Controls Engineering from Scratch

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0.1 Introduction

The starting point for learning control theory is a difficult point to choose. The overhead in terms of math for learning about controls can be daunting at times, so dismissing some of the foundation-level skills can be very tempting. I would be remiss to not stress the importance of this foundation. In many ways, learning controls is like adding a rich set of powerful tools to an theoretical, analytical, and design-based toolbox. Without paying at least some attention to the details of the foundation, the tools may take a very long time to make sense. Additionally, these tools are changing their appearances and their uses rapidly, and more and more tools are joining their ranks as the horizon of controls research pushes forward. We never know where the next big advance in control theory will come from, and leaving out parts of the foundation may make it very difficult to think of the unconventional methods and approaches needed to disrupt this field. I encourage the reader to move carefully through the introductory topics, no matter how elementary they may seem. In return, I will try my hardest to make the explanations concise, clear, to mention why the skills are important and what they might be used for, and to not belabor any points too much.

0.2 Overview

We will begin with a brief (I really do mean brief) introduction to set theory. This material is often overlooked or completely left out in the various math sequences that lead to controls. There are many, many parts of set theory that we will not cover, but we will cover the crucial components that allow us to learn these parts at another time. We will resort to some loose definitions at times so hopefully we can also avoid the pedantry traditionally associated with learning this material.

1 Sets and Functions

1.1 Sets

Mathematicians have been debating for quite some time about the countless implications of defining "set theory" in what seems like an arbitrarily high number of different, particular ways. To avoid spending too much time on this item, let us define a "set" in the following way:

Set A collection of unique objects

The objects being unique suggests that an object cannot be repeated in a set. This definition will certainly displease most mathematical purists, but it will also get us safely through the fundamentals of set theory needed to understand the tools of controls. This definition implies that a set can be of arbitrary length, and its elements can be of arbitrary type. We can delve deeper into the topic and derive all sorts of particular restrictions on the types of sets we can define using this definition, but that is better suited for a textbook chapter on set theory. For now, let us continue with this definition. Some important sets include the empty set, natural numbers, integers, rational numbers, irrational numbers, and real numbers.

1.2 The Empty Set

The empty set is a set without any elements (note that this is not the same as the set containing only 0). The empty set can be a difficult topic to grasp at first. The example that helps me is the set

$$S = \{1, 2\}$$

Removing the last element in the set yields

$$S' = \{1\}$$

It is very clear that both S and S' are sets by every aspect of the definition we proposed. Removing the last element, however, yields a set with nothing in it.

$$S'' = \{\} = \emptyset$$

The key point here is that S'' is still in fact a set by our definition. Its elements have the same type as the elements of S and S', the concept of this set is only somewhat convoluted because it happens to have zero elements versus two or one. The empty set may not seem very useful, but it is hugely important in defining, analyzing, and operating on non-empty sets. The other sets in our list of important sets require a little more knowledge to create. We need to introduce the idea of functions.

1.3 Functions

A function is a way of creating a directed association between two sets in an unambiguous way. Saying that the association is directed means that a function, f, may create associations from a set A to a set B. A different function entirely (known as an inverse, which may or may not exist) creates associations in the opposite direction (from a set B to a set A). From this we can define a function as the following:

Function A mapping from one set to another set

Once again, this definition neglects some of the nuance to the idea of a function, but it will serve its purpose to give us a concrete understanding of the math behind control theory. A less verbose way of describing these associations is by describing the function in the following way:

$$f : A \to B$$

This is read as "f maps A to B." To describe which particular elements of the sets are associated, we introduce the following notation:

$$f(a) = b, a \in A, b \in B$$

This is read as "f of a equals b where a is an element of A and b is an element of B." Let us look at an example of a function. Let $A = \{1, 2, 3\}$ and $B = \{2, 4, 6\}$. Let f be defined as $f : A \to B$, such that $f(1) = 2, f(2) = 4, f(3) = 6$. By inspection, it seems this function maps elements of A to elements of B whose values are two times the value of their corresponding element in A. This function might be called a *doubling* function. Let us look at another example of a function. Let $A = \{\text{"}\textit{new}\text{"}, \text{"}\textit{kind}\text{"}, \text{"}of\text{"}, \text{"}\textit{set}\text{"}\}$, and $B = \{2, 3, 4\}$. Let g be defined as $g : A \to B$ such that $g(\text{"}\textit{new}\text{"}) = 3, g(\text{"}\textit{kind}\text{"}) = 4, g(\text{"}of\text{"}) = 2, g(\text{"}\textit{set}\text{"}) = 3$. By inspection, it seems that this function maps elements of A to elements of B whose values represent the number of characters of their corresponding element in A. Notice that two elements of A, "*new*" and "*set*", both map to the element 3 in B. This does not make g any less of a function, but it does have implications as to what kind of inverse may be defined (we will revisit this point later). Also, notice that A and B look fundamentally different. A has four elements where B only has three, and elements of A seem to be integers where elements of B seem to be strings. Let us, for now, depart from these specific examples and make some key abstract observations about all functions under our definition.

- A function is an operation that transforms elements of one set into elements of another. These two sets are known as the domain and range of the function. A function maps *from* its domain *to* its range.

- A function must be defined for every element in the domain. No matter the type or number of elements in the domain and in the range, the function must map each element of the domain to an element of the range. However, it is not true that each element of the range must be mapped to by an element of the domain.

- The mapping must be unambiguous. This means that each element of the domain is mapped to one and only one element of the range. However, it is important to note that two or more elements of the domain may map to the same element in the range. This is necessary for the function to be defined for each element in the domain in cases where the domain has more elements than the range (like in the function g described above).

Any function we discuss under this definition must satisfy each of these rules, but there are no other restrictions on the domain, range, and mapping from the domain to the range for the function. There are many properties of the domain, range, and mapping of a function that can be used to characterize the function (cardinality of the domain and range, injectivity, surjectivity, bijectivity, etc.). We will cover some of these when we study the linear algebra needed for controls. Now that we know how to define a function, however, let us return to sets.

1.4 Subsets and Sets of Sets

By our definition of a set, the elements of a set can be of any type (including types that are different from other elements of the same set). This implies that the elements of a set may themselves be sets. An example of this would be the following: $S_0 = \emptyset, S_1 = \{1\}, S_2 = \{2\}, S_3 = \{3\}, S_4 = \{1,2\}, S_5 = \{1,3\}, S_6 = \{2,3\}, S_7 = \{1,2,3\}, S = \{S_0, S_1, S_2, S_3, S_4, S_5, S_6, S_7\}$. S is a set whose elements are themselves sets. Let us study carefully the sets which define S. All the elements of sets S_1, S_2, S_3, S_4, S_5 and S_6 are all also elements of S_7. This demonstrates the idea of a subset.

Subset A is a subset of B if and only if each element of A is also an element of B

Once again, we can write this in a less verbose way with the following notation:

$$A \subseteq B \iff \forall a \in A, a \in A \implies a \in B$$

This is read as "A is a subset of B if and only if for each element a in A, a being in A implies a is in B." This means that in our example, sets S_1 through S_6 are subsets of S_7. By this definition of a subset, however, S_0 and S_7 are necessarily also subsets of S_7. This can be generalized to say that for any set A, the empty set as well as the set A itself are subsets of A. These statements may seem odd, but it is necessary that both of these statements be logically true. Let us make one last observation with this example. The set S is the set of all subsets of S_7. This set has a name, it is denoted as the *powerset* of S_7, or, symbolically, 2^{S_7}.

1.5 Equality of Sets

It is important to recognize when two sets are equivalent.

Set Equals Sets A and B are equal if and only if $A \subseteq B$ and $B \subseteq A$

We can write this concisely as $A = B \iff (A \subseteq B) \wedge (B \subseteq A)$. Note that this implies the sets $A = 1, 2, 3$ and $B = 3, 2, 1$ are equal. It is important when constructing a powerset to remember that ordering of elements in a set does not matter; two equal sets with their elements in different orders should not be two separate elements of a powerset.

1.6 n-tuples and the Cartesian Product

Before we can introduce the idea of the Cartesian product, we must first introduce the idea of an ordered pair. An ordered pair is similar to a set of two elements; however, the definition of set equality we just defined does not extend to ordered pairs. This is because the ordered pairs (a, b) and (c, d) are only equal if $a = c$ and $b = d$. The ordered pair can be generalized to an n-tuple, or an ordered collection of n elements. We say that two n-tuples, $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$, are only equal if for each $a_i \in A$ and $b_i \in B$ where $i \in \{1, \ldots, n\}$, $a_i = b_i$. Additionally, an element of a tuple is not restricted to be unique. This means that an element of a tuple may be repeated. But for now, let us return to the Cartesian product. The Cartesian product of sets A and B is the set of all ordered pairs of the elements of A and B. We denote this with the following notation:

$$A \times B = \{(a, b) | a \in A, b \in B\}$$

This is read as "the Cartesian product of A and B is defined as the set of all ordered pairs (a, b) where a is an element of A and b is an element of B." For example, if $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$, $A \times B = \{(1, 4), (1, 5), (1, 6), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6)\}$. The Cartesian product is very useful in defining functions which map elements of two different sets to elements of the range. In cases like these, the domain is defined as the Cartesian product of the two sets being mapped from. For example, we could define the set $C = \{5, \ldots, 9\}$ and the function $f : A \times B \to C$ where f is defined as $f(a, b) = a + b$. $f(a, b)$ is read as "f of a and b." The generalization of ordered pairs as n-tuples is similar to the generalization of the Cartesian product of two sets to the Cartesian product of two or more sets. The Cartesian product of n sets $A_1, \ldots, A_n$ is denoted as $A_1 \times \cdots \times A_n$.

1.7 Set Union

The union of two sets is defined as follows:

$$A \cup B = \{x | (x \in A) \vee (x \in B)\}$$

This is read as "A union B is defined as the set of all elements x where x is in A or x is in B." As an example, let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$. Then $A \cup B = \{1, 2, 3, 4, 5\}$. An important observation is that if an element exists in both sets, the union of the sets only includes the element once. This is because, by our definition of a set, each element must be unique. Abstractly, the union of two sets is the set of all elements in either of the two sets. Two important set unions include the union of a set A with the empty set, and the union of a set A with itself. In these two cases $A \cup \emptyset = A$ and $A \cup A = A$. In fact, for sets A and B where $B \subseteq A$, $A \cup B = A$. This last example shows that determining a set union is a simple way to verify if one set is a subset of the other.

1.8 Set Intersection

The intersection of two sets is defined as follows:

$$A \cap B = \{x | (x \in A) \wedge (x \in B)\}$$

This is read as "A intersect B is defined as the set of all elements x where x is in A and x is in B." Let A and B be the same sets as defined in the set union example. Then $A \cap B = \{3\}$. Abstractly, the intersection of two sets is the set of all elements in both of the two sets. Two important set intersections include the intersection of a set A with the empty set, and the intersection of a set A with itself. In these two cases $A \cap \emptyset = \emptyset$ and $A \cap A = A$. In fact, for sets A and B where $B \subseteq A$, $A \cap B = B$. This last example shows that determining a set intersection is a simple way to verify if one set is a subset of the other. Note that the past two sections read very similarly. Be sure to note the key differences between set unions and set intersections.

1.9 Set Minus

Set minus is defined as follows:

$$A \setminus B = \{x | (x \in A) \wedge (x \notin B)\}$$

This is read as "A set minus B is defined as the set of all elements x where x is in A and x is not in B." Let A and B be the same sets as defined in the set union and set intersection example. Then $A \setminus B = \{1, 2\}$. Notice that in this example

$$(A \setminus B) \cup (A \cap B) = A$$

This is actually true for all sets A and B. Abstractly, a set set minus another set is the set of all elements in the first set but not in the second. Two important examples of this are a set A set minus the empty set, and a set A set minus itself. In these two cases $A \setminus \emptyset = A$ and $A \setminus A = \emptyset$.

1.10 Note on the Set of all Sets

Note that with our definition of a set, it is difficult to categorize set union, set intersection, and set minus as functions. This is because the domain and range are very difficult to define. We could attempt to define the set of all sets, a set with infinitely many elements, each of which are sets themselves. Let S be the set of all sets. Now we could attempt to define the domain as the Cartesian product of the set of all sets with itself, $S \times S$, since set union, intersection, and minus all form a set from two sets. This implies that the range would be the set of all sets. We could attempt to define a set union function $F : S \times S \to S$ as $F(A, B) = A \cup B, \forall A, B \in S$. We could define similar functions for set intersection and set minus. However, as mentioned before, the existence of the domain and range is difficult to prove with our definition of a set. This is because the set of all sets would, by definition, have to include itself. Our definition is not robust enough to handle this logical corner case. I should mention that this is not a point that should be overlooked without a second thought, so take a moment to think about the implications of defining such a set. But once again, we will leave this detail to the math textbooks. In the meantime, interpreting set union, intersection, and minus simply as sets will give us a deep enough understanding for our purposes. At this point we have learned enough about set theory to begin defining the sets we will be using to study linear algebra, calculus, and differential equations, the topics which will give us enough background to learn about systems and control theory. The first pass at defining these sets will be to define the natural numbers, integers, rational numbers, irrational numbers, and real numbers.

1.11 Natural Numbers and Integers

The natural numbers are defined as follows:

$$\mathbb{N} = \{0, 1, 2, 3, 4, 5, \dots\}$$

An important thing to note is the first element of this set, 0. By many definitions of the natural numbers, 0 is not included. We will include it in our definition to reveal some information about how this set is created. Including 0 in this set makes it *closed under addition*. This means that the sum of any two elements in the set of natural numbers is also in the set of natural numbers. This suggests that the set of natural numbers is infinitely large. An easy way to prove this is to assume the contrary, that the set of natural numbers is finite. This means that there is a largest natural number. But the sum of this number and any other natural number besides zero will be a larger natural number not already included in the finite set, contradicting our assumption that the set is finite. This key point of the proof also shows how the set of natural numbers can be created. Start with the set $\mathbb{N} = \{0\}$. Redefine $\mathbb{N}$ as the union of $\mathbb{N}$ and the set of the currently largest element of $\mathbb{N} + 1$. In this case, since the largest element of $\mathbb{N}$ is 0, we redefine $\mathbb{N}$ as $\mathbb{N} \cup \{1\} = \{0, 1\}$. We continue this process in iterations, meaning we continue to redefine $\mathbb{N}$ in this same manner, each time adding the largest element of $\mathbb{N} + 1$ to $\mathbb{N}$. This process does not terminate since the set of natural numbers is infinitely long. The set of integers is very similar to the set of natural numbers, with one key difference. The integers are defined as follows:

$$\mathbb{Z} = \{\dots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \dots\}$$

The main difference between the two sets is the inclusion of negative numbers in the set of integers. Note that the natural numbers are a subset of the integers.

1.12 Rational, Irrational, and Real Numbers

The rational numbers are defined as follows:

$$\mathbb{Q} = \{p/q | p, q \in \mathbb{Z}, q \neq 0\}$$

Note that this set, like the natural numbers and the integers, is infinitely large. Both the natural numbers and the integers are subsets of the rational numbers. The rational numbers are also closed under addition. The irrational numbers are a little more difficult to define. By their name, it may not come as a surprise to know that they are related in some way to the rational numbers. Specifically, their intersection with the rational numbers is the empty set, and their union with the rational numbers is the set of real numbers, which we have not defined yet. Any real number can be represented in writing by an integer followed by a possibly infinite sequence of decimals. The set of real numbers is the set of all possible combinations of these numbers, and is typically denoted with the letter, $\mathbb{R}$. Note that this does not impose any constraints on the sequence of decimals in our definition. We can use the sequence of decimals to draw a distinction between the rational and irrational numbers. As previously mentioned, the sequence may or may not terminate. This seems like a logical place to start, and we might guess that rational numbers are composed of finite sequences of decimals while irrational numbers are composed infinite sequences. However, we can quickly see that the decimal representation of rational numbers is not necessarily finite. $1/3$, for example, can be represented as $0.3333\ldots$. However, we can characterize the infinite sequences of decimals associated with rational numbers by the fact that they are periodic sequences. Stated in another way, for any rational number, there exists a finite sequence of decimals that repeats indefinitely. This pattern may begin after a finite sequence of seemingly patternless decimals, but once the repetition starts it never stops. From this observation, we can define the irrational numbers as infinite non-repeating sequences of decimals. A somewhat standard notation for irrational numbers does not exist, so let us denote them as $\mathbb{Q}^c$, where the superscript c indicates that this set is the *complement* of $\mathbb{Q}$. We can think of the rational and irrational numbers as complementary sets, since they share no common elements and their union makes up the set of all real numbers. We can state these facts with the following two expressions:

$$\mathbb{Q} \cap \mathbb{Q}^c = \emptyset$$

$$\mathbb{Q} \cup \mathbb{Q}^c = \mathbb{R}$$

1.13 Summary

Let us take a moment to review some of the key ideas we learned in this section and see how they fit into the next sections as well. We learned a loose definition of a set and a function that will allow us to explore vector spaces, bases, and linear transformations in linear algebra, limits and linearization in calculus, solution spaces and transforms in differential equations, as well as strengthen our understanding of countless other subjects in these sections. We have not yet been exposed to all the sets we will work with, but the understanding drawn from this section should allow us to build up more complicated and abstract sets. Many of the complex problems we will encounter will have a way of being decomposed into simple problems of clear sets and functions. Being able to understand the sets we work with in controls applications and how we operate on these sets is crucial in devising clever ways to analyze and control systems.

2 Linear Algebra

Coming soon

3 Calculus

Coming soon

4 Differential Equations

Coming soon

5 LTI Systems

Coming soon