Made by Qingkai Shi
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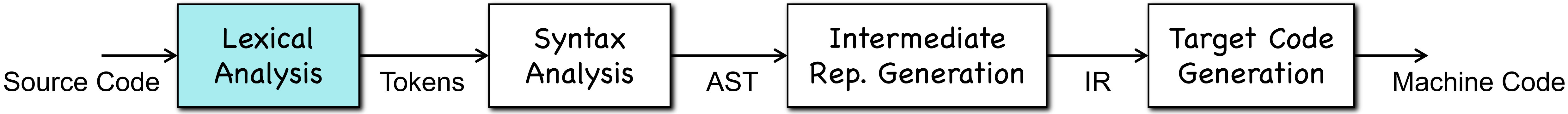
Chapter 3-1
Finite Automata

Lexical Analysis



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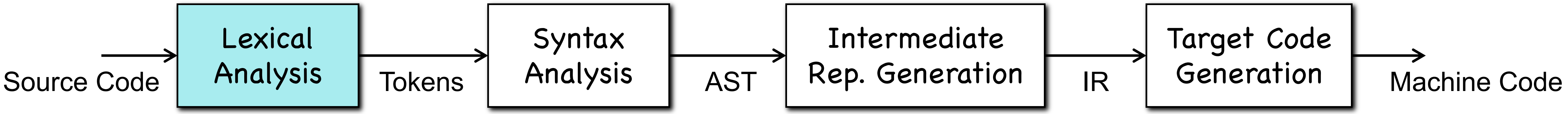

- Find **lexemes** according to **patterns**, and create **tokens**
  - Lexeme – a character string
  - Pattern – regular expression (lexical errors if no patterns matched)
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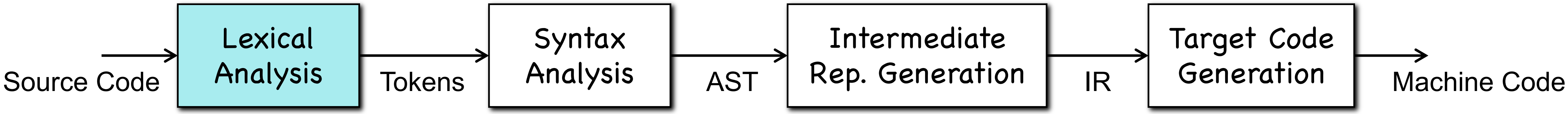

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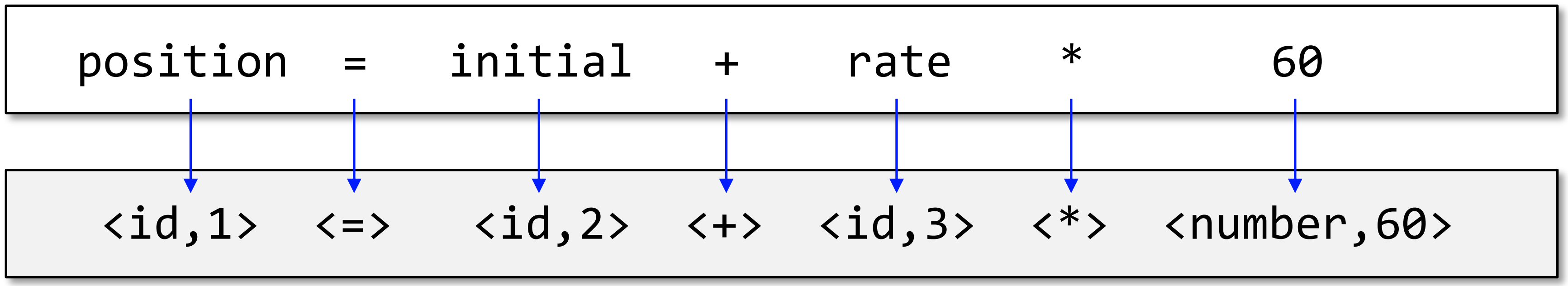

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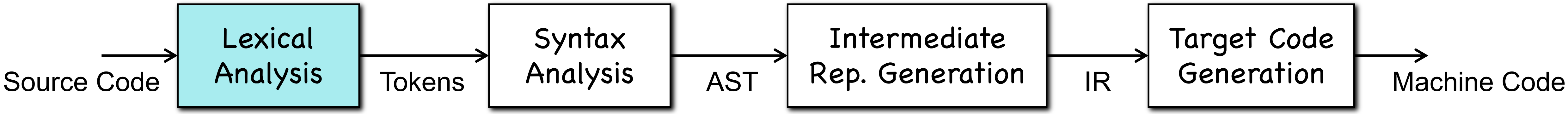

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key	name	…
1	position	…
2	initial	…
…		…

symbol table

```
position   =   initial   +   rate   *   60

<id,1>  <=>  <id,2>  <+>  <id,3>  <*>  <number,60>
```

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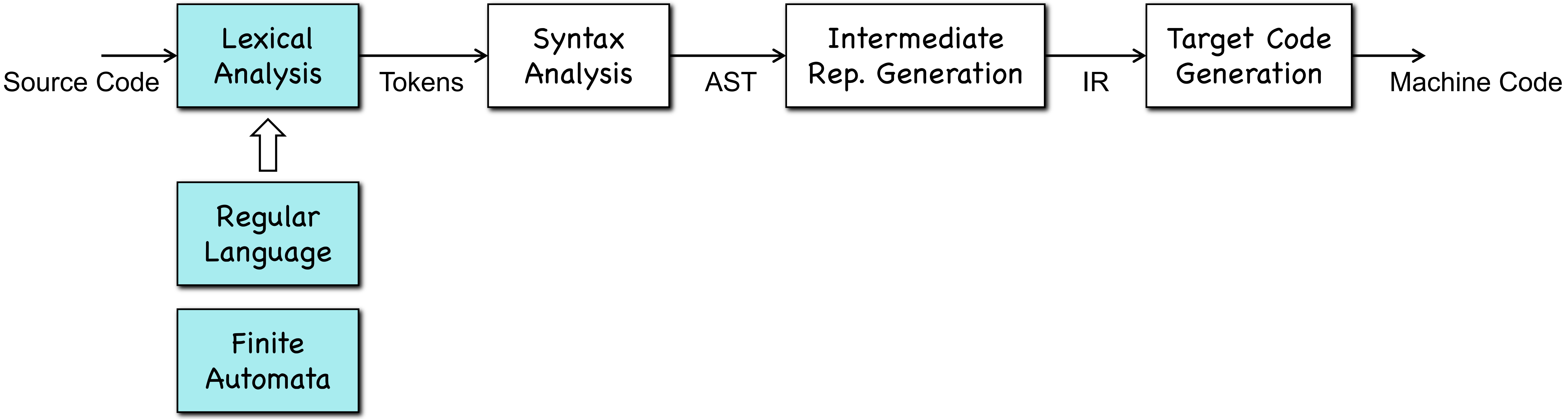

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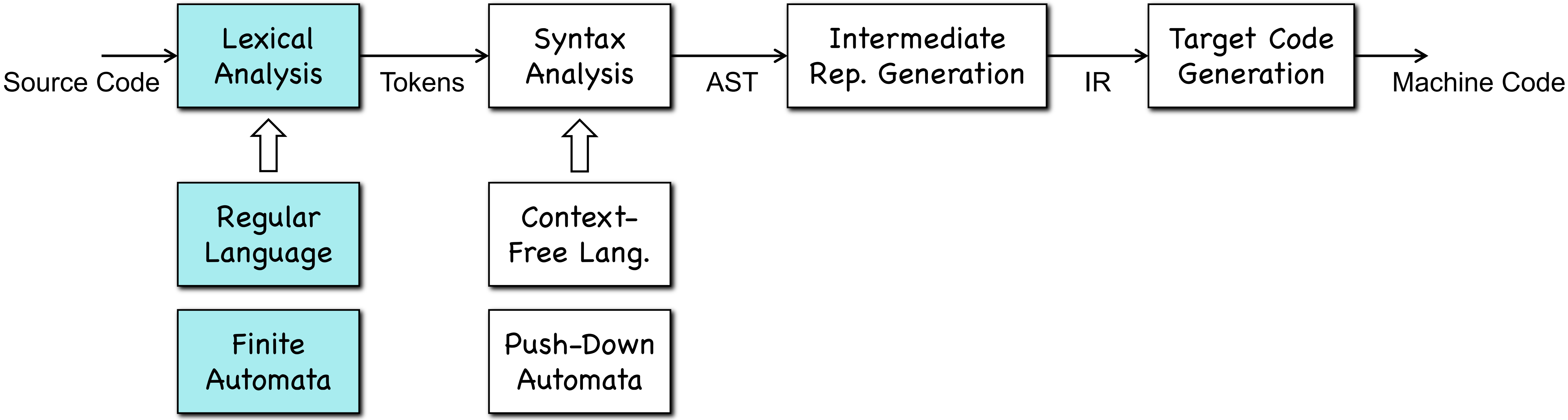

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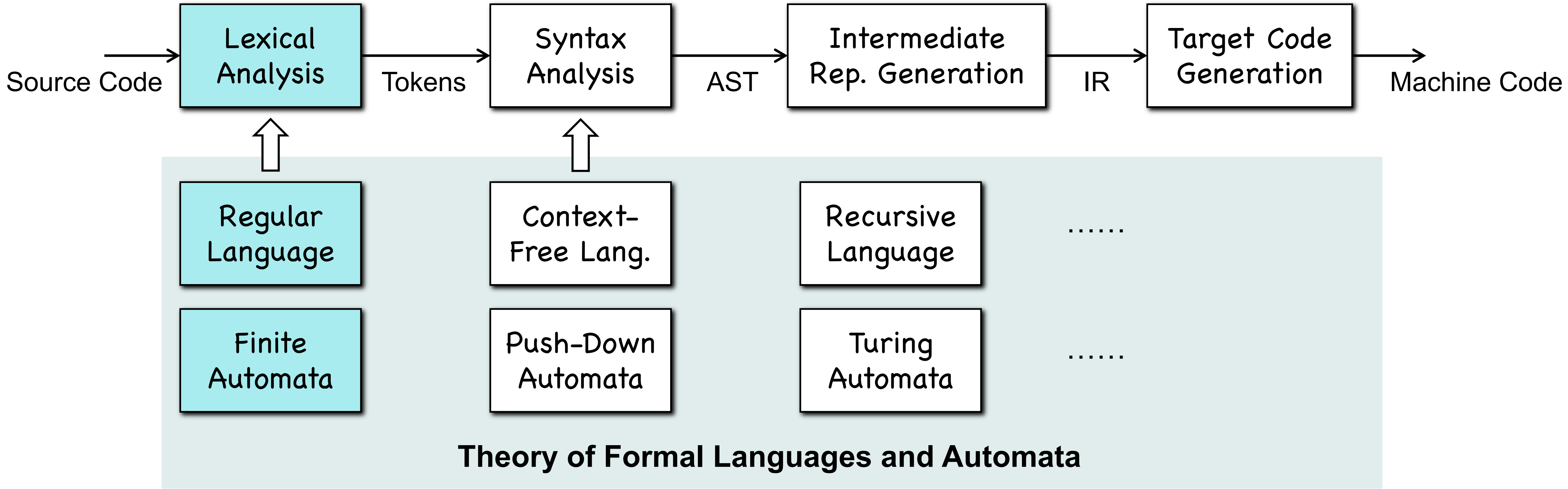

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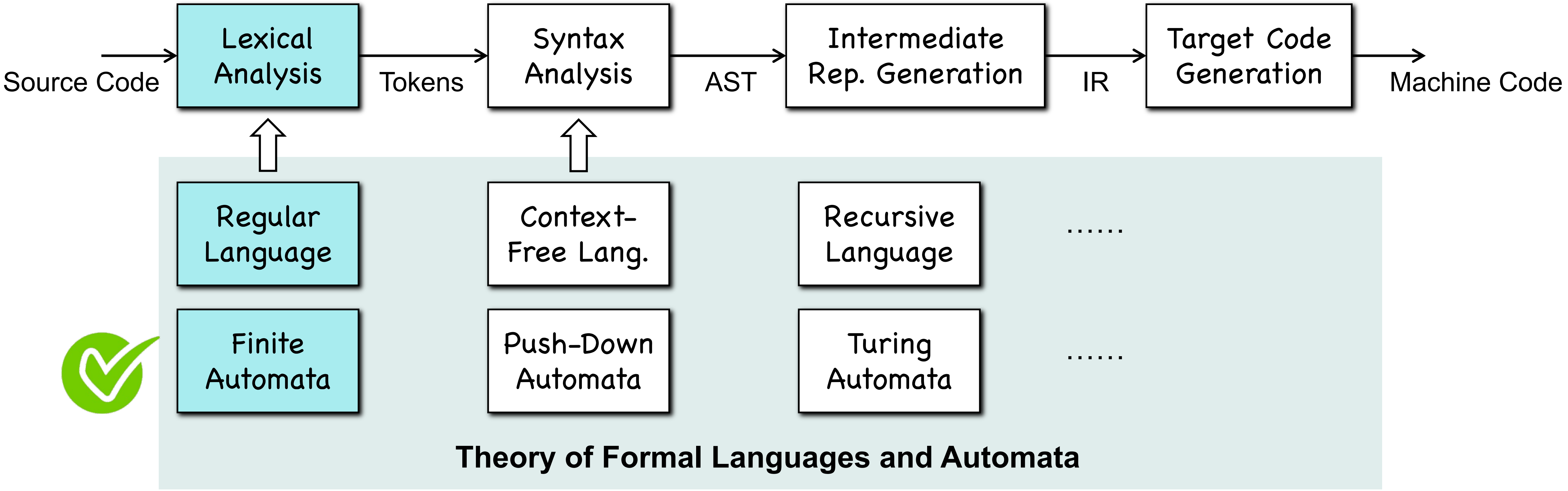

PART I: Math Preliminaries

Alphabet and Strings

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  - string example: cat, dog, …
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- **Example**: consider a small alphabet $\boldsymbol{\Sigma}$ = { a, b }
  - strings: a, b, ab, aab, aabb, …
  - a language is any subset of { a, b, ab, aab, aabb, … }

String Operations

- Concatenation
  - $w_1 = aabb;\ w_2 = bbaa;$ → $w_1w_2 = aabbbbaa;$

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  - $w = a_1a_2a_3a_4;$ → $w^R = a_4a_3a_2a_1;$
- Length
  - $w = a_1a_2a_3a_4;$ → $|w| = 4;$

Empty String and Sub-String

- **Empty String**: ϵ
  - $|\epsilon| = 0$
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- **Prefix and Suffix**: $w = uv$, where u is prefix and v is suffix
  - prefix of abb includes: ϵ, a, ab, abb
  - suffix of abb includes: abb, bb, b, ϵ

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Not ∅ !!!

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Operations on Languages

- Usual set operations
  - $L_1 = \{a, b\}; L_2 = \{a, ab\}; \Rightarrow L_1 \cup L_2 = \{a, b, ab\}$
  - $L_1 = \{a, b\}; L_2 = \{a, ab\}; \Rightarrow L_1 \cap L_2 = \{a\}$
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- Concatenation: $L_1 L_2 = \{xy : x \in L_1, y \in L_2\}$

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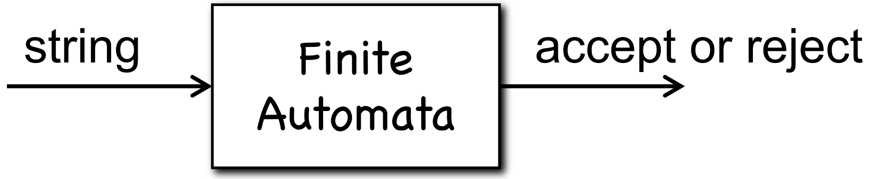
- **Quiz 1**: $L = \{a, b\}$; $\ L^3 = ?$
- **Quiz 2**: $L = \{a^n b^n : n \geq 0\}$; $L^2 = ?$

PART II:
Deterministic Finite Automata

Finite Automata

- Input a string, output “accept” or “reject”



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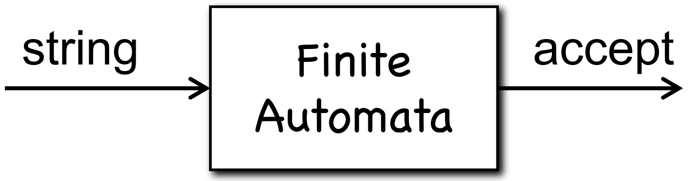

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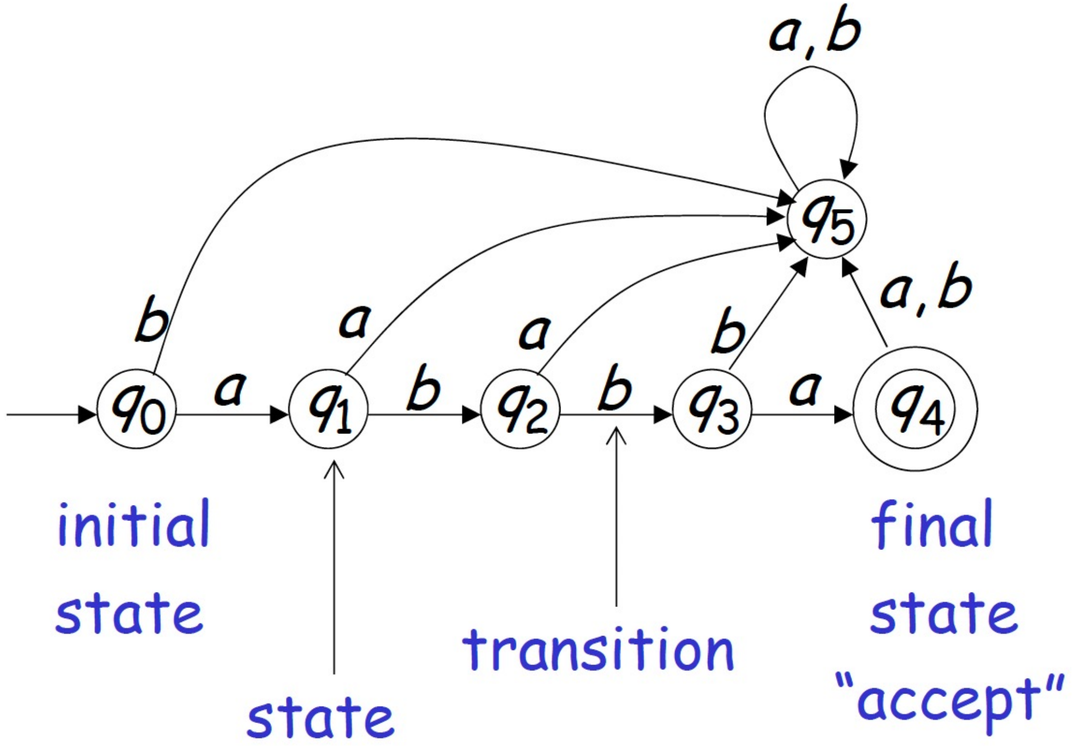

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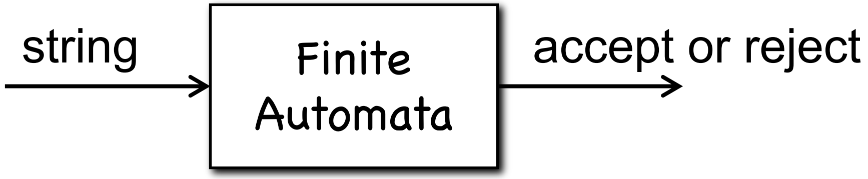

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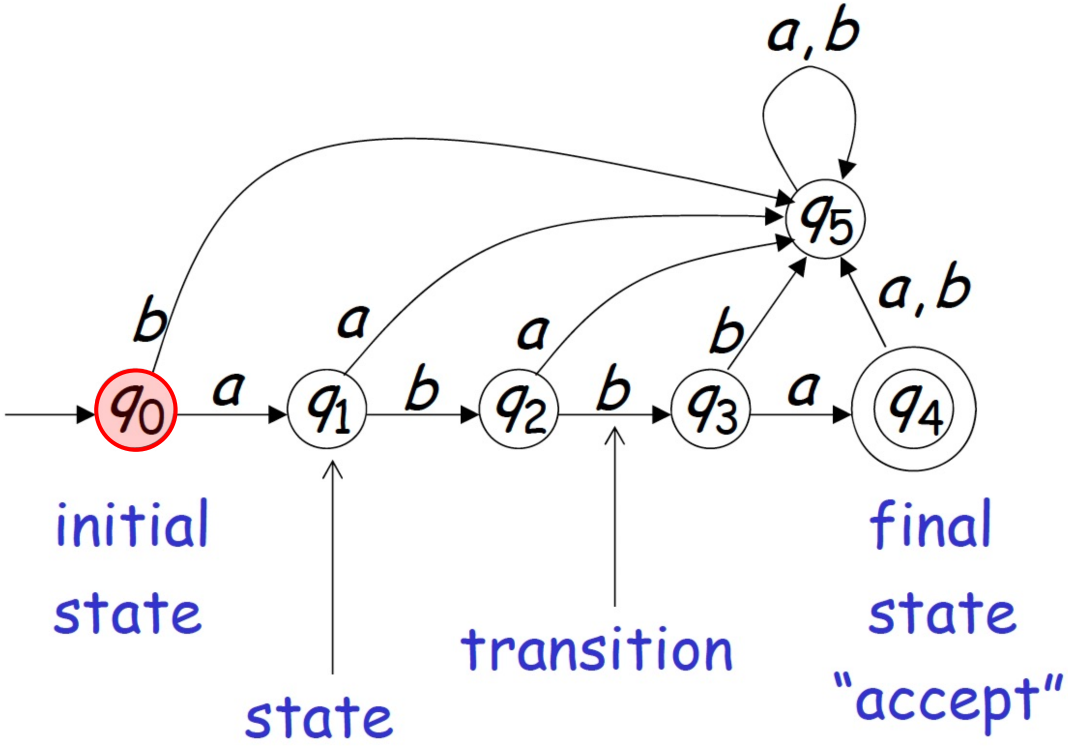

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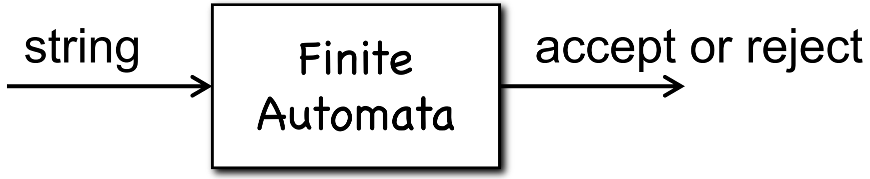

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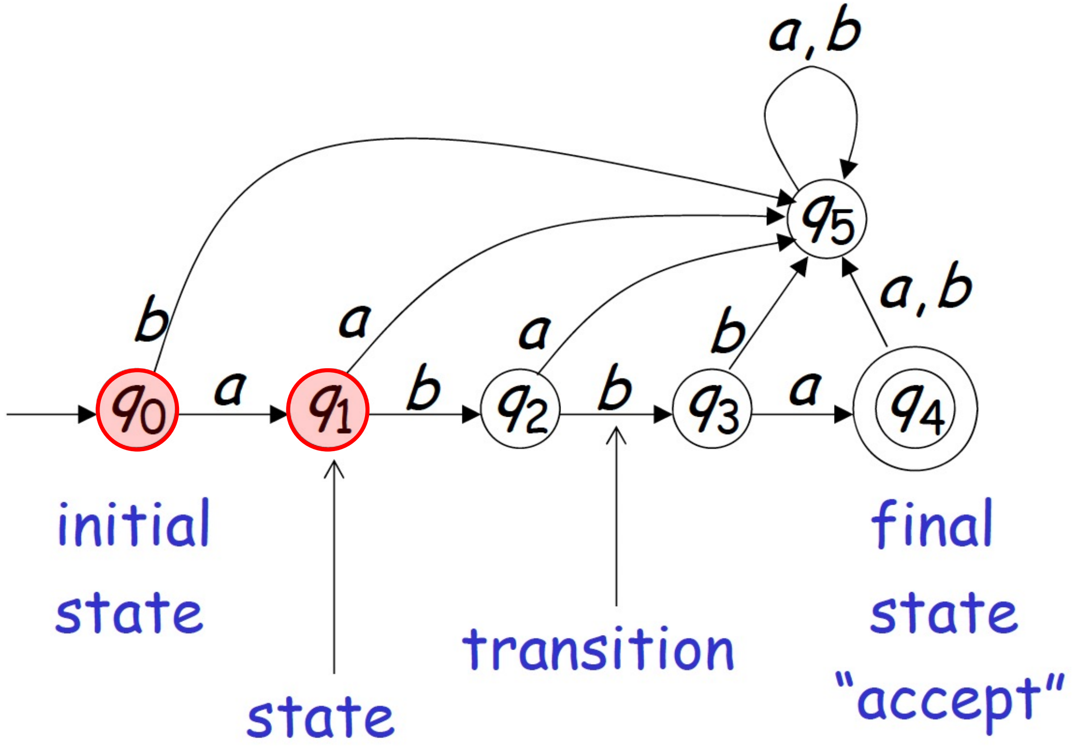

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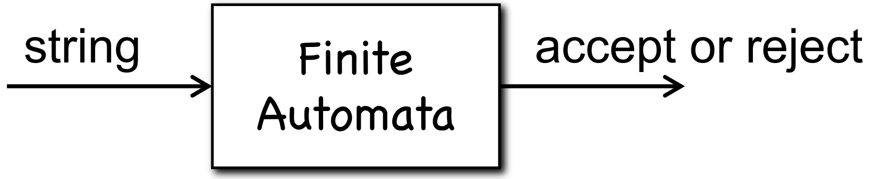

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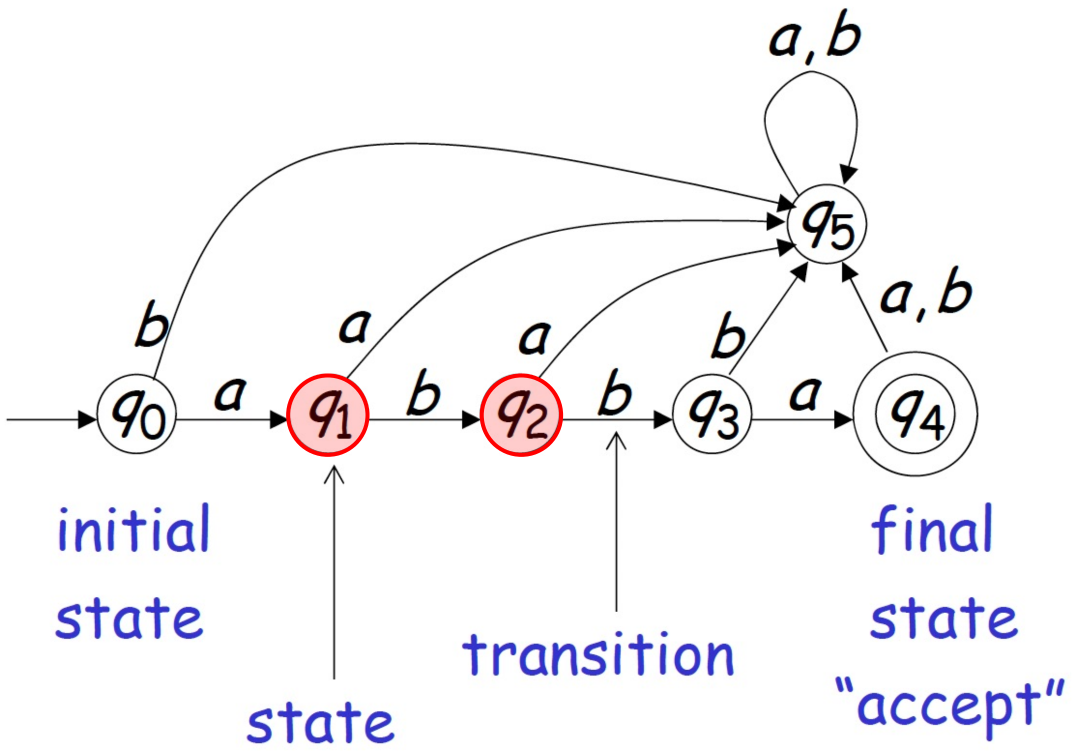

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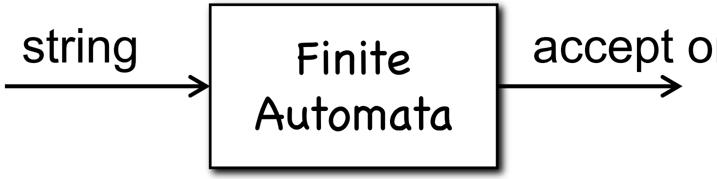

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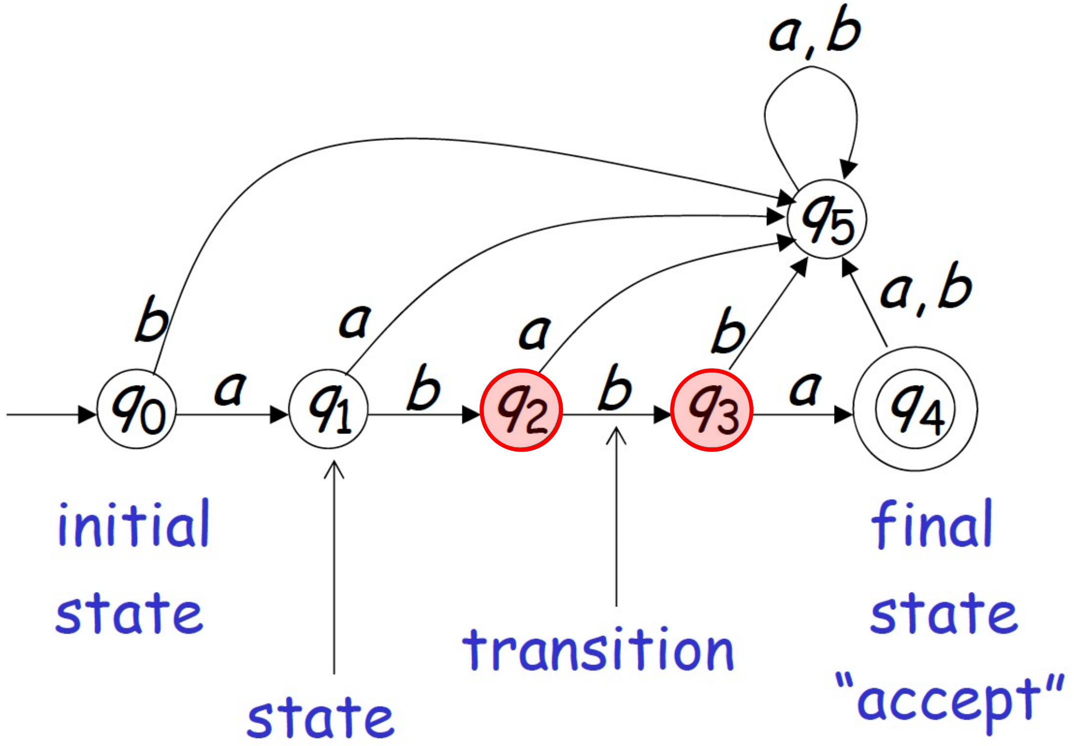

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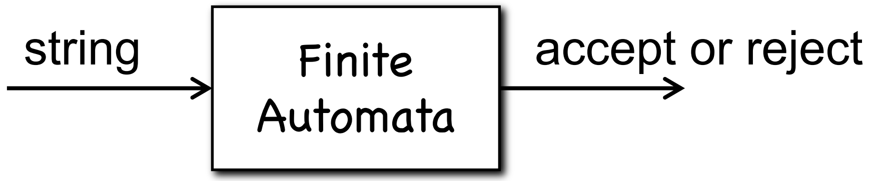

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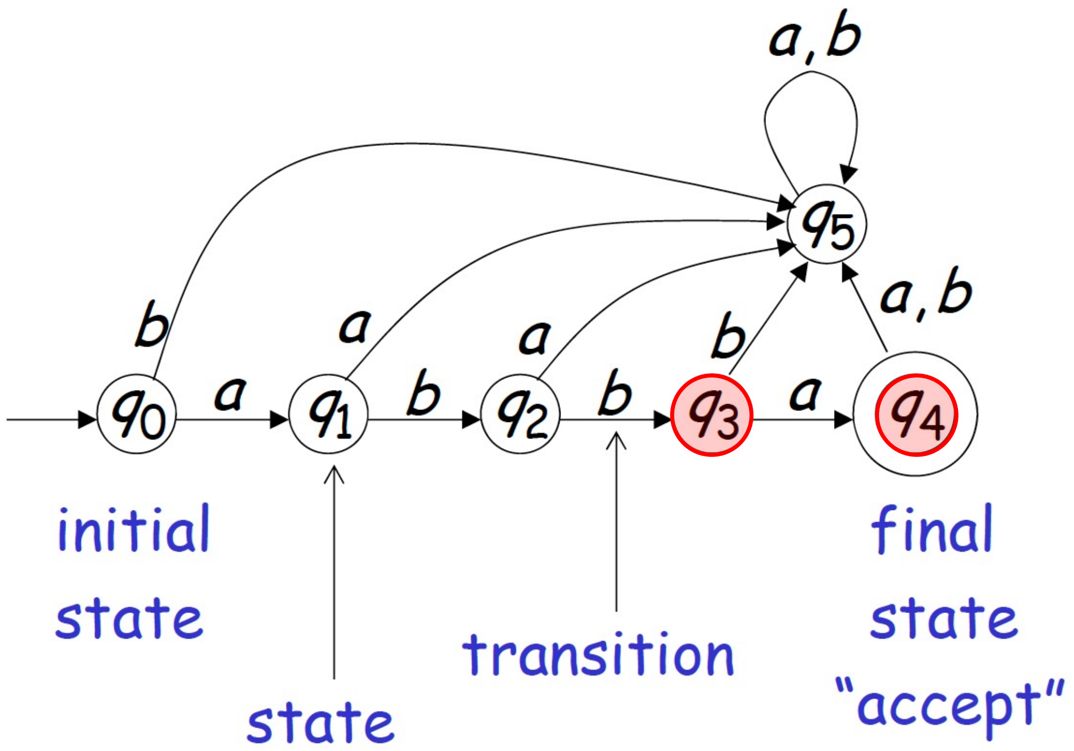

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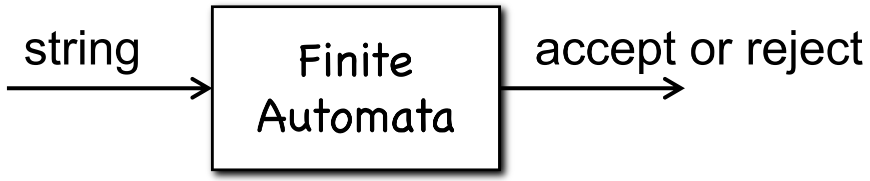

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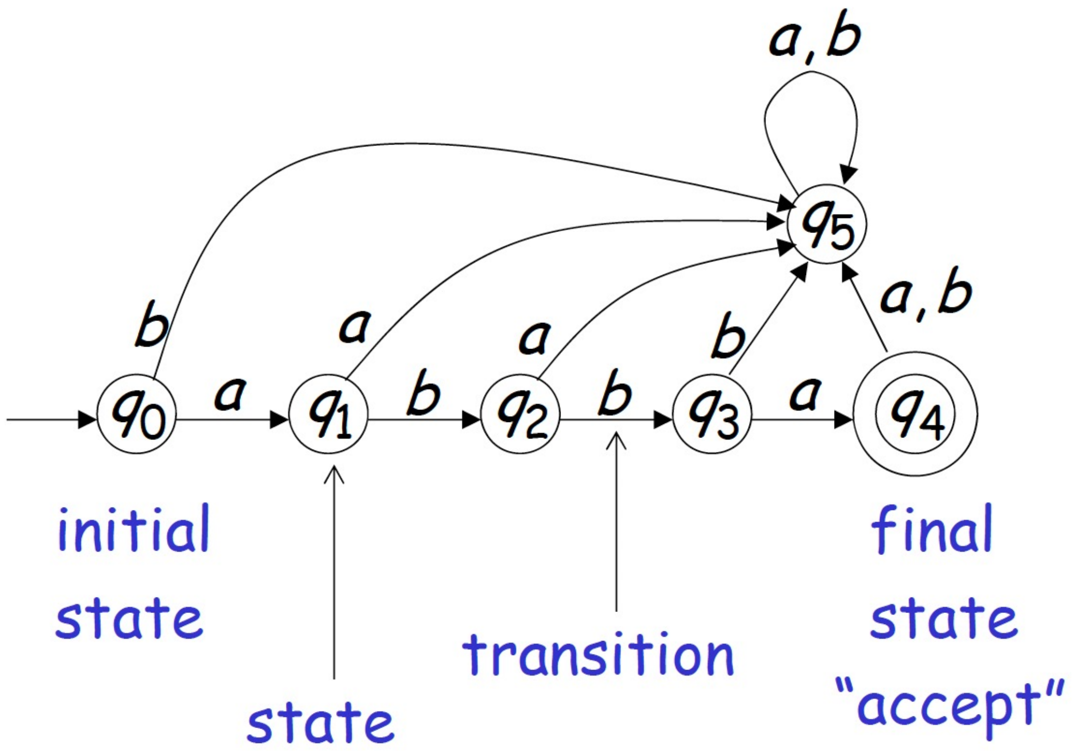

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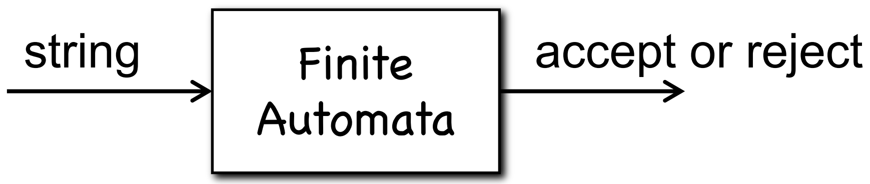

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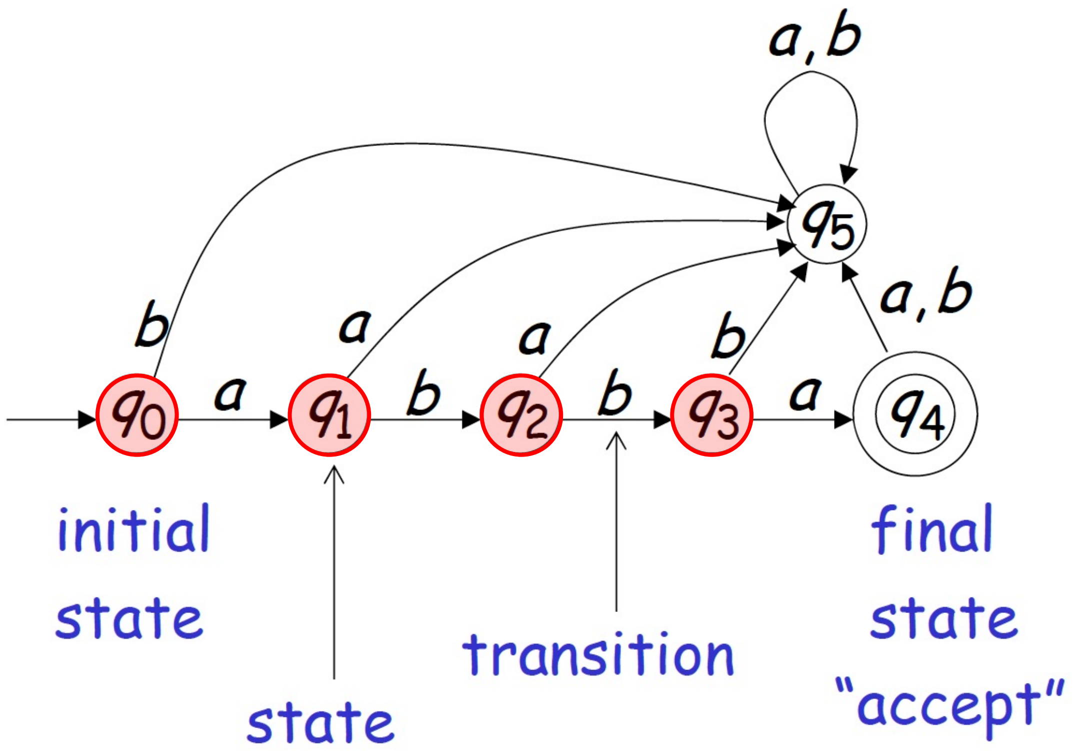

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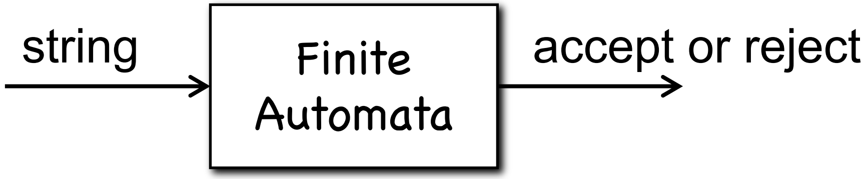

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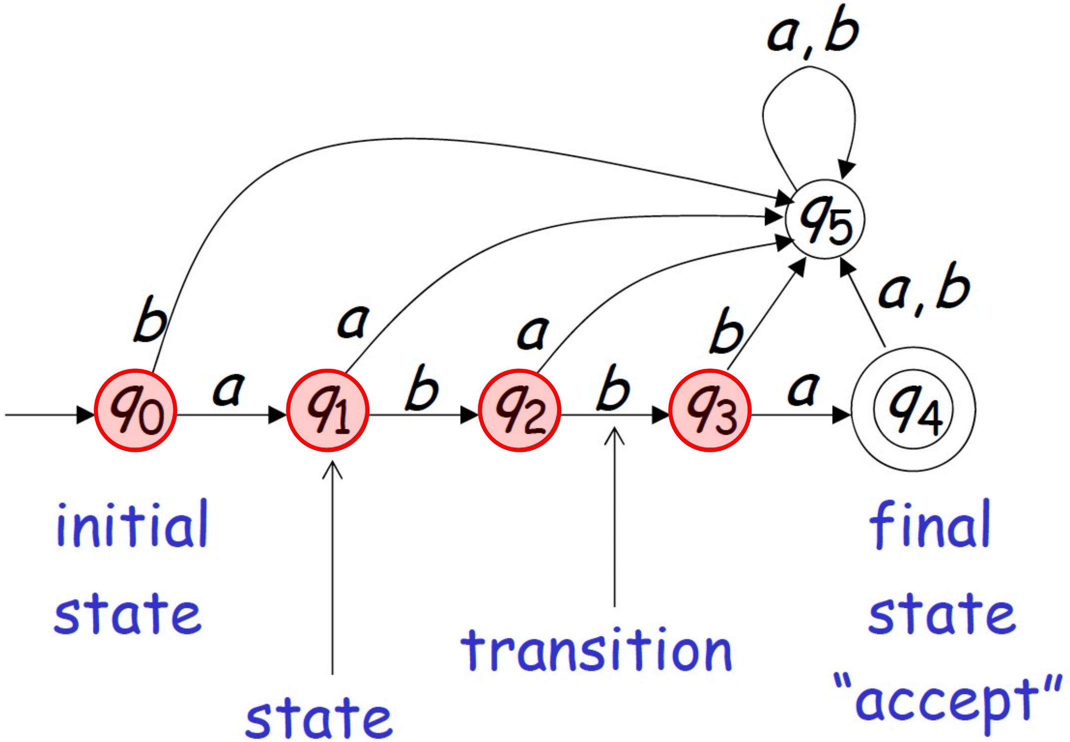
- **Deterministic** Finite Automata!



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- An extension of transition: $\delta^*: Q \times \Sigma^* \longmapsto$ Q
  - $\delta^*(q, abba) = q'; \quad \delta^*(q, \epsilon) = q;$

Defining a Language by DFA

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    q = q0;
    while (true) {
        c = read(w); if (c == None) { print(q ∊ F ? “accept” : “reject”); }
    
    
    
    
    
    
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    q = q₀;
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        switch(q) {
        case q₀:
        case q₁:
        case q₂:
        case ……
        }
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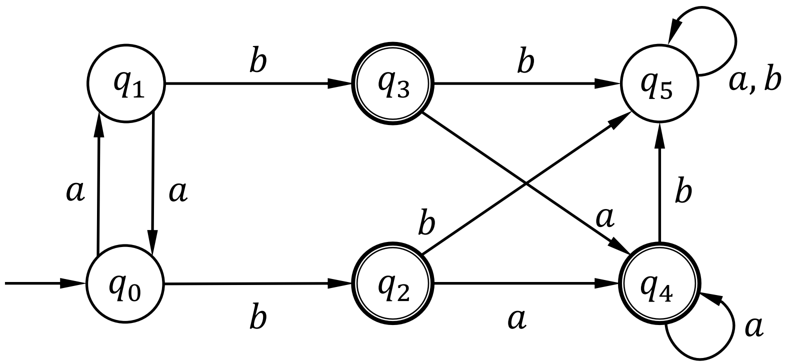
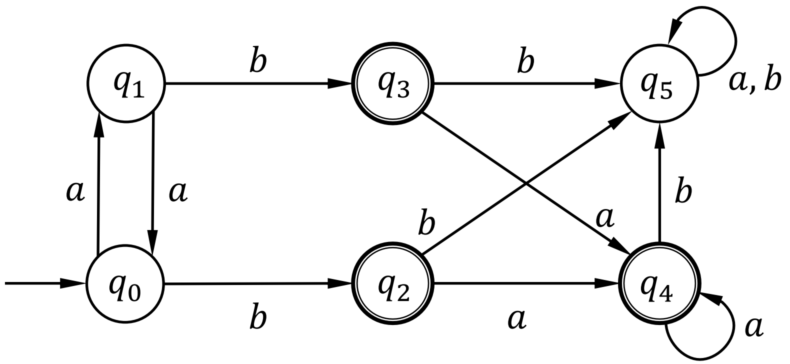


	a	b
q_0	q_1	q_2
q_1	q_0	q_3
q_2	q_4	q_5
q_3	q_4	q_5
q_4	q_4	q_5
q_5	q_5	q_5

DFA Minimization

- Minimizing DFA can improve the efficiency of computation

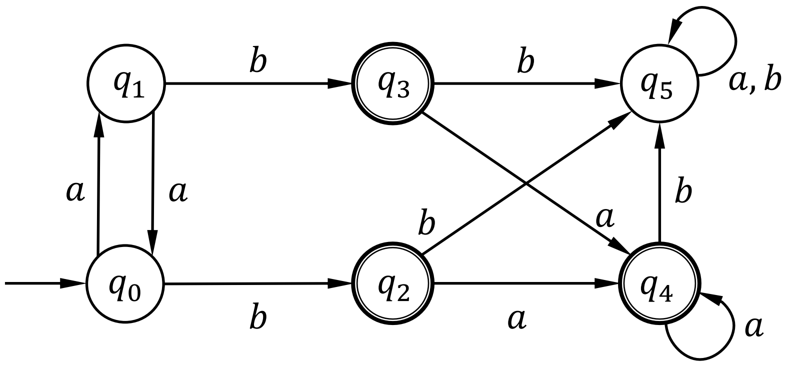


	a	b
q_0	q_1	q_2
q_1	q_0	q_3
q_2	q_4	q_5
q_3	q_4	q_5
q_4	q_4	q_5
q_5	q_5	q_5

Step 1	$\{ q_0, q_1, q_5 \}, \{q_2, q_3, q_4\}$	Distinguish final and non-final states
Step 2		
Step 3		

DFA Minimization

- Minimizing DFA can improve the efficiency of computation

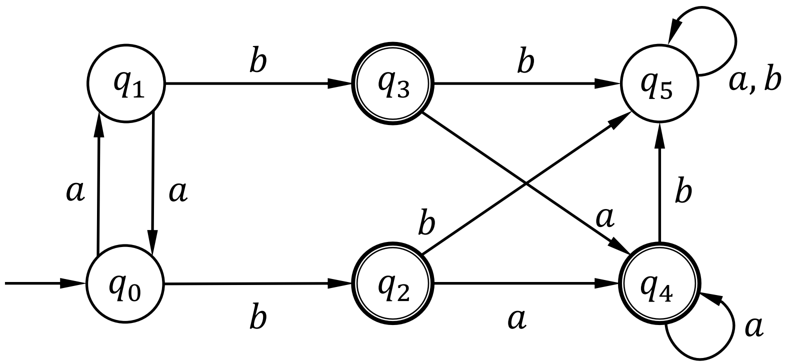


	a	b
q_0	q_1	q_2
q_1	q_0	q_3
q_2	q_4	q_5
q_3	q_4	q_5
q_4	q_4	q_5
q_5	q_5	q_5

Step 1	$\{ q_0, q_1, q_5 \}, \{q_2, q_3, q_4\}$	Distinguish final and non-final states
Step 2	$\{ q_0, q_1$	$\delta(q_0, a/b)$ and $\delta(q_1, a/b)$ are in the same set
Step 3		

DFA Minimization

- Minimizing DFA can improve the efficiency of computation

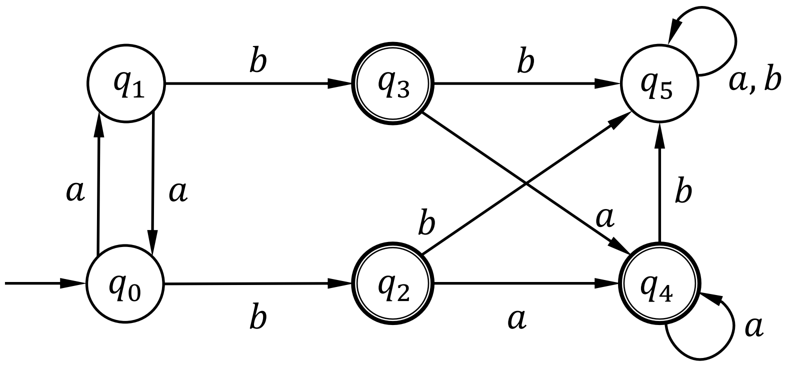


	a	b
q_0	q_1	q_2
q_1	q_0	q_3
q_2	q_4	q_5
q_3	q_4	q_5
q_4	q_4	q_5
q_5	q_5	q_5

Step 1	$\{ q_0, q_1, q_5 \}, \{q_2, q_3, q_4\}$	Distinguish final and non-final states
Step 2	$\{ q_0, q_1 \}, \{ q_5 \}$	$\delta(q_0, a/b)$ and $\delta(q_1, a/b)$ are in the same set, but $\delta(q_5, a/b)$ not
Step 3		

DFA Minimization

- Minimizing DFA can improve the efficiency of computation

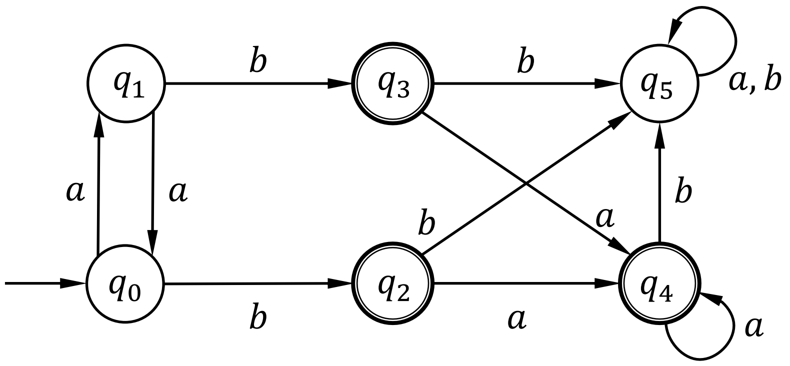


	a	b
q_0	q_1	q_2
q_1	q_0	q_3
q_2	q_4	q_5
q_3	q_4	q_5
q_4	q_4	q_5
q_5	q_5	q_5

Step 1	$\{ q_0, q_1, q_5 \}, \{q_2, q_3, q_4\}$	Distinguish final and non-final states
Step 2	$\{ q_0, q_1 \}, \{ q_5 \}, \{q_2, q_3, q_4\}$	$\delta(q_0, a/b)$ and $\delta(q_1, a/b)$ are in the same set, but $\delta(q_5, a/b)$ not
Step 3		

DFA Minimization

- Minimizing DFA can improve the efficiency of computation

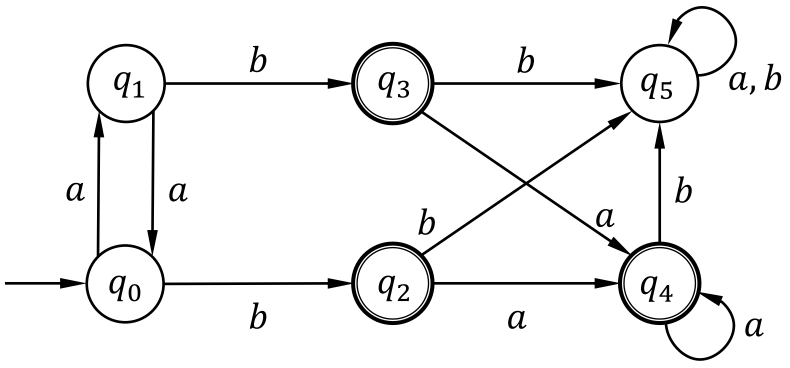


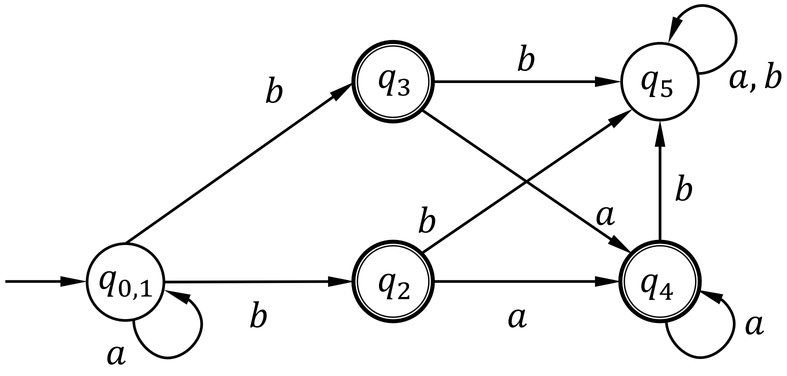
	a	b
q_0	q_1	q_2
q_1	q_0	q_3
q_2	q_4	q_5
q_3	q_4	q_5
q_4	q_4	q_5
q_5	q_5	q_5

Step 1	$\{q_0, q_1, q_5\}, \{q_2, q_3, q_4\}$	Distinguish final and non-final states
Step 2	$\{q_0, q_1\}, \{q_5\}, \{q_2, q_3, q_4\}$	$\delta(q_0, a/b)$ and $\delta(q_1, a/b)$ are in the same set, but $\delta(q_5, a/b)$ not
Step 3	$\{q_0, q_1\}, \{q_5\}, \{q_2, q_3, q_4\}$	The result does not change and the algorithm completes

DFA Minimization

- Minimizing DFA can improve the efficiency of computation

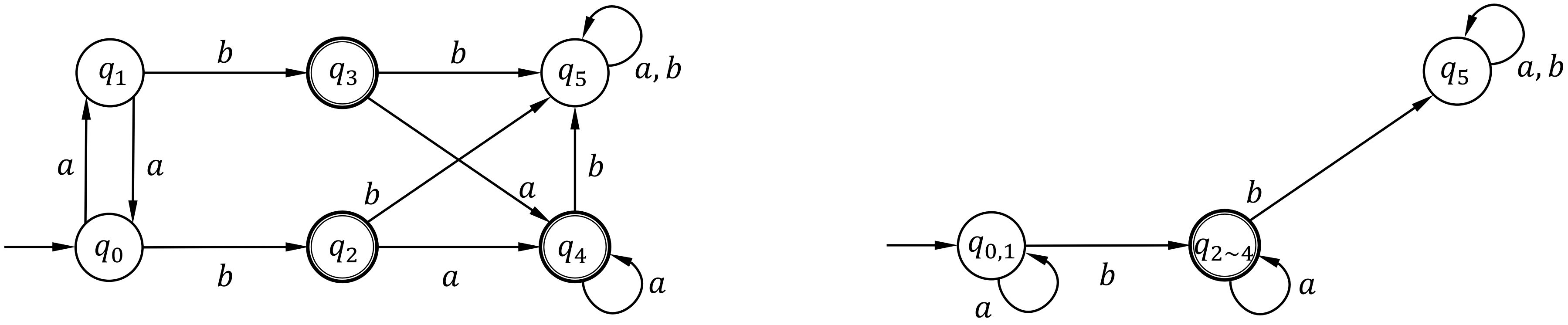




Step 1	$\{q_0, q_1, q_5\}, \{q_2, q_3, q_4\}$	Distinguish final and non-final states
Step 2	$\{q_0, q_1\}, \{q_5\}, \{q_2, q_3, q_4\}$	$\delta(q_0, a/b)$ and $\delta(q_1, a/b)$ are in the same set, but $\delta(q_5, a/b)$ not
Step 3	$\{q_0, q_1\}, \{q_5\}, \{q_2, q_3, q_4\}$	The result does not change and the algorithm completes

DFA Minimization

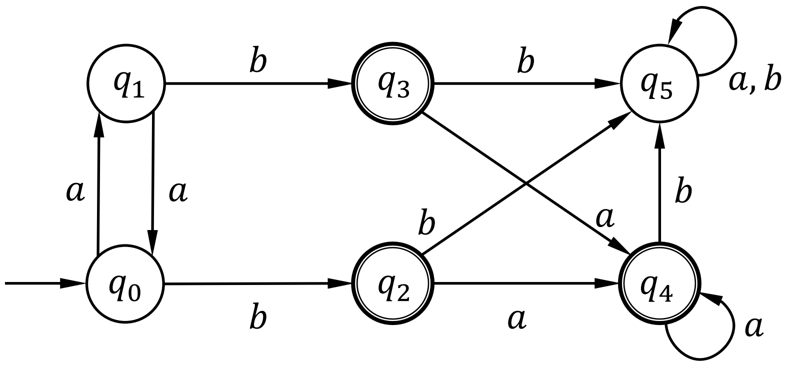
- Minimizing DFA can improve the efficiency of computation

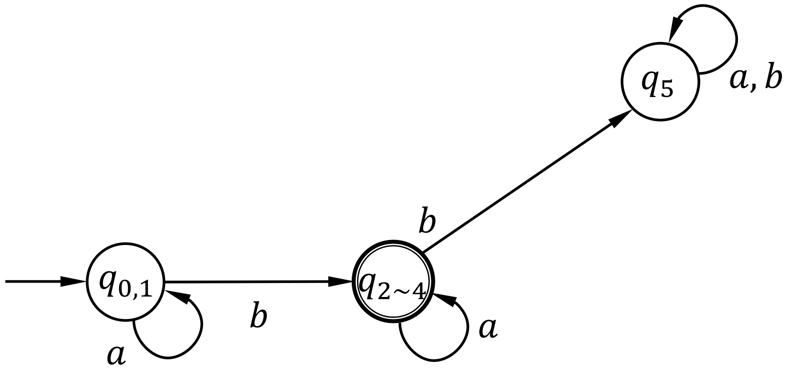


Step 1	$\{q_0, q_1, q_5\}, \{q_2, q_3, q_4\}$	Distinguish final and non-final states
Step 2	$\{q_0, q_1\}, \{q_5\}, \{q_2, q_3, q_4\}$	$\delta(q_0, a/b)$ and $\delta(q_1, a/b)$ are in the same set, but $\delta(q_5, a/b)$ not
Step 3	$\{q_0, q_1\}, \{q_5\}, \{q_2, q_3, q_4\}$	The result does not change and the algorithm completes

DFA Minimization

- Minimizing DFA can improve the efficiency of computation





- **We remove all unreachable states before the above steps**

DFA Minimization

- Best Average Complexity: $O(n \log \log n)$!

ELSEVIER

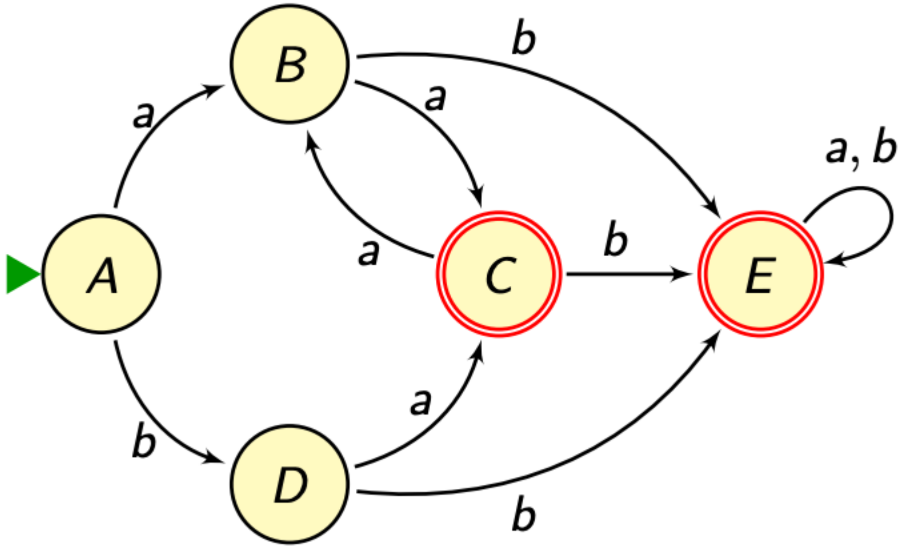
Theoretical Computer Science

Volume 417, 3 February 2012, Pages 50-65

Average complexity of Moore's and Hopcroft's algorithms

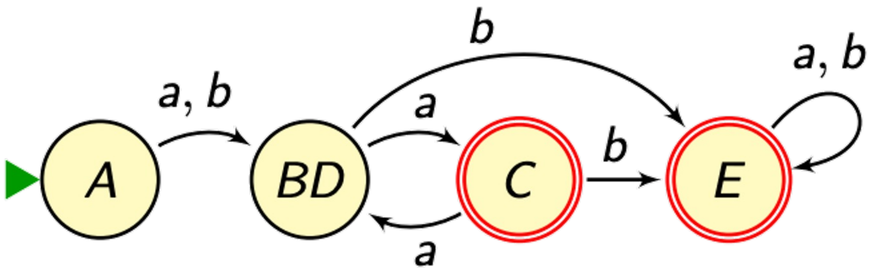
Julien David

DFA Minimization



Have a Try!!

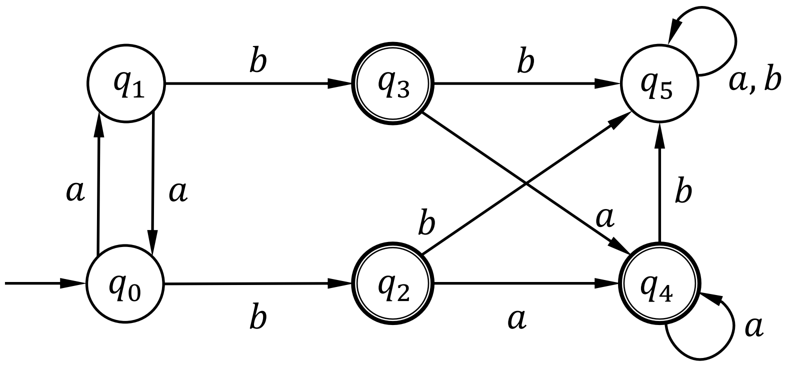
DFA Minimization

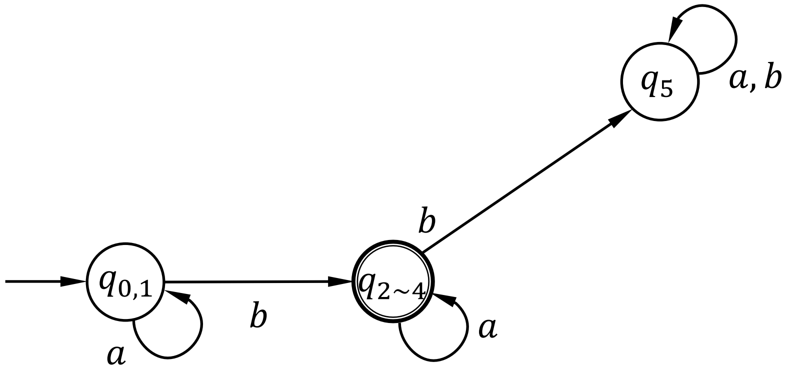


Solution

DFA Bi-Simulation

- Checking the equivalence of the DFAs, i.e., $L(M_1) = L(M_2)$





DFA Bi-Simulation

- Checking the equivalence of the DFAs, i.e., $L(M_1) = L(M_2)$
  - $L(M) = \{w \in \Sigma^*: \delta^*(q_0, w) \subseteq F\}$

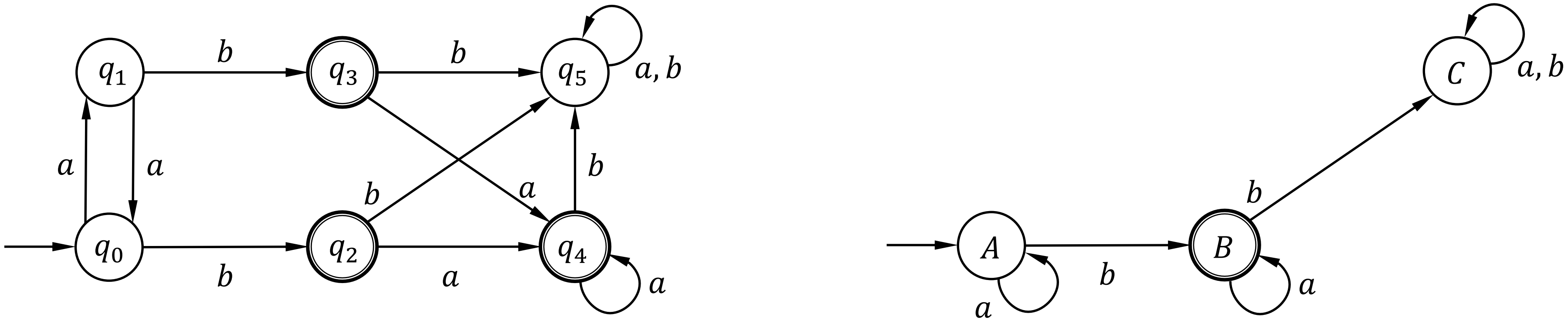
DFA Bi-Simulation

- Checking the equivalence of the DFAs, i.e., $L(M_1) = L(M_2)$
  - $L(M) = \{w \in \Sigma^*: \delta^*(q_0, w) \subseteq F\}$

- Given an input string, let M_1 and M_2 evaluate it at the same time, M_1 reaches a final state if and only if M_2 reaches a final state

DFA Bi-Simulation

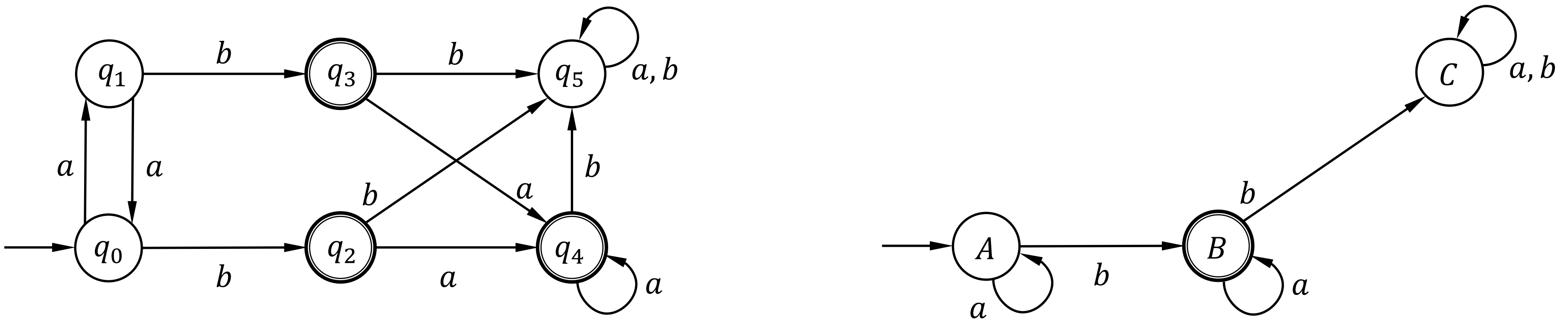
- Checking the equivalence of the DFAs, i.e., $L(M_1) = L(M_2)$



State Pairs	a	b	
$\{ q_0, A \}$			
			M_1 reaches final state if and only if M_2 reaches final state

DFA Bi-Simulation

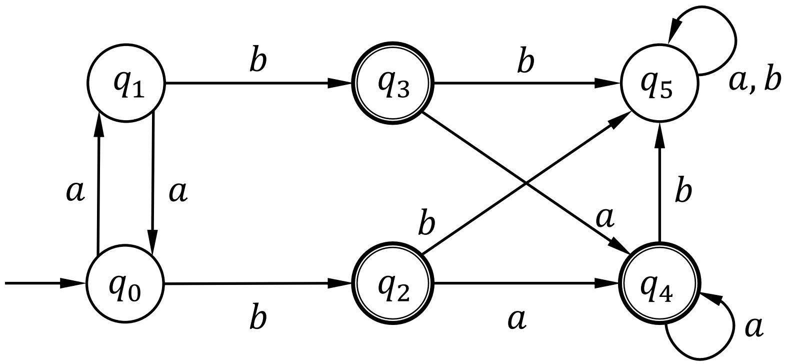
- Checking the equivalence of the DFAs, i.e., $L(M_1) = L(M_2)$

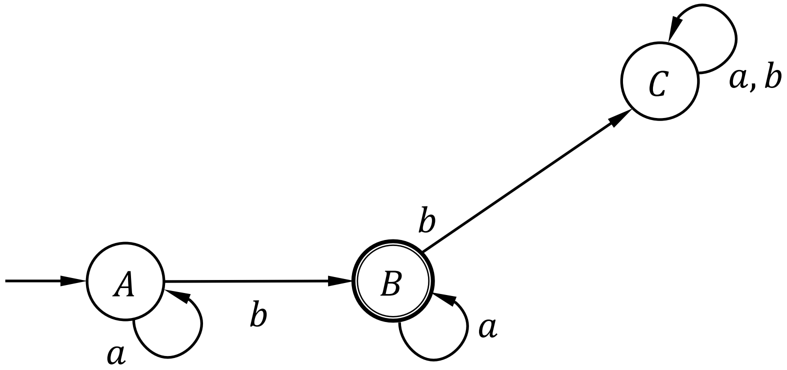


State Pairs	a	b	
$\{ q_0, A \}$	$\{ q_1, A \}$	$\{ q_2, B \}$	
			M_1 reaches final state if and only if M_2 reaches final state

DFA Bi-Simulation

- Checking the equivalence of the DFAs, i.e., $L(M_1) = L(M_2)$

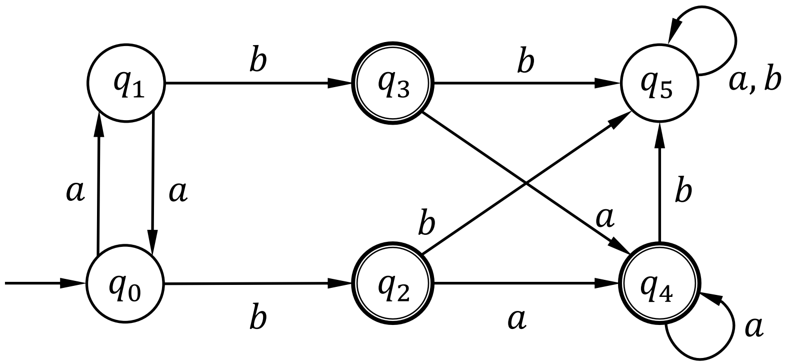


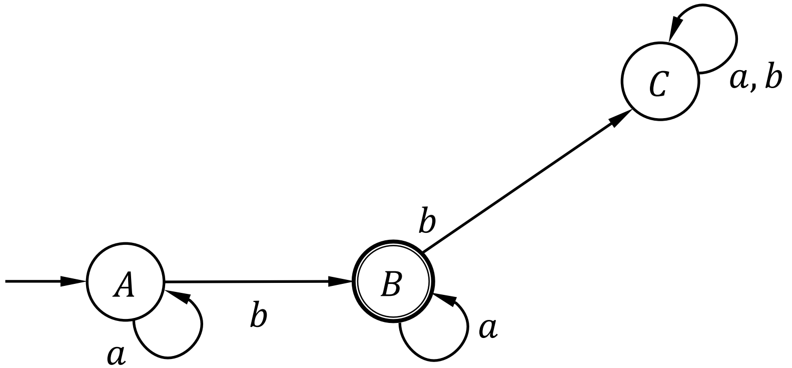


State Pairs	a	b	
$\{ q_0, A \}$	$\{ q_1, A \}$	$\{ q_2, B \}$	
$\{ q_1, A \}$			
$\{ q_2, B \}$			M_1 reaches final state if and only if M_2 reaches final state

DFA Bi-Simulation

- Checking the equivalence of the DFAs, i.e., $L(M_1) = L(M_2)$

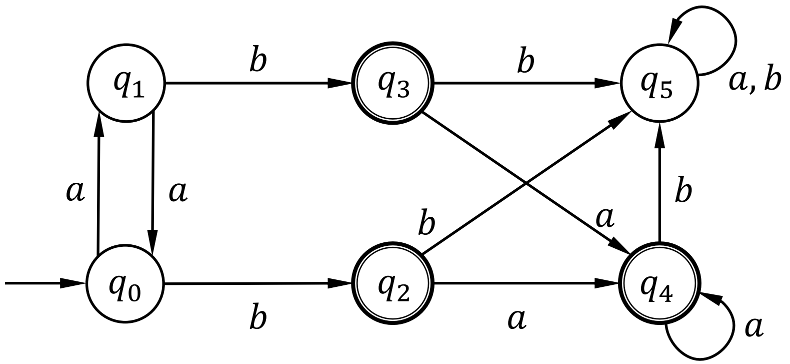


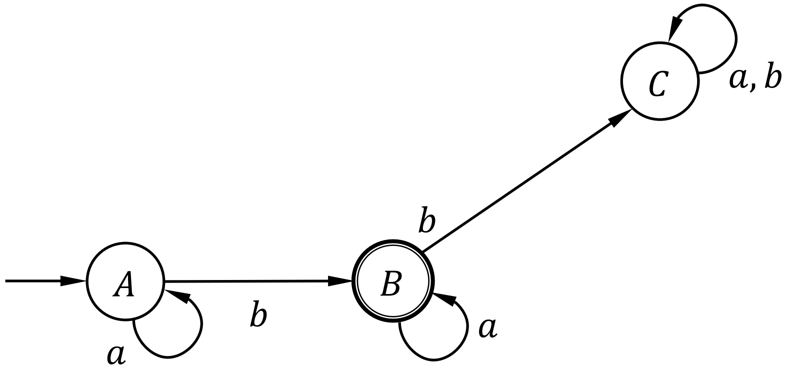


State Pairs	a	b	
$\{ q_0, A \}$	$\{ q_1, A \}$	$\{ q_2, B \}$	
$\{ q_1, A \}$	$\{ q_0, A \}$	$\{ q_3, B \}$	M_1 reaches final state if and only if M_2 reaches final state
$\{ q_2, B \}$			

DFA Bi-Simulation

- Checking the equivalence of the DFAs, i.e., $L(M_1) = L(M_2)$

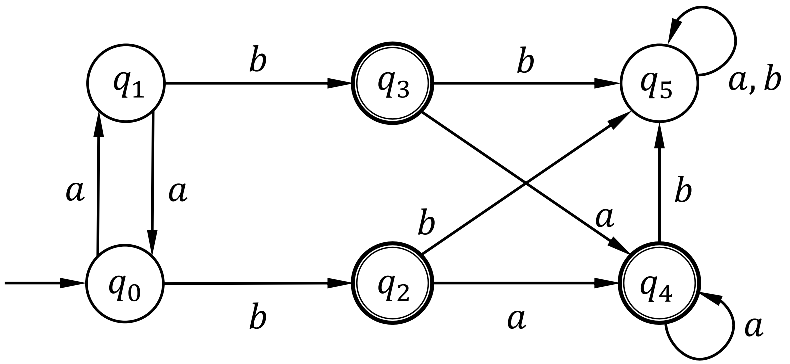


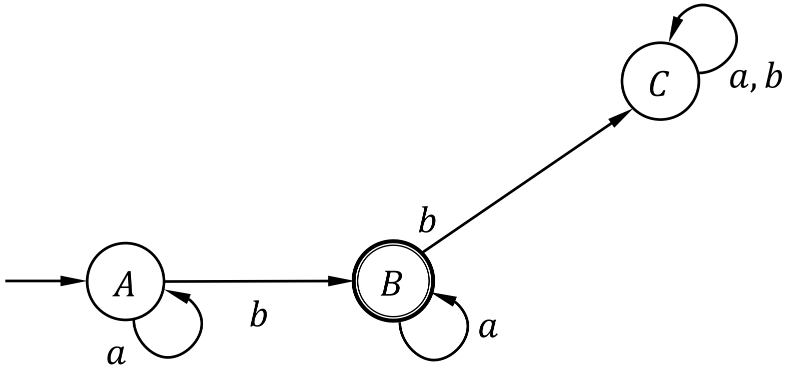


State Pairs	a	b	
$\{ q_0, A \}$	$\{ q_1, A \}$	$\{ q_2, B \}$	
$\{ q_1, A \}$	$\{ q_0, A \}$	$\{ q_3, B \}$	M_1 reaches final state if and only if M_2 reaches final state
$\{ q_2, B \}$			
$\{ q_3, B \}$			

DFA Bi-Simulation

- Checking the equivalence of the DFAs, i.e., $L(M_1) = L(M_2)$

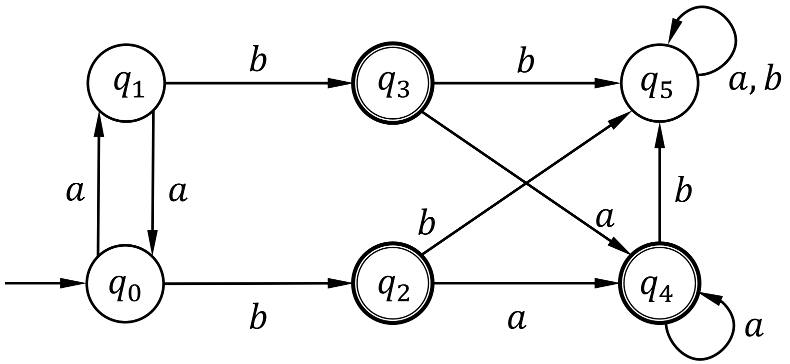


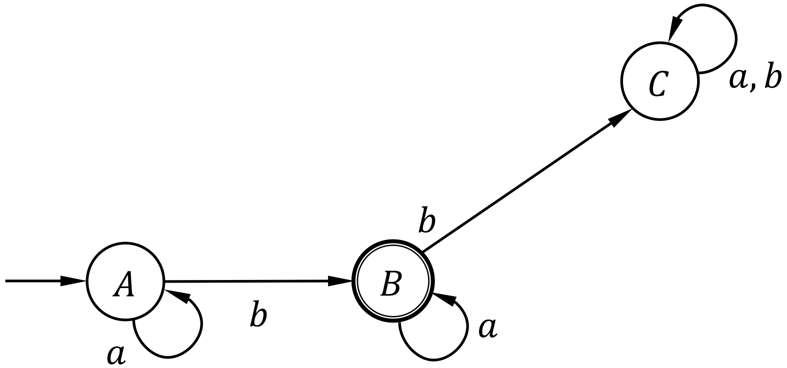


State Pairs	a	b	
$\{q_0, A\}$	$\{q_1, A\}$	$\{q_2, B\}$	
$\{q_1, A\}$	$\{q_0, A\}$	$\{q_3, B\}$	
$\{q_2, B\}$	$\{q_4, B\}$	$\{q_5, C\}$	M_1 reaches final state if and only if M_2 reaches final state
$\{q_3, B\}$			

DFA Bi-Simulation

- Checking the equivalence of the DFAs, i.e., $L(M_1) = L(M_2)$

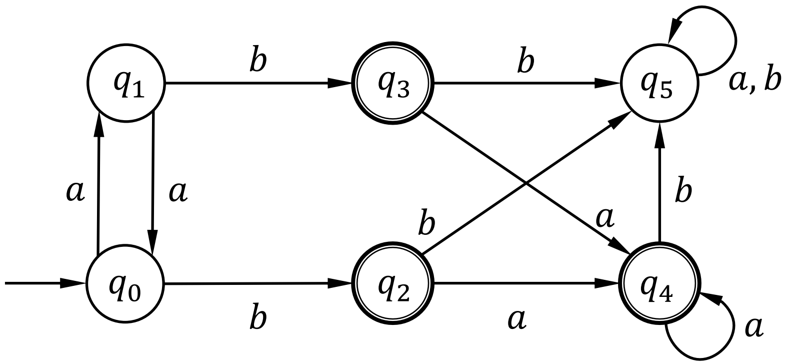


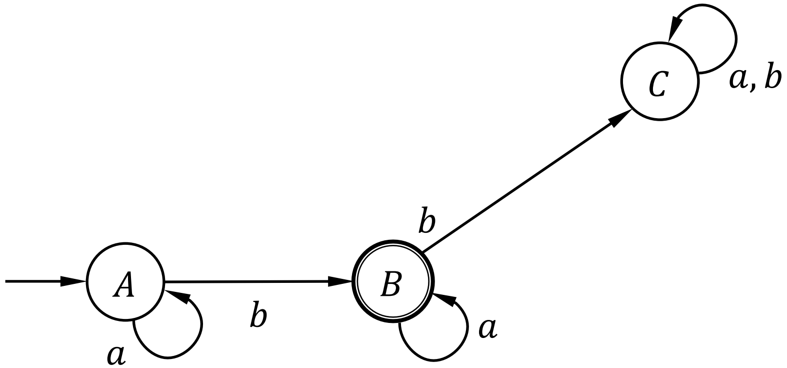


State Pairs	a	b	
$\{ q_0, A \}$	$\{ q_1, A \}$	$\{ q_2, B \}$	
$\{ q_1, A \}$	$\{ q_0, A \}$	$\{ q_3, B \}$	M_1 reaches final state if
$\{ q_2, B \}$	$\{ q_4, B \}$	$\{ q_5, C \}$	and only if
$\{ q_3, B \}$			M_2 reaches final state
$\{ q_4, B \}$			
$\{ q_5, C \}$			

DFA Bi-Simulation

- Checking the equivalence of the DFAs, i.e., $L(M_1) = L(M_2)$

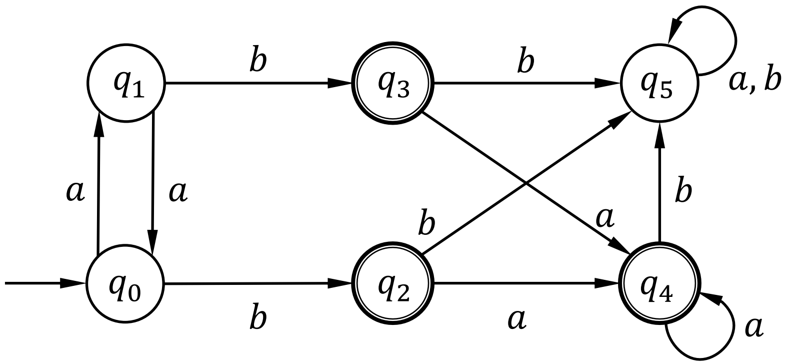


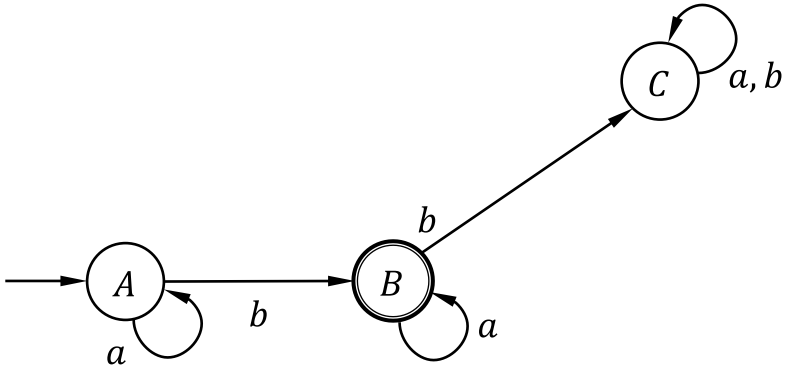


State Pairs	a	b	
$\{q_0, A\}$	$\{q_1, A\}$	$\{q_2, B\}$	
$\{q_1, A\}$	$\{q_0, A\}$	$\{q_3, B\}$	
$\{q_2, B\}$	$\{q_4, B\}$	$\{q_5, C\}$	M_1 reaches final state if and only if M_2 reaches final state
$\{q_3, B\}$	$\{q_4, B\}$	$\{q_5, C\}$	
$\{q_4, B\}$			
$\{q_5, C\}$			

DFA Bi-Simulation

- Checking the equivalence of the DFAs, i.e., $L(M_1) = L(M_2)$

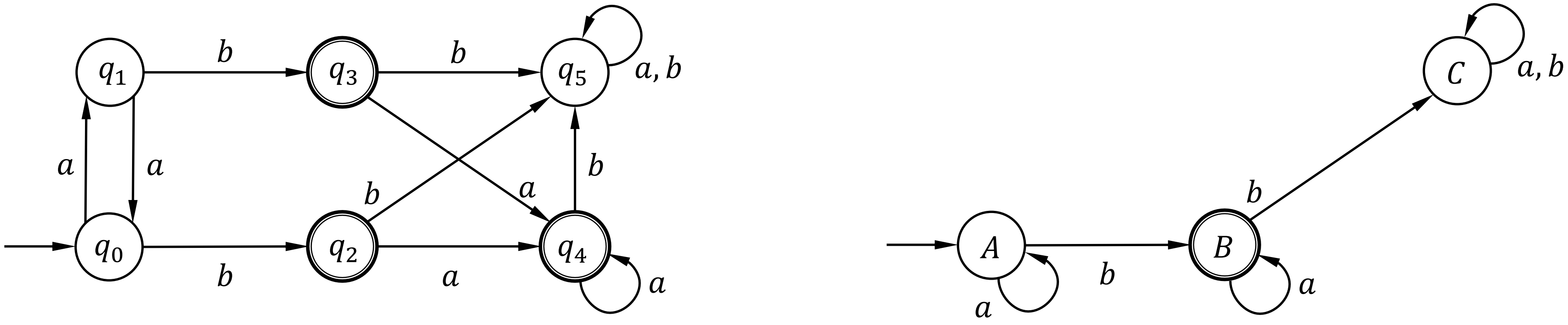




State Pairs	a	b	
$\{q_0, A\}$	$\{q_1, A\}$	$\{q_2, B\}$	
$\{q_1, A\}$	$\{q_0, A\}$	$\{q_3, B\}$	
$\{q_2, B\}$	$\{q_4, B\}$	$\{q_5, C\}$	M_1 reaches final state if and only if M_2 reaches final state
$\{q_3, B\}$	$\{q_4, B\}$	$\{q_5, C\}$	
$\{q_4, B\}$	$\{q_4, B\}$	$\{q_5, C\}$	
$\{q_5, C\}$			

DFA Bi-Simulation

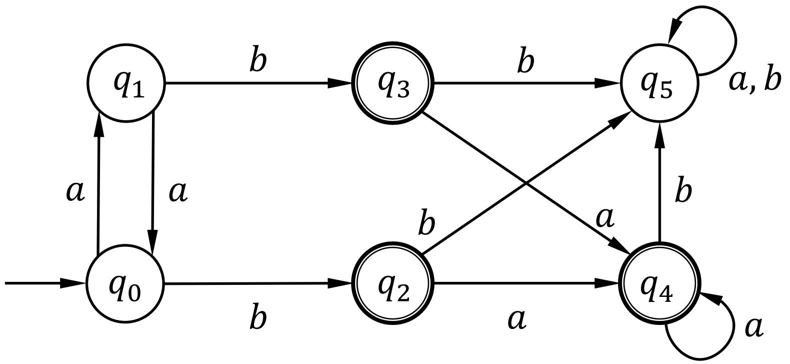
- Checking the equivalence of the DFAs, i.e., $L(M_1) = L(M_2)$

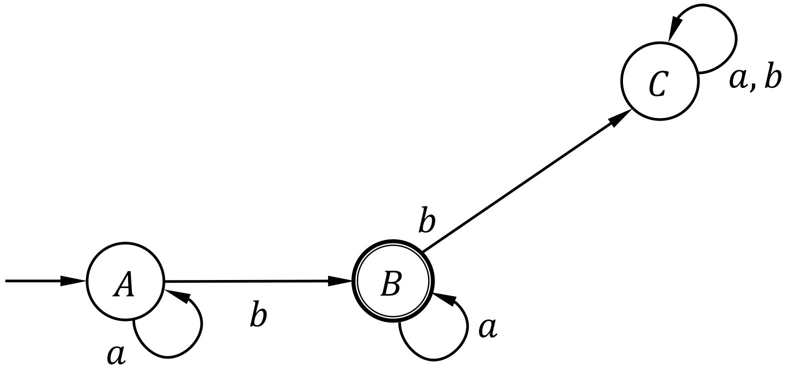


State Pairs	a	b	
$\{q_0, A\}$	$\{q_1, A\}$	$\{q_2, B\}$	
$\{q_1, A\}$	$\{q_0, A\}$	$\{q_3, B\}$	
$\{q_2, B\}$	$\{q_4, B\}$	$\{q_5, C\}$	M_1 reaches final state if
$\{q_3, B\}$	$\{q_4, B\}$	$\{q_5, C\}$	and only if
$\{q_4, B\}$	$\{q_4, B\}$	$\{q_5, C\}$	M_2 reaches final state
$\{q_5, C\}$	$\{q_5, C\}$	$\{q_5, C\}$	

DFA Bi-Simulation

- Checking the equivalence of the DFAs, i.e., $L(M_1) = L(M_2)$





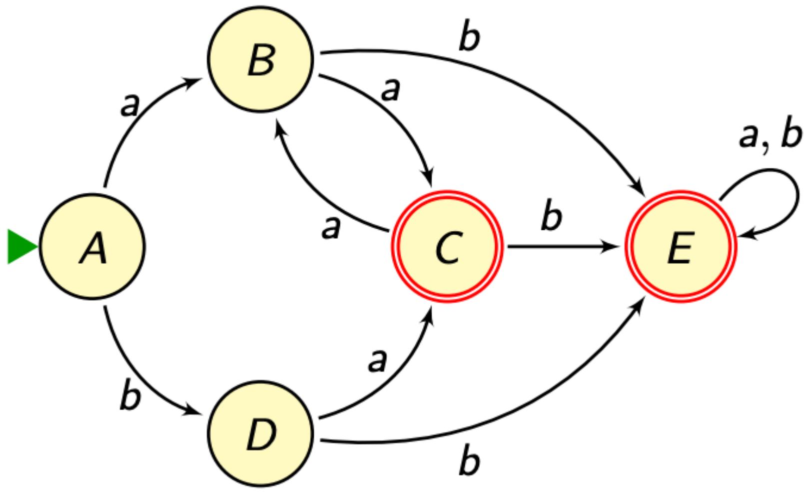
State Pairs	a	b
$\{q_0, A\}$	$\{q_1, A\}$	$\{q_2, B\}$
$\{q_1, A\}$	$\{q_0, A\}$	$\{q_3, B\}$
$\{q_2, B\}$	$\{q_4, B\}$	$\{q_5, C\}$
$\{q_3, B\}$	$\{q_4, B\}$	$\{q_5, C\}$
$\{q_4, B\}$	$\{q_4, B\}$	$\{q_5, C\}$
$\{q_5, C\}$	$\{q_5, C\}$	$\{q_5, C\}$

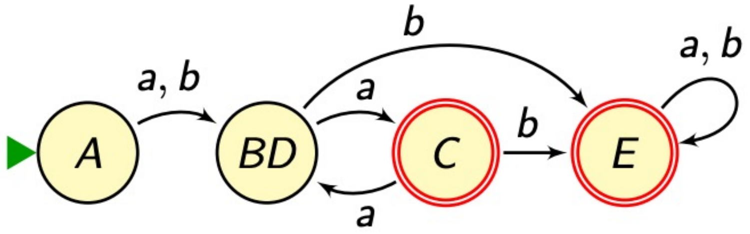
M_1 reaches final state if

and only if

M_2 reaches final state

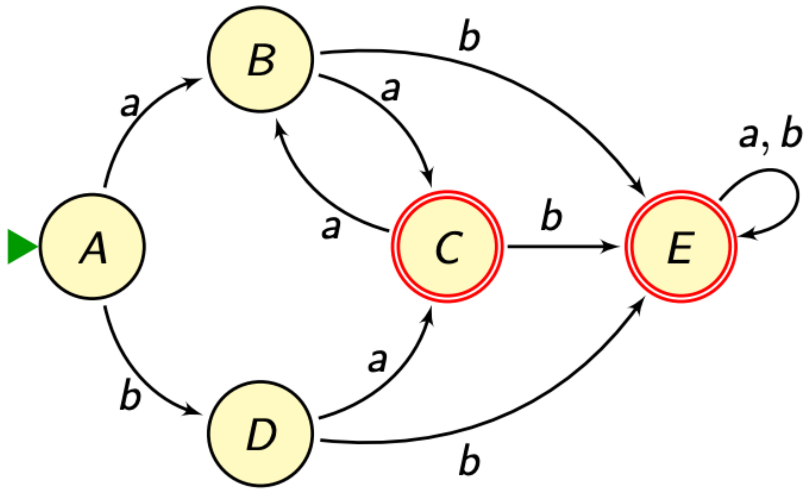
DFA Bi-Simulation

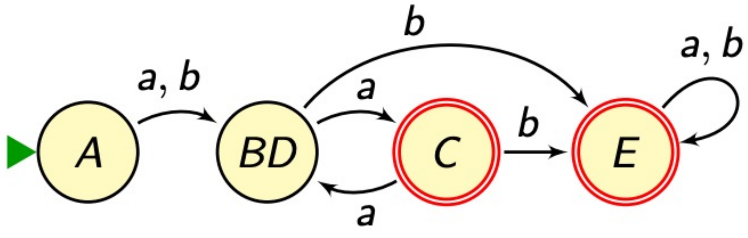




Have a Try!!

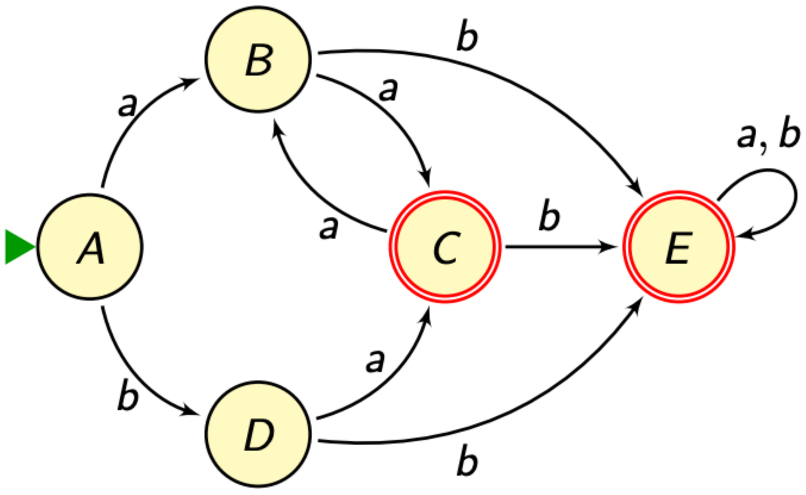
DFA Bi-Simulation

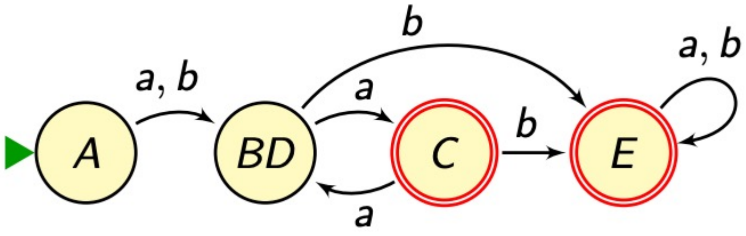




State Pairs	a	b	
(*A*, *A*’)	(*B*, *BD*’)	(*D*, *BD*’)	
			M_1 reaches final state if and only if M_2 reaches final state

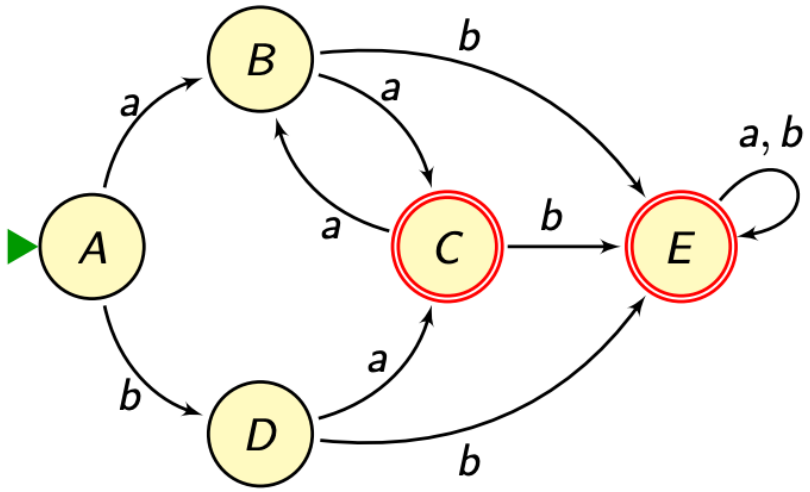
DFA Bi-Simulation

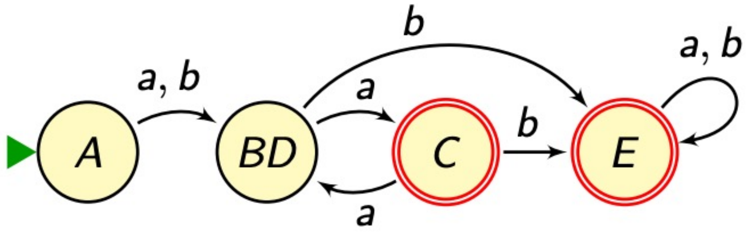




State Pairs	a	b	
(A, A')	(B, BD')	(D, BD')	
(B, BD')			M_1 reaches final state if and only if M_2 reaches final state
(D, BD')			

DFA Bi-Simulation

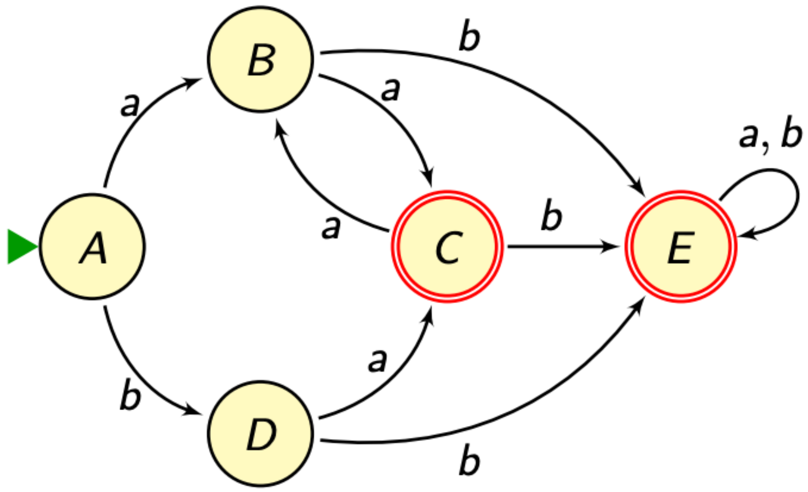


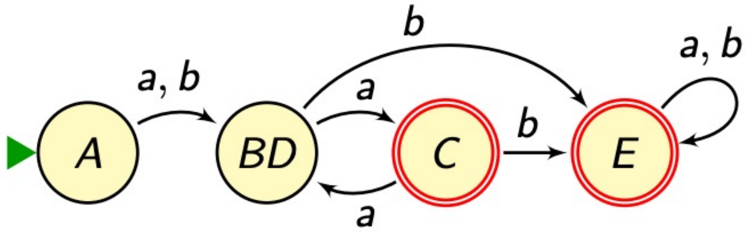


State Pairs	a	b	
(*A*, *A*')	(*B*, *BD*')	(*D*, *BD*')	
(*B*, *BD*')	(*C*, *C*')	(*E*, *E*')	
(*D*, *BD*')			

M_1 reaches final state if
and only if
M_2 reaches final state

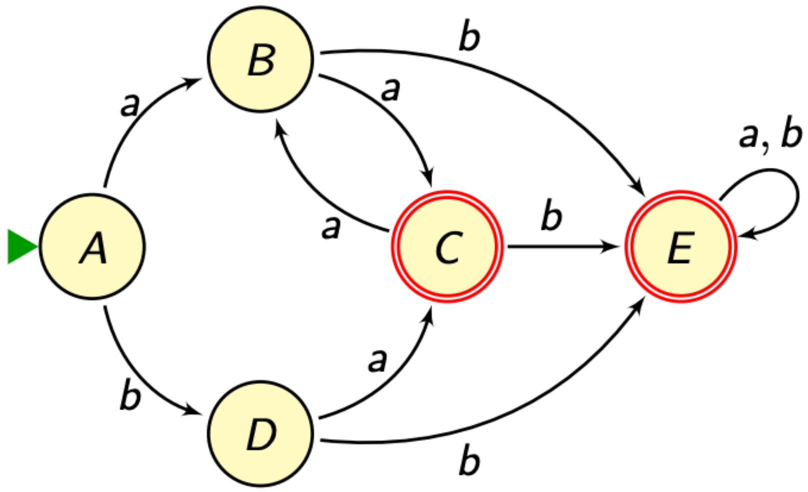
DFA Bi-Simulation

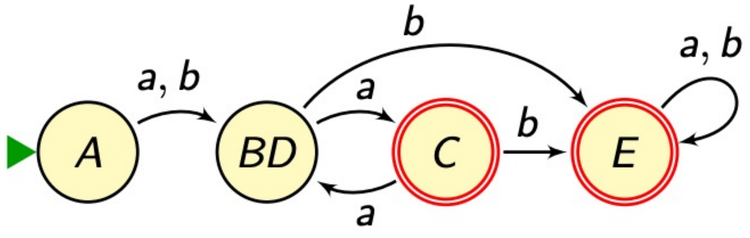




State Pairs	a	b	
(A, A')	(B, BD')	(D, BD')	
(B, BD')	(C, C')	(E, E')	
(D, BD')			M_1 reaches final state if and only if M_2 reaches final state
(C, C')			
(E, E')			

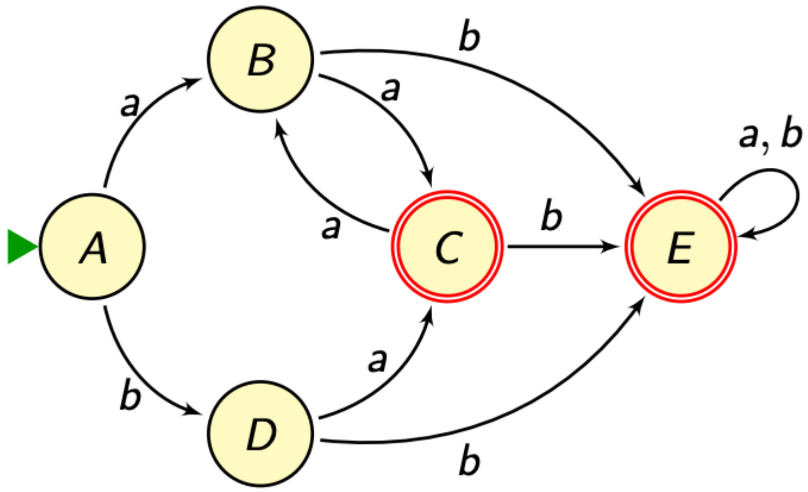
DFA Bi-Simulation

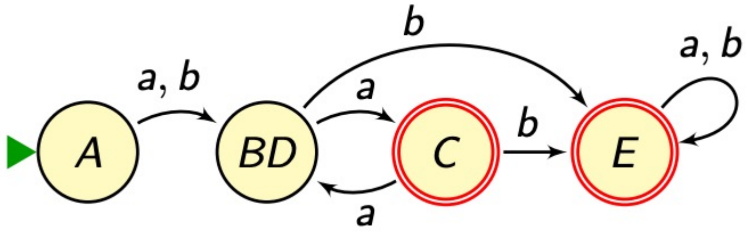




State Pairs	a	b	
(*A*, *A*’)	(*B*, *BD*’)	(*D*, *BD*’)	
(*B*, *BD*’)	(*C*, *C*’)	(*E*, *E*’)	M_1 reaches final state if
(*D*, *BD*’)	(*C*, *C*’)	(*E*, *E*’)	and only if
(*C*, *C*’)			M_2 reaches final state
(*E*, *E*’)			

DFA Bi-Simulation





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(A, A')	(B, BD')	(D, BD')	
(B, BD')	(C, C')	(E, E')	M_1 reaches final state if
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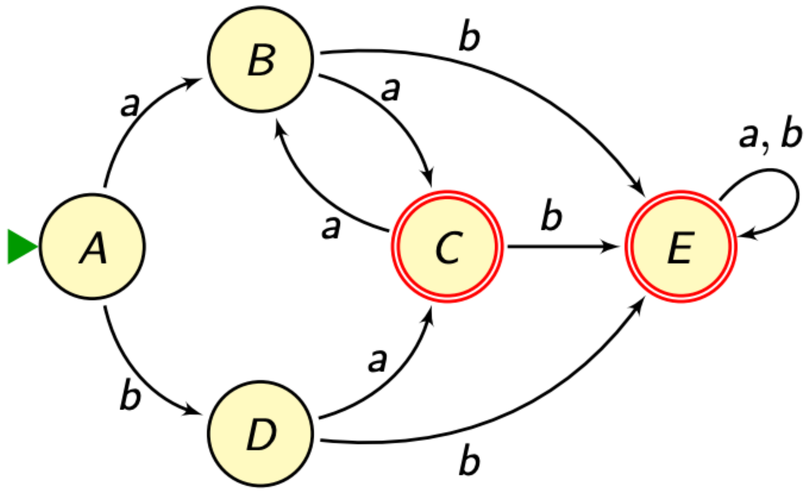

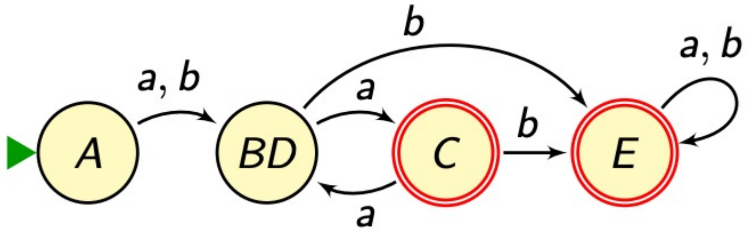

State Pairs	a	b	
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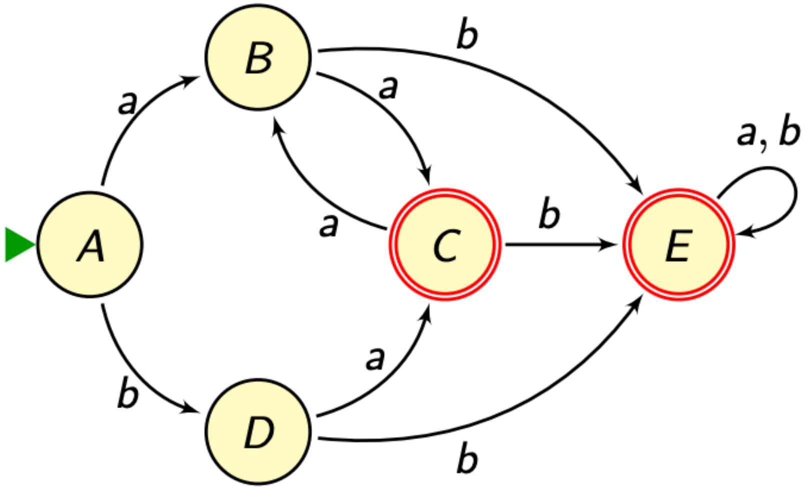
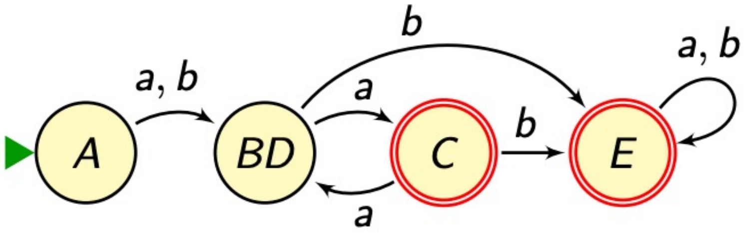
(B, BD')	(C, C')	(E, E')	M_1 reaches final state if and only if M_2 reaches final state
(D, BD')	(C, C')	(E, E')	
(C, C')	(B, BD')	(E, E')	
(E, E')	(E, E')	(E, E')	

DFA Bi-Simulation



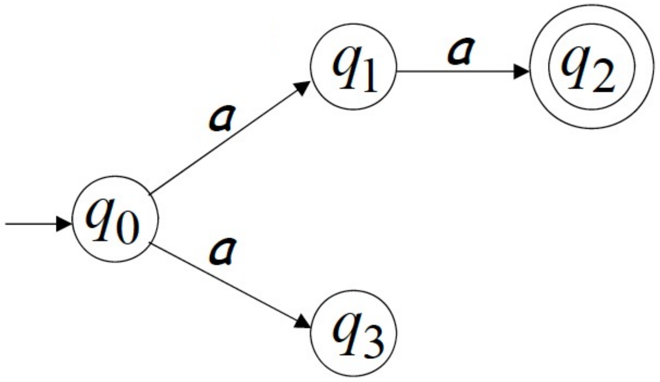


State Pairs	a	b	
(*A*, *A*’)	(*B*, *BD*’)	(*D*, *BD*’)	
(*B*, *BD*’)	(*C*, *C*’)	(*E*, *E*’)	M_1 reaches final state if
(*D*, *BD*’)	(*C*, *C*’)	(*E*, *E*’)	and only if
(*C*, *C*’)	(*B*, *BD*’)	(*E*, *E*’)	M_2 reaches final state
(*E*, *E*’)	(*E*, *E*’)	(*E*, *E*’)	

PART III:
Non-deterministic Finite Automata

Non-deterministic Finite Automata

- There are multiple choices of state transition



Non-deterministic Finite Automata

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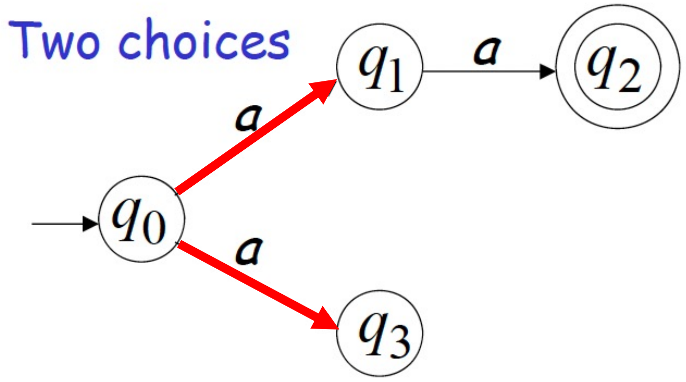

- Given a string, *aa*, there are two choices for the first transition

Non-deterministic Finite Automata

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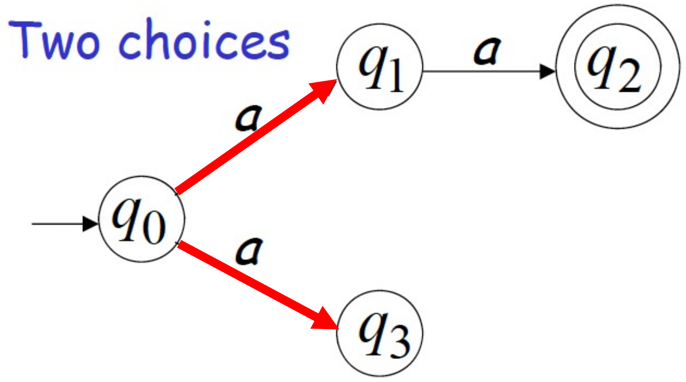
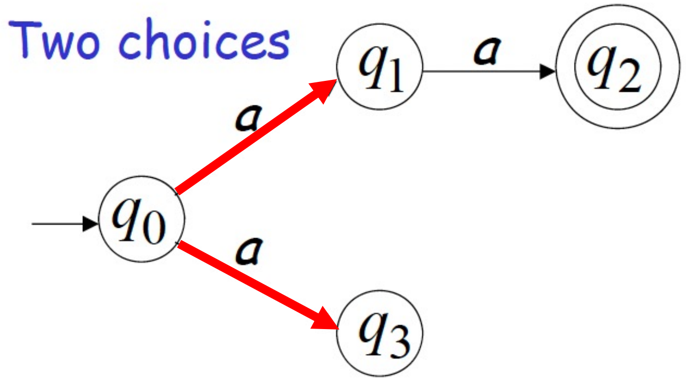


- Given a string, *aa*, there are two choices for the first transition
- If one choice does not work, we need try others

Non-deterministic Finite Automata

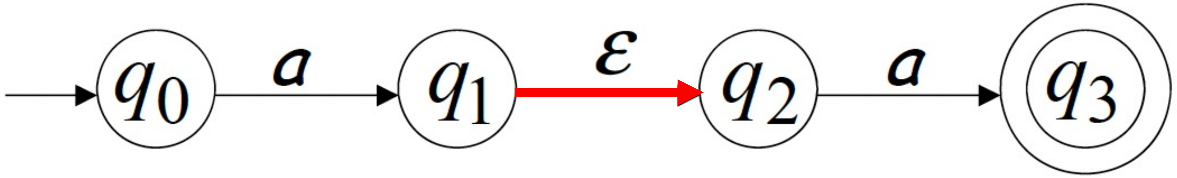
- There are multiple choices of state transition



- Given a string, *aa*, there are two choices for the first transition
- A string, e.g., *aa*, can be accepted by NFA as long as there exists one computation of the NFA accepts the string

ϵ-Transition

- Transition to the next states without reading any inputs



- NFA also allows ϵ-transitions!

Recap: Definition of DFA

- A DFA is a five-tuple: $(Q, \Sigma, \delta, q_0, F)$
  - Q: A finite set of states
  - Σ: A finite set of input characters, i.e., an alphabet
  - $\delta: Q \times \Sigma \longmapsto$ Q: Transition function, e.g., $\delta(q, a) = q'$
  - $q_0 \in Q$: The start state
  - $F \subseteq Q$: A finite subset of final states

Definition of NFA

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  - Σ: A finite set of input characters, i.e., an alphabet
  - $\delta: Q \times (\Sigma \cup \{\epsilon\}) \mapsto \cancel{Q} 2^{Q}$: Transition function, e.g., $\delta(q, a) = \{q', q''\}$
  - $q_0 \in Q$: The start state
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ϵ-Closure

- ϵ-*closure(q)* returns all states *q* can reach via ϵ-transitions, including *q* itself

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  - $q' \in \delta^*(q, w)$, where $w \in \Sigma^*$, if and only if
    - (1) there's a walk from q to q'' with w
    - (2) $q' \in \epsilon-closure(q'')$

Defining a Language by NFA

- **Recap: language defined by DFA**
- Take a DFA $M = (Q, \Sigma, \delta, q_0, F)$, language can be accepted by the DFA is written as $L(M) = \{w \in \Sigma^*: \delta^*(q_0, w) \subseteq F\}$

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- Take a NFA $M = (Q, \Sigma \cup \{\epsilon\}, \delta, q_0, F)$, language can be accepted by the NFA is written as $L(M) = \{w \in \Sigma^*: \delta^*(q_0, w) \cap F \neq \emptyset\}$

NFA = DFA

- Every DFA is trivially an equivalent NFA
- Every NFA N can be converted to an equivalent DFA D

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- Every NFA N can be converted to an equivalent DFA D
  - Every string accepted by the NFA is accepted by the DFA
  - Every string rejected by the NFA is rejected by the DFA
  - i.e., $L(N) = L(D)$

From NFA to DFA

- Given an NFA $M = (Q, \Sigma \cup \{\epsilon\}, \delta, q_0, F)$
  - **Step 1:** initialize a DFA with a start state $\{\ q_0\ \}$, which is a set of NFA states

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DFA state is a subset of NFA states

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$$\text{let } S = \bigcup \begin{cases} \delta^*(q_i, a \in \Sigma) \\ \delta^*(q_j, a \in \Sigma) \\ \ldots \\ \delta^*(q_m, a \in \Sigma) \end{cases}$$

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if any q_i is a final state, S is a final state of the DFA

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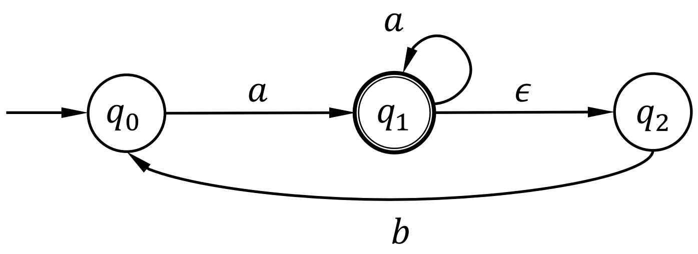
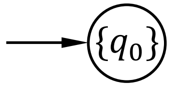
For any input string, the DFA and the NFA reach the final state at the same time, thus DFA=NFA

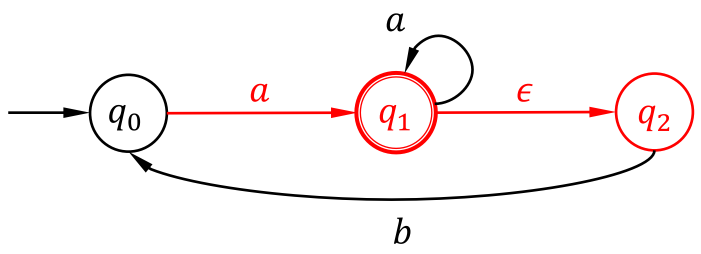
  - **Step 3**: add transition $\delta(\{\, q_i, q_j, \ldots, q_m \,\}, a) = S$ in the DFA

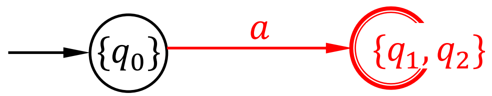
From NFA to DFA





From NFA to DFA





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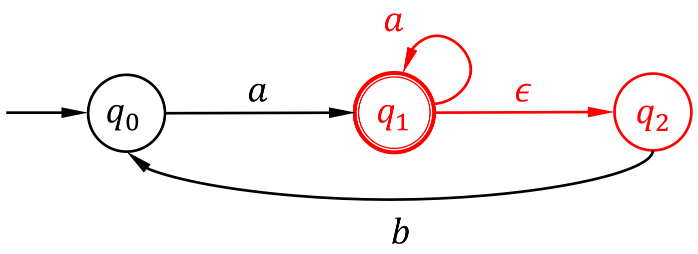

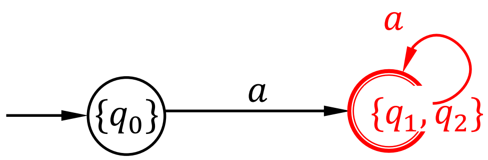

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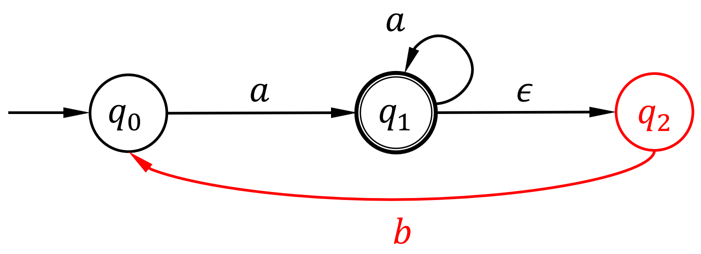

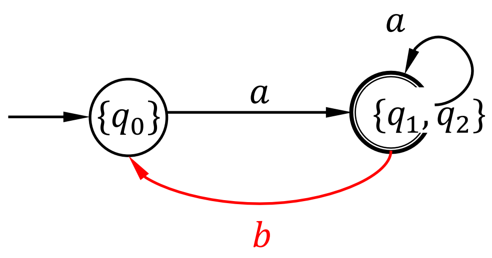

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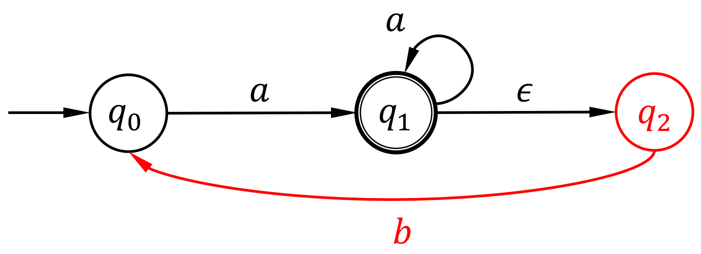
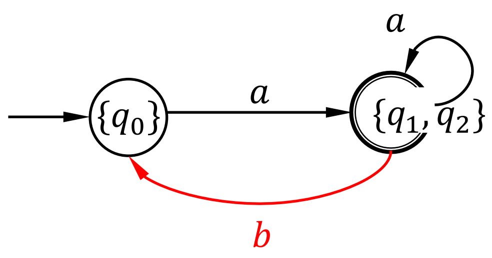
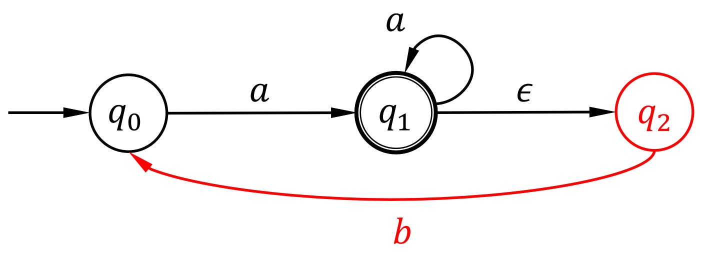
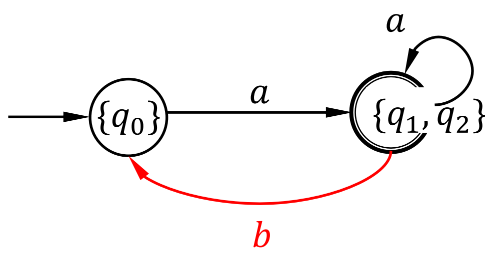




Because the DFA states consist of sets of NFA states, an *n*-state NFA may be converted to a DFA with at most 2^n states. For every *n*, there exist *n*-state NFAs such that every subset of states is reachable from the initial subset, so that the converted DFA has exactly 2^n states, giving $\Theta(2^n)$ worst-case time complexity.

From NFA to DFA





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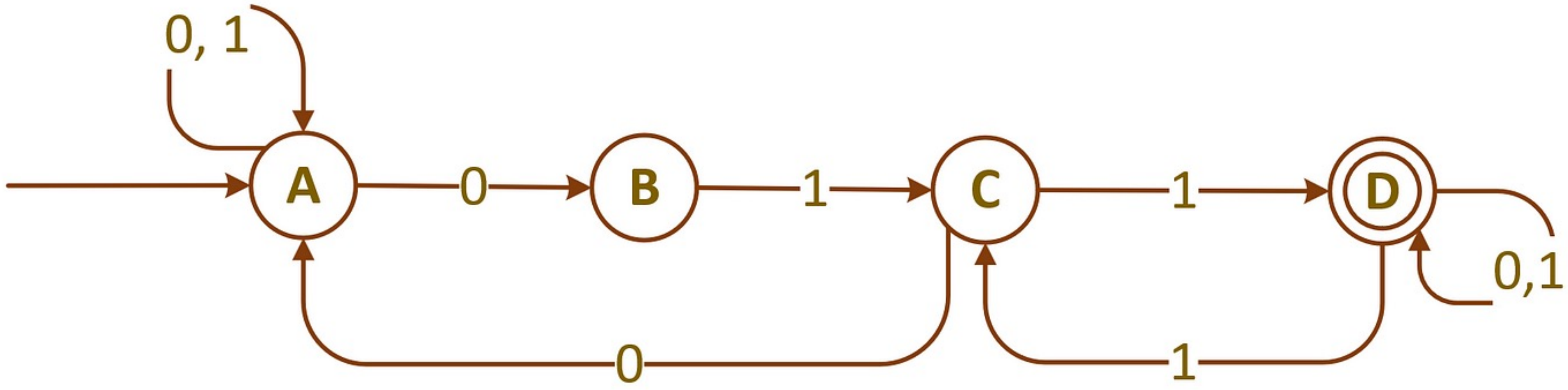
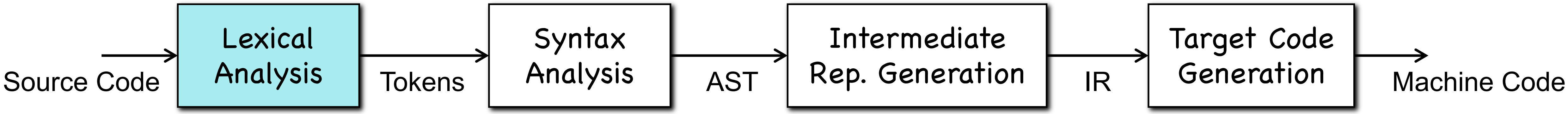
When converting an NFA to a DFA, there is no guarantee that we will have a smaller DFA.

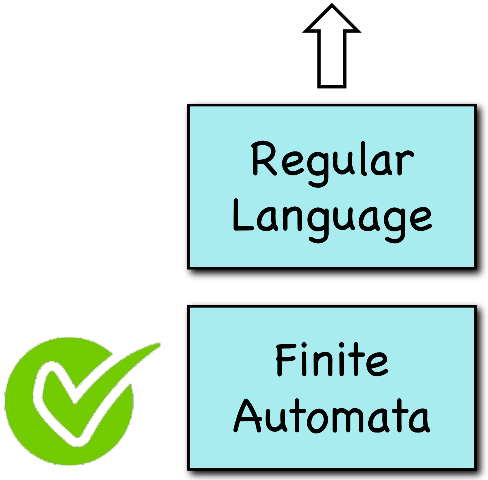
From NFA to DFA



Have a Try!!

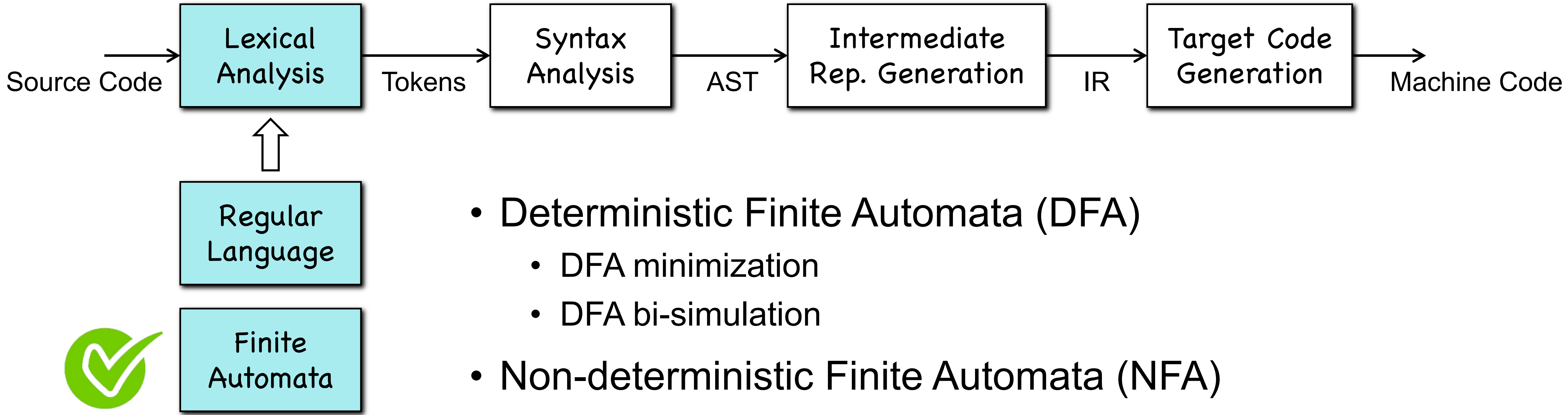
Summary





- Deterministic Finite Automata (DFA)
  - DFA minimization
  - DFA bi-simulation
- Non-deterministic Finite Automata (NFA)
  - NFA = DFA
  - NFA → DFA
- Languages defined by DFA/NFA

Summary



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THANKS!