1 Relevant Definitions

1.1 Definition 1

Defining $|\cdot|$ on PIEOs With Regards to Pushes

Definition 1A: *Given a predicate $f \in \mathcal{F}$, element i and data d, if d satisfies f, then pushing i to p with data d increments the number of instances of i that satisfy f.*

$$\frac{p \in \text{PIEO} \qquad f \in \mathcal{F} \qquad n \in \mathbb{N} \qquad r \in \text{Rank} \qquad |p|_{i,f} = n \qquad \text{PUSH}(p, i, d, r) = p' \qquad f(d)}{|p'|_{i,f} = n + 1}$$

Definition 1B: *Given a predicate $f \in \mathcal{F}$, element i and data d, if d does not satisfy f, then pushing i to p with data d preserves the number of instances of i that satisfy f.*

$$\frac{p \in \text{PIEO} \qquad f \in \mathcal{F} \qquad n \in \mathbb{N} \qquad r \in \text{Rank} \qquad |p|_{i,f} = n \qquad \text{PUSH}(p, i, d, r) = p' \qquad \neg f(d)}{|p'|_{i,f} = n}$$

Definition 1C: *Given elements i, j and data $d \in \mathcal{D}$, pushing i to p with data d does not modify how many times j appears in p.*

$$\frac{p \in \text{PIEO} \qquad f \in \mathcal{F} \qquad n \in \mathbb{N} \qquad r \in \text{Rank} \qquad |p|_{j,f} = n \qquad \text{PUSH}(p, i, d, r) = p' \qquad i \neq j}{|p'|_{j,f} = n}$$

2 Relevant Lemmas

2.1 Lemma 1

Given $f \in \mathcal{F}, pkt \in \text{Pkt}, d \in \mathcal{D}, pt \in \text{Path}, t \in \text{Topo}$ and $q \in \text{PIEOTree}(t)$:

$$\begin{cases} f(d) \implies |push(q, pkt, d, pt)|_f = |q|_f + 1 \\ \neg f(d) \implies |push(q, pkt, d, pt)|_f = |q|_f \end{cases}$$

Proof: We proceed about this proof by Structural Induction over the derivation of $|\cdot|$ as follows:

Base Case: $q = \text{Leaf}(p)$ for some PIEO p.

By definition 2.1, we have that $|\text{Leaf}(p)|_{i,f} = |p|_{i,f}$, where $|p|_{i,f}$ is the number of occurrences of i in PIEO p that satisfy predicate f. The definition of push in 1.4 gives the following:

$$\frac{\text{PUSH}(p, i, d, r) = p'}{\text{push}(\text{Leaf}(p), pkt, d, r) = \text{Leaf}(p')}$$

Where $\text{Leaf}(p')$ is the result of pushing into $\text{Leaf}(p)$.

By definition 1A, we now know that $f(d) \implies |p'|_f = |p|_f + 1$.

By definition 1B, we now know that $\neg f(d) \implies |p'|_f = |p|_f$.

By definition 2.1, we know that $|\text{Leaf}(p')|_f = |p'|_f$ and $|\text{Leaf}(p)|_f = |p|_f$.

Combining these together, we obtain the following:

$$\begin{cases} f(d) \implies |\text{Leaf}(p')|_f = |\text{Leaf}(p)|_f + 1 \\ \neg f(d) \implies |\text{Leaf}(p')|_f = |\text{Leaf}(p)|_f \end{cases}$$

Thus, we have proven our base case.

Inductive Case: Assume an arbitrary PIEO p, set of subtrees qs, node $q = \text{Internal}(qs, p)$, index i, data d and packet pkt. Also assume a set of paths $pts : |pts| = |qs|$.

Inductive Hypothesis: $\forall 1 \leq j \leq |qs|$:

$$\begin{cases} f(d) \implies |push(qs[j], pkt, d, pts[j])|_f = |qs[j]|_f + 1 \\ \neg f(d) \implies |push(qs[j], pkt, d, pts[j])|_f = |qs[j]|_f \end{cases}$$

We now push a packet pkt with an arbitrary index i and rank r.

Let $q' = push(q, pkt, d, (i, r) :: pts[i])$.

Show:

$$\begin{cases} f(d) \implies |q'|_f = |q|_f + 1 \\ \neg f(d) \implies |q'|_f = |q|_f \end{cases}$$

Recall Definition 1.4 for push on Internal Nodes:

$$\frac{\text{push}(qs[i], pkt, d, pt) = q' \qquad \text{PUSH}(p, i, d, r) = p'}{\text{push}(\text{Internal}(qs, p), pkt, d, (i, r) :: pt) = \text{Internal}(qs[q'/i], p')}$$

Recall Definition 2.1 for $|\cdot|_f$ on Internal Nodes:

$$m = |\text{Internal}(qs, p)|_f = \sum_{j=1}^{|qs|} |qs[j]|_f$$

Thus, we can further conclude the following:

$$|\text{push}(\text{Internal}(qs, p), pkt, d, (i, r) :: pt)|_f = \sum_{j=1}^{|qs|} |qs[q'/i][j]|_f$$

From our Inductive Hypothesis, we know the following:

$$\begin{cases} f(d) \implies |push(qs[i], pkt, d, pts[i])|_f = |qs[i]|_f + 1 \\ \neg f(d) \implies |push(qs[i], pkt, d, pts[i])|_f = |qs[i]|_f \end{cases}$$

From the definition of push, pushing changes only $qs[i]$, while the remaining subtrees remain the same. Subsequently, we can assert that $\forall 1 \leq j \leq |qs|, i \neq j \implies |qs[j]|_f = |qs[q'/i][j]|_f$. With this in mind, we make the following claim:

$$|\text{push}(\text{Internal}(qs, p), pkt, (i, r) :: pt)|_f = \sum_{j=1}^{|qs|} |qs[j]|_f - |qs[i]|_f + |qs[q'/i][i]|_f$$

Substituting what we know gives the following:

$$\begin{cases} f(d) \implies |\text{push}(\text{Internal}(qs, p), pkt, (i, r) :: pt)|_f = |\text{Internal}(qs, p)|_f - |qs[i]|_f + |qs[i]|_f + 1 \\ \neg f(d) \implies |\text{push}(\text{Internal}(qs, p), pkt, (i, r) :: pt)|_f = |\text{Internal}(qs, p)|_f - |qs[i]|_f + |qs[i]|_f \end{cases}$$

$$\implies \begin{cases} f(d) \implies |\text{push}(\text{Internal}(qs, p), pkt, (i, r) :: pt)|_f = |q|_f + 1 \\ \neg f(d) \implies |\text{push}(\text{Internal}(qs, p), pkt, (i, r) :: pt)|_f = |q|_f \end{cases}$$

$$\implies \begin{cases} f(d) \implies |q'|_f = |q|_f + 1 \\ \neg f(d) \implies |q'|_f = |q|_f \end{cases}$$

With this, we have shown the desired equality, and proven our Inductive case.

Thus, it follows that Lemma 1 holds.

3 Proofs For Well-Formedness

3.1 Proof of Lemma 2.2.1

Well-Formedness Is Preserved in PIEO Trees Upon Pushes

We proceed with this proof by inducting upon the definition of push for PIEO Trees.

Base Case : $Leaf(p)$

By definition 1.4 of push for PIEOs, we have the following for Leaf nodes:

$$\frac{\text{PUSH}(p, pkt, d, r) = p'}{\text{push}(\text{Leaf}(p), pkt, d, r) = \text{Leaf}(p')}$$

It follows that after pushing, the resultant tree is expressed as Leaf(p') for some packet p'. By definition 2.1, we know the following holds for any arbitrary PIEO p, under any predicate $f \in \mathcal{F}$:

$$\frac{}{\vdash_f \text{Leaf}(p)}$$

We now have: $\forall(p, pkt, d, f, r), \vdash_f \text{push}(\text{Leaf}(p), pkt, d, r)$

With this, our Base Case is proven.

Inductive Case: $q = \text{Internal}(qs, p), f \in \mathcal{F}, \vdash_f q$

Inductive Hypothesis: $\forall 1 \leq i \leq |qs|. \vdash_f \text{push}(qs[i], pkt, d, pt)$

Recall Definition 2.1 for $\vdash_f$ on Internal Nodes, given a predicate f:

$$\frac{\forall 1 \leq i \leq |qs|. \vdash_f qs[i] \wedge |p|_{i,f} = |qs[i]|_f}{\vdash_f \text{Internal(qs, p)}}$$

Recall Definition 1.4 for push on Internal Nodes:

$$\frac{\text{push}(qs[i], pkt, d, pt) = q' \qquad \text{PUSH}(p, i, d, r) = p'}{\text{push}(\text{Internal}(qs, p), pkt, d, (i, r) :: pt) = \text{Internal}(qs[q'/i], p')}$$

Now, let $q' = \text{push}(qs[i], pkt, d, pt)$, and let $p' = \text{PUSH}(p, i, d, r)$.

Let f be any totally-ordered predicate.

Show: $\vdash_f q \implies \vdash_f \text{Internal}(qs[q'/i], p')$.

Using definition 2.1 and our Inductive Hypothesis, we can conclude that $\vdash_f q'$. Furthermore, note that the only subtree to be modified in qs is in $qs[i]$, per the definition of push.

Now we make the following claim:

(1) $\vdash_f$ Internal(qs, p)	*By definition*
(2) $\implies \forall 1 \leq j \leq \lvert qs \rvert.\ \vdash_f qs[j] \wedge \lvert p \rvert_{j,f} = \lvert qs[j] \rvert_f$	*Inversion Lemma (Definition 2.1)*
(3) $\implies \lvert p \rvert_{i,f} = \lvert qs[i] \rvert_f$	*Instance of universal quantifier (2)*
(4.1) $f(d) \implies \lvert p' \rvert_{i,f} = \lvert p \rvert_{i,f} + 1$	*Definition 1A*
(4.2) $\neg f(d) \implies \lvert p' \rvert_{i,f} = \lvert p \rvert_{i,f}$	*Definition 1B*
(5.1) $f(d) \implies \lvert q' \rvert_f = \lvert qs[i] \rvert_f + 1$	*Lemma 1*
(5.2) $\neg f(d) \implies \lvert q' \rvert_f = \lvert qs[i] \rvert_f$	*Lemma 1*
(6) $\lvert p \rvert_{i,f} + 1 = \lvert qs[i] \rvert_f + 1$	*Addition Property of Equality (3)*
(7) $\implies \lvert p' \rvert_{i,f} = \lvert q' \rvert_f$	*Substitution (3 - 6)*
(8)$\forall 1 \leq j \leq \lvert p' \rvert, i \neq j \implies \lvert p' \rvert_{j,f} = \lvert p \rvert_{j,f}$	*Definition 1C*
(9) $\implies \forall 1 \leq j \leq \lvert p' \rvert, i \neq j \implies \lvert p' \rvert_{j,f} = \lvert qs[j] \rvert_f$	*By (2) and (8)*
(10) $\implies \forall 1 \leq j \leq \lvert p' \rvert, \lvert p' \rvert_{j,f} = \lvert qs[q'/i][j] \rvert_f$	*By (7) and (9)*
(11)$\forall 1 \leq j \leq \lvert qs \rvert. \vdash_f qs[q'/i][j]$	*Inductive Hypothesis*
(12) $\implies \forall 1 \leq j \leq \lvert qs \rvert. \vdash_f qs[q'/i][j] \wedge \lvert p' \rvert_{j,f} = \lvert qs[q'/i][j] \rvert_f$	*By (10) and (11)*
(13) $\implies \vdash_f$ Internal$(qs[q'/i], p')$	*Definition of $\vdash$ given f*

With this, we have proven our Inductive Statement and completed the proof. We have shown that pushing to an arbitrary PIEO Tree preserves its well-formedness.