Homework 3– Due April 24, 2020

Please turn in a PDF that includes any code used to complete the following problems. (Note that the PDF will be the only version of your code that you turn in.) Questions and deliverables that should be included with your submission are shown in **bold**.

There are many correct ways to numerically solve problems 1c and 2a-2d in python. One way involves using a black-box numerical integrator like a scipy.integrate.quad and a function that solves for x for *any* value of t. Another way involves solving for a discrete number N of states $x(t)$ in advance and numerically integrating using a method like Riemann integration. While the first method is more accurate, the second is faster and approaches the accuracy of the first approach as $N \to \infty$. This homework is simple enough that computational efficiency should not be a large issue. However, you should be aware of both approaches and that the second approach is preferred as problem complexity increases. For both approaches, we recommend you use scipy.integrate.solve_ivp as we have found that Euler integration introduces a lot of error. If you take the second approach, please chose an N large enough that solutions are accurate to 2 significant digits. Hint: we used a value of $N \in (100, 2000)$.

1. (50 pts) Consider the system $\dot{x} = -Sin[x]$ with $x(0) = x_0$. You are going to optimize $J = \int_0^T (e^{-t} - x(t))^2 dt$ over x_0 with $T = 10$.

   (a) Turn in a **figure** with e^{-t} and $x(t)$ on the same plot with $x(0) = 0.8$. **What does it look like a good initial choice would be for x_0?**

   (b) Turn in a **plot** of J versus x_0 from 0.9 to 1.1. **Where do you roughly expect the minimum to be?**

   (c) Create a function that provides the *gradient* of J. To accomplish this, you will need to numerically solve for the 1-D state transition matrix for the linearization of the dynamics. Hint: $A(t) = \frac{\partial \dot{x}}{\partial x}$. **Evaluate the gradient at $x_0 = 1.0$.**

   (d) Use your optimization routine from the last homework and the gradient of J to minimize J with a tolerance of $\|\nabla J\| < 10^{-6}$, starting with the initial iterate $x_0 = 1.0$. **What is the optimal value of x_0?**

2. (50 pts) Consider the following system:

$$\dot{x} = \begin{cases} f_1(x) & if \quad t \leq \tau \\ f_2(x) & if \quad t > \tau \end{cases} \qquad x(0) = x_0$$

   where

$$f_1 = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix} x, \quad f_2 = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} x, \quad x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

   The control is the switching time τ. The cost we would like to minimize for this problem is

$$J(\tau) = \int_0^{0.5} \|x(t)\|_2^2 dt$$

   (remember that $\|\cdot\|_2$ is the Euclidean norm).

   (a) **Plot** $J(\tau)$ for $\tau \in (0, 0.5)$.

   (b) **Find a formula for the derivative $\frac{\partial J(\tau)}{\partial \tau}$.**

   (c) **Evaluate the derivative at $\tau = 0.25$.**

(d) **Find an approximation of the optimal value for** τ using $\tau = 0.25$ as your initial iterate and the following update rule:

$$\tau_{i+1} = \tau_i - \gamma \nabla J(\tau_i).$$

You will need to chose a value for γ, the magnitude of the step taken in the direction of the gradient.