PIEO Trees for Fun and Profit

We assume familiarity with [MLF$^+$23], adopt its notational conventions, and borrow many of its definitions!

1 Structure & Semantics

Definition 1.1. For sets S, D, and predicates F over D, let **PIEO**(S, D, F) denote the set of *PIEO*s that
  (1) hold entries in S, decorated with meta-data in D
  (2) are ordered by **Rk**
  (3) support predicates in F
  (4) admit partial functions

$$\text{pop} : \textbf{PIEO}(S, D, F) \times F \rightharpoonup S \times \textbf{PIEO}(S, D, F)$$

$$\text{push} : \textbf{PIEO}(S, D, F) \times S \times D \times \textbf{Rk} \rightarrow \textbf{PIEO}(S, D, F)$$

For $p \in \textbf{PIEO}(S, D, F)$, $s \in S$, and $f \in F$, we write
  (1) $|p|$ for the number of entries in p
  (2) $|p|_s$ for the number of times s occurs in p
  (3) $|p|_{s,f}$ for the number of times s occurs in p with associated d such that $f(d)$

We fix an opaque set **Data** and a collection $\mathcal{F}$ of predicates defined on it. These predicates come with a total order $\leq$ and the property that, $\forall d \in \textbf{Data}$ and $f, f' \in \mathcal{F}$, $f \leq f' \wedge f(d) \implies f'(d)$.

Definition 1.2. The set of *PIEO trees* over $t \in \textbf{Topo}$, denoted **PIEOTree**(t), is defined inductively by

$$\frac{p \in \textbf{PIEO}(\textbf{Pkt}, \textbf{Data}, \mathcal{F})}{\text{Leaf}(p) \in \textbf{PIEOTree}(*)} \qquad \frac{n \in \mathbb{N} \qquad ts \in \textbf{Topo}^n \qquad p \in \textbf{PIEO}(\{1, \ldots, n\}, \textbf{Data}, \mathcal{F}) \qquad \forall i \in [1, n].\ qs[i] \in \textbf{PIEOTree}(ts[i])}{\text{Internal}(qs, p) \in \textbf{PIEO}(\text{Node}(ts))}$$

Definition 1.3. Define pop : **PIEOTree**$(t) \times \mathcal{F} \rightharpoonup \textbf{Pkt} \times \textbf{PIEOTree}(t)$ by

$$\frac{\text{pop}(p, f) = (\text{pkt}, p')}{\text{pop}(\text{Leaf}(p), f) = (\text{pkt}, \text{Leaf}(p'))} \qquad \frac{\text{pop}(p, f) = (i, p') \qquad \text{pop}(qs[i], f) = (\text{pkt}, q')}{\text{pop}(\text{Internal}(qs, p), f) = (\text{pkt}, \text{Internal}(qs[q'/i], p'))}$$

Definition 1.4. Define push : **PIEOTree**$(t) \times \textbf{Pkt} \times \textbf{Data} \times \textbf{Path}(t) \rightarrow \textbf{PIEOTree}(t)$ by

$$\frac{\text{push}(p, \text{pkt}, d, r) = p'}{\text{push}(\text{Leaf}(p), \text{pkt}, d, r) = \text{Leaf}(p')} \qquad \frac{\text{push}(p, i, d, r) = p' \qquad \text{push}(qs[i], \text{pkt}, d, pt) = q'}{\text{push}(\text{Internal}(qs, p), \text{pkt}, d, (i, r) :: pt) = \text{Internal}(qs[q'/i], p')}$$

Definition 1.5. Let $t \in \textbf{Topo}$. A *control* over t is a triple (s, q, z), where $s \in \text{St}$ is the *current state*, q is a PIEO tree of topology t, and $z : \text{St} \times \textbf{Pkt} \rightarrow \textbf{Data} \times \textbf{Path}(t) \times \text{St}$ is a function called the *scheduling transaction*. We write Control(t) for the set of controls over t.

Definition 1.6. Define $|\cdot| : \textbf{PIEOTree}(t) \rightarrow \mathbb{N}$ by

$$|\text{Leaf}(p)| = |p| \qquad\qquad |\text{Internal}(qs, p)| = \sum_{i=1}^{|qs|} |qs[i]|$$

We say that $q \in \textbf{PIEOTree}(t)$ is *well-formed* w.r.t $f \in \mathcal{F}$, denoted $\vdash_f q$, if it adheres to the following rules.

$$\frac{}{\vdash_f \text{Leaf}(p)} \qquad \frac{\forall i \in [1, |qs|]\ \vdash_f qs[i] \wedge |p|_i = |qs[i]|}{\vdash_f \text{Internal}(qs, p)}$$

We say q is well-formed, denoted $\vdash q$, if for all $f \in \mathcal{F}$, $\vdash_f q$.

References

[MLF+23] Anshuman Mohan, Yunhe Liu, Nate Foster, Tobias Kappé, and Dexter Kozen. Formal abstractions for packet scheduling, 2023.