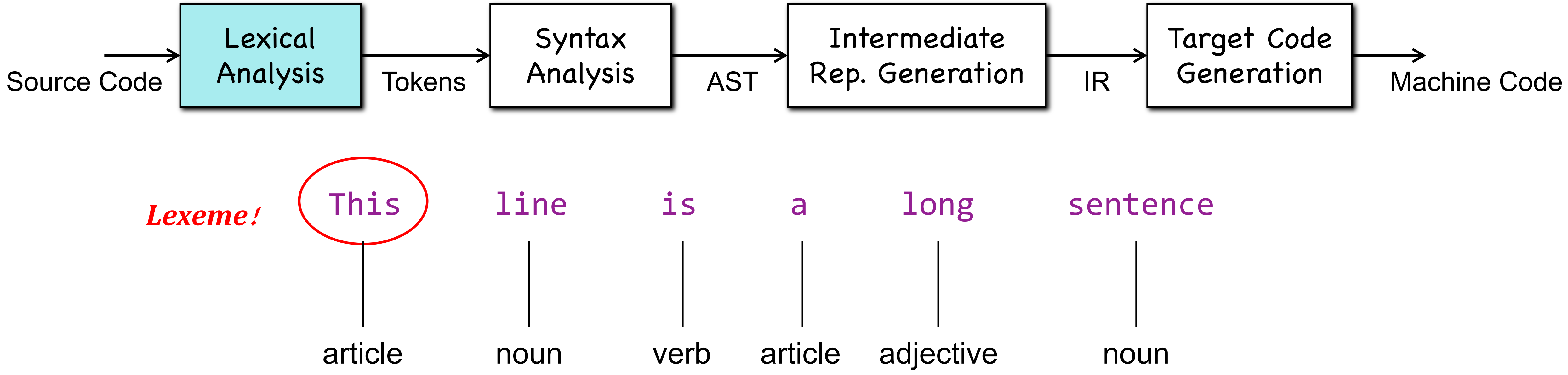
Chapter 4-1
Context-Free Language

Natural Language Perspective



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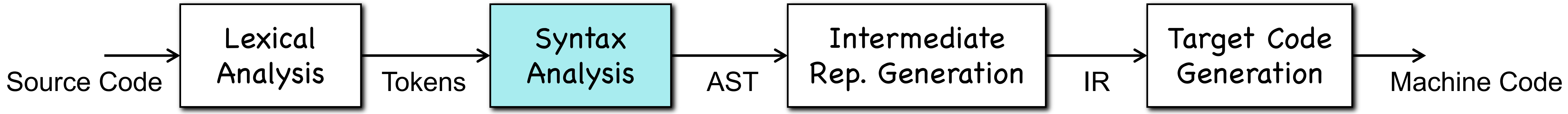
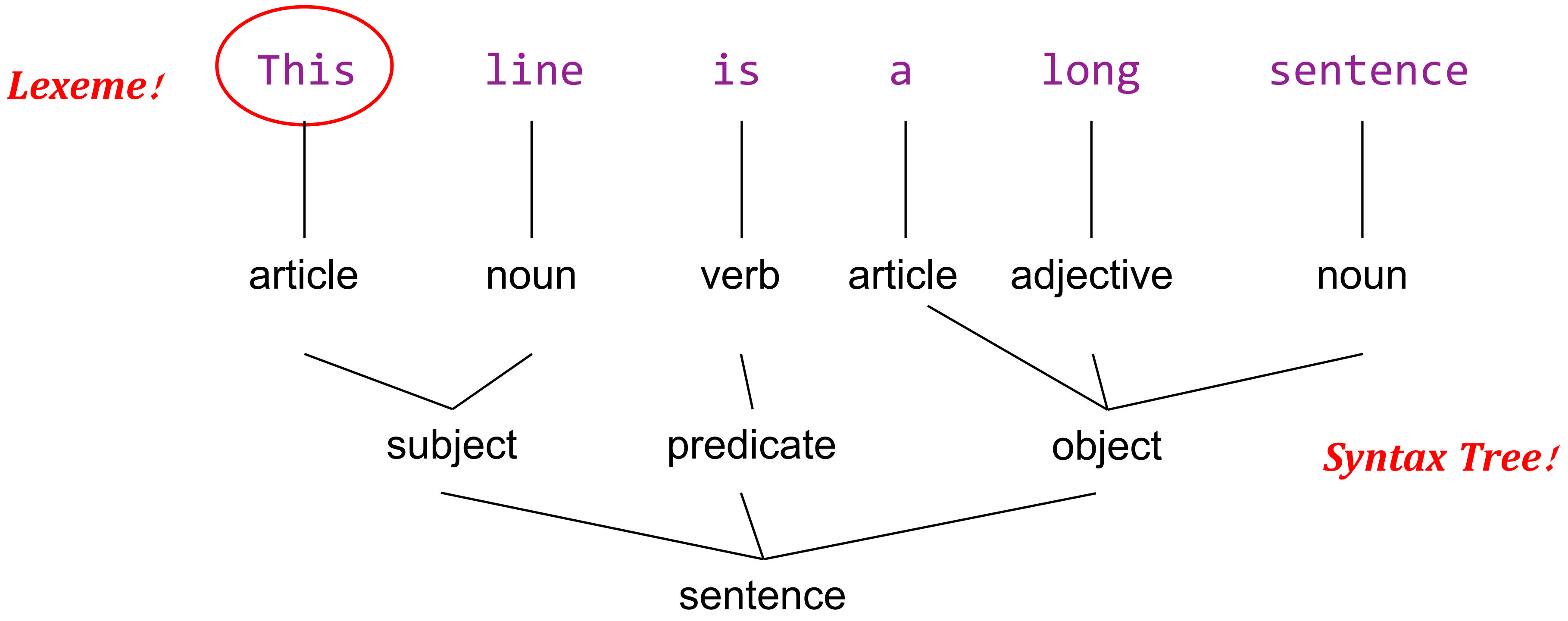
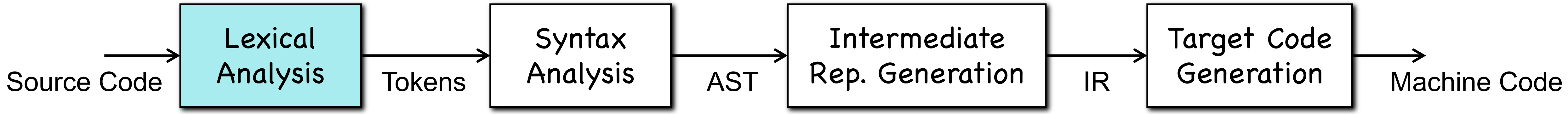




Programming Lang Perspective




```
position   =   initial   +   rate   *   60
<id,1>   <=>   <id,2>   <+>   <id,3>   <*>   <number,60>
```

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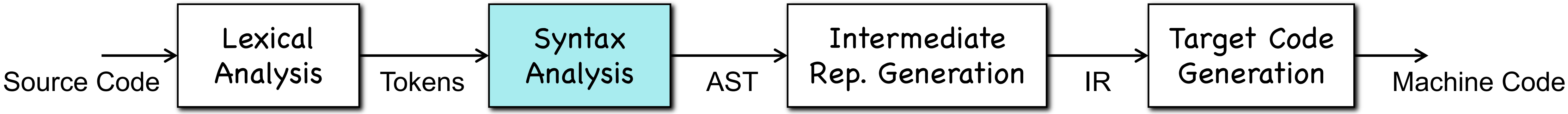
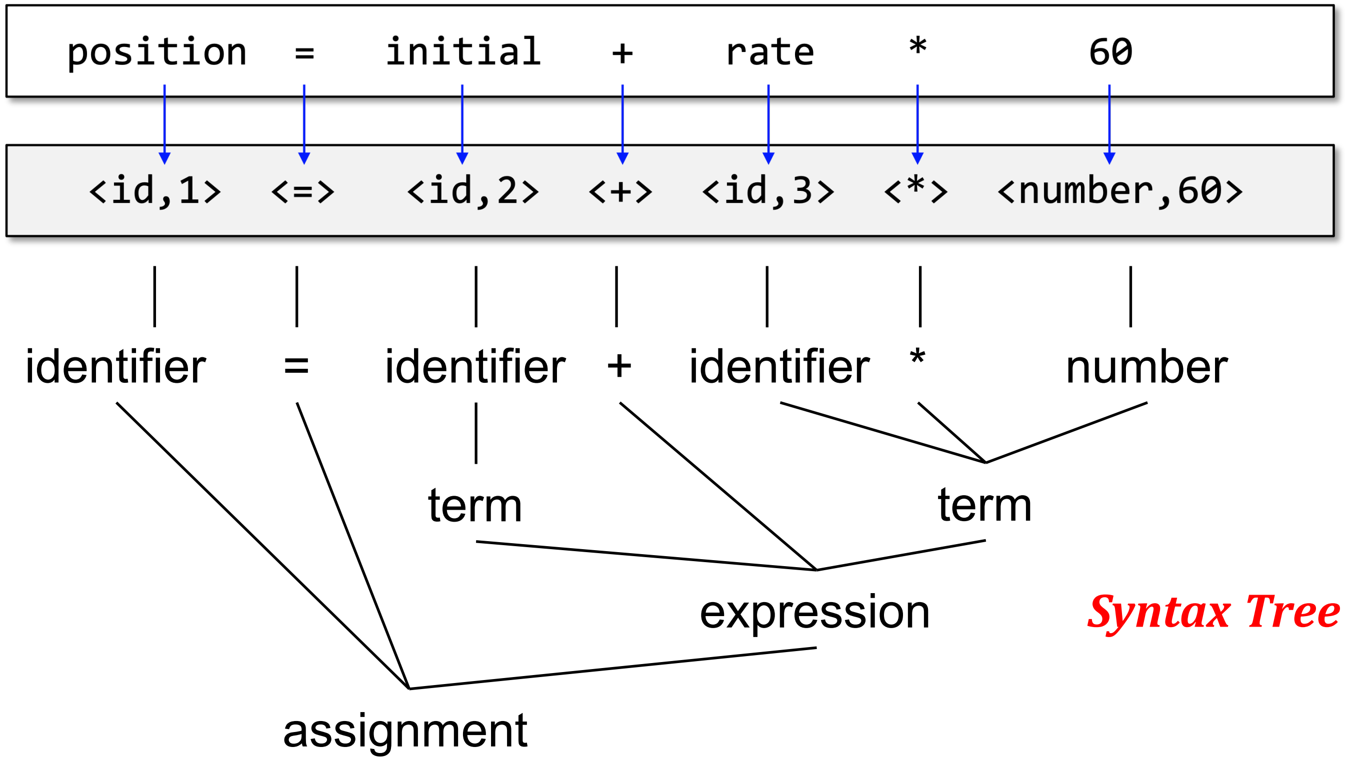
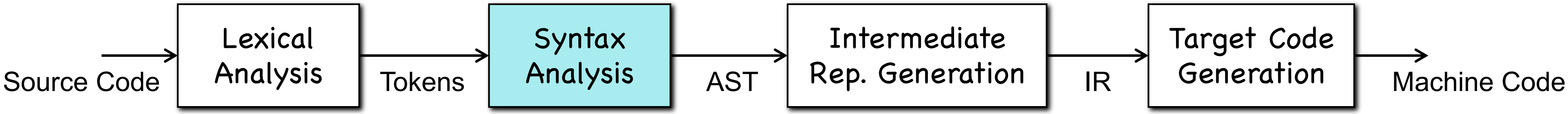
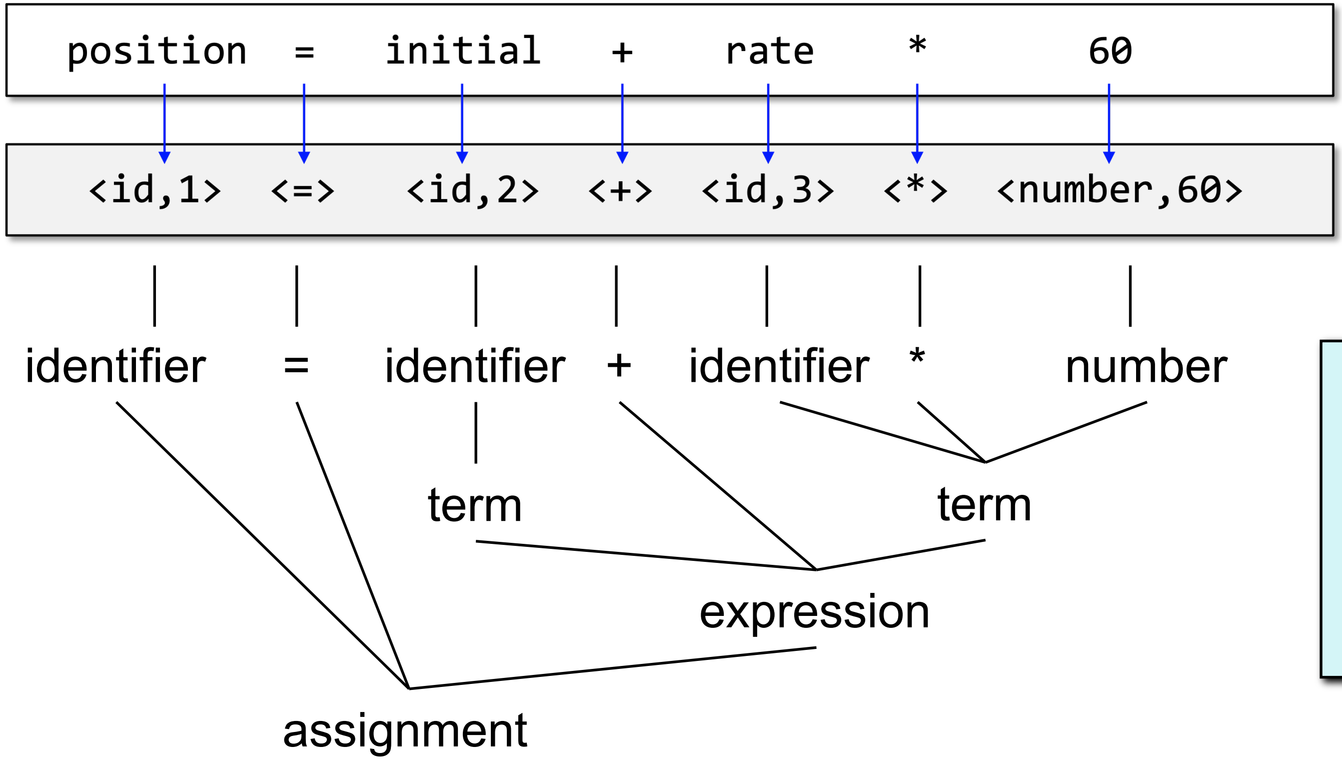




Syntax Tree!

Programming Lang Perspective





assignment → identifier = expression

expression → term + term

term → identifier

| identifier * number

Programming Lang Perspective

Source Code → Lexical Analysis → Tokens → Syntax Analysis → AST → Intermediate Rep. Generation → IR → Target Code Generation → Machine Code

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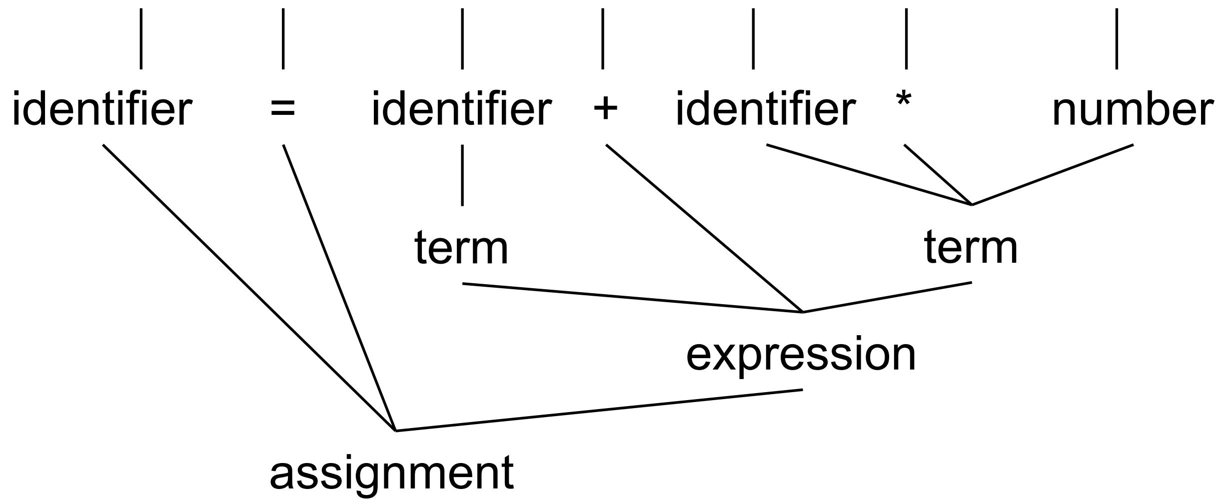

Context-Free Grammar!

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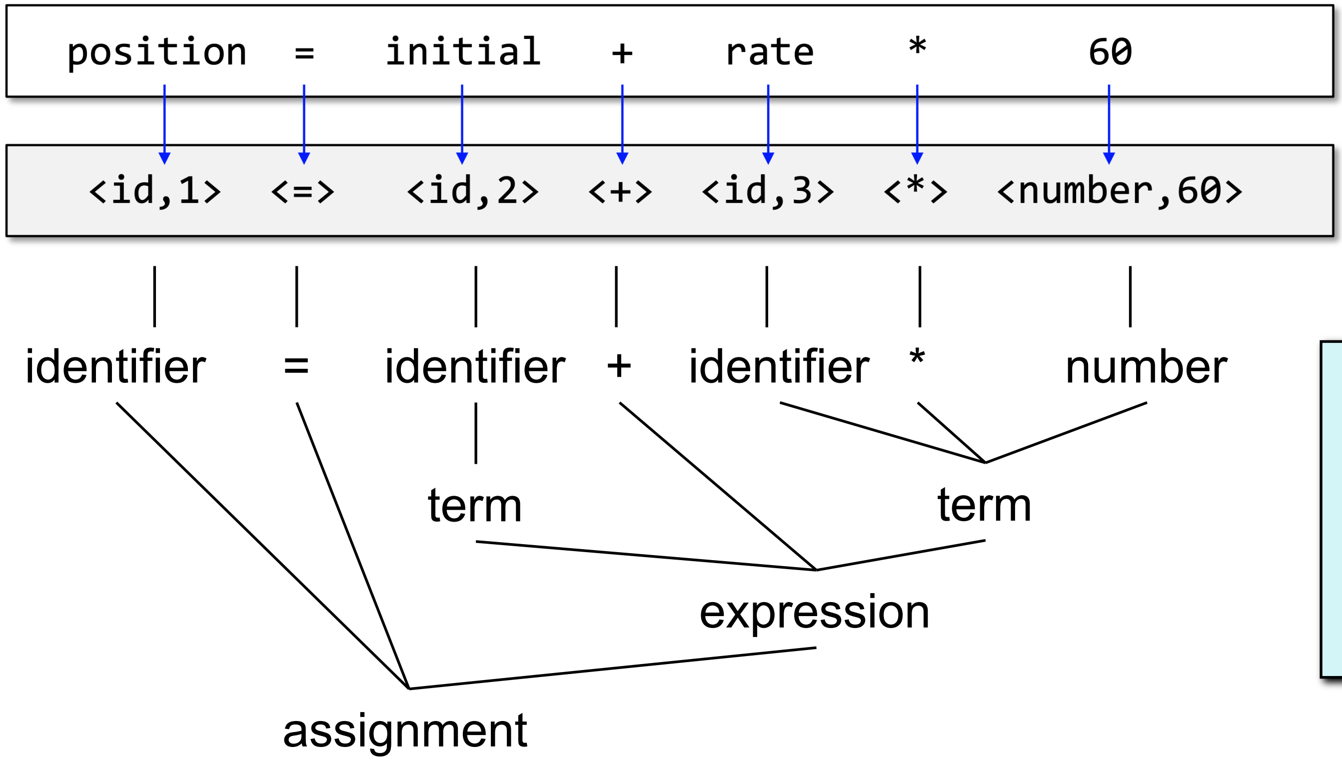

Context-Free Grammar!
Context-Free Language!

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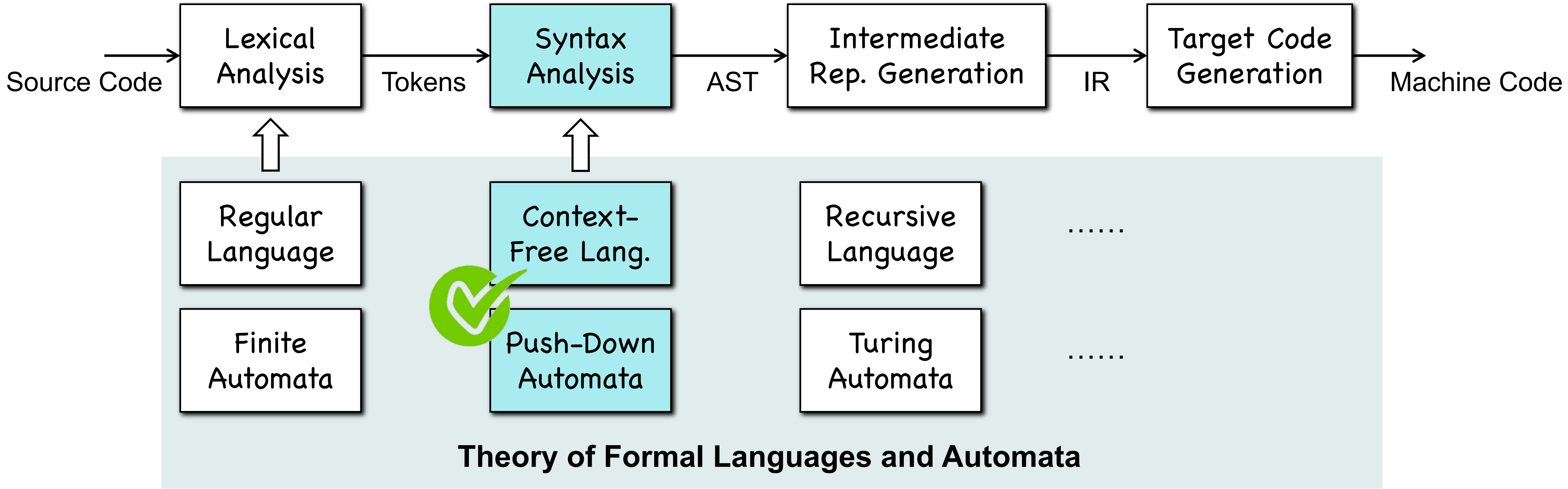
Syntax Analysis



PART I: Context-Free Language

Context Free Grammar (CFG)

- A context-free grammar is a tuple $G = (N, T, S, P)$
  - N: a finite set of non-terminals
  - T: a finite set of terminals, such that $N \cap T = \emptyset$

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Derivation ($\Rightarrow_G$)

- Assume: $A \rightarrow \gamma$
- Derivation: $\alpha A \beta \Rightarrow_G \alpha \gamma \beta$

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  - $S \Rightarrow^* aaabbb$
  - $\underbrace{S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb}_{S \to aSb} \underbrace{\Rightarrow aaabbb}_{S \to \epsilon}$

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- **Context-Free Language**: $L(G) = \{w \in T^* \colon S \Rightarrow^* w\}$

Left-Most Derivation ($\Rightarrow_{lm}$)

- In every step derivation, we replace the left-most non-terminal

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- Left-most derivation for $v * (v + d)$

$$\boxed{\begin{aligned} &E \rightarrow EOE \\ &E \rightarrow (E) \\ &E \rightarrow v \mid d \\ &O \rightarrow + \mid \ * \end{aligned}}$$

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$E \Rightarrow_{lm} EOE \Rightarrow_{lm} vOE$

$E \to EOE$
$E \to (E)$
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$$E \Rightarrow_{lm} EOE \Rightarrow_{lm} vOE \Rightarrow_{lm} v * E$$

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$$\boxed{\begin{array}{l} E \to EOE \\ E \to (E) \\ E \to v \mid d \\ O \to + \mid * \end{array}}$$

- Written as $E \Rightarrow_{lm}^{*} v * (v + d)$

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- Right-most derivation for $v * (v + d)$

$$E \Rightarrow_{rm} EOE \Rightarrow_{rm} EO(E) \Rightarrow_{rm} EO(EOE) \Rightarrow_{rm} EO(EOd) \Rightarrow_{rm} EO(E + d)$$
$$\Rightarrow_{rm} EO(v + d) \Rightarrow_{rm} E * (v + d) \Rightarrow_{rm} v * (v + d)$$

$$E \to EOE$$
$$E \to (E)$$
$$E \to v \mid d$$
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- Written as $E \Rightarrow_{rm}^{*} v * (v + d)$

Parse/Syntax Trees

- Derivation is a procedure of building a parse tree

$$E \Rightarrow_{lm} EOE \Rightarrow_{lm} vOE \Rightarrow_{lm} v * E \Rightarrow_{lm} v * (E) \Rightarrow_{lm} v * (EOE)$$
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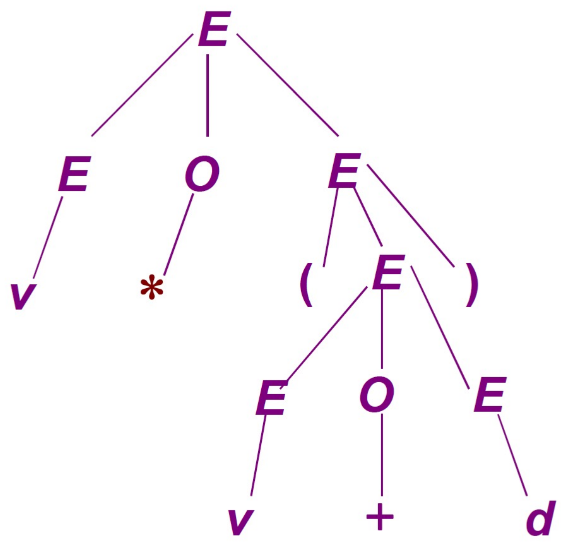

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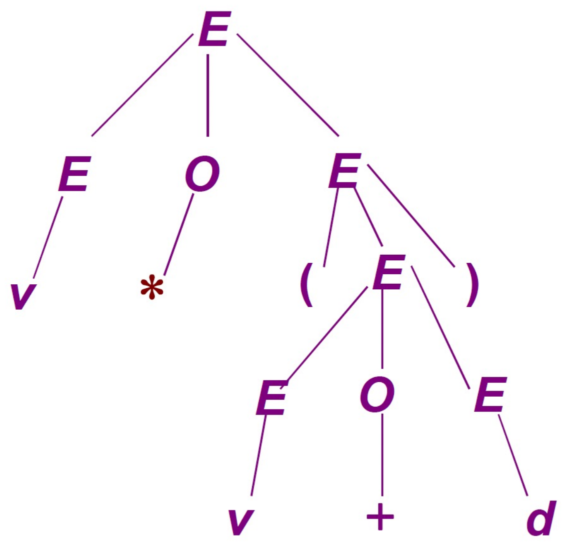


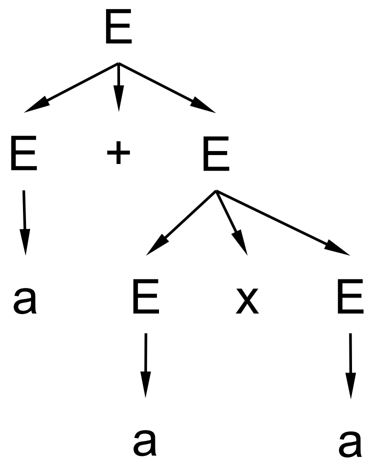
- Parse tree:
  - Every leaf is a terminal
  - Every non-leaf node is a non-terminal

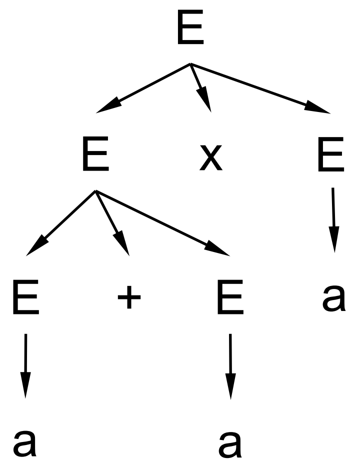
Ambiguity

- $E \rightarrow E + E \mid E \times E \mid a$
- $w = a + a \times a$

$$E \Rightarrow E + E \Rightarrow E + E \times E \Rightarrow \cdots \Rightarrow w$$

$$E \Rightarrow E \times E \Rightarrow E + E \times E \Rightarrow \cdots \Rightarrow w$$





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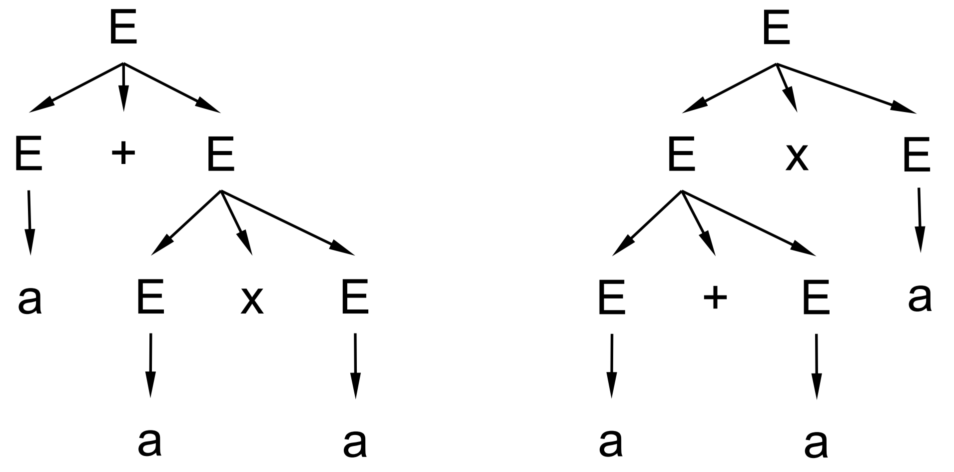

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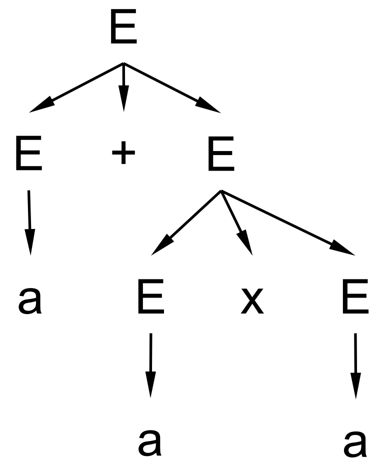

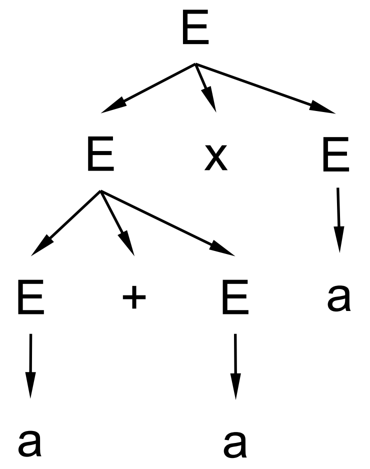

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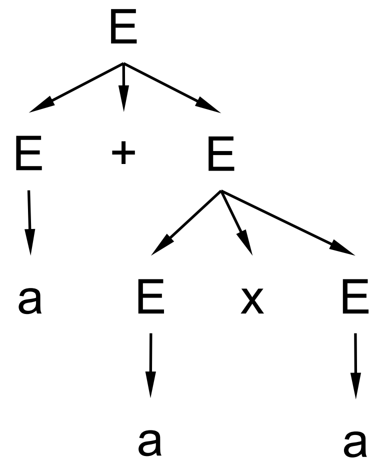
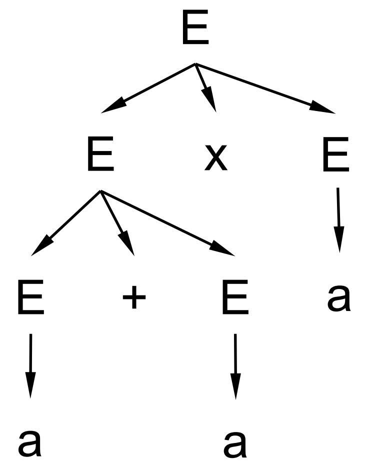
2 + 2 x 2 = 6



2 + 2 x 2 = 8



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- There is not a general technique to eliminate the ambiguity!
- There is not an algorithm that can decide if a grammar is ambiguous (This is an undecidable problem!)

Eliminating Ambiguity

- Lifting operators with higher priority
- $E \rightarrow E + E \mid E \times E \mid a$

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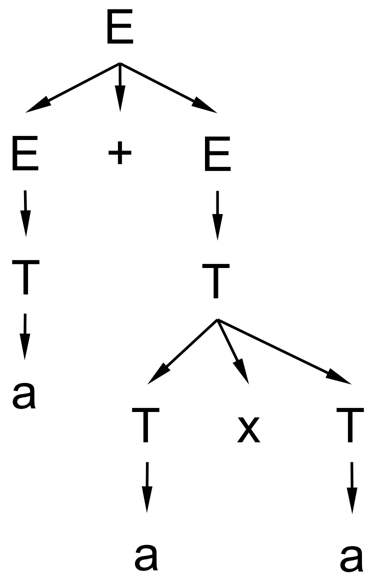

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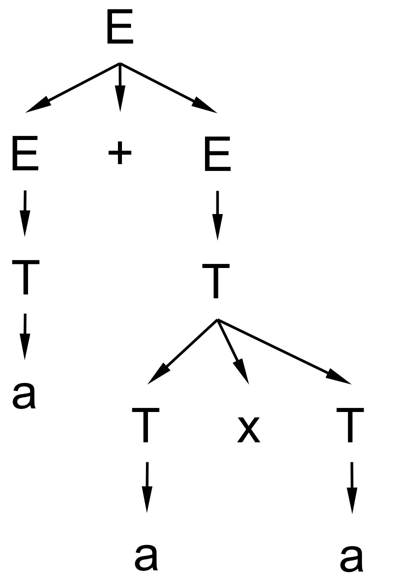

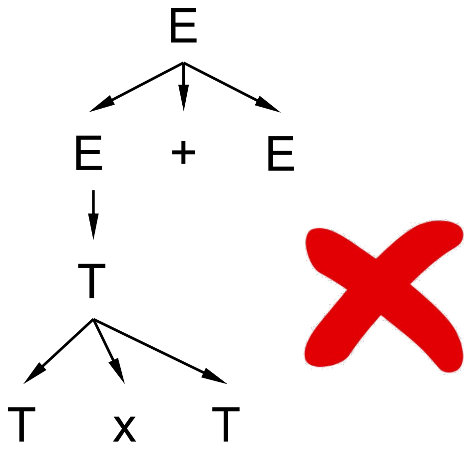

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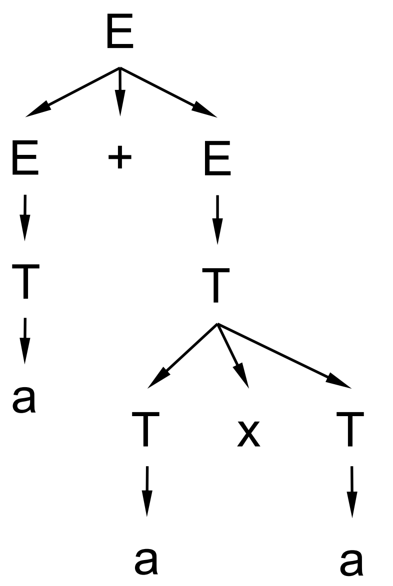
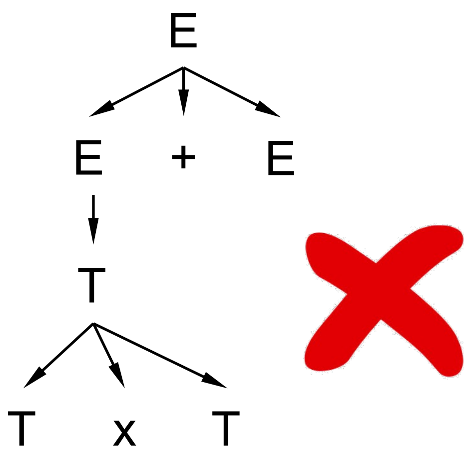
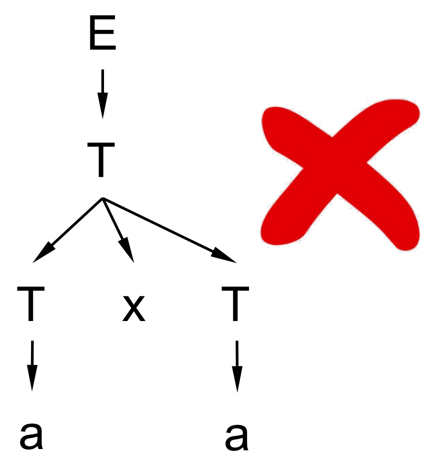
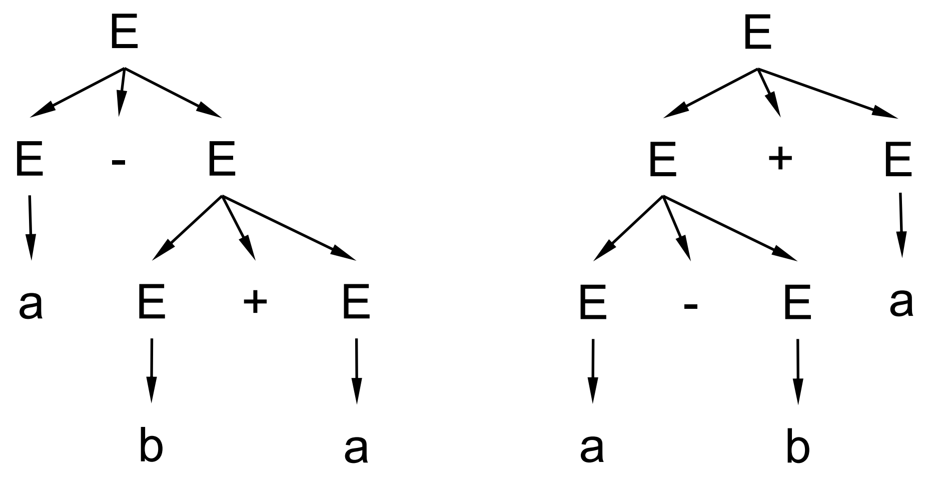






Eliminating Ambiguity

- $E \to E + E \mid E - E \mid T;$
- $T \to T \times T \mid a \mid b$
- ----------------
- $w = a - b + a$



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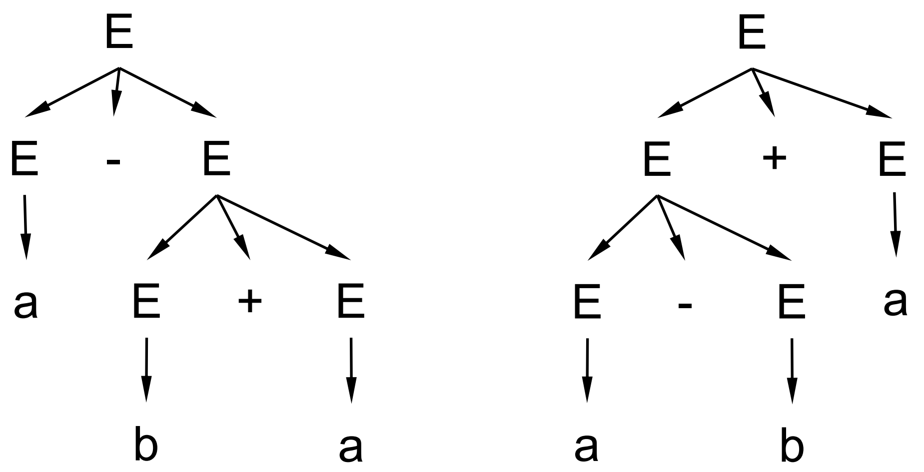

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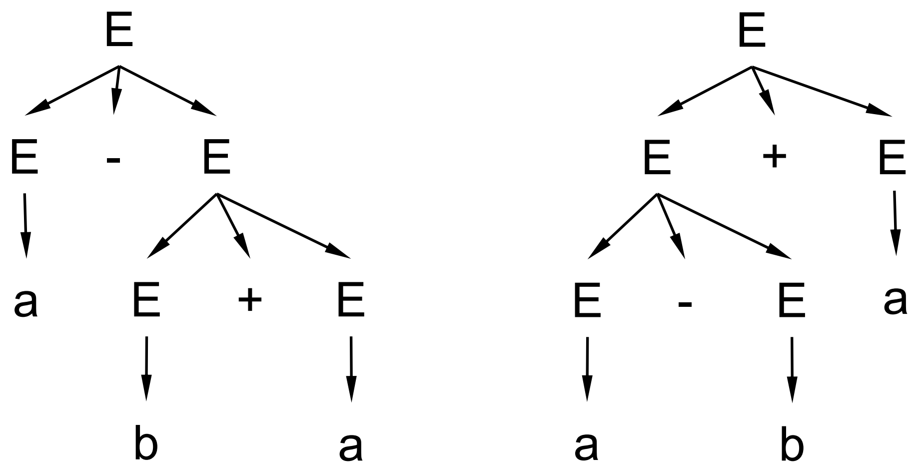
- $E \rightarrow E + T \mid E - T \mid T; \quad T \rightarrow T \times T \mid a \mid b$



Eliminating Ambiguity

$$
\begin{array}{lcl}
\mathit{stmt} & \rightarrow & \textbf{if } \mathit{expr} \textbf{ then } \mathit{stmt} \\
& | & \textbf{if } \mathit{expr} \textbf{ then } \mathit{stmt} \textbf{ else } \mathit{stmt} \\
& | & \textbf{other}
\end{array}
$$

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 & | & \textbf{if } expr \textbf{ then } stmt \textbf{ else } stmt \\
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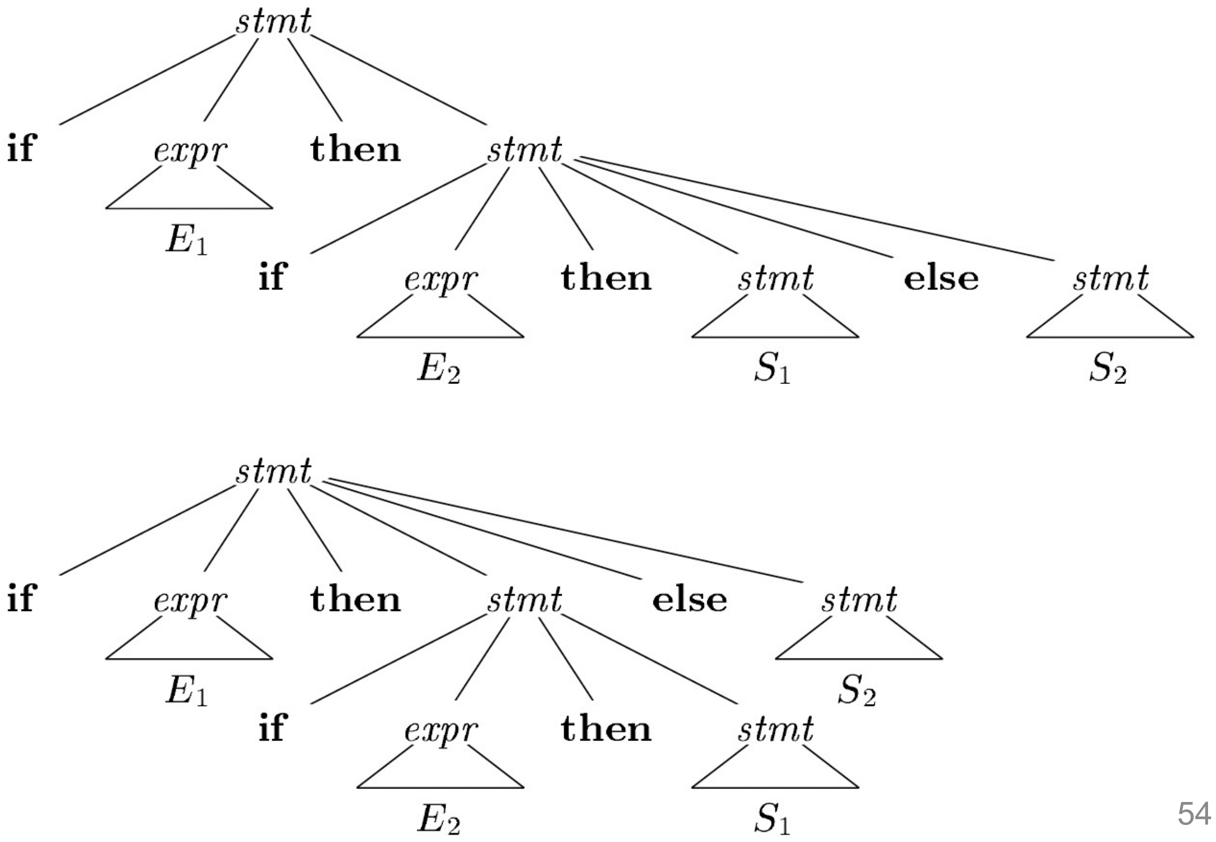
\end{array}
$$

if E_1 **then if** E_2 **then** S_1 **else** S_2

Eliminating Ambiguity

$$
\begin{array}{lcl}
\mathit{stmt} & \rightarrow & \textbf{if } \mathit{expr} \textbf{ then } \mathit{stmt} \\
 & | & \textbf{if } \mathit{expr} \textbf{ then } \mathit{stmt} \textbf{ else } \mathit{stmt} \\
 & | & \textbf{other}
\end{array}
$$

if E_1 **then** **if** E_2 **then** S_1 **else** S_2



Eliminating Ambiguity

stmt →

$$
\begin{array}{rcl}
stmt & \rightarrow & matched\_stmt \\
 & | & open\_stmt \\
matched\_stmt & \rightarrow & \textbf{if}\ expr\ \textbf{then}\ matched\_stmt\ \textbf{else}\ matched\_stmt \\
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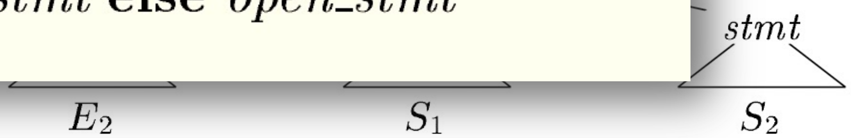
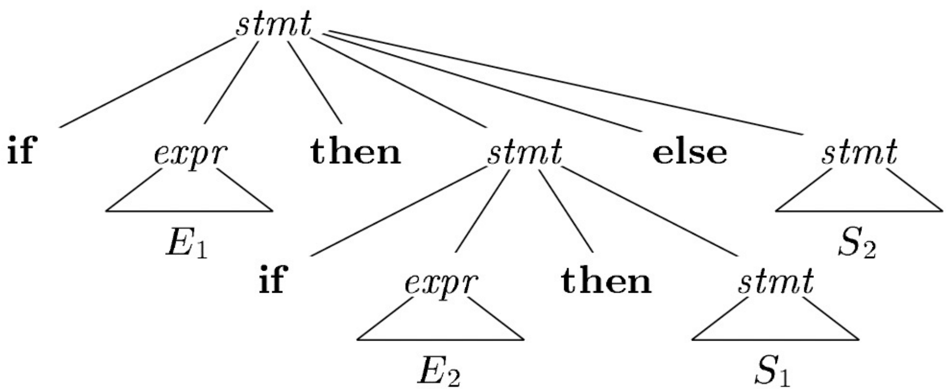
\end{array}
$$

if E_1 **th**





Eliminating Ambiguity

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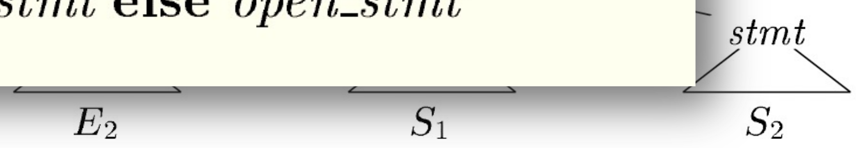

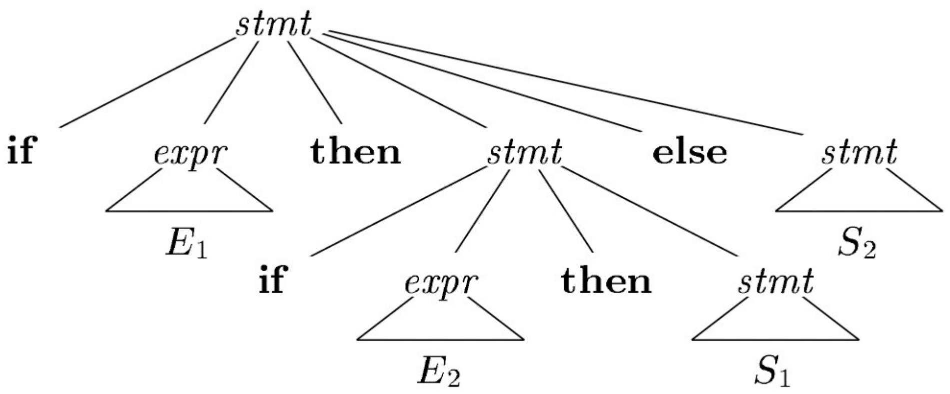

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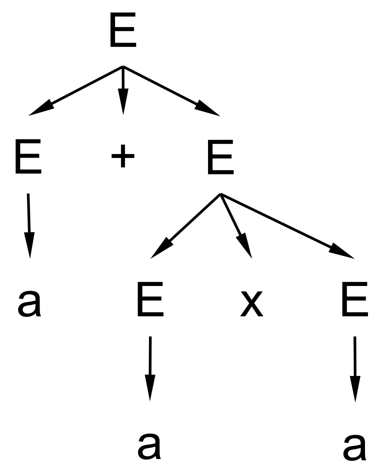
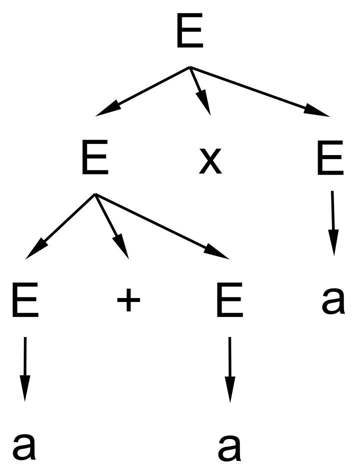
- $w = a + a \times a$





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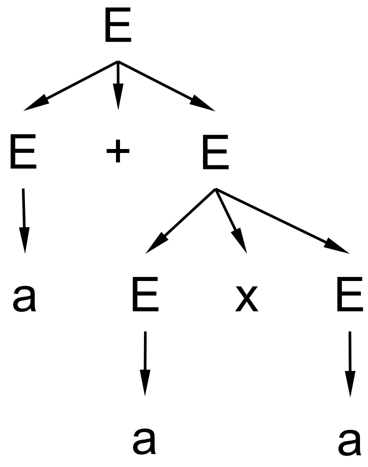

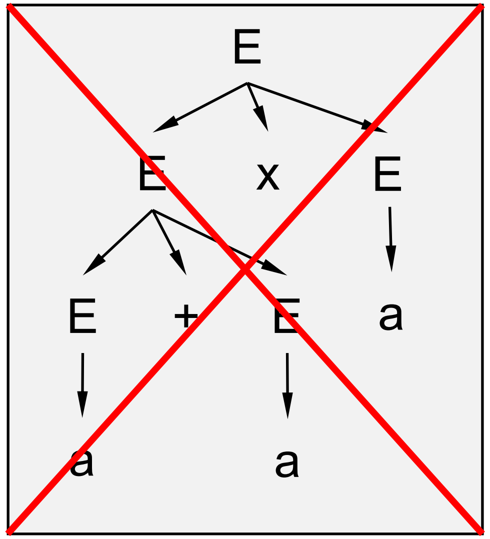

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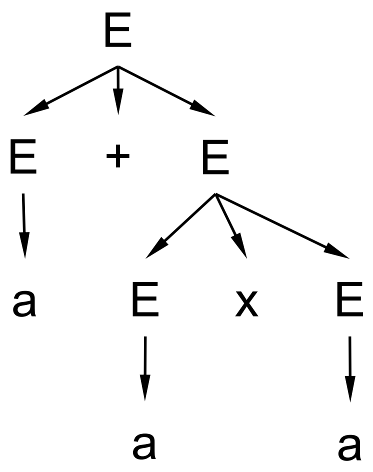
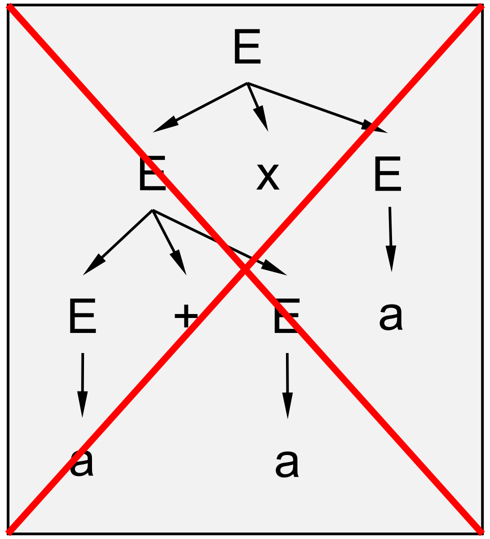
Let us always try production rules in order!





Exercises

$S \to aSb \mid \epsilon$ is the grammar for $a^n b^n$

- Write a context-free grammar for the following languages
  - 0^*1^*

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PART II: Push-Down Automata

Recap: Context-Free Language

- $L(G) = \{w \in T^*: S \Rightarrow^* w\}$ is a context-free language

Push-Down Automata (PDA)

- Regular language = DFA/NFA
- Context-free language = PDA = NFA + Stack (z_0)

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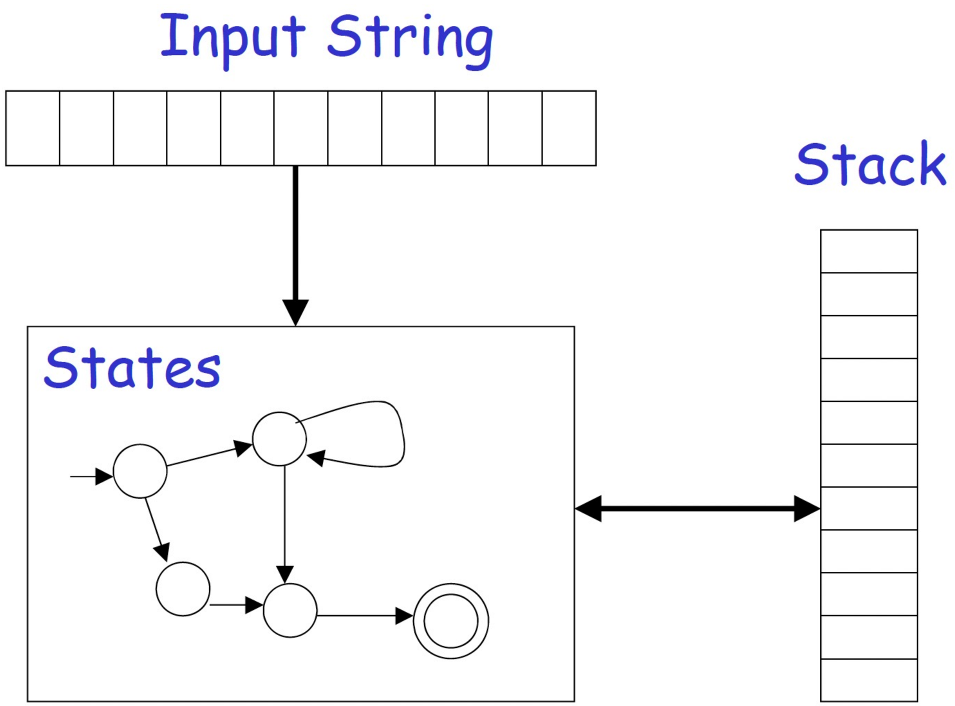

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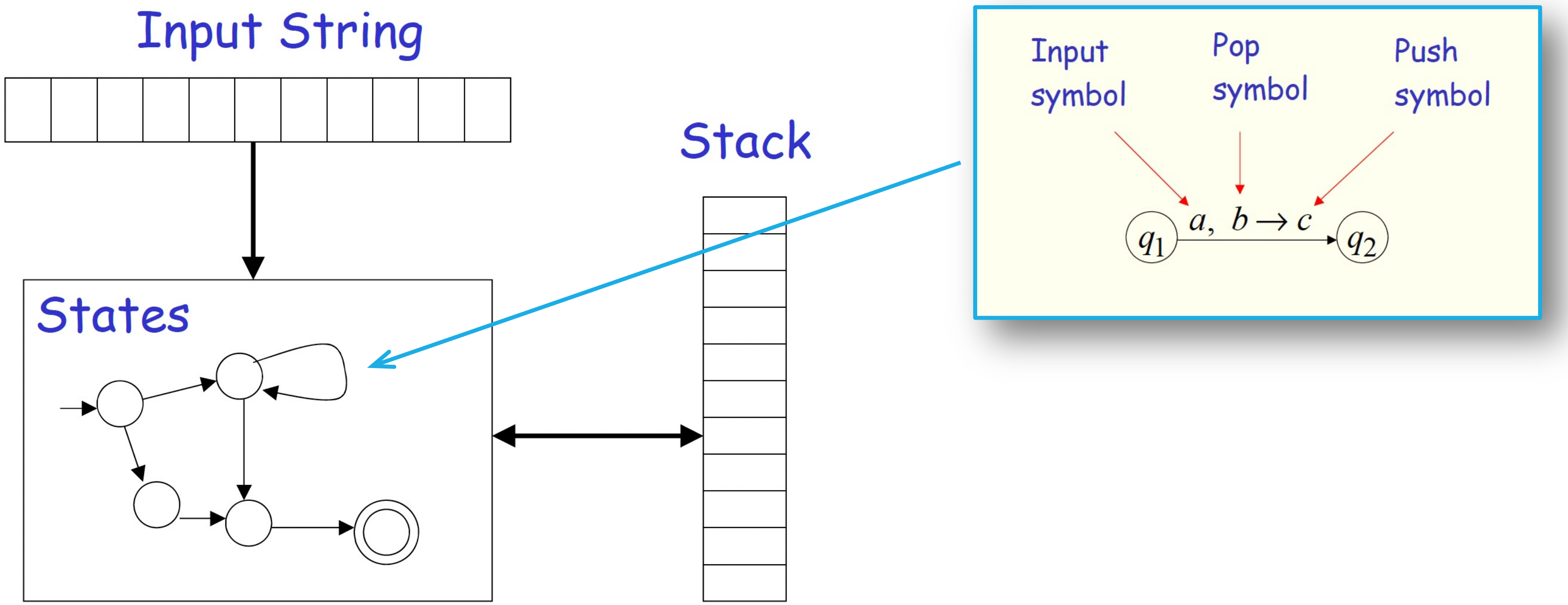

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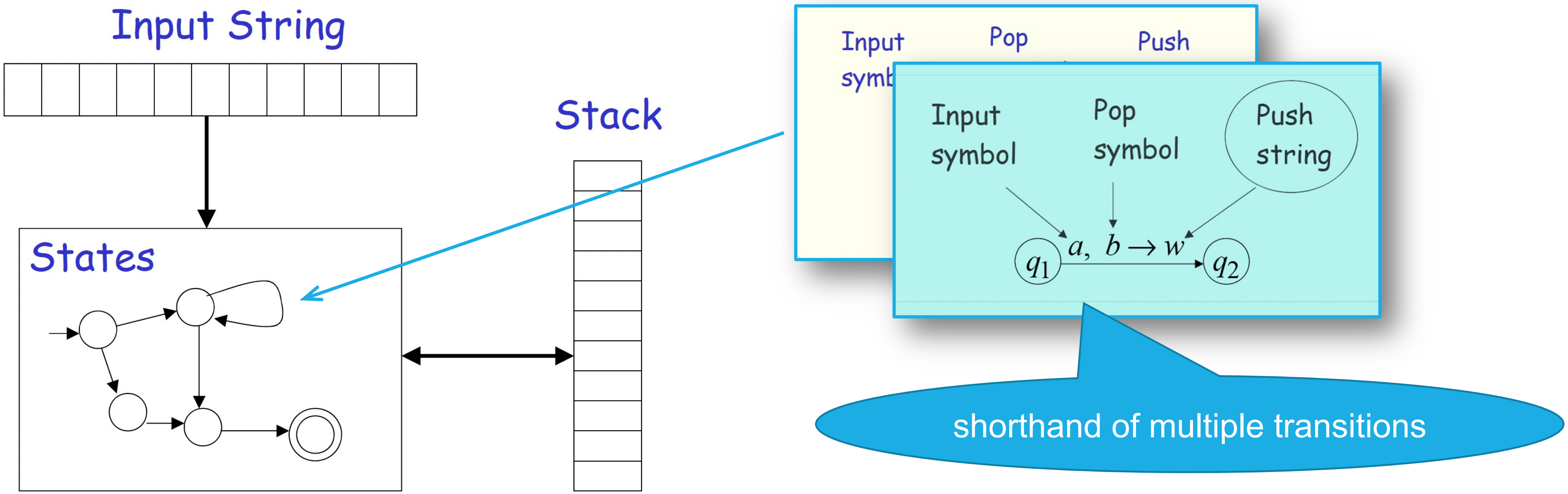

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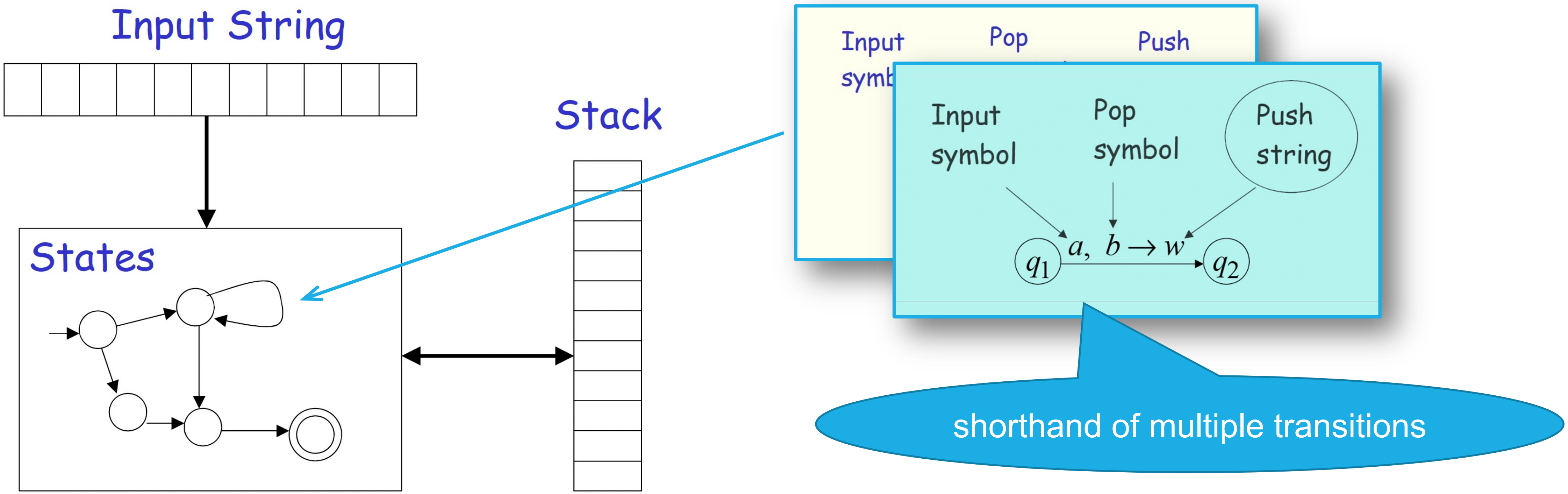

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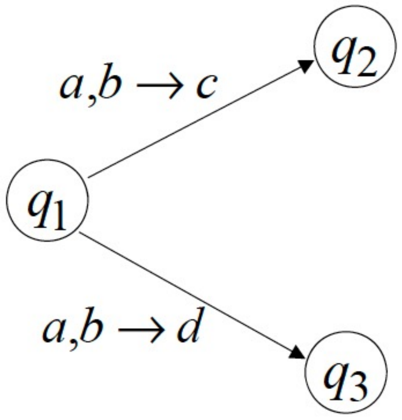

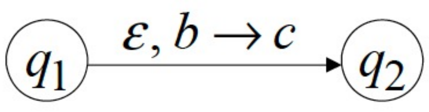

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  - $F \subseteq Q$: set of final states

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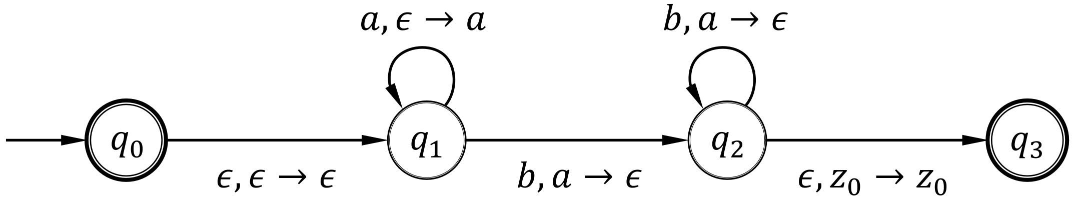

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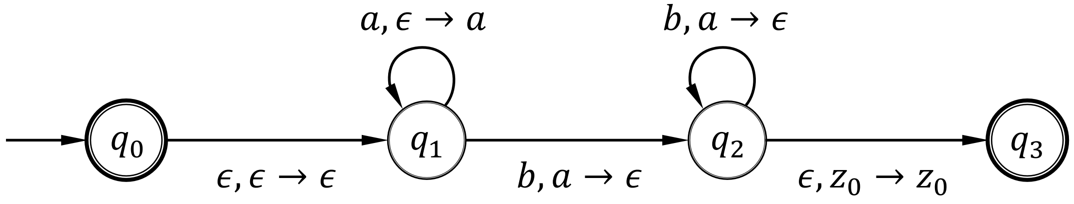
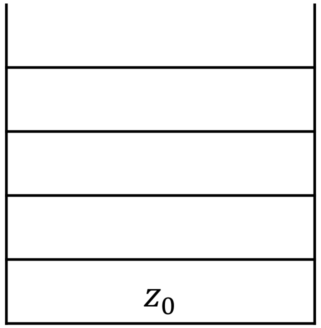


a	a	a	b	b	b

- Let's check how it accepts the string $aaabbb$



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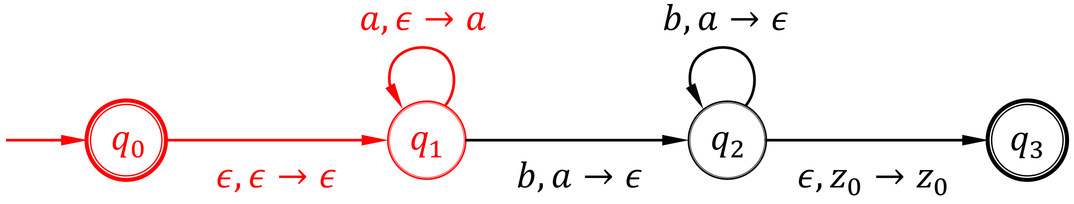

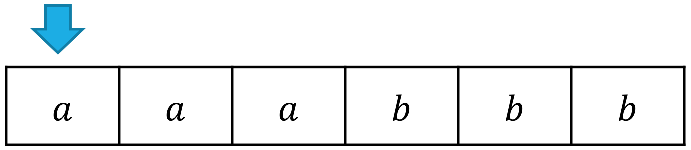

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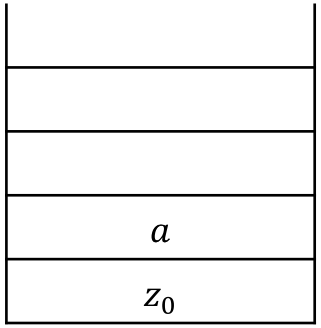

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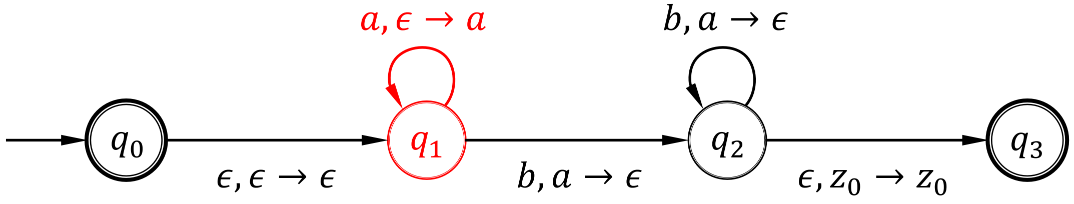

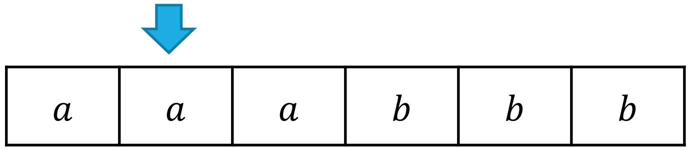

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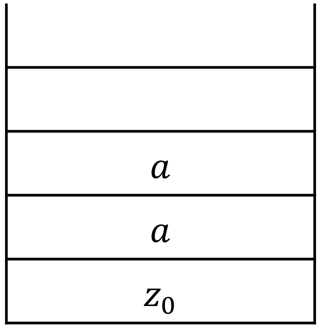

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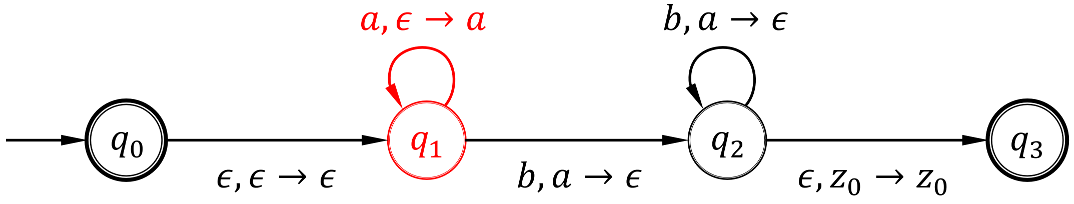

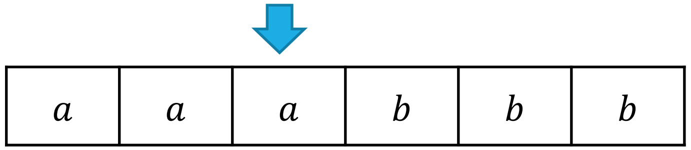

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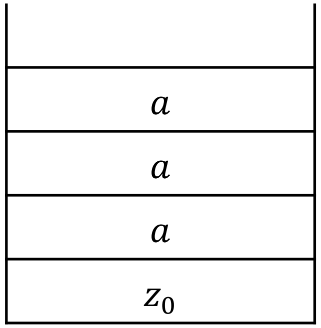

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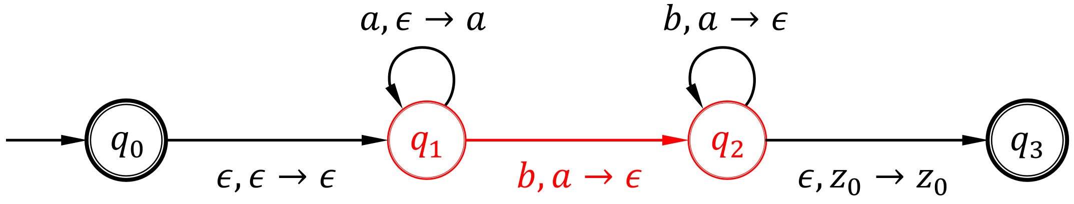

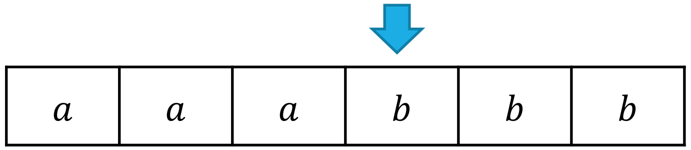

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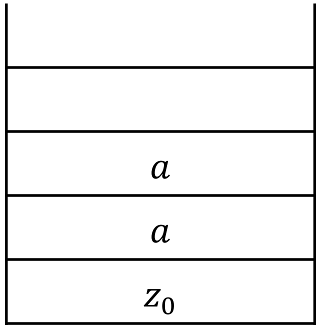

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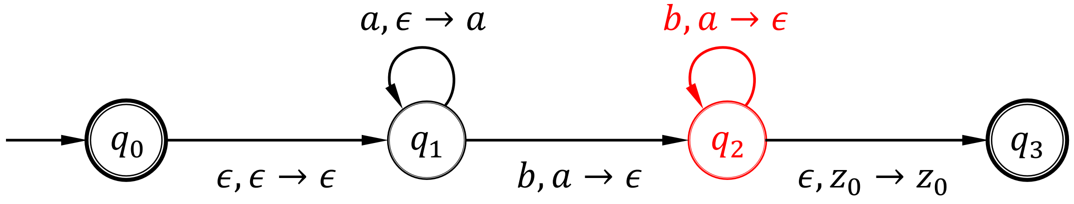

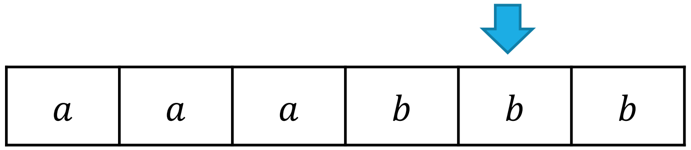

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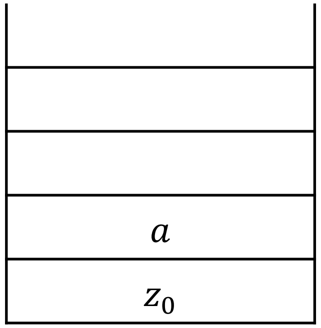

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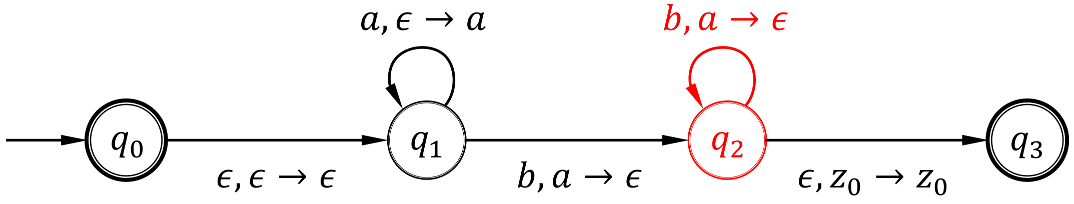

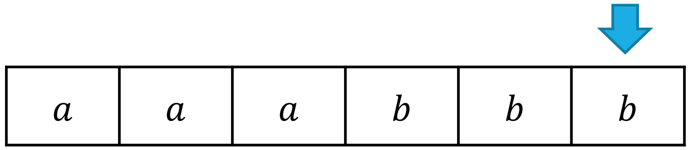

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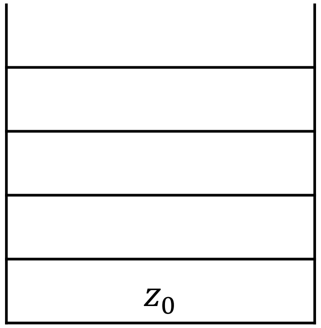

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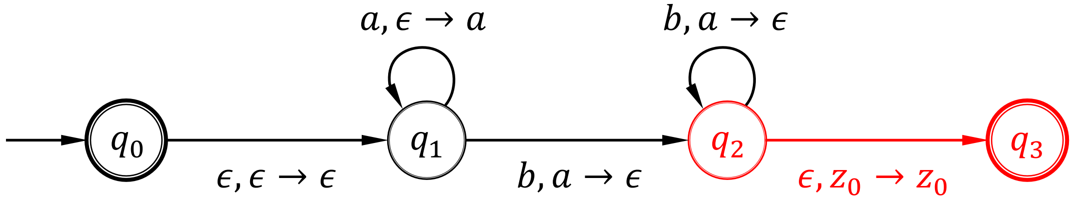

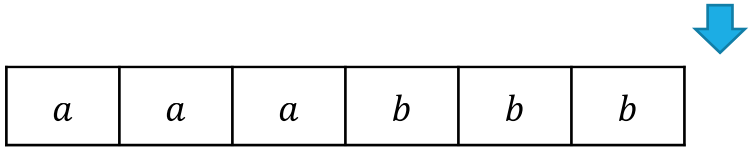

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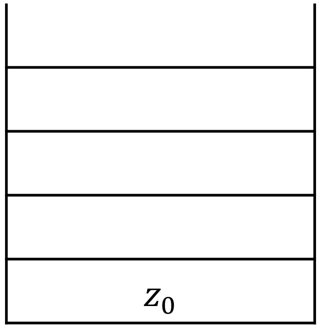

Instantaneous Description

- An instantaneous description of a PDA is a triple

$$(q,\ w,\ \gamma)$$

Current state

Remaining unread input

Remaining stack contents

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Write $\vdash$ when M is clear

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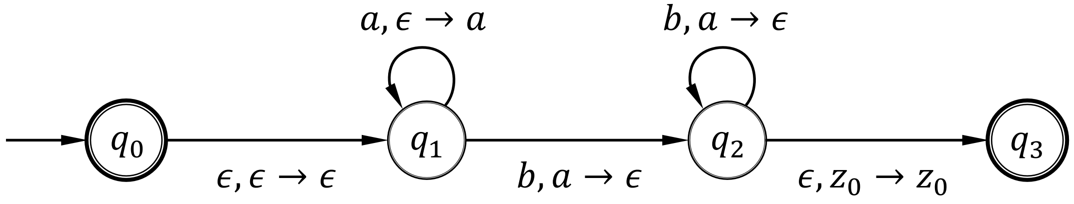

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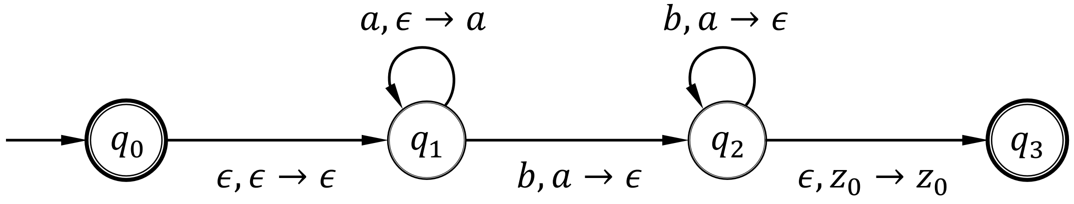

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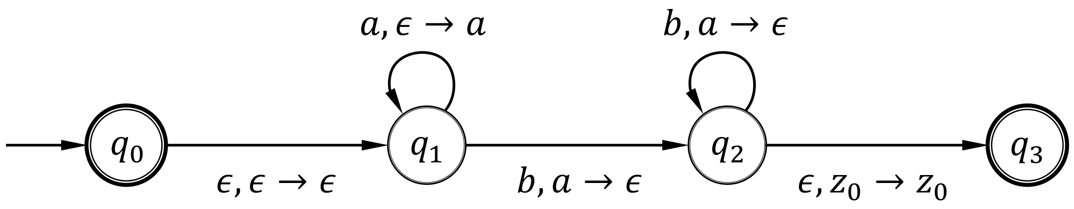

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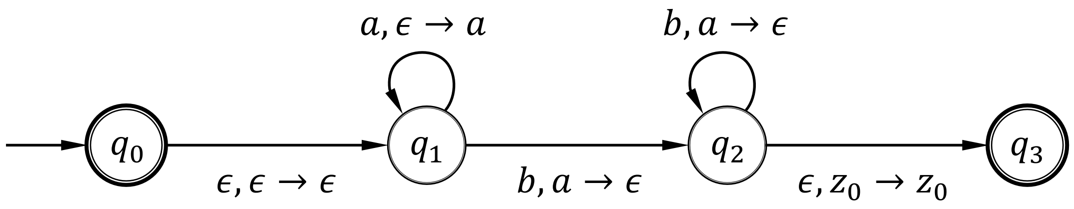

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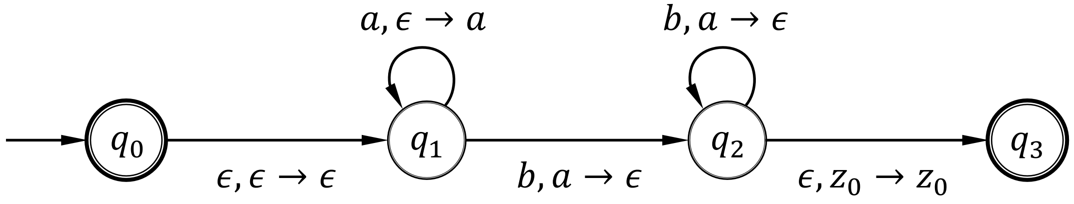

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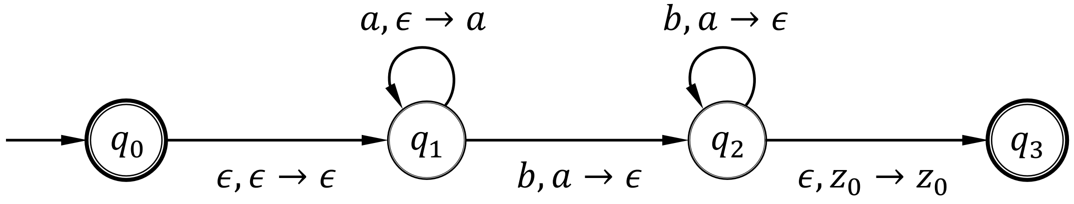

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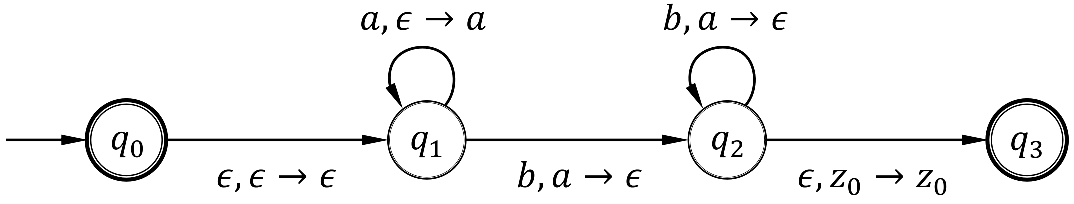

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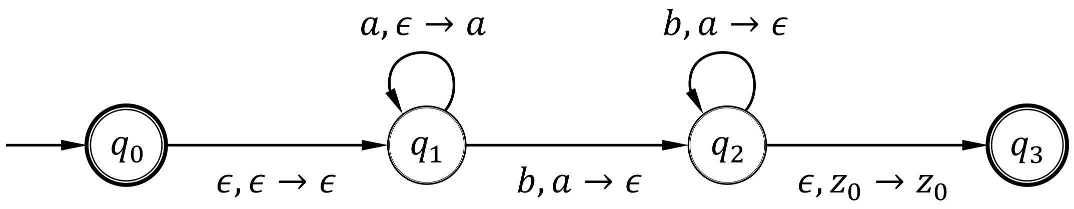

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Equivalent definitions
(1) acceptance by final states
(2) acceptance by empty stack

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- If there is a PDA that accepts strings by final states, there is a PDA accepting the strings by empty stack, vice versa.

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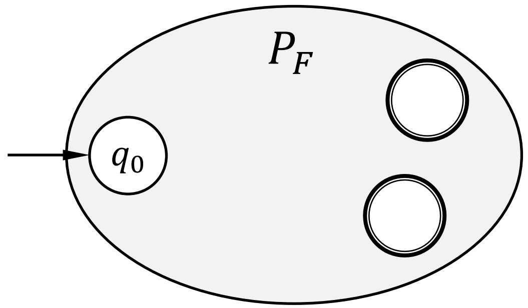
- From final states (P_F) to empty stack ($P_\emptyset$)



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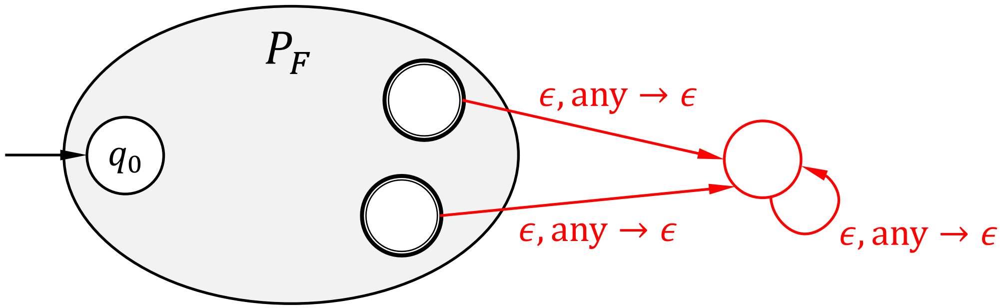

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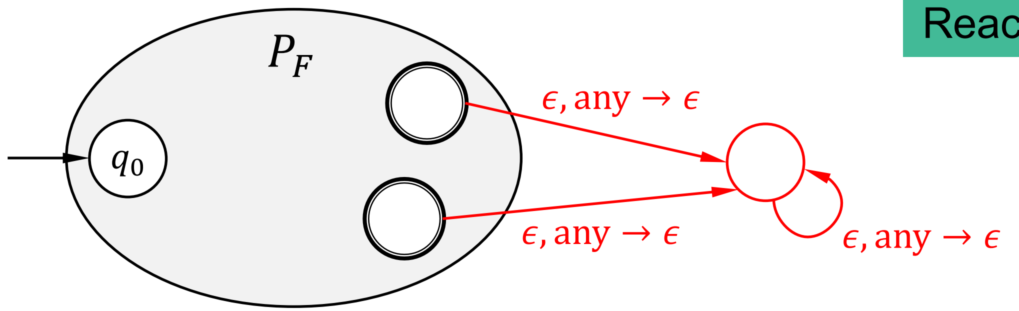

Reaching Final States => Reaching Empty Stack

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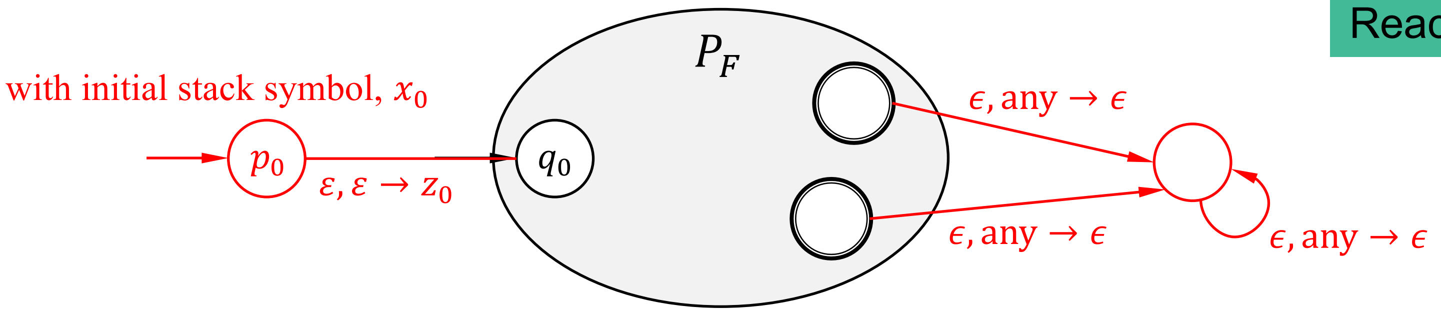

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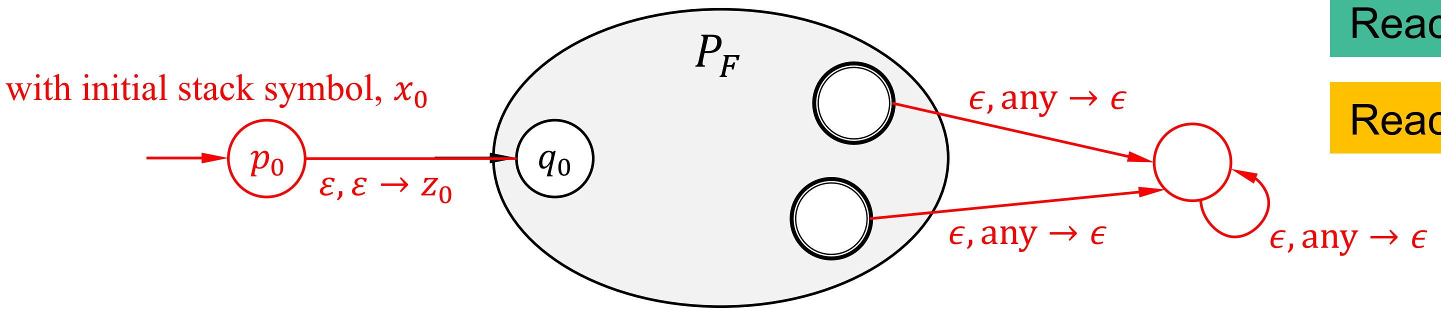

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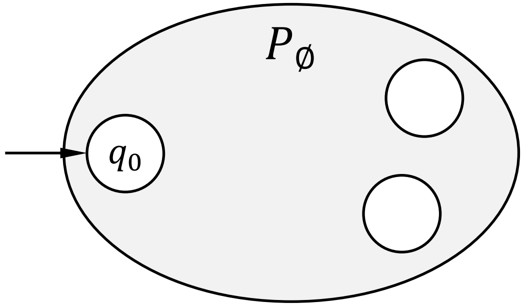

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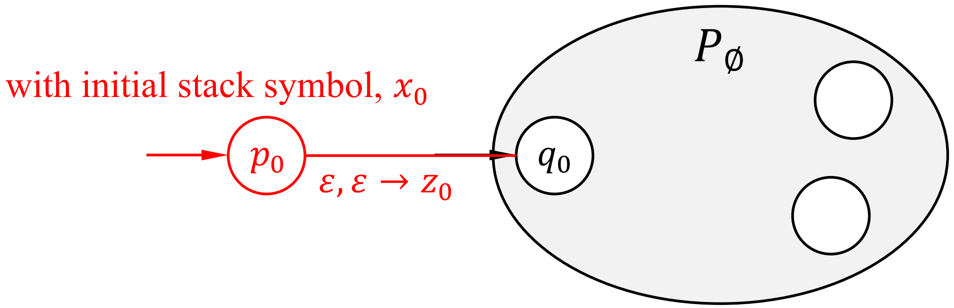

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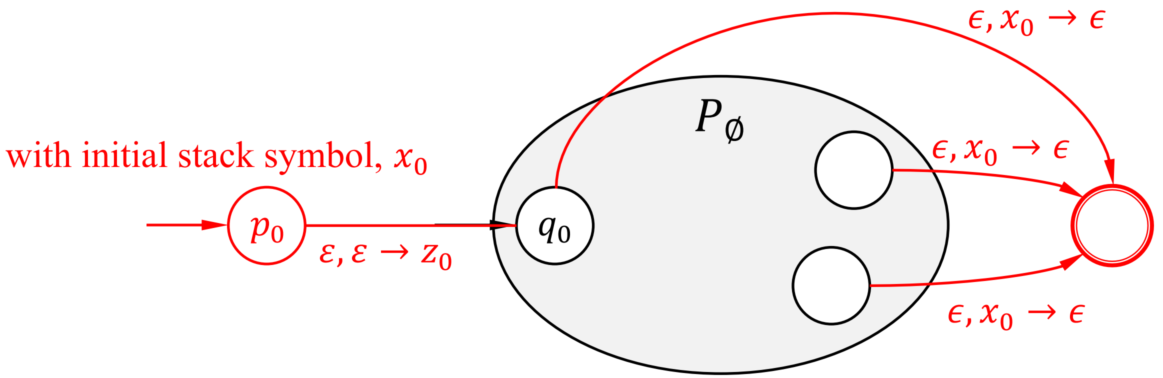

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Exercises

- Design a PDA to accept $a^i b^j c^k$ where $i = j$ or $j = k$, and write the formal definition of the PDA

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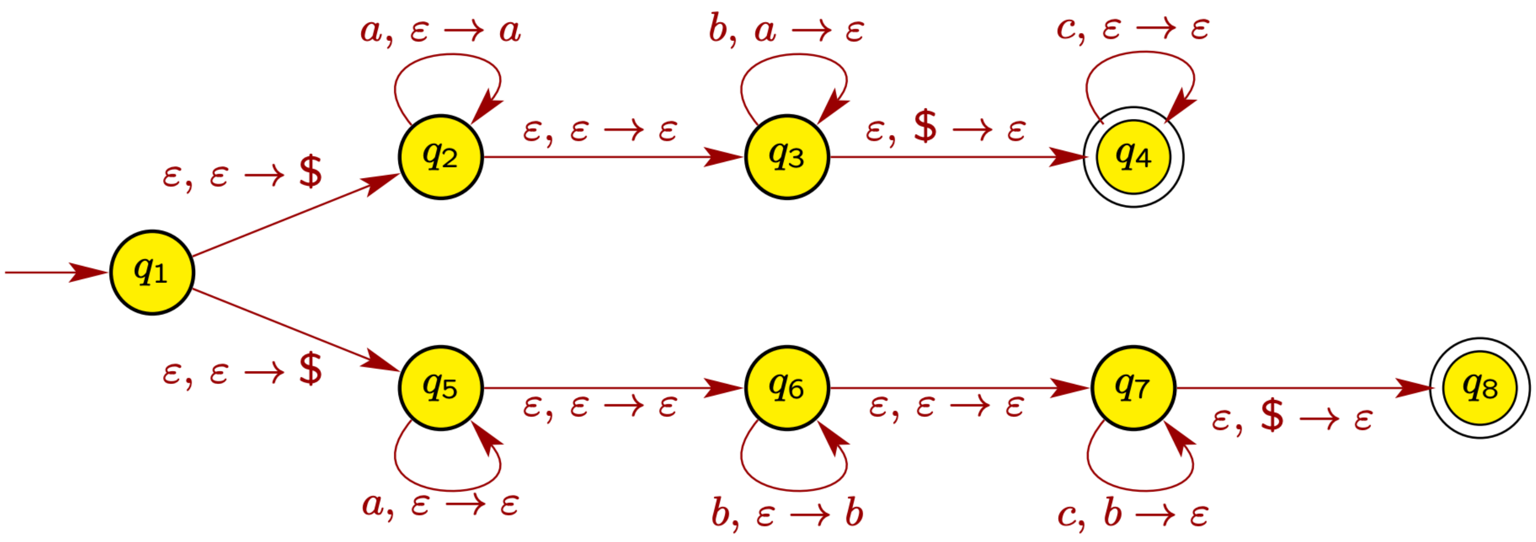

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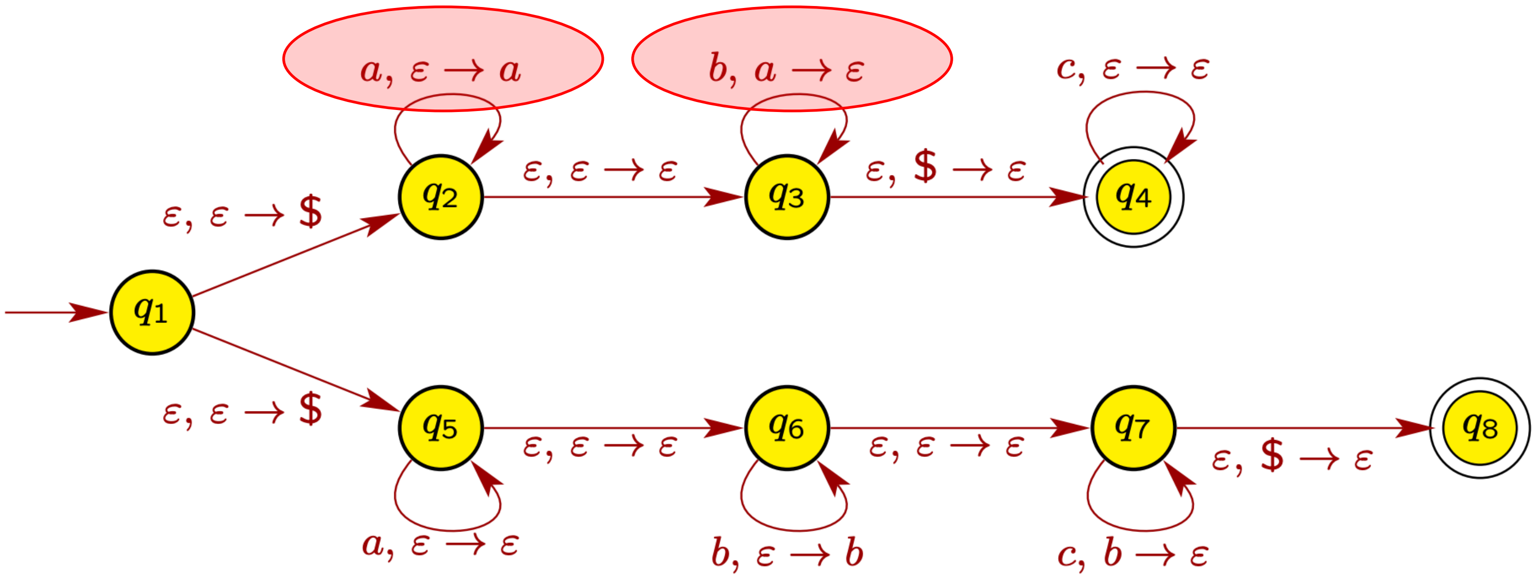

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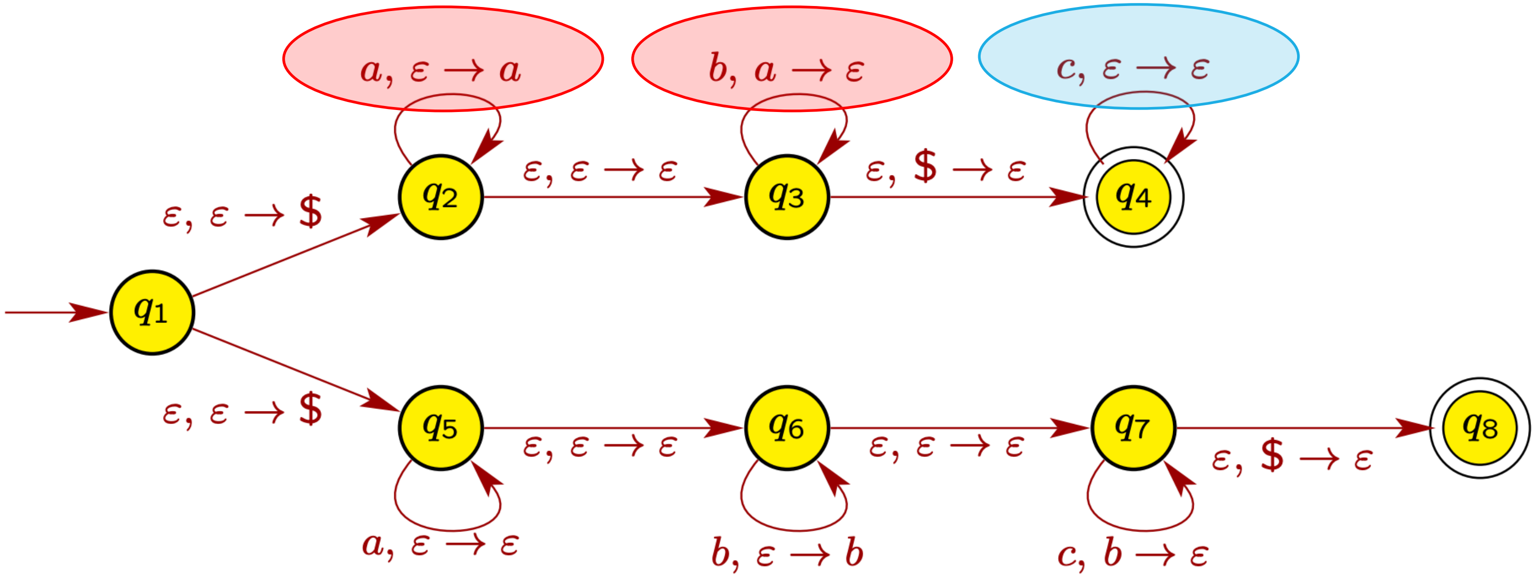

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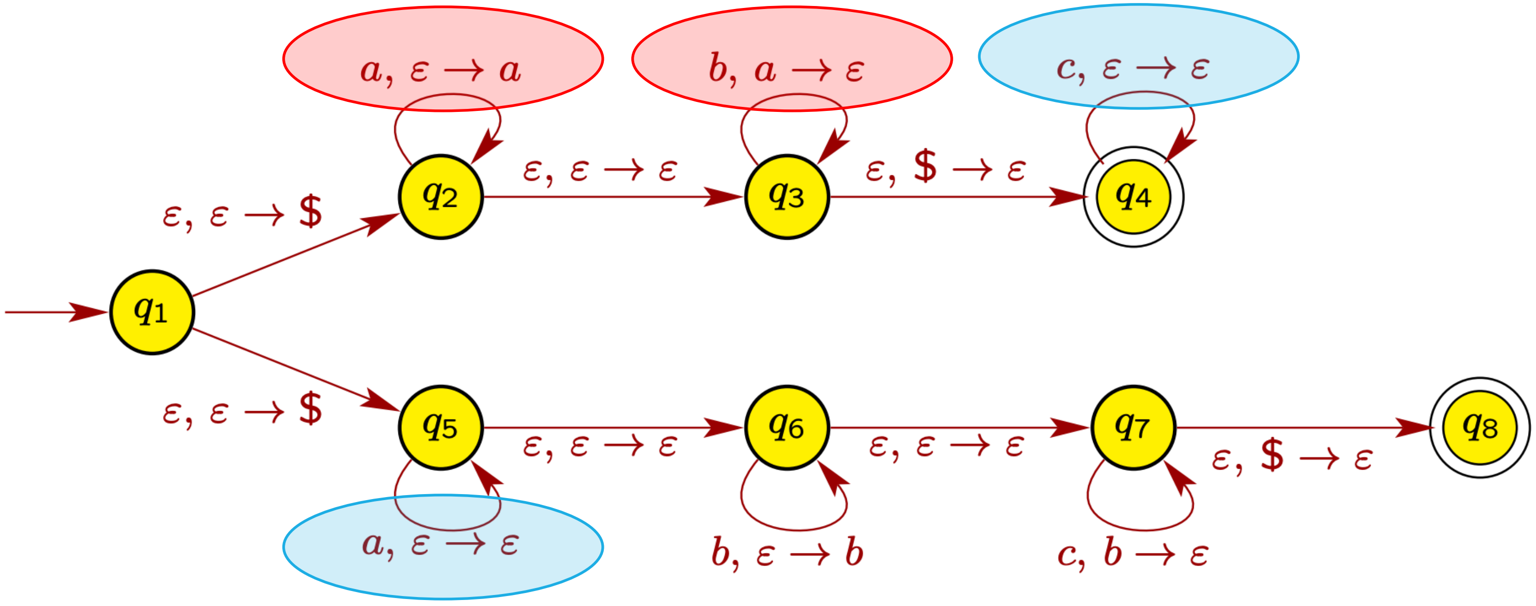

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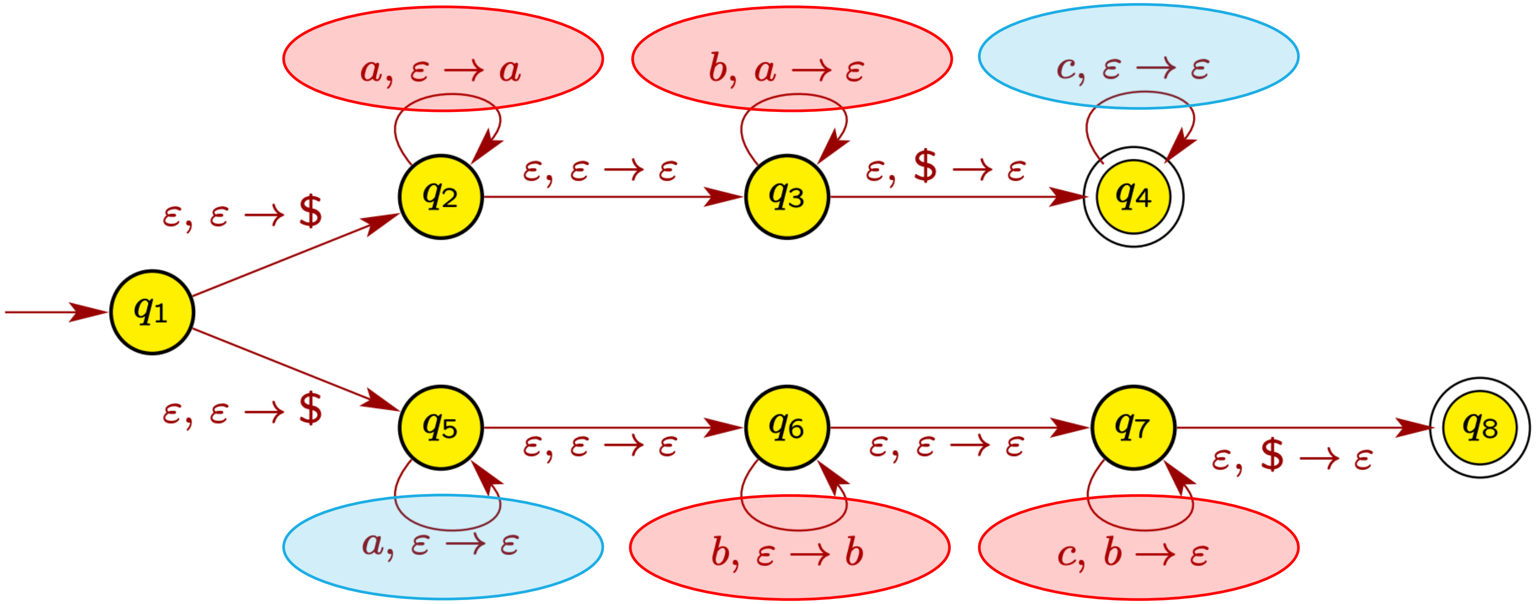

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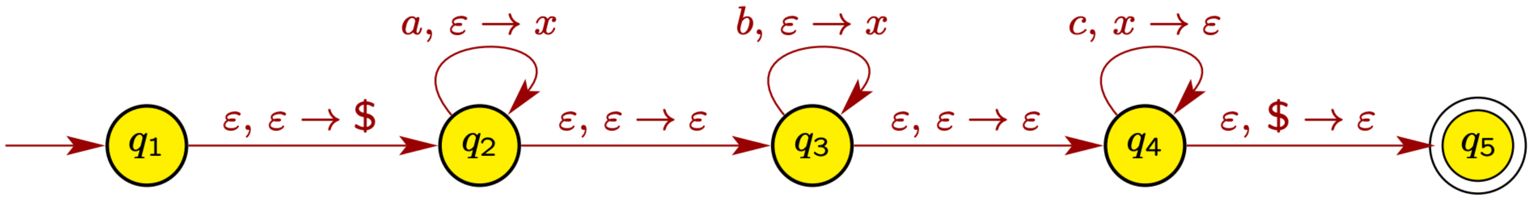

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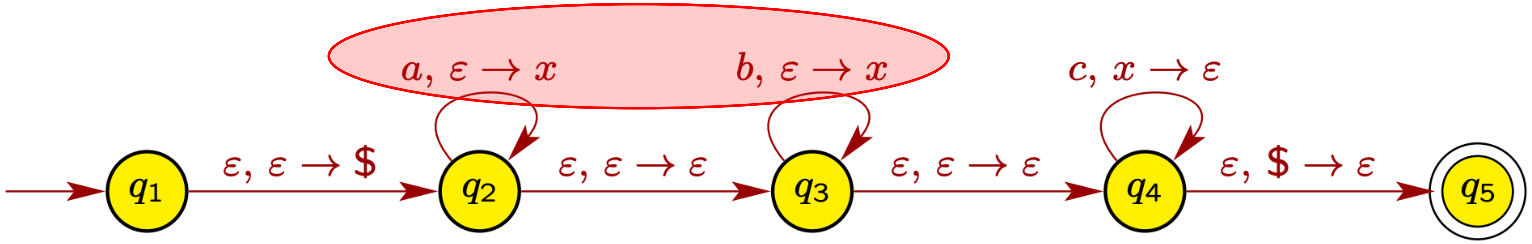

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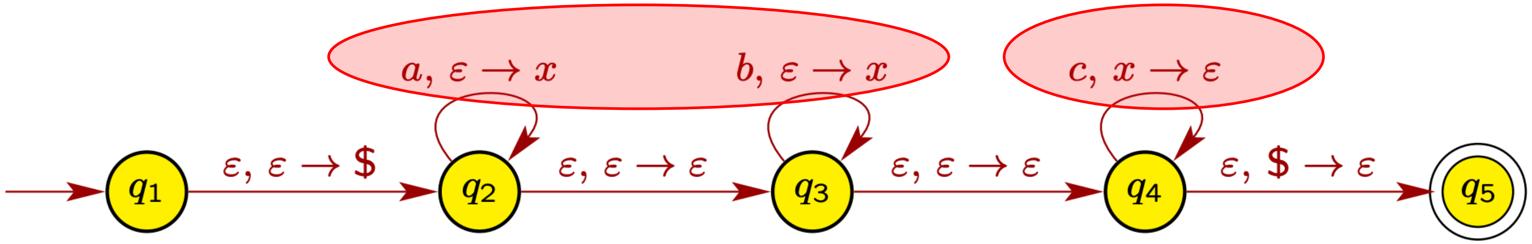

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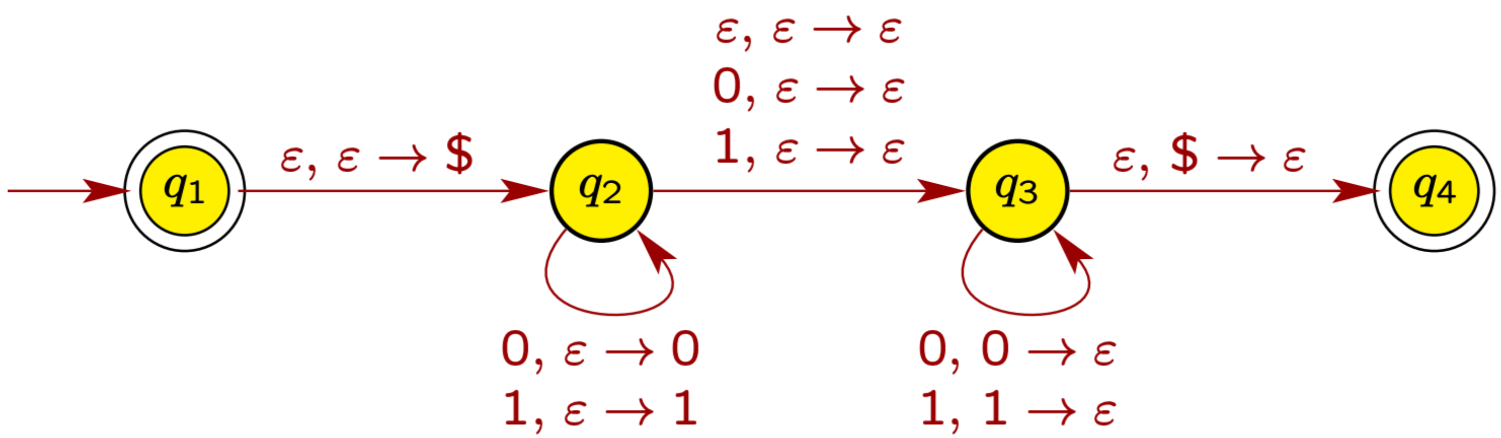

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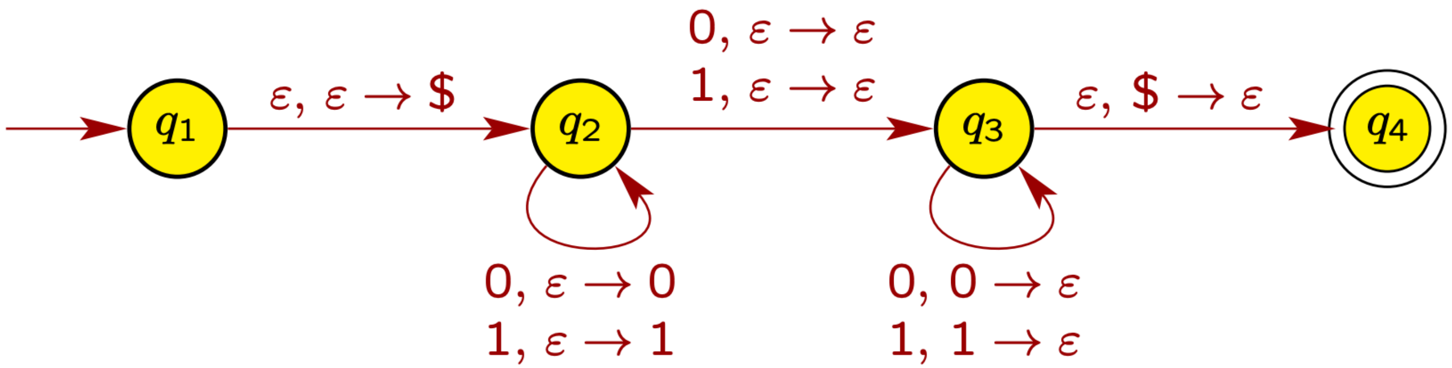

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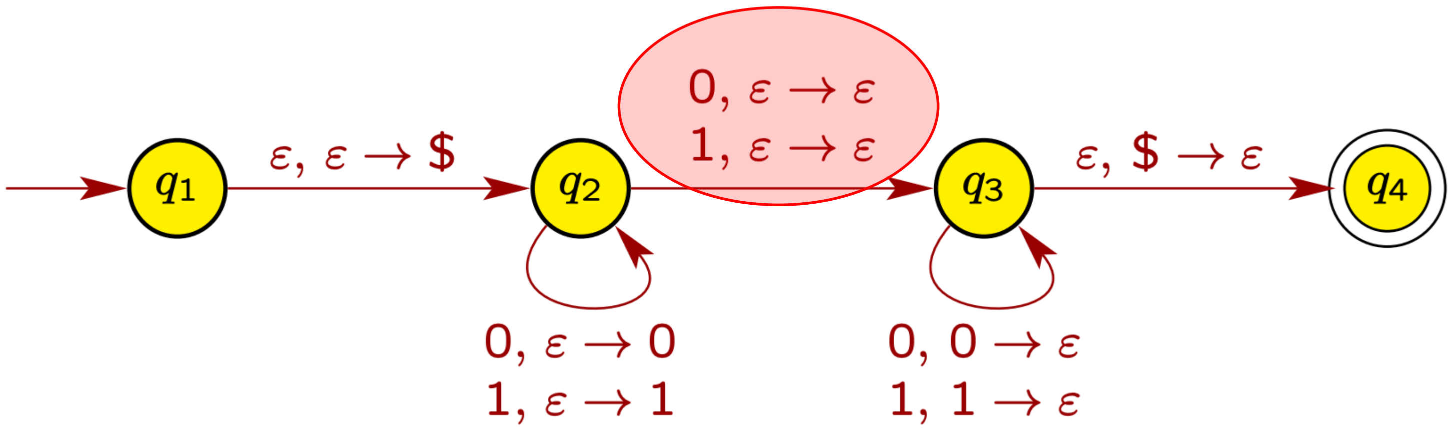

Exercises

- Prove that, for any PDA, there is an equivalent PDA consisting of only two stack symbols

PART III: CFG = PDA

Equivalence of PDA and CFG

- CFG $\subseteq$ PDA
- PDA $\subseteq$ CFG

CFG ⊆ PDA

- Given any CFG, $G = (N, T, P, S)$, we can build an equivalent PDA that accepts string by empty stack, $(\{q\}, T, N \cup T, \delta, q, S, \emptyset)$, where

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  - (2) for any terminal $a \in T$, $\delta(q, a, a) = \{(q, \epsilon)\}$

Keep the Top-of-Stack value as the symbol to consume or derive

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The current Top-of-Stack value

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The current Top-of-Stack value

Replace it with β to simulate the derivation

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Derivation does not read any input symbol

The current Top-of-Stack value

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Read and Pop the Top-of-Stack value *a*

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- **Example**: $S \to aSb \mid \epsilon$, the grammar for $a^n b^n$

Exercise: Try to draw the PDA!

CFG ⊆ PDA

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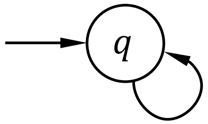

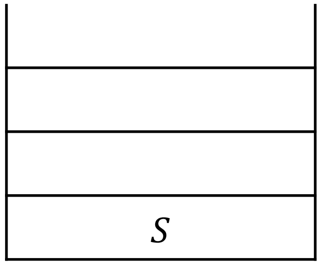

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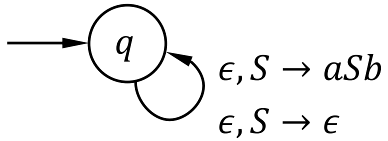

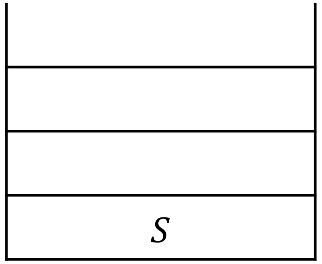

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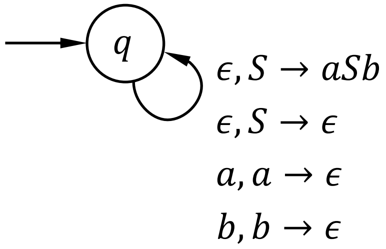

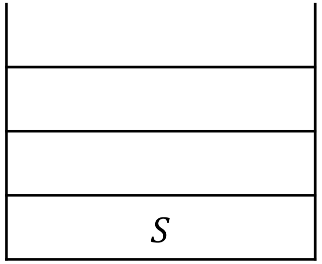

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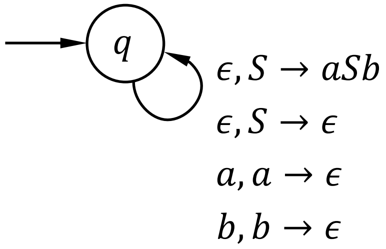

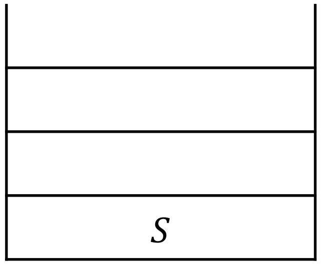

a	a	b	b

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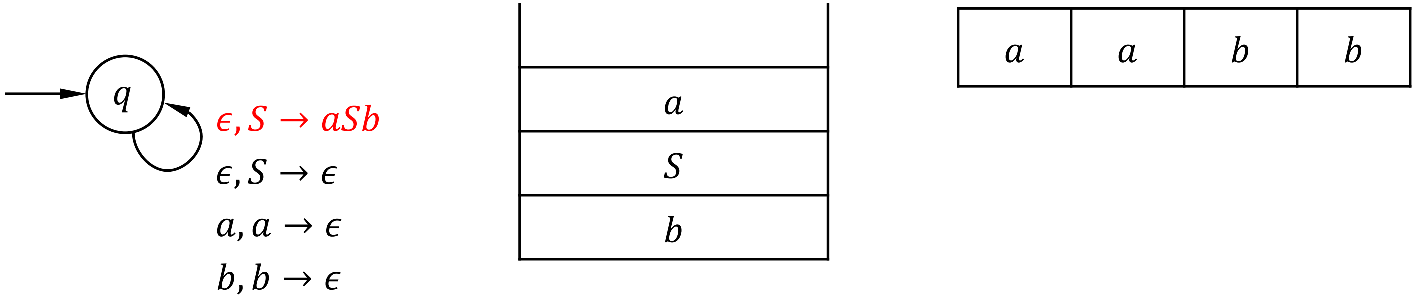

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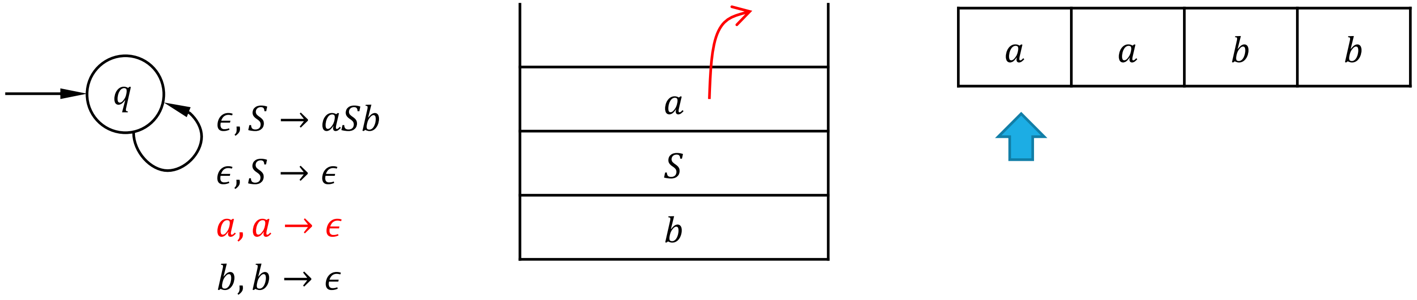

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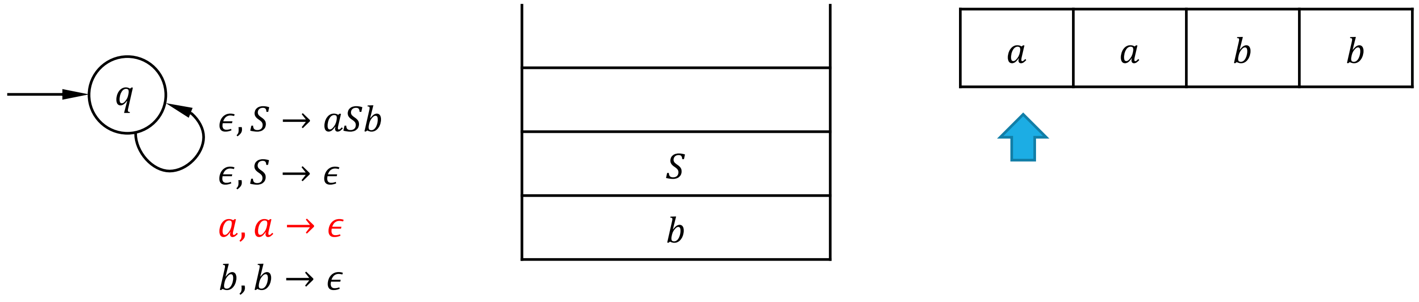

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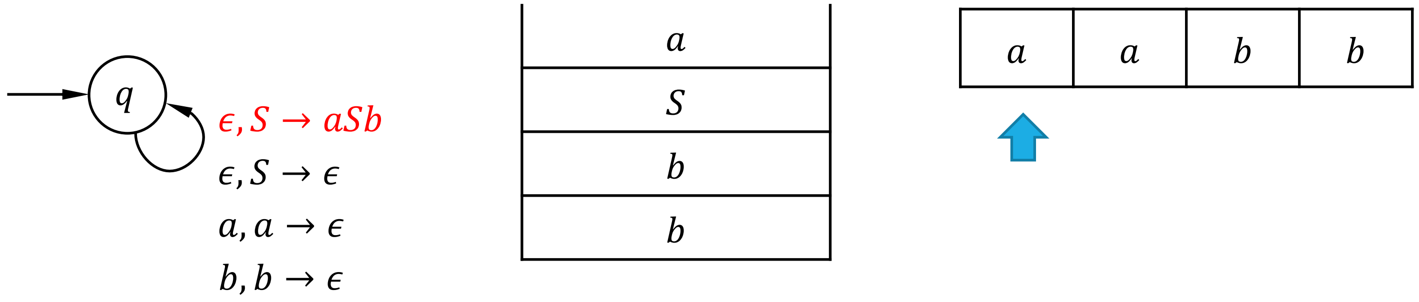

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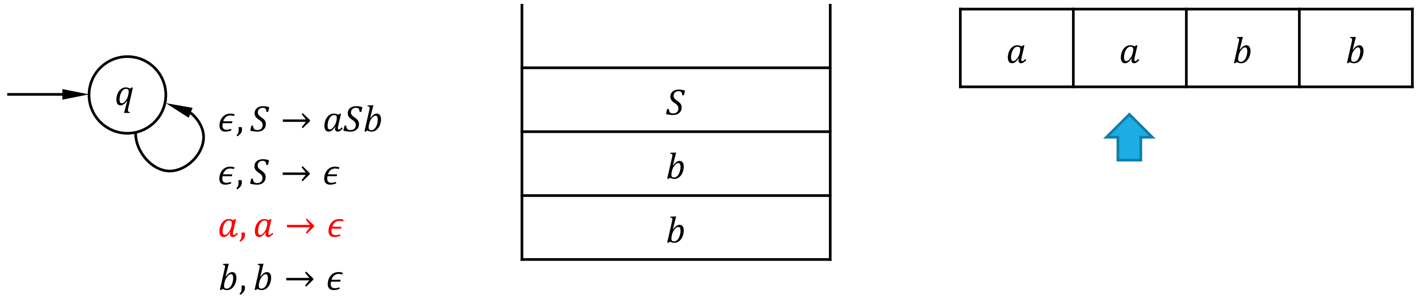

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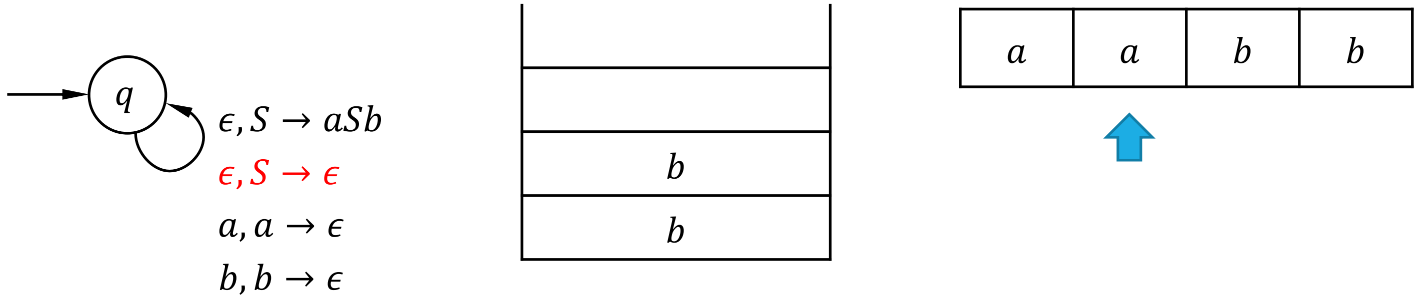

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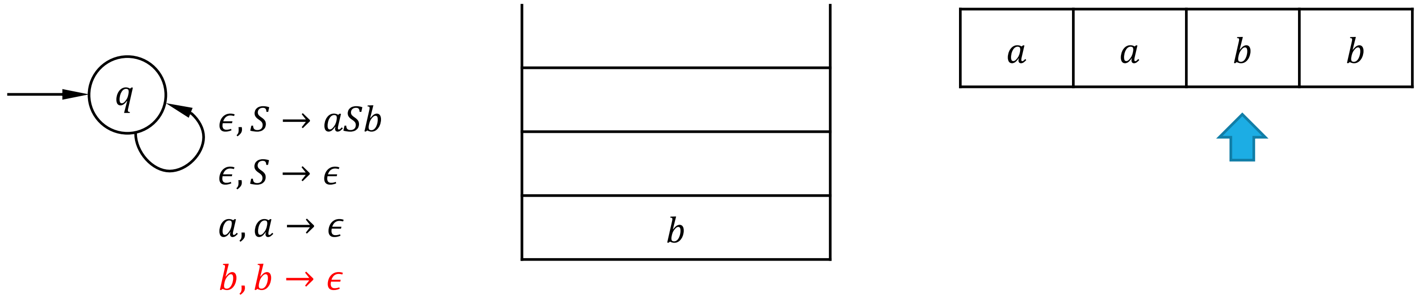

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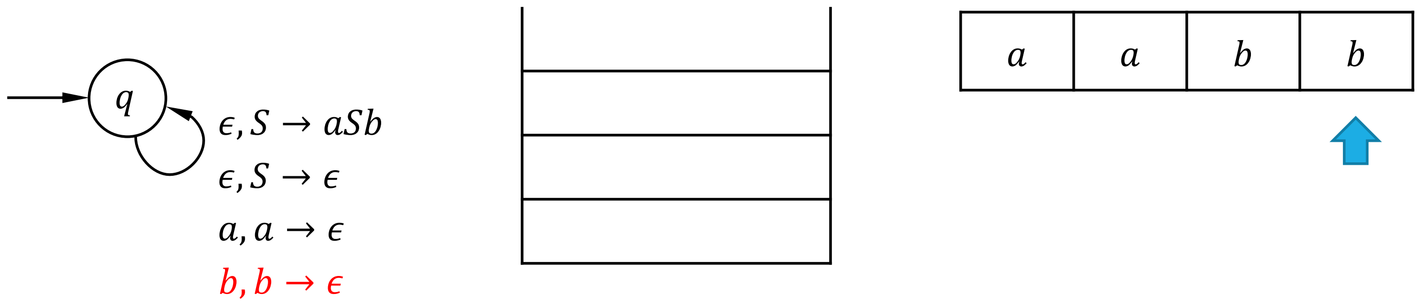

CFG ⊆ PDA

Exercise: Try to prove the equivalence!

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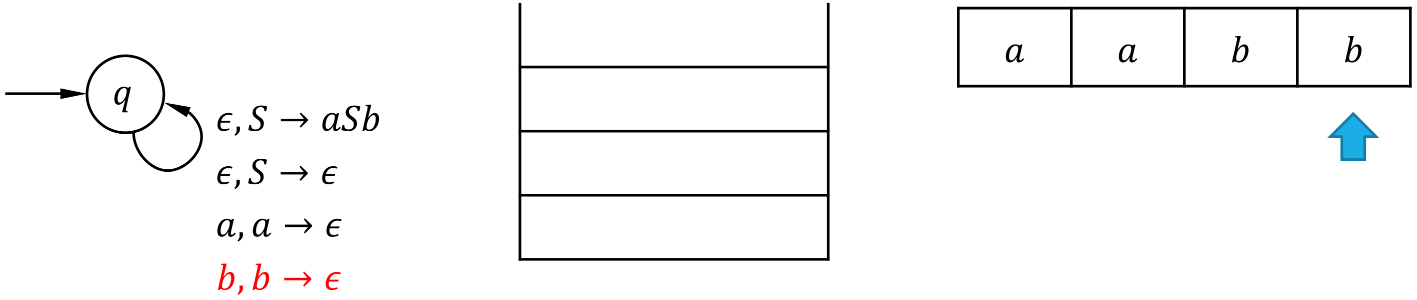

PDA $\subseteq$ CFG

- Given any PDA, $(Q, \Sigma, \Gamma, \delta, q_0, z_0, \emptyset)$, we can build an equivalent CFG (N, Σ, P, S), where $N = \{S\} \cup \{N_{pXq} : p, q \in Q, X \in \Gamma\}$

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From p to q, pop X

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Pop z$_0$, leading to empty stack

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Pop z_0, leading to empty stack

Strings accepted by empty stack = Strings derived from S

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Pop X, consuming *a*

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Pop X, consuming *a*, pushing $X_1X_2 \ldots X_k$

PDA ⊆ CFG

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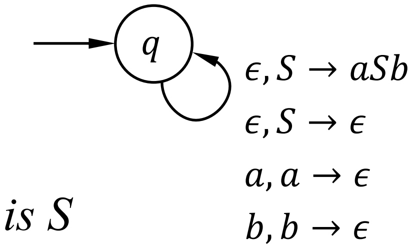

initial stack symbol is S

Exercise: Try to write the CFG!

PDA ⊆ CFG

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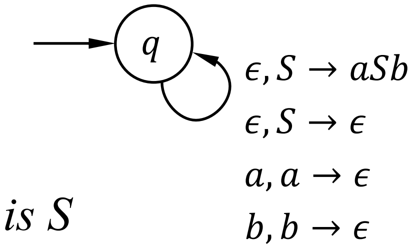

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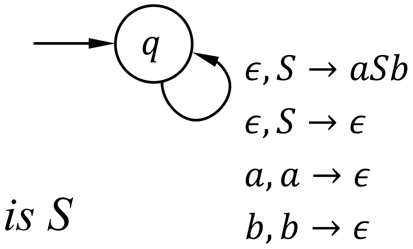

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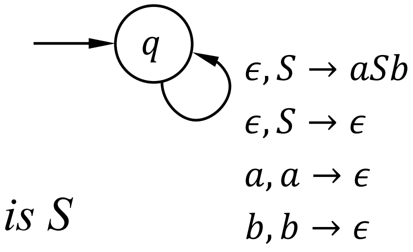

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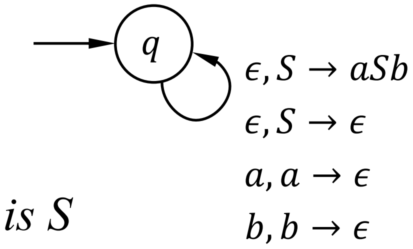

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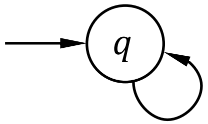

$\epsilon, S \to aSb$
$\epsilon, S \to \epsilon$
$a, a \to \epsilon$
$b, b \to \epsilon$

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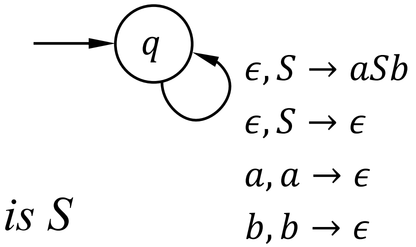

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$S \to \epsilon$
$S \to aSb$

PDA $\subseteq$ CFG

Exercise: Try to prove the equivalence!

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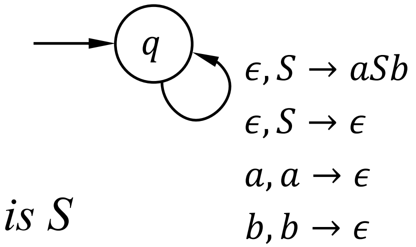

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Deterministic PDA (DPDA)

- DFA/NFA $\subseteq$ DPDA $\subseteq$ NPDA

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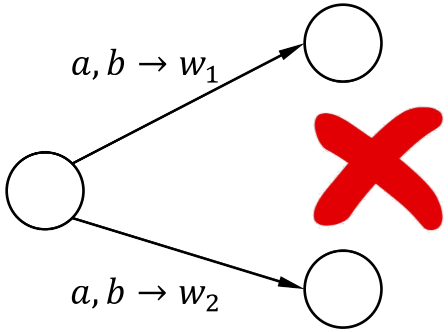

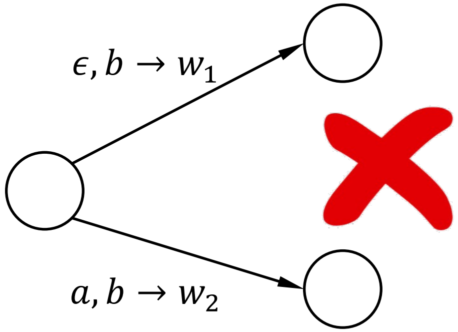

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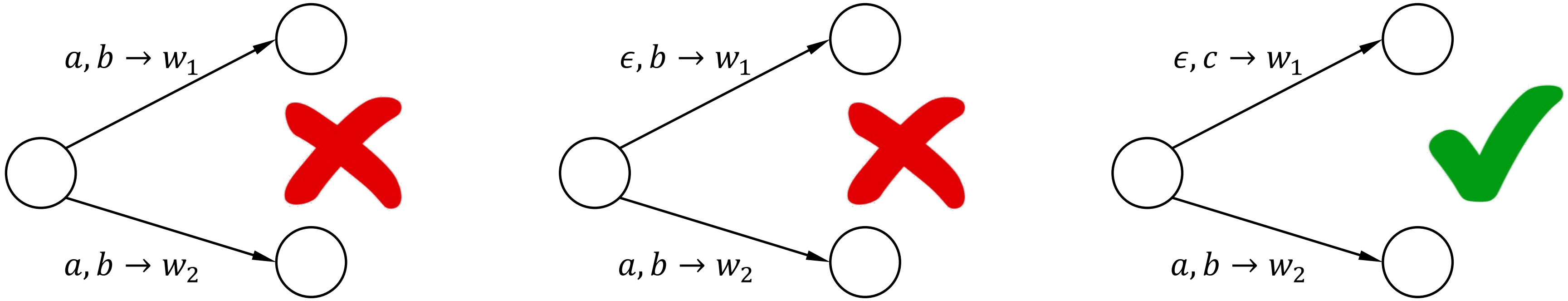

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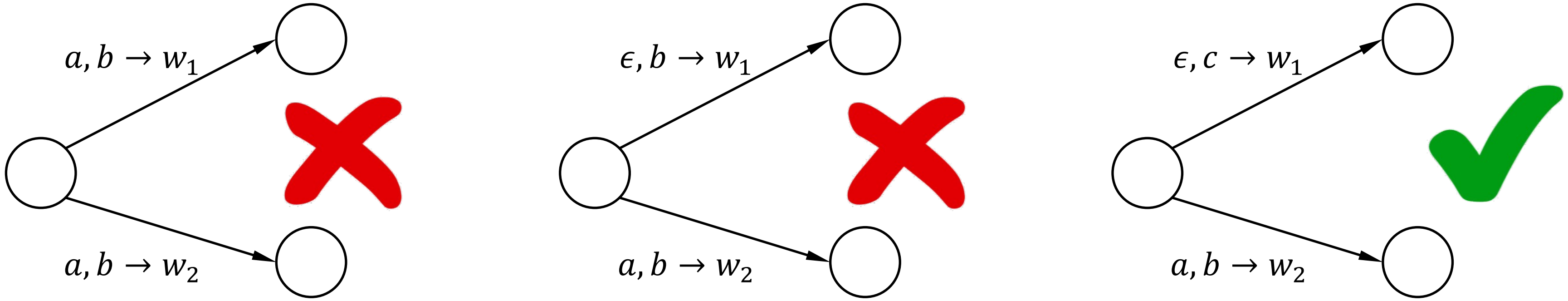

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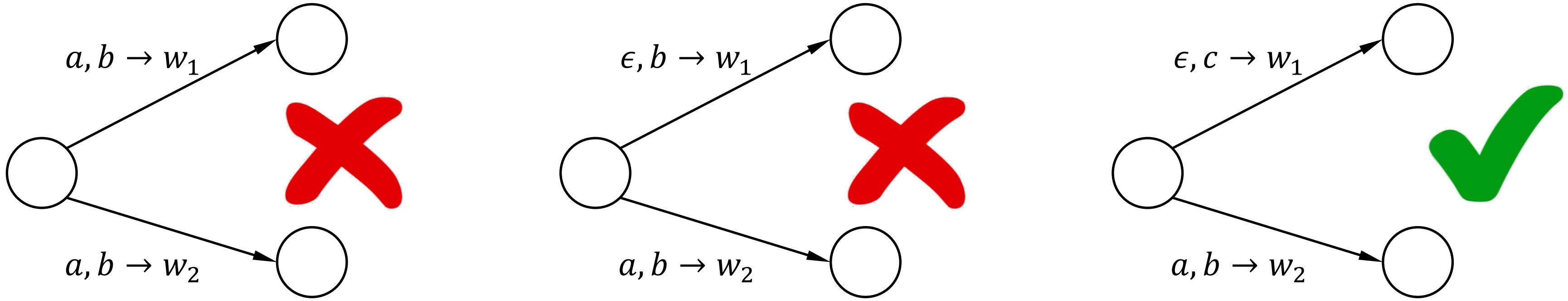

- A CFL must have a NPDA, but may not have a DPDA
- A DPDA language has a CFG without ambiguity

PART IV: Properties of CFL

Closure Properties

- Given two CFL, L_1 and L_2, the following are CFL
  - $L_1 \cup L_2$
  - $L_1 L_2$
  - L_1^*
  - L_1^R

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$S_1 \to \cdots ; \cdots$

$S_2 \to \cdots ; \cdots$

$S \to S_1 \mid S_2$

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Given the CFG of L_1, (N, T, P, S), the grammar of L_1^R is (N, T, P^R, S), where

$$P^R = \{A \to \alpha^R : A \to \alpha \in P\}$$

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  - $L_1 \cap L_2$????

Intersection of CFL

Assume $\{a^n b^n c^n\}$ is not a CFL

- Given two CFL, L_1 and L_2, the following may not be CFL
  - $L_1 \cap L_2$
  - $\overline{L_1}$
  - $L_1 - L_2$

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Example: $L_1 = \{a^n b^n c^m\}$, $L_2 = \{a^n b^m c^m\}$

$L_1 \cap L_2 = \{a^n b^n c^n\}$ *(not CFL, let's prove later)*

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Assume $\{a^n b^n c^n\}$ is not a CFL

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- This is an exercise!

Application of the Properties

- Prove a language is a CFL or not

Application of the Properties

- Prove that $L = \{a^n b^n : n \neq 100, n \geq 0\}$ is a CFL

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Application of the Properties

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  - $L_1 = \{a^{100} b^{100}\}$ is regular
  - $\overline{L_1} = \{a, b\}^* - \{a^{100} b^{100}\}$ is regular
  - $L = \{a^n b^n\} \cap \overline{L_1}$ is context-free as we know $\{a^n b^n\}$ is context-free

Application of the Properties

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  - The assumption is wrong

Decidable Properties

- **Empty Language Question:**
  - Given a CFG, is the CFL empty?
- **Infinite Language Question:**
  - Given a CFG, is the CFL infinite?
- **Membership Question:**
  - Given a CFG, is a string belongs to the CFL?

Decidable Properties

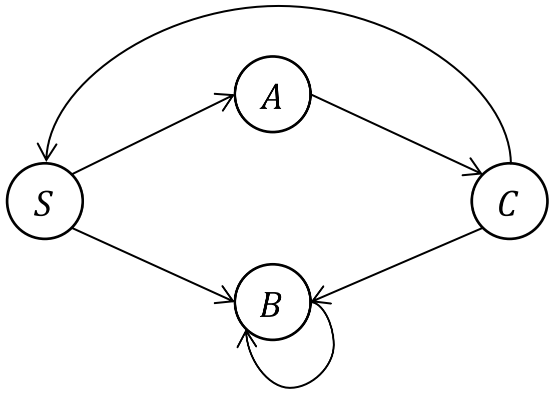
- **Empty Language Question:**
  - Given a CFG, is the CFL empty?

- **Algorithm**:
  - Check if the start non-terminal is not used, or useless, e.g., $S \to S$

Decidable Properties

- **Infinite Language Question:**
  - Given a CFG, is the CFL infinite?

$$S \to AB$$
$$A \to aCb \mid a$$
$$B \to bB \mid bb$$
$$C \to cBS$$



- **Algorithm**:
  - Remove useless non-terminals
  - Remove unit and epsilon productions
  - Create a dependency graph for remaining non-terminals
  - Check if the graph has a circle

Decidable Properties

- **Membership Question:**
  - Given a CFG, is a string belongs to the CFL?
- **Algorithm 1:**
  - Create an NPDA of the CFG
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We will come back to this (CFL-reachability) question when introducing the middle end of compilers

Undecidable Properties

- Check if a CFG is not ambiguous
- Check if a CFG has inherent ambiguity
- Check if the intersection of two CFLs is empty
- Check if two CFLs are equivalent
- Check if a CFL is equivalent to Σ^*

PART V: Pumping Lemma for CFL

Recap: Intersection of CFL

- Given two CFL, L_1 and L_2, the following may not be CFL
  - $L_1 \cap L_2$

Example: $L_1 = \{a^n b^n c^m\}$, $L_2 = \{a^n b^m c^m\}$

$L_1 \cap L_2 = \{a^n b^n c^n\}$ *(let's prove it)*

Chomsky Normal Form

- We can rewrite a CFG in many standard forms, e.g., CNF
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- ⇒ The parse tree is a binary tree (a tree node has ≤ 2 children)

Observation

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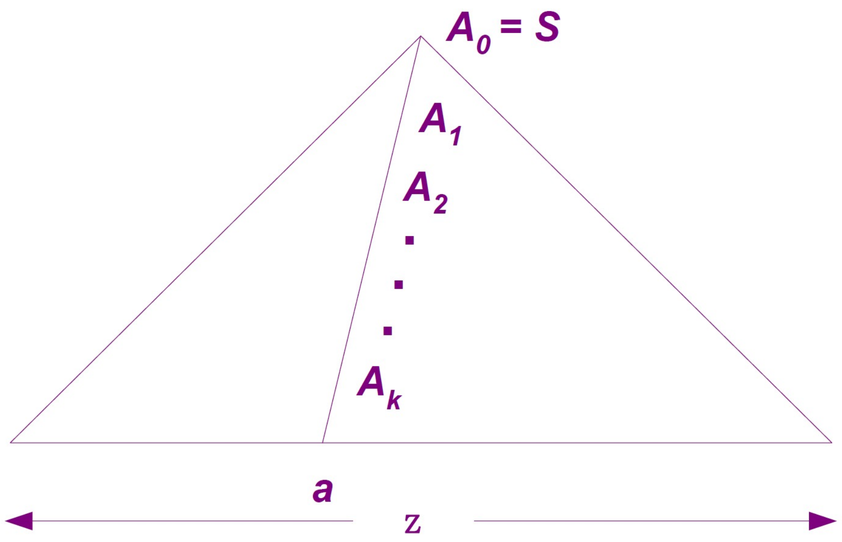

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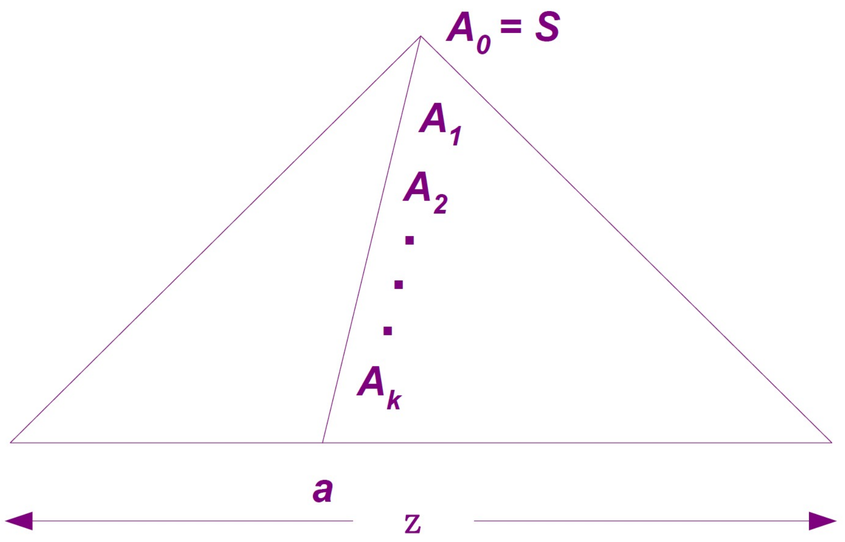

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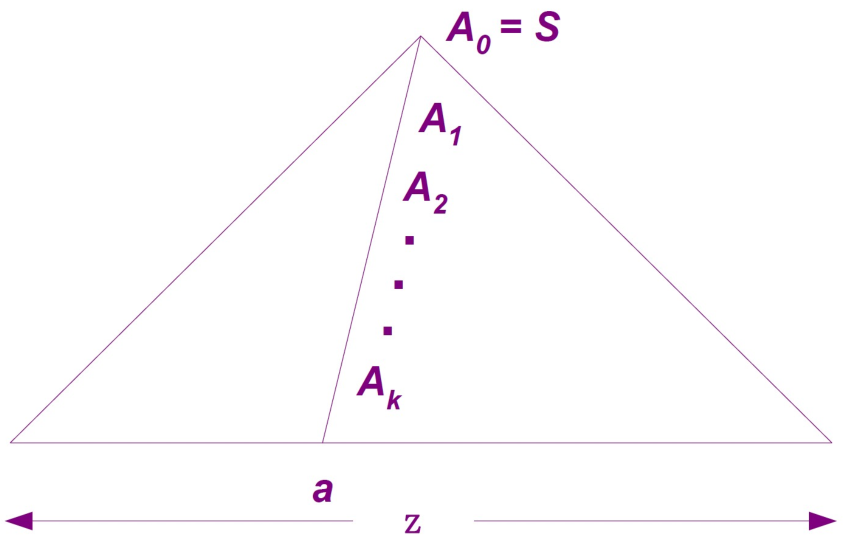

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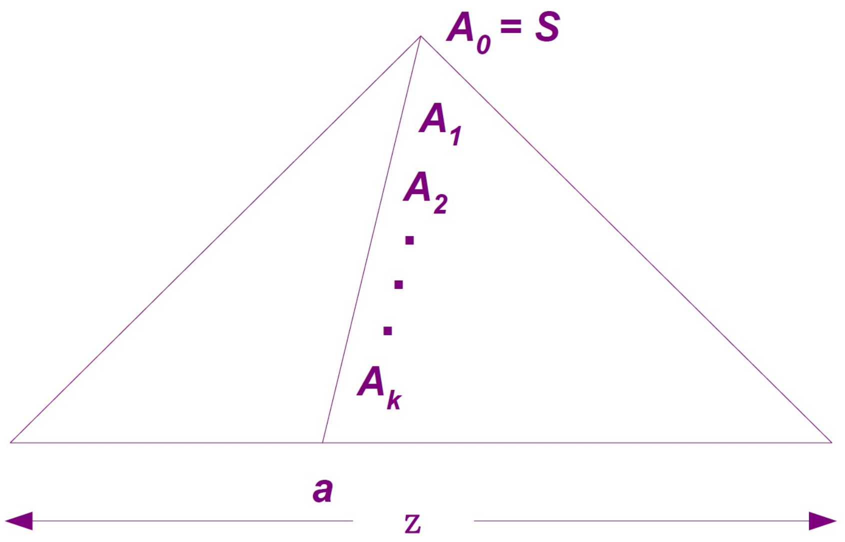

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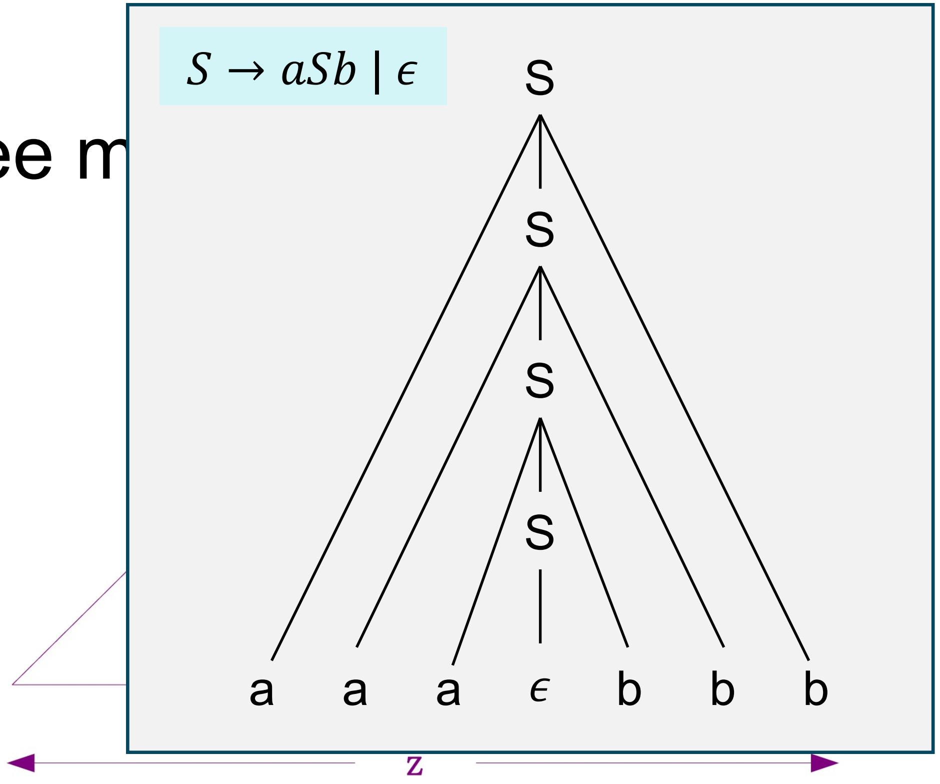

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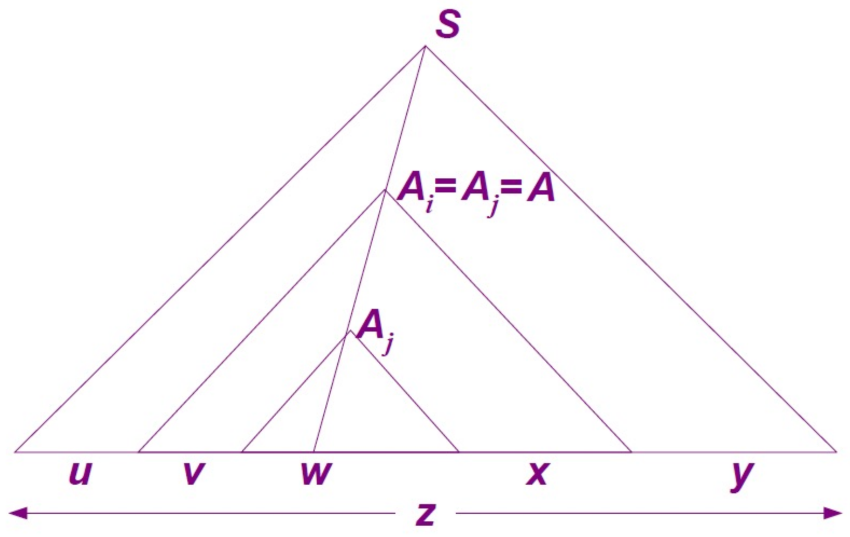

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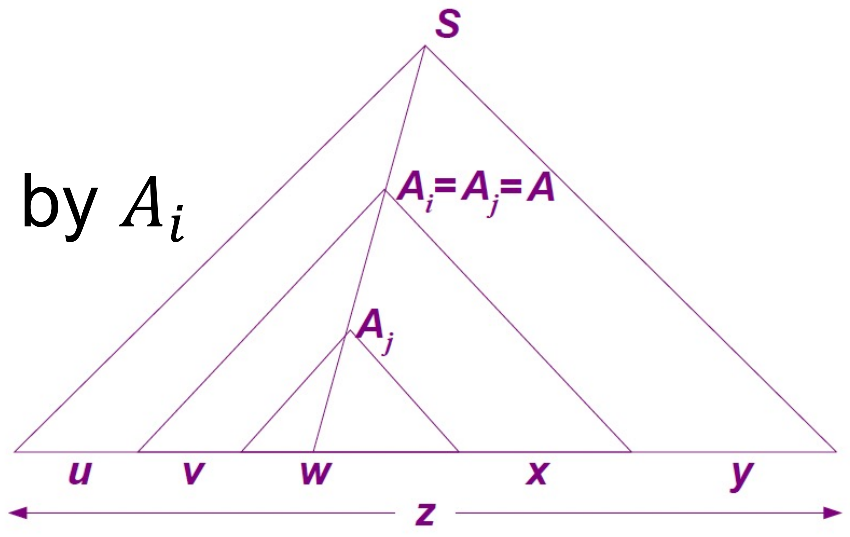

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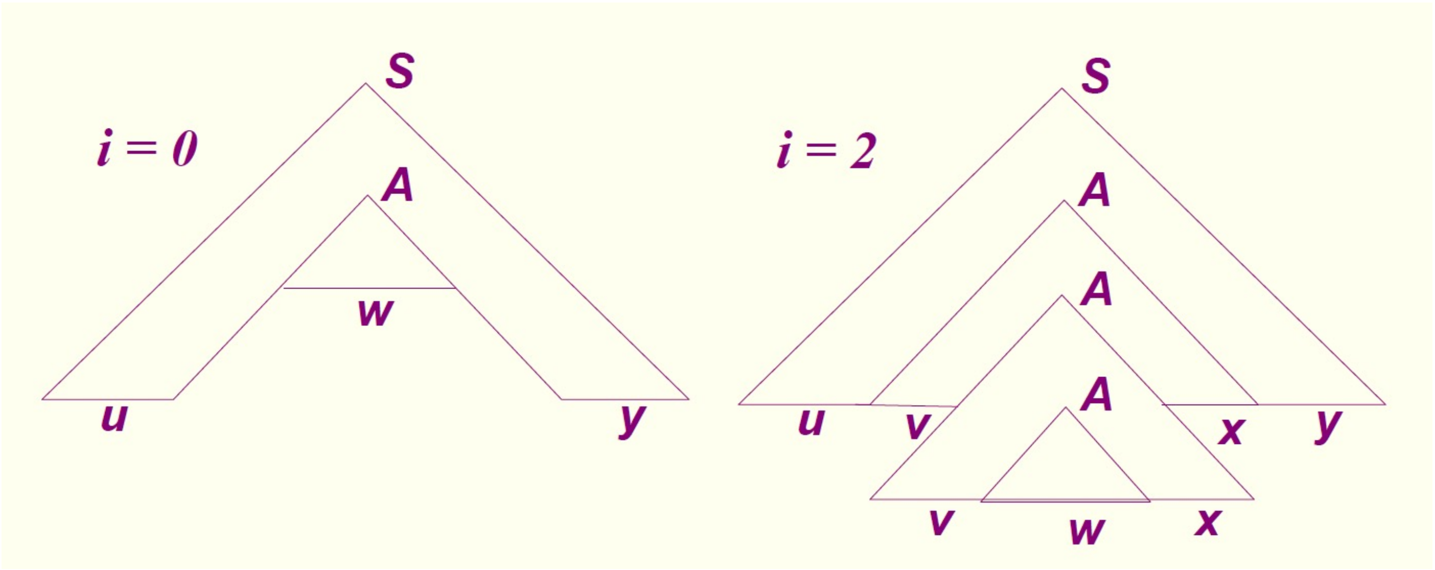
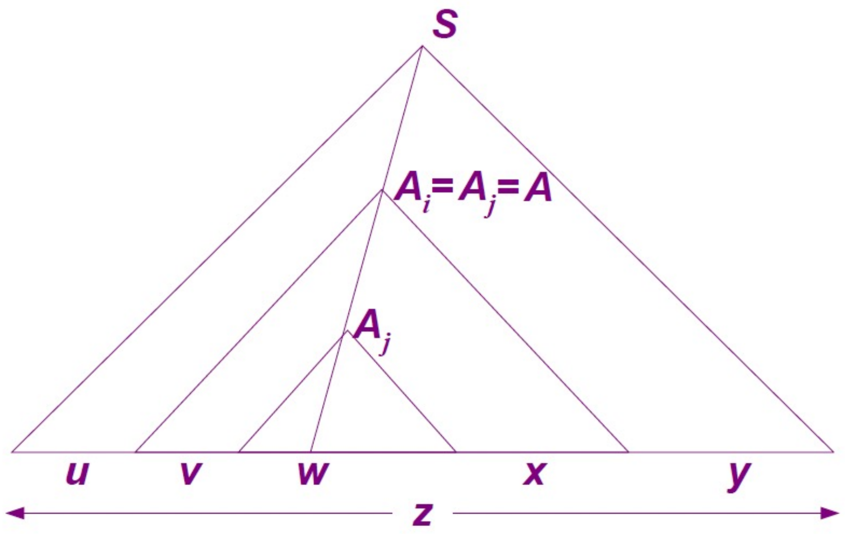
- Write $z = uvwxy$, where vwx is produced by A_i
- Check $z_i = uv^iwx^iy$, which belongs to the CFL for any $i \geq 0$





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- To prove a language is not a CFL, show that we cannot split a string in such a manner.

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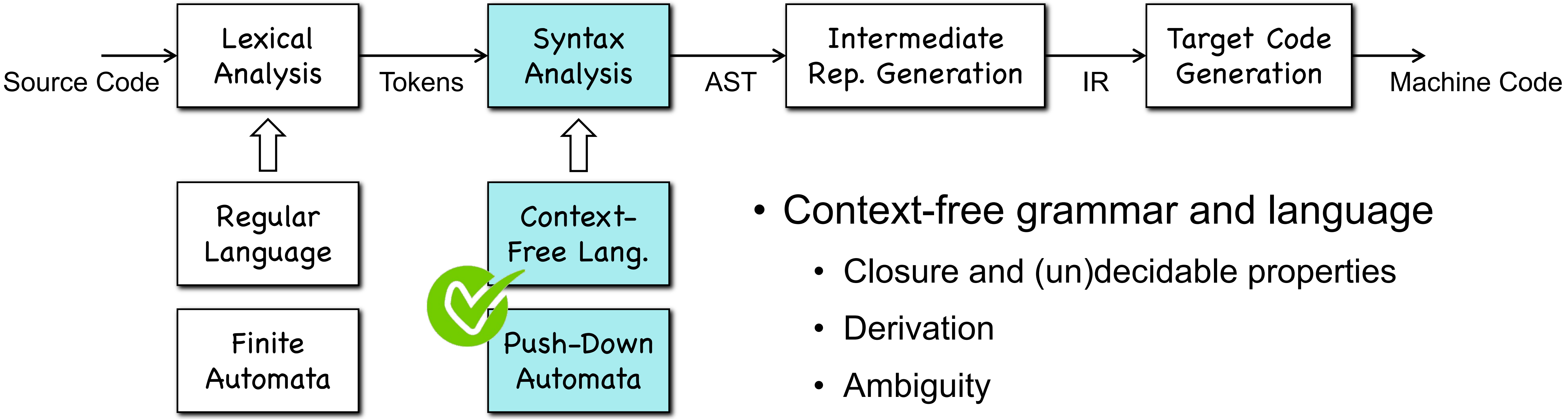
  - $n^2 + 1 \leq |z_2| \leq n^2 + n < (n+1)^2$

Summary



- Context-free grammar and language
  - Closure and (un)decidable properties
  - Derivation
  - Ambiguity
- PDA vs. NPDA vs. DPDA vs. CFL
- The Pumping Lemma for CFL

THANKS!