1 Preservation of Well-Formedness In PIFO Trees

This set of definitions and proofs completes the proof sketch outlined in Section 3.3, Definition 3.8 of Formal Abstractions In Packet Scheduling. In particular, it fleshes out the relationships between the functions push, pop and $|\cdot|$, in order to formally define (and by extension, prove) the preservation of well-formedness in PIFO Trees.

This defines how pushing and popping modifies length in PIFOs, and supplies lemmas (with proof) clarifying how length changes with regards to pushing and popping in PIFO Trees. This then proves preservation of well-formedness in PIFO Trees following push and pop operations.

This also lays some of the foundation for proving for well-formedness in PIEO Trees. Proofs of pushing translate almost one-to-one for PIEO Trees, while popping follows as an extension of this system, but with an inculsion of a totally ordered predicate.

2 Relevant Definitions and Lemmas

2.1 Definition 1

Defining $|\cdot|$ on PIFOs With Regards to Pushes

Definition 1A: *Pushing an element to a PIFO increments how many times it appears*

$$\frac{p \in \text{PIFO} \qquad |p|_i = n \qquad n \in \mathbb{N} \qquad \text{PUSH}(p, i, r) = p' \qquad n' = n + 1}{|p'|_i = n'}$$

Definition 1B: *Pushing an element to a PIFO does not change how much others appear*

$$\frac{p \in \text{PIFO} \qquad |p|_i = n \qquad n \in \mathbb{N} \qquad \text{PUSH}(p, j, r) = p' \qquad i \neq j}{|p'|_i = n}$$

Definition 1C: *Pushing an element to a PIFO increments its size*

$$\frac{p \in \text{PIFO} \qquad |p| = n \qquad n \in \mathbb{N} \qquad \text{PUSH}(p, i, r) = p' \qquad n' = n + 1}{|p'| = n'}$$

2.2 Definition 2

Defining $|\cdot|$ on PIFOs With Regards to Pops

Definition 2A: *Popping an element from a PIFO decrements how many times it appears*

$$\frac{p \in \text{PIFO} \qquad |p|_i = n \qquad n \in \mathbb{N} \qquad \text{POP}(p) = (i, p') \qquad n' = n - 1}{|p'|_i = n'}$$

Definition 2B: *Popping an element from a PIFO does not change how much others appear*

$$\frac{p \in \text{PIFO} \qquad |p|_i = n \qquad n \in \mathbb{N} \qquad \text{POP}(p) = (j, p') \qquad i \neq j}{|p'|_i = n}$$

Definition 2C: *Popping an element from a PIFO decrements its size*

$$\frac{p \in \text{PIFO} \qquad |p| = n \qquad n \in \mathbb{N} \qquad \text{POP}(p) = (i, p') \qquad n' = n - 1}{|p'| = n'}$$

3 Lemmas 1 and 2

Defining $|\cdot|$ on PIFO Trees With Regards to Pushes and Pops

3.1 Lemma 1: Pushing to a PIFO Tree increments its size

$$\frac{q \in \text{PIFOTree} \qquad |q| = n \qquad n \in \mathbb{N} \qquad \text{push}(q, pkt, p) = q' \qquad n' = n + 1}{|q'| = n'}$$

Proof. We proceed upon this proof by structural induction over the definition of $|\cdot|$ as follows.

Base Case: $q = \text{Leaf}(p)$ for some PIFO p.

By definition 3.8, we have that $|\text{Leaf}(p)|_i = |p|_i$, where $|p|_i$ is the number of occurrences of i in PIFO p. The definition of push in 3.6 gives the following:

$$\frac{\text{PUSH}(p, pkt, r) = p'}{\text{push}(\text{Leaf}(p), pkt, r) = \text{Leaf}(p')}$$

Where $\text{Leaf}(p')$ is the result of pushing into $\text{Leaf}(p)$.

By definition 1A, we now know that $|p'|_i = |p|_i + 1$.

By definition 3.8, we know that $|\text{Leaf}(p')|_i = |p'|_i$ and $|\text{Leaf}(p)|_i = |p|_i$.

Combining these together, we obtain the following:

$$|\text{Leaf}(p')| = |\text{Leaf}(p)| + 1$$

Thus, we have proven our base case.

Inductive Case:

Assume an arbitrary PIFO p, Node $q = \text{Internal}(qs, p)$, packet pkt, path pt, index i and rank r.

Inductive Hypothesis: $|qs[i]| = n \implies |\text{push}(qs[i], pkt, pt)| = n + 1$

Show: $|\text{Internal}(qs, p)| = m \implies |\text{push}(\text{Internal}(qs, p), pkt, (i, r) :: pt)| = m + 1$

Note that push is defined for internal nodes as follows:

$$\frac{\text{push}(qs[i], pkt, pt) = q' \qquad \text{PUSH}(p, i, r) = p'}{\text{push}(\text{Internal}(qs, p), pkt, (i, r) :: pt) = \text{Internal}(qs[q'/i], p')}$$

Since q is a PIFO Tree, we know from Definition 3.8 the following:

$$m = |q| = \sum_{j=1}^{|qs|} |qs[j]|$$

Thus, we can further conclude the following:

$$|\text{push}(\text{Internal}(qs, p), pkt, (i, r) :: pt)| = \sum_{j=1}^{|qs|} |qs[q'/i][j]|$$

From our Inductive Hypothesis, we know that $|q'| = |\text{push}(qs[i], pkt, pt| = n + 1$.

From the definition of push, pushing changes only $qs[i]$, while the remaining subtrees remain the same. Subsequently, we can assert that $\forall 1 \leq j \leq |qs|, i \neq j \implies |qs[j]| = |qs[q'/i][j]$. With this in mind, we make the following claim:

$$|\text{push}(\text{Internal}(qs, p), pkt, (i, r) :: pt)| = \sum_{j=1}^{|qs|} |qs[j]| - |qs[i]| + |qs[q'/i][i]|$$

Substituting what we know, this gives us the following:

$$|\text{push}(\text{Internal}(qs, p), pkt, (i, r) :: pt)| = m - n + n + 1$$

$$\implies |\text{push}(\text{Internal}(qs, p), pkt, (i, r) :: pt)| = m + 1$$

With this, we have shown the desired equality, and proven our Inductive case.

Thus, it follows that Lemma 1 holds.

3.2 Lemma 2: Popping from a PIFO Tree decrements its size

$$\frac{q \in \text{PIFOTree} \qquad |q| = n \qquad n \in \mathbb{N} \qquad (pkt, q') = \text{pop}(q) \qquad n' = n - 1}{|q'| = n'}$$

Proof. We proceed upon this proof by structural induction over the definition of $|\cdot|$ as follows.

Base Case: $q = \text{Leaf}(p)$ for some PIFO p.

By definition 3.8, we have that $|\text{Leaf}(p)|_i = |p|_i$, where $|p|_i$ is the number of occurrences of i in PIFO p. The definition of pop in 3.6 gives the following:

$$\frac{\text{POP}(p) = p'}{\text{pop}(\text{Leaf}(p)) = \text{Leaf}(pkt, p')}$$

Where $\text{Leaf}(p')$ is the result of popping from $\text{Leaf}(p)$.

By definition 2A, we now know that $|p'|_i = |p|_i - 1$.

By definition 3.8, we know that $|\text{Leaf}(p')|_i = |p'|_i$ and $|\text{Leaf}(p)|_i = |p|_i$.

Combining these together, we obtain the following:

$$|\text{Leaf}(p')| = |\text{Leaf}(p)| - 1$$

Thus, we have proven our base case.

Inductive Case:

Assume an arbitrary PIFO p, Node $q = \text{Internal}(qs, p)$ and index i.

Inductive Hypothesis: $|qs[i]| = n \implies |q'| = n - 1$ where $(pkt, q') = \text{pop}(qs[i])$.

Show: $|\text{Internal}(qs, p)| = m \implies |q''| = m - 1$ where $(pkt, q'') = \text{pop}(\text{Internal}(qs, p)$.

Note that pop is defined for internal nodes as follows:

$$\frac{\text{POP(p) = (i, p')} \qquad \text{pop(qs[i]) = (pkt, q')}}{\text{pop}(\text{Internal}(qs, p)) = (pkt, \text{Internal}(qs[q'/i], p'))}$$

Since q is a PIFO Tree, we know from Definition 3.8 the following:

$$m = |q| = \sum_{j=1}^{|qs|} |qs[j]|$$

Thus, we can further conclude the following:

$$(pkt, q'') = \text{pop}(\text{Internal}(qs, p) \implies |q''| = \sum_{j=1}^{|qs|} |qs[q'/i][j]|$$

From our Inductive Hypothesis, we know that $|q'| = n - 1$ where $pkt, q' = \text{pop}(qs[i])$.

From the definition of pop, popping changes only $qs[i]$, while the remaining subtrees remain the same. Subsequently, we can assert that $\forall 1 \leq j \leq |qs|, i \neq j \implies |qs[j]| = |qs[q'/i][j]$. With this in mind, we make the following claim:

$$(pkt, q'') = \text{pop}(\text{Internal}(qs, p)) \implies |q''| = \sum_{j=1}^{|qs|} |qs[j]| - |qs[i]| + |qs[q'/i][i]|$$

Substituting what we know, this gives us the following:

$$(pkt, qs'') = \text{pop}(\text{Internal}(qs, p)) \implies |q''| = m - n + n - 1$$

$$\implies (pkt, qs'') = \text{pop}(\text{Internal}(qs, p)) \implies |qs''| = m - 1$$

With this, we have shown the desired equality, and proven our Inductive case.

Thus, it follows that Lemma 2 holds.

4 Proofs For Lemma 3.9 (Well-Formedness)

4.1 Proof of Lemma 3.9.1

Well-Formedness Is Preserved in PIFO Trees Upon Pushes

We proceed with this proof by inducting upon the definition of push for PIFO Trees.

Base Case : $Leaf(p)$

By definition 3.6 of push, we have the following for Leaf nodes:

$$\frac{\text{PUSH}(p, pkt, r) = p'}{\text{push}(\text{Leaf}(p), pkt, r) = \text{Leaf}(p')}$$

It follows that after pushing, the resultant tree is expressed as $\text{Leaf}(p')$ for some PIFO p' By definition 3.8 (cited above), we know the following holds for any arbitrary PIFO p:

$$\frac{}{\vdash \text{Leaf}(p)}$$

We now have: $\forall p, pkt, r, \vdash \text{push}(\text{Leaf}(p), pkt, r)$

With this, our Base Case is proven.

Inductive Case: $q = \text{Internal}(qs, p), \vdash q$

Inductive Hypothesis: $\forall 1 \leq i \leq |qs|. \vdash \text{push}(qs[i], pkt, pt)$

Recall Definition 3.8 for $\vdash$ on Internal Nodes:

$$\frac{\forall 1 \leq i \leq |qs|. \vdash qs[i] \wedge |p|_i = |qs[i]|}{\vdash \text{Internal(qs, p)}}$$

Recall Definition 3.6 for push on Internal Nodes:

$$\frac{\text{push}(qs[i], pkt, pt) = q' \qquad \text{PUSH}(p, i, r) = p'}{\text{push}(\text{Internal}(qs, p), pkt, (i, r) :: pt) = \text{Internal}(qs[q'/i], p')}$$

Now, let $q' = \text{push}(qs[i], pkt, pt)$, and let $p' = \text{PUSH}(p, i, r)$.

Show: $\vdash q \implies \vdash \text{Internal}(qs[q'/i], p')$.

Using definition 3.6 and our Inductive Hypothesis, we can conclude that $\vdash q'$. Furthermore, note that the only subtree to be modified in qs is in $qs[i]$, per the definition of push.

Now we make the following claim:

(1) $\vdash \text{Internal}(qs, p)$	*By definition*
(2) $\implies \forall 1 \leq j \leq \lvert qs\rvert. \ \vdash qs[j] \wedge \lvert p\rvert_j = \lvert qs[j]\rvert$	*Inversion Lemma (Definition 3.8)*
(3) $\implies \lvert p\rvert_i = \lvert qs[i]\rvert$	*Instance of universal quantifier (2)*
(4)$\lvert p'\rvert_i = \lvert p\rvert_i + 1$	*Definition 1A*
(5)$\lvert q'\rvert = \lvert qs[i]\rvert + 1$	*Lemma 1*
(6) $\implies \lvert p\rvert_i + 1 = \lvert qs[i]\rvert + 1$	*Addition Property of Equality*
(7) $\implies \lvert p'\rvert_i = \lvert q'\rvert$	*Substitution (4, 5, 6)*
(8)$\forall 1 \leq j \leq \lvert p'\rvert, i \neq j \implies \lvert p'\rvert_j = \lvert p\rvert_j$	*Definition 1B*
(9) $\implies \forall 1 \leq j \leq \lvert p'\rvert, i \neq j \implies \lvert p'\rvert_j = \lvert qs[j]\rvert$	*By (2) and (8)*
(10) $\implies \forall 1 \leq j \leq \lvert p'\rvert, \lvert p'\rvert_j = \lvert qs[q'/i][j]\rvert$	*By (7) and (9)*
(11)$\forall 1 \leq j \leq \lvert qs\rvert. \vdash qs[q'/i][j]$	*Inductive Hypothesis*
(12) $\implies \forall 1 \leq j \leq \lvert qs\rvert. \vdash qs[q'/i][j] \wedge \lvert p'\rvert_j = \lvert qs[q'/i][j]\rvert$	*By (10) and (11)*
(13) $\implies \vdash \text{Internal}(qs[q'/i], p')$	*Definition of $\vdash$*

With this, we have proven our Inductive Statement and completed the proof. We have shown that pushing to an arbitrary PIFO Tree preserves its well-formedness.

4.2 Proof of Lemma 3.9.2

Well-Formedness Is Preserved in PIFO Trees Upon Pops

We proceed with this proof by inducting upon the definition of pop for PIFO Trees.

Base Case : $Leaf(p)$

By definition 3.4 of pop, the following holds for Leaf nodes:

$$\frac{\text{POP}(p) = (pkt, p')}{\text{pop}(\text{Leaf}(p)) = (pkt, \text{Leaf}(p'))}$$

It follows that after popping, the resultant tree is of form Leaf(p') for some PIFO p'. By definition 3.8 (cited above), we know the following holds for any arbitrary PIFO p:

$$\frac{}{\vdash \text{Leaf}(p)}$$

We now have: $\forall p, \vdash p'$ where $(pkt, p') = \text{pop}(\text{Leaf}(p))$.

With this, our Base Case is proven.

Inductive Case: $q = \text{Internal}(qs, p), \vdash q$

Inductive Hypothesis: $\forall 1 \leq j \leq |qs|. \vdash qs[j]'$, where $(pkt[j], qs[j]') = \text{pop}(qs[j])$

Recall Definition 3.8 for $\vdash$ on Internal Nodes:

$$\frac{\forall 1 \leq i \leq |qs|. \vdash qs[i] \wedge |p|_i = |qs[i]|}{\vdash \text{Internal(qs, p)}}$$

Recall Definition 3.6 for pop on Internal Nodes:

$$\frac{\text{POP(p) = (i, p')} \qquad \text{pop(qs[i]) = (pkt, q')}}{\text{pop}(\text{Internal}(qs, p)) = (pkt, \text{Internal}(qs[q'/i], p'))}$$

Now, let $(pkt, q') = \text{pop}(qs[i])$, and let $(i, p') = \text{POP}(p)$.

Show: $\vdash q \implies \vdash \text{Internal}(qs[q'/i], p')$.

Using definition 3.6 and our Inductive Hypothesis, we can conclude that $\vdash q'$. Furthermore, note that the only subtree to be modified in qs is in $qs[i]$, per the definition of pop.

Now we make the following claim:

(1) $\vdash \text{Internal}(qs, p)$	*By definition*
(2) $\implies \forall 1 \leq j \leq \lvert qs\rvert.\ \ \vdash qs[j] \wedge \lvert p\rvert_j = \lvert qs[j]\rvert$	*Inversion Lemma (Definition 3.8)*
(3) $\implies \lvert p\rvert_i = \lvert qs[i]\rvert$	*Instance of universal quantifier (2)*
(4)$\lvert p'\rvert_i = \lvert p\rvert_i - 1$	*Definition 1A*
(5)$\lvert q'\rvert = \lvert qs[i]\rvert - 1$	*Lemma 1*
(6) $\implies \lvert p\rvert_i - 1 = \lvert qs[i]\rvert - 1$	*Subtraction Property of Equality*
(7) $\implies \lvert p'\rvert_i = \lvert q'\rvert$	*Substitution (4, 5, 6)*
(8)$\forall 1 \leq j \leq \lvert p'\rvert, i \neq j \implies \lvert p'\rvert_j = \lvert p\rvert_j$	*Definition 1B*
(9) $\implies \forall 1 \leq j \leq \lvert p'\rvert, i \neq j \implies \lvert p'\rvert_j = \lvert qs[j]\rvert$	*By (2) and (8)*
(10) $\implies \forall 1 \leq j \leq \lvert p'\rvert, \lvert p'\rvert_j = \lvert qs[q'/i][j]\rvert$	*By (7) and (9)*
(11)$\forall 1 \leq j \leq \lvert qs\rvert. \vdash qs[q'/i][j]$	*Inductive Hypothesis*
(12) $\implies \forall 1 \leq j \leq \lvert qs\rvert. \vdash qs[q'/i][j] \wedge \lvert p'\rvert_j = \lvert qs[q'/i][j]\rvert$	*By (10) and (11)*
(13) $\implies \vdash \text{Internal}(qs[q'/i], p')$	*Definition of* $\vdash$

With this, we have proven our Inductive Statement and completed the proof. We have shown that popping from an arbitrary PIFO Tree preserves its well-formedness.