Data Space to Surface Transformations

Chuck Rino
Vista Research

November 2007

Abstract

Derivation of general formulas for transforming camera or radar data-space coordinates to hopothesided surface sphere or ellipsoid source.

1 Introduction

This note disscusses the problem of mapping sensor reports in data-space coordinates to geodetic coordinates. We assume that the source location and its orientation are known at the time of each report. Each pixel in a camera focal plane, for example, translates to a ray direction that connects to a source of illumination. If the source lies on a known surface, the problem reduces to determining the intercept point of a ray and a surface. The radar problem is a little more complicated, because we usually know only range and azimuth. If we know that the radar target lies on a surface the problem reduces to finding the depression angle in the range-azimuth plane that connects the range to the surface. If the surface is sphere or ellipsoid representing the earth's surface or a parallel surface of constant altitude, the problem admits a closed form solution, although for the ellipsoid a search is more robust. A search must be used if the surface is a terrain elevation map.

2 Formulation

The idealized problem reduces to calculating the intercepts of a line with an ellipsoidal surface. The sphere is a special case. Assume the line eminates from a known source location ($\mathbf{R}_0$) in an XYZ coordinate system with axes along the semimajor (X) and semiminor (Y) axes of the ellipsoid that defines the surface. A ray intercepting the surface is defined by a unit direction vector $\mathbf{v}$ and the distances $r_{1,2}$ along $\mathbf{v}$ to the possible interception points. Only the nearest intercept is usually of interest. The problem has several variants depending on the information available about the intercepting ray. For example, if only the range to the surface is known, the possible solutions lie on the intersections of a cone with the ellipsoidal surface. If the direction vector $\mathbf{v}$ is specified the unknowns are the solutions to a quadratic equation. In all cases intersection points are defined by the relation

$$\mathbf{R} = \mathbf{R}_0 + r\mathbf{v}. \tag{1}$$

Intermediate situations involve a know range r and one of the two angles that define $\mathbf{v}$. In a topocentric system with origin at $\mathbf{R}_0$, for example

$$\mathbf{v}_{\mathrm{tcs}} = [\sin\theta\sin\phi; \sin\theta\cos\phi; \cos\theta],$$

where ϕ is the bearing angle and θ is the polar angle. Note that the only polar angles that define intercepting rays lie in the range $\pi/2 < \theta \leq \pi$. Suppose we know r and ϕ, but not θ. The transformation of $\mathbf{v}_{\mathrm{tcs}}$ to the earth centered fixed system is a rotation Ω.

Let $\mathbf{r}_s = [x_s, y_s, z_s]^T$, represent a point on the surface. The transformation

$$\Lambda = \begin{bmatrix} 1/a & 0 & 0 \\ 0 & 1/a & 0 \\ 0 & 0 & 1/b \end{bmatrix} \tag{2}$$

is such that $\|\Lambda\mathbf{r}_s\| = 1$. From (1)

$$\mathbf{R}_0 + r\mathbf{v} = \mathbf{R}. \tag{3}$$

In the transformed coordinate system

$$\mathbf{\Lambda R} = \mathbf{\Lambda R}_0 + r\mathbf{\Lambda v} \tag{4}$$

It follows that

$$|\mathbf{\Lambda R}_0|^2 + r^2 |\mathbf{\Lambda v}|^2 + 2r\mathbf{\Lambda R}_0 \cdot \mathbf{\Lambda v} = 1 \tag{5}$$

If $\mathbf{v}$ is known (5) is a quadratic equation with solutions $r_{1,2}$.

The same equation can be solved if range and azimuth are known. To carry the unknown elevation into (5) explicitly, it is necessary to introduce the rotation $\mathbf{\Omega}$ that transforms $\mathbf{v}_{\mathrm{tcs}}$ to ecf coordinates. Because the matrix $\mathbf{\Omega}$ depends only on the source location it is known a priori. With

$$\mathbf{v} = \mathbf{\Omega}\mathbf{v}_{\mathrm{tcs}} \tag{6}$$

and the azimuth angle specified, solving (5) is a well-posed problem, but the computation is tedious.

3 Solution

Substituting for $\mathbf{v}$,

$$
\begin{aligned}
& a^2 \left|\boldsymbol{\Lambda}\boldsymbol{\Omega}\mathbf{v}_{\mathrm{tcs}}\right|^2 = \\
& \left(\sin\theta\left(\Omega_{11}\sin\phi + \Omega_{12}\cos\phi\right) + \Omega_{13}\cos\theta\right)^2 \\
& + \left(\sin\theta\left(\Omega_{21}\sin\phi + \Omega_{22}\cos\phi\right) + \Omega_{23}\cos\theta\right)^2 \\
& + \left(\sin\theta\left(\Omega_{31}\sin\phi + \Omega_{32}\cos\phi\right) + \Omega_{33}\cos\theta\right)^2 (a/b)^2 \\
= \; & \sin^2\theta \left(\left(A_1^2 - \Omega_{13}^2\right) + \left(A_2^2 - \Omega_{23}^2\right) + \left(A_3^2 - \Omega_{33}^2\right)(a/b)^2\right) \\
& + 2\sin\theta\cos\theta \left(A_1\Omega_{13} + A_2\Omega_{23} + A_3\Omega_{33}(a/b)^2\right) \\
& + \left(\Omega_{13}^2 + \Omega_{23}^2 + \Omega_{33}^2 (a/b)^2\right) \\
= \; & \alpha\sin^2\theta + \beta\sin\theta\cos\theta + \gamma
\end{aligned}
$$

where

$$
\begin{array}{cc}
A_1 & \Omega_{11}\sin\phi + \Omega_{12}\cos\phi \\
A_2 & \Omega_{21}\sin\phi + \Omega_{22}\cos\phi \\
A_3 & \Omega_{31}\sin\phi + \Omega_{32}\cos\phi \\
\alpha & \begin{array}{c} \left(A_1^2 - \Omega_{13}^2\right) \\ + \left(A_2^2 - \Omega_{23}^2\right) \\ + \left(A_3^2 - \Omega_{33}^2\right)(a/b)^2 \end{array} \\
\beta & 2\left(A_1\Omega_{13} + A_2\Omega_{23} + A_3\Omega_{33}(a/b)^2\right) \\
\gamma & \Omega_{13}^2 + \Omega_{23}^2 + \Omega_{33}^2 (a/b)^2
\end{array}
$$

Similarly,

$$
\begin{aligned}
a^2 \boldsymbol{\Lambda}\mathbf{R}_0 \cdot \boldsymbol{\Lambda}\mathbf{v} \; = \; & \mathbf{X}_0\left(A_1\sin\theta + \Omega_{13}\cos\theta\right) \\
& + \mathbf{Y}_0\left(A_2\sin\theta + \Omega_{23}\cos\theta\right) \\
& + \mathbf{Z}_0\left(A_3\sin\theta + \Omega_{33}\cos\theta\right)(a/b)^2 \\
= \; & \sin\theta\left[\mathbf{X}_0 A_1/a^2 + \mathbf{Y}_0 A_2/a^2 + \mathbf{Z}_0 A_3 (a/b)^2\right] \\
& + \cos\theta\left[\left(\mathbf{X}_0\Omega_{13} + \mathbf{Y}_0\Omega_{23}\right)/a^2 + \mathbf{Z}_0\Omega_{33}(a/b)^2\right] \\
= \; & \zeta\sin\theta + \eta\cos\theta. \qquad (7)
\end{aligned}
$$

where

$$
\begin{array}{cc}
\zeta & \mathbf{X}_0 A_1 + \mathbf{Y}_0 A_2 + \mathbf{Z}_0 A_3 (a/b)^2 \\
\eta & \mathbf{X}_0\Omega_{13} + \mathbf{Y}_0\Omega_{23} + \mathbf{Z}_0\Omega_{33}(a/b)^2
\end{array}
$$

$$
a^2 \left|\boldsymbol{\Lambda}\mathbf{R}_0\right|^2 = D^2 \qquad (8)
$$

where

$$D^2 \quad X_0^2 + Y_0^2 + Z_0^2\,(a/b)^2\,.$$

Rewriting (5)

$$r^2\left(a^2\,|\mathbf{\Lambda v}|^2\right) + 2r\left(a^2\mathbf{\Lambda R}_0\cdot\mathbf{\Lambda v}\right) + (\left(a^2\,|\mathbf{\Lambda R}_0|^2\right) - a^2) = 0 \tag{9}$$

Substituting from the definitions above to isolate the θ-dependt terms

$$\begin{aligned} r^2\left(\alpha\sin^2\theta + \beta\sin\theta\cos\theta + \gamma\right) \\ +2r\left(\zeta\sin\theta + \eta\cos\theta\right) \\ +D^2 - a^2 &= 0. \end{aligned} \tag{10, 11}$$

Note that for a solution to exist, θ must lie between $\pi/2$ and π. To make the constraint on θ more transparent let $\theta = \pi/2 + \epsilon$. It follows that $\sin(\pi/2+\epsilon) = \cos(\epsilon)$, and $\cos(\pi/2+\epsilon) = -\sin(\epsilon)$ with $0 < \epsilon \le \pi/2$.

$$\begin{aligned} r^2\left(\alpha\cos^2\epsilon - \beta\sin\epsilon\cos\epsilon + \gamma\right) \\ +2r\left(\zeta\cos\epsilon - \eta\sin\epsilon\right) \\ +D^2 - a^2 &= 0. \end{aligned} \tag{12}$$

Note that

$$\left(\zeta\cos\epsilon - \eta\sin\epsilon\right)^2 = \left(D^2 - a^2\right)\left(\alpha\cos^2\epsilon - \beta\sin\epsilon\cos\epsilon + \gamma\right) \tag{13}$$

determines the smallest value of ϵ for which a solution exists. The two solutions for $\cos\epsilon$ or $\sin\epsilon$ are symmetric about 0. The range to the surface at either solution point marks the geometric horizon. Elsewhere there are two solutions, but only the solution between the nadir and the geometric horizon is visible to the source.

In principle both (12) or (10) and (13) can be solved analytically as a quartic equations in $\cos\epsilon$ or $\sin\epsilon$ or by a search. However, if ϵ is small one can use the small angle approximations $\cos\epsilon \approx (1-\epsilon^2/2)$ and $\sin\epsilon \approx \epsilon$, which implies that $\cos^2\epsilon = (1-\epsilon^2/2)^2 \approx 1-\epsilon^2$ and $\sin\epsilon\cos\epsilon \approx \epsilon$. Substituting into (12) and manipulating the results:

$$\begin{aligned} r^2\left(\alpha\left(1-\epsilon^2\right) - \beta\epsilon + \gamma\right) \\ +2r\left(\zeta\left(1-\epsilon^2/2\right) - \eta\epsilon\right) \\ +D^2 - a^2 &= 0. \end{aligned} \tag{14}$$

$$\begin{aligned} r^2\alpha - r^2\alpha\epsilon^2 - r^2\beta\epsilon + r^2\gamma \\ +2r\zeta - 2r\zeta\epsilon^2/2 - 2r\eta\epsilon \\ +D^2 - a^2 &= 0. \end{aligned} \tag{15}$$

$$\begin{aligned} \left(r^2\alpha + r\zeta\right)\epsilon^2 + \left(r^2\beta + 2r\eta\right)\epsilon \\ -\left(r^2\gamma + 2r\zeta + r^2\alpha + D^2 - a^2\right) &= 0. \end{aligned} \tag{16}$$

The final quadratic equation for ϵ can be solved directly.

For a sphere, $\alpha = \beta = 0$, and $\gamma = 1$, whereby

$$
\begin{aligned}
r^2 & & \quad (17)\\
+2r\,(\zeta \sin\theta + \eta\cos\theta) & & \\
+D^2 - a^2 &= 0. & \quad (18)
\end{aligned}
$$

which gives the same solution as

$$\cos\theta = -\frac{r^2 + R_0^2 - 1}{2r(R_0 + H)},$$

which can be derived from the cosine law

$$r^2 + (R_0 + H)^2 - 2r(R_0 + H)\cos\theta = R_0.$$