Project Description

Francesco Sasso

I. Discrete-Time Kalman Filter Algorithm

The problem we are seeking to solve is the continual estimation of a set of parameters whose values change over time. Updating is achieved by combining a set of observations or measurements $m(t)$ which contain information about the signal of interest $x(t)$. The role of the estimator is to provide an estimate $\hat{x}(t+\tau)$ at some time $t+\tau$. In our case of study, the signal will be:

$$x(t) = \begin{bmatrix} p_1(t) \\ p_2(t) \\ v_1(t) \\ v_2(t) \end{bmatrix} \in \mathbb{R}^4,$$

where (p_1, p_2) represent the position of a point particle, while (v_1, v_2) represent its velocity. In the following, we will present the *discrete-time Kalman filter*, which will be used to solve the above mentioned problem.

Notation: Before going on, we introduce some useful ingredients:

$x_t \in \mathbb{R}^4$	system state vector at time t
$m_t \in \mathbb{R}^4$	observation vector at time t
$u_t \in \mathbb{R}^4$	control vector
$\hat{x}_{t\|s} \in \mathbb{R}^4$	estimation of x_t based on time s, with $s \leq t$
$P_{t\|s} \in \mathbb{R}^{4\times 4}$	covariance matrix of $(x_t, \hat{x}_{t\|s})$ given the measurements $m_1, \ldots, m_s$, with $s \leq t$
$K_t \in \mathbb{R}^{4\times 4}$	Kalman gain matrix at time t
$A_t \in \mathbb{R}^{4\times 4}$	state transition matrix
$B_t \in \mathbb{R}^{4\times 4}$	input transition matrix
$H_t \in \mathbb{R}^{4\times 4}$	output transition matrix
$Q_t \in \mathbb{R}^{4\times 4}$	process noise transition matrix
$R_t \in \mathbb{R}^{4\times 4}$	measurement noise transition matrix

At this point, we are ready to describe the algorithm: it generates a sequence $\{(\hat{x}_{t|t}, P_{t|t})\}_{t\geq 0}$ by means of three steps: an initialization step, a prediction step, and a correction step.

Initialization: Set the elements $(\hat{x}_{0|0}, P_{0|0})$.

Prediction: Predicts the state and variance at time $t+1$ dependent on information at time t:

$$\begin{aligned} \hat{x}_{t+1|t} &= A_t \hat{x}_{t|t} + B_t u_t \\ P_{t+1|t} &= A_t P_{t|t} A_t^\top + Q_t \end{aligned}$$

Correction: Corrects the state and variance using a combination of the predicted state and the observation m_{t+1}:

$$\begin{aligned} K_{t+1} &= P_{t+1|t} H_{t+1}^\top (H_t P_{t+1|t} H_t^\top + R_{t+1})^{-1} \\ \hat{x}_{t+1|t+1} &= \hat{x}_{t+1|t} + K_{t+1}(m_{t+1} - H_{t+1}\hat{x}_{t+1|t}) \\ P_{t+1|t+1} &= (I_4 - K_{t+1}H_{t+1})P_{t+1|t}, \end{aligned}$$

where I_4 is the identity matrix of order 4.

Francesco Sasso is with the Department of Engineering, Università del Salento, Lecce, Italy, francesco.sasso@unisalento.it.

After the correction step, the prediction step can be looped in order to obtain new prediction of the state at times $t+2,\ldots$; however, these last are not based on new measurements of the state.

In the practical implementation of the just described algorithm, we used as initial elements:

$$\hat{x}_{0|0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad P_{0|0} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Moreover, we used the following matrices:

$$A_t = \begin{bmatrix} 1 & 0 & .2 & 0 \\ 0 & 1 & 0 & .2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B_t = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad H_t = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad Q_t = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & .1 & 0 \\ 0 & 0 & 0 & .1 \end{bmatrix}, \quad R_t = \begin{bmatrix} .1 & 0 & 0 & 0 \\ 0 & .1 & 0 & 0 \\ 0 & 0 & .1 & 0 \\ 0 & 0 & 0 & .1 \end{bmatrix}.$$