

Homework 1

APPM 4720/5720 Scientific Machine Learning, Fall 2024

Due date: Friday, Aug. 30 '24, before midnight, via Gradescope

Instructor: Prof. Becker
Revision date: 8/23/2024

Theme: Review of numerical analysis

Instructions Collaboration with your fellow students is OK and in fact recommended, although direct copying is not allowed. The internet **is allowed for basic tasks** (e.g., looking up definitions on wikipedia, looking at documentation, looking at basic tutorials) but it is not permissible to search for solutions to the exact problem or to *post* requests for help on forums such as <http://math.stackexchange.com/>.

Problem 1: Form a group of between 3 and 5 students, including at least one student enrolled in 4720 and at least one student enrolled in 5720. Write down the names of your group members.

Problem 2: Let $d = 500$ and $\mathbf{x}_0 \in \mathbb{R}^d$ be the all ones vector. For any $n \in \mathbb{N}$, let $G \in \mathbb{R}^{n \times d}$ be a matrix where each entry is drawn i.i.d. from a standard normal distribution, and similarly let $\mathbf{z} \in \mathbb{R}^n$ also have i.i.d. standard normal entries.

Define $A = G + (1 - 10^{-7})\tilde{G}$ where the j^{th} column of \tilde{G} is the $(j - 1)^{\text{th}}$ column of G , done circularly (so the 1st column of \tilde{G} is the last column of G). See `np.roll` in Python or `circshift` in Julia and Matlab

Let $\sigma = 0.1$ and define $\mathbf{b} = A\mathbf{x}_0 + \sigma\mathbf{z}$.

- Write code that returns the least-squares solution $\hat{\mathbf{x}} \stackrel{\text{def}}{=} \arg\min_{\mathbf{x}} \left(f(\mathbf{x}) \stackrel{\text{def}}{=} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2 \right)$, and turn in a PDF of your code, and write a few sentences about how you know that your code is correct (test it on $n = 1000$). *You are allowed to use external linear algebra libraries; you don't have to do this by hand!* Your code should solve the problem in $\mathcal{O}(n)$ time (to be specific, $\mathcal{O}(nd^2)$).
- What are the values of $f(\hat{\mathbf{x}})$ and $f(\mathbf{x}_0)$? (as before, use $n = 1000$).
- Provide numerical evidence that your solver is indeed $\mathcal{O}(n)$. *This is the main part of the problem; think carefully about how to demonstrate this.*

Problem 3: Finite differences. Let $f(x) = \sin(x)$ and $x_0 = 2$. Approximate the derivative of f at x_0 using a finite difference scheme with stepsize h (e.g., the simplest such scheme is forward differences, $f'(x_0) \approx h^{-1}(f(x_0 + h) - f(x_0))$). Turn in a PDF of your code and a plot of the error for different values of h , and comment on whether you can achieve 10^{-14} error.
To calculate the error, you can use the fact that f' is known. When choosing values of h , choose reasonable values, and also decide how to scale your plot (i.e., make some axes logarithmic?).