Computation Tree Logic (CTL)

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Programme for the upcoming lectures

- Introducing CTL
- Basic Algorithms for CTL
- Basic Decision Diagrams

LTL and CTL

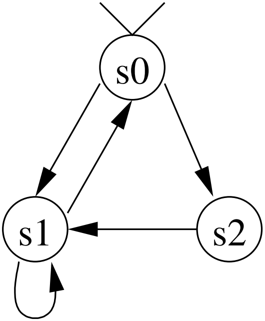
- *LTL* *(linear-time logic)*
  - Describes properties of individual executions.
  - Semantics defined as a set of executions.
- *CTL* *(computation tree logic)*
  - Describes properties of a *computation tree* : formulas can reason about many executions at once. (CTL belongs to the family of *branching-time logics*.)
  - Semantics defined in terms of states.

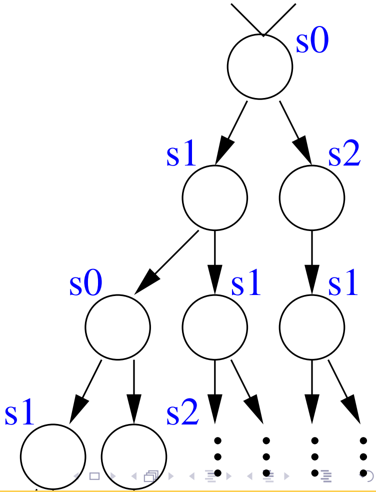
Computation tree

- Let $\mathcal{T} = \langle S, \rightarrow, s^0 \rangle$ be a transition system.
  Intuitively, the *computation tree* of $\mathcal{T}$ is the acyclic unfolding of $\mathcal{T}$.
- Formally, we can define the unfolding as the least (possibly infinite) transition system $\langle U, \rightarrow', u^0 \rangle$ with a labelling $l\colon U \rightarrow S$ such that
  - $u^0 \in U$ and $l(u^0) = s^0$;
  - if $u \in U$, $l(u) = s$, and $s \rightarrow s'$ for some u, s, s', then there is $u' \in U$ with $u \rightarrow' u'$ and $l(u') = s'$;
  - u^0 does not have a direct predecessor, and all other states in U have exactly one direct predecessor.
- *Note :* For model checking CTL, the construction of the computation tree will not be necessary. However, this definition serves to clarify the concepts behind CTL.

Computation tree : Example

- A transition system and its computation tree (labelling *l* given in blue) :





CTL : Overview

- *CTL* = *Computation-Tree Logic*
- Combines temporal operators with quantification over runs

Operators have the following form :

Q T

there exists an execution	**E**	**X**	**next**
for all executions	**A**	**F**	**finally**
		G	**globally**
		U	**until**

(and possibly others)

CTL : Syntax

- We define a minimal syntax first. Later we define additional operators with the help of the minimal syntax.
- Let AP be a set of atomic propositions : The set of CTL formulas over AP is as follows :
  - if $a \in AP$, then a is a CTL formula ;
  - if ϕ_1, ϕ_2 are CTL formulas, then so are
    $\neg\phi_1, \qquad \phi_1 \vee \phi_2, \qquad \mathbf{EX}\,\phi_1, \qquad \mathbf{EG}\,\phi_1, \qquad \phi_1\,\mathbf{EU}\,\phi_2$

Exemples of expressions :

CTL : Semantics

- Let $\mathcal{K} = (S, \rightarrow, s^0, AP, \nu)$ be a Kripke structure. AP is the set of atomic proposition and $\nu : AP \rightarrow P(S)$ assign atomic propositions to states.
- Runs are path within the transition system : $\rho : \mathbb{N} \rightarrow S$
- We define the semantic of every CTL formula ϕ over AP w.r.t. $\mathcal{K}$ as a set of states $[\![\phi]\!]_{\mathcal{K}}$, as follows :

$$
\begin{array}{rcl}
[\![a]\!]_{\mathcal{K}} & = & \nu(a) \qquad \text{for } a \in AP \\
[\![\neg\phi_1]\!]_{\mathcal{K}} & = & S \setminus [\![\phi_1]\!]_{\mathcal{K}} \\
[\![\phi_1 \vee \phi_2]\!]_{\mathcal{K}} & = & [\![\phi_1]\!]_{\mathcal{K}} \cup [\![\phi_2]\!]_{\mathcal{K}} \\
[\![\mathbf{EX}\ \phi_1]\!]_{\mathcal{K}} & = & \{ s \mid \text{there is a } t \text{ s.t. } s \rightarrow t \text{ and } t \in [\![\phi_1]\!]_{\mathcal{K}} \} \\
[\![\mathbf{EG}\ \phi_1]\!]_{\mathcal{K}} & = & \{ s \mid \text{there is a run } \rho \text{ with } \rho(0) = s \\
 & & \qquad \text{and } \rho(i) \in [\![\phi_1]\!]_{\mathcal{K}} \text{ for all } i \geq 0 \} \\
[\![\phi_1\ \mathbf{EU}\ \phi_2]\!]_{\mathcal{K}} & = & \{ s \mid \text{there is a run } \rho \text{ with } \rho(0) = s \text{ and } k \geq 0 \text{ s.t.} \\
 & & \qquad \rho(i) \in [\![\phi_1]\!]_{\mathcal{K}} \text{ for all } i < k \text{ and } \rho(k) \in [\![\phi_2]\!]_{\mathcal{K}} \}
\end{array}
$$

Remarks

- We say that $\mathcal{K}$ satisfies ϕ (denoted $\mathcal{K} \models \phi$) iff $s^0 \in [\![\phi]\!]_{\mathcal{K}}$.
- We declare two formulas equivalent (written $\phi_1 \equiv \phi_2$) iff for every Kripke structure $\mathcal{K}$ we have $[\![\phi_1]\!]_{\mathcal{K}} = [\![\phi_2]\!]_{\mathcal{K}}$.
- In the following, we omit the index $\mathcal{K}$ from $[\![\cdot]\!]_{\mathcal{K}}$ if $\mathcal{K}$ is understood.

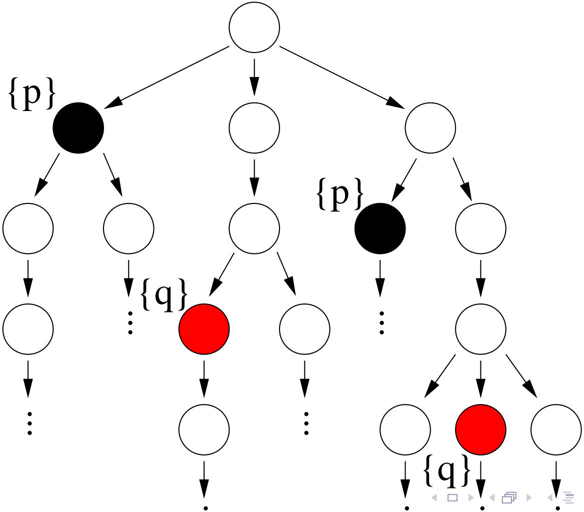
CTL : Extended syntax

$$
\begin{array}{rcl}
\phi_1 \wedge \phi_2 & \equiv & \neg(\neg\phi_1 \vee \neg\phi_2) \\
\mathbf{AX}\,\phi & \equiv & \neg\,\mathbf{EX}\,\neg\phi \\
\mathbf{true} & \equiv & a \vee \neg a \\
\mathbf{AG}\,\phi & \equiv & \neg\,\mathbf{EF}\,\neg\phi \\
\mathbf{false} & \equiv & \neg\mathbf{true} \\
\mathbf{AF}\,\phi & \equiv & \neg\,\mathbf{EG}\,\neg\phi \\
\phi_1\ \mathbf{EW}\ \phi_2 & \equiv & \mathbf{EG}\,\phi_1 \vee (\phi_1\ \mathbf{EU}\ \phi_2) \\
\phi_1\ \mathbf{AW}\ \phi_2 & \equiv & \neg(\neg\phi_2\ \mathbf{EU}\ \neg(\phi_1 \vee \phi_2)) \\
\mathbf{EF}\,\phi & \equiv & \mathbf{true}\ \mathbf{EU}\ \phi \\
\phi_1\ \mathbf{AU}\ \phi_2 & \equiv & \mathbf{AF}\,\phi_2 \wedge (\phi_1\ \mathbf{AW}\ \phi_2))
\end{array}
$$

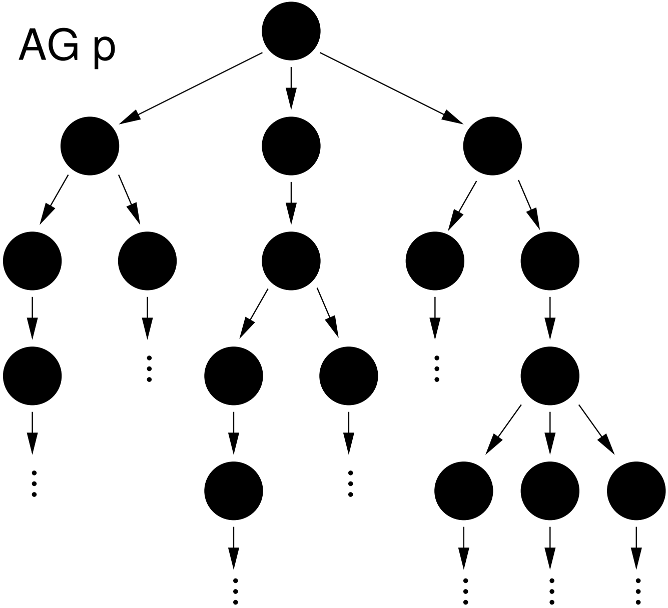
Other logical and temporal operators (e.g. $\rightarrow$), **ER**, **AR**, ... may also be defined.

CTL : Examples

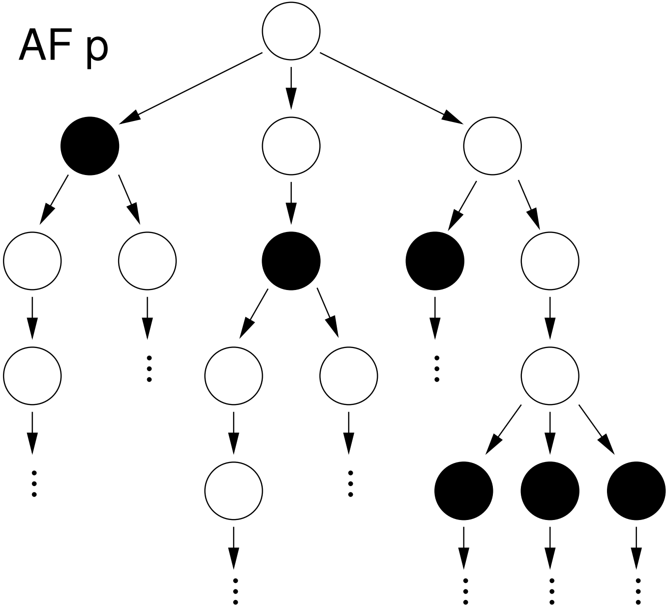
- We use the following computation tree as a running example (with varying distributions of red and black states) :



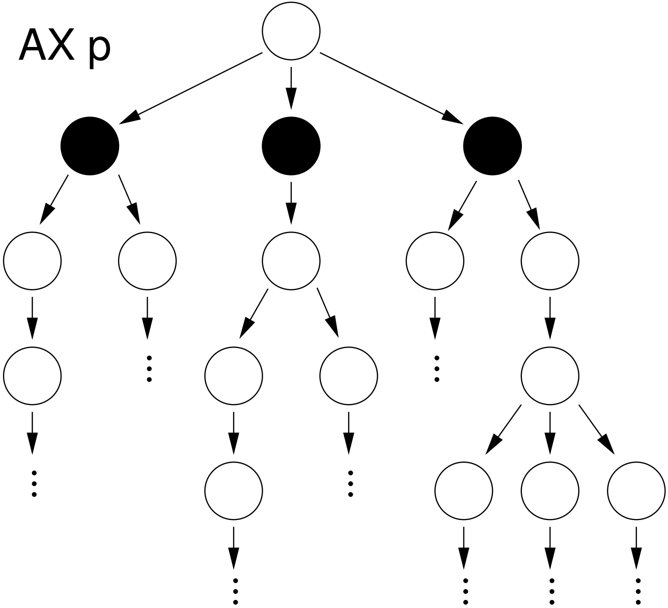
AG p



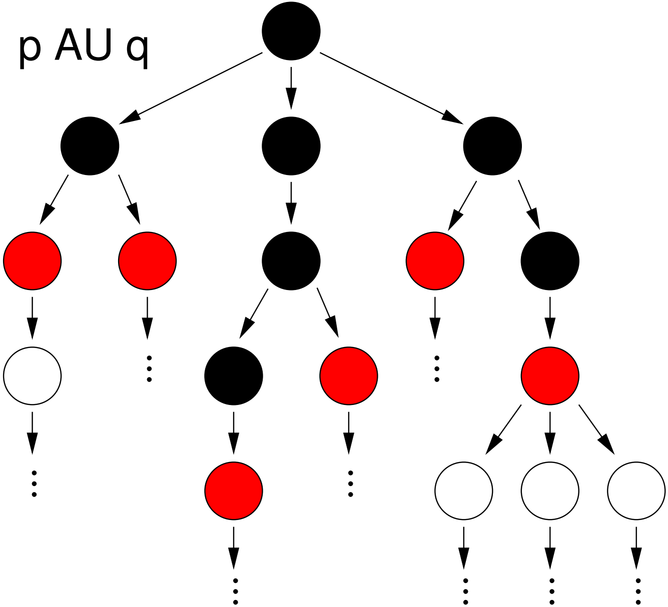
AF p



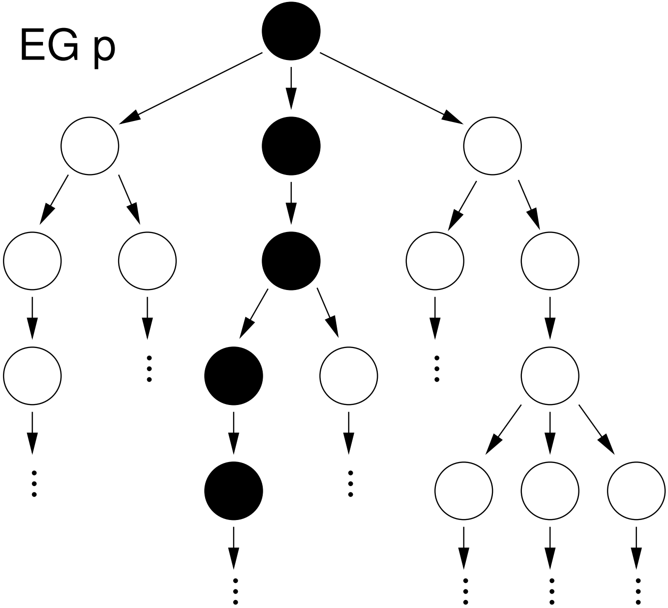
AX p



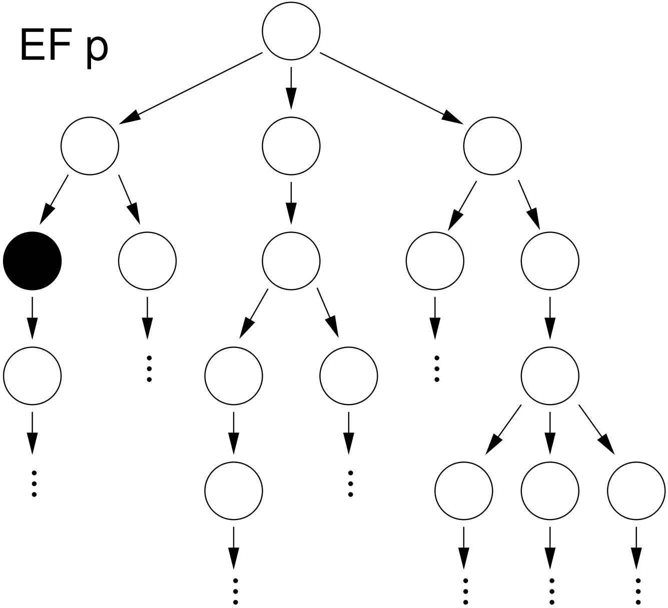
p AU q



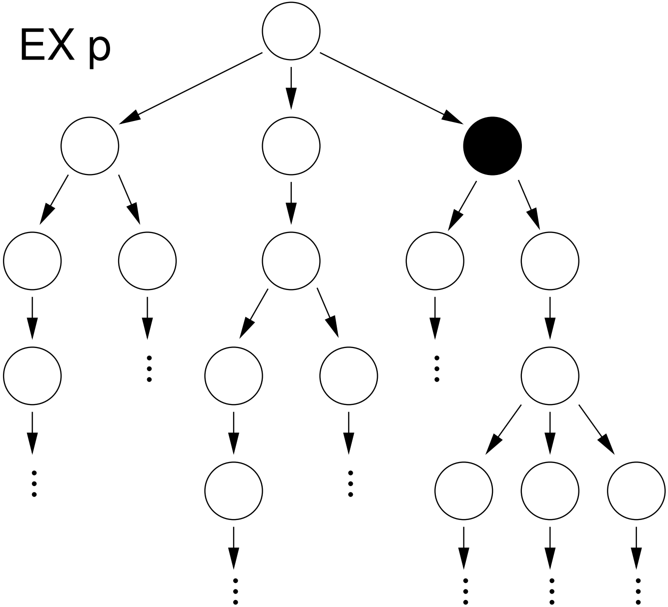
EG p



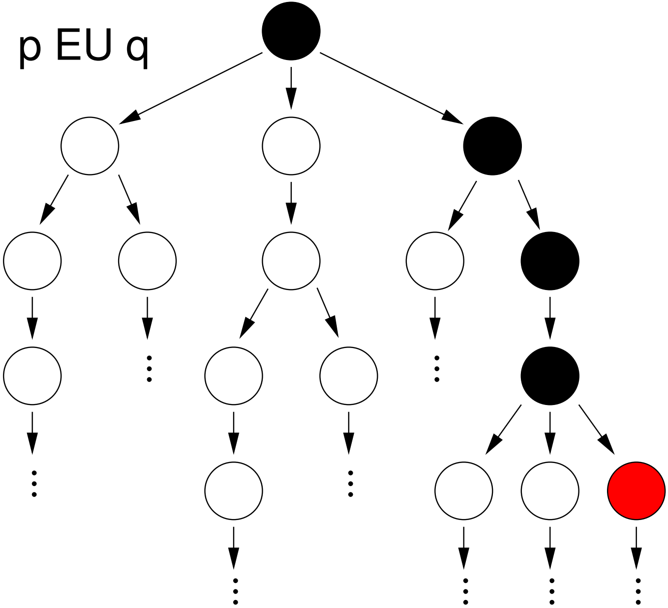
EF p



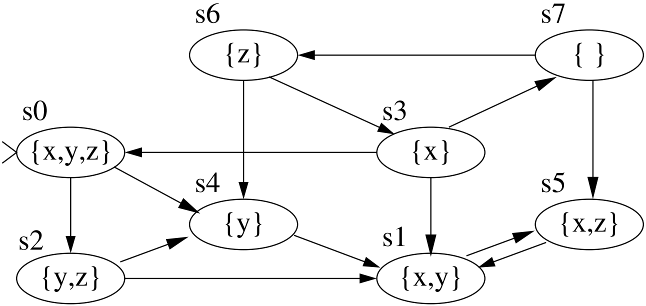
EX p



p EU q

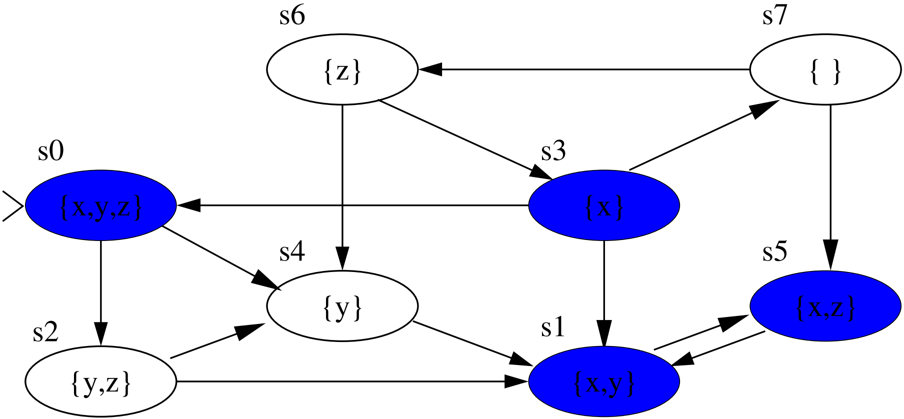


Solving nested formulas : Is $s_0 \in [\![\mathbf{AF\ AG}\ x]\!]$?

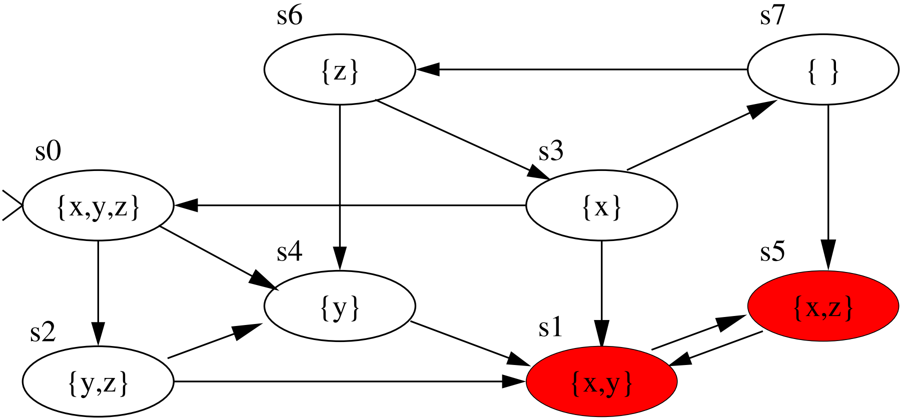


- To compute the semantics of formulas with nested operators, we first compute the states satisfying the innermost formulas ; then we use those results to solve progressively more complex formulas.
- In this example, we compute $[\![x]\!]$, $[\![\mathbf{AG}\ x]\!]$, and $[\![\mathbf{AF\ AG}\ x]\!]$, in that order.

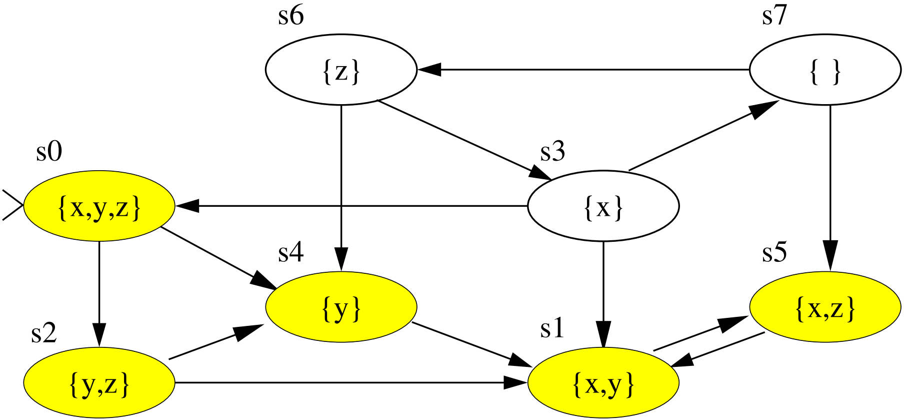
Bottom-up method (1) : Compute⟦x⟧



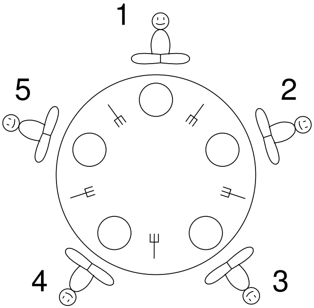
Bottom-up method (2) : Compute⟦**AG** x⟧



Bottom-up method (3) : Compute⟦**AF AG** x⟧



Example : Dining Philosophers



- Five philosophers are sitting around a table, taking turns at thinking and eating.
- We shall express a couple of properties in CTL. Let us assume the following atomic propositions :
  - $e_i \;\widehat{=}\;$ philosopher i is currently eating
  - $f_i \;\widehat{=}\;$ philosopher i has just finished eating

Expected properties

- “Philosophers 1 and 4 will never eat at the same time.”
-
-

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$$\mathbf{AG}\, \neg(e_1 \wedge e_4)$$

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-

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$$\mathbf{AG}(f_4 \rightarrow (\neg e_4 \;\mathbf{AW}\; e_3))$$

- "Philosopher 2 will be the first to eat."

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- "Philosopher 2 will be the first to eat."

$$\neg(e_1 \vee e_3 \vee e_4 \vee e_5)\ \mathbf{AU}\ e_2$$

Expressiveness of CTL and LTL (if you hear about it)

- CTL and LTL have a large overlap, i.e. properties expressible in both logics.
- CTL considers the whole computation tree whereas LTL only considers individual runs. Thus CTL allows to reason about the *branching behaviour*, considering multiple possible runs at once.
- Even though CTL considers the whole computation tree, its state-based semantics. expressible in LTL but not in CTL.
- Also, *fairness conditions* are not directly expressible in CTL
- However there is another way to extend CTL with fairness conditions.

Conclusion

- *Conclusion* : The expressiveness of CTL allows to express complex properties such as livens or safety properties.
- *Remark* : There is a logic called CTL^* that combines the expressiveness of CTL and LTL. However, we will not deal with it in this course.