$$= \frac{b\left(1-\frac{a^2}{b^2}\right)+\frac{2a^2}{b}}{1+\frac{a^2}{b^2}}$$

$$= \frac{b-\frac{a^2}{b}+\frac{2a^2}{b}}{1+\frac{a^2}{b^2}}$$

$$= \frac{\frac{b^2+a^2}{b}}{\frac{b^2+a^2}{b^2}}$$

$$= \frac{b^2+a^2}{b} \times \frac{b^2}{b^2+a^2}$$

$$= b$$

$$= \text{RHS}$$

EXERCISE 4.4

1 **(a)** $1+\tan^2\left(\frac{\pi}{2}-\alpha\right) = 1 + \frac{\sin^2\left(\frac{\pi}{2}-\alpha\right)}{\cos^2\left(\frac{\pi}{2}-\alpha\right)}$

$$= \frac{\cos^2\left(\frac{\pi}{2}-\alpha\right)+\sin^2\left(\frac{\pi}{2}-\alpha\right)}{\cos^2\left(\frac{\pi}{2}-\alpha\right)}$$

$$= \frac{1}{\sin^2\alpha}$$

$$= \operatorname{cosec}^2\alpha$$

(b) $1-\cos^2(\pi+\theta) = 1-\cos^2\theta$

$$= \sin^2\theta$$

(c) $\sin\theta\cos\left(\frac{\pi}{2}-\theta\right)+\cos\theta\sin\left(\frac{\pi}{2}-\theta\right)$

$$= \sin\theta\sin\theta + \cos\theta\cos\theta$$

$$= 1$$

(d) $2\cos^2\frac{\pi}{6} - 1 = \cos\frac{\pi}{3}$

$$= \frac{1}{2}$$

(e) $1-\sin\theta\cos\left(\frac{\pi}{2}-\theta\right) = 1-\sin\theta\sin\theta$

$$= \cos^2\theta$$

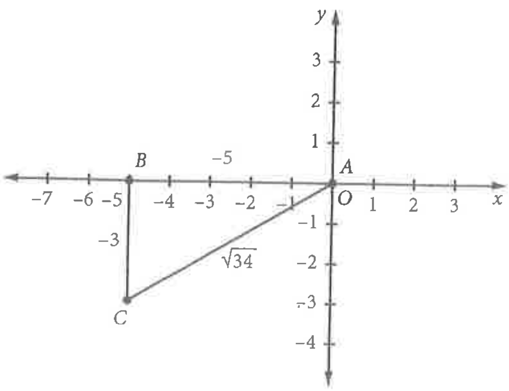
(f) $\sin(\pi-\theta)\cos\phi - \cos(\pi-\theta)\sin\phi$

$$= \sin(\pi-\theta-\phi)$$

$$= \sin(\pi-[\theta+\phi])$$

$$= \sin(\theta+\phi)$$

3 $\tan\theta = \frac{3}{5}$



(a) $\sin\theta = \frac{-3}{\sqrt{34}}$

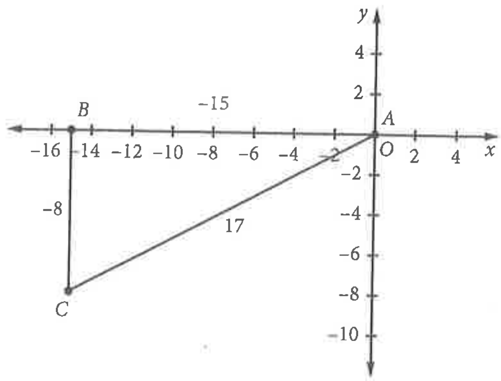
(b) $\cos\theta = \frac{-5}{\sqrt{34}}$

(c) $\cos 2\theta = 2\cos^2\theta - 1$

$$= \frac{2\times 25}{34} - 1$$

$$= \frac{8}{17}$$

5 $\operatorname{cosec}\alpha = -\frac{17}{8}$



(a) $\cot\alpha = \frac{15}{8}$

(b) $\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$

$$= \frac{2\times\frac{8}{15}}{1-\frac{64}{225}}$$

$$= \frac{240}{161}$$

7 $\tan x = \frac{5}{4}$, $\tan y = \frac{1}{9}$

$$\tan(x-y) = \frac{\tan x - \tan y}{1+\tan x\tan y}$$

$$= \frac{\frac{5}{4}-\frac{1}{9}}{1+\frac{5}{4}\times\frac{1}{9}}$$

$$= 1$$

So: $x-y = \frac{\pi}{4}$ (because both x and y are in the first quadrant)

9 $\text{LHS} = \sin\left(\theta + \frac{\pi}{6}\right)\sin\left(\theta - \frac{\pi}{6}\right)$

$= \left(\sin\theta\cos\frac{\pi}{6} + \cos\theta\sin\frac{\pi}{6}\right)\left(\sin\theta\cos\left(\frac{\pi}{6}\right) - \cos\theta\sin\left(\frac{\pi}{6}\right)\right)$

$= \left(\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta\right)\left(\frac{\sqrt{3}}{2}\sin\theta - \frac{1}{2}\cos\theta\right)$

$= \frac{3}{4}\sin^2\theta - \frac{1}{4}\cos^2\theta$

$= \frac{3}{4}\sin^2\theta - \frac{1}{4}(1 - \sin^2\theta)$

$= \sin^2\theta - \frac{1}{4}$

$= \text{RHS}$

11 (a) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\sin(A-B) = \sin A\cos B - \cos A\sin B$

$\sin(A+B) + \sin(A-B) = \sin A\cos B + \cos A\sin B + \sin A\cos B - \cos A\sin B$

$= 2\sin A\cos B$

(b) Solving $\theta = A + B$ and $\phi = A - B$ for A and B gives:

$A = \frac{\theta+\phi}{2}$ and $B = \frac{\theta-\phi}{2}$

$\sin\theta + \sin\phi = \sin(A+B) + \sin(A-B)$

$= 2\sin A\cos B$ [from (a)]

$= 2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$

13 $4\tan(\alpha-\beta) = 3\tan\alpha$

$\frac{4(\tan\alpha - \tan\beta)}{1+\tan\alpha\tan\beta} = 3\tan\alpha$

$(4\tan\alpha - 4\tan\beta) = 3\tan\alpha + 3\tan^2\alpha\tan\beta$

$\tan\alpha = \tan\beta(3\tan^2\alpha + 4)$

$\frac{\sin\alpha}{\cos\alpha} = \tan\beta\left(\frac{3\sin^2\alpha}{\cos^2\alpha} + \frac{4\cos^2\alpha}{\cos^2\alpha}\right)$

$\sin\alpha\cos\alpha = \tan\beta(3 - 3\cos^2\alpha + 4\cos^2\alpha)$

$\frac{1}{2}\sin 2\alpha = \tan\beta(3 + \cos^2\alpha)$

$\sin 2\alpha = \tan\beta(6 + 2\cos^2\alpha - 1 + 1)$

$\sin 2\alpha = \tan\beta(7 + \cos 2\alpha)$

$\tan\beta = \frac{\sin 2\alpha}{7 + \cos 2\alpha}$ (as required)

15 $\tan\theta = t$ so $\sin 2\theta = \frac{2t}{1+t^2}$ and $\cos 2\theta = \frac{1-t^2}{1+t^2}$

$(k+1)\sin 2\theta + (k-1)\cos 2\theta = k+1$

$\frac{(k+1)2t}{1+t^2} + \frac{(k-1)(1-t^2)}{1+t^2} = k+1$

$2tk + 2t + k - kt^2 - 1 + t^2 = k + kt^2 + 1 + t^2$

$0 = 2kt^2 - t(2k+2) + 2$

$0 = (2kt-2)(t-1)$

$t = \frac{1}{k}$ or 1

17 $\cos\theta = \frac{l^2 - m^2}{l^2 + m^2} = \frac{1 - \left(\frac{m}{l}\right)^2}{1 + \left(\frac{m}{l}\right)^2}$

So: $\tan\theta = \frac{2\left(\frac{m}{l}\right)}{1 - \left(\frac{m}{l}\right)^2} = \frac{2lm}{l^2 - m^2}$

and: $\sin\theta = \dfrac{2\left(\frac{m}{l}\right)}{1+\left(\frac{m}{l}\right)^2} = \dfrac{2lm}{l^2+m^2}$

and: $\sin 2\theta = 2\times\dfrac{2lm}{l^2+m^2}\times\dfrac{l^2-m^2}{l^2+m^2}$

$$= \frac{4lm(l^2-m^2)}{(l^2+m^2)^2}$$

19 $\text{LHS} = 4\sin\theta\sin\left(\theta-\frac{\pi}{3}\right)\sin\left(\theta-\frac{2\pi}{3}\right)$

$$= 4\sin\theta\left(\frac{1}{2}\sin\theta-\frac{\sqrt{3}}{2}\cos\theta\right)\left(\frac{-1}{2}\sin\theta-\frac{\sqrt{3}}{2}\cos\theta\right)$$

$$= -4\sin\theta\left(\frac{1}{2}\sin\theta-\frac{\sqrt{3}}{2}\cos\theta\right)\left(\frac{1}{2}\sin\theta+\frac{\sqrt{3}}{2}\cos\theta\right)$$

$$= -4\sin\theta\left(\frac{1}{4}\sin^2\theta-\frac{3}{4}\cos^2\theta\right)$$

$$= -4\sin\theta\left(\frac{1}{4}\sin^2\theta-\frac{3}{4}+\frac{3}{4}\sin^2\theta\right)$$

$$= -4\sin\theta\left(\sin^2\theta-\frac{3}{4}\right)$$

$$= 3\sin\theta-4\sin^3\theta$$

and: $\text{RHS} = \sin(2\theta+\theta)$

$$= \sin 2\theta\cos\theta+\cos 2\theta\sin\theta$$
$$= 2\sin\theta\cos^2\theta+(1-2\sin^2\theta)\sin\theta$$
$$= 2\sin\theta-2\sin^3\theta+\sin\theta-2\sin^3\theta$$
$$= 3\sin\theta-4\sin^3\theta$$

So: $4\sin\theta\sin\left(\theta-\frac{\pi}{3}\right)\sin\left(\theta-\frac{2\pi}{3}\right) = \sin 3\theta$ (as required)

EXERCISE 4.5

1 (a) $\sin\theta = \dfrac{\sqrt{3}}{2}$

$\theta = \dfrac{\pi}{3}, \dfrac{2\pi}{3}$

(b) $\tan x = -1$

$x = \dfrac{3\pi}{4}, \dfrac{7\pi}{4}$

(c) $\cos x = -0.5$

$x = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$

(d) $\tan\theta = \dfrac{1}{\sqrt{3}}$

$\theta = \dfrac{\pi}{6}, \dfrac{7\pi}{6}$

(e) $\sin 2\theta = -\dfrac{1}{2}$ $\quad 0\le\theta\le 2\pi$ so $0\le 2\theta\le 4\pi$

$2\theta = \dfrac{7\pi}{6}, \dfrac{11\pi}{6}, \dfrac{19\pi}{6}, \dfrac{23\pi}{6}$

$\theta = \dfrac{7\pi}{12}, \dfrac{11\pi}{12}, \dfrac{19\pi}{12}, \dfrac{23\pi}{12}$

(f) $\operatorname{cosec}\theta = -2$

$\sin\theta = -\dfrac{1}{2}$

$\theta = \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

(g) $\cot 2x = \sqrt{3}$ $\quad 0\le x\le 2\pi$ so $0\le 2x\le 4\pi$

$\tan 2x = \dfrac{1}{\sqrt{3}}$

$2x = \dfrac{\pi}{6}, \dfrac{7\pi}{6}, \dfrac{13\pi}{6}, \dfrac{19\pi}{6}$

$x = \dfrac{\pi}{12}, \dfrac{7\pi}{12}, \dfrac{13\pi}{12}, \dfrac{19\pi}{12}$

(h) $\sec 2\theta = \sqrt{2}$ $\quad 0\le\theta\le 2\pi$ so $0\le 2\theta\le 4\pi$

$\cos 2\theta = \dfrac{1}{\sqrt{2}}$

$2\theta = \dfrac{\pi}{4}, \dfrac{7\pi}{4}, \dfrac{9\pi}{4}, \dfrac{15\pi}{4}$

$\theta = \dfrac{\pi}{8}, \dfrac{7\pi}{8}, \dfrac{9\pi}{8}, \dfrac{15\pi}{8}$

3 C is the correct alternative.

$\sin 2\theta = -\dfrac{1}{\sqrt{2}}$ $\quad 0\le\theta\le 2\pi$ so $0\le 2\theta\le 4\pi$

$2\theta = \dfrac{5\pi}{4}, \dfrac{7\pi}{4}, \dfrac{13\pi}{4}, \dfrac{15\pi}{4}$

$\theta = \dfrac{5\pi}{8}, \dfrac{7\pi}{8}, \dfrac{13\pi}{8}, \dfrac{15\pi}{8}$

A incorrect (see working given)

B incorrect (see working given)

C correct (see working given)

D incorrect (see working given)

5 (a) $\sin\left(\theta+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}} \quad 0\le\theta\le 2\pi$

So: $\frac{\pi}{4}\le\theta+\frac{\pi}{4}\le 2\pi+\frac{\pi}{4}$

$\theta+\frac{\pi}{4}=\frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}$

$\theta=0, \frac{\pi}{2}, 2\pi$

(b) $\tan\left(\theta-\frac{\pi}{3}\right)=-\sqrt{3} \quad 0\le\theta\le 2\pi$

So: $-\frac{\pi}{3}\le\theta-\frac{\pi}{3}\le 2\pi-\frac{\pi}{3}$

$\theta-\frac{\pi}{3}=-\frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3}$

$\theta=0, \pi, 2\pi$

(c) $\cos\left(2x+\frac{\pi}{3}\right)=\frac{1}{2} \quad 0\le x\le 2\pi$

So: $\frac{\pi}{3}\le 2x+\frac{\pi}{3}\le 4\pi+\frac{\pi}{3}$

$2x+\frac{\pi}{3}=\frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}$

$x=0, \frac{2\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$

(d) $\sin\left(2x-\frac{\pi}{6}\right)=\frac{1}{2} \quad 0\le x\le 2\pi$

So: $-\frac{\pi}{6}\le 2x-\frac{\pi}{6}\le 4\pi-\frac{\pi}{6}$

$2x-\frac{\pi}{6}=\frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$

$x=\frac{\pi}{6}, \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}$

(e) $\tan\left(2\theta-\frac{\pi}{4}\right)=-1 \quad 0\le\theta\le 2\pi$

So: $-\frac{\pi}{4}\le 2\theta-\frac{\pi}{4}\le 4\pi-\frac{\pi}{4}$

$2\theta-\frac{\pi}{4}=-\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}$

$\theta=0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

(f) $\cos\left(2x-\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2} \quad 0\le x\le 2\pi$

So: $-\frac{\pi}{3}\le 2x-\frac{\pi}{3}\le 4\pi-\frac{\pi}{3}$

$2x-\frac{\pi}{3}=-\frac{\pi}{6}, \frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}$

$x=\frac{\pi}{12}, \frac{\pi}{4}, \frac{13\pi}{12}, \frac{5\pi}{4}$

(g) $\sin 2\theta=-\frac{1}{\sqrt{2}} \quad 0\le\theta\le 2\pi$

So: $0\le 2\theta\le 4\pi$

$2\theta=\frac{5\pi}{4}, \frac{7\pi}{4}, \frac{13\pi}{4}, \frac{15\pi}{4}$

$\theta=\frac{5\pi}{8}, \frac{7\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$

(h) $\tan x=1 \quad 0\le x\le 2\pi$

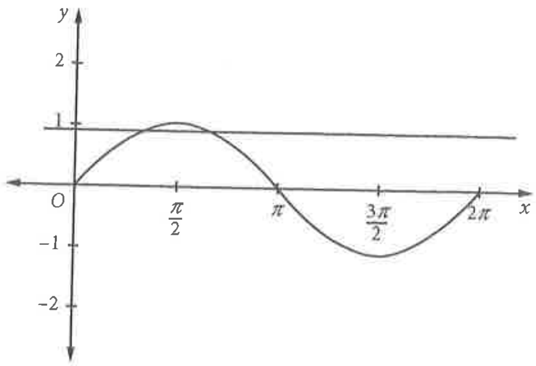
$x=\frac{\pi}{4}, \frac{5\pi}{4}$

7 D is the correct alternative.

$\sin x\le\frac{\sqrt{3}}{2}$

Let $\sin x=\frac{\sqrt{3}}{2}$. Then $x=\frac{\pi}{3}, \frac{2\pi}{3}$.

Answer: $0\le x\le\frac{\pi}{3}, \frac{2\pi}{3}\le x\le 2\pi$



A incorrect (see working given)

B incorrect (see working given)

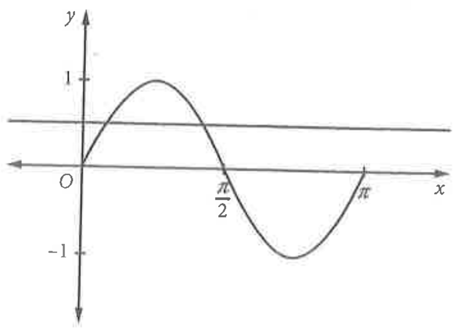
C incorrect (see working given)

D correct (see working given)

9 (a) $\sin 2x\ge\frac{1}{2}, \quad 0<x<\pi$ so $0<2x<2\pi$

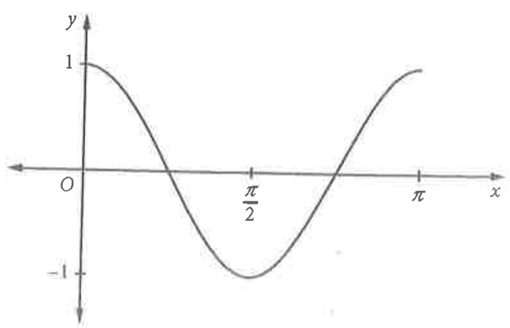
Let $\sin 2x=\frac{1}{2}$: $2x=\frac{\pi}{2}, \frac{5\pi}{6}$ so $x=\frac{\pi}{12}, \frac{5\pi}{12}$

Answer: $\frac{\pi}{12}\le x\le\frac{5\pi}{12}$



(b) $\cos 2x\le 0, \quad 0<x<\pi$ so $0<2x<2\pi$

Let $\cos 2x=0$: $2x=\frac{\pi}{2}, \frac{3\pi}{2}$ so $x=\frac{\pi}{4}, \frac{3\pi}{4}$



Answer: $\frac{\pi}{4}\le x\le\frac{3\pi}{4}$

EXERCISE 4.6

1 (a) $\tan^2 x - 1 = 0$

$\tan x = 1$ or $\tan x = -1$

$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

(b) $\sin^2 x - \sin x = 0$

$\sin x(\sin x - 1) = 0$

$\sin x = 0$ or $\sin x = 1$

$x = 0, \frac{\pi}{2}, \pi, 2\pi$

(c) $\cos^2\theta - 2\cos\theta + 1 = 0$

$(\cos\theta - 1)^2 = 0$

$\cos\theta = 1$

$\theta = 0, 2\pi$

(d) $\sqrt{3}\tan^2 x + \tan x = 0$

$\tan x(\sqrt{3}\tan x + 1) = 0$

$\tan x = 0$ or $\tan x = -\frac{1}{\sqrt{3}}$

$x = 0, \frac{5\pi}{6}, \pi, \frac{11\pi}{6}, 2\pi$

(e) $\sin^2\theta = \frac{1}{4}$

$\sin\theta = \pm\frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

(f) $\sin^2 x - \sin x\cos x = 0$

$\sin x(\sin x - \cos x) = 0$

$\sin x = 0$ or $\tan x = 1$

$x = 0, \frac{\pi}{4}, \pi, \frac{5\pi}{4}, 2\pi$

3 (a) $2\cos^2\theta - 3\cos\theta - 2 = 0$

$(2\cos\theta + 1)(\cos\theta - 2) = 0$

$2\cos\theta + 1 = 0$ or $\cos\theta - 2 = 0$

(Reject second solution because $|\cos\theta| \le 1$.)

$\cos\theta = -\frac{1}{2}$

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

(b) $2\cos^2\theta + \sin\theta = 1$

$2(1 - \sin^2\theta) + \sin\theta = 1$

$2 - 2\sin^2\theta + \sin\theta = 1$

$2\sin^2\theta - \sin\theta - 1 = 0$

$(2\sin\theta + 1)(\sin\theta - 1) = 0$

If: $2\sin\theta + 1 = 0$

$2\sin\theta = -1$

$\sin\theta = -\frac{1}{2}$

Quadrants 3 and 4 with a key angle of $\frac{\pi}{6}$:

$\frac{7\pi}{6}$ and $\frac{11\pi}{6}$

If: $\sin\theta - 1 = 0$

$\sin\theta = 1$

$\theta = \frac{\pi}{2}$

Solution: $\frac{\pi}{2}, \frac{7\pi}{6}$ and $\frac{11\pi}{6}$

(c) $2\sin^2\theta - 3\cos\theta = 2$

$2(1 - \cos^2\theta) - 3\cos\theta - 2 = 0$

$-2\cos^2\theta - 3\cos\theta = 0$

$-\cos\theta(2\cos\theta + 3) = 0$

$\cos\theta = 0$ (Reject $2\cos\theta + 3 = 0$ because $|\cos\theta| \le 1$.)

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

(d) $(2\cos x + 1)(\sin x - 1) = 0$

$\cos x = -\frac{1}{2}$ or $\sin x = 1$

$x = \frac{2\pi}{3}, \frac{4\pi}{3}$ or $x = \frac{\pi}{2}$

5 A is the correct alternative.

$3\sin^2\theta - 4\cos\theta + 1 = 0$

$3 - 3\cos^2\theta - 4\cos\theta + 1 = 0$

$3\cos^2\theta + 4\cos\theta - 4 = 0$

$(3\cos\theta - 2)(\cos\theta + 2) = 0$

$\cos\theta = \frac{2}{3}$ or $\cos\theta = -2$ (Reject second solution because $|\cos\theta| \le 1$.)

$\theta = 0.841, 2\pi - 0.841$

$\theta = 0.841, 5.442$

A correct (see working given)

B incorrect (see working given)

C incorrect (see working given)

D incorrect (see working given)

7 (a) $t^2 - 3 = 0$

$t^2 = 3$

$t = \pm\sqrt{3}$

So: $\tan\frac{\theta}{2} = \pm\sqrt{3}$

If: $\tan\frac{\theta}{2} = \sqrt{3}$

$\frac{\theta}{2} = \frac{\pi}{3}$

Quadrant 3 solution:

$\frac{\theta}{2} = \frac{4\pi}{3}$

$\theta = \frac{8\pi}{3}$

But this is outside the stated range.

If $\tan\frac{\theta}{2} = -\sqrt{3}$, quadrant 2 solution will be:

$\frac{\theta}{2} = \frac{2\pi}{3}$

$\theta = \frac{4\pi}{3}$

Quadrant 4 solution will be outside the range.

Solution: $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$

(b) $t^2 + 2t - 3 = 0$

$(t + 3)(t - 1) = 0$

$t = -3$ or 1

If $t = -3$, this gives a key angle of 1.249 045.

Quadrant 2 solution:

$$\frac{\theta}{2} = \pi - 1.249\,045$$
$$= 1.892\,547$$
$$\theta = 3.785\,094$$

If $t = 1$:

$$\frac{\theta}{2} = \frac{\pi}{4}$$
$$\theta = \frac{\pi}{2}$$

Solution: $\frac{\pi}{2}$ and 3.79

(c) $t^2 - t - 6 = 0$

$(t-3)(t+2) = 0$

$t = 3$ or -2

If $t = 3$: $\frac{\theta}{2} = 1.249\,045\,777$

So: $\theta = 2.498\,091\,555$

If $t = -2$: $\frac{\theta}{2} = \pi - 1.107\,148\,718$

$= 2.034\,443\,936$

$\theta = 4.068\,988\,787\,2$

Solution: 2.50 and 4.07

EXERCISE 4.7

1 (a)
$$\cos 2\theta = \cos\theta$$
$$2\cos^2\theta - 1 - \cos\theta = 0$$
$$(2\cos\theta + 1)(\cos\theta - 1) = 0$$
$$\cos\theta = 1 \quad \text{or} \quad \cos\theta = -\frac{1}{2}$$
$$\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$$

(b)
$$2\cos 2\theta = 4\cos\theta - 3$$
$$4\cos^2\theta - 2 - 4\cos\theta + 3 = 0$$
$$4\cos^2\theta - 4\cos\theta + 1 = 0$$
$$(2\cos\theta - 1)^2 = 0$$
$$\cos\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

(c)
$$3\tan 2\theta = 2\tan\theta$$
$$\frac{6\tan\theta}{1-\tan^2\theta} = 2\tan\theta$$
$$6\tan\theta = 2\tan\theta - 2\tan^3\theta$$
$$2\tan^3\theta + 4\tan\theta = 0$$
$$2\tan\theta(\tan^2\theta + 2) = 0$$
$\tan\theta = 0 \quad \tan^2\theta + 2 = 0$ has no solution
$$\theta = 0, \pi, 2\pi$$

(d)
$$\tan\theta + \frac{2}{\tan\theta} = 3$$
$$\tan^2\theta + 2 - 3\tan\theta = 0$$
$$(\tan\theta - 2)(\tan\theta - 1) = 0$$
$$\tan\theta = 2 \quad \text{or} \quad \tan\theta = 1$$
$$\theta = 45°, 63°26', 243°26', 225°$$

3 (a) $\cos 2x \cos\frac{\pi}{6} - \sin 2x \sin\frac{\pi}{6} = \frac{1}{2}$
$$\cos\left(2x + \frac{\pi}{6}\right) = \frac{1}{2}$$
$$2x + \frac{\pi}{6} = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$$
$$x = \frac{\pi}{12}, \frac{3\pi}{4}, \frac{13\pi}{12}, \frac{7\pi}{4}$$

(b) $\sin 2x \cos\frac{\pi}{3} + \cos 2x \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$
$$\sin\left(2x + \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$
$$2x + \frac{\pi}{3} = -\frac{5\pi}{3}, -\frac{4\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}$$
$$x = -\pi, -\frac{5\pi}{6}, 0, \frac{\pi}{6}, \pi$$

5
$$\tan\theta = \sin 2\theta$$
$$\frac{\sin\theta}{\cos\theta} = 2\sin\theta\cos\theta$$
$$\sin\theta = 2\sin\theta(1 - \sin^2\theta)$$
$$2\sin^3\theta - \sin\theta = 0$$
$$\sin\theta(2\sin^2\theta - 1) = 0$$
$$\sin\theta = 0 \quad \text{or} \quad \sin\theta = \pm\frac{1}{\sqrt{2}}$$
$$\theta = 0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}, 2\pi$$

7 $\sin 3x\cos x - \cos 3x \sin x = \frac{\sqrt{3}}{2}$
$$\sin(3x - x) = \frac{\sqrt{3}}{2}$$
$$\sin 2x = \frac{\sqrt{3}}{2}$$
$$2x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$$
$$x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$$

EXERCISE 4.8

1 (a) $\sin x + \cos x \equiv r\sin(x + \alpha)$

$\equiv r\sin x\cos\alpha + r\cos x\sin\alpha$

Equating coefficients gives:

$r\cos\alpha = 1$ [1]

$r\sin\alpha = 1$ [2]

Squaring equations [1] and [2] and adding gives:

$r^2(\cos^2\alpha + \sin^2\alpha) = 2$

$r = \sqrt{2}$

So: $\cos\alpha = \frac{1}{\sqrt{2}}$ and $\sin\alpha = \frac{1}{\sqrt{2}}$

So $\alpha = \frac{\pi}{4}$.

Hence $\sin x + \cos x = \sqrt{2}\,\sin\left(x + \frac{\pi}{4}\right)$.

(b) $3\sin x + \sqrt{3}\cos x \equiv r\sin(x + \alpha)$

$\equiv r\sin x\cos\alpha + r\cos x\sin\alpha$

Equating coefficients gives:

$r\cos\alpha = 3$ [1]

$r\sin\alpha = \sqrt{3}$ [2]

Squaring equations [1] and [2] and adding gives:

$r^2(\cos^2\alpha + \sin^2\alpha) = 12$

$r = 2\sqrt{3}$

So: $\cos\alpha = \frac{\sqrt{3}}{2}$ and $\sin\alpha = \frac{1}{2}$

So $\alpha = \frac{\pi}{6}$.

Hence $3\sin x + \sqrt{3}\cos x = 2\sqrt{3}\sin\left(x + \frac{\pi}{6}\right)$.

(c) $5\sin x + 12\cos x \equiv r\sin(x + \alpha)$

$\equiv r\sin x\cos\alpha + r\cos x\sin\alpha$

Equating coefficients gives:

$r\cos\alpha = 5$ [1]

$r\sin\alpha = 12$ [2]

Squaring equations [1] and [2] and adding gives:

$r^2(\cos^2\alpha + \sin^2\alpha) = 169$

$r = 13$

So: $\cos\alpha = \frac{5}{13}$ and $\sin\alpha = \frac{12}{13}$

So $\alpha = 67°23'$.

Hence $5\sin x + 12\cos x = 13\sin(x + 67°23')$.

(d) $2\sin x + \cos x \equiv r\sin(x + \alpha)$

$\equiv r\sin x\cos\alpha + r\cos x\sin\alpha$

Equating coefficients gives:

$r\cos\alpha = 2$ [1]

$r\sin\alpha = 1$ [2]

Squaring equations [1] and [2] and adding gives:

$r^2(\cos^2\alpha + \sin^2\alpha) = 5$

$r = \sqrt{5}$

So: $\cos\alpha = \frac{2}{\sqrt{5}}$ and $\sin\alpha = \frac{1}{\sqrt{5}}$

So $\alpha = 26°34'$.

Hence $2\sin x + \cos x = \sqrt{5}\sin(x + 26°34')$.

3 (a) $\cos x + \sin x \equiv r\cos(x - \alpha)$

$\equiv r\cos x\cos\alpha + r\sin x\sin\alpha$

Equating coefficients gives:

$r\cos\alpha = 1$ [1]

$r\sin\alpha = 1$ [2]

Squaring equations [1] and [2] and adding gives:

$r^2(\cos^2\alpha + \sin^2\alpha) = 2$

$r = \sqrt{2}$

So: $\cos\alpha = \frac{1}{\sqrt{2}}$ and $\sin\alpha = \frac{1}{\sqrt{2}}$

So $\alpha = \frac{\pi}{4}$.

Hence $\cos x + \sin x = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right)$.

(b) $24\cos x + 7\sin x \equiv r\cos(x - \alpha)$

$\equiv r\cos x\cos\alpha + r\sin x\sin\alpha$

Equating coefficients gives:

$r\cos\alpha = 24$ [1]

$r\sin\alpha = 7$ [2]

Squaring equations [1] and [2] and adding gives:

$r^2(\cos^2\alpha + \sin^2\alpha) = 625$

$r = 25$

So: $\cos\alpha = \frac{24}{25}$ and $\sin\alpha = \frac{7}{25}$

So $\alpha = 16°16'$.

Hence $24\cos x + 7\sin x = 25\cos(x - 16°16')$.

(c) $2\cos x + 2\sqrt{3}\sin x \equiv r\cos(x - \alpha)$

$\equiv r\cos x\cos\alpha + r\sin x\sin\alpha$

Equating coefficients gives:

$r\cos\alpha = 2$ [1]

$r\sin\alpha = 2\sqrt{3}$ [2]

Squaring equations [1] and [2] and adding gives:

$r^2(\cos^2\alpha + \sin^2\alpha) = 16$

$r = 4$

So: $\cos\alpha = \frac{1}{2}$ and $\sin\alpha = \frac{\sqrt{3}}{2}$

So $\alpha = \frac{\pi}{3}$.

Hence $2\cos x + 2\sqrt{3}\sin x = 4\cos\left(x - \frac{\pi}{3}\right)$.

(d) $3\cos x + 2\sin x \equiv r\cos(x - \alpha)$

$\equiv r\cos x\cos\alpha + r\sin x\sin\alpha$

Equating coefficients gives:

$r\cos\alpha = 3$ [1]

$r\sin\alpha = 2$ [2]

Squaring equations [1] and [2] and adding gives:

$r^2(\cos^2\alpha + \sin^2\alpha) = 13$

$r = \sqrt{13}$

So: $\cos\alpha = \frac{3}{\sqrt{13}}$ and $\sin\alpha = \frac{2}{\sqrt{13}}$

So $\alpha = 33°41'$.

Hence $3\cos x + 2\sin x = \sqrt{13}\cos(x - 33°41')$.

5 **B** is the correct alternative.

$8\sin x - 15\cos x \equiv r\sin(x - \alpha)$

$\equiv r\sin x\cos\alpha - r\cos x\sin\alpha$

Equating coefficients gives:

$r\cos\alpha = 8$ [1]

$r\sin\alpha = 15$ [2]

Squaring equations [1] and [2] and adding gives:

$r^2(\cos^2\alpha + \sin^2\alpha) = 289$

$r = 17$

So: $\cos\alpha = \frac{8}{17}$ and $\sin\alpha = \frac{15}{17}$

So $\alpha = 61°56'$.

Hence $8\sin x - 15\cos x = 17\sin(x - 61°56')$.

A incorrect (see working given)

B correct (see working given)

C incorrect (see working given)

D incorrect (see working given)

7 (a) $\cos x + \sin x = 1$

So: $\sqrt{2}\cos\left(x - \frac{\pi}{4}\right) = 1$ [see working in **3(a)** above]

$$\cos\left(x - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$x - \frac{\pi}{4} = -\frac{\pi}{4}, \frac{\pi}{4}, \frac{7\pi}{4}$$

$$x = 0, \frac{\pi}{2}, 2\pi$$

(b) $\cos x + \sqrt{3}\sin x \equiv r\cos(x - \alpha)$

$\equiv r\cos x\cos\alpha + r\sin x\sin\alpha$

Equating coefficients gives:
$r\cos\alpha = 1$ [1]
$r\sin\alpha = \sqrt{3}$ [2]
Squaring equations [1] and [2] and adding gives:
$r^2(\cos^2\alpha + \sin^2\alpha) = 4$
$r = 2$

So: $\cos\alpha = \frac{1}{2}$ and $\sin\alpha = \frac{\sqrt{3}}{2}$

So $\alpha = \frac{\pi}{3}$.

Hence $\cos x + \sqrt{3}\sin x = 2\cos\left(x - \frac{\pi}{3}\right)$.

$$\cos x + \sqrt{3}\sin x = 2$$
$$2\cos\left(x - \frac{\pi}{3}\right) = 2$$
$$\cos\left(x - \frac{\pi}{3}\right) = 1$$
$$x - \frac{\pi}{3} = 0$$
$$x = \frac{\pi}{3}$$

(c) $3\cos x + 2\sin x \equiv r\cos(x - \alpha)$
$\equiv r\cos x\cos\alpha + r\sin x\sin\alpha$
Equating coefficients gives:
$r\cos\alpha = 3$ [1]
$r\sin\alpha = 2$ [2]
Squaring equations [1] and [2] and adding gives:
$r^2(\cos^2\alpha + \sin^2\alpha) = 13$
$r = \sqrt{13}$

So: $\cos\alpha = \frac{3}{\sqrt{13}}$ and $\sin\alpha = \frac{2}{\sqrt{13}}$

So $\alpha = 33°41'$.

Hence $3\cos x + 2\sin x = \sqrt{13}\cos(x - 33°41')$.

$$3\cos x + 2\sin x = \sqrt{13}$$
$$\sqrt{13}\cos(x - 33°41') = \sqrt{13}$$
$$\cos(x - 33°41') = 1$$
$$x - 33°41' = 0$$
$$x = 33°41'$$

(d) $3\sin x - \sqrt{3}\cos x \equiv r\sin(x - \alpha)$
$\equiv r\sin x\cos\alpha - r\cos x\sin\alpha$
Equating coefficients gives:
$r\cos\alpha = 3$ [1]
$r\sin\alpha = \sqrt{3}$ [2]
Squaring equations [1] and [2] and adding gives:
$r^2(\cos^2\alpha + \sin^2\alpha) = 12$
$r = 2\sqrt{3}$

So: $\cos\alpha = \frac{\sqrt{3}}{2}$ and $\sin\alpha = \frac{1}{2}$

So $\alpha = \frac{\pi}{6}$.

Hence $3\sin x - \sqrt{3}\cos x = 2\sqrt{3}\sin\left(x - \frac{\pi}{6}\right)$.

$$3\sin x - \sqrt{3}\cos x = \sqrt{3}$$
$$2\sqrt{3}\sin\left(x - \frac{\pi}{6}\right) = \sqrt{3}$$
$$\sin\left(x - \frac{\pi}{6}\right) = \frac{1}{2}$$
$$x - \frac{\pi}{6} = \frac{\pi}{6}, \frac{5\pi}{6}$$
$$x = \frac{\pi}{3}, \pi$$

(e) $6\sin x + 8\cos x \equiv r\sin(x + \alpha)$
$\equiv r\sin x\cos\alpha + r\cos x\sin\alpha$
Equating coefficients gives:
$r\cos\alpha = 6$ [1]
$r\sin\alpha = 8$ [2]
Squaring equations [1] and [2] and adding gives:
$r^2(\cos^2\alpha + \sin^2\alpha) = 100$
$r = 10$

So: $\cos\alpha = \frac{3}{5}$ and $\sin\alpha = \frac{4}{5}$

So $\alpha = 53°8'$.

Hence $6\sin x + 8\cos x = 10\sin(53°8')$.

$$6\sin x + 8\cos x = -5$$
$$10\sin(x + 53°8') = -5$$
$$\sin(x + 53°8') = -\frac{1}{2}$$
$$x + 53°8' = 210°, 330°$$
$$x = 156°52', 276°52'$$

(f) $4\cos x + 3\sin x \equiv r\cos(x - \alpha)$
$\equiv r\cos x\cos\alpha + r\sin x\sin\alpha$
Equating coefficients gives:
$r\cos\alpha = 4$ [1]
$r\sin\alpha = 3$ [2]
Squaring equations [1] and [2] and adding gives:
$r^2(\cos^2\alpha + \sin^2\alpha) = 25$
$r = 5$

So: $\cos\alpha = \frac{4}{5}$ and $\sin\alpha = \frac{3}{5}$

So $\alpha = 36°52'$.

Hence $4\cos x + 3\sin x = 5\cos(x - 36°52')$.

$$4\cos x + 3\sin x = -1$$
$$5\cos(x - 36°52') = -1$$
$$\cos(x - 36°52') = -\frac{1}{5}$$
$$x - 36°52' = 101°32', 258°28'$$
$$x = 138°24', 295°20'$$

(g) $\cos x - \sqrt{3}\sin x \equiv r\cos(x + \alpha)$
$\equiv r\cos x\cos\alpha - r\sin x\sin\alpha$
Equating coefficients gives:
$r\cos\alpha = 1$ [1]
$r\sin\alpha = \sqrt{3}$ [2]
Squaring equations [1] and [2] and adding gives:
$r^2(\cos^2\alpha + \sin^2\alpha) = 4$
$r = 2$

So: $\cos\alpha = \frac{1}{2}$ and $\sin\alpha = \frac{\sqrt{3}}{2}$

So $\alpha = \frac{\pi}{3}$.

Hence $\cos x - \sqrt{3}\sin x = 2\cos\left(x + \frac{\pi}{3}\right)$.

$$\cos x - \sqrt{3}\sin x = 2$$
$$2\cos\left(x + \frac{\pi}{3}\right) = 2$$
$$\cos\left(x + \frac{\pi}{3}\right) = 1$$
$$x + \frac{\pi}{3} = 2\pi$$
$$x = \frac{5\pi}{3}$$

(h) $\cos x - \sin x \equiv r\cos(x+\alpha)$

$\equiv r\cos x\cos\alpha - r\sin x\sin\alpha$

Equating coefficients gives:

$r\cos\alpha = 1$ [1]

$r\sin\alpha = 1$ [2]

Squaring equations [1] and [2] and adding gives:

$r^2(\cos^2\alpha + \sin^2\alpha) = 2$

$r = \sqrt{2}$

So: $\cos\alpha = \frac{1}{\sqrt{2}}$ and $\sin\alpha = \frac{1}{\sqrt{2}}$

So $\alpha = \frac{\pi}{4}$.

Hence $\cos x - \sin x = \sqrt{2}\cos\left(x+\frac{\pi}{4}\right)$.

$\cos x - \sin x = -1$

$\sqrt{2}\cos\left(x+\frac{\pi}{4}\right) = -1$

$\cos\left(x+\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

$x+\frac{\pi}{4} = \frac{3\pi}{4}, \frac{5\pi}{4}$

$x = \frac{\pi}{2}, \pi$

(i) $3\sin x + 4\cos x \equiv r\sin(x+\alpha)$

$\equiv r\sin x\cos\alpha + r\cos x\sin\alpha$

Equating coefficients gives:

$r\cos\alpha = 3$ [1]

$r\sin\alpha = 4$ [2]

Squaring equations [1] and [2] and adding gives:

$r^2(\cos^2\alpha + \sin^2\alpha) = 25$

$r = 5$

So: $\cos\alpha = \frac{3}{5}$ and $\sin\alpha = \frac{4}{5}$

So $\alpha = 53°8'$.

Hence $3\sin x + 4\cos x = 5\sin(x + 53°8')$.

$3\sin x + 4\cos x = -2$

$5\sin(x+53°8') = -2$

$\sin(x+53°8') = -\frac{2}{5}$

$x + 53°8' = -23°35', 203°35'$

$x = -76°43', 150°27'$

(j) $\sqrt{2}\sin x - \cos x \equiv r\sin(x-\alpha)$

$\equiv r\sin x\cos\alpha - r\cos x\sin\alpha$

Equating coefficients gives:

$r\cos\alpha = \sqrt{2}$ [1]

$r\sin\alpha = 1$ [2]

Squaring equations [1] and [2] and adding gives:

$r^2(\cos^2\alpha + \sin^2\alpha) = 3$

$r = \sqrt{3}$

So: $\cos\alpha = \frac{\sqrt{2}}{\sqrt{3}}$ and $\sin\alpha = \frac{1}{\sqrt{3}}$

So $\alpha = 35°16'$.

Hence $\sqrt{2}\sin x - \cos x = \sqrt{3}\sin(x - 35°16')$.

$\sqrt{2}\sin x - \cos x = 1.5$

$\sqrt{3}\sin(x-35°16') = 1.5$

$\sin(x-35°16') = \frac{\sqrt{3}}{2}$

$x - 35°16' = 60°, 120°$

$x = 95°16', 155°16'$

9 $\cos x + \sin x \equiv r\cos(x-\alpha)$

$\equiv r\cos x\cos\alpha + r\sin x\sin\alpha$

Equating coefficients gives:

$r\cos\alpha = 1$ [1]

$r\sin\alpha = 1$ [2]

Squaring equations [1] and [2] and adding gives:

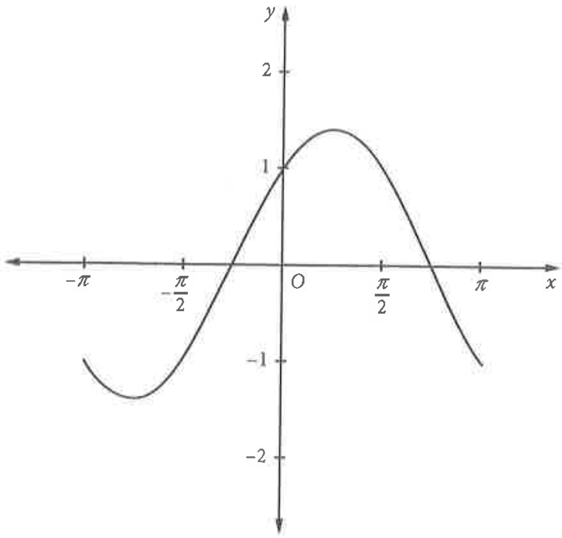
$r^2(\cos^2\alpha + \sin^2\alpha) = 2$

$r = \sqrt{2}$

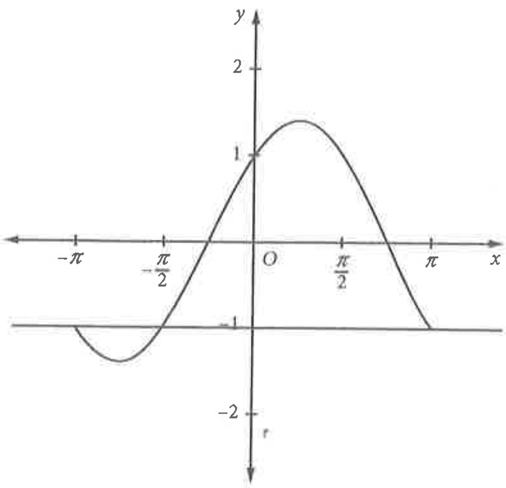
So: $\cos\alpha = \frac{1}{\sqrt{2}}$ and $\sin\alpha = \frac{1}{\sqrt{2}}$

So $\alpha = \frac{\pi}{4}$.

Hence $\cos x + \sin x = \sqrt{2}\cos\left(x-\frac{\pi}{4}\right)$.



(a)



From the graph: $x = -\pi, -\frac{\pi}{2}, \pi$

(b) $x = \pi$ or $-\frac{\pi}{2} \le x \le \pi$

EXERCISE 4.9

1 (a) $\sin x = 1$

$\sin x = \sin\frac{\pi}{2}$

$x = n\pi + (-1)^n \times \frac{\pi}{2}$

(b) $\cos x = 0$

$\cos x = \cos\frac{\pi}{2}$

$x = 2n\pi \pm \frac{\pi}{2}$

(c) $\tan x = -1$

$\tan x = \tan -\frac{\pi}{4}$

$x = n\pi - \frac{\pi}{4}$

(d) $\sqrt{3}\operatorname{cosec} x = 2$

$\sin x = \frac{\sqrt{3}}{2}$

$\sin x = \sin\frac{\pi}{3}$

$x = n\pi + (-1)^n \times \frac{\pi}{3}$

(e) $\sec x = -2$

$\cos x = -\frac{1}{2}$

$\cos x = \cos\frac{2\pi}{3}$

$x = 2n\pi \pm \frac{2\pi}{3}$

(f) $\cot x = \sqrt{3}$

$\tan x = \frac{1}{\sqrt{3}}$

$\tan x = \tan\frac{\pi}{6}$

$x = n\pi + \frac{\pi}{6}$

(g) $2\sin\left(\theta - \frac{\pi}{6}\right) = -1$

$\sin\left(\theta - \frac{\pi}{6}\right) = -\frac{1}{2}$

$\theta - \frac{\pi}{6} = n\pi + (-1)^n \times -\frac{\pi}{6}$

$\theta = n\pi + (-1)^n \times -\frac{\pi}{6} + \frac{\pi}{6}$

(h) $\cos\frac{\theta}{2} = 1$

$\cos\frac{\theta}{2} = \cos 0$

$\theta = 4n\pi$

(i) $\sin^2 x = \frac{1}{2}$

$\sin x = \frac{1}{\sqrt{2}}$ or $\sin x = -\frac{1}{\sqrt{2}}$

$\sin x = \sin\frac{\pi}{4}$ or $\sin x = \sin -\frac{\pi}{4}$

$x = n\pi + (-1)^n \times \frac{\pi}{4}$ or $x = n\pi - (-1)^n \times \frac{\pi}{4}$

$x = n\pi \pm \frac{\pi}{4}$

(j) $\sin x = 0.3894$

Using a calculator, the solution in quadrant 1 is:
$x = 0.399\,98\ldots$

Rounding to $x = 0.4$, the general solution is:
$n\pi + (-1)^n \times 0.4$

3 (a) $\cos^2 x - 2\cos x + 1 = 0$

$(\cos x - 1)^2 = 0$

$\cos x = 1$

$\cos x = \cos 0$

$x = 2n\pi$

(b) $\sin x(\sin x - 1) = 0$

$\sin x = 0$ or $\sin x = 1$

$\sin x = \sin 0$ $\quad \sin x = \sin\frac{\pi}{2}$

$x = n\pi$ or $x = n\pi + (-1)^n \times \frac{\pi}{2}$

(c) $\cos 2\theta = \sin\theta$

$1 - 2\sin^2\theta = \sin\theta$

$2\sin^2\theta + \sin\theta - 1 = 0$

$(2\sin\theta - 1)(\sin\theta + 1) = 0$

$\sin\theta = \frac{1}{2}$ or $\sin\theta = -1$

$\sin\theta = \sin\frac{\pi}{6}$ or $\sin\theta = \sin -\frac{\pi}{2}$

$\theta = n\pi + (-1)^n \times \frac{\pi}{6}$ or $\theta = n\pi + (-1)^n \times -\frac{\pi}{2}$

(d) $\sin^2 x = 1 - \cos x$

$1 - \cos^2 x = 1 - \cos x$

$\cos^2 x - \cos x = 0$

$\cos x(\cos x - 1) = 0$

$\cos x = 0$ or $\cos x = 1$

$x = 2n\pi \pm \frac{\pi}{2}$ or $x = 2n\pi$

(e) $2\cos^2\theta - 1 - \cos\theta - 2 = 0$

$2\cos^2\theta - \cos\theta - 3 = 0$

$(2\cos\theta - 3)(\cos\theta + 1) = 0$

$\cos\theta = -1$ [Reject $(2\cos\theta - 3) = 0$ because $|\cos\theta| \le 1$.]

$\theta = 2n\pi \pm \pi$

(f) $\frac{2\tan x}{1 - \tan^2 x} = \frac{1}{\tan x}$

$2\tan^2 x = 1 - \tan^2 x$

$3\tan^2 x - 1 = 0$

$\tan x = \pm\frac{1}{\sqrt{3}}$

$x = n\pi + \frac{\pi}{6}$ or $x = n\pi - \frac{\pi}{6}$

$x = n\pi \pm \frac{\pi}{6}$

(g) $2\cos^2 x - 1 - \cos x = 0$

$(2\cos x + 1)(\cos x - 1) = 0$

$\cos x = -\frac{1}{2}$ or $\cos x = 1$

$x = 2n\pi \pm \frac{2\pi}{3}$ or $x = 2n\pi$

(h) $2\sin x\cos x = 1$

$\sin 2x = 1$

$2x = n\pi + (-1)^n \times \frac{\pi}{2}$

$x = \frac{n\pi}{2} + (-1)^n \times \frac{\pi}{4}$

(i) $\tan^2 x - \tan x = 0$

$\tan x(\tan x - 1) = 0$

$\tan x = 0$ or $\tan x = 1$

$x = n\pi$ or $x = n\pi + \frac{\pi}{4}$

5 (a) $\sin 2\theta = -\frac{1}{\sqrt{2}}$

$2\theta = n\pi + (-1)^n \times \frac{5\pi}{4}$

$\theta = \frac{n\pi}{2} + (-1)^n \times \frac{5\pi}{8}$

(b) $\tan\left(\theta - \frac{\pi}{3}\right) = -\sqrt{3}$

$\theta - \frac{\pi}{3} = n\pi - \frac{\pi}{3}$

$\theta = n\pi$

(c) $\cos\left(2x + \frac{\pi}{6}\right) = 0.5$

$2x + \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{3}$

$2x = 2n\pi \pm \frac{\pi}{3} - \frac{\pi}{6}$

$x = n\pi \pm \frac{\pi}{6} - \frac{\pi}{12}$

(or $x = n\pi - \frac{\pi}{4}$ and $x = n\pi + \frac{\pi}{12}$)

(d) $\frac{\sin\theta}{\cos\theta} = 2\sin\theta\cos\theta$

$\sin\theta = 2\sin\theta\cos^2\theta$

$\sin\theta(2\cos^2\theta - 1) = 0$

$\sin\theta = 0$ or $\cos\theta = \pm\frac{1}{\sqrt{2}}$

$\theta = n\pi$ or $\theta = 2n\pi \pm \frac{\pi}{4}$ or $\theta = 2n\pi \pm \frac{3\pi}{4}$

(e) $\tan\theta = \frac{1}{\tan\theta}$

$\tan^2\theta = 1$

$\tan\theta = 1$ or $\tan\theta = -1$

$\theta = n\pi + \frac{\pi}{4}$ or $\theta = n\pi - \frac{\pi}{4}$

An equivalent alternative answer is $\theta = \frac{n\pi}{2} - \frac{\pi}{4}$.

(f) $\sqrt{3}\sin x - \cos x = 1$

$\sqrt{3}\sin x - \cos x \equiv r\sin(x - \alpha)$

$\equiv r\sin x\cos\alpha - r\cos x\sin\alpha$

Equating coefficients gives:

$r\cos\alpha = \sqrt{3}$ [1]

$r\sin\alpha = 1$ [2]

Squaring equations [1] and [2] and adding gives:

$r^2(\cos^2\alpha + \sin^2\alpha) = 4$

$r = 2$

So: $\cos\alpha = \frac{\sqrt{3}}{2}$ and $\sin\alpha = \frac{1}{2}$

So $\alpha = \frac{\pi}{6}$.

Hence $\sqrt{3}\sin x - \cos x = 2\sin\left(x - \frac{\pi}{6}\right)$.

$\sqrt{3}\sin x - \cos x = 1$

$2\sin\left(x - \frac{\pi}{6}\right) = 1$

$\sin\left(x - \frac{\pi}{6}\right) = \frac{1}{2}$

$x - \frac{\pi}{6} = n\pi + (-1)^n \times \frac{\pi}{6}$

$x = n\pi + (-1)^n \times \frac{\pi}{6} + \frac{\pi}{6}$

7 (a) $\cos 4x + \cos 2x = 0$

$\cos(3x + x) + \cos(3x - x) = 0$

$\cos 3x\cos x - \sin 3x\sin x$

$+ \cos 3x\cos x + \sin 3x\sin x = 0$

$2\cos 3x\cos x = 0$

$\cos 3x = 0$ or $\cos x = 0$

$3x = 2n\pi \pm \frac{\pi}{2}$ or $x = 2n\pi \pm \frac{\pi}{2}$

$x = \frac{2n\pi}{3} \pm \frac{\pi}{6}$ or $x = 2n\pi \pm \frac{\pi}{2}$

Note that the second set of values for x is included in the first set of values, so the final answer can be $x = \frac{2n\pi}{3} \pm \frac{\pi}{6}$.

An alternative solution could be:

$2\cos^2 2x - 1 + \cos 2x = 0$

$2\cos^2 2x + \cos 2x - 1 = 0$

$(2\cos 2x - 1)(\cos 2x + 1) = 0$

$\cos 2x = \frac{1}{2}$ or $\cos 2x = -1$

$2x = 2n\pi \pm \frac{\pi}{3}$ or $2x = 2n\pi \pm \pi$

$x = n\pi \pm \frac{\pi}{6}$ or $x = n\pi \pm \frac{\pi}{2} = \frac{(2n+1)\pi}{2}$

(The answers are equivalent.)

(b) $\cos 3\theta = \cos\theta$

$\cos(2\theta + \theta) = \cos(2\theta - \theta)$

$\cos 2\theta\cos\theta - \sin 2\theta\sin\theta$

$= \cos 2\theta\cos\theta + \sin 2\theta\sin\theta$

$2\sin 2\theta\sin\theta = 0$

$\sin 2\theta = 0$ or $\sin\theta = 0$

$\theta = \frac{n\pi}{2}$ or $\theta = n\pi$

The second set of values is included in the first set of values, so the final answer can be $\theta = \frac{n\pi}{2}$.

(c) $2(2\cos^2\theta - 1) = 4\cos\theta - 3$

$4\cos^2\theta - 2 - 4\cos\theta + 3 = 0$

$4\cos^2\theta - 4\cos\theta + 1 = 0$

$(2\cos\theta - 1) = 0$

$\theta = 2n\pi \pm \frac{\pi}{3}$

(d) $\frac{6\tan\theta}{1 - \tan^2\theta} = 2\tan\theta$

$6\tan\theta = 2\tan\theta - 2\tan^3\theta$

$$\sin\frac{5x}{2}\cos\frac{x}{2}+\cos\frac{5x}{2}\sin\frac{x}{2} = \sin\frac{5x}{2}\cos\frac{x}{2}-\cos\frac{5x}{2}\sin\frac{x}{2}$$

$$2\cos\frac{5x}{2}\sin\frac{x}{2}=0$$

$$\cos\frac{5x}{2}=0 \quad \text{or} \quad \sin\frac{x}{2}=0$$

$$\frac{5x}{2}=2n\pi\pm\frac{\pi}{2} \quad \text{or} \quad \frac{x}{2}=n\pi$$

Answer: $x=\frac{4n\pi}{5}\pm\frac{\pi}{5}$ or $x=2n\pi$

An alternative solution could be:

$\sin 3x=\sin 2x$

$3x=m\pi+(-1)^m\times 2x$

If m is even, let $m=2n$.

$3x=2n\pi+(-1)^{2n}\times 2x$

$3x=2n\pi+2x$

$x=2n\pi$

If m is odd, let $m=2n+1$.

$3x=(2n+1)\pi+(-1)^{2n+1}\times 2x$

$3x=2n\pi+\pi-2x$

$5x=2n\pi+\pi$

$x=\frac{2n\pi}{5}+\frac{\pi}{5}$

Answer: $x=2n\pi$ or $x=\frac{2n\pi}{5}+\frac{\pi}{5}$

(The answers are equivalent.)

CHAPTER REVIEW 4

1 (a) $\tan\frac{\theta}{2}=t$

Then: $\cos\frac{\theta}{2}=\frac{1}{\sqrt{t^2+1}}$ and $\sin\frac{\theta}{2}=\frac{t}{\sqrt{t^2+1}}$

$$\frac{1-t^2}{1+t^2}=\frac{1}{1+t^2}-\frac{t^2}{1+t^2}$$

$$=\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}$$

$$=\cos\theta$$

(b) $$\frac{\tan\theta-\tan\frac{\pi}{6}}{1+\tan\frac{\pi}{6}\tan\theta}=\tan\left(\theta-\frac{\pi}{6}\right)$$

3 (a) $2\sqrt{3}(\cos\theta\cos\frac{\pi}{6}-\sin\theta\sin\frac{\pi}{6})-2\cos\theta$

$=2\sqrt{3}(\cos\theta\times\frac{\sqrt{3}}{2}-\sin\theta\times\frac{1}{2})-2\cos\theta$

$=3\cos\theta-\sqrt{3}\sin\theta-2\cos\theta$

$=\cos\theta-\sqrt{3}\sin\theta$

$=2(\frac{1}{2}\cos\theta-\frac{\sqrt{3}}{2}\sin\theta)$

$=2(\cos\theta\cos\frac{\pi}{3}-\sin\theta\sin\frac{\pi}{3})$

$=2\cos(\theta+\frac{\pi}{3})$

(b) $2\sqrt{3}(\cos\theta\cos\frac{\pi}{6}-\sin\theta\sin\frac{\pi}{6})-2\cos\theta=1$

$2\cos(\theta+\frac{\pi}{3})=1$

$\cos(\theta+\frac{\pi}{3})=\frac{1}{2}$

$\theta+\frac{\pi}{3}=\frac{\pi}{3},\ 2\pi-\frac{\pi}{3},\ 2\pi+\frac{\pi}{3}$

$\theta=0,\ \frac{4\pi}{3},\ 2\pi$

But $0<\theta<2\pi$ so $\theta=\frac{4\pi}{3}$

5 (a) LHS $=\frac{\cos\theta}{1+\sin\theta}$

$=\frac{\cos\theta\,(1-\sin\theta)}{1-\sin^2\theta}$

$=\frac{\cos\theta\,(1-\sin\theta)}{\cos^2\theta}$

$=\frac{1-\sin\theta}{\cos\theta}$

$=\sec\theta-\tan\theta$

$=$ RHS

(b) RHS $=\frac{1-\cos 2\theta}{1+\cos 2\theta}$

$=\frac{1-(1-2\sin^2\theta)}{1+(2\cos^2\theta-1)}$

$=\frac{\sin^2\theta}{\cos^2\theta}$

$=\tan^2\theta$

$=$ LHS

7 $\tan^2\theta=\frac{1-\cos 2\theta}{1+\cos 2\theta}$

$\tan^2\frac{\pi}{8}=\frac{1-\cos\frac{\pi}{4}}{1+\cos\frac{\pi}{4}}$

$=\left(1-\frac{1}{\sqrt{2}}\right)\div\left(1+\frac{1}{\sqrt{2}}\right)$

$=\frac{\sqrt{2}-1}{\sqrt{2}}\div\frac{\sqrt{2}+1}{\sqrt{2}}$

$=\frac{\sqrt{2}-1}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}+1}$

$=\frac{\sqrt{2}-1}{\sqrt{2}+1}\times\frac{\sqrt{2}-1}{\sqrt{2}-1}$

$=3-2\sqrt{2}$

Let $a-b=\tan\frac{\pi}{8}$, where a is rational and b is irrational.

$3-2\sqrt{2}=(a-b)^2$

$=a^2-2ab+b^2$

So: $a^2+b^2=3$ [1]

and: $2ab=2\sqrt{2}$

$\therefore\ b=\frac{\sqrt{2}}{a}$

Substitute for b in equation [1]:

$a^2+\frac{2}{a^2}=3$

$a^4+2=3a^2$

$a^4-3a^2+2=0$

$(a^2-1)(a^2-2)=0$

$a=1$ or $a=-1$ or $a^2-2=0$

(Reject $a^2-2=0$, it has no rational solution.)

$$2\tan^3\theta + 4\tan\theta = 0$$
$$2\tan\theta(\tan^2\theta + 2) = 0$$
$\tan\theta = 0$ so $\theta = n\pi$ (Reject $\tan^2\theta + 2 = 0$, which has no solution.)

(e) $\tan\left(2\theta - \frac{\pi}{4}\right) = -1$
$$2\theta - \frac{\pi}{4} = n\pi - \frac{\pi}{4}$$
$$\theta = \frac{n\pi}{2}$$

(f) $\cos\left(2x - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$
$$2x - \frac{\pi}{3} = 2n\pi \pm \frac{\pi}{6}$$
$$2x = 2n\pi \pm \frac{\pi}{6} + \frac{\pi}{3}$$
$$x = n\pi \pm \frac{\pi}{12} + \frac{\pi}{6}$$

(g) $\sin\theta - \cos\theta = 1$

$\sin\theta - \cos\theta \equiv r\sin(\theta - \alpha)$

$\equiv r\sin\theta\cos\alpha - r\cos\theta\sin\alpha$

Equating coefficients gives:

$r\cos\alpha = 1$ [1]

$r\sin\alpha = 1$ [2]

Squaring equations [1] and [2] and adding gives:

$r^2(\cos^2\alpha + \sin^2\alpha) = 2$

$r = \sqrt{2}$

So: $\cos\alpha = \frac{1}{\sqrt{2}}$ and $\sin\alpha = \frac{1}{\sqrt{2}}$

So $\alpha = \frac{\pi}{4}$.

Hence $\sin\theta - \cos\theta = \sqrt{2}\sin\left(\theta - \frac{\pi}{4}\right)$.

$\sin\theta - \cos\theta = 1$
$$\sqrt{2}\sin\left(\theta - \frac{\pi}{4}\right) = 1$$
$$\sin\left(\theta - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$
$$\theta - \frac{\pi}{4} = n\pi + (-1)^n \times \frac{\pi}{4}$$
$$\theta = n\pi + (-1)^n \times \frac{\pi}{4} + \frac{\pi}{4}$$

(h) $2(1 - \cos^2 x) + \cos x - 1 = 0$
$$2\cos^2 x - \cos x - 1 = 0$$
$$(2\cos x + 1)(\cos x - 1) = 0$$
$\cos x = -\frac{1}{2}$ or $\cos x = 1$

$x = 2n\pi \pm \frac{2\pi}{3}$ or $x = 2n\pi$

(i) $\cos x - \sqrt{3}\sin x = 1$

$\cos x - \sqrt{3}\sin x \equiv r\cos(x + \alpha)$

$\equiv r\cos x\cos\alpha - r\sin x\sin\alpha$

Equating coefficients gives:

$r\cos\alpha = 1$ [1]

$r\sin\alpha = \sqrt{3}$ [2]

Squaring equations [1] and [2] and adding gives:

$r^2(\cos^2\alpha + \sin^2\alpha) = 4$

$r = 2$

So: $\cos\alpha = \frac{1}{2}$ and $\sin\alpha = \frac{\sqrt{3}}{2}$

So $\alpha = \frac{\pi}{3}$.

Hence $\cos x - \sqrt{3}\sin x = 2\cos\left(x + \frac{\pi}{3}\right)$.

$\cos x - \sqrt{3}\sin x = 1$
$$2\cos\left(x + \frac{\pi}{3}\right) = 1$$
$$\cos\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$$
$$x + \frac{\pi}{3} = 2n\pi \pm \frac{\pi}{3}$$
$$x = 2n\pi \pm \frac{\pi}{3} - \frac{\pi}{3}$$

9 (a) $\tan\theta(\tan^2\theta - 1) = 0$

$\tan\theta = 0$ or $\tan\theta = 1$ or $\tan\theta = -1$

$\theta = n\pi$ or $\theta = n\pi + \frac{\pi}{4}$ or $\theta = n\pi - \frac{\pi}{4}$

(b) $\frac{\sin x}{\cos x} = \sin x$

$\sin x = \sin x\cos x$

$\sin x(\cos x - 1) = 0$

$\sin x = 0$ or $\cos x = 1$

$x = n\pi$ or $x = 2n\pi$

Answer: $x = n\pi$ (as this includes $x = 2n\pi$)

(c) $\frac{1}{\cos 2x} = \frac{1}{\sin 2x}$

$\sin 2x = \cos 2x$

$\tan 2x = 1$
$$2x = n\pi + \frac{\pi}{4}$$
$$x = \frac{n\pi}{2} + \frac{\pi}{8}$$

(d) $2\sin\theta\cos\theta = \frac{\sin\theta}{\cos\theta}$

$2\sin\theta\cos^2\theta - \sin\theta = 0$

$\sin\theta(2\cos^2\theta - 1) = 0$

$\sin\theta = 0$ or $\cos\theta = \frac{1}{\sqrt{2}}$ or $\cos\theta = -\frac{1}{\sqrt{2}}$

$\theta = n\pi$ or $\theta = 2n\pi \pm \frac{\pi}{4}$ or $\theta = 2n\pi \pm \frac{3\pi}{4}$

(e) $\sin 2x + \sin 4x = \sin(3x - x) + \sin(3x + x)$

$= \sin 3x\cos x - \cos 3x\sin x + \sin 3x\cos x + \cos 3x\sin x$

$= 2\sin 3x\cos x$

So $\sin 2x + \sin 4x = \sin 3x$ becomes:

$2\sin 3x\cos x = \sin 3x$

$2\sin 3x\cos x - \sin 3x = 0$

$\sin 3x(2\cos x - 1) = 0$

$\sin 3x = 0$ or $\cos x = \frac{1}{2}$

$3x = n\pi$ or $x = 2n\pi \pm \frac{\pi}{3}$

$x = \frac{n\pi}{3}$ or $x = 2n\pi \pm \frac{\pi}{3}$

(f) $\sin 3x = \sin 2x$
$$\sin\left(\frac{5x}{2} + \frac{x}{2}\right) = \sin\left(\frac{5x}{2} - \frac{x}{2}\right)$$

If $a = 1$: $b = \sqrt{2}$

So: $\tan\frac{\pi}{8} = 1 - \sqrt{2}$

(But this is a negative number, so reject.)

If $a = -1$: $b = -\sqrt{2}$

So: $\tan\frac{\pi}{8} = -1 + \sqrt{2} = \sqrt{2} - 1$

The answer is $\sqrt{2} - 1$.

9
$$\cos 3\theta + \cos 2\theta + \cos\theta = 0$$
$$4\cos^3\theta - 3\cos\theta + 2\cos^2\theta - 1 + \cos\theta = 0$$
$$4\cos^3\theta + 2\cos^2\theta - 2\cos\theta - 1 = 0$$
$$2\cos^2\theta(2\cos\theta + 1) - (2\cos\theta + 1) = 0$$
$$(2\cos^2\theta - 1)(2\cos\theta + 1) = 0$$
$$\cos\theta = \pm\frac{1}{\sqrt{2}} \quad \text{or} \quad \cos\theta = -\frac{1}{2}$$
$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{2\pi}{3}, \frac{4\pi}{3}$$

11
$$\tan(\alpha + \beta + \gamma) = \frac{\tan(\alpha+\beta)+\tan\gamma}{1-\tan(\alpha+\beta)\tan\gamma}$$
$$= \frac{\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}+\tan\gamma}{1-\dfrac{(\tan\alpha+\tan\beta)}{1-\tan\alpha\tan\beta}\tan\gamma}$$
$$= \frac{\tan\alpha + \tan\beta + \tan\gamma(1 - \tan\alpha\tan\beta)}{1 - \tan\alpha\tan\beta - (\tan\alpha + \tan\beta)\tan\gamma}$$
$$= \frac{\tan\alpha+\tan\beta+\tan\gamma-\tan\alpha\tan\beta\tan\gamma}{1-\tan\alpha\tan\beta-\tan\alpha\tan\gamma-\tan\beta\tan\gamma} \qquad [1]$$

If $\tan\alpha$, $\tan\beta$ and $\tan\gamma$ are roots then:

$(x - \tan\alpha)(x - \tan\beta)(x - \tan\gamma) = 0$

$x^3 - x^2(\tan\alpha + \tan\beta + \tan\gamma)$

$+ x(\tan\alpha\tan\beta + \tan\alpha\tan\gamma + \tan\beta\tan\gamma)$

$- \tan\alpha\tan\beta\tan\gamma = 0$

Equating coefficients with

$x^3 - (a + 1)x^2 + (c - a)x - c = 0$ gives:
$$\tan\alpha + \tan\beta + \tan\gamma = a + 1$$
$$\tan\alpha\tan\beta + \tan\alpha\tan\gamma + \tan\beta\tan\gamma = c - a$$
$$\tan\alpha\tan\beta\tan\gamma = c$$

Substituting these values into equation [1] gives:
$$\tan(\alpha + \beta + \gamma) = \frac{a+1-c}{1-(c-a)}$$
$$= \frac{1+a-c}{1-c+a}$$
$$= 1$$

So: $\alpha + \beta + \gamma = n\pi + \frac{\pi}{4}$ (as required)