Solving an “unsolvable” projectile-motion problem

David Montalvo

Citation: The Physics Teacher **37**, 226 (1999); doi: 10.1119/1.880233
View online: http://dx.doi.org/10.1119/1.880233
View Table of Contents: http://aapt.scitation.org/toc/pte/37/4
Published by the American Association of Physics Teachers

N O T E

Solving an "Unsolvable" Projectile-Motion Problem

David Montalvo, Upper Merion Area High School, 435 Crossfield Road, King of Prussia, PA 19406

"Some people will never learn anything; for this reason, because they understand everything too soon."
—Alexander Pope (1688-1744)

If there were any rules about teaching physics listed anywhere, "Don't assign problems to the students that you, yourself, cannot solve" would probably top the list. And yet this bias is an unfortunate one: a closer examination of some of these problems can lead to new and valuable understandings.

My good friend and colleague Pete Vreeland once gave this seemingly simple projectile-motion problem to his honors students:

> *The Gunner's Problem.* You are manning a piece of field artillery that can fire a projectile at a speed v at any angle between 0° and 90°. You wish to hit a tank that is charging at your position at a speed u. At what angle should you aim if the tank is a horizontal distance D away?

The distance the projectile travels in the x-direction and the distance the tank travels must add up to D upon impact. Assuming impact at time t:

$$D = (v\cos\theta)t + ut \qquad (1)$$

From the equations for kinematics, we can also solve for the time t:

$$t = (2v\sin\theta)/g \qquad (2)$$

Substituting Eq. (2) into Eq. (1), and multiplying through by g, gives:

$$Dg = 2v^2\sin\theta\cos\theta + 2uv\sin\theta \qquad (3)$$

Unfortunately, we have run into a small mathematical obstacle in that Eq. (3) isn't explicitly solvable for θ. Even using the well-known trigonometric identity to substitute $\sin(2\theta)$ for $2\sin\theta\cos\theta$ won't help. At this juncture, we might be inclined to despair while wondering how it came to be that the military, for whom this problem is a practical one, could ever hit a moving target! All is not lost, however. With the help of those ubiquitous graphing calculators, so newly embraced by math departments everywhere, we can find solutions for θ. First rearrange Eq. (3):

$$2v^2\sin\theta\cos\theta + 2uv\sin\theta - Dg = 0 \qquad (4)$$

Then type in the equation as a function and use the trace feature to find the "zeroes." First, however, values must be chosen for the constants. Let us take 1000 m for the target distance, 200 m/s for the projectile speed, 10 m/s for the target speed, and 10 m/s^2 for the acceleration due to gravity. After dividing through by 2000 we have:

$$40\sin\theta\cos\theta + 2\sin\theta - 5 = 0 \qquad (5)$$

Now type the equation into the graphing calculator in the following form:

$$Y = 40\sin X\cos X + 2\sin X - 5 \qquad (6)$$

The zeroes of Eq. (6) will give us our answer! Setting the X-range from 0 to 90 (making sure that our calculator is in "degrees" mode), we "zero in" on two solutions: approximately 7° and 86°. Why two angles? The smaller angle corresponds to a "quick hit," which ensures that the "hang time" will be short. We can suppose this to be the best choice when dealing with an armed opponent! The large angle conceivably could be a better alternative when faced with a foe whose forward face is heavily armored, but who is vulnerable from above. Needless to say, the military employs computers that use numerical methods of approximating solutions similar to those the calculator uses.

Although we've solved the problem as originally stated, we've hardly even begun to explore the possibilities of the situation; there is much to be learned. The function is of the form:

$$Y = A\sin X\cos X + B\sin X - C \qquad (7)$$

where A, B, and C are positive constants. Between X = 0 and 90, the function is concave downward, crossing the y-axis at –C, which in our case is equal to $-Dg$. The smaller the range to the target D, the more the curve is shifted upward, guaranteeing that the solution angles will be nearly 0° and close to or beyond 90°. (Try it!) This is, of course, a tough situation for the gunner: either fire almost directly at the target due to its proximity, or actually fire *backwards* at a high angle in order to hit *after* the target has overrun your position! On the other hand, if D is increased, the curve shifts downward and the solution angles approach each other, which eventually leads to the critical situation where only one angle will hit the target (the curve's maximum just touches the x-axis). If D is increased any more, no angle will hit the target: its position remains beyond the range of your projectile, despite the target's approach; the gunner had better wait to fire! If we hold D constant, and instead vary the target or projectile speed, we notice that the curve changes shape, while still crossing the y-axis at $-Dg$. Larger speeds for either the target or projectile cause the curve to peak higher and open out farther in the $+x$-direction. (Try it!) Thus the target angles again approach 0° or 90° (and beyond); a fast projectile or target essentially presents a problem similar to a close target.

It would seem that the possibilities inherent in this initially troublesome situation (which we almost shied away from as unsolvable) are endless and quite educational. We soon realize the possibilities provided by the graphing calculator, as well: we can analyze functions that contain implicit solutions, manipulate them, and find solutions for them, while having the students interpret changes in the shape of the functions due to changes in any given parameter. Actually, you need not even attack an implicit function; you could graph the "normal" y-as-a-function-of-x projectile motion equation, and see the effect of changing any of the parameters. You could even make the important experimental point: vary only one parameter at a time because varying two parameters at once gives results that are hard to analyze.

Nor have we exhausted the initial situation! For instance, let the projectile's angle be fixed, and let its *speed* be variable, as an arrow shot from a bow: *The Yeoman's Problem.* Glancing at Eq. (4), and keeping in mind that v is now the variable, we see that it is quadratic. Calling upon the quadratic formula, and discarding the negative solution, it can be shown that

$$v = (1/2\cos\theta)[(u^2 + 2Dg\cot\theta)^{1/2} - u] \qquad (8)$$

Or perhaps you are interested in the target speed u required to meet the projectile: *The Wide Receiver's Problem*. Solving (4) for u gives

$$u = (Dg/2v\sin\theta) - v\cos\theta \qquad (9)$$

which just begs for analysis! What does the $Dg/2v\sin\theta$ represent?

Oddly enough, reversing the original problem, i.e., letting the moving object fire the projectile at a stationary target, is far simpler. The reader will easily show that the "normal" projectile-motion equations apply with only a relatively small modification to the projectile's initial x-direction speed!