Preservation Of Well-Formedness In PIFO Trees

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1 Relevant Definitions and Lemmas

1.1 Definition 1

Defining $|\cdot|$ ***on PIFOs With Regards to Pushes***

Definition 1A: *Pushing an element to a PIFO increments how many times it appears*

$$\frac{p \in \text{PIFO} \qquad |p|_i = n \qquad n \in \mathbb{N} \qquad \text{PUSH}(p, i, r) = p' \qquad n' = n + 1}{|p'|_i = n'}$$

Definition 1B: *Pushing an element to a PIFO does not change how much others appear*

$$\frac{p \in \text{PIFO} \qquad |p|_i = n \qquad n \in \mathbb{N} \qquad \text{PUSH}(p, j, r) = p' \qquad i \neq j}{|p'|_i = n}$$

Definition 1C: *Pushing an element to a PIFO increments its size*

$$\frac{p \in \text{PIFO} \qquad |p| = n \qquad n \in \mathbb{N} \qquad \text{PUSH}(p, i, r) = p' \qquad n' = n + 1}{|p'| = n'}$$

1.2 Definition 2

Defining $|\cdot|$ ***on PIFOs With Regards to Pops***

Definition 2A: *Popping an element from a PIFO decrements how many times it appears*

$$\frac{p \in \text{PIFO} \qquad |p|_i = n \qquad n \in \mathbb{N} \qquad \text{POP}(p) = (i, p') \qquad n' = n - 1}{|p'|_i = n'}$$

Definition 2B: *Popping an element from a PIFO does not change how much others appear*

$$\frac{p \in \text{PIFO} \qquad |p|_i = n \qquad n \in \mathbb{N} \qquad \text{POP}(p) = (j, p') \qquad i \neq j}{|p'|_i = n}$$

Definition 2C: *Popping an element from a PIFO decrements its size*

$$\frac{p \in \text{PIFO} \qquad |p| = n \qquad n \in \mathbb{N} \qquad \text{POP}(p) = (i, p') \qquad n' = n - 1}{|p'| = n'}$$

2 Lemmas 1 and 2

Defining $|\cdot|$ on PIFO Trees With Regards to Pushes and Pops

2.1 Lemma 1:

Pushing to a PIFO Tree increments its size

$$\frac{q \in \text{PIFOTree} \qquad |q| = n \qquad n \in \mathbb{N} \qquad \text{push}(q, pkt, p) = q' \qquad n' = n + 1}{|q'| = n'}$$

Proof. We proceed upon this proof by structural induction over the definition of $|\cdot|$ as follows.

Base Case: $q = \text{Leaf}(p)$ for some packet p.

By definition 3.8, we have that $|\text{Leaf}(p)|_i = |p|_i$, where $|p|_i$ is the number of occurrences of i in PIFO p. The definition of *push* in 3.6 gives the following:

$$\frac{\text{PUSH}(p, pkt, r) = p'}{push(\text{Leaf}(p), pkt, r) = \text{Leaf}(p')}$$

Where $\text{Leaf}(p')$ is the result of *push*ing into $\text{Leaf}(p)$.

By definition 1A, we now know that $|p'|_i = |p|_i + 1$.

By definition 3.8, we know that $|\text{Leaf}(p')|_i = |p'|_i$ and $|\text{Leaf}(p)|_i = |p|_i$.

Combining these together, we obtain the following:

$$|\text{Leaf}(p')| = |\text{Leaf}(p)| + 1$$

Thus, we have proven our base case.

Inductive Case:

Assume an arbitrary PIFO p, Node $q = \text{Internal}(qs, p)$, packet pkt, path pt, index i and rank r.

Inductive Hypothesis: $|qs[i]| = n \implies |push(qs[i], pkt, pt)| = n + 1$

Show: $|\text{Internal}(qs, p)| = m \implies |\text{push}(\text{Internal}(qs, p), pkt, (i, r) :: pt)| = m + 1$

Note that *push* is defined for internal nodes as follows:

$$\frac{\text{push}(qs[i], pkt, pt) = q' \qquad \text{PUSH}(p, i, r) = p'}{\text{push}(\text{Internal}(qs, p), pkt, (i, r) :: pt) = \text{Internal}(qs[q'/i], p')}$$

Since q is a valid PIFO, we know from Definition 3.8 the following:

$$m = |q| = \sum_{j=1}^{|qs|} |qs[j]|$$

Thus, we can further conclude the following:

$$|\text{push}(\text{Internal}(qs, p), pkt, (i, r) :: pt)| = \sum_{j=1}^{|qs|} |qs[q'/i][j]|$$

From our Inductive Hypothesis, we know that $|q'| = |push(qs[i], pkt, pt| = n + 1$.

From the definition of *push*, pushing changes only $qs[i]$, while the remaining subtrees remain the same. Subsequently, we can assert that $\forall 1 \leq j \leq |qs|, i \neq j \implies |qs[j]| = |qs[q'/i][j]$. With this in mind, we make the following claim:

$$|\text{push}(\text{Internal}(qs, p), pkt, (i, r) :: pt)| = \sum_{j=1}^{|qs|} |qs[j]| - |qs[i]| + |qs[q'/i][i]|$$

Substituting what we know, this gives us the following:

$$|\text{push}(\text{Internal}(qs, p), pkt, (i, r) :: pt)| = m - n + n + 1$$

$$\implies |\text{push}(\text{Internal}(qs, p), pkt, (i, r) :: pt)| = m + 1$$

With this, we have shown the desired equality, and proven our Inductive case.

Thus, it follows that Lemma 1 holds.

2.2 Lemma 2:

Popping from a PIFO Tree decrements its size

$$\frac{q \in \text{PIFOTree} \qquad |q| = n \qquad n \in \mathbb{N} \qquad (pkt, q') = \text{pop}(q) \qquad n' = n - 1}{|q'| = n'}$$

Proof. We proceed upon this proof by structural induction over the definition of $|\cdot|$ as follows.

Base Case: $q = \text{Leaf}(p)$ for some packet p.

By definition 3.8, we have that $|\text{Leaf}(p)|_i = |p|_i$, where $|p|_i$ is the number of occurrences of i in PIFO p. The definition of *pop* in 3.6 gives the following:

$$\frac{\text{POP}(p) = p'}{\mathit{pop}(\text{Leaf}(p)) = \text{Leaf}(pkt, p')}$$

Where $\text{Leaf}(p')$ is the result of *pop*ping from $\text{Leaf}(p)$.

By definition 2A, we now know that $|p'|_i = |p|_i - 1$.

By definition 3.8, we know that $|\text{Leaf}(p')|_i = |p'|_i$ and $|\text{Leaf}(p)|_i = |p|_i$.

Combining these together, we obtain the following:

$$|\text{Leaf}(p')| = |\text{Leaf}(p)| - 1$$

Thus, we have proven our base case.

Inductive Case:

Assume an arbitrary PIFO p, Node $q = \text{Internal}(qs, p)$ and index i.

Inductive Hypothesis: $|qs[i]| = n \implies |q'| = n - 1$ where $(pkt, q') = \mathit{pop}(qs[i])$.

Show: $|\text{Internal}(qs, p)| = m \implies |q''| = m - 1$ where $(pkt, q'') = \text{pop}(\text{Internal}(qs, p)$.

Note that *pop* is defined for internal nodes as follows:

$$\frac{\text{POP(p) = (i, p')} \qquad \text{pop(qs[i]) = (pkt, q')}}{\text{pop}(\text{Internal}(qs, p)) = (pkt, \text{Internal}(qs[q'/i], p'))}$$

Since q is a valid PIFO, we know from Definition 3.8 the following:

$$m = |q| = \sum_{j=1}^{|qs|} |qs[j]|$$

Thus, we can further conclude the following:

$$(pkt, q'') = \text{pop}(\text{Internal}(qs, p) \implies |q''| = \sum_{j=1}^{|qs|} |qs[q'/i][j]|$$

From our Inductive Hypothesis, we know that $|q'| = n - 1$ where $pkt, q' = \text{pop}(qs[i])$.

From the definition of *pop*, popping changes only $qs[i]$, while the remaining subtrees remain the same. Subsequently, we can assert that $\forall 1 \leq j \leq |qs|, i \neq j \implies |qs[j]| = |qs[q'/i][j]$. With this in mind, we make the following claim:

$$(pkt, q'') = \text{pop}(\text{Internal}(qs, p)) \implies |q''| = \sum_{j=1}^{|qs|} |qs[j]| - |qs[i]| + |qs[q'/i][i]|$$

Substituting what we know, this gives us the following:

$$(pkt, qs'') = \text{pop}(\text{Internal}(qs, p)) \implies |q''| = m - n + n - 1$$

$$\implies (pkt, qs'') = \text{pop}(\text{Internal}(qs, p)) \implies |qs''| = m - 1$$

With this, we have shown the desired equality, and proven our Inductive case.

Thus, it follows that Lemma 2 holds.

3 Proofs For Lemma 3.9 (Well-Formedness)

3.1 Proof of Lemma 3.9.1

Well-Formedness Is Preserved in PIFO Trees Upon Pushes

We proceed with this proof by inducting upon the definition of *push* for PIFO Trees.

Base Case : $Leaf(p)$

By definition 3.6 of *push*, we have the following for Leaf nodes:

$$\frac{\text{PUSH}(p, pkt, r) = p'}{push(\text{Leaf}(p), pkt, r) = \text{Leaf}(p')}$$

It follows that after pushing, the resultant tree is expressed as Leaf(p') for some packet p' By definition 3.8 (cited above), we know the following holds for any arbitrary PIFO p:

$$\frac{}{\vdash \text{Leaf}(p)}$$

We now have: $\forall p, pkt, r, \vdash push(\text{Leaf}(p), pkt, r)$

With this, our Base Case is proven.

Inductive Case: $q = \text{Internal}(qs, p), \vdash q$

Inductive Hypothesis: $\forall 1 \leq i \leq |qs|. \vdash push(qs[i], pkt, pt)$

Recall Definition 3.8 for $\vdash$ on Internal Nodes:

$$\frac{\forall 1 \leq i \leq |qs|. \vdash qs[i] \wedge |p|_i = |qs[i]|}{\vdash \text{Internal(qs, p)}}$$

Recall Definition 3.6 for *push* on Internal Nodes:

$$\frac{\text{push}(qs[i], pkt, pt) = q' \qquad \text{PUSH}(p, i, r) = p'}{\text{push}(\text{Internal}(qs, p), pkt, (i, r) :: pt) = \text{Internal}(qs[q'/i], p')}$$

Now, let $q' = push(q, pkt, (i, r) :: pt)$, and let $p' = \text{PUSH}(p, i, r)$.

Show: $\vdash q \implies \vdash \text{Internal}(qs[q'/i], p')$.

Using definition 3.6 and our Inductive Hypothesis, we can conclude that $\vdash q'$. Furthermore, note that the only subtree to be modified in qs is in $qs[i]$, per the definition of *push*.

Now we make the following claim:

(1) $\vdash \text{Internal}(qs, p)$	*By definition*
(2) $\implies \forall 1 \leq j \leq \|qs\|. \ \vdash qs[j] \wedge \|p\|_j = \|qs[j]\|$	*Inversion Lemma (Definition 3.8)*
(3) $\implies \|p\|_i = \|qs[i]\|$	*Instance of universal quantifier (2)*
(4) $\|p'\|_i = \|p\|_i + 1$	*Definition 1A*
(5) $\|q'\| = \|qs[i]\| + 1$	*Lemma 1*
(6) $\implies \|p\|_i + 1 = \|qs[i]\| + 1$	*Addition Property of Equality*
(7) $\implies \|p'\|_i = \|q'\|$	*Substitution (4, 5, 6)*
(8) $\forall 1 \leq j \leq \|p'\|, i \neq j \implies \|p'\|_j = \|p\|_j$	*Definition 1B*
(9) $\implies \forall 1 \leq j \leq \|p'\|, i \neq j \implies \|p'\|_j = \|qs[j]\|$	*By (2) and (8)*
(10) $\implies \forall 1 \leq j \leq \|p'\|, \|p'\|_j = \|qs[q'/i][j]\|$	*By (7) and (9)*
(11) $\forall 1 \leq j \leq \|qs\|. \vdash qs[q'/i][j]$	*Inductive Hypothesis*
(12) $\implies \forall 1 \leq j \leq \|qs\|. \vdash qs[q'/i][j] \wedge \|p'\|_j = \|qs[q'/i][j]\|$	*By (10) and (11)*
(13) $\implies \vdash \text{Internal}(qs[q'/i], p')$	*Definition of* $\vdash$

With this, we have proven our Inductive Statement and completed the proof. We have shown that pushing to an arbitrary PIFO Tree preserves its well-formedness.

3.2 Proof of Lemma 3.9.2

Well-Formedness Is Preserved in PIFO Trees Upon Pops

We proceed with this proof by inducting upon the definition of *pop* for PIFO Trees.

Base Case : $Leaf(p)$

By definition 3.4 of *pop*, the following holds for Leaf nodes:

$$\frac{\mathrm{POP}(p) = (pkt, p')}{pop(\mathrm{Leaf}(p)) = (pkt, \mathrm{Leaf}(p'))}$$

It follows that after popping, the resultant tree is of form $\mathrm{Leaf}(p')$ for some packet p'. By definition 3.8 (cited above), we know the following holds for any arbitrary PIFO p:

$$\frac{}{\vdash \mathrm{Leaf}(p)}$$

We now have: $\forall p, \vdash p'$ where $(pkt, p') = pop(\mathrm{Leaf}(p))$.

With this, our Base Case is proven.