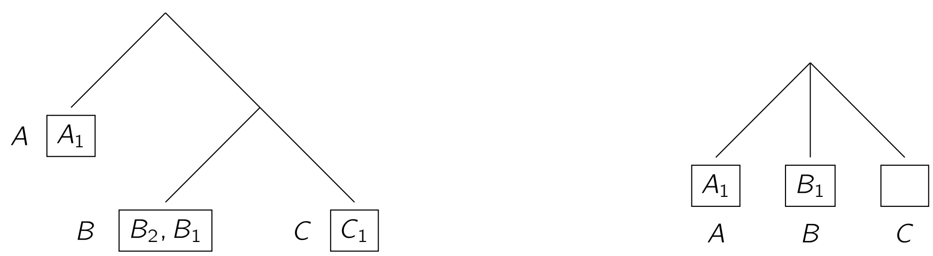
Dequeue-Side Semantics

Disclaimer: we assume familiarity with [MLF+23], adopt its notational conventions, and steal its definitions! Further, we work with a specific subset of *Rio*, denoted **Rio**, namely

$$\frac{c \in \textbf{Class}}{\textbf{edf}[c], \textbf{fifo}[c] \in \textbf{Rio}}\ \text{set2stream} \qquad\qquad \frac{n \in \mathbb{N} \qquad rs \in \textbf{Rio}^n}{\textbf{strict}[rs], \textbf{rr}[rs] \in \textbf{Rio}}\ \text{stream2stream}$$

where **Class** is an opaque collection of *classes*.

1 Structure and Semantics of Rio Trees



(a) Over topology $\text{Node}(*, \text{Node}(*, *))$ (b) Over topology $\text{Node}(*, *, *)$

Figure 1. Rio trees decorated by classes A, B, and C

Definition 1.1. For topology $t \in \textbf{Topo}$, the set $\textbf{RioTree}(t)$ of *Rio trees* over t is defined by

$$\frac{p \in \textbf{PIFO}(\textbf{Pkt}) \qquad c \in \textbf{Class}}{\text{Leaf}(p, c) \in \textbf{RioTree}(*)} \qquad\qquad \frac{ts \in \textbf{Topo}^n \qquad \forall 1 \leq i \leq n.\ qs[i] \in \textbf{RioTree}(ts[i])}{\text{Internal}(qs) \in \textbf{RioTree}(\text{Node}(ts))}$$

These are trees with leaves decorated by both classes and PIFOs.

Definition 1.2. For topology $t \in \textbf{Topo}$, the set $\textbf{OrdTree}(t)$ of *ordered trees* over t is defined by

$$\frac{}{\text{Leaf} \in \textbf{OrdTree}(*)} \qquad\qquad \frac{\begin{array}{c} ts \in \textbf{Topo}^n \qquad rs \in \textbf{Rk}^n \\ \forall 1 \leq i < j \leq n.\ rs[i] \neq rs[j] \qquad \forall 1 \leq i \leq n.\ os[i] \in \textbf{OrdTree}(ts[i]) \end{array}}{\text{Internal}(rs, os) \in \textbf{OrdTree}(\text{Node}(ts))}$$

These are trees with each internal node's child given a *unique* rank, thereby inducing a total ordering of children.

Let $\textbf{flow} : \textbf{Pkt} \to \textbf{Class}$ be an opaque mapping from packets to the class they belong to (flow inference).

Definition 1.3. For $t \in \textbf{Topo}$, define $\text{push} : \textbf{RioTree}(t) \times \textbf{Pkt} \times \textbf{Rk} \rightharpoonup \textbf{RioTree}(t)$ such that

$$\frac{\textbf{flow}(\text{pkt}) = c \qquad \text{push}(p, \text{pkt}, r) = p'}{\text{push}(\text{Leaf}(p, c), \text{pkt}, r) = \text{Leaf}(p', c)} \qquad \frac{\forall 1 \leq i \leq |qs|.\ \ \text{push}(qs[i], \text{pkt}, r) = qs'[i]}{\text{push}(\text{Internal}(qs), \text{pkt}, r) = \text{Internal}(qs')} \qquad \frac{\textbf{flow}(\text{pkt}) \neq c}{\text{push}(\text{Leaf}(p, c), \text{pkt}, r) = \text{Leaf}(p, c)}$$

Informally, we recursively push to all subtrees but only the PIFOs on leaves with the packet's flow are updated.

Definition 1.4. For $t \in \textbf{Topo}$, define $\text{pop} : \textbf{RioTree}(t) \times \textbf{OrdTree}(t) \rightharpoonup \textbf{Pkt} \times \textbf{RioTree}(t)$ such that

$$\frac{\text{pop}(p) = (\text{pkt}, p')}{\text{pop}(\text{Leaf}(p, c), \text{Leaf}) = (\text{pkt}, \text{Leaf}(p', c))} \qquad \frac{\begin{array}{c} \text{pop}(qs[i], os[i]) = (\text{pkt}, q) \\ \forall 1 \leq j \leq |qs|.\ j \neq i \land \text{pop}(qs[j], os[j]) = (\text{pkt}', q') \implies rs[i] < rs[j] \end{array}}{\text{pop}(\text{Internal}(qs), \text{Internal}(rs, os)) = (\text{pkt}, \text{Internal}(qs[q/i]))}$$

Informally, we recursively pop the smallest ranked, poppable subtree.

2 Modelling Scheduling Algorithms

Like [MLF+23], we model scheduling algorithms through a *control*, a thin-layer over the tree policies run on. They keep track of

(1) a state from a fixed collection **St**
(2) an underlying tree of buffered packets
(3) scheduling transactions that update state and construct auxillary structures to push or pop the tree

They come in two flavors: Rio & PIFO Controls. The former determines the order to forward packets at dequeue while latter does so at enqueue.

2.1 Controls

$$\frac{\mathsf{z}_{\text{pre-push}}(s, \mathsf{pkt}) = (r, s') \qquad \mathsf{push}(q, \mathsf{pkt}, r) = q'}{\mathsf{push}((s, q, \mathsf{z}_{\text{pre-push}}, \mathsf{z}_{\text{pre-pop}}, \mathsf{z}_{\text{post-pop}}), \mathsf{pkt}) = (s', q', \mathsf{z}_{\text{pre-push}}, \mathsf{z}_{\text{pre-pop}}, \mathsf{z}_{\text{post-pop}})} \text{ RioCtrl-Push}$$

$$\frac{\mathsf{z}_{\text{pre-pop}}(s) = (o, s') \qquad \mathsf{pop}(q, o) = (\mathsf{pkt}, q') \qquad \mathsf{z}_{\text{post-pop}}(s', \mathsf{pkt}) = s''}{\mathsf{pop}((s, q, \mathsf{z}_{\text{pre-push}}, \mathsf{z}_{\text{pre-pop}}, \mathsf{z}_{\text{post-pop}})) = (\mathsf{pkt}, (s'', q', \mathsf{z}_{\text{pre-push}}, \mathsf{z}_{\text{pre-pop}}, \mathsf{z}_{\text{post-pop}}))} \text{ RioCtrl-Pop}$$

$$\frac{\mathsf{z}_{\text{pre-push}}(s, \mathsf{pkt}) = (pt, s') \qquad \mathsf{push}(q, \mathsf{pkt}, pt) = q'}{\mathsf{push}((s, q, \mathsf{z}_{\text{pre-push}}, \mathsf{z}_{\text{post-pop}}), \mathsf{pkt}) = (s', q', \mathsf{z}_{\text{pre-push}}, \mathsf{z}_{\text{post-pop}})} \text{ PIFOCtrl-Push}$$

$$\frac{\mathsf{pop}(q) = (\mathsf{pkt}, q') \qquad \mathsf{z}_{\text{post-pop}}(s, \mathsf{pkt}) = s'}{\mathsf{pop}((s, q, \mathsf{z}_{\text{pre-push}}, \mathsf{z}_{\text{post-pop}})) = (\mathsf{pkt}, (s', q', \mathsf{z}_{\text{pre-push}}, \mathsf{z}_{\text{post-pop}}))} \text{ PIFOCtrl-Pop}$$

Figure 2. Pushing and Popping Controls

Definition 2.1. Let $t \in \textbf{Topo}$ and *scheduling transactions* $\mathsf{z}_{\text{pre-push}}$ and $\mathsf{z}_{\text{post-pop}}$ be partial functions

$$\begin{aligned} \mathsf{z}_{\text{pre-push}} &: \textbf{St} \times \textbf{Pkt} \rightharpoonup \textbf{Path}(t) \times \textbf{St} \\ \mathsf{z}_{\text{post-pop}} &: \textbf{St} \times \textbf{Pkt} \rightharpoonup \textbf{St} \end{aligned}$$

Define $\textbf{PIFOControl}(t, \mathsf{z}_{\text{pre-push}}, \mathsf{z}_{\text{post-pop}})$ to be the set of quadruples

$$(s, q, \mathsf{z}_{\text{pre-push}}, \mathsf{z}_{\text{post-pop}})$$

where $s \in \textbf{St}$ and well-formed $q \in \textbf{PIFOTree}(t)$.

Definition 2.2. Let $t \in \textbf{Topo}$ and *scheduling transactions* $\mathsf{z}_{\text{pre-push}}, \mathsf{z}_{\text{pre-pop}}, \mathsf{z}_{\text{post-pop}}$ be partial functions

$$\begin{aligned} \mathsf{z}_{\text{pre-push}} &: \textbf{St} \times \textbf{Pkt} \rightharpoonup \textbf{Rk} \times \textbf{St} \\ \mathsf{z}_{\text{pre-pop}} &: \textbf{St} \rightharpoonup \textbf{OrdTree}(t) \times \textbf{St} \\ \mathsf{z}_{\text{post-pop}} &: \textbf{St} \times \textbf{Pkt} \rightharpoonup \textbf{St} \end{aligned}$$

Define $\textbf{RioControl}(t, \mathsf{z}_{\text{pre-push}}, \mathsf{z}_{\text{pre-pop}}, \mathsf{z}_{\text{post-pop}})$ to be the set of quintuples

$$(s, q, \mathsf{z}_{\text{pre-push}}, \mathsf{z}_{\text{pre-pop}}, \mathsf{z}_{\text{post-pop}})$$

where $s \in \textbf{St}$ and $q \in \textbf{RioTree}(t)$.

Both controls admit push and pop operations:

$$\begin{aligned} \mathsf{push} &: \textbf{PIFOControl}(t, \mathsf{z}_{\text{pre-push}}, \mathsf{z}_{\text{post-pop}}) \times \textbf{Pkt} \rightharpoonup \textbf{PIFOControl}(t, \mathsf{z}_{\text{pre-push}}, \mathsf{z}_{\text{post-pop}}) \\ \mathsf{pop} &: \textbf{PIFOControl}(t, \mathsf{z}_{\text{pre-push}}, \mathsf{z}_{\text{post-pop}}) \rightharpoonup \textbf{Pkt} \times \textbf{PIFOControl}(t, \mathsf{z}_{\text{pre-push}}, \mathsf{z}_{\text{post-pop}}) \\ \mathsf{push} &: \textbf{RioControl}(t, \mathsf{z}_{\text{pre-push}}, \mathsf{z}_{\text{pre-pop}}, \mathsf{z}_{\text{post-pop}}) \times \textbf{Pkt} \rightharpoonup \textbf{RioControl}(t, \mathsf{z}_{\text{pre-push}}, \mathsf{z}_{\text{pre-pop}}, \mathsf{z}_{\text{post-pop}}) \\ \mathsf{pop} &: \textbf{RioControl}(t, \mathsf{z}_{\text{pre-push}}, \mathsf{z}_{\text{pre-pop}}, \mathsf{z}_{\text{post-pop}}) \rightharpoonup \textbf{Pkt} \times \textbf{RioControl}(t, \mathsf{z}_{\text{pre-push}}, \mathsf{z}_{\text{pre-pop}}, \mathsf{z}_{\text{post-pop}}) \end{aligned}$$

Their semantics are written out in full in Figure 2.

2.2 Simulations

Definition 2.3. For $t \in \textbf{Topo}$, define $\mathsf{empty}_t \in \textbf{PIFOTree}(t)$ such that

$$\frac{p \in \textbf{PIFO}(\textbf{Pkt}) \qquad \mathsf{pop}(p) \text{ is undefined}}{\mathsf{empty}_* = \mathsf{Leaf}(p)} \qquad \frac{\begin{array}{cc} ts \in \textbf{Topo}^n & p \in \textbf{PIFO}(\{1, \ldots, n\}) \\ \mathsf{pop}(p) \text{ is undefined} & \forall 1 \leq i \leq n.\ qs[i] = \mathsf{empty}_{ts[i]} \end{array}}{\mathsf{empty}_{\mathsf{Node}(ts)} = \mathsf{Internal}(p, qs)}$$

Informally, empty_t is a PIFO tree of topology t, with empty PIFOs at all nodes.

Definition 2.4. Define the $\rightarrow \subseteq \textbf{PIFOControl}(t, \mathsf{z}_{\text{pre-push}}, \mathsf{z}_{\text{post-pop}}) \times \textbf{PIFOControl}(t, \mathsf{z}_{\text{pre-push}}, \mathsf{z}_{\text{post-pop}})$ by

$$\frac{\mathsf{pkt} \in \textbf{Pkt} \qquad \mathsf{push}(c, \mathsf{pkt}) = c'}{c \rightarrow c'}\ \text{Step-Push} \qquad \frac{\mathsf{pop}(c) = (\mathsf{pkt}, c')}{c \rightarrow c'}\ \text{Step-Pop}$$

i.e. $c \rightarrow c'$ if c' is a push or pop from c. We write $\rightarrow^*$ for the reflexive transitive closure of $\rightarrow$.

Definition 2.5. A partial function

$$f : \textbf{PIFOControl}(t, \mathsf{z}_{\text{pre-push}}, \mathsf{z}_{\text{post-pop}}) \rightharpoonup \textbf{RioControl}(t', \mathsf{z}'_{\text{pre-push}}, \mathsf{z}'_{\text{pre-pop}}, \mathsf{z}'_{\text{post-pop}})$$

and control $\mathsf{c}_{\text{init}} \in \mathrm{dom}\, f$ form a *simulation* if the following conditions are satisfied:

(1) The PIFO tree in c_{init} is empty, i.e. $\mathsf{c}_{\text{init}} = (s, \mathsf{empty}_t, \mathsf{z}_{\text{pre-push}}, \mathsf{z}_{\text{post-pop}})$.
(2) When $\mathsf{c}_{\text{init}} \rightarrow^* c_1$ and $f(c_1) = c_2$, we can guarantee the following:

(a) $$\mathsf{pop}(c_1) \text{ is undefined} \implies \mathsf{pop}(c_2) \text{ is undefined}$$

(b) $$\mathsf{pop}(c_1) = (\mathsf{pkt}, c_1') \implies \mathsf{pop}(c_2) = (\mathsf{pkt}, c_2') \wedge f(c_1') = c_2'$$

(c) $$\mathsf{push}(c_1, \mathsf{pkt}) = c_1' \implies \mathsf{push}(c_2, \mathsf{pkt}) = c_2' \wedge f(c_1') = c_2'$$

Definition 2.6. For $t \in \textbf{Topo}$, define the set $\textbf{CTopo}(t)$ such that

$$\frac{c \in \textbf{Class}}{c \in \textbf{CTopo}(*)} \qquad \frac{\forall 1 \leq i \leq |ts|.\ \tau s[i] \in \textbf{CTopo}(ts[i])}{\mathsf{Node}(\tau s) \in \textbf{CTopo}(\mathsf{Node}(ts))}$$

These are topologies decorated with classes at the leaves.

As we usually use t to denote elements of **Topo**, we'll use τ for elements of **CTopo**.

Definition 2.7. For $t \in \textbf{Topo}$, define $\mathsf{proj} : \textbf{CTopo}(t) \times \textbf{PIFOTree}(t) \rightarrow \textbf{RioTree}(t)$ such that

$$\frac{}{\mathsf{proj}(c, \mathsf{Leaf}(p)) = \mathsf{Leaf}(c, p)} \qquad \frac{\forall 1 \leq i \leq |qs|.\ \mathsf{proj}(\tau s[i], qs[i]) = qs'[i]}{\mathsf{proj}(\mathsf{Node}(\tau s), \mathsf{Internal}(p, qs)) = \mathsf{Internal}(qs')}$$

3 Example Controls & Simulations

For all further discussion, we make transparent our previously opaque collections:

$$\textbf{St} = \{s \mid s \text{ is dictionary mapping } \textbf{string} \to \textbf{float}\} \qquad \textbf{Rk} = \textbf{Class} = \mathbb{N} \qquad \mathrm{Im}(\textbf{flow}) = [0, n-1]$$

for some $n \in \mathbb{N}$.

3.1 Round-Robin

```
def z_pre-push(st, pkt):
    r = arrival_time(pkt)
    f = flow(pkt)
    rank_ptr = "r_" + str(f)
    r_i = int(st[rank_ptr])
    st["rank_ptr"] += float(n)
    return ((f, r_i) :: r, st)

def z_post-pop(st, pkt):
    f = flow(pkt)
    turn = int(s["turn"])
    i = turn
    while i != f:
        st["r_" + str(i)] += n
        i = (i + 1) % n
    st["turn"] = float((f + 1) % n)
    if turn >= st["turn"]:
        st["cycle"] += 1
    return st
```

(a) Round-Robin PIFO Control

```
def z_pre-push(st, pkt):
    return arrival_time(pkt)

def z_pre-pop(st):
    turn = int(s["turn"])
    rs = []
    for i in range(n):
        # Python %, e.g. -2 % 3 == 1, not -2
        rs.append((i - turn) % n)
    return Internal(rs, [Leaf] * n)

def z_post-pop(st, pkt):
    f = flow(pkt)
    st["turn"] = float((f + 1) % n)
    return st
```

(b) Round-Robin Rio Control

Figure 3. Scheduling Transactions

Let's put our theory to use by constructing PIFO and Rio controls for $\textbf{rr}[(\textbf{FIFO}[0], \textbf{FIFO}[1], \ldots, \textbf{FIFO}[n-1])]$. Both controls use the same underlying topologies, namely

$$t = \mathrm{Node}(\underbrace{*, *, \ldots, *}_{n \text{ times}}) \in \textbf{Topo} \qquad \tau = \mathrm{Node}((*, 0), (*, 1), \ldots, (*, n-1)) \in \textbf{CTopo}$$

Figure 3 describes their scheduling transactions in pseudocode. Therefore, we have the materials to define

$$\textbf{PIFOControl}(t, z_{\text{pre-push}}, z_{\text{post-pop}}) \qquad \text{and} \qquad \textbf{RioControl}(t, z'_{\text{pre-push}}, z'_{\text{pre-pop}}, z'_{\text{post-pop}})$$

I.e. the collection of PIFO and Rio controls for our program. Let's find a simulation between them!

Definition 3.1. Let $c_{\text{RR}} = (s, \text{empty}_t, z_{\text{pre-push}}, z_{\text{post-pop}}) \in \textbf{PIFOControl}(t, z_{\text{pre-push}}, z_{\text{post-pop}})$, where

$$s[\text{"turn"}] = 0 \qquad s[\text{"cycle"}] = 1 \qquad s[\text{"}r\_\text{"} + \mathrm{str}(i)] = i$$

for all $0 \le i \le n-1$

Definition 3.2. For set S, define ranks : $\textbf{PIFO}(S) \times S \to \mathcal{M}(\textbf{Rk})$[1] such that

$$\frac{\mathrm{pop}(p') = (j, p) \qquad m = \min(\mathrm{ranks}(p', j)) \qquad i = j}{\mathrm{ranks}(p, i) = \mathrm{ranks}(p', i) - \{m\}} \qquad \frac{\mathrm{pop}(p') = (j, p) \qquad i \neq j}{\mathrm{ranks}(p) = \mathrm{ranks}(p', i)}$$

$$\frac{\mathrm{push}(p', j, r) = p \qquad i = j}{\mathrm{ranks}(p, i) = \mathrm{ranks}(p', i) + \{r\}} \qquad \frac{\mathrm{pop}(p) \text{ is undefined}}{\mathrm{ranks}(p, i) = \{\}} \qquad \frac{\mathrm{push}(p', j, r) = p \qquad i \neq j}{\mathrm{ranks}(p, i) = \mathrm{ranks}(p', i)}$$

Informally, ranks(p, i) is the multiset of ranks with which i lives in PIFO p.

[1] We use $\mathcal{M}(X)$ to denote the collection of multisets with entries in set X.

Lemma 3.3. *If* $c_{RR} \to^* c = (s, q, z_{\text{pre-push}}, z_{\text{post-pop}})$*, then both* $z_{\text{pre-push}}$ *and* $z_{\text{post-pop}}$ *are defined on*

$$\{\text{pkt} \in \mathbf{Pkt} \mid (s, \text{pkt})\}$$

Proof Sketch. This follows by induction on $\to^*$ and the following observations:

- $z_{\text{pre-push}}$ and $z_{\text{post-pop}}$ are only undefined if they access unbound keys in the state.
- All keys in s accessed by $z_{\text{pre-push}}$ and $z_{\text{post-pop}}$ are defined in c_{RR}'s state.
- Neither $z_{\text{pre-push}}$ nor $z_{\text{post-pop}}$ unbind keys in the state, and, therefore, if

$$(s, q, z_{\text{pre-push}}, z_{\text{post-pop}}) \to (s', q', z_{\text{pre-push}}, z_{\text{post-pop}})$$

all keys bound in s are bound s'. □

Lemma 3.4. *If* $c_{RR} \to^* c = (s, q, z_{\text{pre-push}}, z_{\text{post-pop}})$ *and* p *is* q*'s root PIFO, i.e.* $q = \text{Internal}(p, qs)$*, then*

$$\text{ranks}(p, i) = \begin{cases} \left\{x \in \mathbf{Rk} \mid x \equiv i \pmod n \wedge n \cdot s[\text{"}cycle\text{"}] \leq x < s[\text{"}r\_\text{"} + \text{str}(i)]\right\} & i < s[\text{"}turn\text{"}] \\ \left\{x \in \mathbf{Rk} \mid x \equiv i \pmod n \wedge n \cdot (s[\text{"}cycle\text{"}] - 1) \leq x < s[\text{"}r\_\text{"} + \text{str}(i)]\right\} & i \geq s[\text{"}turn\text{"}] \end{cases}$$

for all $0 \leq i \leq n - 1$.

Proof Sketch. This follows by induction on $\to^*$ and careful analysis of lines 5-7 of $z_{\text{post-pop}}$ in Figure 3a. □

See Appendix A for a complete proofs of these lemmas.

Theorem 3.5. *Consider the function*

$$f : \mathbf{PIFOControl}(t, z_{\text{pre-push}}, z_{\text{post-pop}}) \to \mathbf{RioControl}(t, z'_{\text{pre-push}}, z'_{\text{pre-pop}}, z'_{\text{post-pop}})$$

such that $f(s, q, z_{\text{pre-push}}, z_{\text{post-pop}}) = (s', \text{proj}(\tau, q), z'_{\text{pre-push}}, z'_{\text{pre-pop}}, z'_{\text{post-pop}})$*, where*

$$s'[x] = \begin{cases} s[x] & x = \text{"}turn\text{"} \\ undefined & otherwise \end{cases}$$

Together f *and* c_{RR} *form a simulation.*

Proof. We'll verify each condition of Definition 2.5 one by one.

(1) By Definition 3.1, the PIFO tree in c_{RR} is empty.

(2) Let $c_{\text{RR}} \to^* c_1 = (s_1, q_1, z_{\text{pre-push}}, z_{\text{pre-pop}})$ and $f(c_1) = c_2 = (s_2, q_2, z_{\text{pre-push}}, z_{\text{pre-pop}}, z_{\text{post-pop}})$.

  (a) Suppose $\text{pop}(c_1)$ is undefined. By Lemma 3.3, $z_{\text{post-pop}}$ is defined on (s_1, pkt) for all $\text{pkt} \in \mathbf{Pkt}$. Therefore, by rule PIFOCtrl-Pop in Figure 2, $\text{pop}(q_1)$ must be undefined. Since q_1 is well-formed, all PIFOs on it are empty. Similarly, all PIFOs on $q_2 = \text{proj}(\tau, q_1)$ are empty. Hence, $\text{pop}(q_2, o)$ is undefined for all $o \in \mathbf{OrdTree}(t)$, i.e. $\text{pop}(c_2)$ is undefined per rule RioCtrl-Pop in Figure 2.

  (b)

  (c) Suppose $\text{push}(c_1, \text{pkt}) = c_1'$. By Lemma 3.3, $z'_{\text{pre-push}}$ is defined on (s_2, pkt) for all $\text{pkt} \in \mathbf{Pkt}$. From inspecting Definition 1.3, $\text{push}(q, \text{pkt}, r)$ is defined for all $q \in \mathbf{RioTree}(t)$, $\text{pkt} \in \mathbf{Pkt}$, and $r \in \mathbf{Rk}$. Therefore, by rule RioCtrl-Push in Figure 2, $\text{push}(c_2, \text{pkt}) = c_2'$ is defined.

□

3.2 Strict

References

[MLF+23] Anshuman Mohan, Yunhe Liu, Nate Foster, Tobias Kappé, and Dexter Kozen. Formal abstractions for packet scheduling, 2023.

A Proofs for Section 3

Lemma 3.4. *If* $c_{RR} \to^* c = (s, q, z_{\text{pre-push}}, z_{\text{post-pop}})$ *and p is q's root PIFO, i.e.* $q = \text{Internal}(p, qs)$*, then*

$$\text{ranks}(p, i) = \begin{cases} \Big\{x \in \mathbf{Rk} \mid x \equiv i \pmod n \wedge n \cdot s[\text{"}cycle\text{"}] \le x < s[\text{"}r\_\text{"} + \text{str}(i)]\Big\} & i < s[\text{"}turn\text{"}] \\ \Big\{x \in \mathbf{Rk} \mid x \equiv i \pmod n \wedge n \cdot (s[\text{"}cycle\text{"}] - 1) \le x < s[\text{"}r\_\text{"} + \text{str}(i)]\Big\} & i \ge s[\text{"}turn\text{"}] \end{cases}$$

for all $0 \le i \le n - 1$.

Proof. We'll proceed by induction on $\to^*$.

(Base Case) Let $c = c_{RR}$ and fix arbitrary $i \in [0, n-1]$.
Recall $q = \text{empty}_t$ by Definition 3.1. Therefore, $\text{pop}(p)$ is undefined by Definition 2.3. Hence,

$$\text{ranks}(p, i) = \{\}$$

by Definition 3.2. Definition 3.1 also insists

$$s[\text{"turn"}] = 0 \qquad s[\text{"}cycle\text{"}] = 1 \qquad s[\text{"}r\_\text{"} + str(i)] = i$$

Hence, $i \ge s[\text{"turn"}]$ and

$$\begin{aligned} &= \begin{cases} \Big\{x \in \mathbf{Rk} \mid x \equiv i \pmod n \wedge n \cdot s[\text{"}cycle\text{"}] \le x < s[\text{"}r\_\text{"} + \text{str}(i)]\Big\} & i < s[\text{"turn"}] \\ \Big\{x \in \mathbf{Rk} \mid x \equiv i \pmod n \wedge n \cdot (s[\text{"}cycle\text{"}] - 1) \le x < s[\text{"}r\_\text{"} + \text{str}(i)]\Big\} & i \ge s[\text{"turn"}] \end{cases} \\ &= \Big\{x \in \mathbf{Rk} \mid x \equiv i \pmod n \wedge n \cdot (s[\text{"}cycle\text{"}] - 1) \le x < s[\text{"}r\_\text{"} + \text{str}(i)]\Big\} \\ &= \Big\{x \in \mathbf{Rk} \mid x \equiv i \pmod n \wedge 0 \le x < i\Big\} = \{\} \end{aligned}$$

Both sides of the desired equality are therefore the empty multiset $\{\}$.

(Inductive Step) Let $c' = (s', q', z_{\text{pre-push}}, z_{\text{post-pop}}) \to c$ and $q' = \text{Internal}(p', qs')$. We have two cases.

(Step-Push) Suppose $c = \text{push}(c', \text{pkt})$ and $f = \textbf{flow}(\text{pkt})$. From inspecting $z_{\text{pre-push}}$ in Figure 3a,

$$\text{push}\Big(p', f, s'[\text{"}r\_\text{"} + \text{str}(f)]\Big) = p \qquad s[x] = \begin{cases} s'[x] + n & x = \text{"}r\_\text{"} + \text{str}(f) \\ s'[x] & \text{otherwise} \end{cases}$$

Hence, by Definition 3.2 and the IH, for $i \ne f$,

$$\begin{aligned} &= \text{ranks}(p, i) = \text{ranks}(p', i) \\ &= \begin{cases} \Big\{x \in \mathbf{Rk} \mid x \equiv i \pmod n \wedge n \cdot s'[\text{"}cycle\text{"}] \le x < s'[\text{"}r\_\text{"} + \text{str}(i)]\Big\} & i < s'[\text{"turn"}] \\ \Big\{x \in \mathbf{Rk} \mid x \equiv i \pmod n \wedge n \cdot (s'[\text{"}cycle\text{"}] - 1) \le x < s'[\text{"}r\_\text{"} + \text{str}(i)]\Big\} & i \ge s'[\text{"turn"}] \end{cases} \\ &= \begin{cases} \Big\{x \in \mathbf{Rk} \mid x \equiv i \pmod n \wedge n \cdot s[\text{"}cycle\text{"}] \le x < s[\text{"}r\_\text{"} + \text{str}(i)]\Big\} & i < s[\text{"turn"}] \\ \Big\{x \in \mathbf{Rk} \mid x \equiv i \pmod n \wedge n \cdot (s[\text{"}cycle\text{"}] - 1) \le x < s[\text{"}r\_\text{"} + \text{str}(i)]\Big\} & i \ge s[\text{"turn"}] \end{cases} \end{aligned}$$

Instead, for $i = f$, let $r = s'[\text{"}r\_\text{"} + \text{str}(i)]$. Definition 3.2 and the IH then once again show

$$\begin{aligned} &= \text{ranks}(p, i) = \text{ranks}(p', i) + \{r\} \\ &= \begin{cases} \Big\{x \in \mathbf{Rk} \mid x \equiv i \pmod n \wedge n \cdot s'[\text{"}cycle\text{"}] \le x < s'[\text{"}r\_\text{"} + \text{str}(i)]\Big\} + \{r\} & i < s'[\text{"turn"}] \\ \Big\{x \in \mathbf{Rk} \mid x \equiv i \pmod n \wedge n \cdot (s'[\text{"}cycle\text{"}] - 1) \le x < s'[\text{"}r\_\text{"} + \text{str}(i)]\Big\} + \{r\} & i \ge s'[\text{"turn"}] \end{cases} \\ &= \begin{cases} \Big\{x \in \mathbf{Rk} \mid x \equiv i \pmod n \wedge n \cdot s'[\text{"}cycle\text{"}] \le x < s'[\text{"}r\_\text{"} + \text{str}(i)] + n\Big\} & i < s'[\text{"turn"}] \\ \Big\{x \in \mathbf{Rk} \mid x \equiv i \pmod n \wedge n \cdot (s'[\text{"}cycle\text{"}] - 1) \le x < s'[\text{"}r\_\text{"} + \text{str}(i)] + n\Big\} & i \ge s'[\text{"turn"}] \end{cases} \\ &= \begin{cases} \Big\{x \in \mathbf{Rk} \mid x \equiv i \pmod n \wedge n \cdot s[\text{"}cycle\text{"}] \le x < s[\text{"}r\_\text{"} + \text{str}(i)]\Big\} & i < s[\text{"turn"}] \\ \Big\{x \in \mathbf{Rk} \mid x \equiv i \pmod n \wedge n \cdot (s[\text{"}cycle\text{"}] - 1) \le x < s[\text{"}r\_\text{"} + \text{str}(i)]\Big\} & i \ge s[\text{"turn"}] \end{cases} \end{aligned}$$

(Step-Pop) Suppose $(\text{pkt}, c) = \text{pop}(c')$ and $f = \textbf{flow}(\text{pkt})$. From inspecting $z_{\text{post-pop}}$ in Figure 3a,

$$\text{pop}(p') = (f, p) \qquad\qquad s[\text{“cycle”}] \in \{s'[\text{“cycle”}], s'[\text{“cycle”}] + 1\}$$

Consider the case where $s[\text{“cycle”}] = s'[\text{“cycle”}]$, i.e. $s'[\text{“turn”}] \le f < f + 1 = s[\text{“turn”}]$.

- For $i < s'[\text{“turn”}]$ or $i \ge s[\text{“turn”}]$, lines 5-7 of $z_{\text{post-pop}}$ in Figure 3a show

$$s[\text{“}r\_\text{”} + \text{str}(i)] = s'[\text{“}r\_\text{”} + \text{str}(i)]$$

Therefore, by Definition 3.2 and the IH,

$$= \text{ranks}(p, i) = \text{ranks}(p', i)$$

$$= \begin{cases} \left\{x \in \textbf{Rk} \mid x \equiv i \pmod n \wedge n \cdot s'[\text{“}cycle\text{”}] \le x < s'[\text{“}r\_\text{”} + \text{str}(i)]\right\} & i < s'[\text{“turn”}] \\ \left\{x \in \textbf{Rk} \mid x \equiv i \pmod n \wedge n \cdot (s'[\text{“}cycle\text{”}] - 1) \le x < s'[\text{“}r\_\text{”} + \text{str}(i)]\right\} & i \ge s'[\text{“turn”}] \end{cases}$$

$$= \begin{cases} \left\{x \in \textbf{Rk} \mid x \equiv i \pmod n \wedge n \cdot s[\text{“}cycle\text{”}] \le x < s[\text{“}r\_\text{”} + \text{str}(i)]\right\} & i < s[\text{“turn”}] \\ \left\{x \in \textbf{Rk} \mid x \equiv i \pmod n \wedge n \cdot (s[\text{“}cycle\text{”}] - 1) \le x < s[\text{“}r\_\text{”} + \text{str}(i)]\right\} & i \ge s[\text{“turn”}] \end{cases}$$

- Suppose $s'[\text{“}turn\text{”}] \le i < f$. By the IH,

$$\text{ranks}(p', j) = \left\{x \in \textbf{Rk} \mid x \equiv j \pmod n \wedge n \cdot (s'[\text{“}cycle\text{”}] - 1) \le x < s'[\text{“}r\_\text{”} + \text{str}(j)]\right\}$$

Since popping p' returned f, it must be a minimally ranked element. However,

$$n \cdot (s'[\text{“}cycle\text{”}] - 1) + i < n \cdot (s'[\text{“}cycle\text{”}] - 1) + f$$

Therefore, $\text{ranks}(p', i) = \{\}$: i.e. there are no numbers congruent to $i \mod n$ in

$$\left[n \cdot (s'[\text{“}cycle\text{”}] - 1), s'[\text{“}r\_\text{”} + \text{str}(i)]\right)$$

The same is therefore true of

$$\left[n \cdot s'[\text{“}cycle\text{”}], s'[\text{“}r\_\text{”} + \text{str}(i)] + n\right)$$

Since lines 5-7 of $z_{\text{post-pop}}$ in Figure 3a show $s[\text{“}r\_\text{”} + \text{str}(i)] = s'[\text{“}r\_\text{”} + \text{str}(i)] + n$,

(By Definition 3.2)
$$\begin{aligned} &= \text{ranks}(p, i) = \text{ranks}(p', i) \\ &= \left\{x \in \textbf{Rk} \mid x \equiv i \pmod n \wedge n \cdot (s'[\text{“}cycle\text{”}] - 1) \le x < s'[\text{“}r\_\text{”} + \text{str}(i)]\right\} \\ &= \{\} \\ &= \left\{x \in \textbf{Rk} \mid x \equiv i \pmod n \wedge n \cdot s'[\text{“}cycle\text{”}] \le x < s'[\text{“}r\_\text{”} + \text{str}(i)] + n\right\} \\ &= \left\{x \in \textbf{Rk} \mid x \equiv i \pmod n \wedge n \cdot s[\text{“}cycle\text{”}] \le x < s[\text{“}r\_\text{”} + \text{str}(i)]\right\} \end{aligned}$$

- For $i = f$, lines 5-7 of $z_{\text{post-pop}}$ in Figure 3a yet again show

$$s[\text{“}r\_\text{”} + \text{str}(i)] = s'[\text{“}r\_\text{”} + \text{str}(i)]$$

Therefore, by Definition 3.2 and the IH,

$$\begin{aligned} &= \text{ranks}(p, i) = \text{ranks}(p') - \{n \cdot (s'[\text{“cycle”}] - 1)\} \\ &= \left\{x \in \textbf{Rk} \mid x \equiv i \pmod n \wedge n \cdot (s'[\text{“cycle”}] - 1) \le x < s'[\text{“}r\_\text{”} + \text{str}(i)]\right\} - \{n \cdot (s'[\text{“cycle”}] - 1)\} \\ &= \left\{x \in \textbf{Rk} \mid x \equiv i \pmod n \wedge n \cdot s'[\text{“cycle”}] \le x < s'[\text{“}r\_\text{”} + \text{str}(i)]\right\} \\ &= \left\{x \in \textbf{Rk} \mid x \equiv i \pmod n \wedge n \cdot s[\text{“cycle”}] \le x < s[\text{“}r\_\text{”} + \text{str}(i)]\right\} \end{aligned}$$

We leave the cases where $s[\text{“cycle”}] = s'[\text{“cycle”}] + 1$ as an exercise as they're much of the same:

(i) $i \in [s[\text{“}turn\text{”}], s'[\text{“}turn\text{”}])$ ($\text{“}r\_\text{”} + \text{str}(i)$ is unchanged)
(ii) $i \in [s'[\text{“}turn\text{”}], n] \cup [0, f)$ ($\text{“}r\_\text{”} + \text{str}(i)$ is incremented by n)
(iii) $i = f$ $\square$