An Introduction to Description Logics

1. Syntax and Semantics

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Contents

The world according to DL

- There are **individuals**
- Individuals may be interconnected through **roles**
- Individuals belong to **concepts**

Similar to RDF/S (resources, properties, classes)

But different

- individuals, classes, and roles are disjoint
  - a class may not be an instance of another class
  - an individual may not have instances

DL languages

a family of logical languages with different

- class constructors
- role constructors

aims

- formalize the knowledge representation languages of the 80-90's (CLASSIC, KL-ONE, semantic networks, ...)
- decidable
  - there is an algorithm to check the consistency of any set of formulas
  - $\Rightarrow$ not as expressive as First order logic

$\mathcal{ALC}$ – concept constructors

Start with a vocabulary of concept names (N_C), role names (N_R) and individual names (N_O)

Concept expressions may be

- the top concept (everything): $\top$
- the bottom (impossible) concept: $\bot$
- a concept name: A
- a conjunction of concepts: $C \sqcap D$
- a disjunction: $C \sqcup D$
- a complement: $\neg C$
- an existential restriction: $\exists R.C$
- a universal restriction: $\forall R.C$

Manchester notation

class name

Building

conjunction

Student **and** Employee

disjunction

Man **or** Woman

complement

not Student

existential restriction

child **some** Student

universal restriction

child **only** Student

Semantics of $\mathcal{ALC}$

An interpretations I consists of

- a domain Δ^{I}
- an interpretation function I such that maps:
  - every *individual name* a to an element $\mathrm{I}(a) \in \Delta^{\mathrm{I}}$
  - every *concept* to a subset of Δ^{I}
  - every *role* to a binary relation on Δ^{I} (a subset of $\Delta^{\mathrm{I}} \times \Delta^{\mathrm{I}}$)

The semantics of non atomic concepts and roles is (recursively) defined in terms of atomic concept and role interpretations.

Semantics of concept expressions

$$
\begin{aligned}
\mathrm{I}(\bot) &= \emptyset \\
\mathrm{I}(\top) &= \Delta^{\mathrm{I}} \\
\mathrm{I}(C \sqcap D) &= \mathrm{I}(C) \cap \mathrm{I}(D) \\
\mathrm{I}(C \sqcup D) &= \mathrm{I}(C) \cup \mathrm{I}(D) \\
\mathrm{I}(\neg C) &= \Delta^{\mathrm{I}} - \mathrm{I}(C) \\
\mathrm{I}(\exists R.C) &= \{x \in \Delta^{\mathrm{I}} : \exists y \colon (x, y) \in \mathrm{I}(R) \text{ and } y \in \mathrm{I}(C)\} \\
\mathrm{I}(\forall R.C) &= \{x \in \Delta^{\mathrm{I}} : \forall y \colon (x, y) \in \mathrm{I}(R) \Rightarrow y \in \mathrm{I}(C)\}
\end{aligned}
$$

Exercise

Consider the vocabulary

concepts: `Human`, `Cat`; roles: `hasParent`

and the interpretation

- $\Delta^I = \{a, b, c, d, e, f, k, l, m\}$
- $I(\text{Human}) = \{a, b, c, d, e\}$
- $I(\text{Cat}) = \{k, l, m\}$
- $I(\text{hasParent}) = \{(a, b), (a, c), (c, d), (b, e), (b, f), (k, m)\}$

What are the formal interpretations (and informal meaning) of

- $\mathtt{Human} \sqcap \mathtt{Cat}$
- $\neg\ \mathtt{Human}$
- $\exists\ \mathtt{hasParent}.\top$
- $\exists\ \mathtt{hasParent}.\mathtt{Cat}$

Exercise (cont.)

If

- $\Delta^{\mathrm{I}} = \{\mathrm{a, b, c, d, e, f}\}$
- $\mathrm{I}(\texttt{Human}) = \{\mathrm{a, b, c, d, e}\}$
- $\mathrm{I}(\texttt{Cat}) = \{\mathrm{k, l, m}\}$
- $\mathrm{I}(\texttt{hasParent}) = \{(\mathrm{a, b}), (\mathrm{a, c}), (\mathrm{c, d}), (\mathrm{b, e}), (\mathrm{b, f}), (\mathrm{k, m})\}$

what are the interpretations of

- $\forall\texttt{.hasParent.Cat}$
- $\exists\ \texttt{hasParent.}\top \sqcap \forall\texttt{.hasParent.Cat}$
- $\exists\ \texttt{hasParent.(}\exists\ \texttt{hasParent.Human)}$
- $\exists\ \texttt{hasParent.(}\forall\ \texttt{hasParent.}\bot\texttt{)}$

More class constructors ($\mathcal{ALCOQ}$)

Number restrictions on properties

≥ n R.C

$$\{x \in \Delta^I : \#\{y | (x, y) \in I(R)\ and\ y \in I(C)\} \geq n\}$$

≤ n R.C

$$\{x \in \Delta^I : \#\{y | (x, y) \in I(R)\ and\ y \in I(C)\} \leq n\}$$

= n R.C ↔ ≥ n R.C and ≤ n R.C

Enumeration of individuals

$$\mathbf{\{i_1, i_2, \ldots, i_n\}}$$

$$\{x \in \Delta: x = I(i_1)\ or\ x = I(i_2)\ or\ \ldots, or\ x = I(in)\}$$

DL Knowledge Base

Vocabulary:

class names, property names, individual names

Terminological axioms (TBox):

provide class definitions and relationships between classes

Role axioms (RBox):

about roles

Assertional axioms (ABox):

about individuals

Terminological Axioms (TBox)

Axioms of the form

$$C \sqsubseteq D$$
$$C \equiv D$$
$$C \text{ disjoint } D$$

Axiom satisfaction by an interpretation I (notation: $\mathrm{I} \vDash \mathrm{Axiom}$)

$$\mathrm{I} \vDash C \sqsubseteq D \text{ if and only if } I(C) \subseteq I(D)$$
$$\mathrm{I} \vDash C \equiv D \text{ if and only if } I(C) = I(D)$$
$$\mathrm{I} \vDash C \text{ disjoint } D \text{ if and only if } I(C) \cap I(D) \neq \emptyset$$

Exercises

Find an interpretation I of the vocabulary

- `Cat`, `Mammal`, `Human` (concept names),
- `hasParent` (role names),
- `felix`, `bob`, `alice` (individual names)

that has $\Delta^{\mathrm{I}}=\{a, b, c, d, e, f\}$ and satisfies the axioms

- $\texttt{Cat} \sqsubseteq \texttt{Mammal}$
- $\texttt{Human} \sqsubseteq \texttt{Mammal}$
- $\texttt{Human} \sqsubseteq \forall\ \texttt{hasParent.Human}$
- $\{\texttt{bob, alice}\} \sqsubseteq \texttt{Human}$
- $\{\texttt{felix}\} \sqsubseteq \texttt{Cat}$
- $\texttt{Cat} \sqsubseteq \exists\ \texttt{hasParent}.\top$
- $\top \sqsubseteq \texttt{Mammal}$

Axioms on roles (RBox) (*ALCHOQ*)

$$P \sqsubseteq R$$

if a is linked to b through P

then a is linked to b through R

$$\mathrm{I} \vDash P \sqsubseteq R \text{ if and only if } \mathrm{I}(P) \subseteq \mathrm{I}(R)$$

Examples

- mother $\sqsubseteq$ parent
- primaryFunction $\sqsubseteq$ function

Axioms on roles: property chains

$$R_1 \circ R_2 \circ \ldots \circ R_n \sqsubseteq P$$

if a is linked to b through a property chain $R_1 \circ R_2 \circ \ldots \circ R_n$
then a is linked to b through P

$$I \vDash R_1 \circ R_2 \circ \ldots \circ R_n \sqsubseteq P$$

if and only if

$$\forall y_0, y_1, \ldots y_n : (\mathrm{y}_0, \mathrm{y}_1) \in \mathrm{I}(R_1), \ldots, (y_{n\text{-}1}, y_n) \in \mathrm{I}(R_n) \rightarrow (y_0, y_n) \in \mathrm{I}(P)$$

Examples

- `parent o parent` $\sqsubseteq$ `grandParent`
- `parent o parent o child o child` $\sqsubseteq$ `cousinOrSiblingOrSelf`

More axioms on roles

functional(R)

$$(x, y) \in I(R) \text{ and } (x, z) \in I(R) \rightarrow y = z$$

inverse(R, S)

$$(x, y) \in I(R) \rightarrow (y, x) \in I(S)$$

symmetric(R)

$$(x, y) \in I(R) \rightarrow (y, x) \in I(R)$$

transitive(R)

$$(x, y) \in I(R) \; and \; (y, z) \in I(R) \rightarrow (x, z) \in I(R)$$

reflexive(R)

$$\forall x \in \Delta^I : (x, x) \in I(R)$$

Examples

functional(`biologicalMother`)

inverse(`mother, child`)

symmetric(`friend`)

transitive(`before`)

reflexive(`knows`)

Assertions on individuals (ABox)

Axioms asserting that

- a named individual belongs to a class

  $$\boldsymbol{C(i)}$$

$I \vDash C(i)$ if and only if $I(i) \in I(C)$

- a named individual is linked to another one through a role

  $$\boldsymbol{R(i, j)}$$

$I \vDash R(i, j)$ if and only if $(I(i), I(j)) \in I(R)$

Assertions on individuals (ABox)

Axioms asserting that

- two named individuals are different

$$i \text{ differentFrom } j$$

$I \models i$ differentFrom j if and only if $I(i) \neq I(j)$

- two named individuals are equal

$$i \text{ sameAs } j$$

$I \models i$ sameAs j if and only if $I(i) = I(j)$

Exercise

Find a minimal interpretation that satisfies

- Man(bob)
- Woman(lisa)
- Human(sam)
- Man $\sqcup$ Woman $\sqsubseteq$ Human
- Man disjointFrom Woman
- hasSibling(bob, lisa)
- symmetric(hasSibling)
- father(lisa, max)
- father(lisa, mix)
- mix differentFrom sam
- functional(father)