we differentiate Eq. (1) with respect to time, we find that the velocities are related by

$$\boldsymbol{v}_{b1}(t) = \boldsymbol{v}_{21}(t) + \boldsymbol{v}_{b2}(t), \tag{2}$$

where $\boldsymbol{v}_{b1}$ is the velocity of b with respect to S_1 and $\boldsymbol{v}_{21}$ is the velocity of S_2 relative to S_1. Equation (2) is just the standard Galilean relative velocity expression. Implicit in this Newtonian treatment is the notion that time is the same for all reference frames. Equations (1) and (2), along with the invariance of time, are the practical forms of the Galilean transformation equations.

If we differentiate Eq. (2) with respect to time, we see that the accelerations relative to the two frames are the same ($\boldsymbol{a}_{b1} = \boldsymbol{a}_{b2}$) as long as the frames' relative velocity $\boldsymbol{v}_{21}$ is constant, which is required by our restriction to inertial reference frames. Consequently, the (net) force exerted on object b is also frame-independent.

We now demonstrate two important (and well-known) facts: (1) the displacement of object b over a time interval from t_i to t_f is not the same for S_1 and S_2, and (2) if we consider two objects (say, b and c), their relative position vector at a particular time is the same in both S_1 and S_2. Using Eq. (1), we see that the displacement $\Delta\boldsymbol{r}_{b1}$ relative to S_1 is related to the displacement $\Delta\boldsymbol{r}_{b2}$ relative to S_2 by

$$\begin{aligned}\Delta\boldsymbol{r}_{b1} &= \boldsymbol{r}_{b1}(t_f) - \boldsymbol{r}_{b1}(t_i) = \boldsymbol{r}_{21}(t_f) + \boldsymbol{r}_{b2}(t_f) - \boldsymbol{r}_{21}(t_i) - \boldsymbol{r}_{b2}(t_i) \\ &= \Delta\boldsymbol{r}_{b2} + \Delta\boldsymbol{r}_{21},\end{aligned} \tag{3}$$

where $\Delta\boldsymbol{r}_{21}$ is the displacement of S_2 with respect to S_1 during the relevant time interval.

Now let's use Eq. (1) to express the relative position of two objects b and c at a single time:

$$\begin{aligned}\boldsymbol{r}_{cb1} &= \boldsymbol{r}_{c1} - \boldsymbol{r}_{b1} \\ &= \boldsymbol{r}_{21} + \boldsymbol{r}_{c2} - (\boldsymbol{r}_{21} + \boldsymbol{r}_{b2}) \\ &= \boldsymbol{r}_{c2} - \boldsymbol{r}_{b2} = \boldsymbol{r}_{cb2}\ .\end{aligned} \tag{4}$$

We see that the relative position vector $\boldsymbol{r}_{cb}$ is frame independent. Eq. (3) will be important when we consider work in the two frames, while Eq. (4) will be important for PE considerations.

Kinetic energy

Let's now consider the change in an object's KE, which in S_1 is given by $K_{b1} = \frac{1}{2} m v_{b1}^2$ with an analogous expression for KE in S_2. It is easy to see that the change in KE over a time interval $\Delta t = t_f - t_i$ will, generally, be different for the two reference frames. Starting with the change in KE ΔK_1 in S_1 and using Eq. (2) yields

$$\begin{aligned}\Delta K_1 &= \tfrac{1}{2} m_b v_{b1}^2(t_f) - \tfrac{1}{2} m_b v_{b1}^2(t_i) = \\ &= \tfrac{1}{2} m_b v_{b2}^2(t_f) - \tfrac{1}{2} m_b v_{b2}^2(t_i) + m_b\left[\boldsymbol{v}_{b2}(t_f) - \boldsymbol{v}_{b2}(t_i)\right]\cdot\boldsymbol{v}_{21}\,,\end{aligned} \tag{5}$$

where we recognize that $\Delta K_2 = \frac{1}{2} m_b v_{b2}^2(t_f) - \frac{1}{2} m_b v_{b2}^2(t_i)$ is the change of the object's KE as viewed from S_2. Thus, we see that

$$\Delta K_1 = \Delta K_2 + m_b\left[\boldsymbol{v}_{b2}(t_f) - \boldsymbol{v}_{b2}(t_i)\right]\cdot\boldsymbol{v}_{21}\,. \tag{6}$$

Note that the last term in Eq. (6) is zero only if the relative velocity of the two frames is zero or orthogonal to the change in momentum of the object and hence orthogonal to the net force acting on the object.

Work and the work-kinetic energy theorem

Let's now calculate the work done on object b as viewed in the two reference frames and see if the results match the appropriate change in KE in each of the frames. Assume for the sake of simplicity a constant force is exerted on object b, resulting in constant acceleration $\boldsymbol{a}$. (Later we shall see that this is not a fundamental restriction). W_{b1} will denote the work done on the object b from the point of view of S_1 with a corresponding expression for S_2. For the time interval Δt, we obtain, using $\boldsymbol{F}_{\text{net on } b} = m_b\boldsymbol{a}$,

$$\begin{aligned}W_{b1} &= m_b\boldsymbol{a}\cdot\left[\boldsymbol{r}_{b1}(t_f) - \boldsymbol{r}_{b1}(t_i)\right] \\ &= m_b\boldsymbol{a}\cdot\left[\boldsymbol{r}_{21}(t_f) + \boldsymbol{r}_{b2}(t_f) - \boldsymbol{r}_{21}(t_i) - \boldsymbol{r}_{b2}(t_i)\right] \\ &= W_{b2} + m_b\boldsymbol{a}\cdot\Delta\boldsymbol{r}_{21}\ .\end{aligned} \tag{7}$$

We see that the two work terms W_{b1} and W_{b2} are not the same unless (1) the acceleration of the object is zero, in which case, there is no net force acting on the object, or (2) if the object's acceleration (or equivalently the net force) is perpendicular to the displacement of S_2 relative to S_1.

Let's call the last term in Eq. (7) "extra work." Note that the last term involves the force exerted on mass b and the displacement of frame 2 with respect to frame 1, rather than the displacement of the object.

We easily see that the last term in Eq. (6) is equal to the "extra work" term in Eq. (7):

$$\begin{aligned}\left[m_b\boldsymbol{v}_{b2}(t_f) - m_b\boldsymbol{v}_{b2}(t_i)\right]\cdot\boldsymbol{v}_{21}(t_f) &= \frac{m_b\left[\boldsymbol{v}_{b2}(t_f) - \boldsymbol{v}_{b2}(t_i)\right]}{\Delta t}\cdot\boldsymbol{v}_{21}(t_f)\,\Delta t \\ &= m_b\boldsymbol{a}\cdot\Delta\boldsymbol{r}_{21}\,.\end{aligned} \tag{8}$$

Note that m_b drops out (not surprisingly) since the KE difference and the "extra work" are due only to a transformation between reference frames and hence should be independent of the object we are tracking.

We see that the difference between the two changes in KE is matched by a corresponding change in the work done on the object, and we are relieved to find that the work-KE theorem holds for each frame

$$\Delta K_1 = W_1 \text{ and } \Delta K_2 = W_2. \tag{9}$$

The important conclusion is that both the change in KE and the work done on the object are frame dependent, but the "law" ($\Delta K = W$) is the same in both frames: the law is form invariant (under Galilean transformations).[8]

Before moving on, we note that the result for the more general case of non-constant acceleration follows by breaking the time interval up into small time steps over which the acceleration is approximately constant. Since the results hold for each time step, they hold for the finite time period as well.

Potential energy

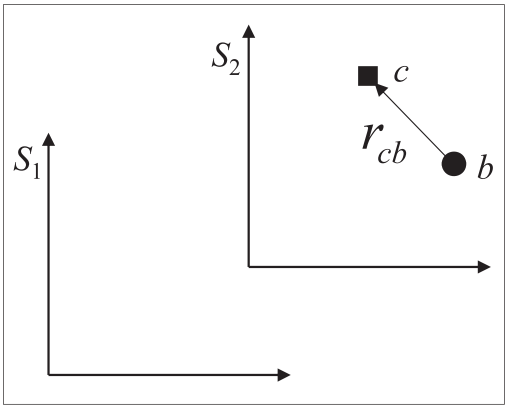

Fig. 2. A system with two objects *b* and *c*. The vector r_{cb} represents the relative position of *c* with respect to *b*. That vector is frame independent as shown in the text.

We now turn our attention to the transformation properties of PE. We will use ΔU to denote the change in PE. We want to see under what conditions the conservation of mechanical energy $\Delta K = -\Delta U$ is applicable in the two reference frames.

For simplicity's sake, let's consider a system consisting of two objects b and c as shown in Fig. 2, interacting with each other through conservative forces with no external forces acting on either of the objects. (The extension to external forces is covered in Ref. 1.) With our new system, there are two work contributions to the "system work" $W^{(s)}$ (the work done by forces internal to the system). For example, in frame S_1 we have $W_1^{(s)} = W_{b1} + W_{c1}$, where

$$W_{b1} = m_b \boldsymbol{a}_b \cdot [\boldsymbol{r}_{b1}(t_f) - \boldsymbol{r}_{b1}(t_i)] \qquad (10)$$
$$W_{c1} = m_c \boldsymbol{a}_c \cdot [\boldsymbol{r}_{c1}(t_f) - \boldsymbol{r}_{c1}(t_i)].$$

Using Newton's third law, we may write the total system work as

$$\begin{aligned} W_1^{(s)} &= m_c \boldsymbol{a}_c \cdot \left[\boldsymbol{r}_{c1}(t_f) - \boldsymbol{r}_{c1}(t_i) - \boldsymbol{r}_{b1}(t_f) + \boldsymbol{r}_{b1}(t_i)\right] \\ &= m_c \boldsymbol{a}_c \cdot \left[\boldsymbol{r}_{cb}(t_f) - \boldsymbol{r}_{cb}(t_i)\right] \\ &= W_2^{(s)}, \end{aligned} \qquad (11)$$

where the last equality follows from the frame independence of the relative position vectors $\boldsymbol{r}_{cb}$. This net system work is frame independent. Defining the change in potential energy as the negative of the system work, we see that

$$\Delta U = -W^{(s)} = -m_c \boldsymbol{a}_c \cdot [\boldsymbol{r}_{cb}(t_f) - \boldsymbol{r}_{cb}(t_i)]. \qquad (12)$$

is frame independent as asserted before. In the "Discussion" section, we will examine how other authors came to the (incorrect) conclusion that potential energy is, in general, frame dependent.

Conservation of mechanical energy

The principle of conservation of mechanical energy relates the change in KE to the change in PE. When there are no external forces acting on the system and all internal forces are conservative, that principle is expressed as

$$\Delta K + \Delta U = 0. \qquad (13)$$

It looks like we are in trouble because we saw that ΔK is frame dependent but ΔU is not. The resolution lies in the recognition that not all contributions to KE behave the same under Galilean transformations. In fact, the key strategy is the traditional separation of KE into two parts: the KE of the center of mass (CM) of the system, and the KE associated with the motion of the system's components with respect to the CM. The KE of the system's CM does change in going from one frame to another, but the KE relative to the CM does not change under a Galilean transformation. We now show that the ΔK that appears in Eq. (13) is the change in the "internal" KE with respect to the CM and that change is frame independent.

We write object b's velocity with respect to S_1 as

$$\boldsymbol{v}_{b1} = \boldsymbol{v}_{b\mathrm{CM}} + \boldsymbol{v}_{\mathrm{CM}1}, \qquad (14)$$

where $\boldsymbol{v}_{b\mathrm{CM}}$ is the velocity of b relative to the system's CM and $\boldsymbol{v}_{\mathrm{CM}1}$ is the velocity of the system's CM relative to S_1. Since the relative positions within the system are frame independent, so are the relative velocities such as $\boldsymbol{v}_{b\mathrm{CM}}$.

Using Eq. (14) and a bit of straightforward algebra, we express the change in the system's KE as seen in S_1 as

$$\begin{aligned} \Delta K_1 &= K_1(t_f) - K_1(t_i) \\ &= \Delta K_{b\mathrm{CM}} + \Delta K_{c\mathrm{CM}} + \left[m_b \boldsymbol{v}_{b\mathrm{CM}}(t_f) - m_b \boldsymbol{v}_{b\mathrm{CM}}(t_i)\right] \cdot \boldsymbol{v}_{\mathrm{CM}1} \\ &\quad + \left[m_c \boldsymbol{v}_{c\mathrm{CM}}(t_f) - m_c \boldsymbol{v}_{c\mathrm{CM}}(t_i)\right] \cdot \boldsymbol{v}_{\mathrm{CM}1}. \end{aligned} \qquad (15)$$

We recognize that the terms in the square brackets in Eq. (15) are just the changes of linear momentum relative to the CM of the system. By the conservation of linear momentum, the sum of the last two terms vanishes and since $\Delta K_{c\mathrm{CM}}$ and $\Delta K_{b\mathrm{CM}}$ are invariant under Galilean transformations, we conclude that $\Delta K_1 = \Delta K_2$, just what we need to "rescue" the principle of conservation of mechanical energy.

Discussion

We have shown explicitly that KE, changes in KE, and work are generally frame-dependent quantities. On the other hand, we have seen that the laws that relate the change in KE to work or to the corresponding change in PE are form invariant under Galilean transformations. The lesson to be drawn from this cautionary tale is that it is wrong in principle to talk about the KE (or the change of KE) of an object and the work done on an object without noting the reference frame being used to describe the situation. However, as long as we restrict ourselves to one (inertial) reference frame, no fundamental issues should arise.

We mentioned previously that several papers[2-4] claim that PE is not invariant under a change of reference frame. Where did the authors go wrong? The difficulty seems to have arisen from their focus on systems in which one of the objects (e.g., Earth or a spring mounted rigidly to Earth) has a mass much larger than the other mass considered. In that case, in a frame 1 in which Earth is initially not moving, its displacement during the fall of the smaller mass is so small that we can neglect the work done on Earth. Now let us consider viewing the situation from a frame 2 moving with respect to Earth. According to Eq. (7), there is a significant displacement of Earth. Of course, there is a force exerted on it (Newton's third law!), so the work done on Earth as viewed in frame 2 is given by $W_{\text{Earth2}} = -\boldsymbol{F}_{\text{onEarth}} \cdot \Delta\boldsymbol{r}_{21}$, which is just what is needed to cancel the "extra work" on the small mass $m\boldsymbol{a} \cdot \Delta\boldsymbol{r}_{21}$. Neglecting the work done on Earth as viewed from frame 2 is what leads to the erroneous claims that the change in PE is frame dependent. The same argument solves the problem of an apparent violation of conservation of energy discussed in Ref. 5.

The analysis presented here could be extended in various ways. Ghanbari[3] shows how work done by non-conservative forces affects the transformation properties in the work-kinetic energy theorem. The extension of these ideas to non-inertial references frames is treated in Ref. 2. Applications to multiparticle systems are treated in Ref. 5, where it is emphasized that it is critical to consider the interactions among the constituents of the system to satisfy conservation of energy.

We end by noting that the frame dependence of KE calls into question the pedagogical strategy[9-11] of treating energy as a substance to aid student understanding of different forms of energy and the transfer of energy within a system or between systems. In fact, Ref. 9 notes that the frame dependence of KE poses difficulties for the substance picture for energy. However, as we noted earlier, for introductory physics situations, there is rarely, if ever, a need to think about KE in different reference frames. As long as we stick to a single reference frame, the substance picture should be fine.

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