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Special acknowledgement to School of Computing, National University of Singapore for allowing Steven to prepare and distribute these teaching materials.

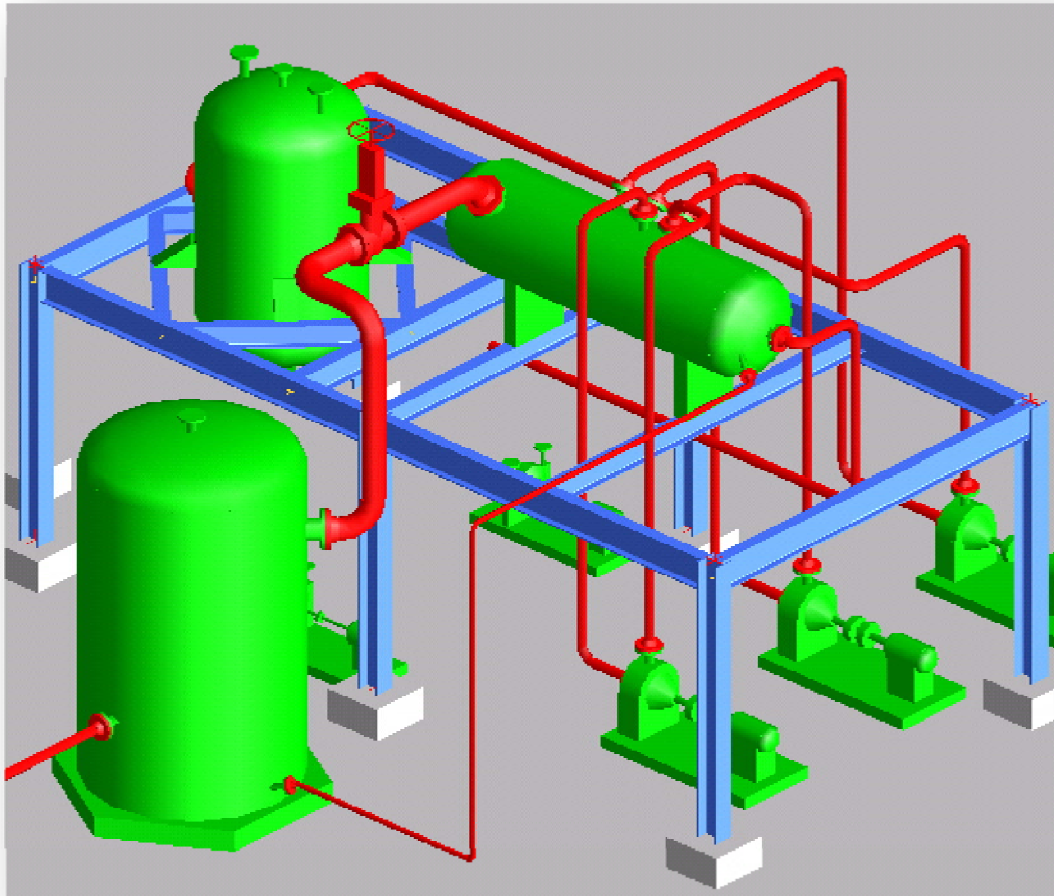
CS3233
Competitive Programming

Dr. Steven Halim

Week 08 – Max Flow and Bipartite Graph
(both are *not* in IOI syllabus 2009)

Outline

- Mini Contest #6 + Break + Discussion + Admins
- Max Flow
  - (We will skip the many variants → read the book!)
- Special Graphs 2
  - Bipartite Graph
  - Finding the bipartite graph in a problem
  - The important Alternating Path algorithm



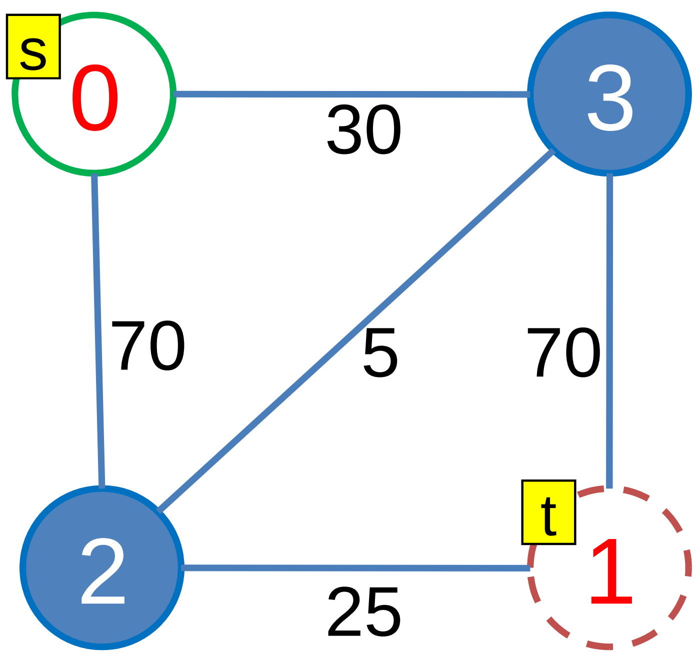
MAX FLOW

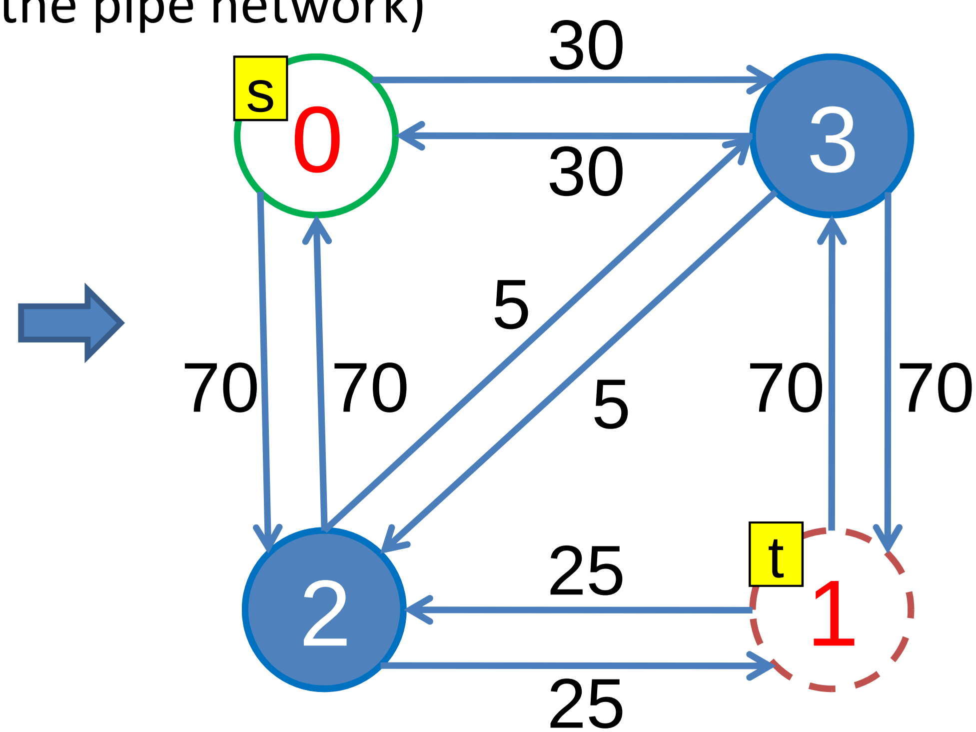
Motivation

- How to solve this UVa problem:
  - 820 (Internet Bandwidth)
    - Similar problems: 259, 753, 10092, 10480, 10511, etc
- Without **Max Flow** algorithms, they look “hard”

Max Flow in a Network

- Imagine **connected**, **weighted**, **directed** graph as *pipe network*
  - The edges are the pipes
  - The vertices are the splitting points
  - There are also two special vertices: source ***s*** & sink ***t***
  - What is the **max flow** (rate) from source ***s*** to sink ***t*** in this graph? (imagine water flowing in the pipe network)



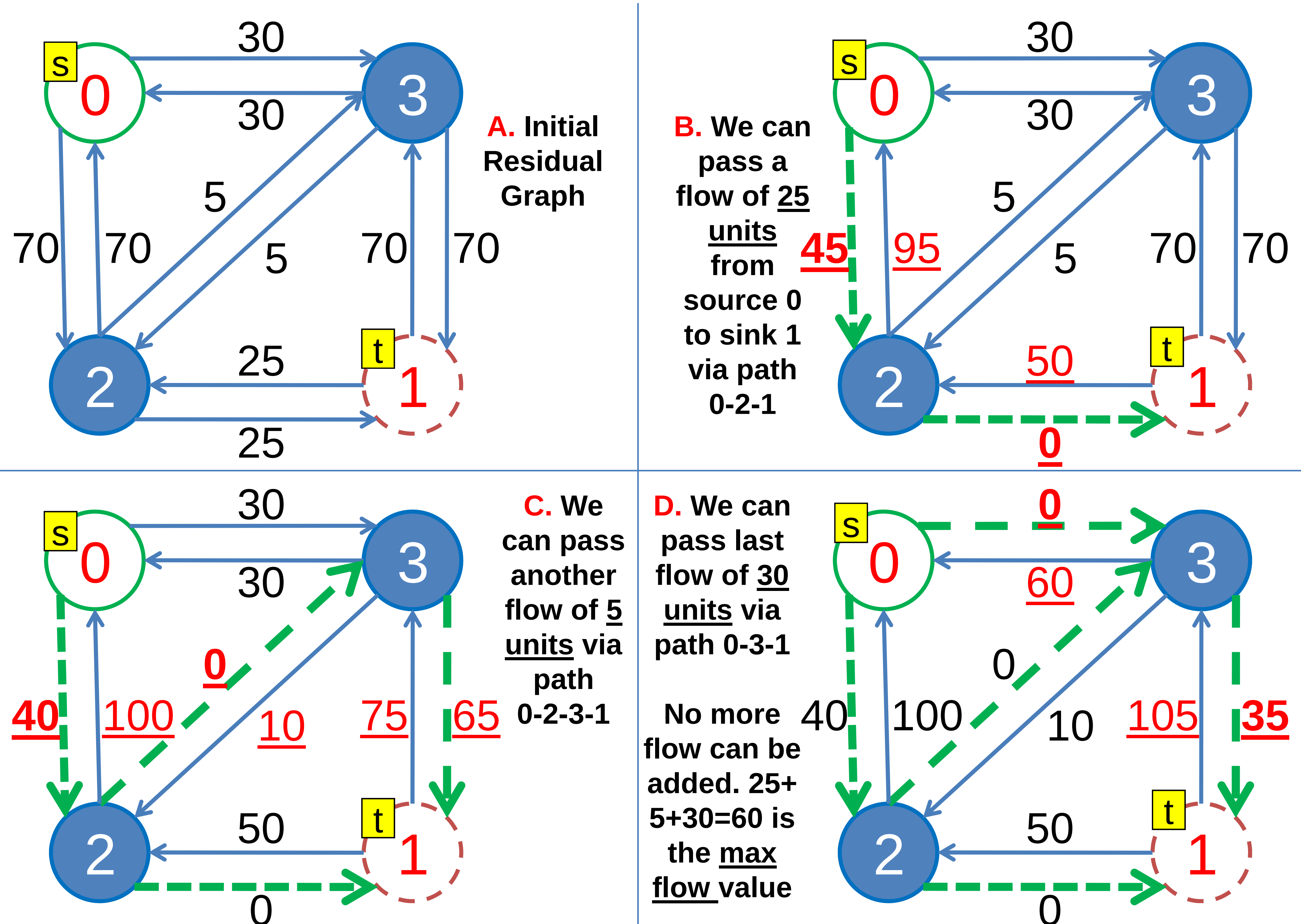


Maximum Flow in Network

- One Solution: Ford Fulkerson's Method

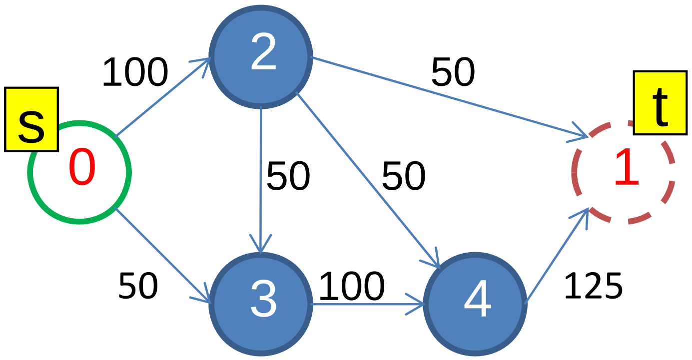
 

  - A surprisingly **simple** *iterative* algorithm
    - Send a flow through path *p* whenever there exists an augmenting path *p* from *s* to *t*
      - *Augmenting path is a path from source s to sink t that pass through positive edges in residual graph*


A. Initial Residual Graph
B. We can pass a flow of 25 units from source 0 to sink 1 via path 0-2-1
C. We can pass another flow of 5 units via path 0-2-3-1
D. We can pass last flow of 30 units via path 0-3-1
No more flow can be added. 25+5+30=60 is the max flow value

UVa 820

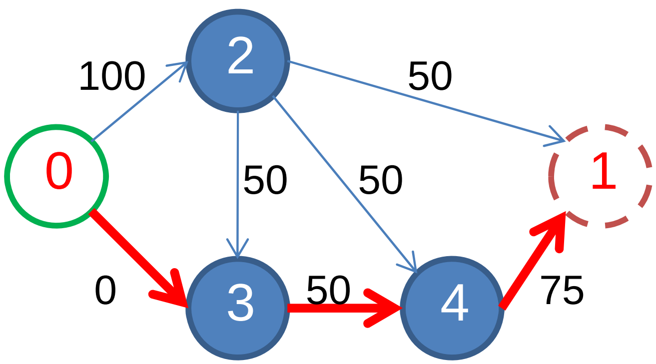
What is the Max Flow value? (1)



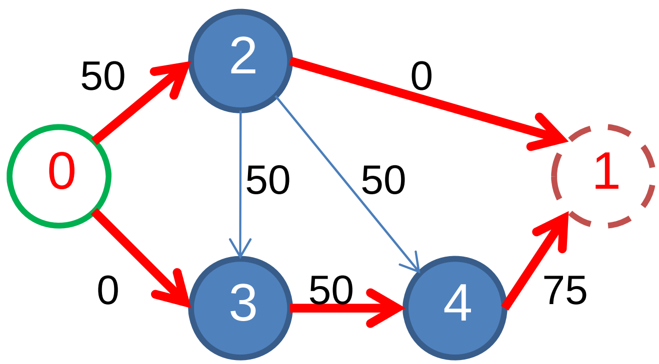
Answer: Maximum flow = 150

Note: **backward edges are not shown…**

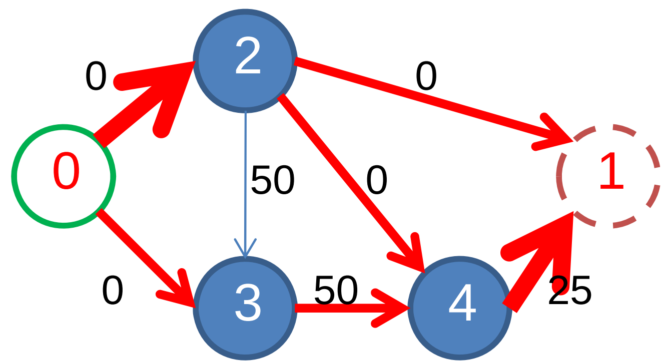
Flow so far = 50



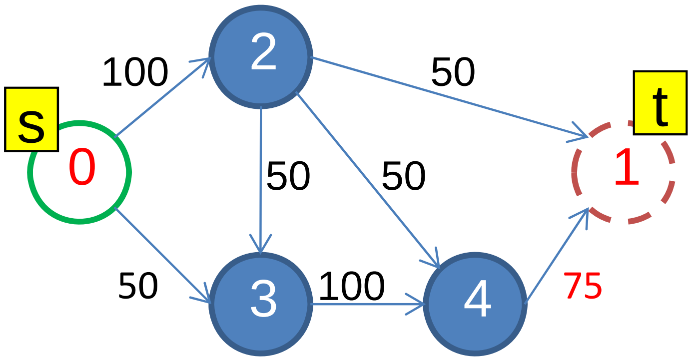
Flow so far = 100



Flow so far = 150



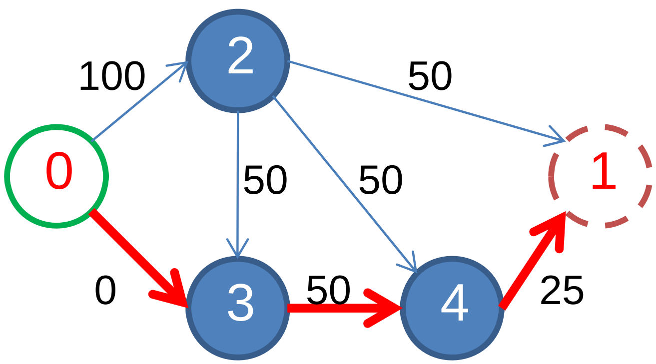
What is the Max Flow value? (2)



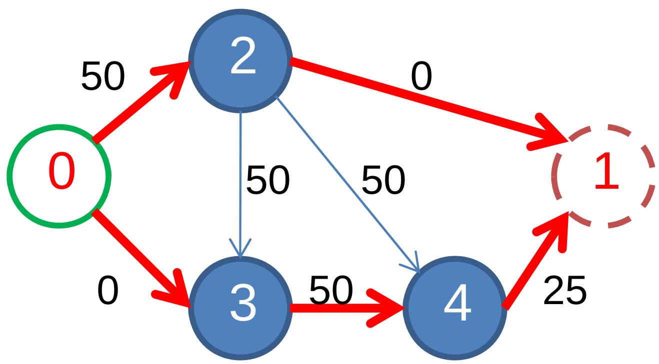
Answer: Maximum flow = 125

Note: **backward edges are not shown…**

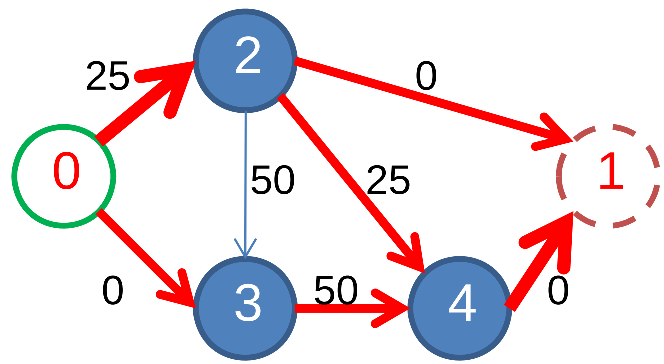
Flow so far = 50



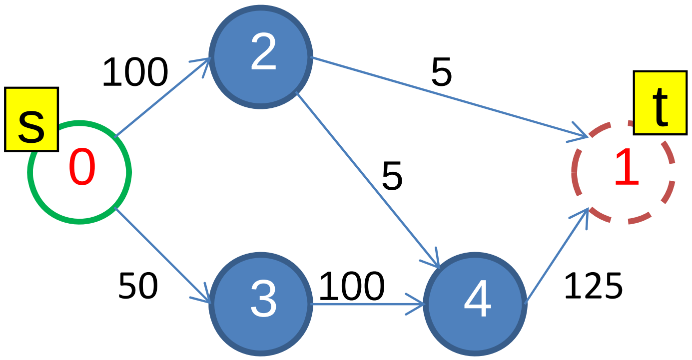
Flow so far = 100



Flow so far = 125



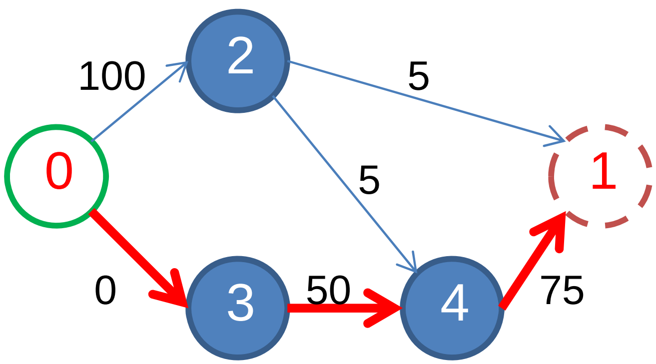
What is the Max Flow value? (3)



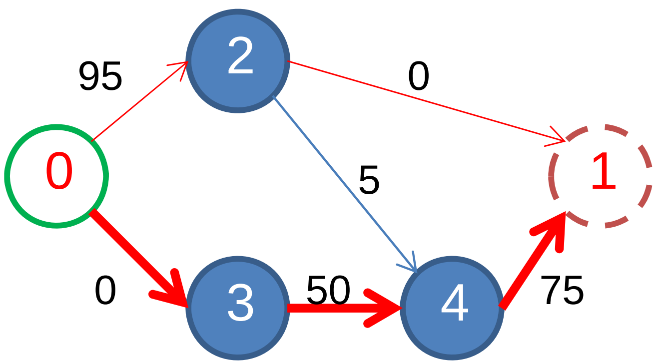
Answer: Maximum flow = 60

Note: **backward edges are not shown…**

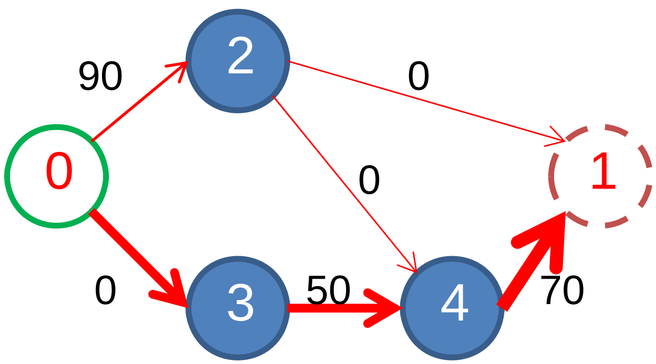
Flow so far = 50



Flow so far = 55



Flow so far = 60



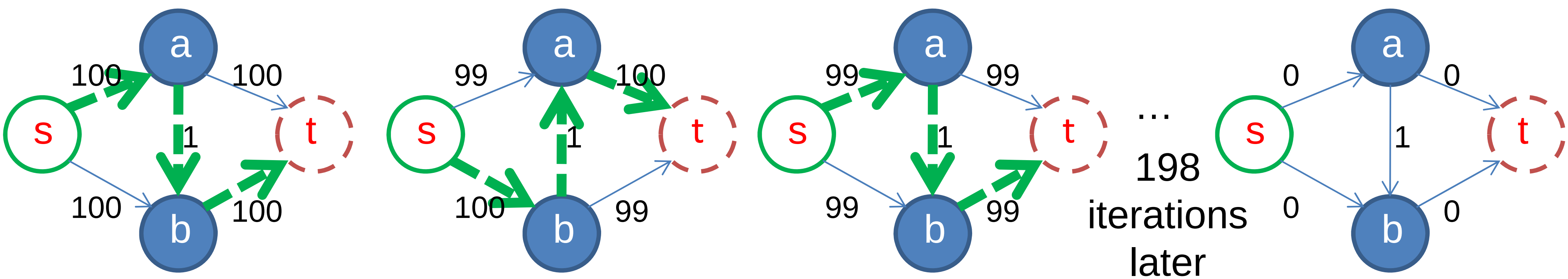
Ford Fulkerson's Method

```
setup directed residual graph
each edge has the same weight with the original graph

mf = 0 // this is an iterative algorithm, mf stands for max_flow
while (there exists an augmenting path p from s to t) {
  // p is a path from s to t that pass through positive edges in residual graph
  augment/send flow f along the path p (s -> ... -> i -> j -> ... t)
    1. find f, the min edge weight along the path p
    2. decrease the weight of forward edges (e.g. i -> j) along path p by f
       reason: obvious, we use the capacities of those forward edges
    3. increase the weight of backward edges (e.g. j -> i) along path p by f
       reason: not so obvious, but this is important for the correctness of Ford
       Fulkerson's method; by increasing the weight/capacity of a backward edge
       (j -> i), we allow later iteration/flow to cancel part of weight/capacity
       used by a forward edge (i -> j) if not all capacity of edge (i -> j) is
       used in the final set of paths that produce the max flow
  mf += f // we can send a flow of size f from s to t, increase mf
}
output mf
```

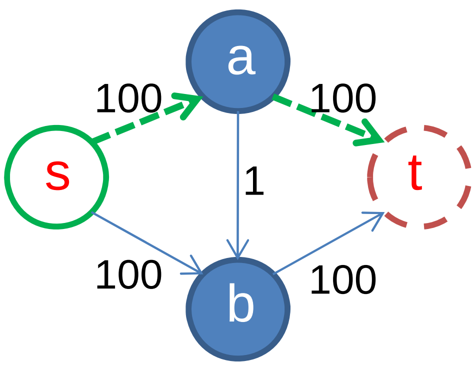
DFS Implementation

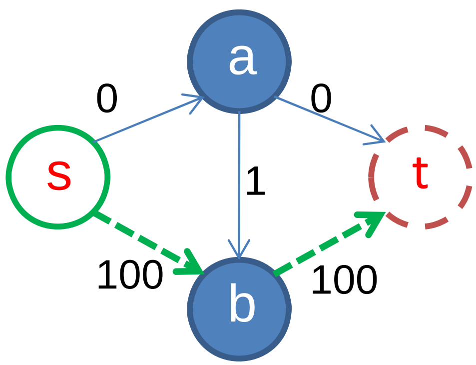
- DFS implementation of Ford Fulkerson's method runs in O(|f*|E) and can be very slow on graph like this:
  - Notice the presence of backward edges (drawn this time)
    - Q: What if we do not use backward edges?



BFS Implementation

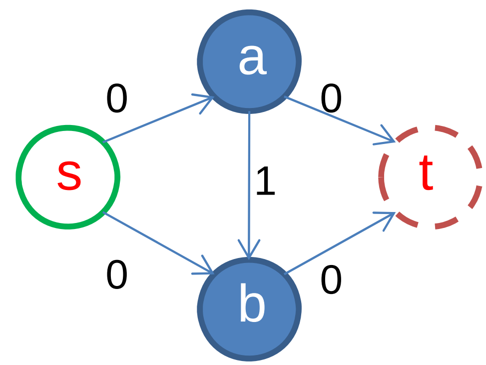
- BFS implementation of Ford Fulkerson's method (called [Edmonds](#) [Karp](#)'s algorithm) runs in $O(VE^2)$





…

After just 2 iterations



Edmonds Karp's (using STL) (1)

```
int res[MAX_V][MAX_V], mf, f, s, t; // global variables
vi p; // note that vi is our shortcut for vector<int>

// traverse the BFS spanning tree as in print_path (section 4.3)
void augment(int v, int minEdge) {
  // reach the source, record minEdge in a global variable 'f'
  if (v == s) { f = minEdge; return; }
  // recursive call
  else if (p[v] != -1) { augment(p[v], min(minEdge, res[p[v]][v]));
  // alter residual capacities
                         res[p[v]][v] -= f; res[v][p[v]] += f; }
}

// in int main()
  // set up the 2d AdjMatrix 'res', 's', and 't' with appropriate values
```

Edmonds Karp's (using STL) (2)

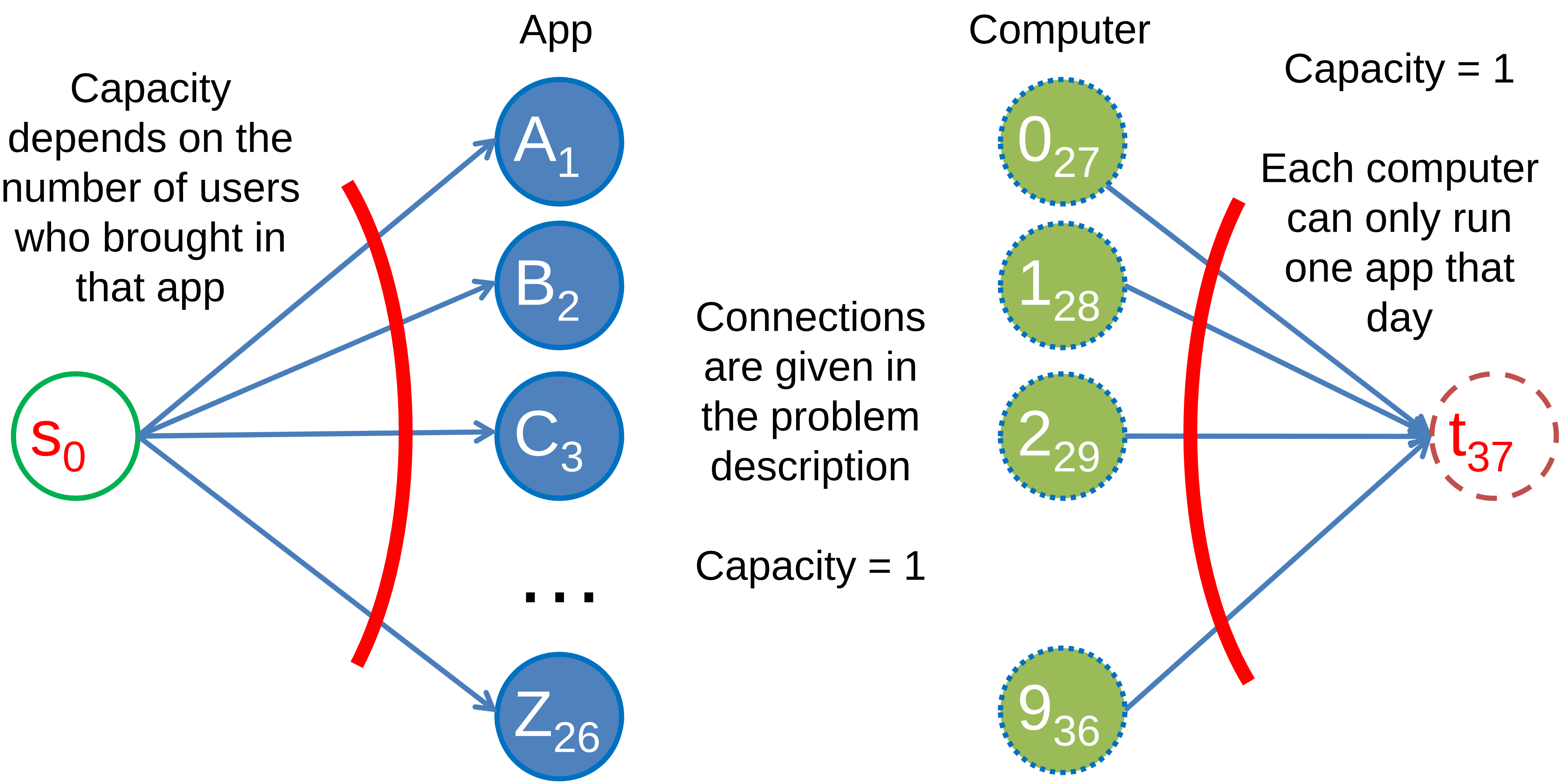
```
mf = 0;
while (1) { // run O(VE * V^2 = V^3*E) Edmonds Karp to solve the Max Flow problem
  f = 0;

  // run BFS, please examine parts of the BFS code that is different than in Section 4.3
  queue<int> q; vi dist(MAX_V, INF); // #define INF 2000000000
  q.push(s); dist[s] = 0;
  p.assign(MAX_V, -1); // (we have to record the BFS spanning tree)
  while (!q.empty()) { // (we need the shortest path from s to t!)
    int u = q.front(); q.pop();
    if (u == t) break; // immediately stop BFS if we already reach sink t
    for (int v = 0; v < MAX_V; v++) // note: enumerating neighbors with AdjMatrix is 'slow'
      if (res[u][v] > 0 && dist[v] == INF) dist[v] = dist[u] + 1, q.push(v), p[v] = u;
  }

  augment(t, INF); // find the min edge weight 'f' along this path, if any
  if (f == 0) break; // if we cannot send any more flow ('f' = 0), terminate the loop
  mf += f; // we can still send a flow, increase the max flow!
}

printf("%d\n", mf); // this is the max flow value of this flow graph
```

UVa 259 – Software Allocation



Other Applications of Max Flow

- Variants:
  - Min Cut
  - Multi-source Multi-sink Max Flow
  - Max Flow with Vertex Capacities
  - Max Independent Path
  - Max Edge-Disjoint Path
  - Min Cost Max Flow (MCMF)
- On "Special Graphs":
  - Max Cardinality Bipartite Matching (MCBM)
  - Max Independent Set on Bipartite Graph (brief)
  - Max Vertex Cover on Bipartite Graph (brief)
  - Min Path Cover on DAG → MCBM on Bipartite Graph

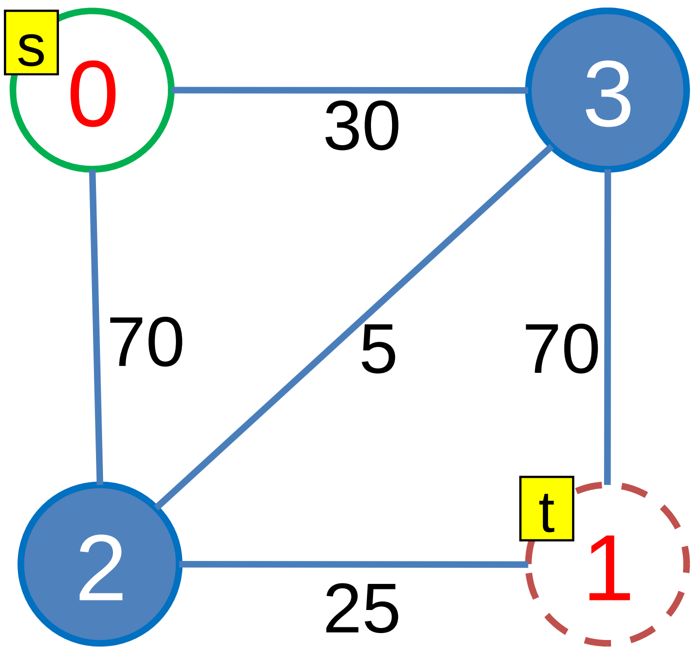
We will focus only on the red ones for this semester's CS3233 ☹

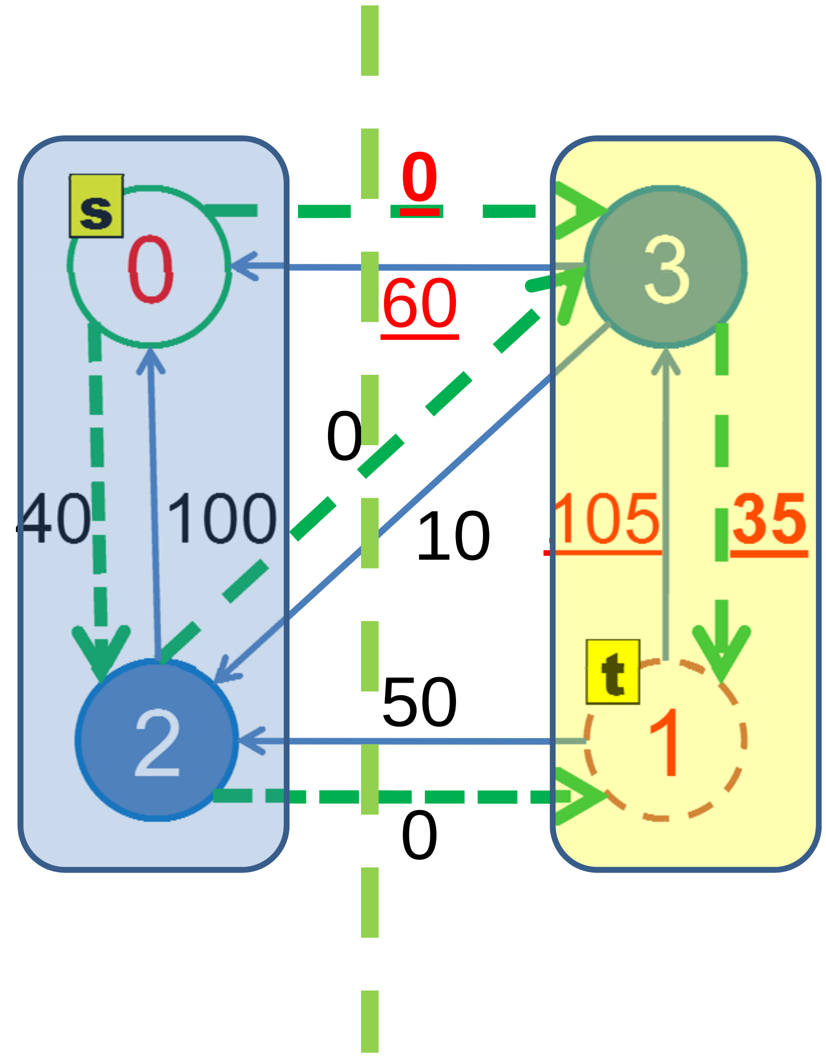
For the rest, read the book!

Max Flow = Min Cut (Dual)

- Min Cut:
  - How to cut the edge(s) of the graph such that we minimizes sum of the edge weights being cut?
    - Note: This is ‘similar’ to finding ‘bridges’, but they are **different**!
- It happens that running max flow algorithm can give us the answer for min cut problem :O
  - Idea: Once we cannot add any more flow (max flow reached), do graph traversal from source s *one more time*
  - All vertices still reachable from s are $\in$ first component, the rest are $\in$ second component, and the cut is obvious
  - Detailed proof is not shown…

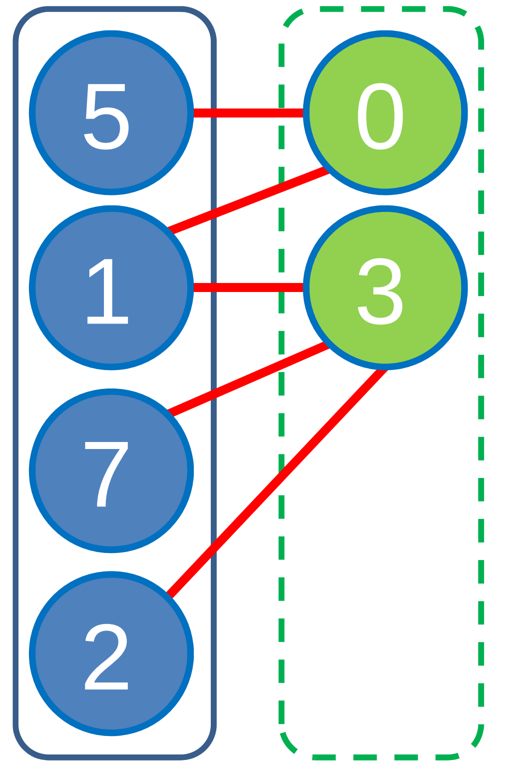
Min Cut Example





Bipartite Graph

SPECIAL GRAPHS 2



Matching
on Bipartite Graph

- In general graph
  - We need to use **Edmonds's Blossom Shrinking** (not discussed today) that is quite hard to code
  - DP + bitmask can only be used if input size is not too large **(N <= 20)**
    - More than that… MLE ☹
- In bipartite graph…
  - Let's see the next few slides ☺

PrimePairs (1)

- Group a list of numbers into pairs s.t the sum of each pair is prime.
- Given the numbers {1, 4, 7, 10, 11, 12}, you could group them as follows:
  1. 1 + 4 = 5   7 + 10 = 17   11 + 12 = 23
  2. 1 + 10 = 11   4 + 7 = 11   11 + 12 = 23
- Task: Given a int[] **numbers**, return a int[] of all the elements in **numbers** that could be paired with **numbers[0]** successfully as part of a *complete* pairing, sorted in ascending order.
- The answer for the example above would be {4, 10}.
  - Even though 1 + 12 is prime, there would be no way to pair the remaining 4 numbers.
- **Constraints:**
  - **numbers** contain an even number of elements in [2 .. 50], inclusive.
  - Each element of **numbers** will be between 1 and 1000, inclusive.
  - Each element of **numbers** will be distinct.

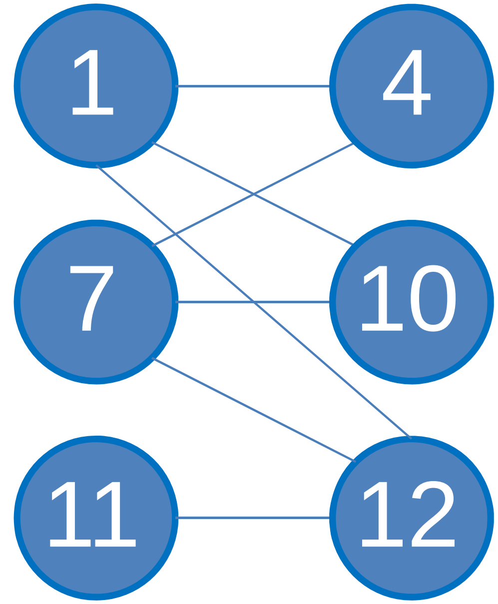
PrimePairs (2)

- Sample test cases (Source: TCO09 Qual 1):
  - Input: {1, 4, 7, 10, 11, 12}, Output: {4, 10}
    - This is the example from the problem statement.
  - Input: {11, 1, 4, 7, 10, 12}, Output: {12}
    - The same numbers, but in a different order.
    - In both of the 2 possible complete pairings, the 11 is paired with the 12.
  - Input: {8, 9, 1, 14}, Output: { }
    - No complete pairings are possible because none of the numbers can be paired with 1.
  - Input: {34, 39, 32, 4, 9, 35, 14, 17}, Output: {9, 39}
  - Input: {941, 902, 873, 841, 948, 851, 945, 854, 815, 898, 806, 826, 976, 878, 861, 919, 926, 901, 875, 864}, Output: {806, 926}

PrimePairs (3)

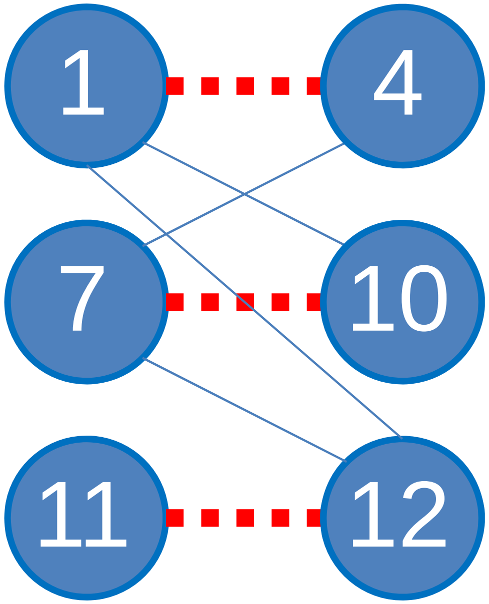
- Is this a Math problem?
  - Yes, there is a bit Math here, we need list of prime…
  - But elements of numbers are ≤ 1000!
    - This is not the major issue
- Complete Search Pairings?
  - ${}_{50}C_2$ for the first pair,
    ${}_{48}C_2$ for the second pair, …,
    until ${}_{2}C_2$ for the last pair?
    - This is too much…
    - This is the major issue

PrimePairs (4)

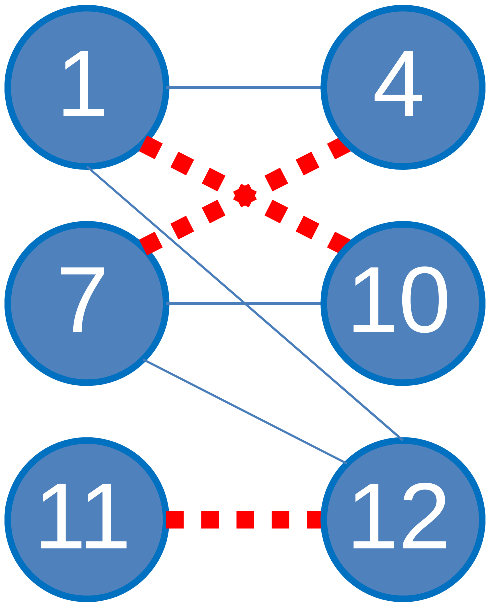


- Any keyword to help?
  - Pairing = Matching, familiar?
- Is this matching bipartite^?
  - YES :D
  - To get a prime, we need to sum 1 odd + 1 even
    - 1 odd + 1 odd = even number → not a prime
    - 1 even + 1 even = even number → not a prime
  - Split odd/even numbers to left/right set
    - Give edge from left to right if left[i] + right[j] = prime

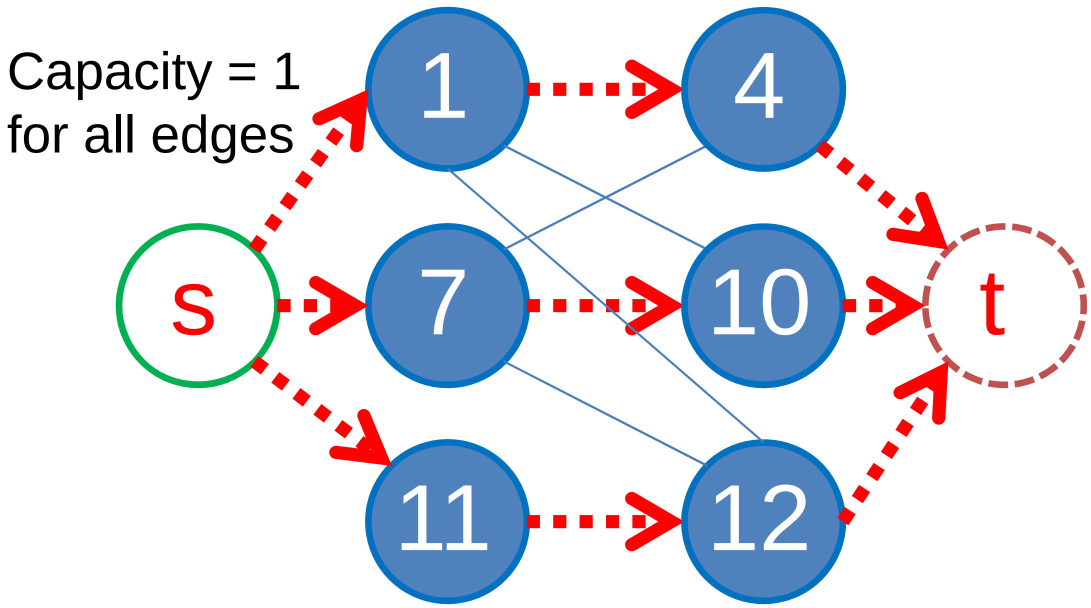
PrimePairs (5)



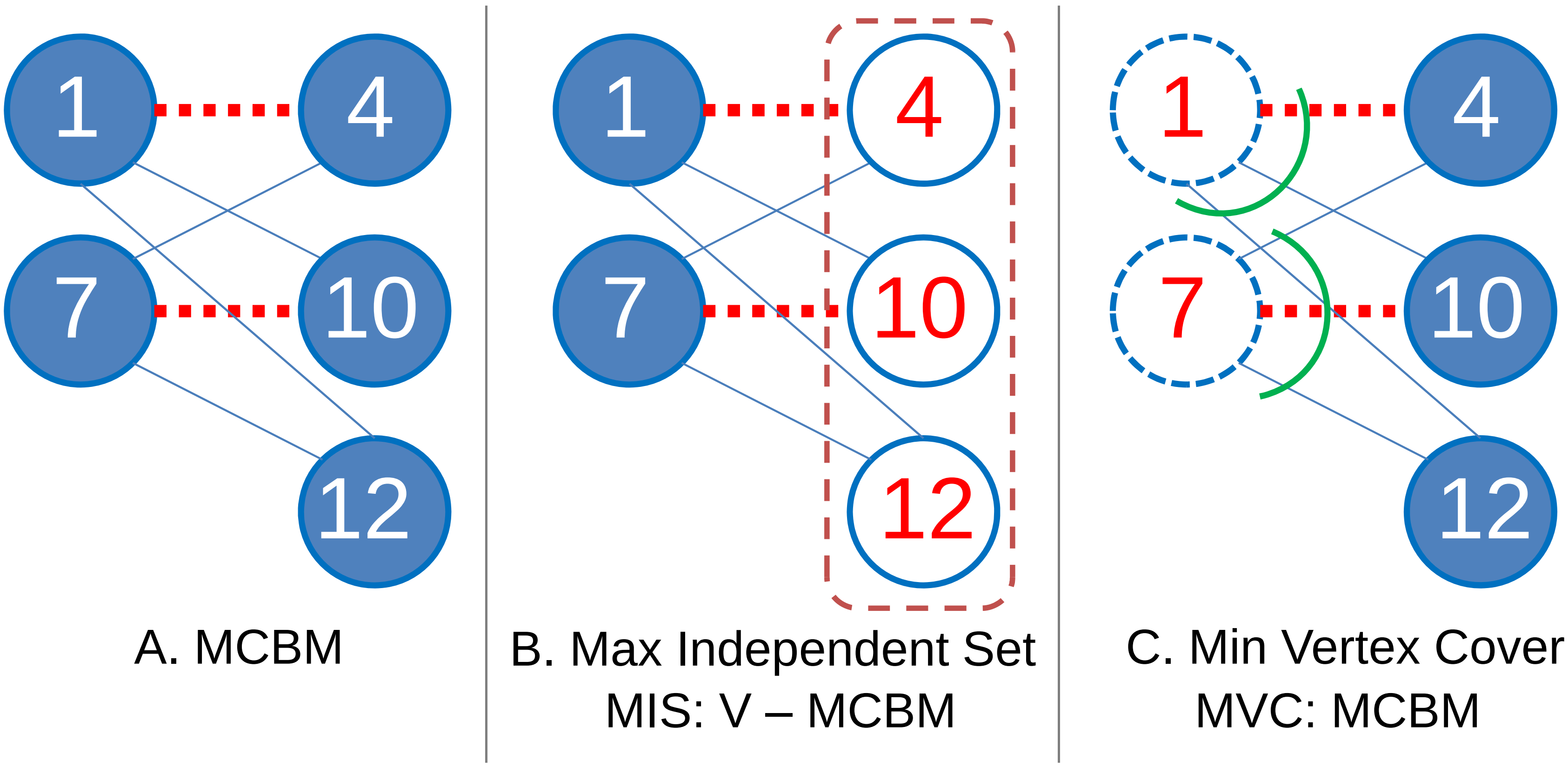
- Solution is then trivial:
  - If size of even and odd set are different
    - Pairing is not possible…
  - Otherwise, if size of both sets are n/2
    - Try to match left[0] with right[k]
      for k = [0 .. n/2 - 1]
      - Do *Bipartite Matching* for the rest
      - If we obtain n/2-1 more matchings
        - » Add right[k] to the answer
    - For this test case,
      we get 1 + 4 and 1 + 10 as the answer



PrimePairs (6)



Can Be Tricky!
Independent Set / Vertex Cover



A. MCBM

B. Max Independent Set
MIS: V – MCBM

C. Min Vertex Cover
MVC: MCBM

(König's theorem)

Graph Theory in ICPC

- Graph problems appear several times in ICPC!
  - Min 1, normally 2, can be 3 out of 10
  - Master all known solutions for classical graph problems
  - Or perhaps combined with DP/Greedy style
- This can move your team nearer to top 10
  - Perhaps rank [11-20] out of 60 now
  - Solving 3-5 problems out of 10

References

- CP2, Chapter 4 ☺
- Introduction to Algorithms, Ch 22,23,24,25,26 (p643-698)
- Algorithm Design, Ch 3,4,6,7 (p337-450)
- Algorithms (Dasgupta et al), Ch 6 & Ch 7
- Algorithms (Sedgewick), Ch 33 & Ch 34
- Algorithms (Alsuwaiyel), Ch 16 & Ch 17
- Programming Challenges, p227-230, Ch 10
- http://www.topcoder.com/tc?module=Static&d1=tutorials&d2=standardTemplateLibrary2
- Internet: TopCoder Max-Flow tutorial, UVa Live Archive, UVa main judge, Felix's blog, Suhendry's blog, Dhaka 2005 solutions, other Max Flow lecture notes, etc…