

1 Relevant Definitions and Lemmas

1.1 Definition 1

Defining $|\cdot|$ on PIFOs With Regards to Pushes

Definition 1A: *Pushing an element to a PIFO increments how many times it appears*

$$\frac{p \in \text{PIFO} \quad |p|_i = n \quad n \in \mathbb{N} \quad \text{PUSH}(p, i, r) = p' \quad n' = n + 1}{|p'|_i = n'}$$

Definition 1B: *Pushing an element to a PIFO does not change how much others appear*

$$\frac{p \in \text{PIFO} \quad |p|_i = n \quad n \in \mathbb{N} \quad \text{PUSH}(p, j, r) = p' \quad i \neq j}{|p'|_i = n}$$

Definition 1C: *Pushing an element to a PIFO increments its size*

$$\frac{p \in \text{PIFO} \quad |p| = n \quad n \in \mathbb{N} \quad \text{PUSH}(p, i, r) = p' \quad n' = n + 1}{|p'| = n'}$$

1.2 Definition 2

Defining $|\cdot|$ on PIFOs With Regards to Pops

Definition 2A: *Popping an element from a PIFO decrements how many times it appears*

$$\frac{p \in \text{PIFO} \quad |p|_i = n \quad n \in \mathbb{N} \quad \text{POP}(p) = (i, p') \quad n' = n - 1}{|p'|_i = n'}$$

Definition 2B: *Popping an element from a PIFO does not change how much others appear*

$$\frac{p \in \text{PIFO} \quad |p|_i = n \quad n \in \mathbb{N} \quad \text{POP}(p) = (j, p') \quad i \neq j}{|p'|_i = n}$$

Definition 2C: *Popping an element from a PIFO decrements its size*

$$\frac{p \in \text{PIFO} \quad |p| = n \quad n \in \mathbb{N} \quad \text{POP}(p) = (i, p') \quad n' = n - 1}{|p'| = n'}$$

2 Lemmas 1 and 2

Defining $|\cdot|$ on PIFO Trees With Regards to Pushes and Pops

2.1 Lemma 1:

Pushing to a PIFO Tree increments its size

$$\frac{q \in \text{PIFOTree} \quad |q| = n \quad n \in \mathbb{N} \quad \text{push}(q, pkt, p) = q' \quad n' = n + 1}{|q'| = n'}$$

Proof. We proceed upon this proof by structural induction over the definition of $|\cdot|$ as follows.

Base Case: $q = \text{Leaf}(p)$ for some packet p .

By definition 3.8, we have that $|\text{Leaf}(p)|_i = |p|_i$, where $|p|_i$ is the number of occurrences of i in PIFO p . The definition of *push* in 3.6 gives the following:

$$\frac{\text{PUSH}(p, pkt, r) = p'}{\text{push}(\text{Leaf}(p), pkt, r) = \text{Leaf}(p')}$$

Where $\text{Leaf}(p')$ is the result of *pushing* into $\text{Leaf}(p)$.

By definition 1A, we now know that $|p'|_i = |p|_i + 1$.

By definition 3.8, we know that $|\text{Leaf}(p')|_i = |p'|_i$ and $|\text{Leaf}(p)|_i = |p|_i$.

Combining these together, we obtain the following:

$$|\text{Leaf}(p')| = |\text{Leaf}(p)| + 1$$

Thus, we have proven our base case.

Inductive Case:

Assume an arbitrary PIFO p , Node $q = \text{Internal}(qs, p)$, packet pkt , path pt , index i and rank r .

Inductive Hypothesis: $|qs[i]| = n \implies |\text{push}(qs[i], pkt, pt)| = n + 1$

Show: $|\text{Internal}(qs, p)| = m \implies |\text{push}(\text{Internal}(qs, p), pkt, (i, r) :: pt)| = m + 1$

Note that *push* is defined for internal nodes as follows:

$$\frac{\text{push}(qs[i], pkt, pt) = q' \quad \text{PUSH}(p, i, r) = p'}{\text{push}(\text{Internal}(qs, p), pkt, (i, r) :: pt) = \text{Internal}(qs[q'/i], p')}$$

Since q is a valid PIFO, we know from Definition 3.8 the following:

$$m = |q| = \sum_{j=1}^{|qs|} |qs[j]|$$

Thus, we can further conclude the following:

$$|\text{push}(\text{Internal}(qs, p), pkt, (i, r) :: pt)| = \sum_{j=1}^{|qs|} |qs[q'/i][j]|$$

From our Inductive Hypothesis, we know that $|q'| = |\text{push}(qs[i], pkt, pt)| = n + 1$.

From the definition of *push*, pushing changes only $qs[i]$, while the remaining subtrees remain the same. Subsequently, we can assert that $\forall 1 \leq j \leq |qs|, i \neq j \implies |qs[j]| = |qs[q'/i][j]|$. With this in mind, we make the following claim:

$$|\text{push}(\text{Internal}(qs, p), pkt, (i, r) :: pt)| = \sum_{j=1}^{|qs|} |qs[j]| - |qs[i]| + |qs[q'/i][i]|$$

Substituting what we know, this gives us the following:

$$\begin{aligned} |\text{push}(\text{Internal}(qs, p), pkt, (i, r) :: pt)| &= m - n + n + 1 \\ \implies |\text{push}(\text{Internal}(qs, p), pkt, (i, r) :: pt)| &= m + 1 \end{aligned}$$

With this, we have shown the desired equality, and proven our Inductive case.

Thus, it follows that Lemma 1 holds.

3 Proofs For Lemma 3.9 (Well-Formedness)

3.1 Proof of Lemma 3.9.1

Well-Formedness Is Preserved in PIFO Trees Upon Pushes

We proceed with this proof by inducting upon the definition of *push* for PIFO Trees.

Base Case : $\text{Leaf}(p)$

By definition 3.6 of *push*, we have the following for Leaf nodes:

$$\frac{\text{PUSH}(p, pkt, r) = p'}{\text{push}(\text{Leaf}(p), pkt, r) = \text{Leaf}(p')}$$

It follows that after pushing, the resultant tree is expressed as $\text{Leaf}(p')$ for some packet p' . By definition 3.8 (cited above), we know the following holds for any arbitrary PIFO p :

$$\overline{\vdash \text{Leaf}(p)}$$

We now have: $\forall p, pkt, r, \vdash \text{push}(\text{Leaf}(p), pkt, r)$

With this, our Base Case is proven.

Inductive Case: $q = \text{Internal}(qs, p), \vdash q$

Inductive Hypothesis: $\forall 1 \leq i \leq |qs|. \vdash \text{push}(qs[i], pkt, pt)$

Recall Definition 3.8 for \vdash on Internal Nodes:

$$\frac{\forall 1 \leq i \leq |qs|. \vdash qs[i] \wedge |p|_i = |qs[i]|}{\vdash \text{Internal}(qs, p)}$$

Recall Definition 3.6 for *push* on Internal Nodes:

$$\frac{\text{push}(qs[i], pkt, pt) = q' \quad \text{PUSH}(p, i, r) = p'}{\text{push}(\text{Internal}(qs, p), pkt, (i, r) :: pt) = \text{Internal}(qs[q'/i], p')}$$

Now, let $q' = \text{push}(q, \text{pkt}, (i, r) :: \text{pt})$, and let $p' = \text{PUSH}(p, i, r)$.

Show: $\vdash q \implies \vdash \text{Internal}(qs[q'/i], p')$.

Using definition 3.6 and our Inductive Hypothesis, we can conclude that $\vdash q'$. Furthermore, note that the only subtree to be modified in qs is in $qs[i]$, per the definition of push .

Now we make the following claim:

(1) $\vdash \text{Internal}(qs, p)$	<i>By definition</i>
(2) $\implies \forall 1 \leq j \leq qs . \vdash qs[j] \wedge p _j = qs[j] $	<i>Inversion Lemma (Definition 3.8)</i>
(3) $\implies p _i = qs[i] $	<i>Instance of universal quantifier (2)</i>
(4) $ p' _i = p _i + 1$	<i>Definition 1A</i>
(5) $ q' = qs[i] + 1$	<i>Lemma 1</i>
(6) $\implies p _i + 1 = qs[i] + 1$	<i>Addition Property of Equality</i>
(7) $\implies p' _i = q' $	<i>Substitution (4, 5, 6)</i>
(8) $\forall 1 \leq j \leq p' , i \neq j \implies p' _j = p _j$	<i>Definition 1B</i>
(9) $\implies \forall 1 \leq j \leq p' , i \neq j \implies p' _j = qs[j] $	<i>By (2) and (8)</i>
(10) $\implies \forall 1 \leq j \leq p' , p' _j = qs[q'/i][j] $	<i>By (7) and (9)</i>
(11) $\forall 1 \leq j \leq qs . \vdash qs[q'/i][j]$	<i>Inductive Hypothesis</i>
(12) $\implies \forall 1 \leq j \leq qs . \vdash qs[q'/i][j] \wedge p' _j = qs[q'/i][j] $	<i>By (10) and (11)</i>
(13) $\implies \vdash \text{Internal}(qs[q'/i], p')$	<i>Definition of \vdash</i>

With this, we have proven our Inductive Statement and completed the proof. We have shown that pushing to an arbitrary PIFO Tree preserves its well-formedness.

3.2 Proof of Lemma 3.9.2

Well-Formedness Is Preserved in PIFO Trees Upon Pops

We proceed with this proof by inducting upon the definition of pop for PIFO Trees.

Base Case : $\text{Leaf}(p)$

By definition 3.4 of pop , the following holds for Leaf nodes:

$$\frac{\text{POP}(p) = (\text{pkt}, p')}{\text{pop}(\text{Leaf}(p)) = (\text{pkt}, \text{Leaf}(p'))}$$

It follows that after popping, the resultant tree is of form $\text{Leaf}(p')$ for some packet p' . By definition 3.8 (cited above), we know the following holds for any arbitrary PIFO p :

$$\overline{\vdash \text{Leaf}(p)}$$

We now have: $\forall p, \vdash p'$ where $(\text{pkt}, p') = \text{pop}(\text{Leaf}(p))$.

With this, our Base Case is proven.