Preservation of Well-Formedness in PIEO Trees

1 Relevant Definitions

Defining $|\cdot|$ on PIEOs With Regards to Pushes

Definition 1A: *Given a predicate $f \in \mathcal{F}$, element i and data d, if d satisfies f, then pushing i to p with data d increments the number of instances of i that satisfy f.*

$$\frac{p \in \text{PIEO} \qquad f \in \mathcal{F} \qquad n \in \mathbb{N} \qquad r \in \text{Rank} \qquad |p|_{i,f} = n \qquad \text{PUSH}(p, i, d, r) = p' \qquad f(d)}{|p'|_{i,f} = n + 1}$$

Definition 1B: *Given a predicate $f \in \mathcal{F}$, element i and data d, if d does not satisfy f, then pushing i to p with data d preserves the number of instances of i that satisfy f.*

$$\frac{p \in \text{PIEO} \qquad f \in \mathcal{F} \qquad n \in \mathbb{N} \qquad r \in \text{Rank} \qquad |p|_{i,f} = n \qquad \text{PUSH}(p, i, d, r) = p' \qquad \neg f(d)}{|p'|_{i,f} = n}$$

Definition 1C: *Given elements i, j and data $d \in \mathcal{D}$, pushing i to p with data d does not modify how many times j appears in p.*

$$\frac{p \in \text{PIEO} \qquad f \in \mathcal{F} \qquad n \in \mathbb{N} \qquad r \in \text{Rank} \qquad |p|_{j,f} = n \qquad \text{PUSH}(p, i, d, r) = p' \qquad i \neq j}{|p'|_{j,f} = n}$$

2 Relevant Lemmas

2.1 Lemma 1

Given $f \in \mathcal{F}, pkt \in \text{Pkt}, d \in \mathcal{D}, pt \in \text{Path}, t \in \text{Topo}$ and $q \in \text{PIEOTree}(t)$:

$$\begin{cases} f(d) \implies |push(q, pkt, d, pt)|_f = |q|_f + 1 \\ \neg f(d) \implies |push(q, pkt, d, pt)|_f = |q|_f \end{cases}$$

Proof: We proceed about this proof by Structural Induction over the derivation of $|\cdot|$ as follows:

Base Case: $q = \text{Leaf}(p)$ for some PIEO p.

By definition 2.1, we have that $|\text{Leaf}(p)|_{i,f} = |p|_{i,f}$, where $|p|_{i,f}$ is the number of occurrences of i in PIEO p that satisfy predicate f. The definition of push in 1.4 gives the following:

$$\frac{\text{PUSH}(p, i, d, r) = p'}{\text{push}(\text{Leaf}(p), pkt, d, r) = \text{Leaf}(p')}$$

Where $\text{Leaf}(p')$ is the result of pushing into $\text{Leaf}(p)$.

By definition 1A, we now know that $f(d) \implies |p'|_f = |p|_f + 1$.

By definition 1B, we now know that $\neg f(d) \implies |p'|_f = |p|_f$.

By definition 2.1, we know that $|\text{Leaf}(p')|_f = |p'|_f$ and $|\text{Leaf}(p)|_f = |p|_f$.

Combining these together, we obtain the following:

$$\begin{cases} f(d) \implies |\text{Leaf}(p')|_f = |\text{Leaf}(p)|_f + 1 \\ \neg f(d) \implies |\text{Leaf}(p')|_f = |\text{Leaf}(p)|_f \end{cases}$$

Thus, we have proven our base case.

Inductive Case: Assume an arbitrary PIEO p, set of subtrees qs, node $q = \text{Internal}(qs, p)$, index i, data d and packet pkt. Also assume a set of paths $pts : |pts| = |qs|$.

Inductive Hypothesis: $\forall 1 \leq j \leq |qs|$:

$$\begin{cases} f(d) \implies |push(qs[j], pkt, d, pts[j])|_f = |qs[j]|_f + 1 \\ \neg f(d) \implies |push(qs[j], pkt, d, pts[j])|_f = |qs[j]|_f \end{cases}$$

We now push a packet pkt with an arbitrary index i and rank r.

Let $q' = push(q, pkt, d, (i, r) :: pts[i])$.

Show:

$$\begin{cases} f(d) \implies |q'|_f = |q|_f + 1 \\ \neg f(d) \implies |q'|_f = |q|_f \end{cases}$$

Recall Definition 1.4 for push on Internal Nodes:

$$\frac{\text{push}(qs[i], pkt, d, pt) = q' \qquad \text{PUSH}(p, i, d, r) = p'}{\text{push}(\text{Internal}(qs, p), pkt, d, (i, r) :: pt) = \text{Internal}(qs[q'/i], p')}$$

Recall Definition 2.1 for $|\cdot|_f$ on Internal Nodes:

$$m = |\text{Internal}(qs, p)|_f = \sum_{j=1}^{|qs|} |qs[j]|_f$$

Thus, we can further conclude the following:

$$|\text{push}(\text{Internal}(qs, p), pkt, d, (i, r) :: pt)|_f = \sum_{j=1}^{|qs|} |qs[q'/i][j]|_f$$

From our Inductive Hypothesis, we know the following:

$$\begin{cases} f(d) \implies |push(qs[i], pkt, d, pts[i])|_f = |qs[i]|_f + 1 \\ \neg f(d) \implies |push(qs[i], pkt, d, pts[i])|_f = |qs[i]|_f \end{cases}$$

From the definition of push, pushing changes only $qs[i]$, while the remaining subtrees remain the same. Subsequently, we can assert that $\forall 1 \leq j \leq |qs|, i \neq j \implies |qs[j]|_f = |qs[q'/i][j]|_f$. With this in mind, we make the following claim:

$$|\text{push}(\text{Internal}(qs, p), pkt, (i, r) :: pt)|_f = \sum_{j=1}^{|qs|} |qs[j]|_f - |qs[i]|_f + |qs[q'/i][i]|_f$$

Substituting what we know gives the following:

$$\begin{cases} f(d) \implies |\text{push}(\text{Internal}(qs, p), pkt, (i, r) :: pt)|_f = |\text{Internal}(qs, p)|_f - |qs[i]|_f + |qs[i]|_f + 1 \\ \neg f(d) \implies |\text{push}(\text{Internal}(qs, p), pkt, (i, r) :: pt)|_f = |\text{Internal}(qs, p)|_f - |qs[i]|_f + |qs[i]|_f \end{cases}$$

$$\implies \begin{cases} f(d) \implies |\text{push}(\text{Internal}(qs, p), pkt, (i, r) :: pt)|_f = |q|_f + 1 \\ \neg f(d) \implies |\text{push}(\text{Internal}(qs, p), pkt, (i, r) :: pt)|_f = |q|_f \end{cases}$$

$$\implies \begin{cases} f(d) \implies |q'|_f = |q|_f + 1 \\ \neg f(d) \implies |q'|_f = |q|_f \end{cases}$$

With this, we have shown the desired equality, and proven our Inductive case.

Thus, it follows that Lemma 1 holds.

3 Proof Of Theorem 2.2

Let $t \in$ Topo, pkt $\in$ Pkt, $d \in \mathcal{D}$, $f, f' \in \mathcal{F}$, and $q \in \text{PIEOTree}(t)$ such that $\vdash_f q$.

1. If $pt \in \text{Path}(t)$, then $\text{push}(q, pkt, d, pt)$ is well-defined and $\vdash_f \text{push}(q, pkt, d, pt)$.

2. If $|q|_{f'} > 0$ and $f' \geq f$, then $\text{pop}(q, f')$ is well-defined and $\vdash_{f'} q'$, where $\text{pop}(q, f') = (pkt, q')$.

3.1 Proof of Theorem 2.2.1

Well-Formedness Is Preserved in PIEO Trees Upon Pushes

We proceed with this proof by inducting upon the definition of push for PIEO Trees.

Base Case : Leaf(p)

By definition 1.4 of push for PIEOs, we have the following for Leaf nodes:

$$\frac{\text{PUSH}(p, pkt, d, r) = p'}{\text{push}(\text{Leaf}(p), pkt, d, r) = \text{Leaf}(p')}$$

It follows that after pushing, the resultant tree is expressed as $\text{Leaf}(p')$ for some packet p'. By definition 2.1, we know the following holds for any arbitrary PIEO p, under any predicate $f \in \mathcal{F}$:

$$\frac{}{\vdash_f \text{Leaf}(p)}$$

We now have: $\forall(p, pkt, d, f, r), \vdash_f \text{push}(\text{Leaf}(p), pkt, d, r)$

With this, our Base Case is proven.

Inductive Case: $q = \text{Internal}(qs, p), f \in \mathcal{F}, \vdash_f q$

Inductive Hypothesis: $\forall 1 \leq i \leq |qs|. \vdash_f \text{push}(qs[i], pkt, d, pt)$

Recall Definition 2.1 for $\vdash_f$ on Internal Nodes, given a predicate f:

$$\frac{\forall 1 \leq i \leq |qs|. \vdash_f qs[i] \wedge |p|_{i,f} = |qs[i]|_f}{\vdash_f \text{Internal(qs, p)}}$$

Recall Definition 1.4 for push on Internal Nodes:

$$\frac{\text{push}(qs[i], pkt, d, pt) = q' \qquad \text{PUSH}(p, i, d, r) = p'}{\text{push}(\text{Internal}(qs, p), pkt, d, (i, r) :: pt) = \text{Internal}(qs[q'/i], p')}$$

Now, let $q' = \text{push}(qs[i], pkt, d, pt)$, and let $p' = \text{PUSH}(p, i, d, r)$.

Let f be any totally-ordered predicate.

Show: $\vdash_f q \implies \vdash_f \text{Internal}(qs[q'/i], p')$.

Using definition 2.1 and our Inductive Hypothesis, we can conclude that $\vdash_f q'$. Furthermore, note that the only subtree to be modified in qs is in $qs[i]$, per the definition of push.

Now we make the following claim:

(1) $\vdash_f$ Internal(qs, p)	*By definition*
(2) $\implies \forall 1 \leq j \leq \lvert qs \rvert.\ \vdash_f qs[j] \wedge \lvert p \rvert_{j,f} = \lvert qs[j] \rvert_f$	*Inversion Lemma (Definition 2.1)*
(3) $\implies \lvert p \rvert_{i,f} = \lvert qs[i] \rvert_f$	*Instance of universal quantifier (2)*
(4.1) $f(d) \implies \lvert p' \rvert_{i,f} = \lvert p \rvert_{i,f} + 1$	*Definition 1A*
(4.2) $\neg f(d) \implies \lvert p' \rvert_{i,f} = \lvert p \rvert_{i,f}$	*Definition 1B*
(5.1) $f(d) \implies \lvert q' \rvert_f = \lvert qs[i] \rvert_f + 1$	*Lemma 1*
(5.2) $\neg f(d) \implies \lvert q' \rvert_f = \lvert qs[i] \rvert_f$	*Lemma 1*
(6) $\lvert p \rvert_{i,f} + 1 = \lvert qs[i] \rvert_f + 1$	*Addition Property of Equality (3)*
(7) $\implies \lvert p' \rvert_{i,f} = \lvert q' \rvert_f$	*Substitution (3 - 6)*
(8)$\forall 1 \leq j \leq \lvert p' \rvert, i \neq j \implies \lvert p' \rvert_{j,f} = \lvert p \rvert_{j,f}$	*Definition 1C*
(9) $\implies \forall 1 \leq j \leq \lvert p' \rvert, i \neq j \implies \lvert p' \rvert_{j,f} = \lvert qs[j] \rvert_f$	*By (2) and (8)*
(10) $\implies \forall 1 \leq j \leq \lvert p' \rvert, \lvert p' \rvert_{j,f} = \lvert qs[q'/i][j] \rvert_f$	*By (7) and (9)*
(11)$\forall 1 \leq j \leq \lvert qs \rvert. \vdash_f qs[q'/i][j]$	*Inductive Hypothesis*
(12) $\implies \forall 1 \leq j \leq \lvert qs \rvert. \vdash_f qs[q'/i][j] \wedge \lvert p' \rvert_{j,f} = \lvert qs[q'/i][j] \rvert_f$	*By (10) and (11)*
(13) $\implies \vdash_f$ Internal$(qs[q'/i], p')$	*Definition of $\vdash_f$*

With this, we have proven our Inductive Statement and completed the proof. We have shown that pushing to an arbitrary PIEO Tree preserves its well-formedness.