

# PIEO Trees for Fun and Profit

We assume familiarity with [?], adopt its notational conventions, and borrow many of its definitions!

## 1 Structure & Semantics

**Definition 1.1.** For sets  $S$ ,  $D$ , and predicates  $F$  over  $D$ , let  $\mathbf{PIEO}(S, D, F)$  denote the set of *PIEOs* that

- (1) hold entries in  $S$ , decorated with meta-data in  $D$
- (2) are ordered by **Rk**
- (3) support predicates in  $F$
- (4) admit partial functions

$$\begin{aligned} \text{pop} &: \mathbf{PIEO}(S, D, F) \times F \rightarrow S \times \mathbf{PIEO}(S, D, F) \\ \text{push} &: \mathbf{PIEO}(S, D, F) \times S \times D \times \mathbf{Rk} \rightarrow \mathbf{PIEO}(S, D, F) \\ \text{proj} &: \mathbf{PIEO}(S, D, F) \times F \rightarrow \mathbf{PIFO}(S) \end{aligned}$$

Maps push and pop are as usual. The *projection*  $\text{proj}(p, f)$  is the PIFO of entries in  $p$  with data satisfying  $f$ . These three maps play nicely together:

$$\begin{aligned} \text{pop}(p, f) \text{ is undefined} &\iff \text{pop}(\text{proj}(p, f)) \text{ is undefined} & (1) \\ \text{pop}(p, f) = (\text{pkt}, p') &\iff \text{pop}(\text{proj}(p, f)) = (\text{pkt}, \text{proj}(p', f)) & (2) \\ \text{proj}(\text{push}(p, s, d, r), f) &= \begin{cases} \text{push}(\text{proj}(p, f), s, r) & f(d) \text{ holds true} \\ \text{proj}(p, f) & \text{otherwise} \end{cases} & (3) \end{aligned}$$

We consider PIEOs  $p, p'$  equal if, for all  $f \in F$ ,  $\text{proj}(p, f) = \text{proj}(p', f)$ , i.e. their projections are always equal. For PIEO  $p$ , entry  $s \in S$ , and predicate  $f \in F$ , we write

- (1)  $|p|$  for the number of entries in  $p$
- (2)  $|p|_f$  for the number of entries in  $p$  satisfying  $f$
- (3)  $|p|_{s,f}$  for the number of times  $s$  occurs in  $p$  with associated  $d \in D$  such that  $f(d)$  holds

We fix an opaque set **Data** and a collection  $\mathcal{F}$  of predicates defined on it. These predicates come with a total order  $\leq$  and the property that,  $\forall d \in \mathbf{Data}$  and  $f, f' \in \mathcal{F}$ ,  $f \leq f' \wedge f(d) \implies f'(d)$ .

**Definition 1.2.** The set of *PIEO trees* over  $t \in \mathbf{Topo}$ , denoted  $\mathbf{PIEOTree}(t)$ , is defined inductively by

$$\frac{p \in \mathbf{PIEO}(\mathbf{Pkt}, \mathbf{Data}, \mathcal{F})}{\text{Leaf}(p) \in \mathbf{PIEOTree}(*)} \quad \frac{n \in \mathbb{N} \quad ts \in \mathbf{Topo}^n \quad p \in \mathbf{PIEO}(\{1, \dots, n\}, \mathbf{Data}, \mathcal{F}) \quad \forall i \in [1, n]. \text{qs}[i] \in \mathbf{PIEOTree}(ts[i])}{\text{Internal}(\text{qs}, p) \in \mathbf{PIEOTree}(\text{Node}(ts))}$$

**Definition 1.3.** Define  $\text{pop} : \mathbf{PIEOTree}(t) \times \mathcal{F} \rightarrow \mathbf{Pkt} \times \mathbf{PIEOTree}(t)$  by

$$\frac{\text{POP}(p, f) = (\text{pkt}, p')}{\text{pop}(\text{Leaf}(p), f) = (\text{pkt}, \text{Leaf}(p'))} \quad \frac{\text{POP}(p, f) = (i, p') \quad \text{pop}(\text{qs}[i], f) = (\text{pkt}, q')}{\text{pop}(\text{Internal}(\text{qs}, p), f) = (\text{pkt}, \text{Internal}(\text{qs}[q'/i], p'))}$$

**Definition 1.4.** Define  $\text{push} : \mathbf{PIEOTree}(t) \times \mathbf{Pkt} \times \mathbf{Data} \times \mathbf{Path}(t) \rightarrow \mathbf{PIEOTree}(t)$  by

$$\frac{\text{PUSH}(p, \text{pkt}, d, r) = p'}{\text{push}(\text{Leaf}(p), \text{pkt}, d, r) = \text{Leaf}(p')} \quad \frac{\text{PUSH}(p, i, d, r) = p' \quad \text{push}(\text{qs}[i], \text{pkt}, d, pt) = q'}{\text{push}(\text{Internal}(\text{qs}, p), \text{pkt}, d, (i, r) :: pt) = \text{Internal}(\text{qs}[q'/i], p')}$$

**Definition 1.5.** Let  $t \in \mathbf{Topo}$ . A *control* over  $t$  is a triple  $(s, q, z)$ , where  $s \in \text{St}$  is the *current state*,  $q$  is a PIEO tree of topology  $t$ , and

$$z : \text{St} \times \mathbf{Pkt} \rightarrow \mathbf{Data} \times \mathbf{Path}(t) \times \text{St}$$

is a function called the *scheduling transaction*.

## 2 Well-Formedness

**Definition 2.1.** Fix  $f \in \mathcal{F}$ . Define  $|\cdot|_f : \mathbf{PIEOTree}(t) \rightarrow \mathbb{N}$  by

$$|\text{Leaf}(p)|_f = |p|_f \qquad |\text{Internal}(qs, p)|_f = \sum_{i=1}^{|qs|} |qs[i]|_f$$

We say  $q \in \mathbf{PIEOTree}(t)$  is *well-formed* w.r.t.  $f$ , denoted  $\models_f q$ , if it adheres to the following rules.

$$\frac{}{\vdash_f \text{Leaf}(p)} \qquad \frac{\forall i \in [1, |qs|], \vdash_f qs[i] \wedge |p|_{i,f} = |qs[i]|_f}{\vdash_f \text{Internal}(qs, p)}$$

We say  $q$  is  $f$ -well-formed, denoted  $\vdash_f q$ , if for all  $f' \geq f$ ,  $\vdash_{f'} q$ .

**Theorem 2.2.** Let  $t \in \mathbf{Topo}$ ,  $\text{pkt} \in \mathbf{Pkt}$ ,  $d \in \mathbf{Data}$ ,  $f, f' \in \mathcal{F}$ , and  $q \in \mathbf{PIEOTree}(t)$  such that  $\vdash_f q$ .

- (1) If  $pt \in \mathbf{Path}(t)$ , then  $\text{push}(q, \text{pkt}, d, pt)$  is well-defined and  $\vdash_f \text{push}(q, \text{pkt}, d, pt)$ .
- (2) If  $|q|_{f'} > 0$  and  $f' \geq f$ , then  $\text{pop}(q, f')$  is well-defined and  $\vdash_{f'} q'$ , where  $\text{pop}(q, f') = (\text{pkt}, q')$ .

*Proof.* TBD

□

### 3 Projection

**Definition 3.1.** For  $f \in \mathcal{F}$ , define  $\text{proj}_f : \mathbf{PIEOTree}(t) \rightarrow \mathbf{PIFOTree}(t)$  by

$$\frac{p' = \text{proj}(p, f)}{\text{proj}_f(\text{Leaf}(p)) = \text{Leaf}(p')} \quad \frac{p' = \text{proj}(p, f) \quad \forall i \in [1, |qs|], \text{ } qs'[i] = \text{proj}_f(qs[i])}{\text{proj}_f(\text{Internal}(qs, p)) = \text{Internal}(qs', p')}$$

**Lemma 3.2.** For  $q, q' \in \mathbf{PIEOTree}(t)$ ,

$$\forall f \in \mathcal{F}, \text{proj}_f(q) = \text{proj}_f(q') \implies q = q'$$

*Proof.* Suppose  $\text{proj}_f(q) = \text{proj}_f(q')$  for all  $f \in \mathcal{F}$ . We'll proceed by induction on  $t$  to show  $q = q'$ .

(Leaf) For  $t = *$ , let  $q = \text{Leaf}(p)$  and  $q' = \text{Leaf}(p')$ . Since

$$\text{proj}_f(q) = \text{Leaf}(\text{proj}(p, f)) = \text{Leaf}(\text{proj}(p', f)) = \text{proj}_f(q')$$

we know  $\text{proj}(p, f) = \text{proj}(p', f)$  for all  $f \in \mathcal{F}$ . By ??,  $p = p'$  and hence  $q = q'$ .

(Node) For  $t = \text{Node}(ts)$  and  $n = |ts|$ , let  $q = \text{Internal}(qs, p)$  and  $q' = \text{Internal}(qs', p')$ . Notice

$$\begin{aligned} \text{proj}(p, f) &= \text{proj}(p', f) \\ \text{proj}_f(qs[i]) &= \text{proj}_f(qs'[i]) \end{aligned} \quad (i = 1, \dots, n)$$

for all  $f \in \mathcal{F}$ . Hence,  $p = p'$  via ?? and  $qs = qs'$  by the inductive hypothesis, i.e.  $q = q'$ . □

**Lemma 3.3.** For  $q \in \mathbf{PIEOTree}(t)$  and  $f \in \mathcal{F}$ ,  $\text{pop}(q, f)$  is undefined if and only if  $\text{pop}(\text{proj}_f(q))$  is undefined.

*Proof.* We'll do induction on  $t$ .

(Leaf) For  $t = *$ , let  $q = \text{Leaf}(p)$  and  $\text{proj}_f(q) = \text{Leaf}(p')$ . By ?? in ??,

$$\begin{aligned} \text{pop}(q, f) \text{ is undefined} &\iff \text{pop}(p, f) \text{ is undefined} \\ &\iff \text{pop}(p') \text{ is undefined} \\ &\iff \text{pop}(\text{proj}_f(q)) \text{ is undefined} \end{aligned}$$

(Node) For  $t = \text{Node}(ts)$ , let  $q = \text{Internal}(qs, p)$  and  $\text{proj}_f(q) = \text{Internal}(qs', p')$ . As before,

$$\text{pop}(p, f) \text{ is undefined} \iff \text{pop}(p') \text{ is undefined}$$

by ?? in ?? and

$$\text{pop}(qs[i], f) \text{ is undefined} \iff \text{pop}(qs'[i]) \text{ is undefined } \forall i \in [1, |ts|]$$

by the inductive hypothesis. Hence, using ?? in ??,

$$\begin{aligned} \text{pop}(q, f) \text{ is undefined} &\iff \text{pop}(p, f) \text{ is undefined} \vee (\text{pop}(p, f) = (i, \_) \wedge \text{pop}(qs[i], f) \text{ is undefined}) \\ &\iff \text{pop}(p') \text{ is undefined} \vee (\text{pop}(p') = (i, \_) \wedge \text{pop}(qs'[i]) \text{ is undefined}) \\ &\iff \text{pop}(\text{proj}_f(q)) \text{ is undefined} \end{aligned}$$

□

**Lemma 3.4.** For  $q \in \mathbf{PIEOTree}(t)$  and  $f \in \mathcal{F}$ ,

$$\text{pop}(q, f) = (\text{pkt}, q') \implies \text{pop}(\text{proj}_f(q)) = (\text{pkt}, \text{proj}_f(q'))$$

*Proof.* More induction on  $t$ !

(Leaf) For  $t = *$ , let

$$\begin{aligned} q &= \text{Leaf}(p_1) & \text{proj}_f(q) &= \text{Leaf}(p_2) \\ q' &= \text{Leaf}(p'_1) & \text{proj}_f(q') &= \text{Leaf}(p'_2) \end{aligned}$$

By ?? in ??,

$$\begin{aligned}
\text{pop}(q, f) = (\text{pkt}, q') &\implies \text{pop}(p_1, f) = (\text{pkt}, p'_1) \\
&\implies \text{pop}(p_2) = \text{pop}(\text{proj}(p_1, f)) = (\text{pkt}, \text{proj}(p'_1, f)) = (\text{pkt}, p'_2) \\
&\implies \text{pop}(\text{proj}_f(q)) = (\text{pkt}, \text{proj}_f(q'))
\end{aligned}$$

(Node) For  $t = \text{Node}(ts)$ , Let

$$\begin{aligned}
q &= \text{Internal}(qs_1, p_1) & \text{proj}_f(q) &= \text{Internal}(qs_2, p_2) \\
q' &= \text{Internal}(qs'_1, p'_1) & \text{proj}_f(q') &= \text{Internal}(qs'_2, p'_2)
\end{aligned}$$

Using ?? in ?? again and the inductive hypothesis,

$$\begin{aligned}
\text{pop}(q, f) = (\text{pkt}, q') &\implies \text{pop}(p_1, f) = (i, p'_1) \wedge \text{pop}(qs_1[i], f) = (\text{pkt}, qs'_1[i]) \\
&\implies \text{pop}(\text{proj}(p_1, f)) = (i, \text{proj}(p'_1, f)) \wedge \text{pop}(\text{proj}_f(qs_1[i])) = (\text{pkt}, \text{proj}_f(qs'_1[i])) \\
&\implies \text{pop}(p_2) = (i, p'_2) \wedge \text{pop}(qs_2[i]) = (\text{pkt}, qs'_2[i]) \\
&\implies \text{pop}(\text{proj}_f(q)) = (\text{pkt}, \text{proj}_f(q'))
\end{aligned}$$

□

**Lemma 3.5.** For  $q \in \mathbf{PIEOTree}(t)$ ,  $\text{pkt} \in \mathbf{Pkt}$ ,  $d \in \mathbf{Data}$ ,  $pt \in \mathbf{Path}(t)$ , and  $f \in \mathcal{F}$ ,

$$\text{proj}_f(\text{push}(q, \text{pkt}, d, pt)) = \begin{cases} \text{push}(\text{proj}_f(q), \text{pkt}, pt) & f(d) \text{ holds true} \\ \text{proj}_f(q) & \text{otherwise} \end{cases}$$

*Proof.* Even more induction on  $t$ !

(Leaf) For  $t = *$ , let  $q = \text{Leaf}(p)$  and  $pt = r$ . By ?? in ??.

$$\begin{aligned}
\text{proj}_f(\text{push}(q, \text{pkt}, d, pt)) &= \text{Leaf}(\text{proj}(\text{push}(p, \text{pkt}, d, r), f)) \\
&= \begin{cases} \text{Leaf}(\text{push}(\text{proj}(p, f), \text{pkt}, r)) & f(d) \text{ holds true} \\ \text{Leaf}(\text{proj}(p, f)) & \text{otherwise} \end{cases} \\
&= \begin{cases} \text{push}(\text{proj}_f(q), \text{pkt}, pt) & f(d) \text{ holds true} \\ \text{proj}_f(q) & \text{otherwise} \end{cases}
\end{aligned}$$

(Node) For  $t = \text{Node}(ts)$ , let  $pt = (i, r) :: pt'$  and

$$\begin{aligned}
q &= \text{Internal}(qs, p) \\
\text{proj}_f(q) &= \text{Internal}(qs', p') \\
\text{push}(\text{proj}_f(q), \text{pkt}, pt) &= \text{Internal}(qs'', p'') \\
\text{proj}_f(\text{push}(q, \text{pkt}, d, pt)) &= \text{Internal}(qs''', p''')
\end{aligned}$$

By ?? in ?? and the inductive hypothesis,

$$\begin{aligned}
p''' &= \begin{cases} \text{push}(\text{proj}(p, f), i, r) & f(d) \text{ holds true} \\ \text{proj}(p, f) & \text{otherwise} \end{cases} \\
&= \begin{cases} \text{push}(p', i, r) & f(d) \text{ holds true} \\ p' & \text{otherwise} \end{cases} = \begin{cases} p'' & f(d) \text{ holds true} \\ p' & \text{otherwise} \end{cases} \\
qs'''[i] &= \begin{cases} \text{push}(\text{proj}_f(qs[i]), \text{pkt}, pt') & f(d) \text{ holds true} \\ \text{proj}_f(qs[i]) & \text{otherwise} \end{cases} \\
&= \begin{cases} \text{push}(qs'[i], \text{pkt}, pt') & f(d) \text{ holds true} \\ qs'[i] & \text{otherwise} \end{cases} = \begin{cases} qs''[i] & f(d) \text{ holds true} \\ qs'[i] & \text{otherwise} \end{cases}
\end{aligned}$$

By inspection of ?? and ??,

$$qs'[j] = qs''[j] = qs'''[j] = \text{proj}_f(qs[j])$$

for all  $j \in [1, |ts|]$  such that  $j \neq i$ . Hence,

$$qs''' = \begin{cases} qs'' & f(d) \text{ holds true} \\ qs' & \text{otherwise} \end{cases}$$

Putting everything together,

$$\begin{aligned} \text{proj}_f(\text{push}(q, \text{pkt}, d, pt)) &= \text{Internal}(qs''', p''') \\ &= \begin{cases} \text{Internal}(qs'', p'') & f(d) \text{ holds true} \\ \text{Internal}(qs', p') & \text{otherwise} \end{cases} \\ &= \begin{cases} \text{push}(\text{proj}_f(q), \text{pkt}, pt) & f(d) \text{ holds true} \\ \text{proj}_f(q) & \text{otherwise} \end{cases} \end{aligned}$$

□

## 4 Embedding & Simulation

**Definition 4.1.** Let  $t_1, t_2 \in \mathbf{Topo}$ . We call a relation  $R \subseteq \mathbf{PIEOTree}(t_1) \times \mathbf{PIEOTree}(t_2)$  a *simulation* if, for all  $\text{pkt} \in \mathbf{Pkt}$ ,  $f \in \mathcal{F}$ , and  $q_1 R q_2$ ,

- (1) If  $\text{pop}(q_1, f)$  is undefined, then so is  $\text{pop}(q_2, f)$
- (2) If  $\text{pop}(q_1, f) = (\text{pkt}, q'_1)$ , then  $\text{pop}(q_2, f) = (\text{pkt}, q'_2)$  such that  $q'_1 R q'_2$ .
- (3) For all  $pt_1 \in \mathbf{Path}(t_1)$  and  $d \in \mathbf{Data}$ , there exists  $pt_2 \in \mathbf{Path}(t_2)$  such that

$$\text{push}(q_1, \text{pkt}, d, pt_1) R \text{push}(q_2, \text{pkt}, d, pt_2)$$

If such a simulation exists, we say that  $q_1$  is *simulated* by  $q_2$ , and we write  $q_1 \preccurlyeq q_2$ .

**Remark 4.2.** For all further discussion, we assume our embeddings are injective.

**Definition 4.3.** For  $t_1, t_2 \in \mathbf{Topo}$ , let  $f$  be an embedding from  $t_1$  to  $t_2$ . We lift  $f$  to a map  $\bar{f}$  from  $\mathbf{PIEOTree}(t_1)$  to  $\mathbf{PIEOTree}(t_2)$  inductively.

- For  $t_1 = *$ , define  $\bar{f}(q) = q$ . This is well-defined by [?, Lemma 5.2].
  - For  $t_1 = \text{Node}(ts_1)$ ,  $n = |ts_1|$ ,  $q = \text{Internal}(qs, p)$ , construct  $\bar{f}_\alpha(q) \in \mathbf{PIEOTree}(t_2/\alpha)$  for each prefix  $\alpha$  of  $f(i)$  for some  $i \in [1, n]$ . Inductively, we'll build up from  $f(i)$ 's to  $\epsilon$  and set  $\bar{f}(q) = \bar{f}_\epsilon(q)$ .
    - Let  $\alpha = f(i)$  for some  $i \in [1, n]$ . We'll set  $\bar{f}_\alpha(q) = \bar{f}_i(qs[i])$ , where  $f_i$  embeds  $t_1/i$  into  $t_2/f(i)$  as per [?, Lemma 5.2]. This well-defined by the injectivity of  $f$ .
    - Let  $\alpha$  point to a transient node, say with  $m$  children. For  $1 \leq j \leq m$  such that  $\alpha \cdot j$  is not a prefix of some  $f(i)$ , define  $\bar{f}(q)_{\alpha \cdot j}$  to be the PIEO tree with empty PIEOs on all leaves and internal nodes. With this and recursion, we know  $\bar{f}(q)_{\alpha \cdot j} \in \mathbf{PIEOTree}(t_2/(\alpha \cdot j))$  for all  $j \in [1, m]$ .
- We create a new PIEO  $p_\alpha$  as follows:
- (1) Start with  $p_\alpha$  empty
  - (2) For each  $i$  in  $p$  such that  $\alpha \cdot j$  is a prefix of  $f(i)$ , push  $j$  into  $p_\alpha$  with  $i$ 's data and rank
- Finally, for all  $j \in [1, m]$ , set  $qs_\alpha[j] = \bar{f}(q)_{\alpha \cdot j}$  and  $\bar{f}(q)_\alpha = \text{Internal}(qs_\alpha, p_\alpha)$ .

**Theorem 4.4.** The following diagram commutes

$$\begin{array}{ccc} \mathbf{PIEOTree}(t_1) & \xrightarrow{\bar{f}} & \mathbf{PIEOTree}(t_2) \\ \downarrow \text{proj}_g & & \downarrow \text{proj}_g \\ \mathbf{PIFOTree}(t_1) & \xrightarrow{\hat{f}} & \mathbf{PIFOTree}(t_2) \end{array}$$

In other words, for  $q \in \mathbf{PIEOTree}(t_1)$  and  $g \in \mathcal{F}$ ,  $\text{proj}_g(\bar{f}(q)) = \hat{f}(\text{proj}_g(q))$ .

*Proof.* We'll proceed by induction on  $t_1$ . Suppose  $t_1 = *$  and  $q = \text{Leaf}(p)$ . By [?, Lemma 5.3],  $t_2 = *$  as well. By ?? and [?, Definition 5.4], both  $\bar{f}$  and  $\hat{f}$  are the identity. Hence,

$$\text{proj}_g(\bar{f}(q)) = \text{proj}_g(q) = \hat{f}(\text{proj}_g(q))$$

Suppose  $t_1 = \text{Node}(ts)$  and  $q = \text{Internal}(qs, p)$ . For any prefix  $\alpha$  of  $f(i)$  for  $i \in [1, |ts|]$ , we'll show

$$\text{proj}_g(\bar{f}(q)_\alpha) = \hat{f}(\text{proj}_g(q))_\alpha \quad (*)$$

by inverse induction on  $\alpha$ . Instantiating ?? with  $\alpha = \epsilon$  yields the desired result.

- For  $\alpha = f(i)$ , ?? holds by the outer inductive hypothesis because

$$\text{proj}_g(\bar{f}(q)_\alpha) = \text{proj}_g(\bar{f}_i(q)) = \hat{f}_i(\text{proj}_g(q)) = \hat{f}(\text{proj}_g(q))_\alpha$$

- Suppose  $\alpha$  is some strict prefix of  $f(i)$ , pointing to a node with  $m$  children. Let

$$\text{proj}_g(\bar{f}(q)) = \text{Internal}(qs', p') \quad \text{and} \quad \hat{f}(\text{proj}_g(q)) = \text{Internal}(qs'', p'')$$

There's two parts to showing ??, namely  $qs' = qs''$  and  $p' = p''$ .

- For all  $j \in [1, m]$ ,

$$\text{proj}_g(\bar{f}(q)_{\alpha \cdot j}) = \hat{f}(\text{proj}_g(q))_{\alpha \cdot j}$$

For  $j$  such that  $\alpha \cdot j$  is a prefix of some  $f(i)$ , this follows from the inner inductive hypothesis. For all other  $j$ , notice the LHS and RHS are both PIFO trees of topology  $t_2$ , with empty PIFOs on all leaves and internal nodes. Hence,  $qs'[j] = qs''[j]$  for all  $j \in [1, m]$ , i.e.  $qs' = qs''$ .

- By inspection, it's clear following the construction for  $p_\alpha$  from ?? and then computing the projection  $\text{proj}(p_\alpha, g)$  yields the same result as following the recipe for  $p_\alpha$  from [?, Definition 5.4] on the projection  $\text{proj}_g(q)$ : that is, exactly when we filter out elements not satisfying  $g$  does not matter. Hence,  $p' = p''$ .

□

**Lemma 4.5.** Let  $t_1, t_2 \in \mathbf{Topo}$  and  $f$  be an embedding of  $t_1$  inside  $t_2$ . For  $g \in \mathcal{F}$ ,

$$\text{pop}(q, g) \text{ is undefined} \implies \text{pop}(\bar{f}(q), g) \text{ is undefined}$$

*Proof.* Suppose  $\text{pop}(q, g)$  is undefined. Applying both ?? and [?, Lemma 5.6], notice  $\text{pop}(\hat{f}(\text{proj}_g(q)))$  is undefined. By ??,  $\hat{f}(\text{proj}_g(q)) = \text{proj}_g(\bar{f}(q))$ . Hence,  $\text{pop}(\text{proj}_g(\bar{f}(q)))$  is undefined. Applying ?? once more,  $\text{pop}(\bar{f}(q), g)$  is undefined. □

**Lemma 4.6.** Let  $t_1, t_2 \in \mathbf{Topo}$  and  $f$  be an embedding of  $t_1$  inside  $t_2$ . For  $g \in \mathcal{F}$ ,

$$\text{pop}(q, g) = (\text{pkt}, q') \implies \text{pop}(\bar{f}(q), g) = (\text{pkt}, \bar{f}(q'))$$

Almost a clone of the proof [?, Lemma 5.7].

*Proof.* We'll proceed by induction on  $t_1$ . Suppose  $t_1 = *$ . By [?, Lemma 5.3],  $t_2 = *$  as well. By ??,  $\bar{f}$  is the identity. Hence,

$$\text{pop}(q, g) = (\text{pkt}, q') \implies \text{pop}(\bar{f}(q), g) = \text{pop}(q, g) = (\text{pkt}, q') = (\text{pkt}, \bar{f}(q'))$$

Suppose  $t_1 = \text{Node}(ts)$ . Let

$$q = \text{Internal}(qs, p) \quad q' = \text{Internal}(qs', p') \quad \text{pop}(p, g) = (j, p')$$

For any prefix  $\alpha$  of some  $f(i)$  (where  $i \in [1, |ts|]$ ), we'll show

$$\begin{aligned} \text{pop}(\bar{f}(q)_\alpha, g) &= (\text{pkt}, \bar{f}(q')_\alpha) & \text{if } \alpha \text{ is a prefix of } f(j) \\ \bar{f}(q)_\alpha &= \bar{f}(q')_\alpha & \text{otherwise} \end{aligned} \quad (\dagger)$$

by inverse induction on  $\alpha$ . Instantiating ?? with  $\alpha = \epsilon$  yields the desired result.

- Suppose  $\alpha = f(i)$ . If  $\alpha$  is a prefix of  $f(j)$ ,  $i = j$  by injectivity and [?, Definition 5.2, Equation (3)]. Recall  $\text{pop}(qs[j], g) = (\text{pkt}, qs'[j])$ . Hence, by the outer inductive hypothesis,

$$\text{pop}(\bar{f}(q)_\alpha, g) = \text{pop}(\bar{f}_i(qs[j]), g) = (\text{pkt}, \bar{f}_i(qs'[j])) = (\text{pkt}, \bar{f}(q')_\alpha)$$

- Once more, suppose  $\alpha = f(i)$ . If  $\alpha$  is not a prefix of  $f(j)$ , then  $i \neq j$ . Since  $qs[i] = qs'[i]$ ,

$$\bar{f}(q)_\alpha = \bar{f}_i(qs[i]) = \bar{f}_i(qs'[i]) = \bar{f}(q')_\alpha$$

- Suppose  $\alpha$  is some strict prefix of  $f(j)$ , pointing to a node with  $m$  children. Let

$$\bar{f}(q)_\alpha = \text{Internal}(qs_\alpha, p_\alpha) \quad \bar{f}(q')_\alpha = \text{Internal}(qs'_\alpha, p'_\alpha)$$

There exists  $k \in [1, m]$  such that  $\alpha \cdot k$  is a prefix of  $f(j)$ . By the inner inductive hypothesis,

$$qs_\alpha[i] = \bar{f}(q)_{\alpha \cdot i} = \bar{f}(q')_{\alpha \cdot i} = qs'_\alpha[i] \text{ for } i \in [1, m] \text{ with } i \neq k \quad (!)$$

$$\text{pop}(qs_\alpha[k], g) = \text{pop}(\bar{f}(q)_{\alpha \cdot k}, g) = (\text{pkt}, \bar{f}(q')_{\alpha \cdot k}) = (\text{pkt}, qs'_\alpha[k])$$

Via the construction in ?? and since  $\text{pop}(p, g) = p'$ ,  $\text{pop}(p_\alpha, g) = (k, p'_\alpha)$ . Putting everything together,  $\text{pop}(\bar{f}(q)_\alpha, g) = (\text{pkt}, \bar{f}(q')_\alpha)$ , as desired.

- Suppose  $\alpha$  is some strict prefix of some  $f(i)$  but not  $f(j)$ , pointing to a node with  $m$  children. Let

$$\bar{f}(q)_\alpha = \text{Internal}(qs_\alpha, p_\alpha) \quad \bar{f}(q')_\alpha = \text{Internal}(qs'_\alpha, p'_\alpha)$$

Since  $p$  and  $p'$  agree on all indices  $i$  such that  $f(i)$  is a child of  $\alpha$ ,  $p_\alpha = p'_\alpha$ . For  $i \in [1, m]$ , since  $\alpha \cdot i$  is not a prefix of  $f(j)$ , the inner inductive hypothesis yields

$$qs_\alpha[i] = \bar{f}(q)_{\alpha \cdot i} = \bar{f}(q')_{\alpha \cdot i} = qs'_\alpha[i] \quad (!!)$$

Putting everything together,  $\bar{f}(q)_\alpha = \bar{f}(q')_\alpha$ , as desired.

**NOTE:** even when  $\alpha \cdot i$  is not a prefix of any  $f(i)$ , ?? and ?? hold! Both  $\bar{f}(q)_{\alpha \cdot i}$  and  $\bar{f}(q')_{\alpha \cdot i}$  would be PIEO trees with empty PIEOs on all leaf and internal nodes.  $\square$

**Lemma 4.7.** Let  $t_1, t_2 \in \mathbf{Topo}$  and  $f$  be an embedding of  $t_1$  inside  $t_2$ . For  $\text{pkt} \in \mathbf{Pkt}$ ,  $d \in \mathbf{Data}$ , and  $pt \in \mathbf{Path}(t_1)$ ,

$$\bar{f}(\text{push}(q, \text{pkt}, d, pt)) = \text{push}(\bar{f}(q), \text{pkt}, d, \tilde{f}(pt))$$

*Proof.* Let  $q_2 = \bar{f}(q_1)$ . For  $g \in \mathcal{F}$  such that  $g(d)$  holds true,

$$\begin{aligned} \text{(by ??)} \quad \text{proj}_g(\bar{f}(\text{push}(q_1, \text{pkt}, d, pt))) &= \hat{f}(\text{proj}_g(\text{push}(q_1, \text{pkt}, d, pt))) \\ \text{(by ??)} &= \hat{f}(\text{push}(\text{proj}_g(q_1), \text{pkt}, pt)) \\ \text{(by [?, Lemma 5.9])} &= \text{push}(\hat{f}(\text{proj}_g(q_1)), \text{pkt}, \tilde{f}(pt)) \\ \text{(by ??)} &= \text{push}(\text{proj}_g(q_2), \text{pkt}, \tilde{f}(pt)) \\ \text{(by ??)} &= \text{proj}_g(\text{push}(q_2, \text{pkt}, d, \tilde{f}(pt))) \end{aligned}$$

For  $g \in \mathcal{F}$  such that  $g(d)$  does not hold true,

$$\begin{aligned} \text{(by ??)} \quad \text{proj}_g(\bar{f}(\text{push}(q_1, \text{pkt}, d, pt))) &= \hat{f}(\text{proj}_g(\text{push}(q_1, \text{pkt}, d, pt))) \\ \text{(by ??)} &= \hat{f}(\text{proj}_g(q_1)) \\ \text{(by ??)} &= \text{proj}_g(q_2) \\ \text{(by ??)} &= \text{proj}_g(\text{push}(q_2, \text{pkt}, d, \tilde{f}(pt))) \end{aligned}$$

Overall,  $\text{proj}_g(\bar{f}(\text{push}(q_1, \text{pkt}, d, pt))) = \text{proj}_g(\text{push}(q_2, \text{pkt}, d, \tilde{f}(pt)))$  for all  $g \in \mathcal{F}$ . Hence,

$$\bar{f}(\text{push}(q_1, \text{pkt}, d, pt)) = \text{push}(q_2, \text{pkt}, d, \tilde{f}(pt))$$

by ??, as desired.  $\square$

**Theorem 4.8.** Let  $t_1, t_2 \in \mathbf{Topo}$ . If  $f$  embeds  $t_1$  into  $t_2$ , then

$$R = \{(q, \bar{f}(q)) \mid q \in \mathbf{PIEOTree}(t_1)\}$$

is a simulation.

*Proof.* By ??, ??, and ??, the conditions from ?? hold.  $\square$