Homework 6– Due May 15, 2020

Please turn in a PDF that includes any written solutions and/or code used to complete the following problems. (Note that the PDF will be the only version of your code that you turn in.) Questions and deliverables that should be included with your submission are shown in **bold**.

1. (20 pts) Demonstrate numerically that one of your optimal solution from Homework 5 (Problem 3 or 4) is indeed optimal. You can do this by evaluating the directional derivative of J at $(x_{sol}(t), u_{sol}(t))$ in 10 different directions $\zeta(t) = (z(t), v(t))$ of your choosing. If you have found the optimizer, these values should be close to zero regardless of the directions you choose. Please choose either your TPBVP or Riccati equation solution. **Turn in:** A table or a easy-to-read list of the 10 directions you chose and the corresponding values of the directional derivative. Hint: A direction you might choose for $v(t)$ is of the form $A\sin(Bt + C) + D$.

2. (30 pts) Compute the control $u(t)$ that minimizes

$$J = \frac{1}{2}\int_0^{10} (x - x_d)^T \begin{bmatrix} 10 & 0 \\ 0 & 1 + \frac{1}{8}\sin t \end{bmatrix} (x - x_d) + u^T[1]u dt +$$

$$\frac{1}{2}(x(10) - x_d(10))^T \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix} (x(10) - x_d(10))$$

subject to the constraint that

$$\dot{x} = \begin{bmatrix} 0 & 1 + \frac{1}{2}\sin t \\ -1 - \frac{1}{2}\cos t & \frac{1}{4}\sin t \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 + \frac{1}{2}\sin t \end{bmatrix} u.$$

where $x_d = [0.1t + 1, -0.2t + 1]^T$ and $x_0 = [1\ 1]^T$ using the Riccati equations.
Hint: Although not necessary, your numerical solutions for P and r will be more accurate if you solve both ODEs simultaneously (within the same solve_ivp function) as opposed to one after the other. **Turn in:** A plot of your solution for $x(t)$ compared to $x_d(t)$ over time. The optimal control signal I get looks like

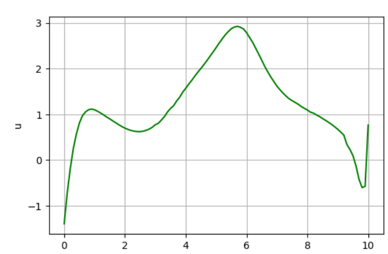

Figure 26: optimal control for time-varying system linear system tracking a reference signal

3. (30 pts) Thus far you have observed that the Pontryagin Maximum's Principle is reasonably good at solving for the optimality conditions using the TPBVP formulation for linear systems. However, we have yet to try solving for the optimal trajectory and control signal using the TPBVP for a *nonlinear* system. For this problem, use the differential drive vehicle from homework 2 with dynamics,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos(\theta)u_1 \\ \sin(\theta)u_1 \\ u_2 \end{bmatrix}, \ (x(0), y(0), \theta(0)) = (0, 0, \pi/2).$$

The desired trajectory of the vehicle is $(x_d, y_d, \theta_d) = (\frac{4}{2\pi}t, 0, \pi/2)$. Compute the control $u(t)$ that minimizes

$$J = \frac{1}{2}\int_0^1 (x - x_d)^T \begin{bmatrix} 1000 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 1 \end{bmatrix} (x - x_d) + u^T \begin{bmatrix} 50 & 0 \\ 0 & 0.1 \end{bmatrix} u dt +$$

$$(x(1) - x_d(1))^T \begin{bmatrix} 10000 & 0 & 0 \\ 0 & 10000 & 0 \\ 0 & 0 & 10 \end{bmatrix} (x(1) - x_d(1))$$

Since this differential drive vehicle is a *nonlinear* system, application of the Pontryagin maximum principle will require you to first linearize the dynamics in order to obtain the matrix A in the dynamics of the co-state variable p. As a reminder, for a function f, compute $A = \frac{\partial f}{\partial x}$ where $\dot{x} = f(x, u)$, $x = [x, y, \theta]^T$, and $u = [u_1, u_2]$. Since both the equations for $\dot{x}$ and $\dot{p}$ will involve u, you will need to do some algebra so that u does not show up in the two point boundary value problem. We recommend building off your solution to Homework 5 problem 4 and using scipy.integrate.solve_bvp. If it helps obtain a reasonable solution, you may change Q, R, and P1. **Turn in: A plot of the optimized trajectory with x on the x-axis and y on the y-axis.** *Hint:* If you cannot solve the two point boundary value problem with $\theta(0) = \frac{\pi}{2}$, try an initial condition of $\theta(0) = \frac{\pi}{4}$. Also, it may help to increase the second diagonal element in the R matrix.

4. (20 pts) In class we discussed how P only exists *near* final time T. For the system described in Problem 3 linearized about trajectory $x = [0, 0, 2\pi t]^T$ (where the vehicle spins in a circle), determine **the time** t in which we are no longer able to solve for P. For this problem, evaluate the system for a 10s horizon as opposed to a 1s horizon in Problem 3 and use Euler integration (backwards in time) as your integration method. *Hint*: One way this can be accomplished is by using Euler integration to step the state transition matrix X for states x in the TPBVP system of ODEs backwards in time until X is no longer invertible.