CHAPTER 9
INTEGRATION TECHNIQUES

EXERCISE 9.1

1 (a)
$$\begin{aligned} u &= x^2 - 1 \\ du &= 2x\,dx \\ \int u^4\,du &= \frac{1}{5}u^5 + C \\ &= \frac{1}{5}(x-1)^5 + C \end{aligned}$$

(b)
$$\begin{aligned} u &= x^3 + 4 \\ du &= 3x^2\,dx \\ \int u^3\,du &= \frac{1}{4}u^4 + C \\ &= \frac{1}{4}(x^3+4)^4 + C \end{aligned}$$

(c)
$$\begin{aligned} u &= x^3 + 1 \\ \frac{1}{3}du &= x^2\,dx \\ \frac{1}{3}\int u^{\frac{1}{2}}\,du &= \frac{2}{9}u^{\frac{3}{2}} + C \\ &= \frac{2}{9}(x^3+1)^{\frac{3}{2}} + C \end{aligned}$$

3 D is the correct alternative.
$$\begin{aligned} u &= 2x + 3 \\ du &= 2\,dx \\ \frac{1}{2}\int u^{-\frac{1}{2}}\,du &= -\frac{1}{4}u^{-2} + C \\ &= \frac{-1}{4(2x+3)^2} + C \end{aligned}$$

5 (a)
$$\begin{aligned} u &= x^3 + 1 \\ du &= 3x^2\,dx \\ \int u^4\,du &= \frac{1}{5}u^5 + C \\ &= \frac{1}{5}(x^3+1)^5 + C \end{aligned}$$

(b)
$$\begin{aligned} u &= 1 - t^2 \\ du &= -2t\,dt \\ -\frac{1}{2}\int u^{-\frac{1}{2}}\,du &= -u^{\frac{1}{2}} + C \\ &= -\sqrt{1-t^2} + C \end{aligned}$$

(c)
$$\begin{aligned} u &= 3x - 5 \\ du &= 3\,dx \\ \frac{1}{3}\int u^{\frac{2}{3}}\,du &= \frac{1}{3}\times\frac{3}{5}u^{\frac{5}{3}} + C \\ &= \frac{1}{5}(3x-5)^{\frac{5}{3}} + C \end{aligned}$$

7 (a)
$$\begin{aligned} u &= y + 1 \\ du &= dy \\ \text{and:}\quad u - 1 &= y \\ \int (u-1)u^{\frac{1}{2}}\,du &= \int (u^{-2} + u^{-3})\,du \\ &= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + C \\ &= \frac{2}{5}(y+1)^{\frac{5}{2}} - \frac{2}{3}(y+1)^{\frac{3}{2}} + C \end{aligned}$$

(b)
$$\begin{aligned} u &= x - 1 \\ du &= dx \\ \text{and:}\quad x &= u + 1 \\ \int \frac{u+1}{u^3}\,du &= \int (u^{-2} + u^{-3})\,du \\ &= -u^{1} - \frac{1}{2}u^{-2} + C \\ &= \frac{-1}{(x-1)} - \frac{1}{2(x-1)^2} + C \end{aligned}$$

(c)
$$\begin{aligned} u &= 2x - 1 \\ du &= 2\,dx \\ \text{and:}\quad x &= \frac{u+1}{2} \\ \frac{1}{2}\int \frac{u+1}{2u^{\frac{1}{2}}}\,du &= \frac{1}{4}\int u^{\frac{1}{2}} + u^{-\frac{1}{2}}\,du \\ &= \frac{1}{4}\left(\frac{2}{3}u^{\frac{3}{2}} + 2u^{\frac{1}{2}}\right) + C \\ &= \frac{1}{6}(2x-1)^{\frac{3}{2}} + \frac{1}{2}(2x-1)^{\frac{1}{2}} + C \end{aligned}$$

9
$$\begin{aligned} u &= x^2 - 4 \\ du &= 2x\,dx \end{aligned}$$

$$y = \frac{1}{2}\int u^{\frac{1}{2}}\,du$$

$$y = \frac{1}{3}u^{\frac{3}{2}} + C$$

$$y = \frac{1}{3}(x^2-4)^{\frac{3}{2}} + C$$

When $y = 2$: $x = \sqrt{5}$

$\therefore C = \frac{5}{3}$

$$y = \frac{1}{3}(x^2-4)^{\frac{3}{2}} + \frac{5}{3}$$

11 $u = t^2 - 2t + 4$
$du = (2t - 2)\,dt$

$$x = \frac{1}{2}\int u^{-\frac{1}{2}}\,du$$
$$= u^{\frac{1}{2}} + C$$
$$= \sqrt{t^2 - 2t + 4} + C$$

When $x = 10$: $t = 0$
$\therefore C = 8$

$$x = \sqrt{t^2 - 2t + 4} + 8$$

13 $u = 2x - 1$
$du = 2\,dx$

$$y = \frac{1}{2}\int u^{\frac{1}{2}}\,du$$
$$y = \frac{1}{3}u^{\frac{3}{2}} + C$$
$$y = \frac{1}{3}(2x - 1)^{\frac{3}{2}} + C$$

When $x = \frac{5}{2}$: $y = 9$

$\therefore C = \frac{19}{3}$

$$y = \frac{1}{3}(2x - 1)^{\frac{3}{2}} + \frac{19}{3}$$

EXERCISE 9.2

1 (a) $u = 1 - x^2$
$du = -2x\,dx$
For $x = 1$, $u = 0$; for $x = 0$, $u = 1$.

$$-\frac{1}{2}\int_1^0 \sqrt{u}\,du = -\frac{1}{2}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^0$$
$$= -\frac{1}{2}\left(0 - \frac{2}{3}\right)$$
$$= \frac{1}{3}$$

(b) $u = 2 - x$
$du = -dx$
and: $x = 2 - u$
For $x = 2$, $u = 0$; for $x = -1$, $u = 3$.

$$-\int_3^0 (2 - u)\sqrt{u}\,du = -\int_3^0 (2u^{\frac{1}{2}} - u^{\frac{3}{2}})\,du$$
$$= -\left[\frac{4}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right]_3^0$$
$$= \frac{2\sqrt{3}}{5}$$

(c) $u = x^2 + 1$
$du = 2x\,dx$
For $x = 2$, $u = 5$; for $x = 0$, $u = 1$.

$$\int_1^5 u^{-\frac{1}{2}}\,du = \left[2u^{\frac{1}{2}}\right]_1^5$$
$$= 2(\sqrt{5} - 1)$$

3 **B** is the correct alternative.
$u = 1 + x^2$
$du = 2x\,dx$
For $x = \sqrt{3}$, $u = 4$; for $x = 0$, $u = 1$.

$$\frac{1}{2}\int_1^4 u^{-\frac{1}{2}}\,du = \frac{1}{2}\left[2u^{\frac{1}{2}}\right]_1^4$$
$$= 1$$

5 (a) $u = 1 + t$
$du = dt$
and: $t = u - 1$
For $t = 1$, $u = 2$; for $t = 0$, $u = 1$.

$$\int_1^2 \frac{u - 1}{\sqrt{u}}\,du = \int_1^2 (u^{\frac{1}{2}} - u^{-\frac{1}{2}})\,du$$
$$= \left[\frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right]_1^2$$
$$= \frac{4 - 2\sqrt{2}}{3}$$

(b) $u = x^3 - 1$
$du = 3x^2\,dx$
For $x = 1$, $u = 0$; for $x = 0$, $u = -1$.

$$\int_{-1}^0 u^4\,du = \left[\frac{1}{5}u^5\right]_{-1}^0$$
$$= \frac{1}{5}$$

(c) $u = a^2 - x^2$
$du = -2x\,dx$
For $x = a$, $u = 0$; for $x = -a$, $u = 0$.

$$-\frac{1}{2}\int_0^0 u^{\frac{1}{2}}\,du = -\frac{1}{2}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^0$$
$$= 0$$

7 (a) $u = y + 1$
$du = dy$
and: $y = u - 1$
For $y = 3$, $u = 4$; for $y = 0$, $u = 1$.

$$\int_1^4 (u - 1)\sqrt{u}\,du = \int_1^4 (u^{\frac{3}{2}} - u^{\frac{1}{2}})\,du$$
$$= \left[\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]_1^4$$
$$= 7\tfrac{11}{15}$$

(b) $u = 16 - x^2$

$du = -2x\,dx$

For $x = 4$, $u = 0$; for $x = 0$, $u = 16$.

$$-\frac{1}{2}\int_{16}^{0} u^{\frac{1}{2}}\,du = -\frac{1}{2}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_{16}^{0}$$
$$= 21\tfrac{1}{3}$$

(c) $u = 3x - 1$

$du = 3\,dx$

and: $x = \dfrac{u+1}{3}$

For $x = 1$, $u = 2$; for $x = -1$, $u = -4$.

$$\frac{1}{3}\int_{-4}^{2}\left[\left(\frac{u+1}{3}\right)u^4\right]du = \frac{1}{9}\int_{-4}^{2}(u^5 + u^4)\,du$$
$$= \frac{1}{9}\left[\frac{1}{6}u^6 + \frac{1}{5}u^5\right]_{-4}^{2}$$
$$= -51\tfrac{1}{5}$$

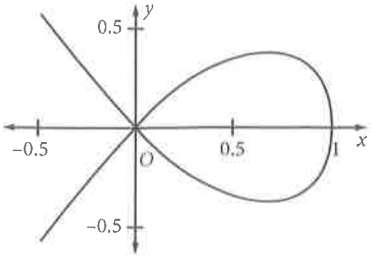
9 $u = 1 - x^2$

$du = -2x\,dx$

For $x = 1$, $u = 0$; for $x = 0$, $u = 1$.

$$A = -\frac{1}{2}\int_{1}^{0} u^{\frac{1}{2}}\,du$$
$$= -\frac{1}{2}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_{1}^{0}$$
$$= \frac{1}{3}\text{units}^2$$

11



$$y = \pm\sqrt{x^2(1-x)}$$

$u = 1 - x$

$du = -dx$

and: $x = 1 - u$

For $x = 1$, $u = 0$; for $x = 0$, $u = 1$.

$$A = 2\int_{0}^{1}(x\sqrt{1-x})\,dx$$
$$= 2\int_{1}^{0}(u^{\frac{3}{2}} - u^{\frac{1}{2}})\,du$$
$$= 2\left[\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]_{1}^{0}$$
$$= \frac{8}{15}\text{units}^2$$

13 $u = x^2 + 4$

$du = 2x\,dx$

For $x = 2\sqrt{3}$, $u = 16$; for $x = 0$, $u = 4$.

$$A = \frac{1}{2}\int_{4}^{16} u^{-\frac{1}{2}}\,du$$
$$= \frac{1}{2}\left[2u^{\frac{1}{2}}\right]_{4}^{16}$$
$$= 2\text{ units}^2$$

EXERCISE 9.3

1 (a) $$2\int\cos^2 x\,dx = 2\times\frac{1}{2}\int(1+\cos 2x)\,dx$$
$$= x + \frac{1}{2}\sin 2x + C$$

(b) $$2\int\sin^2 x\,dx = 2\times\frac{1}{2}\int(1-\cos 2x)\,dx$$
$$= x - \frac{1}{2}\sin 2x + C$$

(c) $$\int\sin^2\frac{x}{2}\,dx = \frac{1}{2}\int(1-\cos x)\,dx$$
$$= \frac{x}{2} - \frac{1}{2}\sin x + C$$

(d) $$2\int\cos^2\frac{x}{2}\,dx = 2\times\frac{1}{2}\int(1+\cos x)\,dx$$
$$= x + \sin x + C$$

(e) $$\int\sin^2 3x\,dx = \frac{1}{2}\int(1-\cos 6x)\,dx$$
$$= \frac{x}{2} - \frac{1}{12}\sin 6x + C$$

(f) $$\int\cos^2 4x\,dx = 2\times\frac{1}{2}\int(1+\cos 8x)\,dx$$
$$= \frac{x}{2} + \frac{1}{16}\sin 8x + C$$

3 C is the correct alternative.

$$V = \pi\int_{0}^{\pi} y^2\,dx$$
$$= \pi\int_{0}^{\pi}\sin^2 x\,dx$$

5 $$V = \pi\int_{0}^{\pi} y^2\,dx$$
$$= \pi\int_{0}^{\pi}\sin^2 x\,dx$$
$$= \pi\times\frac{1}{2}\int_{0}^{\pi}(1-\cos 2x)\,dx$$
$$= \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_{0}^{\pi}$$
$$= \frac{\pi^2}{2}\text{ units}^3$$

7 $\dfrac{dy}{dx} = 2\int \cos^2 x\,dx$

$$= 2 \times \frac{1}{2}\int (1+\cos 2x)\,dx$$

$$= x + \frac{1}{2}\sin 2x + C$$

When $x = \dfrac{\pi}{2}$: $\dfrac{dy}{dx} = 0$

$$\therefore C = -\frac{\pi}{2}$$

$$\frac{dy}{dx} = x + \frac{1}{2}\sin 2x - \frac{\pi}{2}$$

$$y = \int\left(x + \frac{1}{2}\sin 2x - \frac{\pi}{2}\right)dx$$

$$= \frac{1}{2}x^2 - \frac{1}{4}\cos 2x - \frac{\pi}{2}x + C$$

When $x = \dfrac{\pi}{2}$: $y = 0$

$$\therefore C = \frac{\pi^2}{8} - 4$$

$$y = \frac{x^2}{2} - \frac{\pi x}{2} - \frac{\cos 2x}{4} + \frac{\pi^2 - 2}{8}$$

9 $V = \pi\displaystyle\int_0^{\frac{\pi}{12}} y^2\,dx$

$$= \pi\int_0^{\frac{\pi}{12}} \sin^2 3x\,dx$$

$$= \pi \times \frac{1}{2}\int_0^{\frac{\pi}{12}} (1 - \cos 6x)\,dx$$

$$= \frac{\pi}{2}\left[x - \frac{1}{6}\sin 6x\right]_0^{\frac{\pi}{12}}$$

$$= \frac{\pi}{2}\left(\frac{\pi}{12} - \frac{1}{6}\sin\frac{1}{2} - 0 + \frac{1}{6}\sin 0\right)$$

$$= \frac{\pi}{24}(\pi - 2)\text{ units}^3$$

11 $V = \pi\displaystyle\int_0^{\frac{\pi}{4}} y^2\,dx$

$$= \pi\int_0^{\frac{\pi}{4}} (\cos^2 x - \sin^2 x)\,dx$$

$$= \pi\int_0^{\frac{\pi}{4}} \cos(2x)\,dx$$

$$= \frac{\pi}{2}[\sin(2x)]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{2}\left(\sin\frac{\pi}{2} - \sin(0)\right)$$

$$= \frac{\pi}{2}\text{ units}^3$$

EXERCISE 9.4

1 (a) Let $u = \cos x$, so $\dfrac{du}{dx} = -\sin x$.

$$\int \sin x\cos^2 x\,dx = -\int u^2\,du$$

$$= -\frac{1}{3}u^3 + C$$

$$= -\frac{1}{3}\cos^3 x + C$$

(b) Let $u = \tan x$, so $\dfrac{du}{dx} = \sec^2 x$.

$$\int \tan x\sec^2 x\,dx = \int u\,du$$

$$= \frac{1}{2}u^2 + C$$

$$= \frac{1}{2}\tan^2 x + C$$

(c) Let $u = \cos x$, so $\dfrac{du}{dx} = -\sin x$.

$$\int \sin x\cos^3 x\,dx = -\int u^3\,du$$

$$= -\frac{1}{4}u^4 + C$$

$$= -\frac{1}{4}\cos^4 x + C$$

(d) Let $u = \sin x$, so $\dfrac{du}{dx} = \cos x$.

$$\int \cos x\sin^4 x\,dx = \int u^4\,du$$

$$= \frac{1}{5}u^5 + C$$

$$= \frac{1}{5}\sin^5 x + C$$

(e) $\displaystyle\int (1+\cos 2x)\sin x\,dx = \int 2\cos^2 x\sin x\,dx$

Let $u = \cos x$, so $\dfrac{du}{dx} = -\sin x$.

$$\int 2\cos^2 x\sin x\,dx = -2\int u^2\,du$$

$$= -\frac{2}{3}u^3 + C$$

$$= -\frac{2}{3}\cos^3 x + C$$

(f) Let $u = \sin x$, so $\dfrac{du}{dx} = \cos x$.

$$\int (\sin x\cos x)\,dx = \int u\,du$$

$$= -\frac{1}{2}u^2 + C$$

$$= \frac{1}{2}\sin^2 x + C$$

EXERCISE 9.5

1 $x = 4\sin\theta$, $dx = 4\cos\theta\,d\theta$

$\sqrt{16-x^2} = \sqrt{16-16\sin^2 x} = 4\cos\theta$

For $x = 4$, $\theta = \frac{\pi}{2}$; for $x = -4$, $\theta = -\frac{\pi}{2}$.

$$\begin{aligned}\int_{-4}^{4}\sqrt{16-x^2}\,dx &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\cos\theta \times 4\cos\theta\,d\theta\\ &= 16\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2\theta\,d\theta\\ &= 16\times\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+\cos 2\theta)\,d\theta\\ &= 8\left[\theta+\frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\\ &= 8\left[\frac{\pi}{2}+\frac{1}{2}\sin\pi-\left(-\frac{\pi}{2}+\frac{1}{2}\sin\pi\right)\right]\\ &= 8\pi\end{aligned}$$

Semicircle with radius 4.

3 $x = 2\tan\theta$, $dx = 2\sec^2\theta\,d\theta$

$4 + x^2 = 4 + 4\tan^2\theta = 4\sec^2\theta$

For $x = 2$, $\theta = \frac{\pi}{4}$; for $x = 0$, $\theta = 0$.

$$\begin{aligned}\int_0^2 \frac{dx}{4+x^2} &= \int_0^{\frac{\pi}{4}} \frac{1}{4\sec^2\theta}2\sec^2\theta\,d\theta\\ &= \frac{1}{2}\int_0^{\frac{\pi}{4}} 1\,d\theta\\ &= \frac{1}{2}[\theta]_0^{\frac{\pi}{4}}\\ &= \frac{\pi}{8}\end{aligned}$$

5 $$\begin{aligned}\int_{\frac{\pi}{8}}^{\frac{\pi}{4}}(\sin x\cos x)^2\,dx &= \int_{\frac{\pi}{8}}^{\frac{\pi}{4}}\left(\frac{1}{2}\sin 2x\right)^2 dx\\ &= \frac{1}{4}\times\frac{1}{2}\int_{\frac{\pi}{8}}^{\frac{\pi}{4}}(1-\cos 4x)\,dx\\ &= \frac{1}{8}\left[x-\frac{1}{4}\sin 4x\right]_{\frac{\pi}{8}}^{\frac{\pi}{4}}\\ &= \frac{1}{8}\left[\frac{\pi}{4}-\frac{1}{4}\sin\pi-\left(\frac{\pi}{8}-\frac{1}{4}\sin\frac{\pi}{2}\right)\right]\\ &= \frac{\pi+2}{64}\end{aligned}$$

7 $x = \cos\theta$, $dx = -\sin\theta\,d\theta$

$\sqrt{1-x^2} = \sqrt{1-\cos^2 x} = \sin\theta$

For $x = 1$, $\theta = 0$; for $x = \frac{1}{2}$, $\theta = \frac{\pi}{3}$.

$$\begin{aligned}\int_{\frac{\pi}{3}}^{0}\frac{\sin\theta}{\cos^2\theta} - \sin\theta\,d\theta &= -\int_{\frac{\pi}{3}}^{0}\tan^2\theta\,d\theta\\ &= -\int_{\frac{\pi}{3}}^{0}(\sec^2\theta-1)\,d\theta\\ &= -\left[\tan\theta-\theta\right]_{\frac{\pi}{3}}^{0}\\ &= -\left[\tan 0 - 0 - \left(\tan\frac{\pi}{3}-\frac{\pi}{3}\right)\right]\\ &= \sqrt{3}-\frac{\pi}{3}\end{aligned}$$

9 $x = 4\sin\theta$, $dx = 4\cos\theta\,d\theta$

$\sqrt{16-x^2} = \sqrt{16-16\sin^2 x} = 4\cos\theta$

For $x = 4$, $\theta = \frac{\pi}{2}$; for $x = 0$, $\theta = 0$.

$$\begin{aligned}\int_0^4 x\sqrt{16-x^2}\,dx &= \int_0^{\frac{\pi}{2}} 4\sin\theta\,4\cos\theta\,4\cos\theta\,d\theta\\ &= 64\int_0^{\frac{\pi}{2}}\sin\theta\cos^2\theta\,d\theta\end{aligned}$$

$u = \cos\theta$, $du = -\sin\theta\,d\theta$

For $\theta = \frac{\pi}{2}$, $u = 0$; for $\theta = 0$, $u = 1$.

$$\begin{aligned}&= -64\int_1^0 u^2\,d\theta\\ &= -64\left[\frac{1}{3}u^3\right]_1^0\\ &= \frac{64}{3}\end{aligned}$$

11 $u = \tan x + 3$, $du = \sec^2 x\,dx$

For $x = \frac{\pi}{4}$, $u = 4$; for $x = 0$, $u = 3$.

$$\begin{aligned}\int_3^4 \frac{1}{u^4}\,du &= \int_3^4 u^{-4}\,du\\ &= \left[-\frac{1}{3}u^{-3}\right]_3^4\\ &= -\frac{1}{3}\left(\frac{1}{64}-\frac{1}{27}\right)\\ &= \frac{37}{5184}\end{aligned}$$

13 $u = \cos x + \sin x$, $du = (-\sin x + \cos x)\,dx$

$$\begin{aligned}\int\frac{\sin x-\cos x}{\sin x+\cos x}\,dx &= -\int\frac{1}{u}\,du\\ &= -\ln|u| + C\\ &= -\ln|(\cos x+\sin x)| + C\end{aligned}$$

CHAPTER REVIEW 9

1 $u = \sqrt{x}$, $du = \dfrac{1}{2\sqrt{x}}\,dx$

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^u\,du$$
$$= 2e^u + C$$
$$= 2e^{\sqrt{x}} + C$$

3 $u = x^3 + 1$, $du = 3x^2\,dx$
For $x = 1$, $u = 2$; for $x = 0$, $u = 1$.

$$\int_0^1 x^2 e^{x^3+1}\,dx = \frac{1}{3}\int_1^2 e^u\,du$$
$$= \frac{1}{3}\left[e^u\right]_1^2$$
$$= \frac{1}{3}(e^2 - e^1)$$
$$= \frac{e^2 - e}{3}$$

5 $u = \log_e x$, $du = \dfrac{1}{x}\,dx$
For $x = e^3$, $u = 3$; for $x = e$, $u = 1$.

$$\int_3^1 \frac{1}{u^2}\,du = \int_3^1 u^{-2}\,du$$
$$= \left[-\frac{1}{u}\right]_1^3$$
$$= -\frac{1}{3} + 1$$
$$= \frac{2}{3}$$

7 (a) $$\int_0^{\frac{\pi}{6}} \sin^2 x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{6}} (1 - \cos 2x)\,dx$$
$$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{6}}$$
$$= \frac{1}{2}\left[\frac{\pi}{6} - \frac{1}{2}\sin\frac{\pi}{3} - \left(0 - \frac{1}{2}\sin 0\right)\right]$$
$$= \frac{2\pi - 3\sqrt{3}}{24}$$

(b) $$\int_0^{\frac{\pi}{6}} \cos^2 2x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{6}} (1 + \cos 4x)\,dx$$
$$= \frac{1}{2}\left[x + \frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{6}}$$
$$= \frac{1}{2}\left[\frac{\pi}{6} + \frac{1}{4}\sin\frac{2\pi}{3} - \left(0 - \frac{1}{4}\sin 0\right)\right]$$
$$= \frac{4\pi + 3\sqrt{3}}{48}$$

9 (a) $$\frac{d}{dx}[e^{2x}(2\sin x - \cos x)]$$
$$= e^{2x}(2\cos x + \sin x) + 2e^{2x}(2\sin x - \cos x)$$
$$= 5e^{2x}\sin x$$

(b) Hence: $\int 5e^{2x}\sin x\,dx = e^{2x}(2\sin x - \cos x) + C$

$$\therefore \int e^{2x}\sin x\,dx = \frac{1}{5}e^{2x}(2\sin x - \cos x) + C$$

11 $u = 25 - x^2$, $du = -2x\,dx$
For $x = 3$, $u = 16$; for $x = 2$, $u = 21$.

$$\int_2^3 \frac{2x}{\sqrt{25 - x^2}}\,dx = -\int_{21}^{16} u^{-\frac{1}{2}}\,du$$
$$= -\left[2u^{\frac{1}{2}}\right]_{21}^{16}$$
$$= -(2\times\sqrt{16} - 2\times\sqrt{21})$$
$$= 2\sqrt{21} - 8$$

13 $$V = \pi\int_0^{\frac{\pi}{6}} y^2\,dx$$
$$= \pi\int_0^{\frac{\pi}{6}} \sin^2 2x\,dx$$
$$= \pi\times\frac{1}{2}\int_0^{\frac{\pi}{6}} (1 - \cos 4x)\,dx$$
$$= \frac{\pi}{2}\left[x - \frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{6}}$$
$$= \frac{\pi}{2}\left[\frac{\pi}{6} - \frac{1}{4}\sin\frac{2\pi}{3} - \left(0 - \frac{1}{4}\sin 0\right)\right]$$
$$= \frac{4\pi^2 - 3\sqrt{3}\pi}{48}\text{ units}^3$$

15 (a) $\sin\theta\cos 2\theta + \cos\theta\sin 2\theta$
$= \sin\theta(1 - 2\sin^2\theta) + \cos\theta(2\cos\theta\sin\theta)$
$= \sin\theta - 2\sin^3\theta + 2\cos^2\theta\sin\theta$
$= \sin\theta - 2\sin^3\theta + 2\sin\theta(1 - \sin^2\theta)$
$= \sin\theta - 2\sin^3\theta + 2\sin\theta - 2\sin^3\theta$
$= 3\sin\theta - 4\sin^3\theta$ (as required)

(b) $$\int (3\sin\theta - 4\sin^3\theta)\,d\theta = \int \sin 3\theta\,d\theta$$
$$= -\frac{1}{3}\cos 3\theta + C$$