Data

$$(X, Y) = \{(x_1, y_1), \ldots, (x_N, y_N)\}$$

$x_i \in \mathbb{R}^D$

$y_i \in \{1, \ldots, L\}$

labelled

$\widetilde{p}_l(x, y)$

unlabelled

$\widetilde{p}_u(x)$

Latent variable

$$Z = \{z_1, \ldots, z_N\}$$

M_1

$p(z) = \mathcal{N}(z|0, I)$

$p_\theta(x|z) = f(x; z, \theta)$

$p(z|x) \propto p(z)\, p_\theta(x|z)$

$z_i \in \mathbb{R}^J$

$q_\phi(z|x) = \mathcal{N}(z|\mu_\phi(x), \text{diag}(\sigma^2_\phi(x)))$

$$q_\phi(z, y|x) = q_\phi(z|x)\, q_\phi(y|x)$$

M_2

$p(y) = \text{Cat}(y|\pi)$

$p(z) = \mathcal{N}(z|0, I)$

$p_\theta(x|y, z) = f(x; y, z, \theta)$

$p(y|x) \propto \int p(z)\, p(y)\, p_\theta(x|y, z)\, dz$

$q_\phi(y|x) = \text{Cat}(y|\pi_\phi(x))$

$q_\phi(z|y, x) = \mathcal{N}(z|\mu_\phi(y, x), \text{diag}(\sigma^2_\phi(x)))$

ELBO M1

$$\log p(x) \geq \mathbb{E}_{q_\phi(z|x)} \log p_\theta(x|z) - KL\left(q_\phi(z|x) \| p(z)\right) = -\mathcal{J}(x)$$

ELBO M2

- labelled data

$$\log p_\theta(x,y) \geq \log p(y) + \mathbb{E}_{q_\phi(z|y,x)}\left[\log p(z) + \log p_\theta(x|y,z) - \log q_\phi(z|y,x)\right] = -\mathcal{L}(x,y)$$

- unlabelled data

$$\log p(x) \geq \mathbb{E}_{q_\phi(z,y|x)}\left[\log p_\theta(x|y,z) - \log q_\phi(z,y|x) + \log p(z) + \log p(y)\right]$$

$$= \sum_y q_\phi(y|x)\left[-\mathcal{L}(x,y) - \log q_\phi(y|x)\right]$$

$$= \sum_y q_\phi(y|x)\left[-\mathcal{L}(x,y)\right] + H(y|x) = -\mathcal{U}(x)$$

- entire dataset

$$\mathcal{J} = \sum_{(x,y)\sim\tilde{p}_l} \mathcal{L}(x,y) + \sum_{x\sim\tilde{p}_u} \mathcal{U}(x)$$

$$\mathcal{J}^\alpha = \mathcal{J} + \alpha\, \mathbb{E}_{\tilde{p}_l(x,y)}\left[-\log q_\phi(y|x)\right]$$

$$\log p_\theta(x,y) \geq \log p(y) + \mathbb{E}_{q_\phi(z|y,x)}\left[\log p(z) + \log p_\theta(x|y,z) - \log q_\phi(z|y,x)\right] = -\mathcal{L}(x,y)$$

$$= \log \text{Cat}(y|\pi) - KL(q_\phi(z|y,x)\,\|\,p(z)) + \mathbb{E}_{q_\phi(z|y,x)} \log p_\theta(x|y,z)$$

$$= \log \pi_{y_i} + \frac{1}{2}\sum_j^J \left(1 + \log \sigma_{ij}^2 - \mu_{ij}^2 - \sigma_{ij}^2\right) + \mathbb{E}_{\mathcal{N}(\epsilon|0,I)} \log p_\theta(x|y,z)$$

Bernoulli

$$\log p_\theta(x|y,z) = \sum_i^D x_i \log p_i + (1-x_i)\log(1-p_i) = -BCE(x,p)$$

$$p = f_\theta(y,z)$$

Gaussian

$$\log p_\theta(x|y,z) = \mathcal{N}(x|\mu, \sigma^2 I)$$

$$\mu = f_\theta^{(1)}(h)$$

$$\log \sigma^2 = f_\theta^{(2)}(h)$$

$$h = f_\theta^{(0)}(y,z)$$

$$-KL\left(q_\phi(z|y,x)\,\|\,p(z)\right)$$

$$p(z) = \mathcal{N}(z|0,I)$$

$$q_\phi(z|y,x) = \mathcal{N}(z|u_\phi(y,x), \mathrm{diag}(\sigma^2_\phi(x)))$$

$$-\int q_\phi(z)\log\frac{q_\phi(z)}{p(z)}dz = -\int q(z)\log q(z) + \int q(z)\log p(z)$$

$$\int q(z)\left[-\frac{J}{2}\log 2\pi - \frac{1}{2}\sum_i^J \log\sigma_i^2 - \sum_i^J \frac{(z_i-u_i)^2}{2\sigma_i^2}\right]dz$$

$$= -\frac{J}{2}\log 2\pi - \frac{1}{2}\sum_i^J \log\sigma_i^2 - \sum_i^J \frac{E_{q(z)}[(z_i-u_i)^2]}{2\sigma_i^2}$$

$$= -\frac{J}{2}\log 2\pi - \frac{1}{2}\sum_i^J 1 + \log\sigma_i^2$$

$$\int q(z)\left[-\frac{J}{2}\log 2\pi - \frac{1}{2}z^Tz\right]dz = -\frac{J}{2}\log 2\pi - \frac{1}{2}\sum_i^J u_i^2 + \sigma_i^2$$

$$-KL = \frac{1}{2}\sum_i^J 1 + \log\sigma_i^2 - u_i^2 - \sigma_i^2$$