# PIEO Trees for Fun and Profit

We assume familiarity with [MLF+23], adopt its notational conventions, and borrow many of its definitions!

## 1 Structure & Semantics

**Definition 1.1.** For sets $S$, $D$, and predicates $F$ over $D$, let $\mathbf{PIEO}(S, D, F)$ denote the set of *PIEO*s that

(1) hold entries in $S$, decorated with meta-data in $D$
(2) are ordered by **Rk**
(3) support predicates in $F$
(4) admit partial functions

$$\begin{aligned} \mathrm{pop} &: \mathbf{PIEO}(S, D, F) \times F \rightharpoonup S \times \mathbf{PIEO}(S, D, F) \\ \mathrm{push} &: \mathbf{PIEO}(S, D, F) \times S \times D \times \mathbf{Rk} \to \mathbf{PIEO}(S, D, F) \end{aligned}$$

For $p \in \mathbf{PIEO}(S, D, F)$, $s \in S$, and $f \in F$, we write

(1) $|p|$ for the number of entries in $p$
(2) $|p|_s$ for the number of times $s$ occurs in $p$
(3) $|p|_{s,f}$ for the number of times $s$ occurs in $p$ with associated $d \in D$ such that $f(d)$ holds

We fix an opaque set **Data** and a collection $\mathcal{F}$ of predicates defined on it. These predicates come with a total order $\leq$ and the property that, $\forall d \in \mathbf{Data}$ and $f, f' \in \mathcal{F}$, $f \leq f' \wedge f(d) \implies f'(d)$.

**Definition 1.2.** The set of *PIEO trees* over $t \in \mathbf{Topo}$, denoted $\mathbf{PIEOTree}(t)$, is defined inductively by

$$\frac{p \in \mathbf{PIEO}(\mathbf{Pkt}, \mathbf{Data}, \mathcal{F})}{\mathrm{Leaf}(p) \in \mathbf{PIEOTree}(*)} \qquad \frac{n \in \mathbb{N} \qquad ts \in \mathbf{Topo}^n \qquad p \in \mathbf{PIEO}(\{1, \ldots, n\}, \mathbf{Data}, \mathcal{F}) \quad \forall i \in [1, n].\ qs[i] \in \mathbf{PIEOTree}(ts[i])}{\mathrm{Internal}(qs, p) \in \mathbf{PIEO}(\mathrm{Node}(ts))}$$

**Definition 1.3.** Define $\mathrm{pop} : \mathbf{PIEOTree}(t) \times \mathcal{F} \rightharpoonup \mathbf{Pkt} \times \mathbf{PIEOTree}(t)$ by

$$\frac{\mathrm{pop}(p, f) = (\mathrm{pkt}, p')}{\mathrm{pop}(\mathrm{Leaf}(p), f) = (\mathrm{pkt}, \mathrm{Leaf}(p'))} \qquad \frac{\mathrm{pop}(p, f) = (i, p') \qquad \mathrm{pop}(qs[i], f) = (\mathrm{pkt}, q')}{\mathrm{pop}(\mathrm{Internal}(qs, p), f) = (\mathrm{pkt}, \mathrm{Internal}(qs[q'/i], p'))}$$

**Definition 1.4.** Define $\mathrm{push} : \mathbf{PIEOTree}(t) \times \mathbf{Pkt} \times \mathbf{Data} \times \mathbf{Path}(t) \to \mathbf{PIEOTree}(t)$ by

$$\frac{\mathrm{push}(p, \mathrm{pkt}, d, r) = p'}{\mathrm{push}(\mathrm{Leaf}(p), \mathrm{pkt}, d, r) = \mathrm{Leaf}(p')} \qquad \frac{\mathrm{push}(p, i, d, r) = p' \qquad \mathrm{push}(qs[i], \mathrm{pkt}, d, pt) = q'}{\mathrm{push}(\mathrm{Internal}(qs, p), \mathrm{pkt}, d, (i, r) :: pt) = \mathrm{Internal}(qs[q'/i], p')}$$

**Definition 1.5.** Let $t \in \mathbf{Topo}$. A *control* over $t$ is a triple $(s, q, z)$, where $s \in \mathrm{St}$ is the *current state*, $q$ is a PIEO tree of topology $t$, and $z : \mathrm{St} \times \mathbf{Pkt} \to \mathbf{Data} \times \mathbf{Path}(t) \times \mathrm{St}$ is a function called the *scheduling transaction*. We write $\mathrm{Control}(t)$ for the set of controls over $t$.

**Definition 1.6.** Define $|\cdot| : \mathbf{PIEOTree}(t) \to \mathbb{N}$ by

$$|\,\mathrm{Leaf}(p)| = |p| \qquad\qquad |\,\mathrm{Internal}(qs, p)| = \sum_{i=1}^{|qs|} |qs[i]|$$

We say that $q \in \mathbf{PIEOTree}(t)$ is *well-formed* w.r.t $f \in \mathcal{F}$, denoted $\vdash_f q$, if it adheres to the following rules.

$$\frac{}{\vdash_f \mathrm{Leaf}(p)} \qquad\qquad \frac{\forall i \in [1, |qs|]\ \vdash_f qs[i] \wedge |p|_i = |qs[i]|}{\vdash_f \mathrm{Internal}(qs, p)}$$

We say $q$ is well-formed, denoted $\vdash q$, if for all $f \in \mathcal{F}$, $\vdash_f q$.

## 2 Embedding and Simulation

**Definition 2.1.** Let $t_1, t_2 \in \textbf{Topo}$. We call a relation $R \subseteq \textbf{PIFOTree}(t_1) \times \textbf{PIFOTree}(t_2)$ a *simulation* if, for all $\mathsf{pkt} \in \textbf{Pkt}$, $f \in \mathcal{F}$, and $q_1 \mathrel{R} q_2$,

(1) If $\mathsf{pop}(q_1, f)$ is undefined, then so is $\mathsf{pop}(q_2, f)$
(2) If $\mathsf{pop}(q_1, f) = (\mathsf{pkt}, q_1')$, then $\mathsf{pop}(q_2) = (\mathsf{pkt}, q_2')$ such that $q_1' \mathrel{R} q_2'$.
(3) For all $pt_1 \in \textbf{Path}(t_1)$ and $d \in \textbf{Data}$, there exists $pt_2 \in \textbf{Path}(t_2)$ such that

$$\mathsf{push}(q_1, \mathsf{pkt}, d, pt_1) \mathrel{R} \mathsf{push}(q_2, \mathsf{pkt}, d, pt_2)$$

# References

[MLF+23] Anshuman Mohan, Yunhe Liu, Nate Foster, Tobias Kappé, and Dexter Kozen. Formal abstractions for packet scheduling, 2023.