

# Homework 5

## APPM 4720/5720 Scientific Machine Learning, Fall 2024

**Due date:** Monday, Sep. 30 '24, before midnight, via Gradescope

**Instructor:** Prof. Becker  
**Revision date:** 9/25/2024

**Theme:** ODE solvers (for a Hamiltonian system)

**Instructions** Collaboration with your fellow students is OK and in fact recommended, although direct copying is not allowed. The internet **is allowed for basic tasks** (e.g., looking up definitions on wikipedia, looking at documentation, looking at basic tutorials) but it is not permissible to search for solutions to the exact problem or to *post* requests for help on forums such as <http://math.stackexchange.com/>.

**Background** The **two-body problem** is about predicting the motion of two objects orbiting each other in space, interacting with each other (due to gravity). It can be **reduced to a pair of one-body problems** and what's left is the **Kepler problem**, as we'll describe below. This has a long history, and for many years people tried to have similarly clean analysis of the three-body problem but without luck (now known to be generally impossible, though numerically it's not an issue).

For us, we have a variable  $\mathbf{x} \in \mathbb{R}^2$  and its corresponding velocity  $\mathbf{v} \in \mathbb{R}^2$  (note: even if we work in 3D, all the motion will stay in a fixed plane, so we can still work in 2D without loss of generality), and assuming the center of mass is at the origin, we have a coupled 4D system of ODEs:

$$\begin{aligned}\frac{d\mathbf{x}}{dt} &= \mathbf{v}(t) \\ \frac{d\mathbf{v}}{dt} &= -\mu \frac{\mathbf{x}(t)}{\|\mathbf{x}(t)\|^3}\end{aligned}\quad (\text{Kepler problem})$$

where  $\mu = 4\pi^2$  (we have used non-dimensionalized units; we will interpret time as being in units of years, and you can think of a distance of 1 being one astronomical unit). Use starting conditions  $\mathbf{x}(0) = [1, 0]$  and  $\mathbf{v}(0) = [0, 1.1\sqrt{\mu}]$ . This should lead to elliptical orbits (if we had used  $\mathbf{v}(0) = [0, \sqrt{\mu}]$  it would be perfectly circular) of period close to 1. We could find a closed form solution (see wikipedia), but for now the main analytic observation is that the force is conservative (there is no friction nor drag) and therefore total energy is conserved. *Recall total energy is potential plus kinetic, with potential being  $-\mu/\|\mathbf{x}\|$  and kinetic being  $\frac{1}{2}\|\mathbf{v}\|^2$ .*

*Comment: this PDF has URL links (designated by colored text), but they may not be clickable in github or Canvas' default viewers. If that's the case, just download the homework PDF and use a normal PDF viewer*

**Problem 1:** Write ODE solvers for the Kepler problem using at least 2 methods: forward Euler, and a symplectic method (e.g., **Semi-implicit Euler method** (also known as symplectic Euler, semi-implicit Euler, Euler–Cromer, and Newton–Størmer–Verlet (NSV)), or **velocity Verlet**).

Turn in the code in a nice PDF.

**Students in 5720** must also implement a third method: backward Euler. *For backward Euler, you have at least three implementation options. You can solve iteratively via Picard iterations, or iteratively via Newton-Raphson, or solve it (mostly) by hand.*

**Problem 2:** Simulate the system for 10 years using all two (or three) solvers you created, with a timestep  $\Delta t$  being a quarter of a day (remember, time is in units of years). Plot the total energy as a function of time for all two (or three) methods. Make some observations about conservation of energy.

- Problem 3: Refinement study.** Take an even smaller time step  $\Delta t$  and solve the ODE; we'll take this as a "ground truth". Then solve the ODE using progressively larger stepsizes (I'd suggest increasing them by a factor of 2 every time), and look at the error compared to the "ground truth" in terms of the mean-squared-error of all the state variables (that is,  $\mathbf{x}$  and  $\mathbf{v}$ ) (*When doing this, don't forget to subset as needed, so that you are matching the right time points*). Repeat this for all two (or three) methods, and turn in a log-log plot of stepsize vs error, and comment on what you think the order of each method is.
- Problem 4:** For at least one of the methods, plot the total energy for a variety of stepsizes, and comment on whether refining the stepsize helps conservation of energy.
- Problem 5: Students in 5720 only** Back to using  $\Delta t$  being a quarter of a day, run the backward Euler method for 11.5 years and comment on what you observe and discuss possible explanations.