# PIEO Trees for Fun and Profit

We assume familiarity with [**?**], adopt its notational conventions, and borrow many of its definitions!

## 1 Structure & Semantics

**Definition 1.1.** For sets $S$, $D$, and predicates $F$ over $D$, let $\mathbf{PIEO}(S, D, F)$ denote the set of *PIEO*s that

(1) hold entries in $S$, decorated with meta-data in $D$
(2) are ordered by **Rk**
(3) support predicates in $F$
(4) admit partial functions

$$\begin{aligned} \mathrm{pop} &: \mathbf{PIEO}(S, D, F) \times F \rightharpoonup S \times \mathbf{PIEO}(S, D, F) \\ \mathrm{push} &: \mathbf{PIEO}(S, D, F) \times S \times D \times \mathbf{Rk} \to \mathbf{PIEO}(S, D, F) \\ \mathrm{proj} &: \mathbf{PIEO}(S, D, F) \times F \to \mathbf{PIFO}(S) \end{aligned}$$

Maps push and pop are as usual. The *projection* $\mathrm{proj}(p, f)$ is the PIFO of entries in $p$ with data satisfying $f$. These three maps play nicely together:

$$\mathrm{pop}(p, f) \text{ is undefined} \iff \mathrm{pop}(\mathrm{proj}(p, f)) \text{ is undefined} \tag{1}$$

$$\mathrm{pop}(p, f) = (\mathrm{pkt}, p') \iff \mathrm{pop}(\mathrm{proj}(p, f)) = (\mathrm{pkt}, \mathrm{proj}(p', f)) \tag{2}$$

$$\mathrm{proj}(\mathrm{push}(p, s, d, r), f) = \begin{cases} \mathrm{push}(\mathrm{proj}(p, f), s, r) & f(d) \text{ holds true} \\ \mathrm{proj}(p, f) & \text{otherwise} \end{cases} \tag{3}$$

We consider PIEOs $p, p'$ equal if, for all $f \in F$, $\mathrm{proj}(p, f) = \mathrm{proj}(p', f)$, i.e. their projections are always equal. For PIEO $p$, entry $s \in S$, and predicate $f \in F$, we write

(1) $|p|$ for the number of entries in $p$
(2) $|p|_f$ for the number of entries in $p$ satisfying $f$
(3) $|p|_{s,f}$ for the number of times $s$ occurs in $p$ with associated $d \in D$ such that $f(d)$ holds

We fix an opaque set **Data** and a collection $\mathcal{F}$ of predicates defined on it. These predicates come with a total order $\leq$ and the property that, $\forall d \in \mathbf{Data}$ and $f, f' \in \mathcal{F}$, $f \leq f' \wedge f(d) \implies f'(d)$.

**Definition 1.2.** The set of *PIEO trees* over $t \in \mathbf{Topo}$, denoted $\mathbf{PIEOTree}(t)$, is defined inductively by

$$\frac{p \in \mathbf{PIEO}(\mathbf{Pkt}, \mathbf{Data}, \mathcal{F})}{\mathrm{Leaf}(p) \in \mathbf{PIEOTree}(*)} \qquad \frac{n \in \mathbb{N} \qquad ts \in \mathbf{Topo}^n \qquad p \in \mathbf{PIEO}(\{1, \ldots, n\}, \mathbf{Data}, \mathcal{F}) \qquad \forall i \in [1, n].\ qs[i] \in \mathbf{PIEOTree}(ts[i])}{\mathrm{Internal}(qs, p) \in \mathbf{PIEO}(\mathrm{Node}(ts))}$$

**Definition 1.3.** Define $\mathrm{pop} : \mathbf{PIEOTree}(t) \times \mathcal{F} \rightharpoonup \mathbf{Pkt} \times \mathbf{PIEOTree}(t)$ by

$$\frac{\mathrm{pop}(p, f) = (\mathrm{pkt}, p')}{\mathrm{pop}(\mathrm{Leaf}(p), f) = (\mathrm{pkt}, \mathrm{Leaf}(p'))} \qquad \frac{\mathrm{pop}(p, f) = (i, p') \qquad \mathrm{pop}(qs[i], f) = (\mathrm{pkt}, q')}{\mathrm{pop}(\mathrm{Internal}(qs, p), f) = (\mathrm{pkt}, \mathrm{Internal}(qs[q'/i], p'))}$$

**Definition 1.4.** Define $\mathrm{push} : \mathbf{PIEOTree}(t) \times \mathbf{Pkt} \times \mathbf{Data} \times \mathbf{Path}(t) \to \mathbf{PIEOTree}(t)$ by

$$\frac{\mathrm{push}(p, \mathrm{pkt}, d, r) = p'}{\mathrm{push}(\mathrm{Leaf}(p), \mathrm{pkt}, d, r) = \mathrm{Leaf}(p')} \qquad \frac{\mathrm{push}(p, i, d, r) = p' \qquad \mathrm{push}(qs[i], \mathrm{pkt}, d, pt) = q'}{\mathrm{push}(\mathrm{Internal}(qs, p), \mathrm{pkt}, d, (i, r) :: pt) = \mathrm{Internal}(qs[q'/i], p')}$$

**Definition 1.5.** Let $t \in \mathbf{Topo}$. A *control* over $t$ is a triple $(s, q, z)$, where $s \in \mathrm{St}$ is the *current state*, $q$ is a PIEO tree of topology $t$, and

$$z : \mathrm{St} \times \mathbf{Pkt} \to \mathbf{Data} \times \mathbf{Path}(t) \times \mathrm{St}$$

is a function called the *scheduling transaction*.

## 2 Well-Formedness

**Definition 2.1.** Fix $f \in \mathcal{F}$. Define $|\cdot|_f : \textbf{PIEOTree}(t) \to \mathbb{N}$ by

$$|\operatorname{Leaf}(p)|_f = |p|_f \qquad\qquad |\operatorname{Internal}(qs, p)|_f = \sum_{i=1}^{|qs|} |qs[i]|_f$$

We say that $q \in \textbf{PIEOTree}(t)$ is *well-formed* w.r.t $f$, denoted $\vdash_f q$, if it adheres to the following rules.

$$\frac{}{\vdash_f \operatorname{Leaf}(p)} \qquad\qquad \frac{\forall i \in [1, |qs|],\ \vdash_f qs[i] \wedge |p|_{i,f} = |qs[i]|_f}{\vdash_f \operatorname{Internal}(qs, p)}$$

We say $q$ is well-formed, denoted $\vdash q$, if there exists $f \in \mathcal{F}$ such that, for all $f' \geq f$, $\vdash_{f'} q$.

**Theorem 2.2.** *Let* $t \in \textbf{Topo}$, $\text{pkt} \in \textbf{Pkt}$, $d \in \textbf{Data}$, $f \in \mathcal{F}$, *and* $q \in \textbf{PIEOTree}(t)$ *such that* $\vdash q$.

(1) *If* $pt \in \textbf{Path}(t)$, *then* $\operatorname{push}(q, \text{pkt}, d, pt)$ *is well-defined and* $\vdash \operatorname{push}(q, \text{pkt}, d, pt)$.

(2) *If* $|q|_f > 0$, *then* $\operatorname{pop}(q, f)$ *is well-defined and* $\vdash q'$, *where* $\operatorname{pop}(q, f) = (\text{pkt}, q')$.

*Proof.* TBD $\square$

## 3 Projection

**Definition 3.1.** For $f \in \mathcal{F}$, define $\mathrm{proj}_f : \textbf{PIEOTree}(t) \to \textbf{PIFOTree}(t)$ by

$$\frac{p' = \mathrm{proj}(p, f)}{\mathrm{proj}_f(\mathrm{Leaf}(p)) = \mathrm{Leaf}(p')} \qquad \frac{p' = \mathrm{proj}(p, f) \qquad \forall i \in [1, |qs|],\ qs'[i] = \mathrm{proj}_f(qs[i])}{\mathrm{proj}_f(\mathrm{Internal}(qs, p)) = \mathrm{Internal}(qs', p')}$$

**Lemma 3.2.** *For $q, q' \in \textbf{PIEOTree}(t)$,*

$$\forall f \in \mathcal{F},\ \mathrm{proj}_f(q) = \mathrm{proj}_f(q') \implies q = q'$$

*Proof.* Suppose $\mathrm{proj}_f(q) = \mathrm{proj}_f(q')$ for all $f \in \mathcal{F}$. We'll proceed by induction on $t$ to show $q = q'$.

(Leaf) For $t = *$, let $q = \mathrm{Leaf}(p)$ and $q' = \mathrm{Leaf}(p')$. Since

$$\mathrm{proj}_f(q) = \mathrm{Leaf}(\mathrm{proj}(p, f)) = \mathrm{Leaf}(\mathrm{proj}(p', f)) = \mathrm{proj}_f(q')$$

we know $\mathrm{proj}(p, f) = \mathrm{proj}(p', f)$ for all $f \in \mathcal{F}$. By **??**, $p = p'$ and hence $q = q'$.

(Node) For $t = \mathrm{Node}(ts)$ and $n = |ts|$, let $q = \mathrm{Internal}(qs, p)$ and $q' = \mathrm{Internal}(qs', p')$. Notice

$$\begin{aligned} \mathrm{proj}(p, f) &= \mathrm{proj}(p', f) \\ \mathrm{proj}_f(qs[i]) &= \mathrm{proj}_f(qs'[i]) \qquad (i = 1, \dots, n) \end{aligned}$$

for all $f \in \mathcal{F}$. Hence, $p = p'$ via **??** and $qs = qs'$ by the inductive hypothesis, i.e. $q = q'$.

□

**Lemma 3.3.** *For $q \in \textbf{PIEOTree}(t)$ and $f \in \mathcal{F}$,* $\mathrm{pop}(q, f)$ *is undefined if and only if* $\mathrm{pop}(\mathrm{proj}_f(q))$ *is undefined.*

*Proof.* We'll do induction on $t$.

(Leaf) For $t = *$, let $q = \mathrm{Leaf}(p)$ and $\mathrm{proj}_f(q) = \mathrm{Leaf}(p')$. By **??** in **??**,

$$\begin{aligned} \mathrm{pop}(q, f) \text{ is undefined} &\iff \mathrm{pop}(p, f) \text{ is undefined} \\ &\iff \mathrm{pop}(p') \text{ is undefined} \\ &\iff \mathrm{pop}(\mathrm{proj}_f(q)) \text{ is undefined} \end{aligned}$$

(Node) For $t = \mathrm{Node}(ts)$, let $q = \mathrm{Internal}(qs, p)$ and $\mathrm{proj}_f(q) = \mathrm{Internal}(qs', p')$. As before,

$$\mathrm{pop}(p, f) \text{ is undefined} \iff \mathrm{pop}(p') \text{ is undefined}$$

by **??** in **??** and

$$\mathrm{pop}(qs[i], f) \text{ is undefined} \iff \mathrm{pop}(qs'[i]) \text{ is undefined } \forall i \in [1, |ts|]$$

by the inductive hypothesis. Hence, using **??** in **??**,

$$\begin{aligned} \mathrm{pop}(q, f) \text{ is undefined} &\iff \mathrm{pop}(p, f) \text{ is undefined} \vee (\mathrm{pop}(p, f) = (i, \_) \wedge \mathrm{pop}(qs[i], f) \text{ is undefined}) \\ &\iff \mathrm{pop}(p') \text{ is undefined} \vee (\mathrm{pop}(p') = (i, \_) \wedge \mathrm{pop}(qs'[i]) \text{ is undefined}) \\ &\iff \mathrm{pop}(\mathrm{proj}_f(q)) \text{ is undefined} \end{aligned}$$

□

**Lemma 3.4.** *For $q \in \textbf{PIEOTree}(t)$ and $f \in \mathcal{F}$,*

$$\mathrm{pop}(q, f) = (\mathrm{pkt}, q') \implies \mathrm{pop}(\mathrm{proj}_f(q)) = (\mathrm{pkt}, \mathrm{proj}_f(q'))$$

*Proof.* More induction on $t$!

(Leaf) For $t = *$, let

$$\begin{aligned} q &= \mathrm{Leaf}(p_1) & \mathrm{proj}_f(q) &= \mathrm{Leaf}(p_2) \\ q' &= \mathrm{Leaf}(p_1') & \mathrm{proj}_f(q') &= \mathrm{Leaf}(p_2') \end{aligned}$$

By **??** in **??**,

$$\begin{aligned}
\mathrm{pop}(q, f) = (\mathrm{pkt}, q') &\implies \mathrm{pop}(p_1, f) = (\mathrm{pkt}, p_1') \\
&\implies \mathrm{pop}(p_2) = \mathrm{pop}(\mathrm{proj}(p_1, f)) = (\mathrm{pkt}, \mathrm{proj}(p_1', f)) = (\mathrm{pkt}, p_2') \\
&\implies \mathrm{pop}(\mathrm{proj}_f(q)) = (\mathrm{pkt}, \mathrm{proj}_f(q'))
\end{aligned}$$

(Node) For $t = \mathrm{Node}(ts)$, Let

$$\begin{aligned}
q &= \mathrm{Internal}(qs_1, p_1) & \mathrm{proj}_f(q) &= \mathrm{Internal}(qs_2, p_2) \\
q' &= \mathrm{Internal}(qs_1', p_1') & \mathrm{proj}_f(q') &= \mathrm{Internal}(qs_2', p_2')
\end{aligned}$$

Using **??** in **??** again and the inductive hypothesis,

$$\begin{aligned}
\mathrm{pop}(q, f) = (\mathrm{pkt}, q') &\implies \mathrm{pop}(p_1, f) = (i, p_1') \wedge \mathrm{pop}(qs_1[i], f) = (\mathrm{pkt}, qs_1'[i]) \\
&\implies \mathrm{pop}(\mathrm{proj}(p_1, f)) = (i, \mathrm{proj}(p_1', f)) \wedge \mathrm{pop}(\mathrm{proj}_f(qs_1[i])) = (\mathrm{pkt}, \mathrm{proj}_f(qs_1'[i])) \\
&\implies \mathrm{pop}(p_2) = (i, p_2') \wedge \mathrm{pop}(qs_2[i]) = (\mathrm{pkt}, qs_2'[i]) \\
&\implies \mathrm{pop}(\mathrm{proj}_f(q)) = (\mathrm{pkt}, \mathrm{proj}_f(q'))
\end{aligned}$$

$\square$

**Lemma 3.5.** *For $q \in \mathbf{PIEOTree}(t)$, $\mathrm{pkt} \in \mathbf{Pkt}$, $d \in \mathbf{Data}$, $pt \in \mathbf{Path}(t)$, and $f \in \mathcal{F}$,*

$$\mathrm{proj}_f(\mathrm{push}(q, \mathrm{pkt}, d, pt)) = \begin{cases} \mathrm{push}(\mathrm{proj}_f(q), \mathrm{pkt}, pt) & f(d) \text{ holds true} \\ \mathrm{proj}_f(q) & \text{otherwise} \end{cases}$$

*Proof.* Even more induction on $t$!

(Leaf) For $t = *$, let $q = \mathrm{Leaf}(p)$ and $pt = r$. By **??** in **??**.

$$\begin{aligned}
\mathrm{proj}_f(\mathrm{push}(q, \mathrm{pkt}, d, pt)) &= \mathrm{Leaf}(\mathrm{proj}(\mathrm{push}(p, \mathrm{pkt}, d, r), f)) \\
&= \begin{cases} \mathrm{Leaf}(\mathrm{push}(\mathrm{proj}(p, f), \mathrm{pkt}, r)) & f(d) \text{ holds true} \\ \mathrm{Leaf}(\mathrm{proj}(p, f)) & \text{otherwise} \end{cases} \\
&= \begin{cases} \mathrm{push}(\mathrm{proj}_f(q), \mathrm{pkt}, pt) & f(d) \text{ holds true} \\ \mathrm{proj}_f(q) & \text{otherwise} \end{cases}
\end{aligned}$$

(Node) For $t = \mathrm{Node}(ts)$, let $pt = (i, r) :: pt'$ and

$$\begin{aligned}
q &= \mathrm{Internal}(qs, p) \\
\mathrm{proj}_f(q) &= \mathrm{Internal}(qs', p') \\
\mathrm{push}(\mathrm{proj}_f(q), \mathrm{pkt}, pt) &= \mathrm{Internal}(qs'', p'') \\
\mathrm{proj}_f(\mathrm{push}(q, \mathrm{pkt}, d, pt)) &= \mathrm{Internal}(qs''', p''')
\end{aligned}$$

By **??** in **??** and the inductive hypothesis,

$$\begin{aligned}
p''' &= \begin{cases} \mathrm{push}(\mathrm{proj}(p, f), i, r) & f(d) \text{ holds true} \\ \mathrm{proj}(p, f) & \text{otherwise} \end{cases} \\
&= \begin{cases} \mathrm{push}(p', i, r) & f(d) \text{ holds true} \\ p' & \text{otherwise} \end{cases} = \begin{cases} p'' & f(d) \text{ holds true} \\ p' & \text{otherwise} \end{cases} \\
qs'''[i] &= \begin{cases} \mathrm{push}(\mathrm{proj}_f(qs[i]), \mathrm{pkt}, pt') & f(d) \text{ holds true} \\ \mathrm{proj}_f(qs[i]) & \text{otherwise} \end{cases} \\
&= \begin{cases} \mathrm{push}(qs'[i], \mathrm{pkt}, pt') & f(d) \text{ holds true} \\ qs'[i] & \text{otherwise} \end{cases} = \begin{cases} qs''[i] & f(d) \text{ holds true} \\ qs'[i] & \text{otherwise} \end{cases}
\end{aligned}$$

By inspection of **??** and **??**,

$$qs'[j] = qs''[j] = qs'''[j] = \mathrm{proj}_f(qs[j])$$

for all $j \in [1, |ts|]$ such that $j \neq i$. Hence,

$$qs''' = \begin{cases} qs'' & f(d) \text{ holds true} \\ qs' & \text{otherwise} \end{cases}$$

Putting everything together,

$$\begin{aligned} \mathrm{proj}_f(\mathrm{push}(q, \mathrm{pkt}, d, pt)) &= \mathrm{Internal}(qs''', p''') \\ &= \begin{cases} \mathrm{Internal}(qs'', p'') & f(d) \text{ holds true} \\ \mathrm{Internal}(qs', p') & \text{otherwise} \end{cases} \\ &= \begin{cases} \mathrm{push}(\mathrm{proj}_f(q), \mathrm{pkt}, pt) & f(d) \text{ holds true} \\ \mathrm{proj}_f(q) & \text{otherwise} \end{cases} \end{aligned}$$

□

# 4 Embedding & Simulation

**Definition 4.1.** Let $t_1, t_2 \in \textbf{Topo}$. We call a relation $R \subseteq \textbf{PIEOTree}(t_1) \times \textbf{PIEOTree}(t_2)$ a *simulation* if, for all pkt $\in \textbf{Pkt}$, $f \in \mathcal{F}$, and $q_1 \; R \; q_2$,

(1) If $\mathrm{pop}(q_1, f)$ is undefined, then so is $\mathrm{pop}(q_2, f)$
(2) If $\mathrm{pop}(q_1, f) = (\mathrm{pkt}, q_1')$, then $\mathrm{pop}(q_2, f) = (\mathrm{pkt}, q_2')$ such that $q_1' \; R \; q_2'$.
(3) For all $pt_1 \in \textbf{Path}(t_1)$ and $d \in \textbf{Data}$, there exists $pt_2 \in \textbf{Path}(t_2)$ such that

$$\mathrm{push}(q_1, \mathrm{pkt}, d, pt_1) \; R \; \mathrm{push}(q_2, \mathrm{pkt}, d, pt_2)$$

If such a simulation exists, we say that $q_1$ is *simulated* by $q_2$, and we write $q_1 \preccurlyeq q_2$.

**Remark 4.2.** For all further discussion, we assume our embeddings are injective.

**Definition 4.3.** For $t_1, t_2 \in \textbf{Topo}$, let $f$ be an embedding from $t_1$ to $t_2$. We lift $f$ to a map $\overline{f}$ from $\textbf{PIEOTree}(t_1)$ to $\textbf{PIEOTree}(t_2)$ inductively.

- For $t_1 = *$, define $\overline{f}(q) = q$. This is well-defined by [**?**, Lemma 5.2].
- For $t_1 = \mathrm{Node}(ts_1)$, $n = |ts_1|$, $q = \mathrm{Internal}(qs, p)$, construct $\overline{f}_\alpha(q) \in \textbf{PIEOTree}(t_2/\alpha)$ for each prefix $\alpha$ of $f(i)$ for some $i \in [1, n]$. Inductively, we'll build up from $f(i)$'s to $\epsilon$ and set $\overline{f}(q) = \overline{f}_\epsilon(q)$.
  - Let $\alpha = f(i)$ for some $i \in [1, n]$. We'll set $\overline{f}_\alpha(q) = \overline{f_i}(qs[i])$, where $f_i$ embeds $t_1/i$ into $t_2/f(i)$ as per [**?**, Lemma 5.2]. This well-defined by the injectivity of $f$.
  - Let $\alpha$ point to a transient node, say with $m$ children. For $1 \leq j \leq m$ such that $\alpha \cdot j$ is not a prefix of some $f(i)$, define $\overline{f}(q)_{\alpha \cdot j}$ to be the PIEO tree with empty PIEOs on all leaves and internal nodes. With this and recursion, we know $\overline{f}(q)_{\alpha \cdot j} \in \textbf{PIEOTree}(t_2/(\alpha \cdot j))$ for all $j \in [1, m]$.
    We create a new PIEO $p_\alpha$ as follows:
    (1) Start with $p_\alpha$ empty
    (2) For each $i$ in $p$ such that $\alpha \cdot j$ is a prefix of $f(i)$, push $j$ into $p_\alpha$ with $i$'s data and rank
    Finally, for all $j \in [1, m]$, set $qs_\alpha[j] = \overline{f}(q)_{\alpha \cdot j}$ and $\overline{f}(q)_\alpha = \mathrm{Internal}(qs_\alpha, p_\alpha)$.

**Theorem 4.4.** *The following diagram commutes*

$$\begin{tikzcd} \textbf{PIEOTree}(t_1) \arrow[r, "\overline{f}"] \arrow[d, "\mathrm{proj}_g"] & \textbf{PIEOTree}(t_2) \arrow[d, "\mathrm{proj}_g"] \\ \textbf{PIFOTree}(t_1) \arrow[r, "\widehat{f}"] & \textbf{PIFOTree}(t_2) \end{tikzcd}$$

*In other words, for $q \in \textbf{PIEOTree}(t_1)$ and $g \in \mathcal{F}$, $\mathrm{proj}_g(\overline{f}(q)) = \widehat{f}(\mathrm{proj}_g(q))$.*

*Proof.* We'll proceed by induction on $t_1$. Suppose $t_1 = *$ and $q = \mathsf{Leaf}(p)$. By [**?**, Lemma 5.3], $t_2 = *$ as well. By **??** and [**?**, Definition 5.4], both $\overline{f}$ and $\widehat{f}$ are the identity. Hence,

$$\mathsf{proj}_g(\overline{f}(q)) = \mathsf{proj}_g(q) = \widehat{f}(\mathsf{proj}_g(q))$$

Suppose $t_1 = \mathsf{Node}(ts)$ and $q = \mathsf{Internal}(qs, p)$. For any prefix $\alpha$ of $f(i)$ for $i \in [1, |ts|]$, we'll show

$$\mathsf{proj}_g(\overline{f}(q)_\alpha) = \widehat{f}(\mathsf{proj}_g(q))_\alpha \tag{$*$}$$

by inverse induction on $\alpha$. Instantiating **??** with $\alpha = \epsilon$ yields the desired result.

- For $\alpha = f(i)$, **??** holds by the outer inductive hypothesis because

$$\mathsf{proj}_g(\overline{f}(q)_\alpha) = \mathsf{proj}_g(\overline{f_i}(q)) = \widehat{f_i}(\mathsf{proj}_g(q)) = \widehat{f}(\mathsf{proj}_g(q))_\alpha$$

- Suppose $\alpha$ is some strict prefix of $f(i)$, pointing to a node with $m$ children. Let

$$\mathsf{proj}_g(\overline{f}(q)) = \mathsf{Internal}(qs', p') \qquad \text{and} \qquad \widehat{f}(\mathsf{proj}_g(q)) = \mathsf{Internal}(qs'', p'')$$

  There's two parts to showing **??**, namely $qs' = qs''$ and $p' = p''$.
  - For all $j \in [1, m]$,

  $$\mathsf{proj}_g(\overline{f}(q)_{\alpha\cdot j}) = \widehat{f}(\mathsf{proj}_g(q))_{\alpha\cdot j}$$

    For $j$ such that $\alpha \cdot j$ is a prefix of some $f(i)$, this follows from the inner inductive hypothesis. For all other $j$, notice the LHS and RHS are both PIFO trees of topology $t_2$, with empty PIFOs on all leaves and internal nodes. Hence, $qs'[j] = qs''[j]$ for all $j \in [1, m]$, i.e. $qs' = qs''$.
  - By inspection, it's clear following the construction for $p_\alpha$ from **??** and then computing the projection $\mathsf{proj}(p_\alpha, g)$ yields the same result as following the recipe for $p_\alpha$ from [**?**, Definition 5.4] on the projection $\mathsf{proj}_g(q)$: that is, exactly when we filter out elements not satisfying $g$ does not matter. Hence, $p' = p''$.

□

**Lemma 4.5.** *Let $t_1, t_2 \in$ **Topo** and $f$ be an embedding of $t_1$ inside $t_2$. For $g \in \mathcal{F}$,*

$$\mathsf{pop}(q, g) \textit{ is undefined} \implies \mathsf{pop}(\overline{f}(q), g) \textit{ is undefined}$$

*Proof.* Suppose $\mathsf{pop}(q, g)$ is undefined. Applying both **??** and [**?**, Lemma 5.6], notice $\mathsf{pop}(\widehat{f}(\mathsf{proj}_g(q)))$ is undefined. By **??**, $\widehat{f}(\mathsf{proj}_g(q)) = \mathsf{proj}_g(\overline{f}(q))$. Hence, $\mathsf{pop}(\mathsf{proj}_g(\overline{f}(q)))$ is undefined. Applying **??** once more, $\mathsf{pop}(\overline{f}(q), g)$ is undefined. □

**Lemma 4.6.** *Let $t_1, t_2 \in$ **Topo** and $f$ be an embedding of $t_1$ inside $t_2$. For $g \in \mathcal{F}$,*

$$\mathsf{pop}(q, g) = (\mathsf{pkt}, q') \implies \mathsf{pop}(\overline{f}(q), g) = (\mathsf{pkt}, \overline{f}(q'))$$

Almost a clone of the proof [**?**, Lemma 5.7].

*Proof.* We'll proceed by induction on $t_1$. Suppose $t_1 = *$. By [**?**, Lemma 5.3], $t_2 = *$ as well. By **??**, $\overline{f}$ is the identity. Hence,

$$\mathsf{pop}(q, g) = (\mathsf{pkt}, q') \implies \mathsf{pop}(\overline{f}(q), g) = \mathsf{pop}(q, g) = (\mathsf{pkt}, q') = (\mathsf{pkt}, \overline{f}(q'))$$

Suppose $t_1 = \mathsf{Node}(ts)$. Let

$$q = \mathsf{Internal}(qs, p) \qquad q' = \mathsf{Internal}(qs', p') \qquad \mathsf{pop}(p, g) = (j, p')$$

For any prefix $\alpha$ of some $f(i)$ (where $i \in [1, |ts|]$), we'll show

$$\begin{aligned} \mathsf{pop}(\overline{f}(q)_\alpha, g) &= (\mathsf{pkt}, \overline{f}(q')_\alpha) && \text{if } \alpha \text{ is a prefix of } f(j) \\ \overline{f}(q)_\alpha &= \overline{f}(q')_\alpha && \text{otherwise} \end{aligned} \tag{$\dagger$}$$

by inverse induction on $\alpha$. Instantiating **??** with $\alpha = \epsilon$ yields the desired result.

- Suppose $\alpha = f(i)$. If $\alpha$ is a prefix of $f(j)$, $i = j$ by injectivity and [**?**, Definition 5.2, Equation (3)]. Recall $\mathsf{pop}(qs[j], g) = (\mathsf{pkt}, qs'[j])$. Hence, by the outer inductive hypothesis,

$$\mathsf{pop}(\overline{f}(q)_\alpha, g) = \mathsf{pop}(\overline{f_j}(qs[j]), g) = (\mathsf{pkt}, \overline{f_j}(qs'[j])) = (\mathsf{pkt}, \overline{f}(q')_\alpha)$$

- Once more, suppose $\alpha = f(i)$. If $\alpha$ is not a prefix of $f(j)$, then $i \neq j$. Since $qs[i] = qs'[i]$,
$$\overline{f}(q)_\alpha = \overline{f}_i(qs[i]) = \overline{f}_i(qs'[i]) = \overline{f}(q')_\alpha$$
- Suppose $\alpha$ is some strict prefix of $f(j)$, pointing to a node with $m$ children. Let
$$\overline{f}(q)_\alpha = \mathsf{Internal}(qs_\alpha, p_\alpha) \qquad \overline{f}(q')_\alpha = \mathsf{Internal}(qs'_\alpha, p'_\alpha)$$
There exists $k \in [1, m]$ such that $\alpha \cdot k$ is a prefix of $f(j)$. By the inner inductive hypothesis,
$$qs_\alpha[i] = \overline{f}(q)_{\alpha\cdot i} = \overline{f}(q')_{\alpha\cdot i} = qs_\alpha[i] \text{ for } i \in [1, m] \text{ with } i \neq k \tag{!}$$
$$\mathsf{pop}(qs_\alpha[k], g) = \mathsf{pop}(\overline{f}(q)_{\alpha\cdot k}, g) = (\mathsf{pkt}, \overline{f}(q')_{\alpha\cdot k}) = (\mathsf{pkt}, qs'[k])$$
Via the construction in **??** and since $\mathsf{pop}(p, g) = p'$, $\mathsf{pop}(p_\alpha, g) = (k, p'_\alpha)$. Putting everything together, $\mathsf{pop}(\overline{f}(q)_\alpha, g) = (\mathsf{pkt}, \overline{f}(q')_\alpha)$, as desired.
- Suppose $\alpha$ is some strict prefix of some $f(i)$ but not $f(j)$, pointing to a node with $m$ children. Let
$$\overline{f}(q)_\alpha = \mathsf{Internal}(qs_\alpha, p_\alpha) \qquad \overline{f}(q')_\alpha = \mathsf{Internal}(qs'_\alpha, p'_\alpha)$$
Since $p$ and $p'$ agree on all indices $i$ such that $f(i)$ is a child of $\alpha$, $p_\alpha = p'_\alpha$. For $i \in [1, m]$, since $\alpha \cdot i$ is not a prefix of $f(j)$, the inner inductive hypothesis yields
$$qs_\alpha[i] = \overline{f}(q)_{\alpha\cdot i} = \overline{f}(q')_{\alpha\cdot i} = qs'_\alpha[i] \tag{!!}$$
Putting everything together, $\overline{f}(q)_\alpha = \overline{f}(q')_\alpha$, as desired.

**NOTE:** even when $\alpha \cdot i$ is not a prefix of any $f(i)$, **??** and **??** hold! Both $\overline{f}(q)_{\alpha\cdot i}$ and $\overline{f}(q')_{\alpha\cdot i}$ would be PIEO trees with empty PIEOs on all leaf and internal nodes. ☐

**Lemma 4.7.** *Let $t_1, t_2 \in$ **Topo** and $f$ be an embedding of $t_1$ inside $t_2$. For $\mathsf{pkt} \in$ **Pkt**, $d \in$ **Data**, and $pt \in$ **Path**$(t_1)$,*
$$\overline{f}(\mathsf{push}(q, \mathsf{pkt}, d, pt)) = \mathsf{push}(\overline{f}(q), \mathsf{pkt}, d, \widetilde{f}(pt))$$

*Proof.* Let $q_2 = \overline{f}(q_1)$. For $g \in \mathcal{F}$ such that $g(d)$ holds true,

$$\begin{aligned}
&(\text{by } \textbf{??}) & \mathsf{proj}_g(\overline{f}(\mathsf{push}(q_1, \mathsf{pkt}, d, pt))) &= \widehat{f}(\mathsf{proj}_g(\mathsf{push}(q_1, \mathsf{pkt}, d, pt))) \\
&(\text{by } \textbf{??}) & &= \widehat{f}(\mathsf{push}(\mathsf{proj}_g(q_1), \mathsf{pkt}, pt)) \\
&(\text{by [\textbf{?}, Lemma 5.9]}) & &= \mathsf{push}(\widehat{f}(\mathsf{proj}_g(q_1)), \mathsf{pkt}, \widetilde{f}(pt)) \\
&(\text{by } \textbf{??}) & &= \mathsf{push}(\mathsf{proj}_g(q_2), \mathsf{pkt}, \widetilde{f}(pt)) \\
&(\text{by } \textbf{??}) & &= \mathsf{proj}_g(\mathsf{push}(q_2, \mathsf{pkt}, d, \widetilde{f}(pt)))
\end{aligned}$$

For $g \in \mathcal{F}$ such that $g(d)$ does not hold true,

$$\begin{aligned}
&(\text{by } \textbf{??}) & \mathsf{proj}_g(\overline{f}(\mathsf{push}(q_1, \mathsf{pkt}, d, pt))) &= \widehat{f}(\mathsf{proj}_g(\mathsf{push}(q_1, \mathsf{pkt}, d, pt))) \\
&(\text{by } \textbf{??}) & &= \widehat{f}(\mathsf{proj}_g(q_1)) \\
&(\text{by } \textbf{??}) & &= \mathsf{proj}_g(q_2) \\
&(\text{by } \textbf{??}) & &= \mathsf{proj}_g(\mathsf{push}(q_2, \mathsf{pkt}, d, \widetilde{f}(pt)))
\end{aligned}$$

Overall, $\mathsf{proj}_g(\overline{f}(\mathsf{push}(q_1, \mathsf{pkt}, d, pt))) = \mathsf{proj}_g(\mathsf{push}(q_2, \mathsf{pkt}, d, \widetilde{f}(pt)))$ for all $g \in \mathcal{F}$. Hence,
$$\overline{f}(\mathsf{push}(q_1, \mathsf{pkt}, d, pt)) = \mathsf{push}(q_2, \mathsf{pkt}, d, \widetilde{f}(pt))$$
by **??**, as desired. ☐

**Theorem 4.8.** *Let $t_1, t_2 \in$ **Topo**. If $f$ embeds $t_1$ into $t_2$, then*
$$R = \{(q, \overline{f}(q)) \mid q \in \textbf{PIEOTree}(t_1)\}$$
*is a simulation.*

*Proof.* By **??**, **??**, and **??**, the conditions from **??** hold. ☐