CHAPTER 10
INVERSE FUNCTIONS

EXERCISE 10.1

1 (a) Straight line with gradient of 1; is one-to-one

(b) Right-hand portion of parabola with turning point at $x = 1$; is one-to-one

(c) The top half of a circle; not one-to-one

(d) Straight line with gradient of -1; is one-to-one

(e) Rectangular hyperbola; is one-to-one

(f) Graph of absolute value function; not one-to-one

(g) Parabola; not one-to-one

(h) Portion of cosine graph from (0, 1) to $\left(\frac{\pi}{2}, 0\right)$; is one-to-one

(i) Graph of cos function for all x; not one-to-one

(j) $f(x) = x^3 - 4x$
$= x(x+2)(x-2)$
This graph has x-intercepts at $x = -2, 0, 2$, so there are turning points between $x = -2$, $x = 0$ and $x = 2$; not one-to-one

3 (a) Let $y = 2x - 4$ $\quad d_f = R$ and $r_f = R$
$x = 2y - 4$ (interchange x and y)
$x + 4 = 2y$
$y = \frac{1}{2}x + 2$
$f^{-1}(x) = \frac{1}{2}x + 2 \quad d_{f^{-1}} = R \quad r_{f^{-1}} = R$

(b) Let $y = x^2 - 1$ $\quad d_f: x \geq 0$ and $r_f: y \geq -1$
$x = y^2 - 1$ (interchange x and y)
$x + 1 = y^2$
$y = \sqrt{x+1}$
$f^{-1}(x) = \sqrt{x+1} \quad d_{f^{-1}}: x \geq -1 \quad r_{f^{-1}}: y \geq 0$

(c) Let $y = \sqrt{x-3}$ $\quad d_g: x \geq 3$ and $r_g: y \geq 0$
$x = \sqrt{y-3}$ (interchange x and y)
$x^2 = y - 3$
$y = x^2 + 3$
$g^{-1}(x) = x^2 + 3 \quad d_{g^{-1}}: x \geq 0 \quad r_{g^{-1}}: y \geq 3$

(d) Let $y = \sqrt{9 - x^2}$ $\quad d_f: -3 \leq x \leq 0$ and $r_f: 0 \leq y \leq 3$
$x = \sqrt{9 - y^2}$ (interchange x and y)
$x^2 = 9 - y^2$
$y^2 = 9 - x^2$
$y = \pm\sqrt{9 - x^2}$
$d_{f^{-1}}: 0 \leq x \leq 3$ and $r_{f^{-1}}: -3 \leq y \leq 0$
So: $f^{-1}(x) = -\sqrt{9 - x^2}$

(e) Let $y = x^3$ $\quad d_f = R$ and $r_f = R$.
$x = y^3$ (interchange x and y)
$y = x^{\frac{1}{3}}$
$f^{-1}(x) = x^{\frac{1}{3}} \quad d_{f^{-1}} = R \quad r_{f^{-1}} = R$

(f) Let $y = (x+2)^2$ $\quad d_f: x \leq -2$ and $r_f: y \geq 0$
$x = (y+2)^2$ (interchange x and y)
$\pm\sqrt{x} = y + 2$
$y = \pm\sqrt{x} - 2$
$d_{f^{-1}}: x \geq 0$ and $r_{f^{-1}}: y \leq -2$
So: $f^{-1}(x) = -\sqrt{x} - 2$

(g) Let $y = x^2 + 2x$ $\quad d_f: x \geq 0$ and $r_f: y \geq 0$
$x = y^2 + 2y$ (interchange x and y)
$x + 1 = y^2 + 2y + 1$
$x + 1 = (y+1)^2$
$\pm\sqrt{x+1} = y + 1$
$d_{f^{-1}}: x \geq 0$ and $r_{f^{-1}}: y \geq 0$
So: $\sqrt{x+1} = y + 1$
$f^{-1}(x) = \sqrt{x+1} - 1$

(h) Let $y = \log_e(x+1)$ $\quad d_f: x > -1$ and $r_f = R$
$x = \log_e(y+1)$ (interchange x and y)
$y + 1 = e^x$
$y = e^x - 1$
$f^{-1}(x) = e^x - 1 \quad d_{f^{-1}} = R \quad r_{f^{-1}}: y > -1$

(i) Let $y = 2 - \sqrt{x-2}$
$d_f: x \geq 2$ and $r_f: y \leq 2$
$x = 2 - \sqrt{y-2}$ (interchange x and y)
$\sqrt{y-2} = 2 - x$
$y - 2 = (2-x)^2$
$y = 4 - 4x + x^2 + 2$
$f^{-1}(x) = x^2 - 4x + 6 \quad d_{f^{-1}}: x \leq 2 \quad r_{f^{-1}}: y \geq 2$

(j) Let $y = \sqrt{(5-x)} - 1$
$d_g: x \leq 5$ and $r_g: y \geq -1$
$x = \sqrt{(5-y)} - 1$ (interchange x and y)
$x + 1 = \sqrt{(5-y)}$
$(x+1)^2 = 5 - y$
$y = 5 - (x+1)^2$
$y = 5 - x^2 - 2x - 1$
$g^{-1}(x) = -x^2 - 2x + 4 \quad d_{g^{-1}}: x \geq -1 \quad r_{g^{-1}}: y \leq 5$

(k) Let $y = 2^{-x}$ $\quad d_f: x > 0$ and $r_f: 0 < y < 1$

$x = 2^{-y}$ (interchange x and y)

$-y = \log_2 x$

$f^{-1}(x) = -\log_2 x \quad d_{f^{-1}}: 0 < x < 1 \quad r_{f^{-1}}: y > 0$

(l) Let $y = \dfrac{1}{x+1}$ $\quad d_h: x > -1$ and $r_h: y > 0$

$x = \dfrac{1}{y+1}$ (interchange x and y)

$y + 1 = \dfrac{1}{x}$

$h^{-1}(x) = \dfrac{1}{x} - 1 \quad d_{h^{-1}}: x > 0 \quad r_{h^{-1}}: y > -1$

5 (a) Let $y = \sqrt{a^2 - x^2}$.

$y^2 = a^2 - x^2$

$x^2 + y^2 = a^2$

So the graph of $f(x)$ is the top half of a circle with centre at $(0, 0)$ and radius a. It has domain $-a \le x \le a$. This is not a one-to-one function and so does not have an inverse.

If the domain were $0 \le x \le a$ or $-a \le x \le 0$ then it would be a one-to-one function and would therefore have an inverse.

(b) The graph of $f(x) = 4 - x^2$ is a parabola with turning point at (0, 4). It is not a one-to-one function and so does not have an inverse.

If the domain were $x \le 0$ or $x \ge 0$ then it would be one-to-one and so the inverse function would exist.

(c) The graph of $f(x) = \dfrac{1}{x^2}$ is symmetrical about $x = 0$ and so is not a one-to-one function. The domain of this function is $x < 0, x > 0$.

If the domain were restricted to either $x < 0$ or $x > 0$ then it would be one-to-one and so the inverse function would exist.

7 **D** is the correct alternative.

Let $y = \sqrt{4 - x^2}$ $\quad d_f: -2 \le x \le 0$ and $r_f: 0 \le y \le 2$

$x = \sqrt{4 - y^2}$ (interchange x and y)

$x^2 = 4 - y^2$

$y^2 = 4 - x^2$

$y = \pm\sqrt{4 - x^2}$

$d_{f^{-1}}: 0 \le x \le 2$ and $r_{f^{-1}}: -2 \le y \le 0$

So: $y = -\sqrt{4 - x^2} \quad 0 \le x \le 2$

A incorrect, this is the equation of the circle, which is not one-to-one

B incorrect, this is the equation of the top half of the circle, which is not one-to-one

C incorrect, this is a one-to-one function but is the wrong part of the circle (see given working)

D correct (see given working)

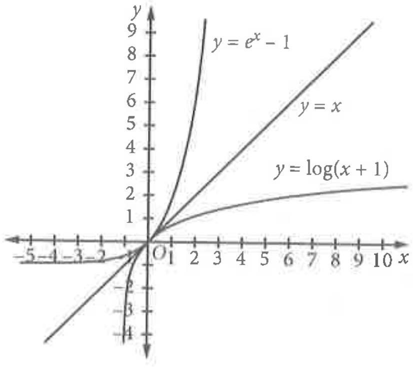
9 (a) $y = e^x - 1$

$\dfrac{dy}{dx} = e^x$

The derivative is positive for all x, so the function is increasing for all x. Hence it is a monotonic increasing function.

(b) Where $x = 0$, $\dfrac{dy}{dx} = e^0 = 1$. Hence the graph of $y = e^x - 1$ has a gradient of 1 at the point $(0, 0)$.

(c) Where $x = 0$, $y = 0$: at the point $(0, 0)$, which is on the line $y = x$, the gradient is 1.



(d) $y = e^x - 1$

$x = e^y - 1$ (interchange x and y)

$x + 1 = e^y$

$y = \log_e(x + 1)$ is the inverse function.

(e) $\log_e(x + 1) < x$ is where the graph of $y = \log_e(x + 1)$ is below the graph of $y = x$. This is where $-1 < x < 0, x > 0$.

EXERCISE 10.2

1 **B** is the correct alternative.

$\cos^{-1} x - \cos^{-1}(-x) = \cos^{-1} x - (\pi - \cos^{-1} x)$

$= 2\cos^{-1} x - \pi$

A incorrect, this would be the answer to $\cos^{-1} x + \cos^{-1}(-x)$

B correct (see working above)

C incorrect (see working above)

D incorrect (see working above)

3 (a) $\cos\left(\sin^{-1}\dfrac{1}{2}\right) = \cos\left(\dfrac{\pi}{6}\right)$

$= \dfrac{\sqrt{3}}{2}$

(b) Because $\tan[\tan^{-1}(x)] = \tan x$ for $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$:

$\tan\left(\tan^{-1}\left[-\dfrac{5}{13}\right]\right) = -\dfrac{5}{13}$

(c) $\tan^{-1}(\tan 245°) = \tan^{-1}(\tan[180 + 65]°)$

$= 65°$

(d) $\cos^{-1}(\cos 540°) = \cos^{-1}(\cos[360 + 180]°)$
$= 180°$

(e) $\cos(\tan^{-1}[-\sqrt{3}]) = \cos\left(-\frac{\pi}{3}\right)$
$= \frac{1}{2}$

(f) $\cos\left(2\sin^{-1}\frac{\sqrt{3}}{2}\right) = \cos\left(\frac{2\pi}{3}\right)$
$= -\frac{1}{2}$

(g) Find $\cos\left(2\cos^{-1}\frac{5}{13}\right)$.

Let $\theta = \cos^{-1}\frac{5}{13}$.

So: $\cos\theta = \frac{5}{13}$

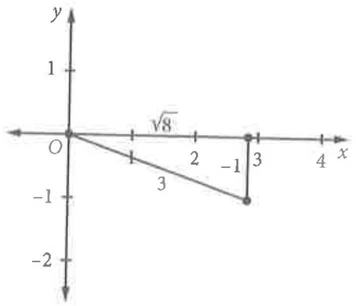
$$\cos\left(2\cos^{-1}\frac{5}{13}\right) = \cos 2\theta$$
$$= 2\cos^2\theta - 1$$
$$= 2\left(\frac{5}{13}\right)^2 - 1$$
$$= -\frac{119}{169}$$

(h) Find $\sec\left(\sin^{-1}\left[-\frac{1}{3}\right]\right)$.

Let $\theta = \sin^{-1}\left(-\frac{1}{3}\right)$.

So: $\sin\theta = -\frac{1}{3}$

and: $\cos\theta = \frac{\sqrt{8}}{3} = \frac{2\sqrt{2}}{3}$

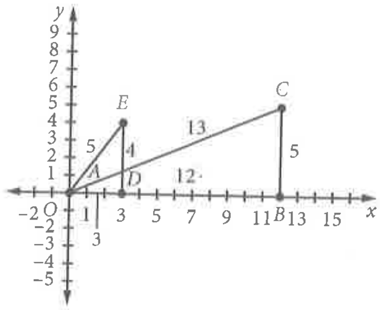


So: $\sec\left(\sin^{-1}\left[-\frac{1}{3}\right]\right) = \dfrac{1}{\cos\left(\sin^{-1}\left[-\frac{1}{3}\right]\right)}$
$$= \frac{1}{\frac{2\sqrt{2}}{3}}$$
$$= \frac{3\sqrt{2}}{4}$$

5 (a) $\sin\left(\sin^{-1}\left[\frac{3}{5}\right]\right) + \sin\left(\sin^{-1}\left[-\frac{3}{5}\right]\right) = \frac{3}{5} - \frac{3}{5}$
$= 0$

(b) $\sin\left(\sin^{-1}\frac{3}{5} + \sin^{-1}\left[-\frac{3}{5}\right]\right)$
$$= \sin\left(\sin^{-1}\frac{3}{5} - \sin^{-1}\frac{3}{5}\right)$$
$$= \sin 0$$
$$= 0$$

(c) $\cos\left[\sin^{-1}\frac{5}{13} + \sin^{-1}\frac{4}{5}\right]$
$$= \cos\left(\sin^{-1}\frac{5}{13}\right)\cos\left(\sin^{-1}\frac{4}{5}\right) - \sin\left(\sin^{-1}\frac{5}{13}\right)\sin\left(\sin^{-1}\frac{4}{5}\right)$$
$$= \frac{12}{13}\times\frac{3}{5} - \frac{5}{13}\times\frac{4}{5}$$
$$= \frac{16}{65}$$



(d) $\sin\left[2\tan^{-1}\frac{4}{3}\right]$
$$= 2\sin\left(\tan^{-1}\frac{4}{3}\right)\cos\left(\tan^{-1}\frac{4}{3}\right)$$
$$= 2\times\frac{4}{5}\times\frac{3}{5}$$
$$= \frac{24}{25}$$

(e) $\cos\left[\tan^{-1}\frac{4}{3} - \cos^{-1}\frac{5}{13}\right]$
$$= \cos\left(\tan^{-1}\frac{4}{3}\right)\cos\left(\cos^{-1}\frac{5}{13}\right) + \sin\left(\tan^{-1}\frac{4}{3}\right)\sin\left(\cos^{-1}\frac{5}{13}\right)$$
$$= \frac{3}{5}\times\frac{5}{13} + \frac{4}{5}\times\frac{12}{13}$$
$$= \frac{63}{65}$$

(f) $\sin\left[\cos^{-1}\frac{3}{5} + \tan^{-1}\left(-\frac{3}{4}\right)\right]$
$$= \sin\left(\cos^{-1}\frac{3}{5}\right)\cos\left(\tan^{-1}\left[-\frac{3}{4}\right]\right) + \cos\left(\cos^{-1}\frac{3}{5}\right)\sin\left(\tan^{-1}\left[-\frac{3}{4}\right]\right)$$
$$= \frac{4}{5}\times\frac{4}{5} + \frac{3}{5}\times -\frac{3}{5}$$
$$= \frac{7}{25}$$

(g) $\tan\left[\tan^{-1}\frac{4}{3} + \tan^{-1}\frac{12}{13}\right]$
$$= \frac{\tan\left(\tan^{-1}\frac{4}{3}\right) + \tan\left(\tan^{-1}\frac{12}{13}\right)}{1 - \tan\left(\tan^{-1}\frac{4}{3}\right)\tan\left(\tan^{-1}\frac{12}{13}\right)}$$
$$= \frac{\frac{4}{3} + \frac{12}{13}}{1 - \frac{4}{3}\times\frac{12}{13}}$$

$$= \frac{\frac{52}{39} + \frac{36}{39}}{1 - \frac{48}{39}}$$

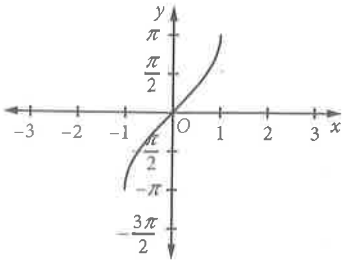
$$= \frac{\frac{88}{39}}{-\frac{9}{39}}$$

$$= -\frac{88}{9}$$

7 **(a)** Domain: $-1 \le x \le 1$

Range: $-\pi \le y \le \pi$

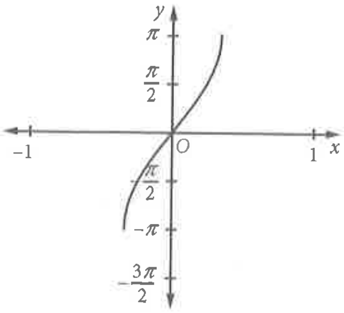
$y = 2\sin^{-1} x$



(b) Domain: $-\frac{1}{3} \le x \le \frac{1}{3}$

Range: $-\pi \le y \le \pi$

$y = 2\sin^{-1} 3x$

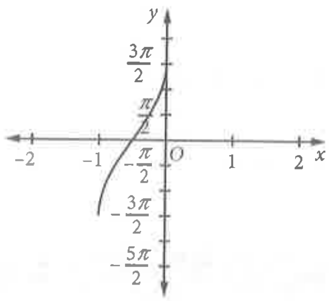


(c) Domain: $-1 \le x \le 0$

Range: $-\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$

$y = 3\sin^{-1}(2x + 1)$

$= 3\sin^{-1}\left(2(x + \frac{1}{2})\right)$

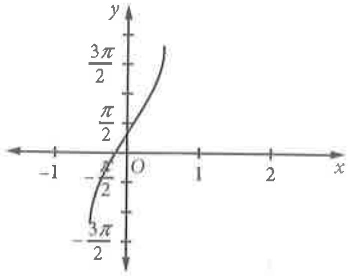


(d) Domain: $-\frac{1}{2} \le x \le \frac{1}{2}$

Range: $-\frac{3\pi}{2} + 1 \le y \le \frac{3\pi}{2} + 1$

$y = 3\sin^{-1}(2x) + 1$

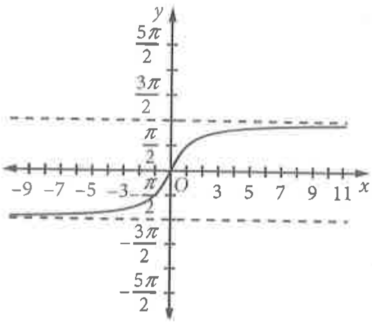
(See graph at top of next column.)



(e) Domain: all real x

Range: $-\pi \le y \le \pi$

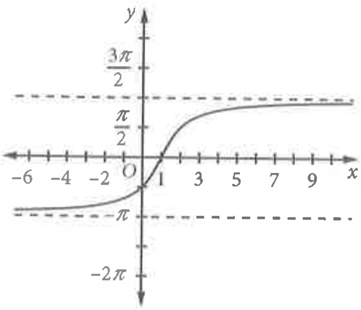
$y = 2\tan^{-1} x$



(f) Domain: all real x

Range: $-\pi \le y \le \pi$

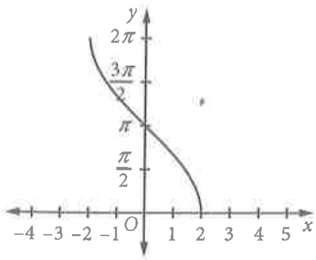
$y = 2\tan^{-1}(x - 1)$



(g) Domain: $-2 \le x \le 2$

Range: $0 \le y \le 2\pi$

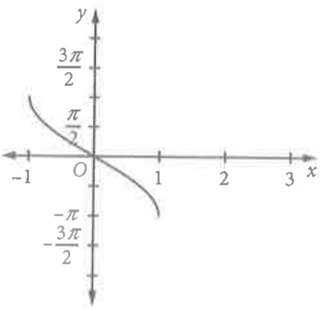
$y = 2\cos^{-1} \frac{x}{2}$



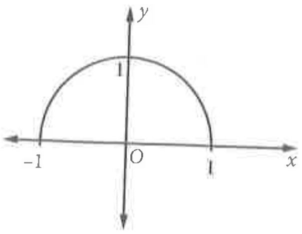
(h) Domain: $-1 \le x \le 1$

Range: $-\pi \le y \le \pi$

$y = 2\sin^{-1}(-x)$



(i) $y = \sin(\cos^{-1} x)$
Let $\alpha = \cos^{-1} x$.
Then: $\cos\alpha = x \quad$ for $0 \le \alpha \le \pi$
Using $\sin^2\alpha + \cos^2\alpha = 1$:
$\sin^2\alpha = 1 - x^2$
$\sin\alpha = \sqrt{1-x^2}$ Use the positive square root because $0 \le \alpha \le \pi$.
So: $y = \sin(\cos^{-1} x)$
$= \sin\alpha$
$= \sqrt{1-x^2}$
The graph of this function is the top half of a circle with radius 1 and centre at (0, 0). The domain is $-1 \le x \le 1$ and the range is $0 \le y \le 1$:



9 (a) $\sin^{-1} x \cos^{-1} x = 0$
$\sin^{-1} x = 0$ or $\cos^{-1} x = 0$
$x = 0$ or $x = 1$
Answer: $x = 0, 1$

(b)
$$\sin^{-1}(1-x) + 2\cos^{-1}(x-1) = \frac{\pi}{2}$$
$$-\sin^{-1}(x-1) + 2\cos^{-1}(x-1) = \frac{\pi}{2}$$
$$-\frac{\pi}{2} + \cos^{-1}(x-1) + 2\cos^{-1}(x-1) = \frac{\pi}{2}$$
(using the result: $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$)
$$3\cos^{-1}(x-1) = \pi$$
$$\cos^{-1}(x-1) = \frac{\pi}{3}$$
$$x - 1 = \frac{1}{2}$$
$$x = \frac{3}{2}$$

(c)
$$\sin^{-1} x \cos^{-1} x = -1$$
$\sin^{-1} x\left(\frac{\pi}{2} - \sin^{-1} x\right) = -1$ (using the result: $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ for $-1 \le x \le 1$)
$(\sin^{-1} x)^2 - \frac{\pi}{2}\sin^{-1} x - 1 = 0$ and
$-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$

Using quadratic formula, approximate values:
$\sin^{-1} x = -0.486\,156, 2.056\,952$
Reject 2.056 952 because it is not in the set of allowable values for $\sin^{-1} x$.
So: $\sin^{-1} x = -0.486\,156$
$x = -0.47$

11 (a) Show that $\sin(\sin^{-1} x - \cos^{-1} x) = 2x^2 - 1$ for $-1 \le x \le 1$.
$\text{LHS} = \sin(\sin^{-1} x - \cos^{-1} x)$
$= \sin(\sin^{-1} x)\cos(\cos^{-1} x) - \cos(\sin^{-1} x)\sin(\cos^{-1} x)$
$= x \times x - \sqrt{1-x^2} \times \sqrt{1-x^2}$
$= x^2 - (1 - x^2)$
$= 2x^2 - 1$
$= \text{RHS}$

(b) $-1 \le x \le 1$
$0 \le x^2 \le 1$
$0 \le 2x^2 \le 2$
$\therefore -1 \le 2x^2 - 1 \le 1$

(c) From (a): $\sin^{-1} x - \cos^{-1} x = \sin^{-1}(2x^2 - 1)$
So: $\sin^{-1} x - \cos^{-1} x = \sin^{-1}(5x - 4)$
is equivalent to: $\sin^{-1}(2x^2 - 1) = \sin^{-1}(5x - 4)$ for $-1 \le x \le 1$:
$2x^2 - 1 = 5x - 4$
$2x^2 - 5x + 3 = 0$
$(2x - 3)(x - 1) = 0$
$x = \frac{3}{2}, 1$
But $-1 \le x \le 1$, so the only solution is $x = 1$.

13 $f(x) = 3\cos^{-1}\left(\frac{x}{2}\right)$ has domain $-2 \le x \le 2$ and range $0 \le y \le 3\pi$.
Let: $y = 3\cos^{-1}\left(\frac{x}{2}\right)$
$x = 3\cos^{-1}\left(\frac{y}{2}\right)$ (interchange x and y)
$\frac{x}{3} = \cos^{-1}\left(\frac{y}{2}\right)$
$2\cos\left(\frac{x}{3}\right) = y$
$f^{-1}(x) = 2\cos\left(\frac{x}{3}\right)$ has domain $0 \le x \le 3\pi$ and range $-2 \le y \le 2$.

EXERCISE 10.3

1 **B** is the correct alternative.
$y = \sin^{-1} 3x$
Let $u = 3x$.
Using the chain rule: $\frac{dy}{dx} = 3 \times \frac{1}{\sqrt{1-u^2}}$
$= \frac{3}{\sqrt{1-9x^2}}$

A incorrect (see given working)
B correct (see given working)
C incorrect (see given working)
D incorrect (see given working)

3 $y = 2\tan^{-1}(2x + 1)$
Let $u = 2x + 1$.
So: $y = 2\tan^{-1} u$

Using the chain rule: $\frac{dy}{dx} = 2 \times \frac{2}{1+u^2}$

$= \frac{4}{1+(2x+1)^2}$

and where $x = -1$:

$\frac{dy}{dx} = \frac{4}{1+(-2+1)^2}$

$= 2$

Where $x = -1$: $y = 2\tan^{-1}(-2+1)$

$= 2\tan^{-1}(-1)$

$= -\frac{\pi}{2}$

Using $(-1, -\frac{\pi}{2})$ and $m = 2$:

$y - y_1 = m(x - x_1)$

$y + \frac{\pi}{2} = 2(x+1)$

$y = 2x + 2 - \frac{\pi}{2}$ is the equation of the tangent at $x = -1$.

5 $y = \cos^{-1} x + \cos^{-1}(-x)$

$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}} + -1 \times \frac{-1}{\sqrt{1-x^2}}$

$= 0$

So the graph of y is a horizontal line and is of the form $y = c$.

Where $x = 0$: $y = \cos^{-1} 0 + \cos^{-1}(0)$

$= \pi$

So, for all x: $y = \pi$

7 (a) $y = x\tan^{-1} x$

$\frac{dy}{dx} = 1 \times \tan^{-1} x + x \times \frac{1}{1+x^2}$

$= \tan^{-1} x + \frac{x}{1+x^2}$

(b) From (a): $\int \left(\tan^{-1} x + \frac{x}{1+x^2} \right) dx = x\tan^{-1} x + C$

$\int \tan^{-1} x\, dx + \int \frac{x}{1+x^2} dx = x\tan^{-1} x + C$

$\int \tan^{-1} x\, dx + \frac{1}{2}\int \frac{2x}{1+x^2} dx = x\tan^{-1} x + C$

$\int \tan^{-1} x\, dx + \frac{1}{2}\log_e(1+x^2) = x\tan^{-1} x + C_1$

$\int \tan^{-1} x\, dx = x\tan^{-1} x - \frac{1}{2}\log_e(1+x^2) + C_1$

(c) Let $u = \log_e x$

$\frac{du}{dx} = \frac{1}{x}$

For $x = e$: $u = 1$

and for $x = 0$: $u = 0$

$\int_1^e \frac{\tan^{-1}(\log_e x)}{x} dx = \int_0^1 \tan^{-1} u\, du$

$= \left[u\tan^{-1} u - \frac{1}{2}\log_e(1+u^2) \right]_0^1$

$= \tan^{-1} 1 - \frac{1}{2}\log_e 2 - 0 + \frac{1}{2}\log_e 1$

$= \frac{\pi}{4} - \frac{1}{2}\log_e 2$

9 $y = 2\sin x + 3\cos x$

$\frac{dy}{dx} = 2\cos x - 3\sin x$

Let $\frac{dy}{dx} = 0$ to find stationary points.

$2\cos x - 3\sin x = 0$

$\tan x = \frac{2}{3}$

$x = \tan^{-1}\frac{2}{3}$

and: $y = 2\sin\left(\tan^{-1}\frac{2}{3}\right) + 3\cos\left(\tan^{-1}\frac{2}{3}\right)$

$= 2 \times \frac{2}{\sqrt{13}} + 3 \times \frac{3}{\sqrt{13}}$ because $0 \le x \le \frac{\pi}{2}$

$= \frac{13}{\sqrt{13}}$

$= \sqrt{13}$

The stationary point is $\left(\tan^{-1}\frac{2}{3}, \sqrt{13}\right)$.

At $x = 0$: $y = 2\sin 0 + 3\cos 0$

$= 3 < \sqrt{13}$

At $x = \frac{\pi}{2}$: $y = 2\sin\frac{\pi}{2} + 3\cos\frac{\pi}{2}$

$= 2 < \sqrt{13}$

Hence the stationary point is a local maximum.

EXERCISE 10.4

1 B is the correct alternative.

Let $u = x + 1$.

Then: $du = dx$

$\int \frac{dx}{\sqrt{16-(x+1)^2}} = \int \frac{du}{\sqrt{16-u^2}}$

$= \sin^{-1}\frac{u}{4} + C$

$= \sin^{-1}\frac{x+1}{4} + C$

A incorrect (see working above)

B correct (see working above)

C incorrect (see working above)

D incorrect (see working above)

3 (a) $\int_0^1 \frac{dx}{\sqrt{4-x^2}} = \left[\sin^{-1}\frac{x}{2} \right]_0^1$

$= \sin^{-1}\frac{1}{2} - \sin^{-1} 0$

$= \frac{\pi}{6}$

(b) $\int_0^{\sqrt{3}} \frac{dx}{x^2+9} = \frac{1}{3}\int_0^{\sqrt{3}} \frac{3dx}{x^2+9}$

$= \frac{1}{3}\left[\tan^{-1}\frac{x}{3} \right]_0^{\sqrt{3}}$

$= \frac{1}{3}\left(\tan^{-1}\frac{\sqrt{3}}{3} - \tan^{-1} 0 \right)$

$= \frac{1}{3} \times \frac{\pi}{6}$

$= \frac{\pi}{18}$

(c) Let $u = 2x$.

Then: $du = 2\,dx$

For $x = \frac{1}{4}$, $u = \frac{1}{2}$; for $x = 0$, $u = 0$.

$$\int_0^{\frac{1}{4}} \frac{dx}{\sqrt{1-4x^2}} = \frac{1}{2}\int_0^{\frac{1}{4}} \frac{2dx}{\sqrt{1-4x^2}}$$
$$= \frac{1}{2}\int_0^{\frac{1}{2}} \frac{du}{\sqrt{1-u^2}}$$
$$= \frac{1}{2}\left[\sin^{-1} u\right]_0^{\frac{1}{2}}$$
$$= \frac{1}{2}\left(\sin^{-1}\frac{1}{2} - \sin^{-1} 0\right)$$
$$= \frac{1}{2} \times \frac{\pi}{6}$$
$$= \frac{\pi}{12}$$

(d) $$\int_{-2}^{2} \frac{dx}{4+x^2} = \frac{1}{2}\int_{-2}^{2} \frac{2\,dx}{4+x^2}$$
$$= \frac{1}{2}\left[\tan^{-1}\frac{x}{2}\right]_{-2}^{2}$$
$$= \frac{1}{2}(\tan^{-1} 1 - \tan^{-1}[-1])$$
$$= \frac{1}{2}\left(\frac{\pi}{4} + \frac{\pi}{4}\right)$$
$$= \frac{\pi}{4}$$

(e) $$\int_{\frac{3}{5}}^{4} \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_{\frac{3}{5}}^{4}$$
$$= \tan^{-1} 4 - \tan^{-1}\frac{3}{5}$$

$$\tan\left(\tan^{-1} 4 - \tan^{-1}\frac{3}{5}\right)$$
$$= \frac{\tan(\tan^{-1} 4) - \tan\left(\tan^{-1}\frac{3}{5}\right)}{1 + \tan(\tan^{-1} 4)\tan\left(\tan^{-1}\frac{3}{5}\right)}$$
$$= \frac{4 - \frac{3}{5}}{1 + 4 \times \frac{3}{5}}$$
$$= 1$$

So: $\tan^{-1} 4 - \tan^{-1}\frac{3}{5} = \tan^{-1} 1$
$$= \frac{\pi}{4}$$

(f) $$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}} = \left[\sin^{-1} x\right]_{-\frac{1}{2}}^{\frac{1}{2}}$$
$$= \sin^{-1}\frac{1}{2} - \sin^{-1}\left(-\frac{1}{2}\right)$$
$$= \frac{\pi}{6} + \frac{\pi}{6}$$
$$= \frac{\pi}{3}$$

(g) $$\int_0^{1} \frac{dx}{\sqrt{2-x^2}} = \left[\sin^{-1}\frac{x}{\sqrt{2}}\right]_0^{1}$$
$$= \sin^{-1}\frac{1}{\sqrt{2}} - \sin^{-1} 0$$
$$= \frac{\pi}{4}$$

(h) $$\int_0^{2} \frac{-1}{\sqrt{16-x^2}}\,dx = \left[\cos^{-1}\frac{x}{4}\right]_0^{2}$$
$$= \cos^{-1}\frac{1}{2} - \cos^{-1} 0$$
$$= \frac{\pi}{3} - \frac{\pi}{2}$$
$$= -\frac{\pi}{6}$$

(i) $$\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}$$
$$= \tan^{-1}\sqrt{3} - \tan^{-1}\frac{1}{\sqrt{3}}$$
$$= \frac{\pi}{3} - \frac{\pi}{6}$$
$$= \frac{\pi}{6}$$

(j) $$\int_0^{\frac{1}{3}} \frac{dx}{1+9x^2} = \frac{1}{3}\int_0^{\frac{1}{3}} \frac{3\,dx}{1+9x^2}$$
$$= \frac{1}{3}\left[\tan^{-1} 3x\right]_0^{\frac{1}{3}}$$
$$= \frac{1}{3}(\tan^{-1} 1 - \tan^{-1} 0)$$
$$= \frac{1}{3}\left(\frac{\pi}{4} - 0\right)$$
$$= \frac{\pi}{12}$$

(k) $$\int_0^{1}\left(\frac{1}{1+x^2} + \frac{x}{1+x^2}\right)dx$$
$$= \int_0^{1}\left(\frac{1}{1+x^2} + \frac{1}{2}\left[\frac{2x}{1+x^2}\right]\right)dx$$
$$= \left[\tan^{-1} x + \frac{1}{2}\log_e(1+x^2)\right]_0^{1}$$
$$= \tan^{-1} 1 - \tan^{-1} 0 + \frac{1}{2}\log_e 2 - \frac{1}{2}\log_e 1$$
$$= \frac{\pi}{4} + \frac{1}{2}\log_e 2$$

(l) $$\int_0^{\sqrt{3}} \frac{dx}{\sqrt{3-x^2}} = \left[\sin^{-1}\frac{x}{\sqrt{3}}\right]_0^{\sqrt{3}}$$
$$= \sin^{-1} 1 - \sin^{-1} 0$$
$$= \frac{\pi}{2}$$

(m) $$\int_{-4}^{4} \frac{dx}{x^2+16} = \frac{1}{4}\int_{-4}^{4} \frac{4\,dx}{x^2+16}$$
$$= \frac{1}{4}\left[\tan^{-1}\frac{x}{4}\right]_{-4}^{4}$$
$$= \frac{1}{4}(\tan^{-1} 1 - \tan^{-1}[-1])$$
$$= \frac{1}{4}\left(\frac{\pi}{4} + \frac{\pi}{4}\right)$$
$$= \frac{\pi}{8}$$

(n) $$\int_0^{\frac{\sqrt{3}}{6}} \frac{-1}{\sqrt{1-9x^2}}\,dx$$

Let $u = 3x$.

Then: $du = 3\,dx$

For $x = \frac{\sqrt{3}}{6}$: $\quad u = \frac{\sqrt{3}}{2}$

For $x = 0$: $\quad u = 0$

$$\text{Then: } \int_0^{\frac{\sqrt{3}}{6}} \frac{-1}{\sqrt{1-9x^2}}\,dx = \frac{1}{3}\int_0^{\frac{\sqrt{3}}{6}} \frac{-3}{\sqrt{1-9x^2}}\,dx$$
$$= \frac{1}{3}\int_0^{\frac{\sqrt{3}}{2}} \frac{-1}{\sqrt{1-u^2}}\,du$$
$$= \frac{1}{3}\left[\cos^{-1} u\right]_0^{\frac{\sqrt{3}}{2}}$$
$$= \frac{1}{3}\left(\cos^{-1}\frac{\sqrt{3}}{2} - \cos^{-1} 0\right)$$
$$= \frac{1}{3}\left(\frac{\pi}{6} - \frac{\pi}{2}\right)$$
$$= -\frac{\pi}{9}$$

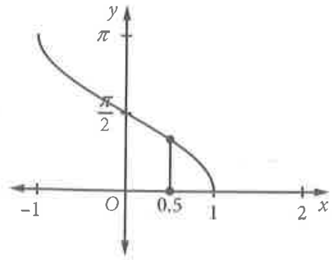
(o) $$\int_0^{\frac{1}{2}} \frac{dx}{1+4x^2} = \int_0^{\frac{1}{2}} \frac{\frac{1}{4}dx}{\frac{1}{4}+x^2}$$
$$= \frac{1}{2}\int_0^{\frac{1}{2}} \frac{\frac{1}{2}dx}{\frac{1}{4}+x^2}$$
$$= \frac{1}{2}\left[\tan^{-1} 2x\right]_0^{\frac{1}{2}}$$
$$= \frac{1}{2}(\tan^{-1} 1 - \tan^{-1} 0)$$
$$= \frac{1}{2}\times\frac{\pi}{4}$$
$$= \frac{\pi}{8}$$

(p) $$\int_0^{\sqrt{2}} \frac{dx}{x^2+2} = \frac{1}{\sqrt{2}}\left[\tan^{-1}\frac{x}{\sqrt{2}}\right]_0^{\sqrt{2}}$$
$$= \frac{1}{\sqrt{2}}(\tan^{-1} 1 - \tan^{-1} 0)$$
$$= \frac{1}{\sqrt{2}}\times\frac{\pi}{4}$$
$$= \frac{\pi\sqrt{2}}{8}$$

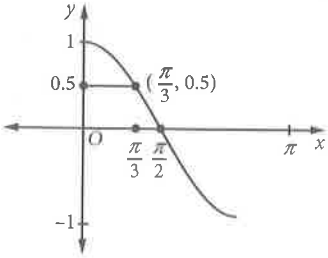
(q) $$\int_{-1}^{1} \frac{dx}{\sqrt{2-x^2}} = \left[\sin^{-1}\frac{x}{\sqrt{2}}\right]_{-1}^{1}$$
$$= \left(\sin^{-1}\frac{1}{\sqrt{2}} - \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)\right)$$
$$= \frac{\pi}{4} + \frac{\pi}{4}$$
$$= \frac{\pi}{2}$$

(r) $$\int_{-5}^{5} \frac{dx}{\sqrt{100-x^2}} = \left[\sin^{-1}\frac{x}{10}\right]_{-5}^{5}$$
$$= \sin^{-1}\frac{1}{2} - \sin^{-1}\left(-\frac{1}{2}\right)$$
$$= \frac{\pi}{6} + \frac{\pi}{6}$$
$$= \frac{\pi}{3}$$

(s) The area given by $\int_0^{\frac{1}{2}} \cos^{-1} x\,dx$ is the area bounded by the axes, the graph of $y = \cos^{-1} x$ and the graph of $x = 0.5$.



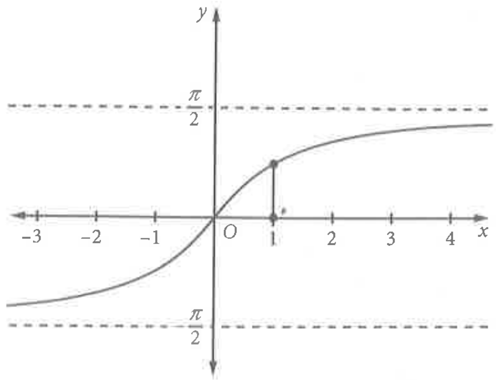
This is the same area as that bounded by the axes, the graph of $y = \cos x$ and the line $y = 0.5$. The intersection of the graphs $y = \cos x$ and $y = 0.5$ is at $\left(\frac{\pi}{3}, \frac{1}{2}\right)$.



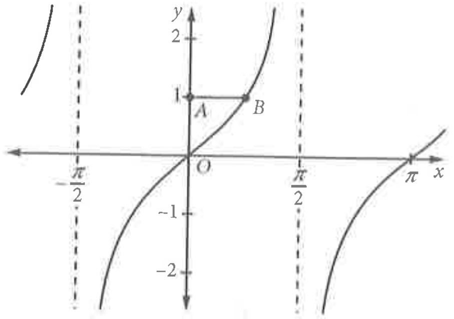
So the required area is given by:

$$\frac{1}{2}\times\frac{\pi}{3} + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos x\,dx = \frac{\pi}{6} + \left[\sin x\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$
$$= \frac{\pi}{6} + \sin\frac{\pi}{2} - \sin\frac{\pi}{3}$$
$$= \frac{\pi}{6} + 1 - \frac{\sqrt{3}}{2}$$

(t) The area given by $\int_0^1 \tan^{-1} x\,dx$ is the area bounded by the x-axis, the graph of $y = \tan^{-1} x$ and the line $x = 1$.



This is equivalent to the area bounded by the y-axis, the graph of $y = \tan x$ and the line $y = 1$. (See next page.)



The point of intersection of the lines $y = 1$ and $y = \tan x$ is $\left(\frac{\pi}{4}, 1\right)$.

The required area is given by:

$1 \times \frac{\pi}{4} - \int_0^{\frac{\pi}{4}} \tan x\,dx = \frac{\pi}{4} - \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x}\,dx$

Let $u = \cos x$.

Then: $du = -\sin x\,dx$

For $x = \frac{\pi}{4}$, $u = \frac{1}{\sqrt{2}}$; for $x = 0$, $u = 1$.

$$\begin{aligned}
\text{So: } \frac{\pi}{4} - \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x}\,dx &= \frac{\pi}{4} + \int_1^{\frac{1}{\sqrt{2}}} \frac{1}{u}\,du \\
&= \frac{\pi}{4} + \log_e \frac{1}{\sqrt{2}} - 0 \\
&= \frac{\pi}{4} + \log_e 2^{-\frac{1}{2}} \\
&= \frac{\pi}{4} - \frac{1}{2}\log_e 2
\end{aligned}$$

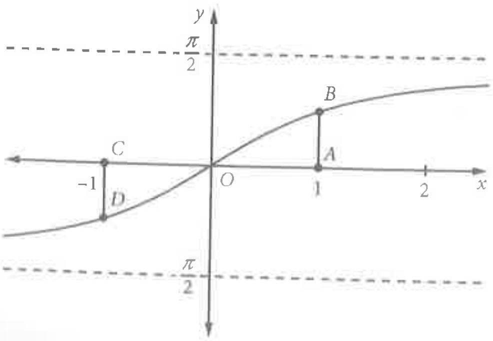
5 $$\begin{aligned}
2\int_0^1 \frac{1}{x^2+1}\,dx &= 2\left[\tan^{-1} x\right]_0^1 \\
&= 2\tan^{-1} 1 \\
&= 2 \times \frac{\pi}{4} \\
&= \frac{\pi}{2}
\end{aligned}$$

7 $y = \frac{1}{\sqrt{1+x^2}}$

Required volume is:

$$\begin{aligned}
\pi\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} y^2\,dx &= \pi\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{1+x^2}\,dx \\
&= \pi\left[\tan^{-1} x\right]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \\
&= \pi\left(\tan^{-1}\sqrt{3} - \tan^{-1}\frac{1}{\sqrt{3}}\right) \\
&= \pi\left(\frac{\pi}{3} - \frac{\pi}{6}\right) \\
&= \frac{\pi^2}{6}
\end{aligned}$$

9



$\int_{-1}^1 \tan^{-1} x\,dx = \int_{-1}^0 \tan^{-1} x\,dx + \int_0^1 \tan^{-1} x\,dx$ and it is evident from the graph of $y = \tan^{-1} x$ that the area above the x-axis is equal to the area below the x-axis, so they cancel each other out, resulting in 0.

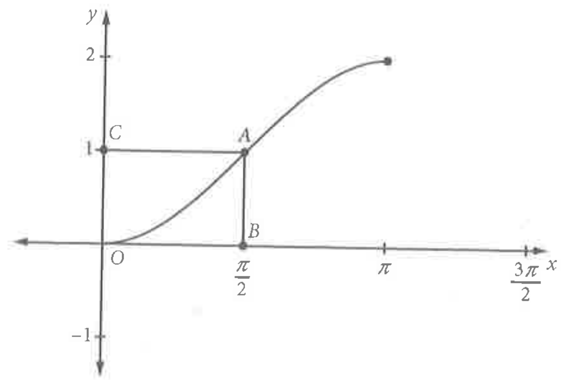
$y = \tan^{-1} x$ is an odd function $f(-x) = -f(x)$ and $\int_{-a}^{a} f(x)\,dx = 0$ for every odd function $f(x)$.

11 $$\begin{aligned}
\pi\int_0^{\frac{\pi}{2}} \sin^2 x\,dx &= \pi\int_0^{\frac{\pi}{2}} (1 - \cos^2 x)\,dx \\
&= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} (1 - \cos 2x)\,dx \\
&= \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}} \\
&= \frac{\pi}{2}\left(\frac{\pi}{2} - \frac{1}{2}\sin\pi - 0 + 0\right) \\
&= \frac{\pi^2}{4}
\end{aligned}$$

When $y = \sin^{-1} x$: $\quad x = \sin y$

$$\begin{aligned}
\text{and: } \pi\int_0^{\frac{\pi}{2}} x^2\,dy &= \pi\int_0^{\frac{\pi}{2}} \sin^2 y\,dy \\
&= \frac{\pi^2}{4}
\end{aligned}$$

13 **(a)**



(b) $$\begin{aligned}
\int_0^{\frac{\pi}{2}} (1 - \cos x)\,dx &= \left[x - \sin x\right]_0^{\frac{\pi}{2}} \\
&= \frac{\pi}{2} - 1
\end{aligned}$$

This is the area bounded by the x-axis, the graph of $f(x) = 1 - \cos x$ and the line AB.

(c) Let $y = 1 - \cos x$.

$x = 1 - \cos y$ (interchange x and y to find the inverse)

$\cos y = 1 - x$

$y = \cos^{-1}(1 - x)$

$f^{-1}(x) = \cos^{-1}(1 - x)$

domain: $0 \le x \le 2$ range: $0 \le y \le \pi$

(d) $$\begin{aligned}
\int_0^1 f^{-1}(x)\,dx &= \int_0^1 \cos^{-1}(1 - x)\,dx \\
&= \frac{\pi}{2} \times 1 - \text{area found in (b)} \\
&= \frac{\pi}{2} - \left(\frac{\pi}{2} - 1\right) \\
&= 1
\end{aligned}$$

This is the area bounded by the y-axis, the graph of $f(x) = 1 - \cos x$ and the line AC.

15 $\displaystyle\int \frac{\sqrt{1-x^2}}{x^2}\,dx$

Let $x = \cos\theta$.

Then: $\sqrt{1-x^2} = \sin\theta$

and: $dx = -\sin\theta\, d\theta$

$$\int \frac{\sqrt{1-x^2}}{x^2}\,dx = \int \frac{\sin\theta \times -\sin\theta}{\cos^2\theta}\,d\theta$$

$$= -\int \tan^2\theta\, d\theta$$

$$= \int 1 - \sec^2\theta\, d\theta$$

$$= \theta - \tan\theta + C$$

$$= \cos^{-1}x - \tan(\cos^{-1}x) + C$$

$$= \cos^{-1}x - \frac{\sqrt{1-x^2}}{x} + C$$

CHAPTER REVIEW 10

1 (a)

$$y = x^2 - 2x - 8$$
$$= x^2 - 2x + 1 - 9$$
$$= (x-1)^2 - 9$$

and: $y = x^2 - 2x - 8$
$$= (x-4)(x+2)$$

Turning point: $(1, 9)$

y-intercept: $(0, -8)$

x-intercepts: $(4, 0)$, $(-2, 0)$

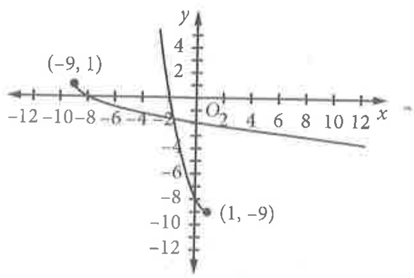
(b) $x \le 1$

(c) For the inverse: $x = (y-1)^2 - 9$

$$x + 9 = (y-1)^2$$
$$y = -\sqrt{x+9} + 1$$
$$f^{-1}(x) = -\sqrt{x+9} + 1$$

(Take the negative square root due to the range $y \le 1$.)

(d)



(e) $f(x)$ and $f^{-1}(x)$ intersect along the line $y = x$, so the x value of the point of intersection can be found from:

$$x = x^2 - 2x - 8$$
$$0 = x^2 - 3x - 8$$

Solving this gives: $x = \dfrac{3 \pm \sqrt{9+32}}{2}$

$$x = \frac{3 \pm \sqrt{41}}{2}$$

But $x \le 1$, so reject $x = \dfrac{3+\sqrt{41}}{2}$.

The x-coordinate of T is $\alpha = \dfrac{3-\sqrt{41}}{2}$.

(f) $f^{-1}(x) = 1 - \sqrt{x+9}$

So: $y = 1 - \sqrt{x+9}$

and: $\dfrac{dy}{dx} = \dfrac{-1}{2\sqrt{x+9}}$

Instead of evaluating this at T, where $x = \dfrac{3-\sqrt{41}}{2}$, use the fact that at the point of intersection of f and f^{-1} the product of the gradients is 1.

For f: $y = x^2 - 2x - 8$ $\quad \dfrac{dy}{dx} = 2x - 2$

Where $x = \dfrac{3-\sqrt{41}}{2}$: $\quad \dfrac{dy}{dx} = 2 \times \dfrac{3-\sqrt{41}}{2} - 2$

$$\frac{dy}{dx} = 1 - \sqrt{41}$$

So: Gradient of f^{-1} at T is $\dfrac{1}{1-\sqrt{41}} = \dfrac{1+\sqrt{41}}{-40}$

$$= -\frac{1+\sqrt{41}}{40}$$

3 (a) $0 < \tan^{-1}\dfrac{3}{4} < \dfrac{\pi}{4}$ and $0 < \tan^{-1}\dfrac{1}{2} < \dfrac{\pi}{4}$

So: $0 < \tan^{-1}\dfrac{3}{4} + \tan^{-1}\dfrac{1}{2} < \dfrac{\pi}{2}$ (the sum is an angle in the first quadrant)

$$\tan\left(\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{2}\right)$$

$$= \frac{\tan\left(\tan^{-1}\frac{3}{4}\right) + \tan\left(\tan^{-1}\frac{1}{2}\right)}{1 - \tan\left(\tan^{-1}\frac{3}{4}\right)\tan\left(\tan^{-1}\frac{1}{2}\right)}$$

$$= \frac{\frac{3}{4} + \frac{1}{2}}{1 - \frac{3}{4} \times \frac{1}{2}} = 2$$

So: $\tan^{-1}\dfrac{3}{4} + \tan^{-1}\dfrac{1}{2} = \tan^{-1}2$ (as required)

(b) $0 < \tan^{-1}\dfrac{3}{4} - \tan^{-1}\dfrac{1}{2} < \dfrac{\pi}{2}$ (the sum is an angle in the first quadrant)

$$\sin\left(\tan^{-1}\frac{3}{4} - \tan^{-1}\frac{1}{2}\right) = \sin\left(\tan^{-1}\frac{3}{4}\right) \times \cos\left(\tan^{-1}\frac{1}{2}\right) - \cos\left(\tan^{-1}\frac{3}{4}\right)\sin\left(\tan^{-1}\frac{1}{2}\right)$$

$$= \frac{3}{5} \times \frac{2}{\sqrt{5}} - \frac{4}{5} \times \frac{1}{\sqrt{5}}$$

$$= \frac{2}{5\sqrt{5}}$$

$$= \frac{2\sqrt{5}}{25}$$

So: $\tan^{-1}\dfrac{3}{4} - \tan^{-1}\dfrac{1}{2} = \sin^{-1}\left(\dfrac{2\sqrt{5}}{25}\right)$

(as required)

5 $y = \tan^{-1}(2x+1)$

When $y = -\dfrac{\pi}{4}$: $\quad -\dfrac{\pi}{4} = \tan^{-1}(2x+1)$

$$\tan\left(-\frac{\pi}{4}\right) = 2x + 1$$
$$2x = -1 - 1$$
$$x = -1$$

$$\frac{dy}{dx}=\frac{2}{1+(2x+1)^2}$$

When $x=-1$, $\frac{dy}{dx}=1$.

The gradient of the normal is -1.

Using $x=-1, y=-\frac{\pi}{4}$, the equation of the normal is:

$$y+\frac{\pi}{4}=-1(x+1)$$

$$y=-x-1-\frac{\pi}{4}$$

7 (a) $y=\sin^{-1}x$

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}$$

(b) $y=\cos^{-1}2x$

$$\frac{dy}{dx}=\frac{-2}{\sqrt{1-4x^2}}$$

(c) $y=\tan^{-1}\frac{x}{2}$

$$\frac{dy}{dx}=\frac{2}{4+x^2}$$

(d) $y=e^x\sin^{-1}x$

$$\frac{dy}{dx}=\frac{e^x}{\sqrt{1-x^2}}+e^x\sin^{-1}x$$

(e) $y=e^{\sin^{-1}x}$

$$\frac{dy}{dx}=\frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$$

(f) $y=\tan^{-1}\frac{2}{x}$

$$\frac{dy}{dx}=-\frac{2}{x^2}\left(\frac{1}{1+\frac{4}{x^2}}\right)$$

$$\frac{dy}{dx}=\frac{-2}{x^2+4}$$

(g) $y=x\cos^{-1}2x$

$$\frac{dy}{dx}=\cos^{-1}2x-\frac{2x}{\sqrt{1-4x^2}}$$

(h) $y=\sqrt{\sin^{-1}x}$

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}\times\frac{1}{2\sqrt{\sin^{-1}x}}$$

$$\frac{dy}{dx}=\frac{1}{2\sqrt{1-x^2}\sqrt{\sin^{-1}x}}$$

(i) $y=\log_e(\tan^{-1}x)^3$

$y=3\log_e(\tan^{-1}x)$

$$\frac{dy}{dx}=\frac{3}{\tan^{-1}x(1+x^2)}$$

9 (a) $\int\frac{dx}{\sqrt{8-x^2}}=\sin^{-1}\frac{x}{2\sqrt{2}}+C$

(b) $\int\frac{dx}{64+x^2}=\frac{1}{8}\tan^{-1}\frac{x}{8}+C$

(c) $\int\frac{4dx}{4+9x^2}=\frac{2}{3}\tan^{-1}\frac{3x}{2}+C$

(d) $\int\frac{dx}{\sqrt{4-8x^2}}=\frac{1}{2\sqrt{2}}\sin^{-1}\frac{2\sqrt{2}x}{2}+C$

$$=\frac{1}{2\sqrt{2}}\sin^{-1}(\sqrt{2}x)+C$$

(e) $\int\frac{dx}{\sqrt{16-(x+2)^2}}=\int\frac{du}{\sqrt{16-u^2}}$

where $u=x+2$, $du=dx$

$$=\sin^{-1}\frac{u}{4}+C$$

$$=\sin^{-1}\frac{x+2}{4}+C$$

11 (a) $\text{LHS}=\frac{x^3+x+2}{1+x^2}$

$$=\frac{x(x^2+1)}{1+x^2}+\frac{2}{1+x^2}$$

$$=x+\frac{2}{1+x^2}$$

$$=\text{RHS}$$

(b) $\int\frac{x^3+x+2}{1+x^2}dx=\int\left(x+\frac{2}{1+x^2}\right)dx$

$$=\frac{1}{2}x^2+2\tan^{-1}x+C$$

13 (a) $\int\frac{dx}{1+x}=\log_e(1+x)+C$

(b) $\int\frac{dx}{1+x^2}=\tan^{-1}x+C$

(c) $\int\frac{x}{1+x^2}dx=\frac{1}{2}\log_e(1+x^2)+C$

(d) $\int\frac{1+x^2}{x}dx=\int\left(\frac{1}{x}+x\right)dx$

$$=\log_e x+\frac{1}{2}x^2+C$$

(e) $\int\frac{x^2}{1+x^2}dx=\int 1-\frac{1}{1+x^2}dx$

$$=x-\tan^{-1}x+C$$

(f) $\int\frac{x^3}{1+x^2}dx=\int x-\frac{x}{1+x^2}dx$

$$=\frac{1}{2}x^2-\frac{1}{2}\log_e(1+x^2)+C$$

(g) $\int\frac{dx}{(1+x)^2}=\frac{-1}{1+x}+C$

(h) $\int\frac{dx}{\sqrt{1+x}}=2(1+x)^{\frac{1}{2}}+C$

$$=2\sqrt{1+x}+C$$

(i) $\int\frac{dx}{\sqrt{1-x^2}}=\sin^{-1}x+C$

(j) $\int\frac{x}{\sqrt{1-x^2}}dx=-\frac{1}{2}\int\frac{-2x}{\sqrt{1-x^2}}dx$

$$=-\frac{1}{2}\int\frac{du}{\sqrt{u}}$$ where $u=1-x^2$, $du=-2x\,dx$

$$=-\frac{1}{2}\times 2u^{\frac{1}{2}}+C$$

$$=-\sqrt{1-x^2}+C$$

(k) $\int\frac{e^x}{2+e^{2x}}dx=\int\frac{1}{2+u^2}du$

where $u=e^x$, $du=e^x\,dx$

$$=\frac{1}{\sqrt{2}}\int\frac{\sqrt{2}}{2+u^2}du$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{u}{\sqrt{2}} + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{e^x}{\sqrt{2}} + C$$

(l) $\int \frac{\tan^{-1} 3x}{1+9x^2} dx = \frac{1}{3} \int u \, du$

where $u = \tan^{-1} 3x$, $du = \frac{3}{1+9x^2} dx$

$$= \frac{1}{3} \times \frac{1}{2} u^2 + C$$

$$= \frac{1}{6} (\tan^{-1} 3x)^2 + C$$

15 (a) The point of intersection of the graphs $y = \tan^2 x$ and $y = \cos^2 x$ occurs where:

$$\cos^2 x = \tan^2 x$$

$$= \sec^2 x - 1$$

$$\cos^4 x = 1 - \cos^2 x$$

$$\cos^4 x + \cos^2 x - 1 = 0$$

$$\cos^2 x = \frac{-1 \pm \sqrt{5}}{2}$$

But: $\cos^2 x > 0$

So: $\cos^2 x = \frac{-1 + \sqrt{5}}{2}$

$$\cos x = \pm \sqrt{\frac{-1 + \sqrt{5}}{2}}$$

But: $0 \le x \le \frac{\pi}{2}$

$\therefore \cos x > 0$

$$\therefore \cos x = \sqrt{\frac{-1 + \sqrt{5}}{2}}$$

and: $x = \cos^{-1} \sqrt{\frac{-1 + \sqrt{5}}{2}}$ (as required)

(b) The required area is:

$$\int_0^a \left(\cos^2 x - \tan^2 x\right) dx$$

$$= \int_0^a \left(\cos^2 x + 1 - \sec^2 x\right) dx$$

where $a = \cos^{-1} \sqrt{\frac{-1 + \sqrt{5}}{2}}$

$$= \int_0^a \left(\frac{1}{2} \cos 2x + \frac{1}{2} + 1 - \sec^2 x\right) dx$$

$$= \left[\frac{1}{4} \sin 2x + \frac{3}{2} x - \tan x\right]_0^a$$

$$= 0.5$$