On estimation of nonparametric regression models with autoregressive and moving average errors

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Abstract

The nonparametric regression model with correlated random errors is a powerful tool for time series forecasting. We are interested in the estimation of such a model under a random design setup, where the random errors are assumed to follow an autoregressive and moving average (ARMA) process, and the covariates can also be correlated. We propose a spline-based method to estimate the mean function and the parameters of the ARMA process. We establish the desirable asymptotic properties of the proposed approach under mild regularity conditions. Extensive simulation studies demonstrate that our proposed method performs well and generates strong evidence supporting the established theoretical results. The proposed method complements the arsenal of tools of nonparametric time series analysis. We further illustrate the practical usefulness of our method by modeling and forecasting the weekly natural gas scraping data for the state of Iowa.

Keywords: nonparametric model with correlated errors; oracally efficient estimation; τ-mixing; splines

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Table 1: Estimation of the parameters of ARMA$(1,1)$ process, when X_t's are serially correlated and satisfy the conditions of Theorem 2.2, and innovations ζ_t's have a t distribution with degrees of freedom ν.

	$(\phi,\theta)=(0.6,0.3)$ $\nu=3$ $g_0(X_t)=f_1(X_t)$					
	$n=500$		$n=1000$		$n=2000$	
	$\hat{\phi}$	$\hat{\theta}$	$\hat{\phi}$	$\hat{\theta}$	$\hat{\phi}$	$\hat{\theta}$
mean	0.5907	0.3160	0.5991	0.3058	0.5989	0.3024
s.d.	0.0489	0.0583	0.0325	0.0386	0.0232	0.0272
	(0.0469)	(0.0559)	(0.0332)	(0.0396)	(0.0235)	(0.0280)
	$(\phi,\theta)=(0.6,0.3)$ $\nu=3$ $g_0(X_t)=f_2(X_t)$					
	$n=500$		$n=1000$		$n=2000$	
	$\hat{\phi}$	$\hat{\theta}$	$\hat{\phi}$	$\hat{\theta}$	$\hat{\phi}$	$\hat{\theta}$
mean	0.5934	0.3182	0.5957	0.3081	0.5987	0.3026
s.d.	0.0505	0.0621	0.0327	0.0393	0.0238	0.0285
	(0.0469)	(0.0559)	(0.0332)	(0.0396)	(0.0235)	(0.0280)
	$(\phi,\theta)=(0.6,0.3)$ $\nu=3$ $g_0(X_t)=f_3(X_t)$					
	$n=500$		$n=1000$		$n=2000$	
	$\hat{\phi}$	$\hat{\theta}$	$\hat{\phi}$	$\hat{\theta}$	$\hat{\phi}$	$\hat{\theta}$
mean	0.5911	0.3151	0.5969	0.3061	0.5983	0.3045
s.d.	0.0478	0.0585	0.0327	0.0407	0.0240	0.0290
	(0.0469)	(0.0559)	(0.0332)	(0.0396)	(0.0235)	(0.0280)
	$(\phi,\theta)=(0.2,-0.5)$ $\nu=3$ $g_0(X_t)=f_1(X_t)$					
	$n=500$		$n=1000$		$n=2000$	
	$\hat{\phi}$	$\hat{\theta}$	$\hat{\phi}$	$\hat{\theta}$	$\hat{\phi}$	$\hat{\theta}$
mean	0.2064	-0.5230	0.2048	-0.5123	0.2001	-0.5041
s.d.	0.1428	0.1273	0.0922	0.0814	0.0675	0.0606
	(0.1315)	(0.1162)	(0.0930)	(0.0822)	(0.0657)	(0.0581)
	$(\phi,\theta)=(0.2,-0.5)$ $\nu=3$ $g_0(X_t)=f_2(X_t)$					
	$n=500$		$n=1000$		$n=2000$	
	$\hat{\phi}$	$\hat{\theta}$	$\hat{\phi}$	$\hat{\theta}$	$\hat{\phi}$	$\hat{\theta}$
mean	0.2185	-0.5320	0.2025	-0.5113	0.2026	-0.5057
s.d.	0.1388	0.1256	0.0956	0.0859	0.0660	0.0581
	(0.1315)	(0.1162)	(0.0930)	(0.0822)	(0.0657)	(0.0581)
	$(\phi,\theta)=(0.2,-0.5)$ $\nu=3$ $g_0(X_t)=f_3(X_t)$					
	$n=500$		$n=1000$		$n=2000$	
	$\hat{\phi}$	$\hat{\theta}$	$\hat{\phi}$	$\hat{\theta}$	$\hat{\phi}$	$\hat{\theta}$
mean	0.2022	-0.5165	0.2035	-0.5098	0.2062	-0.5099
s.d.	0.1444	0.1300	0.1009	0.0901	0.0646	0.0562
	(0.1315)	(0.1162)	(0.0930)	(0.0822)	(0.0657)	(0.0581)

Table 2: Estimation of the parameters of AR(2) process, when X_t's are serially correlated and satisfy the conditions of Theorem 2.2, and innovations ζ_t's have a t distribution with degrees of freedom ν.

	$(\phi_1,\phi_2)=(0.4,0.2)$ $\nu=3$ $g_0(X_t)=f_1(X_t)$					
	$n=500$		$n=1000$		$n=2000$	
	$\hat{\phi}$	$\hat{\theta}$	$\hat{\phi}$	$\hat{\theta}$	$\hat{\phi}$	$\hat{\theta}$
mean	0.4002	0.1948	0.3996	0.1965	0.3992	0.1986
s.d.	0.0451	0.4610	0.03144	0.0337	0.0234	0.0237
	(0.0438)	(0.0438)	(0.0310)	(0.0310)	(0.0219)	(0.0219)
	$(\phi_1,\phi_2)=(0.4,0.2)$ $\nu=3$ $g_0(X_t)=f_2(X_t)$					
	$n=500$		$n=1000$		$n=2000$	
	$\hat{\phi}$	$\hat{\theta}$	$\hat{\phi}$	$\hat{\theta}$	$\hat{\phi}$	$\hat{\theta}$
mean	0.4007	0.1935	0.3980	0.1969	0.3990	0.1988
s.d.	0.0451	0.0461	0.0309	0.0307	0.0219	0.0222
	(0.0438)	(0.0438)	(0.0310)	(0.0310)	(0.0219)	(0.0219)
	$(\phi_1,\phi_2)=(0.4,0.2)$ $\nu=3$ $g_0(X_t)=f_3(X_t)$					
	$n=500$		$n=1000$		$n=2000$	
	$\hat{\phi}$	$\hat{\theta}$	$\hat{\phi}$	$\hat{\theta}$	$\hat{\phi}$	$\hat{\theta}$
mean	0.4007	0.1936	0.3993	0.1995	0.3963	0.1991
s.d.	0.0448	0.0450	0.0311	0.0312	0.0226	0.0225
	(0.0438)	(0.0438)	(0.0310)	(0.0310)	(0.0219)	(0.0219)
	$(\phi_1,\phi_2)=(0.5,0.1)$ $\nu=3$ $g_0(X_t)=f_1(X_t)$					
	$n=500$		$n=1000$		$n=2000$	
	$\hat{\phi}$	$\hat{\theta}$	$\hat{\phi}$	$\hat{\theta}$	$\hat{\phi}$	$\hat{\theta}$
mean	0.4827	0.0976	0.4995	0.0965	0.4990	0.0989
s.d.	0.0458	0.0467	0.0321	0.0316	0.0227	0.0229
	(0.0445)	(0.0445)	(0.0314)	(0.0314)	(0.0222)	(0.0222)
	$(\phi_1,\phi_2)=(0.5,0.1)$ $\nu=3$ $g_0(X_t)=f_2(X_t)$					
	$n=500$		$n=1000$		$n=2000$	
	$\hat{\phi}$	$\hat{\theta}$	$\hat{\phi}$	$\hat{\theta}$	$\hat{\phi}$	$\hat{\theta}$
mean	0.5002	0.0958	0.4992	0.0977	0.4991	0.0979
s.d.	0.0451	0.0473	0.0331	0.0324	0.0232	0.0237
	(0.0445)	(0.0445)	(0.0314)	(0.0314)	(0.0222)	(0.0222)
	$(\phi_1,\phi_2)=(0.5,0.1)$ $\nu=3$ $g_0(X_t)=f_3(X_t)$					
	$n=500$		$n=1000$		$n=2000$	
	$\hat{\phi}$	$\hat{\theta}$	$\hat{\phi}$	$\hat{\theta}$	$\hat{\phi}$	$\hat{\theta}$
mean	0.4987	0.0924	0.5009	0.0961	0.4995	0.0987
s.d.	0.0453	0.0461	0.0324	0.0318	0.0234	0.0230
	(0.0445)	(0.0445)	(0.0314)	(0.0314)	(0.0222)	(0.0222)

Table 3: Estimation of the parameters of MA(2) process, when X_t's are serially correlated and satisfy the conditions of Theorem 2.2, and innovations ζ_t's have a t distribution with degrees of freedom ν.

	$(\theta_1, \theta_2) = (0.4, 0.2)$ $\nu = 3$ $g_0(X_t) = f_1(X_t)$					
	$n = 500$		$n = 1000$		$n = 2000$	
	$\hat{\phi}$	$\hat{\theta}$	$\hat{\phi}$	$\hat{\theta}$	$\hat{\phi}$	$\hat{\theta}$
mean	0.4046	0.2025	0.4030	0.2031	0.4014	0.2014
s.d.	0.0439	0.0457	0.0322	0.0317	0.0227	0.0208
	(0.0438)	(0.0438)	(0.0310)	(0.0310)	(0.0219)	(0.0219)
	$(\theta_1, \theta_2) = (0.4, 0.2)$ $\nu = 3$ $g_0(X_t) = f_2(X_t)$					
	$n = 500$		$n = 1000$		$n = 2000$	
	$\hat{\phi}$	$\hat{\theta}$	$\hat{\phi}$	$\hat{\theta}$	$\hat{\phi}$	$\hat{\theta}$
mean	0.4028	0.2026	0.4021	0.2002	0.4010	0.2007
s.d.	0.0452	0.0453	0.0312	0.0315	0.0216	0.0222
	(0.0438)	(0.0438)	(0.0310)	(0.0310)	(0.0219)	(0.0219)
	$(\theta_1, \theta_2) = (0.4, 0.2)$ $\nu = 3$ $g_0(X_t) = f_3(X_t)$					
	$n = 500$		$n = 1000$		$n = 2000$	
	$\hat{\phi}$	$\hat{\theta}$	$\hat{\phi}$	$\hat{\theta}$	$\hat{\phi}$	$\hat{\theta}$
mean	0.4046	0.2025	0.4029	0.2010	0.4020	0.2009
s.d.	0.0455	0.0465	0.0312	0.0316	0.0222	0.0218
	(0.0438)	(0.0438)	(0.0310)	(0.0310)	(0.0219)	(0.0219)
	$(\theta_1, \theta_2) = (-0.2, -0.4)$ $\nu = 3$ $g_0(X_t) = f_1(X_t)$					
	$n = 500$		$n = 1000$		$n = 2000$	
	$\hat{\phi}$	$\hat{\theta}$	$\hat{\phi}$	$\hat{\theta}$	$\hat{\phi}$	$\hat{\theta}$
mean	-0.2140	-0.4155	-0.2072	-0.4071	-0.2036	-0.4034
s.d.	0.0470	0.0454	0.0305	0.0300	0.0201	0.0207
	(0.0410)	(0.0410)	(0.0290)	(0.0290)	(0.0205)	(0.0205)
	$(\theta_1, \theta_2) = (-0.2, -0.4)$ $\nu = 3$ $g_0(X_t) = f_2(X_t)$					
	$n = 500$		$n = 1000$		$n = 2000$	
	$\hat{\phi}$	$\hat{\theta}$	$\hat{\phi}$	$\hat{\theta}$	$\hat{\phi}$	$\hat{\theta}$
mean	-0.2163	-0.4160	-0.2073	-0.4092	-0.2042	-0.4030
s.d.	0.0470	0.0436	0.0307	0.0296	0.0210	0.0207
	(0.0410)	(0.0410)	(0.0290)	(0.0290)	(0.0205)	(0.0205)
	$(\theta_1, \theta_2) = (-0.2, -0.4)$ $\nu = 3$ $g_0(X_t) = f_3(X_t)$					
	$n = 500$		$n = 1000$		$n = 2000$	
	$\hat{\phi}$	$\hat{\theta}$	$\hat{\phi}$	$\hat{\theta}$	$\hat{\phi}$	$\hat{\theta}$
mean	-0.2150	-0.4158	-0.2083	-0.4070	-0.2034	-0.4032
s.d.	0.0450	0.0457	0.0300	0.0295	0.0207	0.0206
	(0.0410)	(0.0410)	(0.0290)	(0.0290)	(0.0205)	(0.0205)

Table 4: Comparing $\hat{g}(\cdot)$ and $g_0(\cdot)$, when ϵ_t's follow an ARMA(1, 1) process, X_t's are serially correlated and satisfy the conditions of Theorem 2.2, and innovations ζ_t's have a t distribution with degrees of freedom ν.

	$(\phi,\theta)=(0.6,0.3)$ $\nu=3$ $g_0(X_t)=f_1(X_t)$					
	$n=500$		$n=1000$		$n=2000$	
	$\rho(g_0,\hat{g})$	$\rho_{19}(g_0,\hat{g})$	$\rho(g_0,\hat{g})$	$\rho_{19}(g_0,\hat{g})$	$\rho(g_0,\hat{g})$	$\rho_{19}(g_0,\hat{g})$
one step	0.3193	0.1550	0.2056	0.0798	0.1608	0.0447
sequential	0.7797	0.3345	0.5655	0.1867	0.4633	0.1193
	$(\phi,\theta)=(0.6,0.3)$ $\nu=3$ $g_0(X_t)=f_2(X_t)$					
	$n=500$		$n=1000$		$n=2000$	
	$\rho(g_0,\hat{g})$	$\rho_{19}(g_0,\hat{g})$	$\rho(g_0,\hat{g})$	$\rho_{19}(g_0,\hat{g})$	$\rho(g_0,\hat{g})$	$\rho_{19}(g_0,\hat{g})$
one step	0.3018	0.1387	0.2584	0.0959	0.1326	0.0462
sequential	0.7461	0.3255	0.7346	0.2885	0.4343	0.1303
	$(\phi,\theta)=(0.6,0.3)$ $\nu=3$ $g_0(X_t)=f_3(X_t)$					
	$n=500$		$n=1000$		$n=2000$	
	$\rho(g_0,\hat{g})$	$\rho_{19}(g_0,\hat{g})$	$\rho(g_0,\hat{g})$	$\rho_{19}(g_0,\hat{g})$	$\rho(g_0,\hat{g})$	$\rho_{19}(g_0,\hat{g})$
one step	0.3003	0.1370	0.2106	0.0770	0.1628	0.0482
sequential	0.8628	0.3414	0.5499	0.1940	0.4890	0.1214
	$(\phi,\theta)=(0.2,-0.5)$ $\nu=3$ $g_0(X_t)=f_1(X_t)$					
	$n=500$		$n=1000$		$n=2000$	
	$\rho(g_0,\hat{g})$	$\rho_{19}(g_0,\hat{g})$	$\rho(g_0,\hat{g})$	$\rho_{19}(g_0,\hat{g})$	$\rho(g_0,\hat{g})$	$\rho_{19}(g_0,\hat{g})$
one step	0.2079	0.0719	0.1440	0.0401	0.1133	0.0232
sequential	0.2365	0.0820	0.1710	0.0489	0.1342	0.0285
	$(\phi,\theta)=(0.2,-0.5)$ $\nu=3$ $g_0(X_t)=f_2(X_t)$					
	$n=500$		$n=1000$		$n=2000$	
	$\rho(g_0,\hat{g})$	$\rho_{19}(g_0,\hat{g})$	$\rho(g_0,\hat{g})$	$\rho_{19}(g_0,\hat{g})$	$\rho(g_0,\hat{g})$	$\rho_{19}(g_0,\hat{g})$
one step	0.2033	0.0727	0.1804	0.0527	0.1056	0.0237
sequential	0.2329	0.0838	0.2067	0.0545	0.1261	0.0283
	$(\phi,\theta)=(0.2,-0.5)$ $\nu=3$ $g_0(X_t)=f_3(X_t)$					
	$n=500$		$n=1000$		$n=2000$	
	$\rho(g_0,\hat{g})$	$\rho_{19}(g_0,\hat{g})$	$\rho(g_0,\hat{g})$	$\rho_{19}(g_0,\hat{g})$	$\rho(g_0,\hat{g})$	$\rho_{19}(g_0,\hat{g})$
one step	0.2022	0.0679	0.1478	0.0422	0.1403	0.0270
sequential	0.2444	0.0843	0.1715	0.0513	0.1667	0.0314

Table 5: Comparing $\hat{g}(\cdot)$ and $g_0(\cdot)$, when ϵ_t's follow an AR(2) process, X_t's are serially correlated and satisfy the conditions of Theorem 2.2, and innovations ζ_t's have a t distribution with degrees of freedom ν.

	$(\phi_1, \phi_2) = (0.4, 0.2)$ $\nu = 3$ $g_0(X_t) = f_1(X_t)$					
	$n = 500$		$n = 1000$		$n = 2000$	
	$\rho(g_0, \hat{g})$	$\rho_{19}(g_0, \hat{g})$	$\rho(g_0, \hat{g})$	$\rho_{19}(g_0, \hat{g})$	$\rho(g_0, \hat{g})$	$\rho_{19}(g_0, \hat{g})$
one step	0.2215	0.1044	0.2028	0.0878	0.1983	0.0751
sequential	0.4083	0.1532	0.3086	0.0996	0.2429	0.0538
	$(\phi_1, \phi_2) = (0.4, 0.2)$ $\nu = 3$ $g_0(X_t) = f_2(X_t)$					
	$n = 500$		$n = 1000$		$n = 2000$	
	$\rho(g_0, \hat{g})$	$\rho_{19}(g_0, \hat{g})$	$\rho(g_0, \hat{g})$	$\rho_{19}(g_0, \hat{g})$	$\rho(g_0, \hat{g})$	$\rho_{19}(g_0, \hat{g})$
one step	0.1848	0.0919	0.1478	0.0534	0.1395	0.0381
sequential	0.4028	0.1590	0.3173	0.0899	0.2500	0.0556
	$(\phi_1, \phi_2) = (0.4, 0.2)$ $\nu = 3$ $g_0(X_t) = f_3(X_t)$					
	$n = 500$		$n = 1000$		$n = 2000$	
	$\rho(g_0, \hat{g})$	$\rho_{19}(g_0, \hat{g})$	$\rho(g_0, \hat{g})$	$\rho_{19}(g_0, \hat{g})$	$\rho(g_0, \hat{g})$	$\rho_{19}(g_0, \hat{g})$
one step	0.1873	0.0965	0.1534	0.0706	0.1575	0.0593
sequential	0.3782	0.1455	0.3234	0.0913	0.2811	0.0594
	$(\phi_1, \phi_2) = (0.5, 0.1)$ $\nu = 3$ $g_0(X_t) = f_1(X_t)$					
	$n = 500$		$n = 1000$		$n = 2000$	
	$\rho(g_0, \hat{g})$	$\rho_{19}(g_0, \hat{g})$	$\rho(g_0, \hat{g})$	$\rho_{19}(g_0, \hat{g})$	$\rho(g_0, \hat{g})$	$\rho_{19}(g_0, \hat{g})$
one step	0.2279	0.1096	0.2248	0.0936	0.2057	0.0617
sequential	0.4556	0.1734	0.3723	0.1240	0.2798	0.0436
	$(\phi_1, \phi_2) = (0.5, 0.1)$ $\nu = 3$ $g_0(X_t) = f_2(X_t)$					
	$n = 500$		$n = 1000$		$n = 2000$	
	$\rho(g_0, \hat{g})$	$\rho_{19}(g_0, \hat{g})$	$\rho(g_0, \hat{g})$	$\rho_{19}(g_0, \hat{g})$	$\rho(g_0, \hat{g})$	$\rho_{19}(g_0, \hat{g})$
one step	0.1877	0.0892	0.1514	0.0539	0.1417	0.0413
sequential	0.4290	0.1688	0.3150	0.1027	0.2789	0.0623
	$(\phi_1, \phi_2) = (0.5, 0.1)$ $\nu = 3$ $g_0(X_t) = f_3(X_t)$					
	$n = 500$		$n = 1000$		$n = 2000$	
	$\rho(g_0, \hat{g})$	$\rho_{19}(g_0, \hat{g})$	$\rho(g_0, \hat{g})$	$\rho_{19}(g_0, \hat{g})$	$\rho(g_0, \hat{g})$	$\rho_{19}(g_0, \hat{g})$
one step	0.2016	0.1035	0.1710	0.0792	0.1625	0.0574
sequential	0.4633	0.1848	0.3252	0.1011	0.3475	0.0396

Table 6: Comparing $\hat{g}(\cdot)$ and $g_0(\cdot)$, when ϵ_t's follow an MA(2) process, X_t's are serially correlated and satisfy the conditions of Theorem 2.2, and innovations ζ_t's have a t distribution with degrees of freedom ν.

	$(\theta_1,\theta_2)=(0.4,0.2)$ $\nu=3$ $g_0(X_t)=f_1(X_t)$					
	$n=500$		$n=1000$		$n=2000$	
	$\rho(g_0,\hat{g})$	$\rho_{19}(g_0,\hat{g})$	$\rho(g_0,\hat{g})$	$\rho_{19}(g_0,\hat{g})$	$\rho(g_0,\hat{g})$	$\rho_{19}(g_0,\hat{g})$
one step	0.2328	0.0884	0.1837	0.0515	0.1612	0.0297
sequential	0.3109	0.1141	0.2489	0.0687	0.2234	0.0400
	$(\theta_1,\theta_2)=(0.4,0.2)$ $\nu=3$ $g_0(X_t)=f_2(X_t)$					
	$n=500$		$n=1000$		$n=2000$	
	$\rho(g_0,\hat{g})$	$\rho_{19}(g_0,\hat{g})$	$\rho(g_0,\hat{g})$	$\rho_{19}(g_0,\hat{g})$	$\rho(g_0,\hat{g})$	$\rho_{19}(g_0,\hat{g})$
one step	0.2471	0.0916	0.1918	0.0598	0.1371	0.0317
sequential	0.3262	0.1171	0.2548	0.0763	0.1990	0.0427
	$(\theta_1,\theta_2)=(0.4,0.2)$ $\nu=3$ $g_0(X_t)=f_3(X_t)$					
	$n=500$		$n=1000$		$n=2000$	
	$\rho(g_0,\hat{g})$	$\rho_{19}(g_0,\hat{g})$	$\rho(g_0,\hat{g})$	$\rho_{19}(g_0,\hat{g})$	$\rho(g_0,\hat{g})$	$\rho_{19}(g_0,\hat{g})$
one step	0.2520	0.0931	0.2124	0.0549	0.1465	0.0328
sequential	0.3296	0.1199	0.2788	0.0717	0.1966	0.0435
	$(\theta_1,\theta_2)=(-0.2,-0.4)$ $\nu=3$ $g_0(X_t)=f_1(X_t)$					
	$n=500$		$n=1000$		$n=2000$	
	$\rho(g_0,\hat{g})$	$\rho_{19}(g_0,\hat{g})$	$\rho(g_0,\hat{g})$	$\rho_{19}(g_0,\hat{g})$	$\rho(g_0,\hat{g})$	$\rho_{19}(g_0,\hat{g})$
one step	0.1877	0.0536	0.1185	0.0316	0.1032	0.0182
sequential	0.2760	0.0825	0.1963	0.0522	0.1640	0.0302
	$(\theta_1,\theta_2)=(-0.2,-0.4)$ $\nu=3$ $g_0(X_t)=f_2(X_t)$					
	$n=500$		$n=1000$		$n=2000$	
	$\rho(g_0,\hat{g})$	$\rho_{19}(g_0,\hat{g})$	$\rho(g_0,\hat{g})$	$\rho_{19}(g_0,\hat{g})$	$\rho(g_0,\hat{g})$	$\rho_{19}(g_0,\hat{g})$
one step	0.1605	0.0536	0.1389	0.0325	0.1017	0.0183
sequential	0.2459	0.0850	0.2189	0.0325	0.1670	0.0302
	$(\theta_1,\theta_2)=(-0.2,-0.4)$ $\nu=3$ $g_0(X_t)=f_3(X_t)$					
	$n=500$		$n=1000$		$n=2000$	
	$\rho(g_0,\hat{g})$	$\rho_{19}(g_0,\hat{g})$	$\rho(g_0,\hat{g})$	$\rho_{19}(g_0,\hat{g})$	$\rho(g_0,\hat{g})$	$\rho_{19}(g_0,\hat{g})$
one step	0.1801	0.0530	0.1349	0.0339	0.1056	0.0178
sequential	0.2791	0.0827	0.2187	0.0564	0.1796	0.0295