QUESTIONS & SOLUTIONS

ST PIUS X COLLEGE
CHATSWOOD

2016 Stage 6 – Year 12

ASSESSMENT TASK #2
Mid-Course Examination

30% of HSC Course Assessment

MATHEMATICS EXTENSION 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen, Black pen is preferred
- Board-approved calculators may be used
- Show all relevant mathematical reasoning and/or calculations

Total Marks – 84

- Attempt Questions 1 – 7
- ***Start each question on a new page***

Question 1 – 12 marks **Start in a new answer booklet**

a. Find the reciprocal of $\left(\frac{1}{a}+\frac{1}{b}\right)$. **2**

b. Solve the following inequality for x, and graph your solution on a number line: $\frac{4}{x}<3$ **3**

c. Find the co-ordinates of the point which divides the join of $(-3,8)$ and $(2,1)$ externally in the ratio $7:2$. **2**

d. Find the acute angle between the lines $x-y+1=0$ and $2y=x+1$, correct to the nearest minute. **2**

e. By using a substitution of $u=e^{2x}$, or otherwise, solve the equation $e^{2x}-2e^{-2x}=1$. **3**

Question 2 – 12 marks **Start in a new answer booklet**

a. Simplify $\frac{6^n - 3^n}{3^n}$. **1**

b. L_1 is the line $3x - 2y + 1 = 0$, and

L_2 is the line $x + y - 5 = 0$.

i. Describe geometrically what the equation **1**

$(3x - 2y + 1) + k(x + y - 5) = 0$ represents for various values of k.

ii. Show that the equation in (i) may be written **1**

$(k + 3)x + (k - 2)y = 5k - 1$.

iii. Solve the equation in (ii) when:

α. $k = 2$; **1**

β. $k = -3$; and **1**

~~γ. explain the significance of the results with regard~~
to the lines L_1 and L_2. **1**

c. $(x - 2)$ is a factor of $P(x) = 3x^3 - 13x^2 + 8x + m$

i. Find the value of m. **1**

ii. Find all the roots of $P(x) = 0$. **3**

d. If α, β, γ are the roots of $5x^3 - 2x^2 - 4x + 7 = 0$,
find the value of $\alpha^2 + \beta^2 + \gamma^2$. **2**

Question 3 – 12 marks **Start in a new answer booklet**

a. Solve the equation $4cos^2\theta = 3$ for $0^\circ \leq \theta \leq 360^\circ$. **3**

b. Prove $sin(A + B)sin(A - B) = sin^2A - sin^2B$. **3**

c.

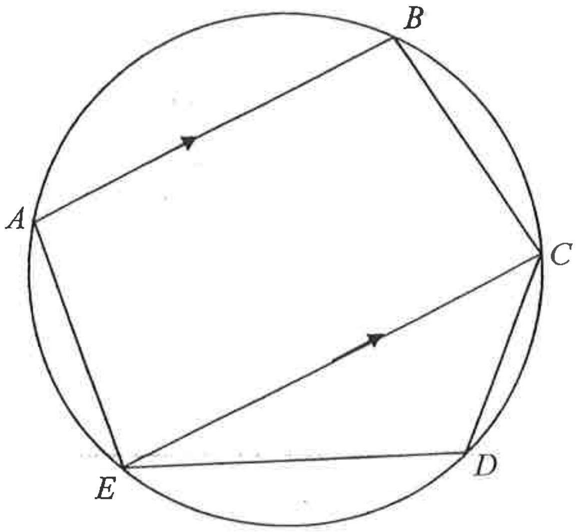


Diagram not to scale

A, B, C, D and E are points on the circumference of a circle such that AB is parallel to EC. **3**

Copy the diagram into you answer booklet.

Prove that $A\hat{D}E = B\hat{D}C$.

d. Consider the statement: $(5^n - 1)$ *is divisible by 4.* **3**

Show that, if the statement is true for $n = k$, then it is true for $n = k + 1$.

Question 4 – 12 marks **Start in a new answer booklet**

a. Find the possible values of k if $y = e^{-kx}$ satisfies the equation **3**

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 35y = 0.$$

b. P is the limiting sum of the GP

$$1 + x + x^2 + \cdots$$

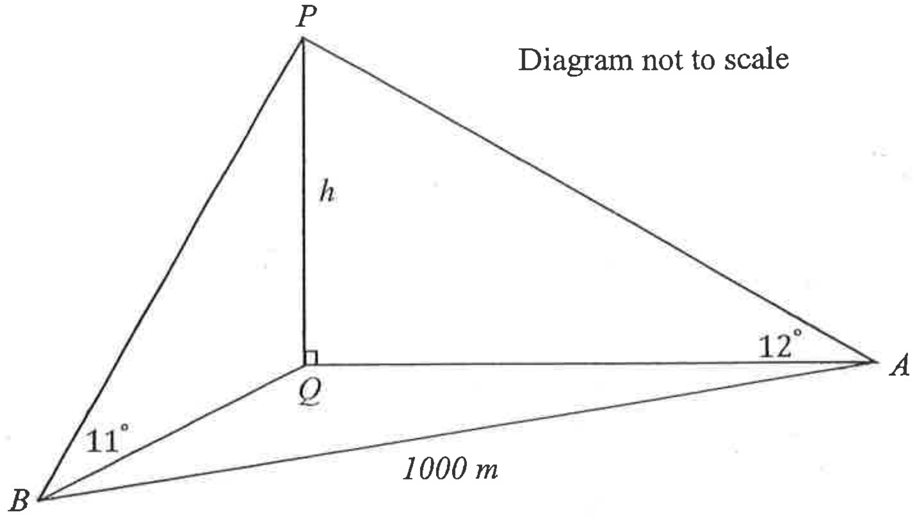
Q is the limiting sum of the GP

$$1 + 2x + 4x^2 + \cdots$$

If $\frac{P}{Q} = \frac{1}{3}$, find x. **3**

Question 4 continues on the next page

c.



The angle of elevation of a tower PQ of height h metres at a point A due east of it is 12°.
From another point B, the bearing of the tower is $051^{\circ}T$ and the angle of elevation is 11°.
The points A and B are 1000 metres apart and on the same level as the base Q of the tower.

i. Show that angle $A\hat{Q}B = 141^{\circ}$. **2**

ii. Consider the triangle APQ and show that $AQ = h\tan 78^{\circ}$. **1**

iii. Find a similar expression for BQ. **1**

iv. Use the cosine rule in the triangle AQB to calculate h to the nearest metre. **2**

Question 5 – 12 marks **Start in a new answer booklet**

a. i. Find $\frac{dy}{dx}$ if $y = x\ lnx$. **1**

ii. Hence, or otherwise, find $\int lnx\ dx$. **2**

b. Prove that the graph of $y = \frac{lnx}{x}$ has one turning point, and find its co-ordinates (in terms of e). **3**

c. Sketch graph of the three functions:

$y = e^{x}$, $y = e^{-x}$ and $y = e^{x} + e^{-x}$ **2**

in the same number plane.

d. Let $S_n = 1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + n(n+1)$, where n is a positive integer.

Prove by mathematical induction that $S_n = \frac{1}{3}n(n+1)(n+2)$. **4**

Question 6 – 12 marks **Start in a new answer booklet**

a. Find the number of terms in the series:

$2 + 4 + 6 + \cdots$ whose sum is 420. **3**

b. Consider the series:

$1 + \frac{1}{x-1} + \frac{1}{(x-1)^2} + \frac{1}{(x-1)^3} + \cdots$ **3**

Find the set of values of x for which this series has a limiting sum.

c. Consider the circle $x^2 + y^2 - 2x - 14y + 25 = 0$.
Show that if the line $y = mx$ intersects the circle at two distinct points, then **3**

$(1 + 7m)^2 - 25(1 + m^2) > 0$.

d. Show that the equation of the tangent to the graph of $y = \frac{e^x}{x}$,
at the point $\left(2, \frac{e^2}{2}\right)$ is $4y = e^2 x$. **3**

Question 7 – 12 marks **Start in a new answer booklet**

a. The point $P(2ap, ap^2)$ lies on the parabola $x^2 = 4ay$. The focus S is the point $(0, a)$. The tangent at P meets the y-axis at Q.

Draw a sketch representing this data.

i. Show that the equation of the tangent at P is $y = px - ap^2$. **1**

ii. Find the co-ordinates of Q. **1**

iii. Prove $SP = SQ$. **2**

iv. Hence show that $P\hat{S}Q + 2S\hat{Q}P = 180^\circ$. **2**

b. Consider the function $y = log_e(1 + x)$.

i. State the domain. **1**

ii. Sketch the graph. **1**

iii. Find the area between the arc of the graph from $x = 0$ to $x = 1$ and the $y -$ axis. **2**

iv. The area in part (iii) is rotated about the $y -$ axis through one complete revolution; determine the volume of the solid generated. **2**

End of Assessment

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2016 HIGHER SCHOOL CERTIFICATE EXAMINATION

REFERENCE SHEET

– Mathematics –

– Mathematics Extension 1 –

– Mathematics Extension 2 –

Mathematics

Factorisation

$$a^2 - b^2 = (a+b)(a-b)$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Angle sum of a polygon

$$S = (n-2) \times 180°$$

Equation of a circle

$$(x-h)^2 + (y-k)^2 = r^2$$

Trigonometric ratios and identities

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} \qquad \operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

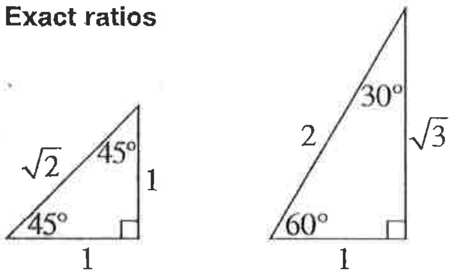
$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}} \qquad \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1$$

Exact ratios



Sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Cosine rule

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Area of a triangle

$$\text{Area} = \frac{1}{2}ab\sin C$$

Distance between two points

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Perpendicular distance of a point from a line

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Slope (gradient) of a line

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Point-gradient form of the equation of a line

$$y - y_1 = m(x - x_1)$$

***n*th term of an arithmetic series**

$$T_n = a + (n-1)d$$

Sum to *n* terms of an arithmetic series

$$S_n = \frac{n}{2}[2a + (n-1)d] \quad \text{or} \quad S_n = \frac{n}{2}(a + l)$$

***n*th term of a geometric series**

$$T_n = ar^{n-1}$$

Sum to *n* terms of a geometric series

$$S_n = \frac{a(r^n - 1)}{r - 1} \quad \text{or} \quad S_n = \frac{a(1 - r^n)}{1 - r}$$

Limiting sum of a geometric series

$$S = \frac{a}{1 - r}$$

Compound interest

$$A_n = P\left(1 + \frac{r}{100}\right)^n$$

Differentiation from first principles

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Derivatives

If $y = x^n$, then $\dfrac{dy}{dx} = nx^{n-1}$

If $y = uv$, then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

If $y = \dfrac{u}{v}$, then $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

If $y = F(u)$, then $\dfrac{dy}{dx} = F'(u)\dfrac{du}{dx}$

If $y = e^{f(x)}$, then $\dfrac{dy}{dx} = f'(x)e^{f(x)}$

If $y = \log_e f(x) = \ln f(x)$, then $\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$

If $y = \sin f(x)$, then $\dfrac{dy}{dx} = f'(x)\cos f(x)$

If $y = \cos f(x)$, then $\dfrac{dy}{dx} = -f'(x)\sin f(x)$

If $y = \tan f(x)$, then $\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$

Solution of a quadratic equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Sum and product of roots of a quadratic equation

$\alpha + \beta = -\dfrac{b}{a}$ $\qquad$ $\alpha\beta = \dfrac{c}{a}$

Equation of a parabola

$$(x-h)^2 = \pm 4a(y-k)$$

Integrals

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C$$

Trapezoidal rule (one application)

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}\left[f(a) + f(b)\right]$$

Simpson's rule (one application)

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Logarithms – change of base

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Angle measure

$180° = \pi$ radians

Length of an arc

$l = r\theta$

Area of a sector

$$\text{Area} = \frac{1}{2}r^2\theta$$

Angle sum identities

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

$$\tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$$

***t* formulae**

If $t = \tan\frac{\theta}{2}$, then

$$\sin\theta = \frac{2t}{1+t^2}$$

$$\cos\theta = \frac{1-t^2}{1+t^2}$$

$$\tan\theta = \frac{2t}{1-t^2}$$

General solution of trigonometric equations

$\sin\theta = a,\quad \theta = n\pi + (-1)^n \sin^{-1} a$

$\cos\theta = a,\quad \theta = 2n\pi \pm \cos^{-1} a$

$\tan\theta = a,\quad \theta = n\pi + \tan^{-1} a$

Division of an interval in a given ratio

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$$

Parametric representation of a parabola

For $x^2 = 4ay$,

$$x = 2at, \quad y = at^2$$

At $(2at, at^2)$,

tangent: $y = tx - at^2$

normal: $x + ty = at^3 + 2at$

At (x_1, y_1),

tangent: $xx_1 = 2a(y + y_1)$

normal: $y - y_1 = -\frac{2a}{x_1}(x - x_1)$

Chord of contact from (x_0, y_0): $xx_0 = 2a(y + y_0)$

Acceleration

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

Simple harmonic motion

$$x = b + a\cos(nt + \alpha)$$

$$\ddot{x} = -n^2(x - b)$$

Further integrals

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

Sum and product of roots of a cubic equation

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

$$\alpha\beta\gamma = -\frac{d}{a}$$

Estimation of roots of a polynomial equation

Newton's method

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Binomial theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Question 1

(a) $\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}$

Reciprocal $= \frac{ab}{a+b}$

(b) $\frac{4}{x} < 3$

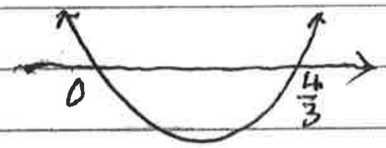
$x \neq 0$

$\frac{4}{x} \times x^2 < 3 \times x^2$

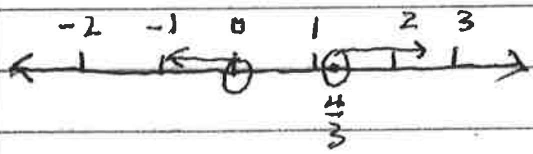
$4x < 3x^2$

$3x^2 - 4x > 0$

$x(3x-4) > 0$



$x < 0, \; x > \frac{4}{3}$



(c) $(-3, 8)$ and $(2, 1)$

ratio $(7:2)$ external

Let 2 be -ive

$P(x,y) = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$

$= \left(\frac{7 \times 2 + -2 \times -3}{7-2}, \frac{7 \times 1 + -2 \times 8}{7-2}\right)$

$= \left(\frac{14+6}{5}, \frac{7-16}{5}\right)$

$= \left(4, -\frac{9}{5}\right)$

(d) $x - y + 1 = 0 \; ; m_1 = 1$

$x - 2y + 1 = 0 \; ; m_2 = \frac{1}{2}$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{1 - \frac{1}{2}}{1 + 1 \times \frac{1}{2}}\right|$

$= \frac{\frac{1}{2}}{\frac{3}{2}}$

$= \frac{1}{3}$

$\theta = 18°26'$

(e) $e^{2x} - 2e^{-2x} - 1 = 0$

mult throughout by e^{2x}

$(e^{2x})^2 - e^{2x} - 2 = 0$

Let $u = e^{2x}$

$u^2 - u - 2 = 0$

$(u-2)(u+1) = 0$

$u = 2$ or $u = -1$

$e^{2x} = 2$ or $e^{2x} = -1$

now $e^{2x} > 0$ for all x hence only solution is:

$e^{2x} = 2$

$\ln 2 = 2x$

$x = \frac{1}{2}\ln 2$.

Approx $0 \cdot 34657359$

Question 2

(a) $\frac{6^n - 3^n}{3^n}$

$= \frac{(3\times 2)^n - 3^n}{3^n}$

$= \frac{3^n(2^n - 1)}{3^n}$

$= 2^n - 1$

~~(b) (i) The lines through the point of intersection of L_1 and L_2~~

(ii) $\cancel{(3x-2y+1) + k(x+y-5) = 0}$

$3x - 2y + 1 + kx + ky - 5k = 0$

$3x + kx + ky - 2y = 5k - 1$

$(k+3)x + (k-2)y = 5k - 1.$

(iii)(a) $k = 2$ | $k = -3$

$5x = 10 - 1$ | $-5y = -15 - 1$

$5x = 9$ | $-5y = -16$

$x = 1\frac{4}{5}$ | $y = 3\frac{1}{5}$

The point of intersection L_1 and L_2

(c) $P(x) = 3x^3 - 13x^2 + 8x + m$

$(x-2)$ is a factor $\therefore P(2) = 0$

$3(2)^3 - 13(2)^2 + 8(2) + m = 0$

$24 - 52 + 16 + m = 0$

$-12 + m = 0$

$m = 12$

$$\begin{array}{r|l} & 3x^2 - 7x - 6 \\ \hline x-2 & 3x^3 - 13x^2 + 8x + 12 \\ & 3x^3 - 6x^2 \\ \hline & -7x^2 + 8x \\ & -7x^2 + 14x \\ \hline & -6x + 12 \\ & -6x + 12 \\ \hline & 0 + 0 \end{array}$$

$\therefore (3x^2 - 7x - 6)$ is a factor

ie let $(3x+2)(x-3) = 0$

$\therefore x = -\frac{2}{3}$ or 3

Ans $x = -\frac{2}{3}, 2$ or 3.

(d) α, β, γ are roots of $5x^3 - 2x^2 - 4x + 7 = 0$

$$\alpha + \beta + \gamma = \frac{2}{5}.$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = -\frac{4}{5}.$$

$$\alpha\beta\gamma = -\frac{7}{5}.$$

$$(\overline{\alpha + \beta} + \gamma)^2 = (\alpha+\beta)^2 + 2(\alpha+\beta)\gamma + \gamma^2.$$

$$= \alpha^2 + 2\alpha\beta + \beta^2 + 2\alpha\gamma + 2\beta\gamma + \gamma^2.$$

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma).$$

$$= \left(\frac{2}{5}\right)^2 - 2\left(-\frac{4}{5}\right)$$

$$= \frac{4}{25} + \frac{8}{5}.$$

$$= \frac{44}{25}.$$

Question 3

(a) $4\cos^2\theta = 3 \qquad 0^\circ \leq \theta \leq 360^\circ$

$\cos^2\theta = \frac{3}{4}$

$\cos\theta = \pm\frac{\sqrt{3}}{2}$.

$\cos\theta$ is both +ive and -ive hence θ lies in all 4 quadrants.

Related acute angle $= 30^\circ$.

Ans $\theta = 30^\circ$

$\theta = 180^\circ - 30^\circ = 150^\circ$

$\theta = 180^\circ + 30^\circ = 210^\circ$.

$\theta = 360^\circ - 30^\circ = 330^\circ$

(b) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$.

$LHS = \sin(A+B)\sin(A-B)$

$= (\sin A\cos B + \cos A\cos B)(\sin A\sin B - \cos A\cos B)$

$= \sin^2 A\cos^2 B - \cos^2 A\sin^2 B$.

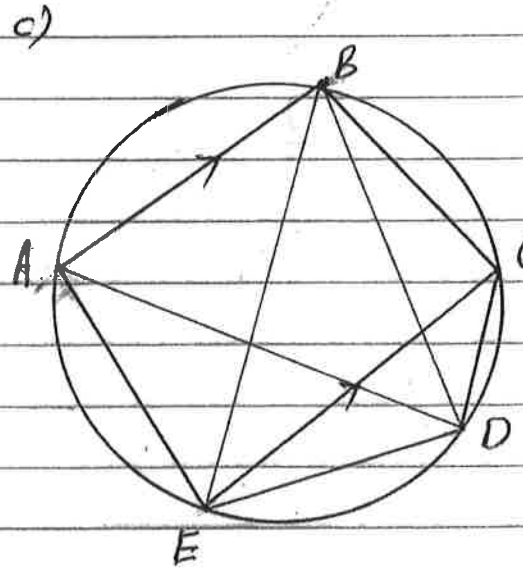
$= \sin^2 A(1-\sin^2 B) - (1-\sin^2 A)\sin^2 B$

$= \sin^2 A - \sin^2 A\sin^2 B - \sin^2 B + \sin^2 A\sin^2 B$

$= \sin^2 A - \sin^2 B$

$= RHS$.

(c)



$\angle ADE = \angle ABE$ (Angles in the same segment subtended by the arc AE)

$\angle ABE = \angle BEC$ (alternate $\angle$'s of $\parallel$ lines AB and EC)

$\angle BEC = \angle BDC$ (Angles in the same segment subtended by the arc BC)

$\therefore \angle ADE = \angle BDC$

(d) Statement $(5^n - 1)$ is divisible by 4.

If true for $n = k$ Then let.

$5^k - 1 = 4A$ where 'A' is an integer.

Consider $n = k+1$.

$5^{k+1} - 1 = 5(5^k) - 1$.

$= 5(4A+1) - 1$ using the assumption

$= 5(4A) + 5 - 1$

$= 4(5A) + 4$

$= 4(5A+1)$

This is divisible by 4.

So if the divisibility holds for $n = k$, then it holds for $n = k+1$.

Question 4

(a) $$\frac{d^2y}{dx^2}+2\frac{dy}{dx}-35y=0$$

$$y=e^{-kx}$$

$$\frac{dy}{dx}=-ke^{-kx}$$

$$\frac{d^2y}{dx^2}=k^2e^{-kx}$$

substitute:

$$k^2e^{-kx}-2ke^{-kx}-35e^{-kx}=0$$

$$e^{-kx}(k^2-2k-35)=0$$ | Note $e^{-kx}>0$ for all x

$$k^2-2k-35=0$$

$$(k-7)(k+5)=0$$

$$k=-5 \text{ or } 7$$

(b) $1+x+x^2+\dots$

$1+2x+4x^2+\dots$

$$S_\infty=\frac{a}{1-r}$$

$$P=\frac{1}{1-x}$$

$$Q=\frac{1}{1-2x}$$

$$\frac{P}{Q}=\frac{1}{3}$$

$$\frac{\frac{1}{1-x}}{\frac{1}{1-2x}}=\frac{1}{3}$$

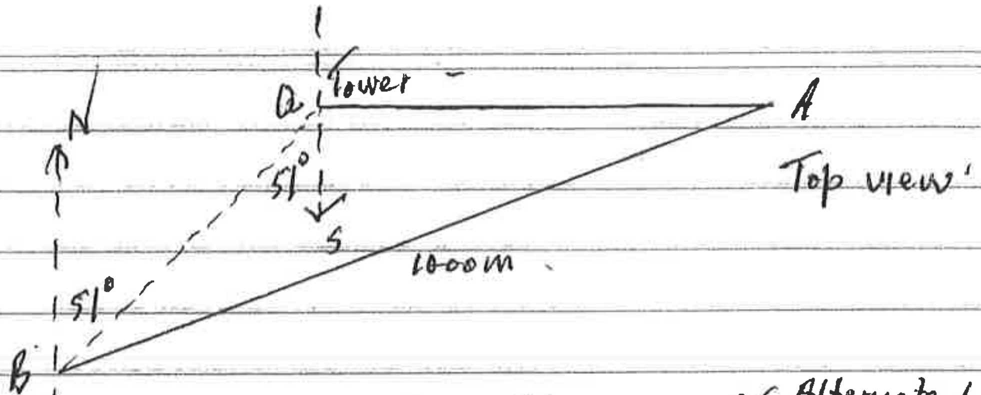
$$\frac{1-2x}{1-x}=\frac{1}{3}$$

$$3-6x=1-x$$

$$2=5x$$

$$x=\frac{2}{5}$$

(c) (i)



$\angle NBQ = 51^\circ$ and $\angle BQS = 51^\circ$ (Alternate $\angle$ of $\parallel$ lines to $\angle NBQ$)

The angle between the south and east directions is $\angle SQA = 90^\circ$

Hence $\angle AQB = \angle AQS + \angle SQB$

$\angle AQB = 141^\circ$

(ii) In right angle Δ APQ, $\angle APQ = 78^\circ$

$$\therefore \frac{AQ}{h} = \tan 78^\circ$$

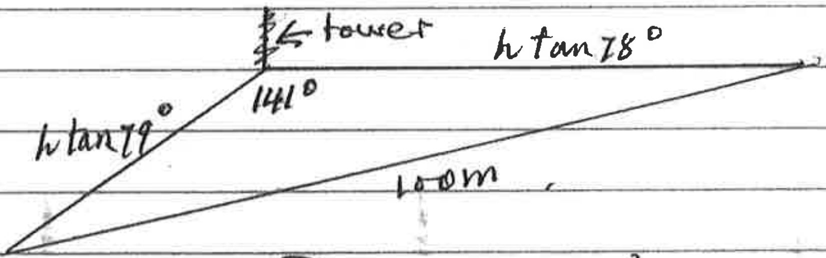
$$AQ = h \tan 78^\circ$$

(iii) similarly, from right angle Δ BPQ, $\angle BPQ = 79^\circ$

$$\text{so } \frac{BQ}{h} = \tan 79^\circ$$

$$BQ = h \tan 79^\circ$$

(iv)



using the cosine-rule:

$$\cos 141^\circ = \frac{(h\tan 79^\circ)^2 + (h\tan 78^\circ)^2 - 1000^2}{2(h\tan 79^\circ)(h\tan 78^\circ)}$$

$$2h\tan 79^\circ \cdot h\tan 78^\circ \cdot \cos 141^\circ = (h\tan 79^\circ)^2 + (h\tan 78^\circ)^2 - 1000^2$$

$$1000^2 = h^2\left[\tan^2 79^\circ + \tan^2 78^\circ - 2\tan 79^\circ \tan 78^\circ \cos 141^\circ\right]$$

$$h^2 = \frac{1000^2}{\tan^2 79^\circ + \tan^2 78^\circ - 2\tan 79^\circ \tan 78^\circ \cos 141^\circ}$$

$h = 108$ m correct to nearest metre.

Question 5

(a) $y = x\ln x$

$\frac{dy}{dx} = 1 \times \ln x + \frac{1}{x} \times x$

$\frac{dy}{dx} = 1 + \ln x$.

(ii) $\ln x = \frac{dy}{dx} - 1$.

Integrate both sides w.r.t x

$\int \ln x \, dx = \int \left(\frac{dy}{dx}\right) dx - \int 1 \, dx$

$= y - x + c$

$= x\ln x - x + c$

(b) $y = \frac{\ln x}{x}$, $x > 0$

$\frac{dy}{dx} = \frac{x \times \frac{1}{x} - \ln x \times 1}{x^2}$

$\frac{dy}{dx} = \frac{1 - \ln x}{x^2}$

$\frac{d^2y}{dx^2} = \frac{x^2 \times -\frac{1}{x} - (1 - \ln x)2x}{x^4}$

$\frac{d^2y}{dx^2} = \frac{-x - 2x + 2x\ln x}{x^4}$

$\frac{d^2y}{dx^2} = \frac{2x\ln x - 3x}{x^4}$

$= \frac{2\ln x - 3}{x^3}$

For stationary points $y' = 0$

$\frac{1 - \ln x}{x^2} = 0$

$1 - \ln x = 0$

$\ln x = 1$

$x = e$

when $x = e$

$y = \frac{\ln e}{e}$

$y = \frac{1}{e}$

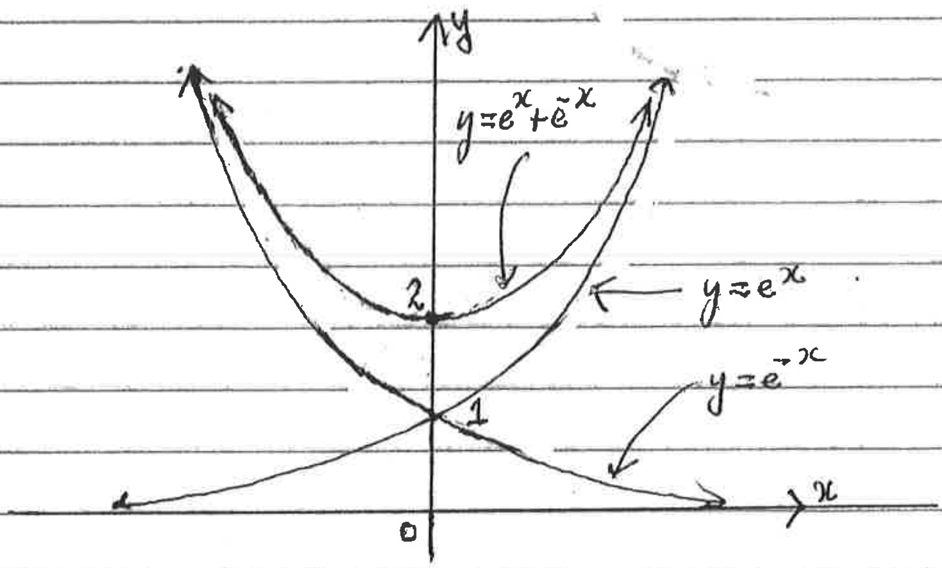
$y'' = \frac{2\ln e - 3}{e^3}$

$y'' = \frac{2 - 3}{e^3}$

$y'' = -\frac{1}{e^3} < 0$ ∴ ✓✓

∴ $\left(e, \frac{1}{e}\right)$ is a local maximum turning point.

(c)



(d) $S_n = 1\times 2 + 2\times 3 + 3\times 4 + \dots + n(n+1)$

Prove $S_n = \frac{1}{3}n(n+1)(n+2)$.

Soln If $T_n = n(n+1)$

Step 1 when $n=1$

$S_1 = \frac{1}{3}\times 1\times 2\times 3$

$S_1 = 1\times 2$ which is true.

Step 2 assume That The sum holds for $n=k$

$\therefore S_k = \frac{1}{3}k(k+1)(k+2)$.

Consider $n=k+1$

$T_{k+1} + S_k = (k+1)(k+2) + \frac{1}{3}k(k+1)(k+2)$

$= \frac{1}{3}(k+1)(k+2)\left[3 + k\right]$

$= \frac{1}{3}(k+1)(k+2)(k+3)$

$= \frac{1}{3}(k+1)\left[(k+1)+1\right]\left[(k+1)+2\right]$

This is the required form ie $(k+1)$ in place of k in the given S_n.

So if the sum holds for $n=k$ then it holds for $n=(k+1)$

Step 3 The sum holds for $n=1$, hence it holds for $n=2$ and $n=3$ and so on for all positive integers 'n'.

Question 6

(a) $2 + 4 + 6 + \ldots$ to a sum of 420

$a = 2$ $\qquad S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

$d = 2$

$n = ?$

$S_n = 420$

$$420 = \frac{n}{2}\left[4 + (n-1)2\right]$$

$$420 = n(2 + n - 1)$$

$$420 = n(n+1)$$

$$n^2 + n - 420 = 0$$

$$(n+21)(n-20) = 0$$

$n = 20$ or $n = -21$ N/A as n is a +ive integer.

$\therefore n = 20$

(b) $1 + \frac{1}{x-1} + \frac{1}{(x-1)^2} + \frac{1}{(x-1)^3} + \ldots$

$a = 1$

$r = \frac{1}{x-1}$

for a limiting sum $|r| < 1$

$$\left|\frac{1}{x-1}\right| < 1 \;, \quad x \neq 1$$

$$\frac{1}{|x-1|} < 1$$

$1 < |x-1|$ ie $|x-1| > 1$

$x - 1 > 1$ or $x - 1 < -1$

$x > 2$ $\qquad x < 0$

(c) $x^2 + y^2 - 2x - 14y + 25 = 0$ --- ①

$y = mx$ ----- ②

Solving : sub ② in ①

$$x^2 + (mx)^2 - 2x - 14(mx) + 25 = 0$$

$$x^2 + m^2x^2 - 2x - 14mx + 25 = 0$$

$$(1+m^2)x^2 - 2(1+7m)x + 25 = 0$$

for intersection with two distinct points

let $\Delta > 0$

ie $b^2 - 4ac > 0$

$$[-2(1+7m)]^2 - 4(1+m^2)(25) > 0$$

$$4(1+7m)^2 - 100(1+m^2) > 0$$

$$(1+7m)^2 - 25(1+m^2) > 0$$

(d). $y = \frac{e^x}{x}$, point $(2, \frac{e^2}{2})$

$$y' = \frac{x \times e^x - e^x \times 1}{x^2}$$

$$y' = \frac{e^x(x-1)}{x^2}$$

when $x = 2$

$$y' = \frac{e^2(1)}{4} = \frac{e^2}{4}$$

$\therefore$ Gradient of tangent $m = \frac{e^2}{4}$

Equation $y - y_1 = m(x - x_1)$

$$y - \frac{e^2}{2} = \frac{e^2}{4}(x - 2)$$

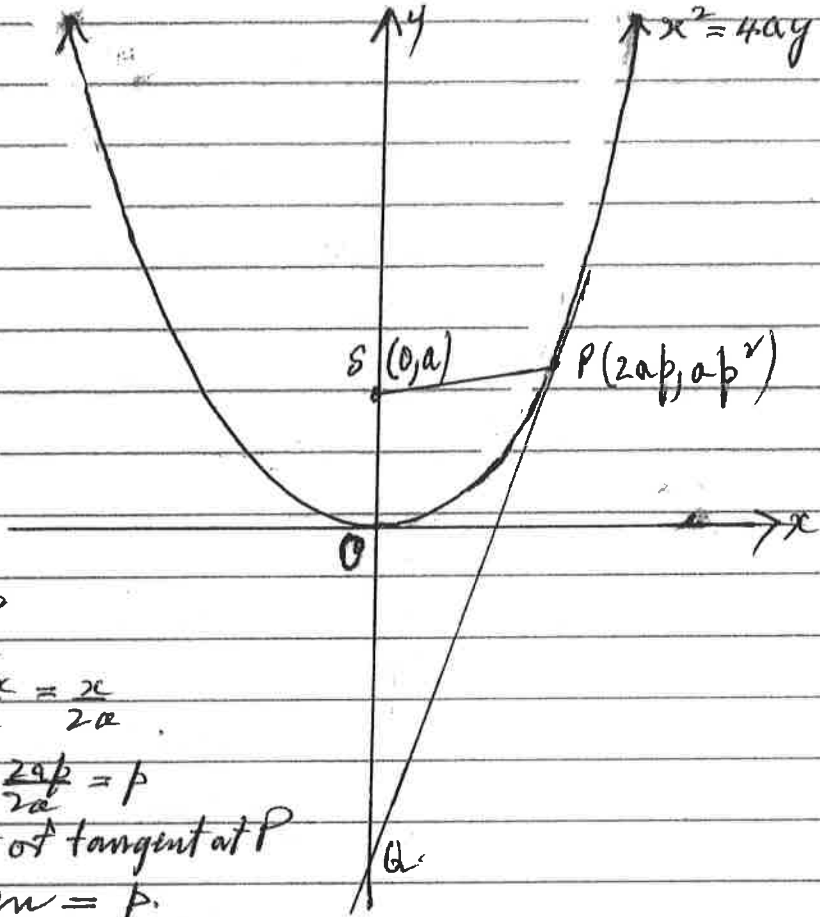
$$y - \frac{e^2}{2} = \frac{e^2}{4}x - \frac{e^2}{2}$$

$$y = \frac{e^2}{4}x$$

ie $4y = e^2x$.

Question 7.

(a)



(i) $y = \frac{x^2}{4a}$

$y' = \frac{2x}{4a} = \frac{x}{2a}$.

at P $y' = \frac{2ap}{2a} = p$

∴ Gradient of tangent at P is $m = p$.

Equation of tangent at P

$y - y_1 = m(x - x_1)$

$y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

$y = px - ap^2$ --- ①

(ii) Let $x = 0$ in ① for point Q.

$y = -ap^2$.

∴ Q is $(0, -ap^2)$

(iii) $SQ = OS + OQ = a + ap^2 = a(p^2+1)$

using distance formula.

$SP^2 = (2ap-0)^2 + (ap^2-a)^2$

$SP^2 = a^2(4p^2 + p^4 - 2p^2 + 1)$

$SP^2 = a^2(p^4 + 2p^2 + 1)$

$SP^2 = a^2(p^2+1)^2$

$SP = a(p^2+1)$

∴ $SP = SQ$.

(iv) With $SP = SQ$ we have ΔSPQ as isosceles

Hence $\angle SQP = \angle SPQ$

and

$\angle PSQ + \angle SQP + \angle SPQ = 180°$

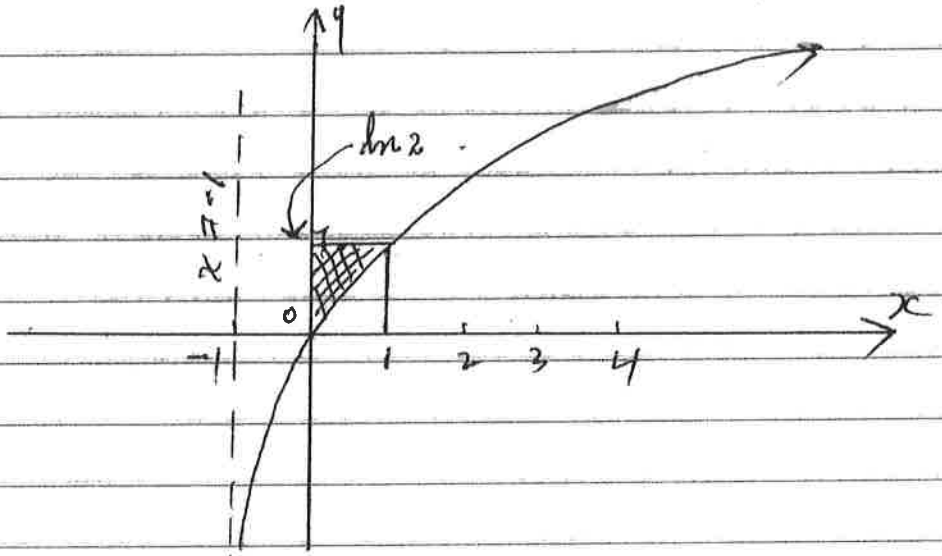
∴ $\angle PSQ + 2\angle SQP = 180°$.

(b) $y = \log_e(1+x)$

(i) Domain : $x + 1 > 0$

$x > -1$ (Reals).

(ii)



(iii) Now $1 + x = e^y$

$x = e^y - 1$

$A = \int_a^b x\,dy$

$A = \int_0^{\ln 2} (e^y - 1)\,dy$

$= \left[e^y - y\right]_0^{\ln 2}$

$= (e^{\ln 2} - \ln 2) - (e^0 - 0)$

$= 2 - \ln 2 - 1$

$= (1 - \ln 2)$ units^2

Approx 0·306852819

For part (iv)

$x^2 = (e^y - 1)^2$

$= e^{2y} - 2e^y + 1$

(iv) $V = \pi \int_a^b x^2\,dy$

$V = \pi \int_0^{\ln 2} (e^{2y} - 2e^y + 1)\,dy$

$= \pi \left[\frac{1}{2}e^{2y} - 2e^y + y\right]_0^{\ln 2}$

$= \pi \left[\left(\frac{1}{2} \times 4 - 2 \times 2 + \ln 2\right) - \left(\frac{1}{2} - 2 + 0\right)\right]$

$= \pi \left(2 - 4 + \ln 2 + \frac{3}{2}\right)$

$= \pi\left(\ln 2 - \frac{1}{2}\right)$ units^3 (Approx 0·606789763...)