Chapter 1

GEOMETRICAL APPLICATIONS OF DIFFERENTIATION

Facts and Formulas

EXERCISE 1

Circle the correct answer.

1. A function is increasing when:

(A) $\frac{dy}{dx} > 0$ (B) $\frac{dy}{dx} = 0$ (C) $\frac{dy}{dx} < 0$

2. A function is decreasing when:

(A) $\frac{dy}{dx} > 0$ (B) $\frac{dy}{dx} = 0$ (C) $\frac{dy}{dx} < 0$

3. A curve is concave upwards at P when:

(A) $\frac{d^2y}{dx^2} > 0$ at P (B) $\frac{d^2y}{dx^2} < 0$ at P (C) $\frac{d^2y}{dx^2} = 0$ at P

4. A curve is concave downwards at P when:

(A) $\frac{d^2y}{dx^2} > 0$ at P (B) $\frac{d^2y}{dx^2} < 0$ at P (C) $\frac{d^2y}{dx^2} = 0$ at P

5. At $x = a$, if $f'(a) = 0$ and $f''(a) > 0$, then there exists at $x = a$:

(A) a maximum turning point. (B) a minimum turning point.

6. At $x = a$, if $f'(a) = 0$ and $f''(a) < 0$, then there exists at $x = a$:

(A) a maximum turning point. (B) a minimum turning point.

7. At $x = a$, if $f''(a) = 0$ and there is a sign change in $f''(x)$ on either side of $x = a$, then there exists at $x = a$:

(A) a point of inflexion. (B) a maximum turning point.
(C) a minimum turning point. (D) a horizontal point of inflexion.

8. At $x = a$, if $f'(a) = 0$, $f''(a) = 0$ and there is a sign change in $f''(x)$ on either side of $x = a$, then there exists at $x = a$:

(A) a point of inflexion. (B) a maximum turning point.
(C) a minimum turning point. (D) a horizontal point of inflexion.

9. Given: (I) maximum turning point (II) minimum turning point
(III) point of inflexion (IV) horizontal point of inflexion

The tangent is parallel to the x axis for,

(A) III only (B) I and II only (C) I, II and IV (D) all of the above

Interpreting the First and Second Derivatives

EXERCISE 2

1. For what values of x are the functions, whose derivatives are given, increasing?

(a) $\frac{dy}{dx} = 3x$

(b) $\frac{dy}{dx} = -2x$

(c) $\frac{dy}{dx} = 9 - 3x$

(d) $\frac{dy}{dx} = x^2 + 3$

(e)* $\frac{dy}{dx} = x^2 - 4$

(f) $\frac{dy}{dx} = (x+2)(x-3)$

(g) $\frac{dy}{dx} = x^2 - 7x + 12$

2. For what values of x are the functions, whose derivatives are given, decreasing?

(a) $\frac{dy}{dx} = -5x$

(b) $\frac{dy}{dx} = 4x$

(c) $\frac{dy}{dx} = -2$

(d) $\frac{dy}{dx} = -x^2$

(e) $\frac{dy}{dx} = x^2 - 1$

(f)* $\frac{dy}{dx} = (x+5)(x-1)$

(g) $\frac{dy}{dx} = x^2 + 8x + 12$

(h) $\frac{dy}{dx} = \frac{-2}{(x-6)^2}$

3. (a) For what values of x is the function $y = 6x^2 - 12x + 8$ increasing?

(b) For what values of x is the function $y = 2x^3 - 6x + 5$ increasing?

(c) For what values of x is the function $y = 4x^2 + 16x - 3$ decreasing?

(d) For what values of x is the function $y = x^3 - 27x + 2$ decreasing?

4. (a) For what values of x is the function $y = 4x^3 - 6x^2 + 7$ concave up?

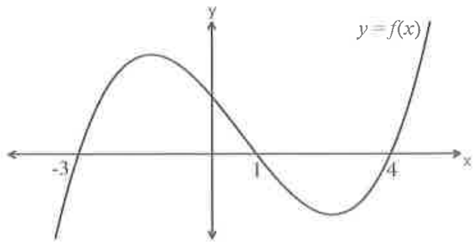
(b) For what values of x is the function $y = 30x^2 - 5x^3$ concave up?

(c)* For what values of x is the function $y = 3x^4 - 72x^2 + 5$ concave down?

(d) For what values of x is the function $y = -x^4$ concave down?

5. If $C = 36m^2 - m^3$ is a cost function, where m is the number of units produced, for what values of m is the cost decreasing?

6.



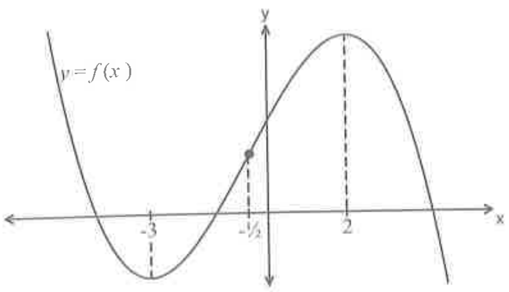
For what values of x is:

(a) $f(x) = 0$?

(b) $f(x) > 0$?

(c) $f(x) < 0$?

7.



(a) What are the x values of the turning points?

For what values of x is:

(b) $f(x)$ increasing?

(c) $f(x)$ decreasing?

(d) $f(x)$ concave up?

(e) $f(x)$ concave down?

Stationary Points and Their Nature

EXERCISE 3

1. Using calculus, locate the turning point of each of the following functions and determine the nature of the point:

(a)* $y = 12x - 2x^2$

(b) $y = 3x^2 - 12x + 10$

(c) $y = -5x^2 + 20x + 1$

2. For the function $y = x^2 - 10x + 21$:

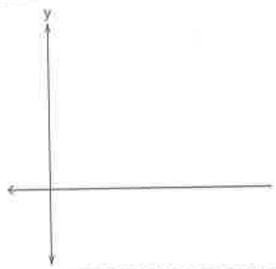
(a) locate the turning point.

(b) determine its nature.

(c) state the y intercept.

(d) state the x intercepts.

(e) sketch the curve, showing all of the above.



3.* For the function $y = -x^2 + 7x + 8$:

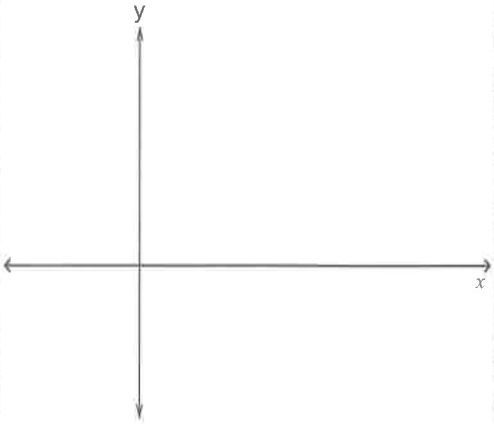
(a) locate the turning point.

(b) determine its nature.

(c) state the y intercept.

(d) state the x intercepts.

(e) sketch the curve, showing all of the above.



Curve Sketching

EXERCISE 4

1. For the function $y = x^3 + 5 \cdot 5x^2 + 10x + 1$:

(a) locate the x values of the turning points.

(b) determine their nature.

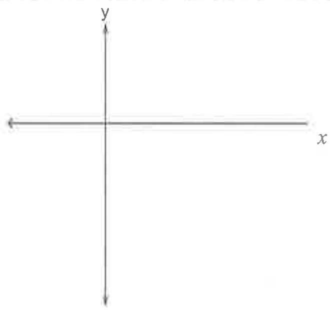
2. For the function $y = x^3 - 5x^2 + 3x - 4$:

(a) locate the turning points.

(b) determine their nature.

(c) state the y intercept.

(d) sketch the curve, showing all of the above.



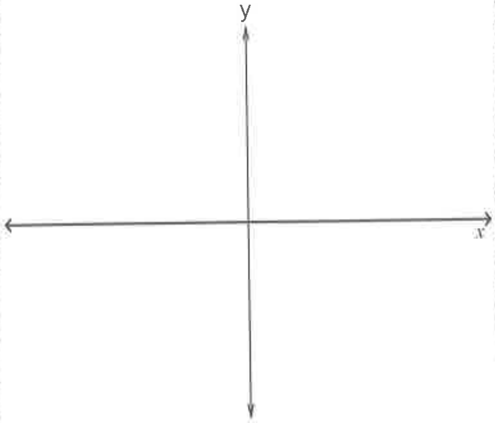
3. For the function $y = 5x^3 - 60x + 1$:

(a) locate the turning points.

(b) determine their nature.

(c) state the y intercept.

(d) sketch the curve, showing all of the above.



(e) What is the maximum value of the function in the domain $-3 \leq x \leq 5$?

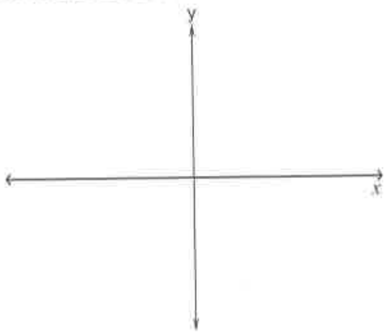
(f) What is the minimum value of the function in the domain $-5 \leq x \leq 3$?

4.* (a) Expand $x(x+3)(x+1)$.

(b) Locate all the turning points on the function $y = \frac{1}{4}x^4 + \frac{4}{3}x^3 + \frac{3}{2}x^2$.

(c) Determine the nature of the turning points.

(d) sketch the curve, showing all of the above.



5. Explain why the function $y = x^3 + 5x$ has no turning points.

6. Find the point on the curve $y = x^3 + 9x^2$ where the concavity changes.

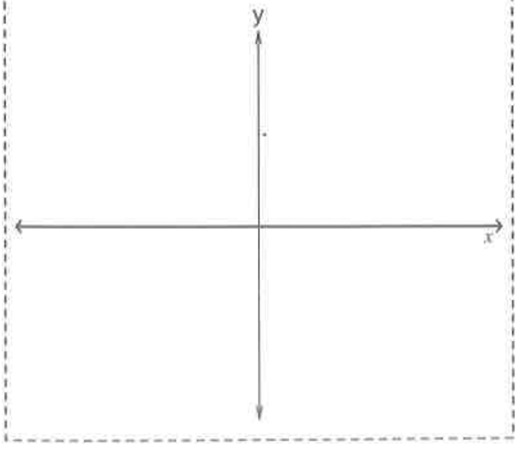
EXERCISE 5

1. For $f(x) = 2x^3 - 54x$:

(a) locate all turning points and determine their nature.

(b) locate any points of inflexion.

(c) sketch the function.

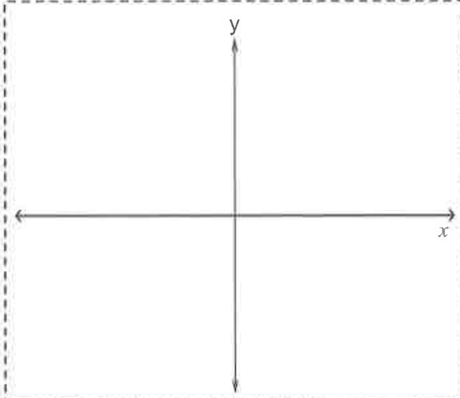


2.* For $f(x) = x^3 - 3x + 5$:

(a) locate all turning points and determine their nature.

(b) locate any points of inflexion.

(c) sketch the function.

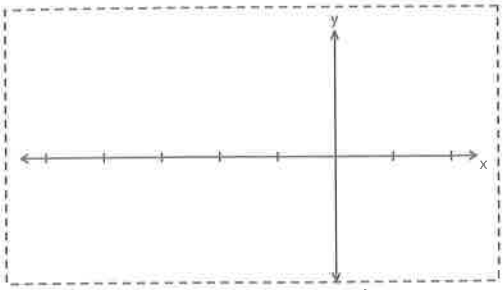


3. For the function $y = x^3 + 6x^2 + 9x + 2$:

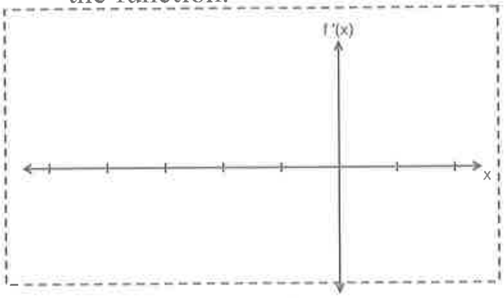
(a) find all turning points and determine their nature.

(b) find any points of inflexion.

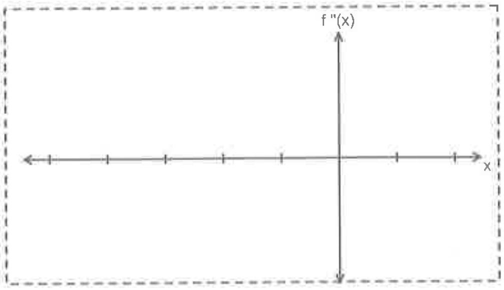
(c) sketch $y = f(x)$ (include the y intercept).



(d) sketch the derivative $f'(x)$ of the function.



(e) sketch the second derivative $f''(x)$ of the function.



(f) Complete: the turning points of $y = f(x)$ are located on the ______ of the graph of the first derivative.

(g) Complete: the point of inflexion of $y = f(x)$ is located on the ______ of the graph of the second derivative.

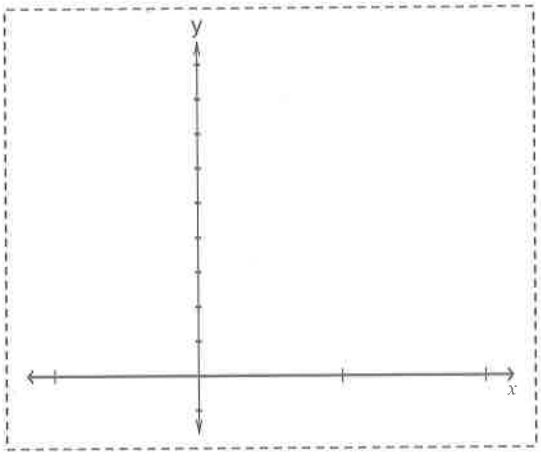
4. (a) On the given set of axes sketch (not to scale) the function which has the following attributes:

$f(0) = 7,\quad f'(0) = 0,\quad f''(0) > 0$

$f(1) = 8,\quad f'(1) = 0,\quad f''(1) < 0$

$f(2) = 7,\quad f'(2) = 0,\quad f''(2) > 0$

Changes in concavity occur when $x = 0.42$ and $x = 1{\cdot}58$.



(b) State the domain for which the curve is concave down.

EXERCISE 6

1. $f(x) = x^3 + 6x^2 + 12x + 7$

(a) $f'(x) =$

(b) $f''(x) =$

(c) Evaluate $f(-2)$

(d) Evaluate $f'(-2)$

(e) Evaluate $f''(-2)$

(f) Evaluate $f''(-1)$

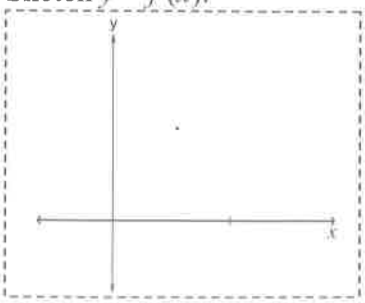
(g) Evaluate $f''(-3)$

(h) Complete:

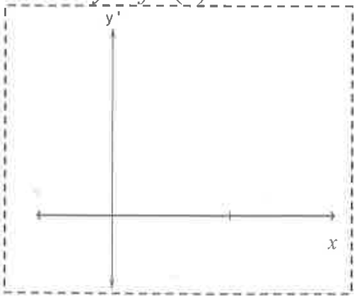
A ☐ point of inflexion occurs at (,) since both $f'(x) =$ ☐ and $f''(x) =$ ☐ at this point, and $f''(x)$ changes ☐ on either side of this point.

2.* (a) Locate stationary point(s) for the function $f(x) = x^3 - 12x^2 + 48x$ and determine the nature of the point(s).

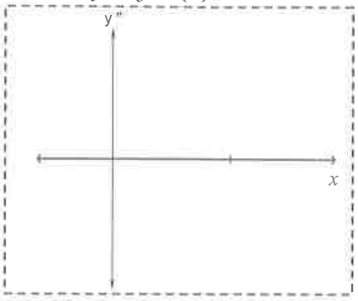
(b) Sketch $y = f(x)$.



(c) Sketch $y = f'(x)$.

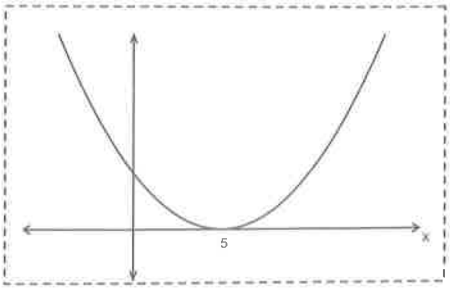


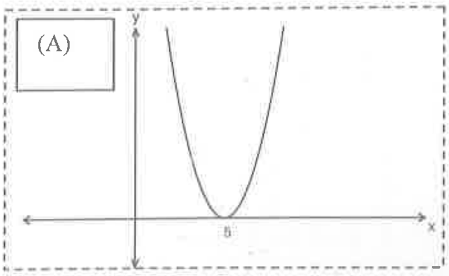
(d) Sketch $y = f''(x)$.

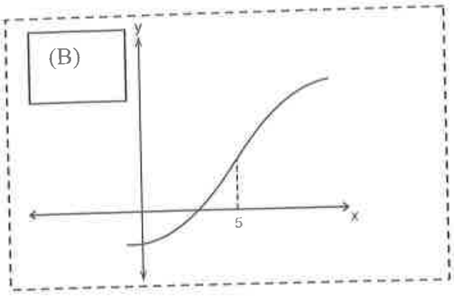


3. (a) For a function $y = f(x)$, $f''(x) = (x+2)^2$. Explain why there is not a point of inflexion at $x = -2$ for this function.

(b) Which of the following graphs (A) or (B) could represent the function whose second derivative has this graph?







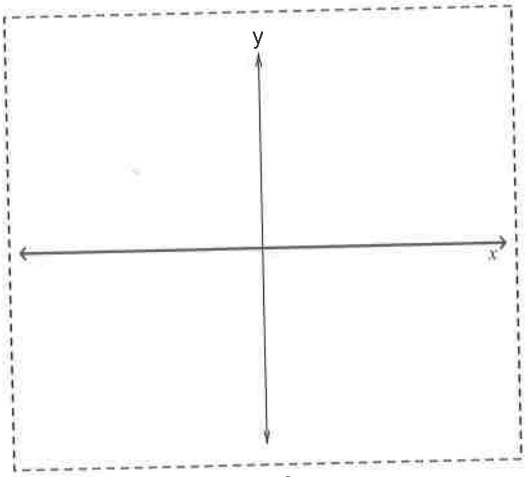
EXERCISE 7

1. For $f(x) = x^4 + \frac{8}{3}x^3 + 1$:

(a) locate all turning points and determine their nature.

(b) locate any points of inflexion.

(c) sketch the function.

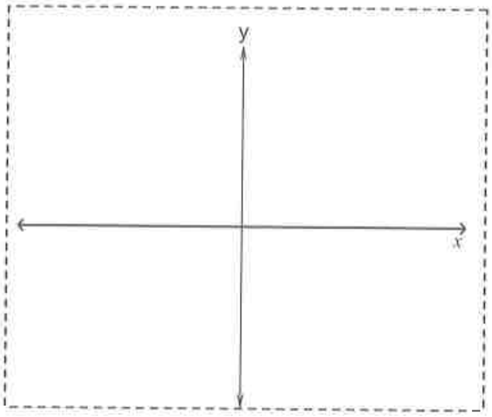


2.* For $f(x) = -3x^4 - 8x^3$:

(a) locate all turning points and determine their nature.

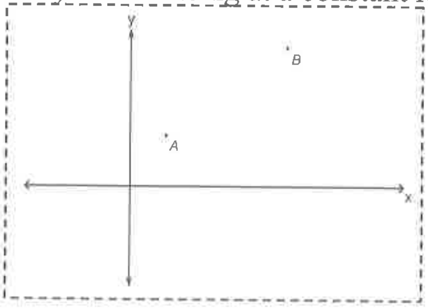
(b) locate any points of inflexion.

(c) sketch the function.

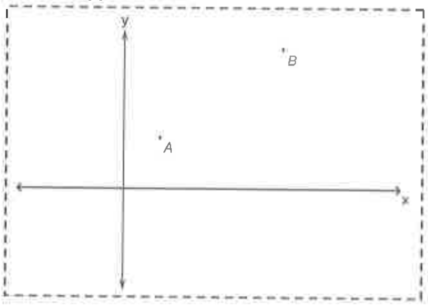


3. Connect the points A and B to demonstrate:

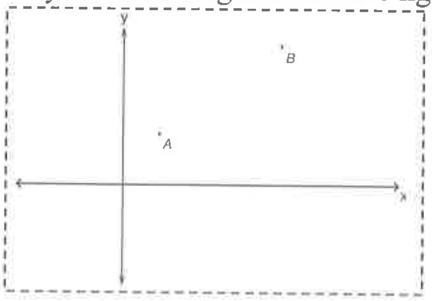
(a) y is increasing at a constant rate.



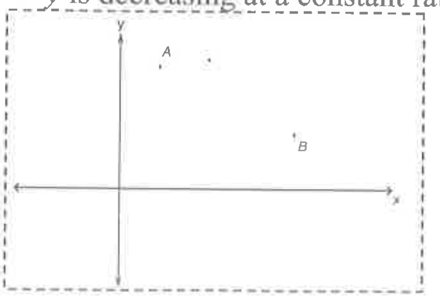
(b) y is increasing at an increasing rate.



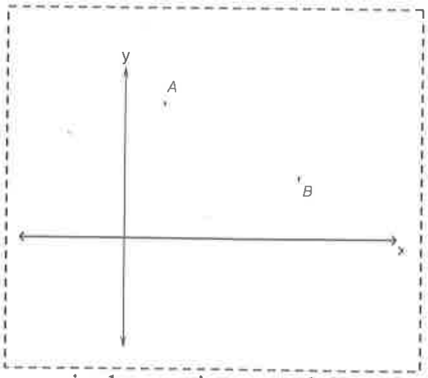
(c) y is increasing at a decreasing rate.



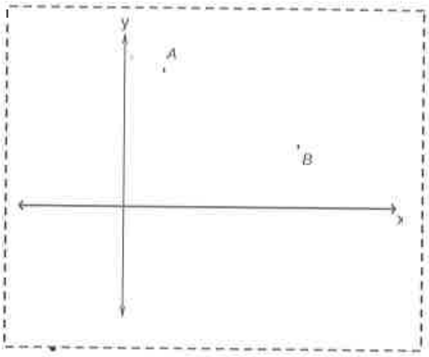
(d) y is decreasing at a constant rate.



(e) y is decreasing at an increasing rate.

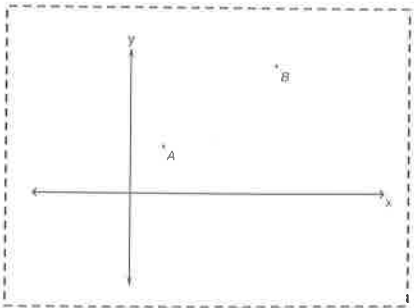


(f) y is decreasing at a decreasing rate.

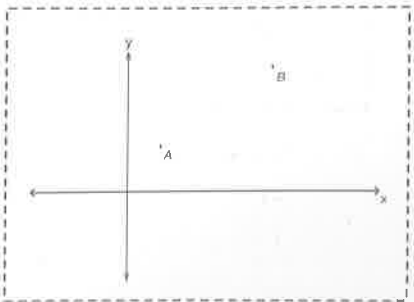


4. Draw the function which passes through A and B and which satisfies the given conditions:

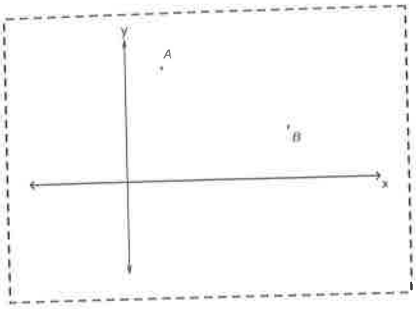
(a) $\frac{dy}{dx} > 0$ and $\frac{d^2y}{dx^2} > 0$



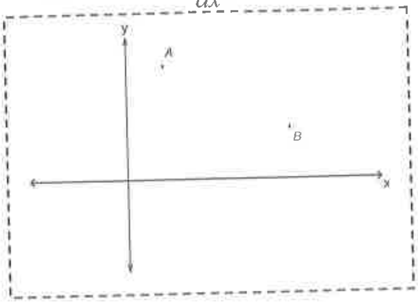
(b) $\frac{dy}{dx} > 0$ and $\frac{d^2y}{dx^2} < 0$



(c) $\frac{dy}{dx} < 0$ and $\frac{d^2y}{dx^2} > 0$



(d) $\frac{dy}{dx} < 0$ and $\frac{d^2y}{dx^2} < 0$



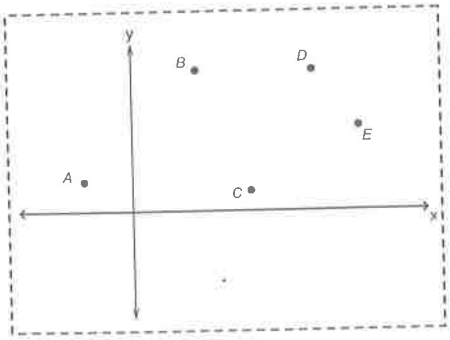
5. Show the following on the set of axes below. The section from:

(a) A to B is increasing at an increasing rate.

(b) B to C is decreasing at an increasing rate.

(c) C to D is increasing at a decreasing rate.

(d) D to E is decreasing at a decreasing rate.



EXERCISE 8

1. For the function $f(x) = \frac{2}{x^2}$:

(a) Use calculus to show that there are no turning points or points of inflexion.

(b) Determine algebraically whether the function is even, odd, or neither.

(c) Complete:

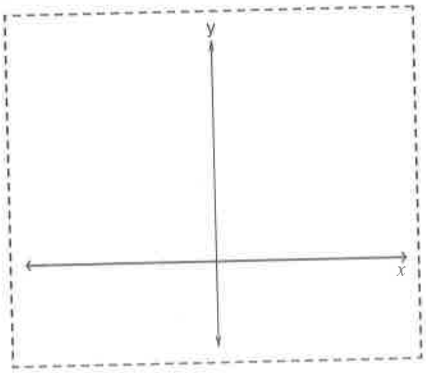
As $x \to \infty$, $f(x) \to$ $\boxed{}$

As $x \to -\infty$, $f(x) \to$ $\boxed{}$

As $x \to 0$, $f(x) \to$ $\boxed{}$

(d) An asymptote occurs when $x = \boxed{}$.

(e) Sketch $y = f(x)$

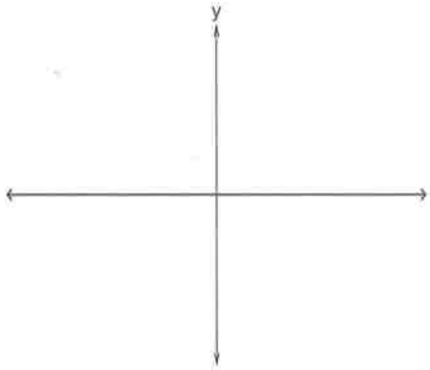


2.* For the function $f(x) = 2x + \dfrac{32}{x}$:

(a) locate all turning points and determine their nature. Check for any points of inflexion.

(b) A vertical asymptote occurs when $x = \boxed{}$.

(c) Draw the graph of $y = 2x$.



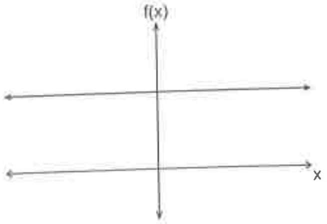
(d) Sketch $f(x) = 2x + \dfrac{32}{x}$ on the graph in part (c).

Matching the graph of a Function with its First Derivative

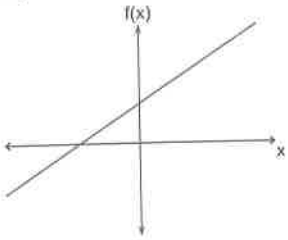
EXERCISE 9

Match a graph of $y = f(x)$ from the first column with a possible graph of its first derivative $y = f'(x)$ in the second column.

1.



2.



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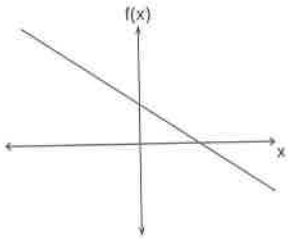

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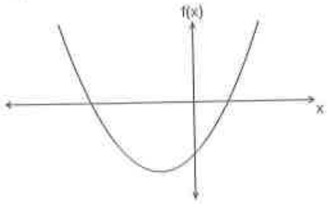
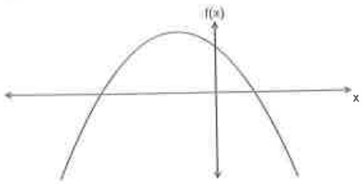


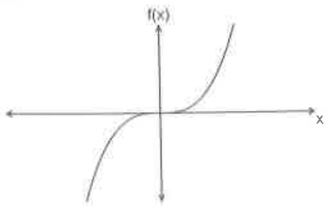
(a quadratic function)

5.



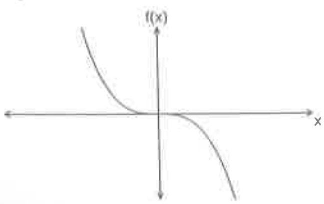
(a quadratic function)

6.



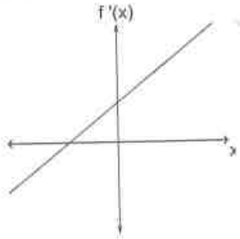
(a cubic function)

7.

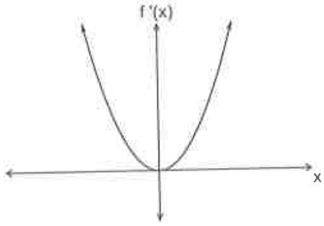


(a cubic function)

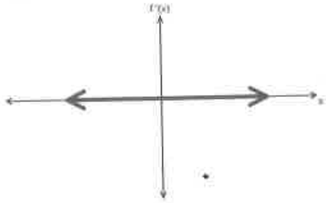
A.



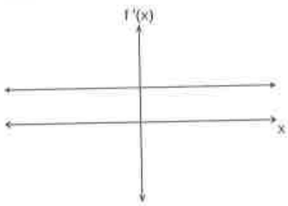
B.



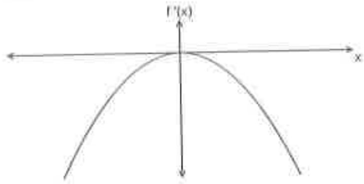
C.



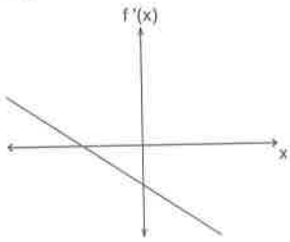
D.



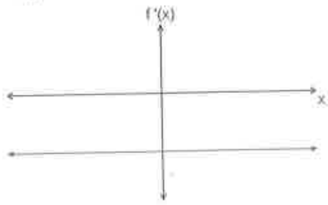
E.



F.



G.



ANSWERS	
1	
2	
3	
4	
5	
6	
7	

Maxima and Minima Problems

EXERCISE 10

1. For each of the following pair of equations, express A in terms of x only:

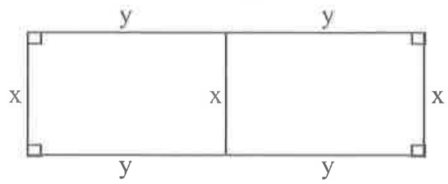
(a) $A = xy$; $y = 9 - 2x$

(b) $A = x^2y$; $y = 10 - x$

(c) $A = 2x + 3y$; $xy = 200$

(d) $A = 3xy$; $x^2 + y^2 = 25$

2. Two identical rectangular paddocks, as shown below, are to be enclosed using 60 metres of fencing.



(a) State the equation for the length of fencing.

(b) Make y the subject of this equation

(c) Show that the total area to be enclosed is $A = 30x - \frac{3x^2}{2}$

(d) Find values for x and y to give the maximum area A.

3.* A farmer decides to fence a rectangular area for storage, using part of the rear wall of his barn as one boundary. He needs an area of 800 m^2. Fencing will cost \$14 per metre.

(a) Determine the dimensions of the enclosure so that the cost of the fencing will be minimised.

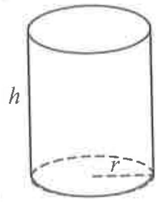
(b) What is the cost of fencing this enclosure?

4. The frame of a rectangular prism (with square base) is to be constructed using 64 cm of wire.

(a) Find the dimensions of the prism that will give the maximum volume for the box.

(b) Calculate the maximum volume of the box.

5. Calculate the radius and height of a closed cylindrical can which will give the minimum surface area if the can has a volume of 2400 cm^3. (4 dec.pl.)



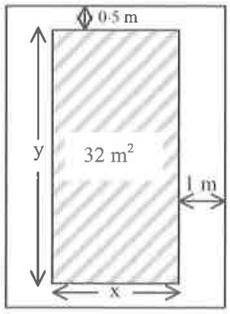
6. The surface area of a closed cylinder is 300π cm^2.

(a) Calculate the radius and height of the cylinder which will give the maximum volume (2 dec.pl.)

(b) Find the maximum volume.

7.* An open rectangular tank is to hold 60 750 litres of water. The base length is to be 3 times its width. Determine the dimensions of the tank so that the surface area is a minimum.

8.* A rectangular garden bed of area 32 m^2 is to have a concrete path 0·5 metres wide above and below it, and a path 1 metre wide each side (shown below). Find the dimensions of the garden bed so that the total area (garden bed and paths) is a minimum.



9.* An object is projected into the air. The height (h metres) of the object after t seconds is given by $h = 60t - 5t^2$

(a) What is the maximum height the object reaches?

(b) How long after reaching its maximum height is the object 100 m above its starting level?

10. The revenue from a company's product is given by $R = 180m + 42m^2 - m^3$ where m is the output (per unit of time). Determine the output that will give maximum revenue.

11.* The cost of operating a truck on a highway is made up of running costs plus the driver's wage. The running costs are $\left(0\cdot 25 + \frac{S}{250}\right)$ dollars per kilometre, where S is the speed of the truck in km/h, and the truck driver's wage is $45 per hour.

(a) Write an expression for the driver's wage for a 750 km trip in terms of S.

(b) Write an equation for the total cost of a 750 km trip.

(c) At what speed should the driver travel to make the 750 km trip as economical as possible?

SUMMARY

1. A function is increasing when $\frac{dy}{dx} \boxed{\quad} 0$

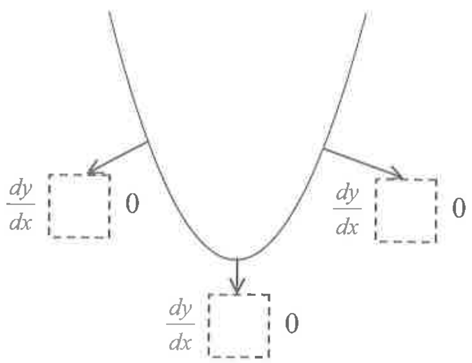
2. A function is decreasing when $\frac{dy}{dx} \boxed{\quad} 0$

3. To determine the nature of a turning point means to determine whether it is a $\boxed{\qquad\qquad}$ or $\boxed{\qquad\qquad}$ turning point.

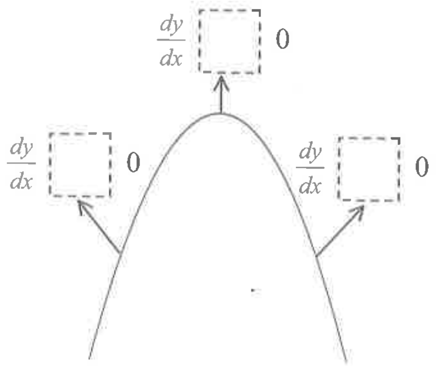
4. A **stationary point** is a point on $y = f(x)$ such that the gradient at the point has a value of $\boxed{\quad}$ and thus the tangent at that point is $\boxed{\qquad\qquad}$ to the x axis.

5. A turning point occurs when $\frac{dy}{dx} = \boxed{\quad}$.

6. Complete, for a minimum turning point:



7. Complete, for a maximum turning point:



8. **Curve sketching techniques**

(a) Find turning points by solving $\frac{dy}{dx} = \boxed{}$.

(b) Determine their nature by considering the $\boxed{}$ of the first derivative either side of the point (as shown in **6.** and **7.** above) *or* by substituting the x value of the turning points into the $\boxed{}$ and considering its sign.

(c) Locate the y intercept by substituting $x = \boxed{}$.

9. If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$ at a point, there is a $\boxed{}$ turning point.

10. If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$ at a point, there is a $\boxed{}$ turning point.

11. A function is concave upwards when $\frac{d^2y}{dx^2} \boxed{} 0$

12. A function is concave downwards when $\frac{d^2y}{dx^2} \boxed{} 0$

13. A point of inflexion is a change in $\boxed{}$

14. A point of inflexion exists when $\frac{d^2y}{dx^2} \boxed{} 0$ and $\boxed{}$

$\boxed{}$

15. A horizontal point of inflexion exists when $\frac{dy}{dx} \boxed{} 0$ and $\frac{d^2y}{dx^2} \boxed{} 0$

and $\boxed{}$

ANSWERS

Geometrical Applications of Differentiation

Exercise 1 (page 1)

1. A **2.** C **3.** A **4.** B **5.** B **6.** A **7.** A
8. D **9.** C

Exercise 2 (page 2)

1.(a) $x > 0$ **(b)** $x < 0$ **(c)** $x < 3$
(d) all real x **(e)** * **(f)** $x < -2$ and $x > 3$
(g) $x > 4$ and $x < 3$

2.(a) $x > 0$ **(b)** $x < 0$ **(c)** all real x
(d) all real x except $x = 0$
(e) $-1 < x < 1$ **(f)** * **(g)** $-6 < x < -2$
(h) all real x except $x = 6$

3.(a) $x > 1$ **(b)** $x > 1$ and $x < -1$ **(c)** $x < -2$
(d) $-3 < x < 3$

4.(a) $x > ½$ **(b)** $x < 2$ **(c)** * **(d)** all real x

5. $m > 24$

6.(a) $-3,\ 1,\ 4$ **(b)** $-3 < x < 1,\ x > 4$
(c) $x < -3,\ 1 < x < 4$

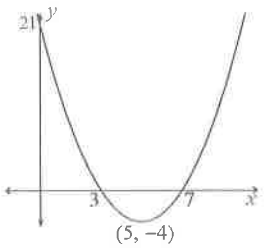
7.(a) -3 and 2 **(b)** $-3 < x < 2$ **(c)** $x < -3$ and $x > 2$
(d) $x < -½$ **(e)** $x > -½$

Exercise 3 (page 4)

1.(a) * **(b)** Min $(2, -2)$ **(c)** Max $(2, 21)$

2.(a) $(5, -4)$ **(b)** Min **(c)** 21 **(d)** 3 and 7

(e)



3. *

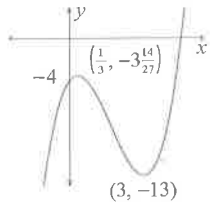
Exercise 4 (page 5)

1.(a) $x = \frac{-5}{3}$ and $x = -2$ **(b)** Min, Max

2.(a) $\left(\frac{1}{3}, -3\frac{14}{27}\right)$, $(3, -13)$. **(b)** Max, Min

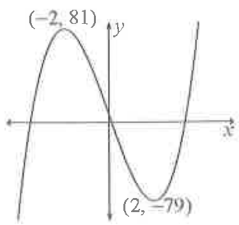
(c) -4

(d)



3.(a) $(2, -79)$ $(-2, 81)$ **(b)** Min, Max **(c)** 1

(d)



(e) 326 **(f)** -324

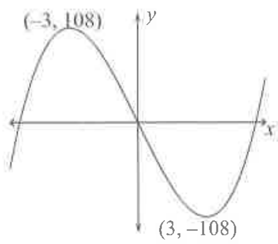
4. *

5. $\frac{dy}{dx} > 0$ for all values of x **6.** $(-3, 54)$

Exercise 5 (page 7)

1.(a) $(-3, 108)$ Max $(3, -108)$ Min **(b)** $(0, 0)$

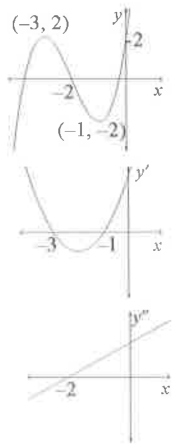
(c)



2. *

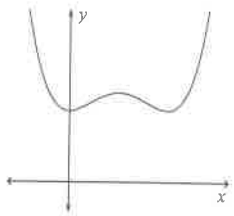
3.(a) $(-3, 2)$ Max $(-1, -2)$ Min **(b)** $(-2, 0)$

(c), (d), (e)



(f) x axis **(g)** x axis

4.(a)



(b) $0{\cdot}42 < x < 1{\cdot}58$

Exercise 6 (page 8)

1.(a) $3x^2 + 12x + 12$ **(b)** $6x + 12$ **(c)** -1

(d) 0 **(e)** 0 **(f)** 6 **(g)** -6

(h) horizontal, $(-2, -1)$, 0, 0, sign

2. *

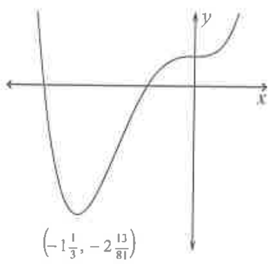
3.(a) no change in sign around $f''(-2)$ **(b)** A

Exercise 7 (page 10)

1.(a) $(-2, -4\tfrac{1}{3})$ Min

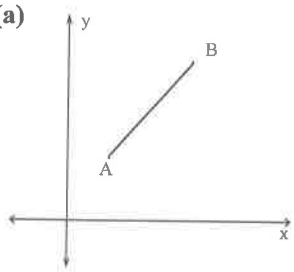
(b) (0, 1) horiz. pt. of inflex. $\left(-1\frac{1}{3}, -2\frac{13}{81}\right)$ pt. of inflex.

(c)

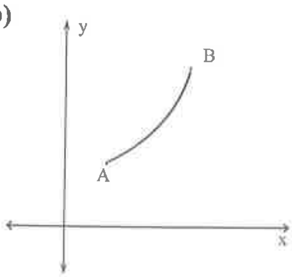


2. *

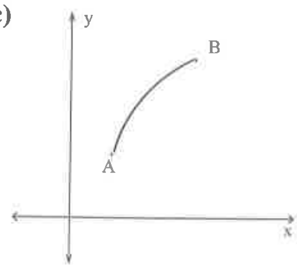
3.(a)



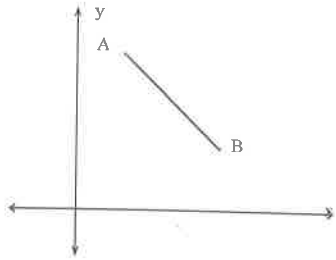
(b)



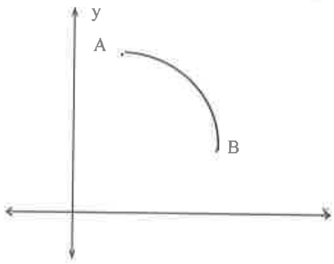
(c)



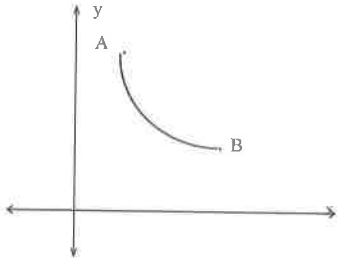
(d)



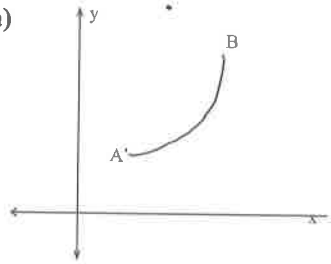
(e)



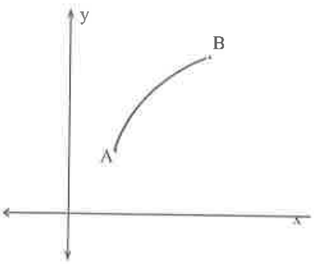
(f)



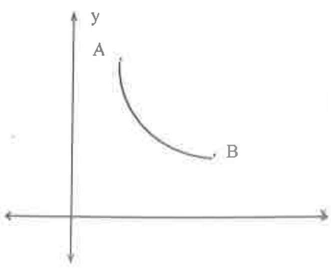
4.(a)



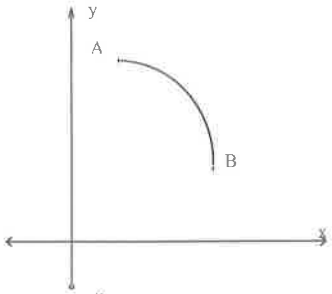
(b)



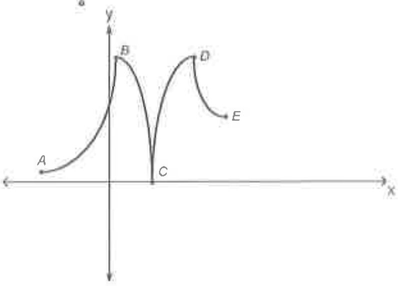
(c)



(d)



5.

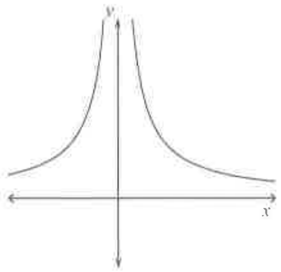


Exercise 8 (page 12)

1.(a) $f'(x) \neq 0,\ f''(x) \neq 0$ **(b)** even

(c) $0, 0, \infty$ **(d)** 0

(e)



2. *

Exercise 9 (page 14)

1. C **2.** D **3.** G **4.** A **5.** F **6.** B **7.** E

Exercise 10 (page 15)

1.(a) $A = 9x - 2x^2$ **(b)** $A = 10x^2 - x^3$

(c) $A = 2x + \frac{600}{x}$ **(d)** $A = \pm 3x\sqrt{25 - x^2}$

2.(a) $60 = 4y + 3x$ **(b)** $y = 15 - \frac{3}{4}x$

(d) $x = 10,\ y = 7 \cdot 5$

3. *

4.(a) base length and height = $5\frac{1}{3}$ cm **(b)** $151{\cdot}7\ \text{cm}^3$

5. $r = 7{\cdot}2557$ cm, $h = 14{\cdot}5112$ cm

6.(a) $r = 7{\cdot}07$ cm, $h = 14{\cdot}15$ cm **(b)** $2222{\cdot}01\ \text{cm}^3$

7. * **8.** * **9.** *

10. 30 **11.** *

Summary (page 19)

1. > **2.** < **3.** Max, Min **4.** 0, parallel

5. 0 **6.** <, =, > **7.** >, =, <

8. (a) 0 **(b)** sign, 2nd derivative **(c)** 0

9. Min **10.** Max **11.** > **12.** <

13. concavity

14. =, there is a change in the sign of $\frac{d^2y}{dx^2}$ on either side of the point.

15. =, =, there is a change in the sign of $\frac{d^2y}{dx^2}$ on either side of the point.

Integration

Exercise 1 (page 21)

1. C **2.** D **3.** A **4.** B **5.** C **6.** B

7. B **8.** D **9.** C **10.** C **11.** C **12.** D

13. C **14.** A

Exercise 2 (page 24)

1.(a) $\frac{x^6}{6} + c$ **(b)** $\frac{p^8}{8} + c$ **(c)** $x^5 + c$

(d) $4x^4 + c$ **(e)** $\frac{-3x^4}{4} + c$ **(f)** $\frac{x^{10}}{2} + c$

(g) $\frac{x^3}{12} + c$ **(h)** $\frac{x^3}{5} + c$

2.(a) $\frac{10x^3}{9} + c$ **(b)** $\pi x^9 + c$ **(c)** $3x + c$

(d) $\pi x + c$ **(e)** $\frac{\pi x^2}{2} + c$ **(f)** $-2m + c$

(g) $\frac{x}{3} + c$

Exercise 3 (page 24)

1.(a) $-\frac{1}{3x^3} + c$ **(b)** $-\frac{3}{x^2} + c$ **(c)** $-\frac{1}{2x^4} + c$

(d) $-\frac{1}{3x} + c$ **(e)** $-\frac{1}{5x^5} + c$ **(f)** $-\frac{1}{2x^6} + c$

(g) *

2.(a) $\frac{5}{7}x^{\frac{7}{5}}$ **(b)** $\frac{5}{6}x^{1\cdot2}$ **(c)** $8x^{\frac{7}{4}}$

WORKED SOLUTIONS

GEOMETRICAL APPLICATIONS OF DIFFERENTIATION

Exercise 2

1(e)

Increasing when $\frac{dy}{dx} > 0$

$$\therefore x^2 - 4 > 0$$

$$x^2 > 4$$

$$x > 2 \text{ and } x < -$$

2(f)

Decreasing when $\frac{dy}{dx} < 0$

$$\therefore (x+5)(x-1) < 0$$

< 0 below x axis

$$-5 < x < 1$$

4(c)

Concave down when $\frac{d^2y}{dx^2} < 0$

Now $\frac{dy}{dx} = 12x^3 - 144x$

$$\frac{d^2y}{dx^2} = 36x^2 - 144$$

$$36x^2 - 144 < 0$$

$$36x^2 < 144$$

$$x^2 < 4$$

$$-2 < x < 2$$

Exercise 3

1(a)

$y = 12x - 2x^2$

St. pts. when $\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = 12 - 4x$$

when $12 - 4x = 0$

$$4x = 12$$

$$x = 3 \rightarrow y = 18$$

now $\frac{d^2y}{dx^2} = -4$ $\quad(< 0 \therefore \text{max})$

Max turning pt. at $(3, 18)$

3 $\quad y = -x^2 + 7x + 8$

$$\frac{dy}{dx} = -2x + 7$$

$$\frac{d^2y}{dx^2} = -2$$

(a) turn. pts. $\frac{dy}{dx} = 0$

$$\therefore -2x + 7 = 0$$

$$x = 3\cdot5 \rightarrow y = 20\cdot25$$

$\therefore$ turn. pt. $(3\cdot5, 20\cdot25)$

(b) $\frac{d^2y}{dx^2} = -2 < 0 \therefore$ Max Turn. Pt.

(c) y intercept when $x = 0$ i.e. $y = 8$

(d) x intercepts when $y = 0$

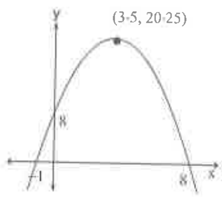
$$-x^2 + 7x + 8 = 0 \text{ (concave down)}$$

$$-[x^2 - 7x - 8] = 0$$

$$-(x-8)(x+1) = 0$$

$\therefore x$ intercepts are $8, -1$

(e)



Exercise 4

4 $\quad x(x+3)(x+1)$

(a) $\quad x(x^2 + 4x + 3)$

$$x^3 + 4x^2 + 3x$$

(b) $\quad y = \frac{1}{4}x^4 + \frac{4x^3}{3} + \frac{3x^2}{2}$

$$\frac{dy}{dx} = x^3 + 4x^2 + 3x$$

St. pts. when $\frac{dy}{dx} = 0$

$$x^3 + 4x^2 + 3x = 0$$

i.e. $x(x+3)(x+1) = 0$

$\therefore x = 0, x = -3$ and $x = -1$

St. Pts. $(0, 0)$ $(-3, -2\frac{1}{4})$, $(-1, \frac{5}{12})$

(c) $\frac{d^2y}{dx^2} = 3x^2 + 8x + 3$

$x = 0$:

$\frac{d^2y}{dx^2} = 3 > 0 \therefore$ Min at $(0, 0)$

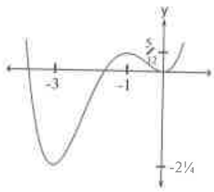
$x = -3$:

$\frac{d^2y}{dx^2} = 6 > 0 \therefore$ Min at $(-3, -2\frac{1}{4})$

$x = -1$:

$\frac{d^2y}{dx^2} = -2 < 0 \therefore$ Max at $(-1\frac{5}{12})$

(d)



Exercise 5

2 $\quad f(x) = x^3 - 3x + 5$

$$f'(x) = 3x^2 - 3$$

$$f''(x) = 6x$$

(a) turn. pts. $f'(x) = 0$

$$\therefore 3x^2 - 3 = 0$$

$$x^2 = 1$$

$$x = \pm 1$$

$\therefore$ turn. pts. $(-1, 7)$ and $(1, 3)$

$f''(-1) < 0 \therefore$ Max. T. Pt. at $(-1, 7)$

$f''(1) > 0 \therefore$ Min. T. Pt. at $(1, 3)$

(b) Pts. of inflexion when $f''(x) = 0$

i.e. $6x = 0$

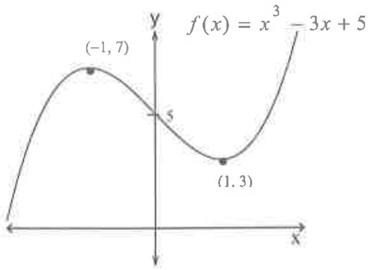
$$x = 0 \rightarrow (0, 5)$$

Check for change in sign either side of $(0, 5)$

$$\left.\begin{matrix} f''(-1) < 0 \\ f''(1) > 0 \end{matrix}\right\} \text{sign changes}$$

$\therefore$ Pt. of Inflex. at $(0, 5)$

(c)



Exercise 6

2(a) $f(x) = x^3 - 12x^2 + 48x$

$f'(x) = 3x^2 - 24x + 48$

St. Pts. when $f'(x) = 0$

i.e. $3x^2 - 24x + 48 = 0$

$x^2 - 8x + 16 = 0$

$(x-4)^2 = 0$

$x = 4 \rightarrow (4,64)$

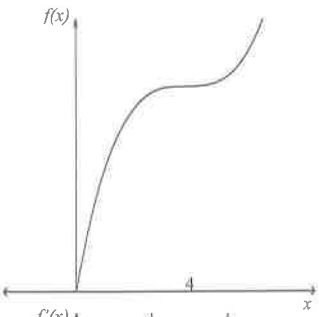
$f''(x) = 6x - 24$

at $(4,64)$ $f''(4) = 0$ $\therefore$ NOT Max/Min

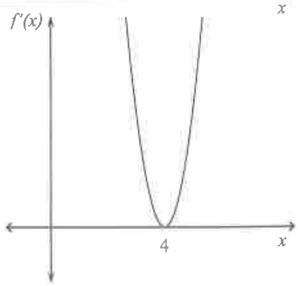
$\left.\begin{array}{l} f''(3) = -6 < 0 \\ f''(5) = 6 > 0 \end{array}\right\}$ concavity changes

$\therefore$ horizontal pt. of inflex. at $(4, 64)$

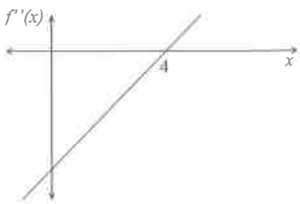
(b)



(c)



(d)



Exercise 7

2 $f(x) = -3x^4 - 8x^3$

$f'(x) = -12x^3 - 24x^2$

$f''(x) = -36x^2 - 48x$

(a) St. Pts when $f'(x) = 0$

i.e. $-12x^3 - 24x^2 = 0$

$-12x^2(x+2) = 0$

$x = 0,\ x = -2$

St. Pts $(0, 0)$ and $(-2, 16)$

Check nature :

$f''(0) = 0$ (possible pt. of inflex.)

$f''(-2) = -48 < 0$ $\therefore$ Max T Pt.

(b) Pts. of inflex. $f''(x) = 0$

$\therefore -36x^2 - 48x = 0$

$-12x(3x+4) = 0$

$x = 0,\ x = -1\frac{1}{3}$

Possible pts. $(0, 0)$ and $(-1\frac{1}{3}, 9\frac{13}{27})$

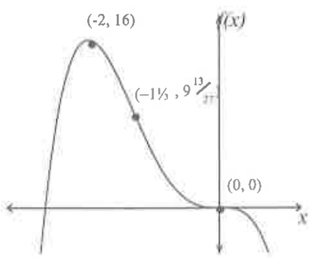
Check for change in concavity

$(0,0) : \begin{cases} f''(-1) > 0 \\ f''(1) < 0 \end{cases}$

change in concavity and since $f'(0) = 0$ and $f''(0) = 0$ there is a horizontal pt. of inflex. at $(0, 0)$

$(-1\frac{1}{3}, 9\frac{13}{27}) : \begin{cases} f''(-1) > 0 \\ f''(-2) < 0 \end{cases}$

change in concavity, there is a pt. of inflex. at $(-1\frac{1}{3}, 9\frac{13}{27})$



Exercise 8

2 $f(x) = 2x + \dfrac{32}{x}$

i.e. $f(x) = 2x + 32x^{-1}$

$f'(x) = 2 - 32x^{-2} = 2 - \dfrac{32}{x^2}$

$f''(x) = 64x^{-3} = \dfrac{64}{x^3}$

St. pts. when $f'(x) = 0$

$2 - \dfrac{32}{x^2} = 0$

$\dfrac{32}{x^2} = 2$

$2x^2 = 32$

$x^2 = 16$

$x = \pm 4$

St.pts at $(-4, -16)$ and $(4, 16)$

$f''(-4) = -0\cdot 5 < 0$

$\therefore$ Max T. Pt. at $(-4, -16)$

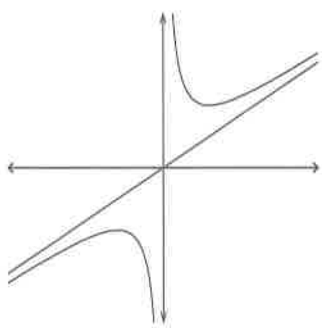
$f''(4) = 0\cdot 5 > 0$

$\therefore$ Min T. Pt. at $(-4, -16)$

Possible Pts. of Inflex. when $f''(x) = 0$, but no solutions for $\dfrac{64}{x^3} = 0$ $\therefore$ No Pts. of Inflex.

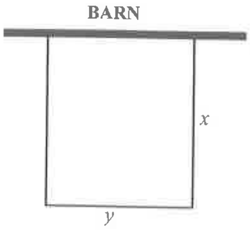
(b) Vertical asymptote when x = 0 (Division by zero not defined)

(c) and **(d)**



Exercise 10

3(a)



$A = xy$

$800 = xy$

$y = \frac{800}{x}$

$C = (y + 2x)(14)$

$C = 14y + 28x$

$C = 14\left(\frac{800}{x}\right) + 28x$

$C = \frac{11200}{x} + 28x$

$C = 11200x^{-1} + 28x$

$C' = -11200x^{-2} + 28$

When $C' = 0$

$0 = -11200x^{-2} + 28$

$0 = \frac{-11200}{x^2} + 28$

$\frac{11200}{x^2} = 28$

$11200 = 28x^2$

$28x^2 = 11200$

$x^2 = \frac{11200}{28}$

$x^2 = 400$

$x = \pm 20$ but $x > 0$

$\therefore x = 20$

$C'' = 22400x^{-3}$

$C'' = 22400(20)^{-3}$

$> 0 \therefore$ Min

when $x = 20 \Rightarrow y = \frac{800}{20}$

$= 40$

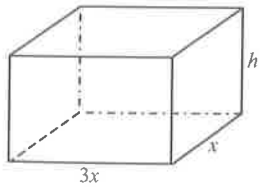
Min cost when $x = 20, y = 40$

(b)

when $x = 20 \Rightarrow C = \frac{11200}{(20)} + 28(20)$

$= 1120$

7



Note : 60 750 L = $60 \cdot 75\ \text{m}^3$

$V = Ah$

$= (3x)(x)h$

$= 3x^2h$

$60 \cdot 75 = 3x^2h$

$\frac{60 \cdot 75}{3x^2} = h$

$h = \frac{60 \cdot 75}{3x^2}$

$A = (3x)(x) + 2(x)(h) + 2(3x)(h)$

$= 3x^2 + 2xh + 6xh$

$= 3x^2 + 8xh$

$= 3x^2 + 8x\left(\frac{60 \cdot 75}{3x^2}\right)$

$= 3x^2 + \frac{486x}{3x^2}$

$= 3x^2 + \frac{162}{x}$

$= 3x^2 + 162x^{-1}$

$A' = 6x - 162x^{-2}$

$A'' = 6 + 324x^{-3}$

When $A' = 0 \Rightarrow 0 = 6x - 162x^{-2}$

$0 = 6x - \frac{162}{x^2} \quad (\times x^2)$

$0 = 6x^3 - 162$

$162 = 6x^3$

$x^3 = \frac{162}{6}$

$x^3 = 27$

$x = 3$

Sub $x = 3$ into A''

$A'' = 6 + 324(3)^{-3} > 0 \therefore$ Min.

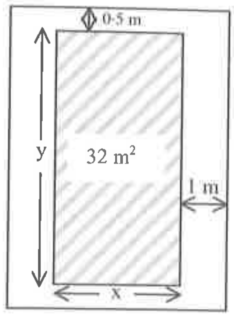
Dimensions :

When $x = 3 \Rightarrow h = \frac{60 \cdot 75}{3(3)^2} = 2 \cdot 25$

When $x = 3 \Rightarrow 3x = 3(3) = 9$

$\therefore$ Tank is $9\,\text{m} \times 3\,\text{m} \times 2 \cdot 25\,\text{m}$

8



Garden bed: $xy = 32 \Rightarrow y = \frac{32}{x}$

Let outer area $= A$

$A = (x + 2)(y + 1)$

$= xy + x + 2y + 2$

$= x\left(\frac{32}{x}\right) + x + 2\left(\frac{32}{x}\right) + 2$

$= 32 + x + \frac{64}{x} + 2$

$= 34 + x + 64x^{-1}$

$\therefore A' = 1 - 64x^{-2} = 1 - \frac{64}{x^2}$

$\therefore A'' = 128x^{-3} = \frac{128}{x^3}$

When $A' = 0$

$0 = 1 - \frac{64}{x^2} \quad (\times x^2)$

$0 = x^2 - 64$

$x^2 = 64$

$x = \pm 8 \quad (x \neq -8)$

$\therefore x = 8$

When $x = 8$:

$A'' = \frac{128}{8^3} > 0 \therefore$ Min.

$y = \frac{32}{8} = 4$

Dimensions $x = 8\,\text{m}, y = 4\,\text{m}$

9(a) $h = 60t - 5t^2$

$h' = 60 - 10t$

$h'' = -10 < 0 \therefore$ Max.

When $h' = 0 : 0 = 60 - 10t$

$10t = 60$

$t = 6$

When $t = 6 : h = 60(6) - 5(6)^2$

$= 360 - 180$

$= 180$

Max. height 180 m

(b) let $h = 100$

$100 = 60t - 5t^2$

$5t^2 - 60t + 100 = 0 \quad (\div 5)$

$t^2 - 12t + 20 = 0$

$(t - 10)(t - 2) = 0$

$t = 2, 10$ but $t = 6$ was max h

$\therefore t \neq 2 \Rightarrow t = 10$

Time after max $h = 10 - 6 = 4$ sec

11(a) $\text{Speed} = \dfrac{\text{Distance}}{\text{Time}}$

i.e. $S = \dfrac{D}{T}$ but $D = 750$

$\therefore S = \dfrac{750}{T} \Rightarrow T = \dfrac{750}{S}$

Wage = \$45 per hour

$\therefore W = 45 \times T$

$= 45 \times \dfrac{750}{S}$

$= \dfrac{33750}{S}$

(b) Let overall cost $= C$

$C = (\text{cost / km}) \times (\text{no. of km}) + \text{Wages}$

$C = \left(0 \cdot 25 + \frac{S}{250}\right) \times 750 + \frac{33750}{S}$

$C = 187 \cdot 5 + 3S + \frac{33750}{S}$

(c)

$C = 187 \cdot 5 + 3S + 33750S^{-1}$

$C' = 3 - 33750S^{-2} = 3 - \dfrac{33750}{S^2}$

$C'' = 67500S^{-3} = \dfrac{67500}{S^3}$

When $C' = 0$

$0 = 3 - \dfrac{33750}{S^2} \quad (\times S^2)$

$0 = 3S^2 - 33750$

$33750 = 3S^2 \quad (\div 3)$

$11250 = S^2$

$S = \pm 106 \cdot 06$ But $S > 0$

$\therefore$ Economical speed is 106 km/h