Fourier Transform Background

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normalization conventions differ, not important as long as you're consistent

FOURIER TRANSFORM "FT"

All variables continuous

Sometimes 2π

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\omega t} f(t)\,dt$$

"$\hat{f} = \mathcal{F}(f)$"

Sometimes $\frac{1}{1}$ or $\frac{1}{2\pi}$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\omega t} \hat{f}(\omega)\,d\omega$$

[* if not integrable, define weakly (for distributions) or via density (for L^2)]

DISCRETE-TIME FOURIER TRANS. "DTFT"

Time is discrete

Frequency periodic

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \sum_{t=-\infty}^{\infty} e^{-i\omega t} f(t)$$

Periodic

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{i\omega t} \hat{f}(\omega)\,d\omega$$

$t \in \mathbb{Z}$

FOURIER SERIES "FS"

Time is periodic

Frequency is discrete

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-i\omega t} f(t)\,dt$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \sum_{\omega=-\infty}^{\infty} e^{i\omega t} \hat{f}(\omega)$$

DISCRETE FOURIER TRANSFORM "DFT"

Time periodic and discrete, i.e. finite

Frequency periodic and discrete, i.e. finite

$$\hat{f}(\omega) = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} e^{-\omega t \frac{2\pi i}{n}} f(t)$$

$$f(t) = \frac{1}{\sqrt{n}} \sum_{\omega=1}^{n} e^{\omega t \frac{2\pi i}{n}} \hat{f}(\omega)$$

Sometimes used to approximate (cts.) FT

All 4 versions share many similar properties,

e.g. all linear,

make differentiation multiplication by a given function, ...

Ex: CONVOLUTION, continuous version

$$(f * g)(x) := \int_{\mathbb{R}} f(y)\, g(x-y)\,dy$$

Thm $\widehat{(f*g)} = \hat{f} \cdot \hat{g}$ i.e. ptwise multiplication

e.g.

$$f * g = \mathcal{F}^{-1}\big(\mathcal{F}(f) \cdot \mathcal{F}(g)\big)$$

via DFT in practice

DFT defines a $n \times n$ orthogonal matrix

e.g. $$\boxed{f} = n \underbrace{\boxed{\text{DFT}}}_{} \cdot \boxed{\hat{f}}$$

So you can compute it in $O(n^2)$ time.

But... the Fast Fourier Transform

is a (collection of) algorithms to do it in $O(n \log n)$ time.

"Top 10 Algs" from 20^{th} century

(modern version: 1965 Cooley + Tukey, but ideas back to Gauss 1805)

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proof of convolution theorem (in limited setting)

$$\hat{f}(\omega) = \int f(x) e^{-2\pi i \omega x} \, dx$$

$$\hat{g}(\omega) = \int g(x) e^{-2\pi i \omega x} \, dx$$

using wikipedia convention for constants

let $h = f * g$, $h(x) = \int f(y) g(x-y) \, dy$

$$\hat{h}(\omega) = \int e^{-2\pi i \omega x} \underbrace{\int f(y) g(x-y) \, dy}_{h} \, dx$$

interchange limits via Fubini's thm (if $f, g \in L^1$)

$$= \int f(y) \left(\underbrace{\int g(x-y) e^{-2\pi i \omega x} dx}_{} \right) dy$$

$d\bar{x} = dx$

change of variables: $\bar{x} = x - y$, $x = \bar{x} + y$

$$= \underbrace{\int g(\bar{x}) e^{-2\pi i \omega \bar{x}} d\bar{x}}_{= \hat{g}(\omega)} \cdot e^{-2\pi i \omega y}$$

$$= \int f(y) \, \hat{g}(\omega) e^{-2\pi i \omega y} \, dy$$

$$= \hat{g}(\omega) \int \underbrace{f(y) e^{-2\pi i \omega y}}_{= \hat{f}(\omega)} \, dy$$

$$= \hat{g}(\omega) \hat{f}(\omega) \quad \square$$

w/ different normalization conventions, you might get a factor of $\sqrt{2\pi}$ somewhere ...

Circulant Matrix

$$\begin{bmatrix} c_0 & c_{n-1} & \ddots & \\ c_1 & c_0 & c_{n-1} & \\ \ddots & c_1 & \ddots & c_{n-1} \\ c_{n-1} & \ddots & c_1 & c_0 \end{bmatrix}$$

Toeplitz Matrix

$$\begin{bmatrix} c_0 & c_{-1} & \ddots & \\ c_1 & c_0 & \ddots & \\ & c_1 & \ddots & c_{-1} \\ c_{n-1} & \ddots & c_1 & c_0 \end{bmatrix}$$

... diagonalized by DFT

(i.e. can invert, find eigenvalues, ...)