Exercises on Description Logics

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1. Interpretations and models

1. Prove that the DL expressions

   E1 = $\forall r.(C \sqcup D)$
   E2 = $(\forall r.C) \sqcup (\forall r.D)$

   are not equivalent by finding an interpretation $\mathcal{I} = (C^{\mathcal{I}}, D^{\mathcal{I}}, r^{\mathcal{I}})$ such that $\mathrm{E1}^{\mathcal{I}} \neq \mathrm{E2}^{\mathcal{I}}$.

2. Find an interpretation $\mathcal{I}$ that satisfies the following axioms:

   1. $A \equiv \forall s.L$,
   2. $L \sqsubseteq (\exists r.A) \sqcap (\forall r.(\exists s.\top))$,
   3. $L(u)$

3. Find an interpretation $\mathcal{I}$ that satisfies the following axioms:

   1. $A \sqsubseteq \forall s.L$,
   2. $L \sqsubseteq (\exists r.A) \sqcap (\forall r.(\exists s.\top))$,
   3. $L(u)$

4. Find an acyclic interpretation that satisfies the above axioms.

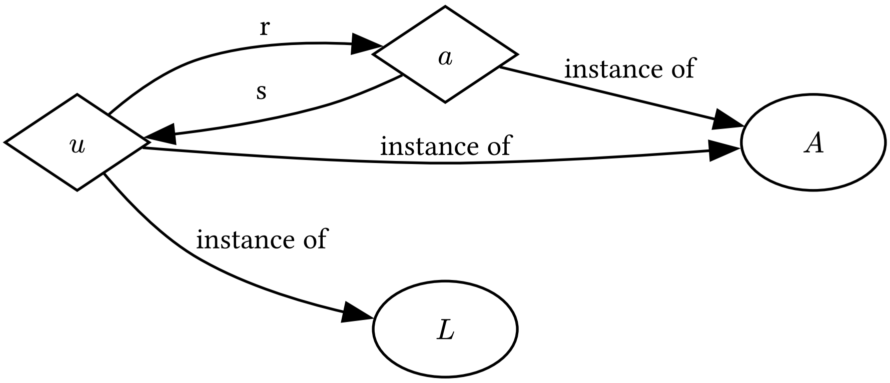
Solutions

1. With the interpretation $\Delta^{\mathcal{I}} = \{a, c, d\}, C^{(\mathcal{I})} = \{c\}, D^{\mathcal{I}} = \{d\}, r^{\mathcal{I}} = \{(a, c), (a, d)\}$ we have

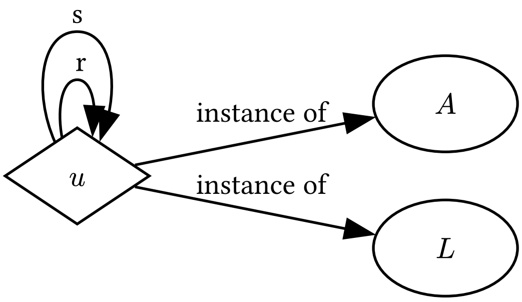
   $A^{\mathcal{I}} = (\forall r.(C \sqcup D))^{\mathcal{I}} = \{a\}$ and

   $B^{\mathcal{I}} = ((\forall r.C) \sqcup (\forall r.D))^{\mathcal{I}} = ((\forall r.C))^{\mathcal{I}} \cup (\forall r.D))^{\mathcal{I}} = \emptyset \cup \emptyset = \emptyset$

2. An interpretation that satisfies the axioms: $\Delta^{\mathcal{I}} = \{u, a\}, L^{\mathcal{I}} = \{u\}, A^{\mathcal{I}} = \{u, a\}, r^{\mathcal{I}} = \{(u, a)\}, s^{\mathcal{I}} = \{(a, u)\}$. Note that u must be in $A^{\mathcal{I}}$ because it trivially satisfies $\forall s.L$.



3. A minimal interpretation that satisfies the axioms: $\Delta^{\mathcal{I}} = \{u\}, L^{\mathcal{I}} = \{u\}, A^{\mathcal{I}} = \{u\}, r^{\mathcal{I}} = \{(u, u)\}, s^{\mathcal{I}} = \{(u, u)\}$



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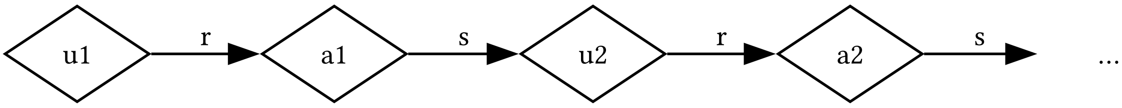
4. An acyclic interpretation: $\Delta^{\mathcal{I}} = \{u_1, u_2, ..., a_1, a_2, ...\}$, $L^{\mathcal{I}} = \{u_1, u_2, ...\}$, $A^{\mathcal{I}} = \{a_1, a_2, ...\}$, $r^{\mathcal{I}} = \{(u_1, a_1), (u_2, a_2), ...\}$, $s^{\mathcal{I}} = \left\{(a_1, u_2), \left(a_2.u_{\{3\}}\right), ...\right\}$. This interpretation is infinite.



2. DL Logical consequences

Source: Rudolph, S. (2011). Foundations of Description Logics. Longman Publishing. http://www.aifb.kit.edu/images/1/19/DL-Intro.pdf

Consider the knowledge base $\mathcal{KB}$ with

RBox

- owns $\sqsubseteq$ caresFor

TBox

- Healthy $\sqsubseteq$ $\neg$ Dead)
- Cat $\sqsubseteq$ Dead $\sqcup$ Alive
- HappyCatOwner $\sqsubseteq$ $\exists$ owns . Cat) $\sqcap$ $\forall$ caresFor . Healthy

ABox

- HappyCatOwner(schrödinger)
- Cat(hbar),
- owns(schrödinger, psi), owns(schrödinger, hbar)

Decide whether the following propositions about the knowledge base are true and give evidence:

1. $\mathcal{KB}$ is satisfiable,
2. $\mathcal{KB} \vDash$ Alive(schrödinger),
3. $\mathcal{KB} \vDash$ Dead $\sqcap$ Alive $\sqsubseteq \bot$,
4. $\mathcal{KB} \vDash$ Alive(psi),
5. $\mathcal{KB} \vDash$ Alive(hbar).

Solutions

1. $\mathcal{KB}$ is satisfiable , with the model $\mathcal{M}$ defined as
   - $\Delta^{\mathcal{M}} = \{s, h, p\}$

- $\mathsf{schrödinger}^{\mathcal{M}} = s$, $\mathsf{psi}^{\mathcal{M}} = p$, $\mathsf{hbar}^{\mathcal{M}} = h$,
- $\mathsf{Cat}^{\mathcal{M}} = \{h, p\}$,
- $\mathsf{HappyCatOwner}^{\mathcal{M}} = \{s\}$
- $\mathsf{Healthy}^{\mathcal{M}} = \{h, p\}$,
- $\mathsf{Alive}^{\mathcal{M}} = \{h\}$,
- $\mathsf{Dead}^{\mathcal{M}} = \{\}$,
- $\mathsf{owns}^{\mathcal{M}} = \{(s, p), (s, h)\}$,
- $\mathsf{caresFor}^{\mathcal{M}} = \{(s, p), (s, h)\}$,

2. $\mathcal{KB} \vDash \mathsf{Alive}(\mathsf{schrödinger})$ is false. For instance in the model $\mathcal{M}$ $s \notin \mathsf{Alive}(\mathsf{schrödinger})^{\mathcal{M}}$
3. $\mathcal{KB} \vDash \mathsf{Dead} \sqcap \mathsf{Alive} \sqsubset \bot$ is false. If we add s to $\mathsf{Alive}^{\mathcal{M}}$ and $\mathsf{Dead}^{\mathcal{M}}$ we obtain a model of $\mathcal{KB}$ in which the interpretation of $\mathsf{Dead} \sqcap \mathsf{Alive}$ is not empty.
4. $\mathcal{KB} \vDash \mathsf{Alive}(\mathsf{psi})$ is false, in the model $\mathcal{M}$ p is not in $\mathsf{Alive}^{\mathcal{M}}$
5. $\mathcal{KB} \vDash \mathsf{Alive}(\mathsf{hbar})$ is true. In every model $\mathcal{J}$ of KB h must be in $\mathsf{Cat}^{\mathcal{J}}$ and in $\mathsf{Healthy}^{\mathcal{J}}$ because of $\mathsf{HappyCatOwner} \sqsubseteq \exists\ \mathsf{owns}.\mathsf{Cat} \sqcap \forall\ \mathsf{caresFor}.\mathsf{Healthy}$ so it must be in $\neg\mathsf{Dead}^{\mathcal{J}}$ bacause of $\mathsf{Healthy} \sqsubseteq \neg\ \mathsf{Dead}$ and hence in $\mathsf{Alive}^{\mathcal{J}}$ because $\mathsf{Cat} \sqsubseteq \mathsf{Dead} \sqcup \mathsf{Alive}$

3. Inference

Use the tableau DL algorithm to show that $(\forall r.(B \sqcap C)) \sqcap (\exists r.(\neg C))$ is not satisfiable

Solution

Let's start with the ABox

$$\{(\forall r.(B \sqcap C)) \sqcap (\exists r.(\neg C))(a)\}$$

Applying the $\sqcap$ rule adds

$$(\forall r.(B \sqcap C))(a),$$
$$(\exists r.(\neg C))(a)$$

The $\exists$ rule adds

$$r(a, b), (\neg C)b$$

The $\forall$ rule adds

$$(B \sqcap C))(b)$$

The $\sqcap$ rule adds

$$B(b),$$
$$C(b)$$

There is a clash between $\neg C(a)$ and $C(a) \Rightarrow$ the starting concept was inconsistent.

4. OWL-RL inference

Use the OWL-RL inference rules to find the ABox logical consequences of the following knowledge bases

$\mathcal{KB}1$:

1. $\exists\ \mathsf{owns}\ .\ \mathsf{Cat} \sqsubseteq \mathsf{CatOwner}$
2. $\exists\ \mathsf{owns}\ .\ \mathsf{Mouse} \sqsubseteq \mathsf{MouseOwner}$

3. $\mathsf{CatOwner} \sqsubseteq \forall\ \mathsf{owns}.\ \mathsf{Cat}$
4. $\mathsf{owns}(\mathsf{felix},\ \mathsf{max})$,
5. $\mathsf{owns}(\mathsf{felix},\ \mathsf{illi})$,
6. $\mathsf{owns}(\mathsf{marta},\ \mathsf{chesee})$,
7. $\mathsf{owns}(\mathsf{marta},\ \mathsf{bob})$,
8. $\mathsf{Cat}(\mathsf{max})$,
9. $\mathsf{Cat}(\mathsf{bob})$
10. $\mathsf{Mouse}(\mathsf{cheese})$

Solution

Rule	Applied on (IF)	Infers (THEN)
some	1, 4, 8	11. $\mathsf{CatOwner}(\mathsf{felix})$
some	1, 7, 9	12. $\mathsf{CatOwner}(\mathsf{marta})$
some	2, 6, 10	13. $\mathsf{MouseOwner}(\mathsf{marta})$
only	3, 11, 5	14, $\mathsf{Cat}(\mathsf{illi})$
only	3, 12, 10	14, $\mathsf{Cat}(\mathsf{cheese})$

So, cheese is both a Cat and a Mouse 🤔

Exercise 6. Closing the world

Consider the knowledge base $\mathcal{KB}$

TBox

1. $\mathsf{BlueRun} \sqsubseteq \mathsf{SkiRun}$
2. $\mathsf{BlueRun} \sqsubseteq \mathsf{SkiRun}\}$,
3. $\mathsf{RedOnlyPlace} \equiv \mathsf{Place} \sqcap \exists\ \mathsf{isEndOf}\ .\ \mathsf{SkiRun}) \sqcap (\forall\ \mathsf{isEndOf}\ .\ \mathsf{RedRun})$

ABox

4. $\mathsf{isEndOf}(\mathsf{p1}, \mathsf{sr1})$,
5. $\mathsf{RedRun}(\mathsf{sr1})$,
6. $\mathsf{Place}(\mathsf{p1})$

What must be added to the ABox and/or TBox to ensure that $\mathcal{KB} \models \mathsf{RedOnlyPlace}(\mathsf{p1})$? Test your answer with a DL reasoner (in Protégé).

Solution

For example:

7. $\mathsf{IsEndOfOnlyOneRun} \equiv\ \leq_1\ \mathsf{isEndOf}\ .\top$
8. $\mathsf{IsEndOfOnlyOneRun}(\mathsf{p1})$