A Reputational Theory of Influencer Marketing*

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Abstract

Social media influencers often endorse products under brand sponsorships without disclosing the nature of their relationship with the brand. This paper explores the incentives for truth-telling when influencers face private sponsorship opportunities and the potential consequences for consumers. While reputation concerns can encourage influencers to be truthful when promoting sponsored products, they may also lead to under-endorsement when no sponsorship opportunity exists. As sponsorship opportunities become more abundant, the quality of information transmission initially decreases before increasing. Therefore, new technologies that efficiently match influencers with sponsors may improve consumer welfare. The Federal Trade Commission's mandatory disclosure rules are also shown to benefit consumers without necessarily harming influencers.

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1 Introduction

Influencer marketing has been increasingly popular these years. Social media influencers, who often specialize in niche areas such as fashion, fitness, or electronics, create content and recommend products to their followers. Marketers offer sponsorships to influencers in exchange for their endorsements, and according to a report published by the Influencer Marketing Hub, the industry is set to grow to approximately $16.4 billion in the US. The report also notes that 90% of market participants believe that influencer marketing is effective.[1] Consumers have embraced this new channel for discovering quality products, with a 2019 survey by CivicScience revealing that 15% of the over 3,000 US consumers surveyed had purchased a product or service based on an influencer or blogger's recommendation on social media. This number increased to 42% among daily Instagram users.

Without voluntary disclosure from influencers, followers often struggle to determine if a particular endorsement is sponsored. In 2017, the US Federal Trade Commission (FTC) began requiring all influencers to disclose brand sponsorships. However, there have only been a few high-profile cases in which influencers were fined for not adhering to the disclosure rules. Boerman et al. (2018) show that there is low regulation compliance. Moreover, Awin.com's 2020 study finds that even among compliant influencers, over three-quarters on Instagram had hidden the disclosure somewhere in the post, such as in a comment. The effectiveness of influencer marketing for both marketers and consumers despite the limited regulation of sponsored endorsements is puzzling.

Despite the inherent conflicts of interest that arise due to sponsorships, influencers can maintain the trust of their followers if they are concerned about their reputation. This paper develops a reputation model to examine how influencers balance their desire for sponsorship revenue with their need to maintain a good reputation. In each period, the influencer comes across a single product and receives a binary signal about its quality, either through a marketer offering a sponsorship or through their own search. The influencer privately knows whether a sponsorship has been offered, but their followers do not. The influencer decides whether to endorse the product based on the sponsorship and the informative signal. After the endorsement, the influencer receives immediate sponsorship revenue, and followers decide whether to purchase the endorsed product. If the influencer chooses not to endorse, they forgo the sponsorship (if there is one), and their followers do not see the product. The true product quality is revealed publicly at the end of each period.

[1]See the Influencer Marketing Benchmark Report 2022, https://influencermarketinghub.com/influencer-marketing-benchmark-report.

In the absence of reputation concerns, a subgame perfect equilibrium exists where no information is transmitted. To discipline the endorsing behavior, I adopt the incomplete information approach widely used in the literature and assume that the influencer could be a commitment type who always endorses honestly. A high reputation attracts more followers in the future and earns deeper trust from them, making marketers willing to offer a higher price for endorsements. Followers update their beliefs about the influencer's type based on their endorsing behavior and the true quality of the product. As a result, the influencer has an incentive to turn down a sponsorship in the current period to maintain their reputation for honesty and gain a larger continuation payoff.

Private sponsorships give rise to two different distortions in communication, even with the disciplining effect of reputation. In particular, there not only exists **over-endorsement** by the sponsored influencer but also **under-endorsement** by the unsponsored influencer. When the influencer is sponsored to promote a product for which she receives a low signal, she dishonestly endorses with strictly positive probability. This finding aligns with the canonical reputation literature (e.g. Benabou and Laroque (1992)) that insiders exploit their information and lie with positive probability when signals are noisy. Surprisingly, the unsponsored influencer chooses not to endorse with positive probability even when she receives a high signal, despite that truthfully endorsing would improve her reputation in expectation. Intuitively, since the sponsored strategic influencer endorses more than an honest influencer would do, the action of endorsing itself becomes an indicative signal of the strategic type, irrespective of the realized product quality. This distortion then decreases the continuation value associated with endorsing while increasing the continuation value associated with not endorsing. Hence, reputation concern motivates the unsponsored influencer to endorse less. This important observation motivates the rest of the paper, in which I further study the role of private sponsorships on the quality of information transmission.

What happens to the credibility of endorsements when the environment changes to make it more likely for influencers to receive sponsorships? The intuition that a higher chance of being sponsored may induce the strategic influencer to over-endorse low-signal products even more, resulting in more severe reputational concerns and thus more under-endorsement of high-signal products, seems to suggest that both changes drive down the credibility of endorsements. However, surprisingly, this intuition turns out to be incorrect. The credibility of endorsements actually varies with the probability of getting sponsored in a non-monotone pattern. As the probability of being sponsored increases, the credibility of endorsements decreases first but increases later. Credibility picks up later because under-endorsement only occurs to the unsponsored influencer, the fraction of which decreases with the probability of being sponsored. Therefore, new technologies that efficiently match influencers with sponsors may enhance consumer welfare, justifying the emergence of business platforms such as the Amazon Influencer Program, where all endorsements are compensated by default.

Next, I consider the effect of the FTC regulation on consumer welfare. I first extend the model such that the influencers can voluntarily disclose their sponsorships without assuming that the disclosure rules are enforced by an outside party. I show that voluntary disclosure does not change the influencer's endorsement credibility. Here, when receiving a low signal about the product, the strategic sponsored influencer randomizes between endorsing with disclosure and endorsing without disclosure with equal probability. The strict enforcement of the disclosure rule has a direct impact on the credibility of endorsements, making the no-disclosure endorsement much more credible and reducing the over-endorsement of low-quality products by sponsored influencers. As a result, unsponsored influencers provide more honest endorsements, leading to an overall improvement in consumer welfare.

[To be added - literature review]

2 Setup

A single influencer (she) makes content to recommend products to a continuum of followers (he, they) of mass $N > 0$. With probability $\varphi \in (0, 1)$, the influencer is privately sponsored by a firm to promote its product. With complementary probability $1 - \varphi$, the influencer is not sponsored but organically comes across a product. Let $s \in S := \{Y, N\}$ denote whether or not the influencer is sponsored. The true quality θ is drawn uniformly from $\Theta := \{H, L\}$, regardless of whether the influencer knows about the product organically or through the sponsor. The influencer privately receives an informative signal $z \in Z := \{h, l\}$ about the quality of the product, such that

$$Pr\,(h|H) = Pr\,(l|L) = \lambda, \tag{1}$$

where $\lambda \in (\frac{1}{2}, 1]$ is the *accuracy* of the signal. The influencer chooses whether to endorse the product after observing the signal. She sends a message m from the message space $M := \{\varnothing, E\}$, where $\varnothing$ is a null message representing no endorsement and E represents an endorsement.[2] When the influencer sends an endorsement, i.e. $m = E$, customers

[2]In Section 4.2 I consider an extension that allows the influencer to attach a sponsorship disclosure statement to an endorsement.

decide whether to purchase the endorsed product; when no product has been endorsed, the customers do not have a purchase decision. Finally, the true quality of the endorsed product θ, becomes publicly known. It is worth noting that when the influencer does not endorse any product, the followers do not know which product the influencer has considered earlier.

Influencer The followers are uncertain about the influencer's honesty. In particular, the influencer can be one of two types in $\Omega = \{\omega^0, \omega^1\}$: with probability $1 - \pi \in (0, 1)$, she is the strategic type ω^1 who freely chooses his strategy as described earlier; with probability π, she is the honest type ω^0 who endorses products only upon receiving high signals and does not endorse upon receiving low signals. Let $\sigma_\pi(m|zs)$ denote the probability of the influencer with reputation π sending $m \in M$ conditional on signal $z \in Z$ and sponsorship indicator $s \in S$. Given strategy σ and prior π in the beginning of the game, the message sent by the influencer and the realization of the true quality θ (if there is an endorsed product) together lead to a Bayesian update to the posterior π'. Throughout this paper, I focus on the behavior of a strategic influencer.

The influencer's payoff is comprised of her received sponsorship, a non-pecuniary return $F(\pi) : [0, 1] \to \mathbb{R}$, and her future reputation payoff $W(\pi') : [0, 1] \to \mathbb{R}$, with F non-decreasing and continuously differentiable in π and W strictly increasing and continuous in π'. I use the non-pecuniary return F to capture any other benefits such as outside opportunities other than product recommendations, ego utility from a good public image, etc. If sponsored in this period, the influencer gets an additional flow payment from the sponsorship (details later) provided that she endorses the sponsored product. The continuation value $W(\pi')$ can be derived by assuming a second period or an infinite-repeated continuation game with the same structure as the static game. The influencer discounts $W(\pi')$ with factor $\delta \in (0, 1)$.

Followers The number of followers who subscribe to an influencer varies with her reputation π, reflecting the notion that followers may subscribe or unsubscribe as their opinion about the influencer changes. The number of followers is given by $N(\pi)$, where $N : [0, 1] \to \mathbb{R}_+$ is non-decreasing and bounded. Followers are initially uninformed about products and have a correct uniform prior, with a belief of $\beta_0 = \frac{1}{2}$ that any product is of high quality. A follower who does not make a purchase obtains a payoff of 0. The payoff for purchasing a high-quality product is 1, and the payoff for purchasing a low-quality product is a random variable $-v < 0$, where v is independently drawn from a continuous and strictly decreasing cumulative distribution function $H(v)$ with support $[\underline{v}, \overline{v}]$.

Based on these specifications, the decision rule for followers is to purchase an endorsed product if their belief β in high quality satisfies the condition

$$\beta + (1-\beta)(-v) \geq 0$$

or equivalently

$$v \leq \frac{\beta}{1-\beta}. \tag{2}$$

Therefore, a fraction of $H(1)$ of the followers are going to purchase the product even without an endorsement. Let $G(v) := H(v) - H(1)$ for all $v \in [\underline{v}, \overline{v}]$, then and a fraction of $G(\frac{\beta}{1-\beta})$ of the followers make purchases if and only if there is an endorsement. I further assume that $\underline{v} \leq 1 < \frac{\lambda}{1-\lambda} < \overline{v}$ to simplify the equilibrium characterization. It ensures that a positive and interior fraction of followers will buy the endorsed product when they believe the influencer has obtained a high signal.

Firms In each period, a new group of firms participate in a sealed-bid auction to determine the price for an endorsement. For simplicity, suppose each firm (or the firms they represent) gains a net profit of 1 for each unit of sales. Due to the competition, the endorsement price will be given by each firm's common true value for the endorsement, i.e. the rational expectation of how many *additional* followers make purchases because of the endorsement. In particular, given a rational expectation β, the price would be given by $N(\pi)R$, where $R = G(\frac{\beta}{1-\beta})$ denotes the *per follower rate.*

3 Equilibrium Characterization

3.1 The Complete-Information Case

As in many settings studied by the reputation literature, reputation concern is instrumental here in ensuring the provision of intertemporal incentives. By endorsing truthfully in every period, the influencer could obtain an endorsement rate of $\overline{R} \equiv G\left(\frac{\lambda}{1-\lambda}\right) > 0$. Canonical results in the reputation literature imply that when $\pi > 0$, a sufficiently patient strategic influencer obtains a total payoff close to $\frac{1}{1-\delta}\left(F(0) + \frac{1}{2}N(0)\varphi\overline{R}\right)$. By contrast, if the influencer is known to be strategic, i.e. $\pi = 0$, then there always exists an equilibrium with no information transmission.

Observation 1. *Suppose $\pi = 0$. There exists a subgame perfect Nash equilibrium σ_0 in which the influencer always endorses when sponsored, never endorses when not sponsored,* *and receives an endorsement rate of* 0.

Note that the influencer is indifferent between all messages when he is not sponsored, resultingin a continuum of equilibria. In the equilibrium σ_0, an endorsement is sent if and only if the influencer is sponsored; as a result, endorsements are uninformative and the firms are unwilling to pay anything.

3.2 Reputation Equilibrium

The public makes two inferences regarding the influencer. Firstly, followers form a posterior belief about the quality of the endorsed product before deciding to make a purchase. Secondly, both customers and firms update their beliefs about the influencer's honesty based on the endorsement (or lack thereof) and the actual quality of the product.

In equilibrium, the public forms a rational expectation about the strategic influencer's behavior and use it to attach probabilities to each message being sent at each state. For each possible quality $\theta \in \{H, L\}$, define $q_\theta := Pr\left(E|\theta\right)$. Given the influencer's strategy σ, these conditional probabilities can be decomposed as follows,

$$q_H = \pi\lambda + (1-\pi)\sum_{Z,S}\sigma\left(E|zs\right)Pr\left(zs|H\right), \tag{3}$$

$$q_L = \pi\left(1-\lambda\right) + (1-\pi)\sum_{Z,S}\sigma\left(E|zs\right)Pr\left(zs|L\right). \tag{4}$$

The quantities q_H and q_L have a natural interpretation in that they measure the distortion in the communication: the probability $1-q_H$ is the probability of *false no-endorsements* while q_L is the probability of *false endorsements*. If the strategic influencer fully mimics an honest type, then false endorsements and no-endorsements fully stem from inaccurate signals, and we have $1-q_H = Pr(\ell|H) = 1-\lambda$ and $q_L = Pr(h|L) = 1-\lambda$.

Inference about product quality Upon observing an endorsement, the customers update their posterior in high quality to

$$\beta = Pr(H|E) = \frac{\frac{1}{2}q_H}{\frac{1}{2}q_H + \frac{1}{2}q_L}.$$

A customer with potential loss v buys the endorsed product if and only if $v \leq \frac{q_H}{q_L}$. The endorsement rate can thus be represented in terms of q_H and q_L,

$$R = G\left(\frac{q_H}{q_L}\right). \tag{5}$$

I term the ratio q_H/q_L the *credibility* of the endorsement. Both the fraction of followers willing to buy and the rate that firms are willing to bid grow higher if the influencer's endorsement is deemed more credible.

Inference about influencer honesty Using Bayes rule, We can write the posteriors following endorsement as

$$\pi^+ = \frac{\pi\lambda}{q_H}, \pi^- = \frac{\pi(1-\lambda)}{q_L}, \pi^\varnothing = \frac{\pi}{2-(q_H+q_L)}. \tag{6}$$

By endorsing a product upon receiving signal h, the influencer obtains an expected payoff

$$V_{hs} = F(\pi) + N(\pi)R \cdot \mathbf{1}_{s=Y} + \delta\left[\lambda W(\pi^+) + (1-\lambda)W(\pi^-)\right], \tag{7}$$

which includes a sponsorship if $s = Y$. Note that continuation value after endorsing can take two values depending on the true product quality and the resulted updated reputation. In contrast, by endorsing a product upon receiving signal ℓ, the influencer obtains a lower expected payoff

$$V_{\ell s} = F(\pi) + N(\pi)R \cdot \mathbf{1}_{s=Y} + \delta\left[\lambda W(\pi^-) + (1-\lambda)W(\pi^+)\right]. \tag{8}$$

The payoff associated with not endorsing is much simpler and given by

$$U = F(\pi) + \delta W(\pi^\varnothing), \tag{9}$$

Note that this payoff is independent from the signal and the true quality.

Equilibrium Characterization The first result delineates the strategic influencer's equilibrium behavior by characterizing four important properties that must hold in equilibrium.

Proposition 1. *Fix $W \in C_{++}$ and $\pi \in (0,1)$. In any static equilibrium σ^* with a positive endorsement rate, the strategic influencer's behavior satisfies the following properties:*

1. *She behaves honestly at both* (ℓ, N) *and* (h, Y),

$$\sigma^*(\varnothing|\ell N) = 1 \text{ and } \sigma^*(E|hY) = 1.$$

2. *She behaves dishonestly with positive probability at both* (ℓ, Y) *and* (h, N),

$$\sigma^*(E|\ell Y) > 0 \text{ and } \sigma^*(\varnothing|hN) > 0.$$

When the influencer receives a high signal as well as a sponsorship opportunity, her interest is fully aligned with her followers' interests—endorsing the product not only earns her sponsorship revenue but also increases her reputation in expectation. Their interests are also aligned when the influencer receives a low signal and no sponsorship opportunity—endorsing the product only harms her reputation without bringing more income. A conflict of interest naturally arises when the influencer is sponsored to promote a low-signal product. No matter how patient the strategic influencer is, the sponsorship motivates her to dishonestly endorse with positive probability. Suppose for contradiction that she does not endorse products following a low signal, then anticipating this, followers would update their beliefs upwards following an endorsement, irrespective of the product's true quality. The strategic influencer would then deviate to endorse the product, knowing that there would be no reputational repercussions.

More surprisingly, she chooses to conceal high-signal products with positive probability when no sponsor shows up, despite that truthfully endorsing would improve her reputation in expectation. Intuitively, since the sponsored strategic influencer endorses more than an honest influencer would do, the action of endorsing itself becomes an indicative signal of the strategic type. This distortion then decreases the continuation value associated with endorsing uniformly regardless of the true product quality while increasing the continuation value associated with not endorsing. Since the unsponsored influencer only cares about her continuation payoff, this reputation concern motivates her to endorse less. This important observation motivates the rest of the paper, in which I further study the role of private sponsorships on the quality of information transmission.

Taken these observations together, Proposition 1 implies that there not only exists **over-endorsement** by the sponsored influencer but also **under-endorsement** by the unsponsored influencer. These two distortions have fundamentally different causes: over-endorsement stems from the sponsorship, while under-endorsement stems from reputational effects that arise due to over-endorsement. These distortions could be reformulated as over-endorsement of low-signal products and under-endorsement of high-signal products.

Since signals are noisy, however, it remains ambiguous whether these distortions lead to higher conditional probabilities of false endorsements and no-endorsements than when the influencer is honest. To see why, suppose both over-endorsement and the under-endorsement reach their maximum level in equilibrium, i.e. $\sigma^*(E|\ell Y) = 1$ and $\sigma^*(\varnothing|hN) = 1$, then we have $q^*_L = \pi(1-\lambda)+(1-\pi)\varphi$ and $1-q^*_H = \pi(1-\lambda)+(1-\pi)(1-\varphi)$. In this case, we either have $1-q^*_H \geq 1-\lambda$ or $q^*_L > 1-\lambda$ but not both at the same time. When sponsorships are abundant ($\varphi > \lambda$), overendorsement by the sponsored influencer dominates underendorsement by the unsponsored influencer, resulting in more products being endorsed overall, in which case the probability of false endorsement q^*_L is higher than $1 - \lambda$ but the probability of false no-endorsement $1-q^*_H$ is lower than $1-\lambda$. Nevertheless, the fact that we may end up with a lower probability of false no-endorsement does not imply that the influencer's endorsement may be more credible than when the influencer is honest. Both over-endorsement of low-signal products and under-endorsement of high-signal products push the credibility ratio q^*_H/q^*_L downward such that it is lower than the benchmark ratio $\lambda/(1-\lambda)$, implying a lower rate $R^* < \overline{R}$.

Proposition 2 further establishes the existence and uniqueness of the static equilibrium. In particular, a positive endorsement price emerges and thus the equilibrium satisfies the properties described by Proposition 1. Depending on the primitives, the strategic influencer could randomize at (ℓ, Y) or (h, N), or play pure actions under both circumstances.

Proposition 2. *Fix $W \in C_{++}$ and $\pi \in (0,1)$. The static game admits a unique equilibrium σ^*, which features a positive endorsement rate $R^* \in (0, \overline{R})$ and has the following properties:*

1. *Fixing other parameters, as the strategic influencer becomes more impatient, she behaves more dishonestly and endorsements become less credible. That is, as δ becomes smaller, $\sigma^*(E|\ell Y)$ and $\sigma^*(\varnothing|hN)$ increase, and the rate R^* decreases.*

2. *Fixing other parameters, there exists a cutoff $\underline{\delta}(\lambda,\varphi) \in (0,1]$ such that: (1) if $\delta > \underline{\delta}(\lambda,\varphi)$, then we have $\sigma^*(E|\ell Y) \in (0,1)$ and $\sigma^*(\varnothing|hN) \in (0, c(\lambda,\varphi))$; (2) if $\delta \leq \underline{\delta}(\lambda,\varphi)$, then $\sigma^*(E|\ell Y) = 1$ and $\sigma^*(\varnothing|hN) = c(\lambda,\varphi) \in (0,1]$, where $c(\lambda,\varphi) = 1$ if λ is sufficiently small and φ is sufficiently large.*

Proposition 2 also shows that the strategic influencer's equilibrium behavior changes with her patience level in an intuitive way. Both over-endorsement by the sponsored influencer and under-endorsement by the unsponsored influencer increase as the discount factor δ decreases. Intuitively, a less patient influencer attaches a lower weight to her continuation payoff and has a stronger incentive to sacrifice her reputation in exchange for sponsorships; thus, low patience leads to more over-endorsement and in turn induces more under-endorsement. Furthermore, if her discount factor is below a cutoff, the influencer endorses with probability 1 whenever she is sponsored, with her under-endorsement of high-signal products also attaining its maximum level. Nevertheless, the maximum $\sigma^*(\varnothing|hN)$ could be bounded away from 1 if signals are accurate and the probability of being sponsored is small, putting an endogenous limit on the magnitude of under-endorsement by the unsponsored influencer. High signal accuracy makes it likely for the good signal to be translated into a realization of high quality, promising a high expected continuation payoff following an honest endorsement. Meanwhile, a lower probability of being sponsored reduces the scope of gaining a high reputation by concealing an unsponsored product, which also makes truthful endorsing more attractive for the unsponsored influencer.

4 Role of Private Sponsorships

In this section, exploiting the characterization of the static equilibrium, I explore the role of private sponsorships by analyzing two comparative statics. In Section 4.1 I investigate how endorsement credibility varies with the probability of the influencer being sponsored and find a non-monotone relationship. In Section 4.2 I consider the effect of the FTC regulation that enforces transparency of sponsorships.

4.1 Are Fewer Sponsorships Better?

What happens when the environment changes in a direction that makes it more likely for the influencer to come across sponsors? Intuitively, a higher chance of being sponsored may induce the strategic influencer to over-endorse low-signal products even more; this then results in more severe reputational concerns and hence more under-endorsement of high-signal products; taken together, both changes drive down endorsement credibility and hurt consumers. Perhaps surprisingly, this intuition turns out to be incorrect. As shown by Proposition 3, credibility and consumer welfare vary with φ in a non-monotone pattern.[3]

Proposition 3. *Fix $W \in C_{++}$ and $\pi \in (0,1)$. There exist $\underline{\varphi}, \overline{\varphi} \in (0,1]$ with $0 < \underline{\varphi} \leq \overline{\varphi} \leq 1$ such that the probability of correct endorsement q^*_H, credibility q^*_H/q^*_L, and total consumer*

[3]Additional complicating effects might arise when W is endogenized, since the change of any exogenous parameter can affect the continuation payoff associated with the Markov Perfect equilibrium.

welfare all strictly decrease in φ *over* $[0, \underline{\varphi}]$ *and strictly increase in* φ *over* $[\overline{\varphi}, 1]$*. Moreover,* $\overline{\varphi} < 1$ *when* λ *is sufficiently close to* $1/2$*.*

When there are no chances of being sponsored ($\varphi = 0$), the strategic influencer is only concerned about her continuation payoff and thus endorses honestly, with credibility attaining its maximum level, $\lambda/(1-\lambda)$. As φ rises slightly from 0, over-endorsement of sponsored low-signal products begins to emerge, further leading to under-endorsement of unsponsored high-signal products. During this initial phase, the sponsored influencer endorses with probability 1, while the unsponsored strategic influencer endorses high-signal products with probability less than 1, i.e. $\sigma^*(E|\ell Y) = 1$ and $\sigma^*(E|hN)$ gradually decreases from 1. Since the credibility of endorsements remains at a high level, the reputational repercussion following endorsing a low-signal product is negligible while the monetary reward is substantial, incentivizing endorsement whenever receiving a sponsorship; on the other hand, as the fraction of the sponsored types increases, the reputational reward associated with not endorsing also becomes larger, resulting in more under-endorsement and lower $\sigma^*(E|hN)$. Overall, more low-signal products are endorsed and more high-signal products are hidden, harming her endorsement's credibility and negatively impacting the expected payoffs of her followers.

However, as φ continues to rise, the pattern could be reversed. When φ is near 1, the unsponsored strategic influencer endorses with probability 0, and the sponsored strategic influencer endorses low-signal products with positive probability, i.e. $\sigma^*(\varnothing|hN) = 1$ and $\sigma^*(\varnothing|\ell Y) > 0$. With noisy signals, the endorsements have low informativeness and thus low credibility, which endogenously discourages the strategic influencer to endorse low-signal products as firms are only willing to offer small sponsorships. In this case, the sponsored influencer randomizes and $\sigma^*(\varnothing|\ell Y) \in (0,1)$ when φ is large. Meanwhile, since the fraction of the sponsored influencers is large, the reputation damage associated with endorsing is substantial, incentiving hiding high-signal unsponsored products. It remains uncertain how $\sigma^*(\varnothing|\ell Y)$ changes with φ: the observation that a larger fraction of high-signal products are sponsored and endorsed implies higher credibility and price, but this counteracts the increasing reputation damage associated with endorsing. If $\sigma^*(\varnothing|\ell Y)$ increases, then this renders endorsements even more credible; if $\sigma^*(\varnothing|\ell Y)$ decreases instead, then this implies the increase in the price must dominate the growing reputation damage. Either way, credibility and price increase with φ when φ is near one, which in turn benefits the followers.

4.2 Policy Experiment

In this section, I study the effect of disclosure rules that require influencers must disclose their sponsorship (if there is one) whenever they endorse a product. This regulation creates two different endorsement messages, the one with or without a disclosure statement, denoted by E_y and E_n. The rate offered by firms now only reflect the additional sales generated by an endorsement with sponsorship disclosure.

Before proceeding to the policy experiment, I first extend the baseline model to allow for both E_y and E_n without forcing transparency. Suppose that influencers can freely choose whether to attach the disclosure statement to their endorsements—they may disguise their sponsorship by sending E_n when $s = Y$, or pretend they receive a sponsorship by sending E_y when $s = N$.

For each endorsement message $m \in \{E_y, E_n\}$ and each possible state $\theta \in \{H, L\}$, we can analogously define $q_\theta^m \coloneqq Pr\left(m|\theta\right)$. Upon receiving an endorsement message $m \in \{E_y, E_n\}$, a customer with potential loss buys the endorsed product if and only if $v \leq \frac{q_H^m}{q_L^m}$. In this case, firms rationally anticipate the strategic influencer's strategy and bid according to the following formula,

$$R = \sum_{\tilde{m} \in \{E_y, E_n\}} Pr\left(m = \tilde{m}|m \neq \varnothing\right) G\left(\frac{q_H^m}{q_L^m}\right). \tag{10}$$

The inferences of the followers can be similarly defined in this extended model. Depending on the influencer's action and the product quality, there are five possible posteriors,

$$\pi^{E_y+} = \frac{\pi\lambda\varphi}{q_H^{E_y}}, \pi^{E_y-} = \frac{\pi\left(1-\lambda\right)\varphi}{q_L^{E_y}}, \tag{11}$$

$$\pi^{E_n+} = \frac{\pi\lambda\left(1-\varphi\right)}{q_H^{E_n}}, \pi^{E_n-} = \frac{\pi\left(1-\lambda\right)\left(1-\varphi\right)}{q_L^{E_n}}, \tag{12}$$

$$\pi^{\varnothing} = \frac{\pi}{2 - \left(q_H^{E_y} + q_L^{E_y} + q_H^{E_n} + q_L^{E_n}\right)}. \tag{13}$$

Lemma 1 shows that without regulation, endorsing with disclosure or without disclosure earn the influencer the same flow payoff and the same continuation value. In addition, E_y and E_n have the same credibility level and induce the same amount of sales. In equilibrium, the strategic sponsored influencer endorses low-signal products by sending both E_y and E_n with equal probability.

Lemma 1. *Fix $W \in C_{++}$ and $\pi \in (0,1)$. In any static equilibrium, we have*

$$\pi^{E_y+} = \pi^{E_n+}, \pi^{E_y-} = \pi^{E_n-}, \text{ and } \frac{q_H^{E_y}}{q_L^{E_y}} = \frac{q_H^{E_n}}{q_L^{E_n}}.$$

With the regulation, a sponsored influencer is prohibited to send E_n. As I will show later, disclosure now makes a difference in equilibrium. To begin with, I show that there again exists a unique static equilibrium, and it satisfies very similar properties as when there is no regulation. Although an unsponsored influencer is not restricted to a particular endorsement message, it turns out that she strictly prefers sending E_n over E_y in equilibrium.

Proposition 4. *Fix $W \in C_{++}$ and $\pi \in (0,1)$. The static game under transparency admits a unique equilibrium σ^T. In equilibrium, the strategic influencer receives a positive rate $R^T \in (0, \overline{R})$ and her behavior satisfies the following properties:*

1. *She behaves honestly at both (ℓ, N) ant (h, Y),*

$$\sigma^*(\varnothing|\ell N) = 1 \text{ and } \sigma^*(E_y|hY) = 1.$$

2. *She behaves dishonestly with positive probability at both (ℓ, Y) and (h, N),*

$$\sigma^*(E_y|\ell Y) > 0 \text{ and } \sigma^*(\varnothing|hN) > 0 = \sigma^T(E_y|hN).$$

Proposition 5 shows that enforcing the disclosure rule increases endorsement quality overall and increase the consumer welfare.

Proposition 5. *Fix $W \in C_{++}$ and $\pi \in (0,1)$. By enforcing the disclosure rule:*

1. *There is less over-endorsement and under-endorsement,*

$$\sigma^T(E_y|\ell Y) < \sigma^*(E|\ell Y) \text{ and } \sigma^T(\varnothing|hN) < \sigma^*(\varnothing|hN).$$

2. *The total welfare of the followers is larger.*

The disclosure rule has a direct impact on the credibility of the no-disclosure endorsement, which is now only used by influencers who receive a high signal. Moreover, since sponsored influencers are no longer permitted to use the no-disclosure endorsement, it becomes more challenging for them to promote low-quality products without detection. This reduction in over-endorsement of low-signal products leads to more honest endorsements by unsponsored influencers. Taken together, consumer welfare increases unambiguously.

5 Extensions

5.1 Endogeneizing the Value of Reputation

In this section, I establish the existence and uniqueness of the desired Markov Perfect equilibrium of the dynamic game under a few additional restrictions over the primitives.

Pick any $\underline{\kappa} > 0$ and define $C_{\underline{\kappa}} \subset C_{++}$ to be the set of continuous and strictly increasing functions with their slope bounded below by $\underline{\kappa}$, i.e.

$$C_{\underline{\kappa}} = \left\{ W \in C_{++} : \max_{\pi^1, \pi^2 \in [0,1]} \left| \frac{W(\pi^1) - W(\pi^2)}{\pi^1 - \pi^2} \right| \geq \underline{\kappa} \right\}.$$

I focus on MPEs with continuation value function W that belongs to the set $C_{\underline{\kappa}}$.

Assumption 1. *The following conditions hold:*

1. $F'(\pi) \geq \underline{\kappa}$*,*
2. $\lim_{\pi \to 0} \frac{N(\pi)}{\pi} \leq \infty$*,*
3. $N(\pi)G(R) \geq N(x\pi)G(R/x)$ *for any* $x > 1$*,* $\pi, x\pi \in [0,1]$ *and* $R, R/x \in [\frac{1-\lambda}{\lambda}, \frac{\lambda}{1-\lambda}]$.

Theorem 1. *Suppose Assumption 1 holds. There exists* $\underline{\lambda} \in (1/2, 1)$ *such that when* $\lambda < \underline{\lambda}$*, the infinite-horizon game admits a unique Markov perfect equilibrium* σ^* *with a value function* $W^* \in C_{\underline{\kappa}}$*. In equilibrium, the endorsement rate* R^*_π *is positive at all reputation levels* π*.*

TO BE CONTINUED...

A Proofs

A.1 Proof of Proposition 1

I first prove the following lemma.

Lemma 2. *Fix any $W \in C_{++}$. In any static equilibrium with $R^* > 0$, we have $\pi^- < \pi^+$.*

Proof. By Eq. (6), $\pi^- \leq \pi^+$ if and only if

$$\frac{\sum_{Z,S} \sigma\left(E|zs\right) Pr\left(zs|H\right)}{\sum_{Z,S} \sigma\left(E|zs\right) Pr\left(zs|L\right)} \leq \frac{\lambda}{1-\lambda}.$$

Notice that the left-hand side is increasing in $\sigma\left(E|hY\right)$, $\sigma\left(E|hN\right)$ and decreasing in $\sigma\left(E|\ell Y\right)$, $\sigma\left(E|\ell N\right)$. Hence, we have that

$$\frac{\sum_{Z,S} \sigma\left(E|zs\right) Pr\left(zs|H\right)}{\sum_{Z,S} \sigma\left(E|zs\right) Pr\left(zs|L\right)} \leq \frac{\lambda\varphi + \lambda\left(1-\varphi\right) + 0 + 0}{\left(1-\lambda\right)\varphi + \left(1-\lambda\right)\left(1-\varphi\right) + 0 + 0} = \frac{\lambda}{1-\lambda},$$

where the equality holds only when $\sigma\left(E|\ell Y\right) = \sigma\left(E|\ell N\right) = 0$, i.e. when endorsements are only sent by an influencer with a high signal.

Suppose for the sake of contradiction that $\pi^+ = \pi^- := \pi^*$. Then it follows that upon receiving a low signal, the influencer sends $\varnothing$ with probability 1, i.e. $\sigma^*\left(\varnothing|\ell s\right) = 1, \forall s \in \{Y, N\}$. By Eqs. (3) and (4), we have

$$q_H^* + q_L^* \leq 1,$$

and so Eq. (6) implies that $\pi^{\varnothing} \leq \pi$, where the equality holds only when $\sigma^*\left(\varnothing|hY\right) = \sigma\left(\varnothing|hN\right) = 0$. Since the prior π is a weighted average of π^* and $\pi^{\varnothing}$, it follows that $\pi^* \geq \pi \geq \pi^{\varnothing}$. In addition, by Eq. (5), the endorsement prices must be positive since $\frac{q_H^*}{q_L^*} = \frac{\pi^-}{\pi^+}\frac{\lambda}{1-\lambda} = \frac{\lambda}{1-\lambda} > 1$. These two observations imply that $\varnothing$ cannot be an optimal message at (ℓ, Y), because

$$U = \delta W\left(\pi^{\varnothing}\right) < R^* + \delta W\left(\pi^*\right) = V_{\ell Y}.$$

This is a contradiction. $\square$

Fix any $W \in C_{++}$, I now prove Proposition 1 in four parts.

I first show $\sigma^*(\varnothing|\ell N) = 1$. Suppose for contradiction that $\sigma^*(\varnothing|\ell N) < 1$ in equilibrium. Then sending E earns the influencer a weakly higher payoff than the payoff of sending $\varnothing$,

$$\delta\left[\lambda W\left(\pi^-\right) + (1-\lambda) W\left(\pi^+\right)\right] \geq \delta W\left(\pi^\varnothing\right). \tag{14}$$

By Lemma 2 and the assumptions that $W \in C_{++}$ and $R^* > 0$, the influencer strictly prefers E to $\varnothing$ at all other circumstances, i.e. $(h,Y), (\ell,Y), (h,N)$. Thus, $q_H^* > Pr(h|H) = \lambda$ and $q_L^* > Pr(h|L) = 1-\lambda$, and so $\pi^\varnothing = \frac{\pi}{2-\left(q_H^*+q_L^*\right)} > \pi, \pi^+ = \frac{\pi\lambda}{q_H^*} < \pi$. But this contradicts Eq. (14).

Next I show $\sigma^*(E|hY) = 1$. Suppose instead that $\sigma^*(\varnothing|hY) > 0$. Since W is strictly increasing and $R^* > 0$, by comparing incentives, we observe that $\varnothing$ is strictly optimal at other circumstances, $(\ell,S), (\ell,N), (h,N)$. However, this implies $\sigma^*(\varnothing|\ell Y) = \sigma^*(\varnothing|\ell N) = 1$, contradicting the implication of Lemma 2.

I now show $\sigma^*(\varnothing|\ell Y) < 1$. Suppose instead that $\sigma^*(\varnothing|\ell Y) = 1$. Again by comparing incentives, we observe that $\varnothing$ is strictly optimal at (ℓ, N), again implying $\sigma^*(\varnothing|\ell Y) = \sigma^*(\varnothing|\ell N) = 1$ and contradicting the implication of Lemma 2.

Finally, I show $\sigma^*(E|hN) < 1$. Suppose instead that $\sigma^*(E|hN) = 1$. By the previous three claims, we know that $\sigma^*(\varnothing|\ell N) = 1$, $\sigma^*(E|hY) = 1$, and $\sigma^*(\varnothing|\ell Y) < 1$, which implies that $\pi^\varnothing > \pi > \pi^+$ and contradicts the optimality of E at (h,N).

A.2 Proof of Proposition 2

I first rule out any static equilibrium with a weakly negative endorsement price. Suppose there exists a static equilibrium $\hat{\sigma}$ with an endorsement price $\hat{R} < 0$. Then the strategic influencer has the strongest incentive to endores at (h,N) and the least incentive at (ℓ,Y). Using a similar argument as in Proposition 1, it is easy to show that $\hat{\sigma}(E|\ell Y) = 0$ and $\hat{\sigma}(E|hN) = 1$. But then

$$\begin{aligned}
\frac{\hat{q}_H}{\hat{q}_L} =& \frac{\pi\lambda + (1-\pi)\sum_{Z,S}\hat{\sigma}(E|zs)\,Pr(zs|H)}{\pi(1-\lambda) + (1-\pi)\sum_{Z,S}\hat{\sigma}(E|zs)\,Pr(zs|L)} \\
=& \frac{\pi\lambda + (1-\pi)\left(\lambda\varphi\hat{\sigma}(E|hY) + \lambda(1-\varphi) + (1-\lambda)(1-\varphi)\hat{\sigma}(E|\ell N)\right)}{\pi(1-\lambda) + (1-\pi)\left((1-\lambda)\varphi\hat{\sigma}(E|hY) + (1-\lambda)(1-\varphi) + \lambda(1-\varphi)\hat{\sigma}(E|\ell N)\right)} \\
\geq& \frac{\pi\lambda + (1-\pi)\left(\lambda(1-\varphi) + (1-\lambda)(1-\varphi)\right)}{\pi(1-\lambda) + (1-\pi)\left((1-\lambda)(1-\varphi) + \lambda(1-\varphi)\right)} > 1,
\end{aligned}$$

implying that $\hat{R} = G\left(\frac{\hat{q}_H}{\hat{q}_L}\right) > 0$—a contradiction. Now suppose there exists a static equilibrium $\hat{\sigma}$ with $\hat{R} = 0$. Then the strategic influencer has identical preferences with or without a sponsorship. Since W is strictly increasing, it follows that the influencer truthfully endorses at all circumstances, again resulting in a positive endorsement price $\hat{R} = \overline{R} = G\left(\frac{\lambda}{1-\lambda}\right)$, so such an equilibrium does not exist. Therefore, the endorsement price must be positive in any static equilibrium.

Proposition 1 shows that a static equilibrium σ^* with a positive endorsement price R^* must satisfy $\sigma^*(\varnothing|\ell N) = 1$, $\sigma^*(E|hY) = 1$, $\sigma^*(E|\ell Y) > 0$, and $\sigma^*(\varnothing|hN) > 0$. So the conditional probabilities of false no-endorsements and false endorsements could be written as

$$\begin{aligned} q_H^* =& \pi\lambda + (1-\pi)\left[\lambda\varphi + \lambda(1-\varphi)\sigma^*(E|hN) + (1-\lambda)\varphi\sigma^*(E|\ell Y)\right], \\ q_L^* =& \pi(1-\lambda) + (1-\pi)\left[(1-\lambda)\varphi + (1-\lambda)(1-\varphi)\sigma^*(E|hN) + \lambda\varphi\sigma^*(E|\ell Y)\right]. \end{aligned}$$

We can solve out $\sigma^*(E|hN)$ and $\sigma^*(E|\ell Y)$ in terms of q_L^*, q_H^*,

$$\begin{aligned} \sigma^*(E|hN) &= \frac{\lambda q_H^* - (1-\lambda)q_L^* - (2\lambda-1)(\pi+\varphi(1-\pi))}{(1-\pi)(2\lambda-1)(1-\varphi)}, \\ \sigma^*(E|\ell Y) &= \frac{\lambda q_L^* - (1-\lambda)q_H^*}{(1-\pi)(2\lambda-1)\varphi}. \end{aligned}$$

Note that there exists a one-to-one mapping between the influencer's endorsement strategy σ^* and (q_L^*, q_H^*). The set of feasible values of $(\sigma^*(E|hN), \sigma^*(E|\ell Y))$, given by $[0,1) \times (0,1]$, then corresponds to a feasible area for (q_L, q_H) denoted by $D \subset [0,1]^2$, where D is given by

$$\begin{aligned} D = \Bigg\{ (q_L, q_H) \in [0,1]^2 : & \frac{(2\lambda-1)(\pi+\varphi(1-\pi))}{\lambda} \le q_H - \frac{1-\lambda}{\lambda}q_L < \frac{2\lambda-1}{\lambda}, \\ & \text{and } 0 < q_H - \frac{\lambda}{1-\lambda}q_L \le \frac{(1-\pi)(2\lambda-1)\varphi}{1-\lambda} \Bigg\}. \end{aligned}$$

Since E is optimal at (ℓ, Y) and $\varnothing$ is optimal at (h, N), we know that $V_{lY} \ge U$ and $V_{hN} \le U$. For convenience, define $f_{hN}, f_{\ell Y} \in [0,1]^2 \to \mathbb{R}$ such that $f_{hN}(q_L, q_H) = \frac{1}{\delta}(V_{hN} - U)$ and $f_{\ell Y}(q_L, q_H) = \frac{1}{\delta}(V_{\ell Y} - U)$. We have

$$\begin{aligned} f_{hN}(q_L, q_H) &= \lambda W\left(\frac{\pi\lambda}{q_H}\right) + (1-\lambda)W\left(\frac{\pi(1-\lambda)}{q_L}\right) - W\left(\frac{\pi}{2-q_H-q_L}\right), \\ f_{\ell Y}(q_L, q_H) &= \frac{1}{\delta}G\left(\frac{q_H}{q_L}\right) + \lambda W\left(\frac{\pi(1-\lambda)}{q_L}\right) + (1-\lambda)W\left(\frac{\pi\lambda}{q_H}\right) - W\left(\frac{\pi}{2-q_H-q_L}\right). \end{aligned}$$

For any static equilibrium σ^* and its associated values of q_H^* and q_L^*, the following observations follow from the optimality of σ^*:

1. $f_{hN}(q_L^*, q_H^*) \leq 0$ and $f_{\ell Y}(q_L^*, q_H^*) \geq 0$
2. $\sigma^*(E|\ell Y) < 1$ implies $f_{\ell Y}(q_L^*, q_H^*) = 0$, while $f_{hN}(q_L^*, q_H^*) < 0$ implies $\sigma^*(\varnothing|hN) = 1$
3. $\sigma^*(\varnothing|hN) < 1$ implies $f_{hN}(q_L^*, q_H^*) = 0$, while $f_{\ell Y}(q_L^*, q_H^*) > 0$ implies $\sigma^*(E|\ell Y) = 1$

With a slight abuse of notation, let's write q_H and q_L as functions of $\sigma(E|hN)$ and $\sigma(E|\ell Y)$. When the strategic influencer endorses truthfully, they take values $q_L(1,0) = 1 - q_H(1,0) = 1 - \lambda$. When playing according to this strategy, the influencer is indifferent between E and $\varnothing$ at (h, N) because

$$f_{hN}(q_L(1,0), q_H(1,0)) = f_{hN}(1-\lambda, \lambda) = 0$$

but she strictly prefers E at (ℓ, Y) because

$$f_{\ell Y}(q_L(1,0), q_H(1,0)) = f_{\ell Y}(1-\lambda, \lambda) = \frac{1}{\delta} G\left(\frac{\lambda}{1-\lambda}\right) > 0.$$

When the strategic influencer endorses all low-signal products when sponsored and does not endorse any high-signal product when unsponsored, they take values $q_L^\dagger := q_L(0,1) = \pi(1-\lambda) + (1-\pi)\varphi$ and $q_H^\dagger := q_H(0,1) = \pi\lambda + (1-\pi)\varphi$. I now prove three claims and together imply Proposition 2.

Claim 1. If $f_{hN}(q_L^\dagger, q_H^\dagger) > 0$, then there exists a unique equilibrium σ^*, in which $R^* > 0$, $\sigma^*(E|hN) \in (0,1)$, and $f_{hN}(q_L^*, q_H^*) = 0$.

Proof. Suppose $f_{hN}(q_L^\dagger, q_H^\dagger) > 0$. If $\sigma^*(E|hN) = 0$, then for any $\sigma(E|\ell Y) \in [0,1]$, we have

$$\begin{aligned}
&f_{hN}\left(q_L(\sigma^*(E|hN), \sigma(E|\ell Y)), q_H(\sigma^*(E|hN), \sigma(E|\ell Y))\right) \\
=&f_{hN}\left(q_L(0, \sigma(E|\ell Y)), q_H(0, \sigma(E|\ell Y))\right) \\
\geq&f_{hN}(q_L^\dagger, q_H^\dagger) > 0,
\end{aligned}$$

implying strict optimality of E at (h, N), contradicting Proposition 1. So in any static equilibrium, $\sigma^*(E|hN) \in (0,1)$ and thus $f_{hN}(q_L^*, q_H^*) = 0$.

Since $f_{hN}(q_L(1,0), q_H(1,0)) = 0$, we have $f_{hN}(q_L(1,1), q_H(1,1)) < 0$. Together with the assumption that $f_{hN}(q_L(0,1), q_H(0,1)) > 0$, we know that there exists $x \in (0,1)$ such that $f_{hN}(q_L(x,1), q_H(x,1)) = 0$.

Case 1. Suppose $f_{\ell Y}(q_L(x,1), q_H(x,1)) \leq 0$. Then since $f_{\ell Y}(q_L(1,0), q_H(1,0)) > 0$, there must exist $(\hat{x}, \hat{y}) \in [x, 1) \times (0,1)$ such that

$$f_{hN}(q_L(\hat{x},\hat{y}), q_H(\hat{x},\hat{y})) = f_{\ell Y}(q_L(\hat{x},\hat{y}), q_H(\hat{x},\hat{y})) = 0.$$

Moreover, there exists only one such solution to $f_{hN}(q_L, q_H) = f_{\ell Y}(q_L, q_H) = 0$ because since $f_{\ell Y}(q_L, q_H)$ is strictly decreasing in both q_H and q_L and $f_{\ell Y}(q_L, q_H) - f_{hN}(q_L, q_H)$ is strictly increasing in q_H but strictly decreasing in q_L.

Let $(\sigma^*(E|hN), \sigma^*(E|\ell Y)) = (\hat{x}, \hat{y})$, then $(q_L^*, q_H^*) := (q_L(\hat{x},\hat{y}), q_H(\hat{x},\hat{y})) \in D$. Observe that σ^* is a static equilibrium with a positive endorsement price because: (1) the influencer is indifferent between E and $\varnothing$ at both (h, N) and (ℓ, Y); (2) the endorsement price $R^* = G\left(\frac{q_H^*}{q_L^*}\right) > G\left(\frac{q_H(0,1)}{q_L(0,1)}\right) > G(1) > 0$; (3) the influencer strictly prefers E at (h, Y) and strictly prefers $\varnothing$ at (ℓ, N) since $R^* > 0$ and W is strictly increasing.

Uniqueness. If there exists another equilibrium σ^{**}, then it must be that $\sigma^{**}(E|hN) \in (0,1)$ and $\sigma^{**}(E|\ell Y) = 1$. Note that $f_{hN}(q_L^{**}, q_H^{**}) = 0$ immediately implies that $\sigma^{**}(E|hN) = x$. However, we then have $f_{\ell Y}(q_L^{**}, q_H^{**}) = f_{\ell Y}(q_L(x,1), q_H(x,1)) < 0$, so σ^{**} cannot be an equilibrium.

Case 2. Now suppose instead $f_{\ell Y}(q_L(x,1), q_H(x,1)) > 0$. Let $(\sigma^*(E|hN), \sigma^*(E|\ell Y)) = (x, 1)$, then σ^* is a static equilibrium with a positive endorsement price because: (1) the influencer is indifferent between E and $\varnothing$ at (h, N); (2) the influencer strictly prefers E at (ℓ, Y) and thus plays E with probability 1; (3) price $R^* = G\left(\frac{q_H^*}{q_L^*}\right) > G\left(\frac{q_H(0,1)}{q_L(0,1)}\right) > 0$; (4) the influencer strictly prefers E at (h, Y) and strictly prefers $\varnothing$ at (ℓ, N) since $R^* > 0$ and W is strictly increasing.

Uniqueness. Suppose there exists another static equilibrium σ^{**}. Then we either have $\sigma^{**}(E|hN) \in (0,1)$ and $\sigma^{**}(E|\ell Y) = 1$, or both $\sigma^{**}(E|hN) \in (0,1)$ and $\sigma^{**}(E|\ell Y) \in (0,1)$. First suppose the first possibility holds, then since $f_{hN}(q_L(\sigma^{**}(E|hN), 1), q_H(\sigma^{**}(E|hN), 1)) = f_{hN}(q_L(x,1), q_H(x,1)) = 0$, it follows that $\sigma^{**}(E|hN) = x$ and $\sigma^{**} = \sigma^*$. Now let's consider the second possibility that the influencer is randomizing at both (h, N) and (ℓ, Y). We must have $f_{\ell Y}(q_L^{**}, q_H^{**}) = f_{hN}(q_L^{**}, q_H^{**}) = 0$. Since $f_{hN}(q_L(x,1), q_H(x,1)) = 0$ and f_{hN} is strictly decreasing in both arguments, $q_L(x,1) - q_L^{**}$ and $q_H(x,1) - q_H^{**}$ must have different signs; also, since $\sigma^{**}(E|\ell Y) < 1$, we have $\sigma^{**}(E|hN) > x$. Suppose $q_L(x,1) < q_L^{**}$ and $q_H(x,1) > q_H^{**}$, then this implies

$$\begin{aligned}\lambda(1-\varphi)x + (1-\lambda)\varphi >& \lambda(1-\varphi)\sigma^{**}(E|hN) + (1-\lambda)\varphi\sigma^{**}(E|\ell Y),\\ (1-\lambda)(1-\varphi)x + \lambda\varphi <& (1-\lambda)(1-\varphi)\sigma^{**}(E|hN) + \lambda\varphi\sigma^{**}(E|\ell Y).\end{aligned}$$

Rearrange,

$$\frac{\lambda(1-\varphi)}{\varphi(1-\lambda)}(\sigma^{**}(E|hN) - x) < 1 - \sigma^{**}(E|\ell Y) < \frac{(1-\lambda)(1-\varphi)}{\lambda\varphi}(\sigma^{**}(E|hN) - x),$$

which yields a contradiction because the left-hand side is larger than the right-hand side. Therefore, we must have $q_L(x,1) > q_L^{**}$ and $q_H(x,1) < q_H^{**}$. However, this then implies

$$\begin{aligned}&f_{\ell Y}(q_L^{**}, q_H^{**})\\ =&f_{\ell Y}(q_L^{**}, q_H^{**}) - f_{hN}(q_L^{**}, q_H^{**})\\ >&f_{\ell Y}(q_L(x,1), q_H(x,1)) - f_{hN}(q_L(x,1), q_H(x,1))\\ =&f_{\ell Y}(q_L(x,1), q_H(x,1)) > 0,\end{aligned}$$

where the first inequality holds since $f_{\ell Y} - f_{hN}$ is strictly increasing in q_H but strictly decreasing in q_L. It contradicts the assumption that $f_{\ell Y}(q_L^{**}, q_H^{**}) = f_{hN}(q_L^{**}, q_H^{**}) = 0$.

□

Claim 2. If $f_{hN}(q_L^\dagger, q_H^\dagger) \leq 0$ and $f_{\ell Y}(q_L^\dagger, q_H^\dagger) < 0$, then there exists a unique equilibrium σ^*, in which $R^* > 0$, $\sigma^*(E|\ell Y) \in (0,1)$, and $f_{\ell Y}(q_L^*, q_H^*) = 0$.

Proof. Suppose $f_{hN}(q_L^\dagger, q_H^\dagger) \leq 0$ and $f_{\ell Y}(q_L^\dagger, q_H^\dagger) < 0$. If $\sigma^*(E|\ell Y) = 1$, then either $\sigma^*(E|hN) \in (0,1)$ and thus $f_{hN}(q_L^*, q_H^*) = 0$, or $\sigma^*(E|hN) = 0$. But notice that for any $\sigma^*(E|hN) \in (0,1)$, we have

$$f_{hN}(q_L(\sigma^*(E|hN),1), q_H(\sigma^*(E|hN),1)) > f_{hN}(q_L^\dagger, q_H^\dagger) = 0,$$

so the first possibility is ruled out. If $\sigma^*(E|hN) = 0$, then

$$f_{\ell Y}(q_L(\sigma^*(E|hN),1), q_H(\sigma^*(E|hN),1)) = f_{\ell Y}(q_L^\dagger, q_H^\dagger) < 0,$$

contradicting the optimality of E at (ℓ, Y). Therefore, in any static equilibrium, $\sigma^*(E|\ell Y) \in$

$(0,1)$ and $f_{\ell Y}(q_L^*, q_H^*) = 0$.

Since $f_{hN}(q_L(1,0), q_H(1,0)) = 0$, we have $f_{hN}(q_L(0,0), q_H(0,0)) > 0$. Together with the assumption that $f_{hN}(q_L(0,1), q_H(0,1)) \leq 0$, we know that there exists $y' \in (0,1)$ such that $f_{hN}(q_L(0,y'), q_H(0,y')) = 0$.

Case 1. Suppose $f_{\ell Y}(q_L(0,y'), q_H(0,y')) \leq 0$. Then since $f_{\ell Y}(q_L(1,0), q_H(1,0)) > 0$, there must exist $(\hat{x}', \hat{y}') \in [0,1) \times (0, y']$ such that

$$f_{hN}(q_L(\hat{x}', \hat{y}'), q_H(\hat{x}', \hat{y}')) = f_{\ell Y}(q_L(\hat{x}', \hat{y}'), q_H(\hat{x}', \hat{y}')) = 0.$$

Analogously, there exists only one such solution. Let $(\sigma^*(E|hN), \sigma^*(E|\ell Y)) = (\hat{x}', \hat{y}')$, then $(q_L^*, q_H^*) := (q_L(\hat{x}', \hat{y}'), q_H(\hat{x}', \hat{y}') \in D$. As in Claim 1, σ^* is a static equilibrium with a positive endorsement price.

Uniqueness. If there exists another equilibrium σ^{**}, then it must be that $\sigma^{**}(E|hN) = 0$ and $\sigma^{**}(E|\ell Y) \in (0,1)$. Note that when $\sigma(E|hN) = 0$, function $f_{\ell Y}$ decreases as $\sigma(E|\ell Y)$ increases, because

$$\begin{aligned} f_{\ell Y}(q_L(0,y), q_H(0,y)) =& \frac{1}{\delta} G\left(\frac{\pi\lambda + (1-\pi)\left(\lambda\varphi + (1-\lambda)\varphi y\right)}{\pi(1-\lambda) + (1-\pi)\left((1-\lambda)\varphi + \lambda\varphi y\right)}\right) + \lambda W\left(\frac{\pi(1-\lambda)}{q_L(0,y)}\right) \\ &+ (1-\lambda) W\left(\frac{\pi\lambda}{q_H(0,y)}\right) - W\left(\frac{\pi}{2 - q_H(0,y) - q_L(0,y)}\right), \end{aligned}$$

and each term in the last line is decreasing in y. Then since

$$\begin{aligned} & f_{\ell Y}(q_L(0, \sigma^{**}(E|\ell Y)), q_H(0, \sigma^{**}(E|\ell Y))) = 0, \\ & f_{\ell Y}(q_L(0,y'), q_H(0,y')) \leq 0, \end{aligned}$$

we infer that $\sigma^{**} < y'$. We then have $f_{hN}(q_L^{**}, q_H^{**}) > f_{\ell Y}(q_L(0,y'), q_H(0,y')) = 0$, a contradiction.

Case 2. Now suppose instead $f_{\ell Y}(q_L(0,y'), q_H(0,y')) > 0$. Since $f_{\ell Y}(q_L(0,1), q_H(0,1)) < 0$, there exists y'' such that $f_{\ell Y}(q_L(0,y''), q_H(0,y'')) = 0$. Let $\sigma^*(E|hN) = 0$ and $\sigma^*(E|\ell Y) = y''$, then σ^* is a static equilibrium with a positive endorsement price because: (1) the influencer is indifferent between E and $\varnothing$ at (ℓ, Y) by definition of y''; (2) the influencer strictly prefers $\varnothing$ at (h, N) and thus plays $\varnothing$ with probability 1; (3) price $R^* = G\left(\frac{q_H^*}{q_L^*}\right) > G\left(\frac{q_H(0,1)}{q_L(0,1)}\right) > 0$; (4) the influencer strictly prefers E at (h, Y) and strictly prefers $\varnothing$ at (ℓ, N) since $R^* > 0$ and W is strictly increasing.

Uniqueness. Suppose there exists another equilibrium σ^{**}. Then we either have $\sigma^{**}(E|hN) = 0$ and $\sigma^{**}(E|\ell Y) \in (0,1)$, or both $\sigma^{**}(E|hN) \in (0,1)$ and $\sigma^{**}(E|\ell Y) \in (0,1)$. First suppose the first possibility holds, then since $f_{\ell Y}(q_L(0,\sigma^{**}(E|\ell Y)), q_H(0,\sigma^{**}(E|\ell Y)) = 0 = f_{\ell Y}(q_L(0,y''), q_H(0,y''))$ and $f_{\ell Y}(q_L(0,y), q_H(0,y))$ strictly decreases in y, it follows that $\sigma^{**}(E|\ell Y) = y''$ and $\sigma^{**} = \sigma^*$. Now let's consider the second possibility that the influencer is randomizing at both (h,N) and (ℓ,Y). We must have $f_{\ell Y}(q_L^{**}, q_H^{**}) = f_{hN}(q_L^{**}, q_H^{**}) = 0$. Since $f_{hN}(q_L(0,y'), q_H(0,y')) = f_{hN}(q_L^{**}, q_H^{**}) = 0$, using a similar argument as in Claim 1, we infer that $q_L(0,y') > q_L^{**}$ and $q_H(0,y') < q_H^{**}$. However, this then implies

$$
\begin{aligned}
&f_{\ell Y}(q_L^{**}, q_H^{**}) \\
=&f_{\ell Y}(q_L^{**}, q_H^{**}) - f_{hN}(q_L^{**}, q_H^{**}) \\
>&f_{\ell Y}(q_L(0,y'), q_H(0,y')) - f_{hN}(q_L(0,y'), q_H(0,y')) \\
=&f_{\ell Y}(q_L(0,y'), q_H(0,y')) > 0,
\end{aligned}
$$

which contradicts the assumption that $f_{\ell Y}(q_L^{**}, q_H^{**}) = f_{hN}(q_L^{**}, q_H^{**}) = 0$.

□

Claim 3. If $f_{hN}(q_L^\dagger, q_H^\dagger) \leq 0$ and $f_{\ell Y}(q_L^\dagger, q_H^\dagger) \geq 0$, then there exists a unique equilibrium σ^*, in which $R^* > 0$, $\sigma^*(E|hN) = 0$, and $\sigma^*(E|\ell Y) = 1$.

Proof. It is straightforward to see that if $\sigma^*(E|hN) = 0$ and $\sigma^*(E|\ell Y) = 1$, then such σ^* is a static equilibrium with a positive price because: (1) the influencer weakly prefers E at (ℓ, Y) and $\varnothing$ at (h,N) since $f_{hN}(q_L^\dagger, q_H^\dagger) \leq 0$ and $f_{\ell Y}(q_L^\dagger, q_H^\dagger) \geq 0$; (2) price $R^* = G(\frac{q_H^\dagger}{q_L^\dagger}) > 0$; (3) the influencer strictly prefers E at (h,Y) and $\varnothing$ at (ℓ, N) since $R^* > 0$ and W is strictly increasing.

Uniqueness. Suppose there exists another equilibrium σ^*. Then at least one of $\sigma^{**}(E|hN)$ and $\sigma^{**}(E|\ell Y)$ is interior and contained in $(0,1)$.

Suppose $\sigma^{**}(E|hN) = 0$ and $\sigma^{**}(E|\ell Y) \in (0,1)$. Then we have $f_{\ell Y}(q_L^{**}, q_H^{**}) = 0$; but since $f_{\ell Y}(q_L(0,y), q_H(0,y))$ strictly decreases in y, this contradicts the assumption that $f_{\ell Y}(q_L^\dagger, q_H^\dagger) \geq 0$.

Next, suppose $\sigma^{**}(E|hN) \in (0,1)$ and $\sigma^{**}(E|\ell Y) = 1$. Then we have $f_{hN}(q_L^{**}, q_H^{**}) = 0$; but since f_{hN} is strictly decreasing in both arguments, this cannot be reconciled with the assumption that $f_{hN}(q_L^\dagger, q_H^\dagger) \leq 0$.

Finally, suppose $\sigma^{**}(E|hN) \in (0,1)$ and $\sigma^{**}(E|\ell Y) \in (0,1)$. Then we have $f_{\ell Y}(q_L^{**}, q_H^{**}) = f_{hN}(q_L^{**}, q_H^{**}) = 0$. As in Claim 2, there exists $y' \in (0,1)$ such that $f_{hN}(q_L(0,y'), q_H(0,y')) = 0$.

Moreover, since $f_{hN}(q_L(0,y'),q_H(0,y')) > f_{\ell Y}(q_L^\dagger,q_H^\dagger) \geq 0$, we can then use an identical argument as in Case 2 of Claim 2 to show that such an equilibrium does not exist.

$\square$

A.3 Proof of Proposition 3

It directly follows from Proposition 1 that the equilibrium strategies σ^* and credibility levels q_L^* and q_H^* are continuous in φ. Define $q_L^* : \varphi \mapsto [0,1]$ and $q_H^* : \varphi \mapsto [0,1]$, which are the static equilibrium credibility levels under φ.

I now define φ^1 and φ^2. First define $q_L^\dagger : \varphi \mapsto [0,1]$ and $q_H^\dagger : \varphi \mapsto [0,1]$ with $q_L^\dagger(\varphi) = \pi(1-\lambda)+(1-\pi)\varphi$ and $q_H^\dagger(\varphi) = \pi\lambda+(1-\pi)\varphi$. Namely, they are the credibility measures when the probability of being sponsored is given by φ and the strategic influencer endorses a product if and only if she is sponsored. Define φ^1 by the following equation,

$$f_{hN}(q_L^\dagger(\varphi^1),q_H^\dagger(\varphi^1)) = 0,$$

where f_{hN} is as defined in the proof of Proposition 2. The equation has a unique and interior solution in $(0,1)$ since f_{hN} is strictly decreasing in φ, $f_{hN}(q_L^\dagger(0),q_H^\dagger(0)) > 0$, and $f_{hN}(q_L^\dagger(1),q_H^\dagger(1)) < 0$. Next, note that $f_{\ell Y}(q_L^\dagger(\varphi),q_H^\dagger(\varphi))$ is also strictly decreasing in φ and thus $f_{\ell Y}(q_L^\dagger(\varphi),q_H^\dagger(\varphi)) = 0$ admits at most one solution. Whether a solution exists depends on the value of π and other parameters. When $\pi = 1$, we have $f_{\ell Y}(q_L^\dagger(\varphi),q_H^\dagger(\varphi)) > 0$ for all φ; by continuity, this observation also holds when π is close to 1, in which case the equation has no solution. If there exists a solution, let φ^2 take its value, i.e.

$$f_{\ell Y}(q_L^\dagger(\varphi^2),q_H^\dagger(\varphi^2)) = 0,$$

otherwise let $\varphi^2 = 1$.

Scenario 1. Consider $\varphi < \varphi^1$. Then $f_{hN}(q_L^\dagger(\varphi),q_H^\dagger(\varphi)) > 0$. By Claim 1 in the proof of Proposition 2, in equilibrium we have $f_{hN}(q_L^*(\varphi),q_H^*(\varphi)) = 0$. There exists $x \in (0,1)$ (vary with φ continuously) such that $f_{hN}(q_L(x,1),q_H(x,1)) = 0$. Since $q_H(x,1) = \frac{\lambda}{1-\lambda}q_L(x,1) - \frac{(1-\pi)(2\lambda-1)}{1-\lambda}\varphi$, as φ increases, $q_L(x,1)$ increases and $q_H(x,1)$ decreases. When $\varphi = 0$, we have $q_L(x,1) = 1-\lambda$ and $q_H(x,1) = \lambda$; when $\varphi = \varphi^1$, we instead have $q_L(x,1) = q_L^\dagger(\varphi^1) > 1-\lambda$ and $q_H(x,1) = q_H^\dagger(\varphi^1) < \lambda$. In addition, (i) when $f_{\ell Y}(q_L(x,1),q_H(x,1)) \leq 0$, the influencer is indifferent between E and $\varnothing$ at (ℓ,Y) as well; (ii) when $f_{\ell Y}(q_L(x,1),q_H(x,1)) \geq 0$ the influencer plays E with probability 1 at (ℓ,Y). Note that with $f_{hN}(q_L(x,1),q_H(x,1)) = 0$, we can equivalently write

$$f_{\ell Y}(q_L(x,1),q_H(x,1)) = \frac{1}{\delta}N(\pi)G\left(\frac{q_H(x,1)}{q_L(x,1)}\right) - (2\lambda-1)\left[W\left(\frac{\pi\lambda}{q_H(x,1)}\right) - W\left(\frac{\pi(1-\lambda)}{q_L(x,1)}\right)\right],$$

which is increasing in $q_H(x,1)$ but decreasing in $q_L(x,1)$, and thus $f_{\ell Y}(q_L(x,1),q_H(x,1))$ decreases in φ. Note that $f_{\ell Y}(q_L(x,1),q_H(x,1))$ is positive when $\varphi = 0$. If $\varphi^1 \leq \varphi^2$, then $f_{\ell Y}(q_L(x,1),q_H(x,1))$ is still weakly positive when $\varphi = \varphi^1$, implying that situation (ii) comes true. If instead $\varphi^2 < \varphi^1$, then $f_{\ell Y}(q_L(x,1),q_H(x,1))$ is negative when $\varphi = \varphi^1$, so there exists a cutoff $\hat{\varphi}^2 \in (0,\varphi^2)$ such that condition (ii) holds when $\varphi \in [0,\hat{\varphi}^2]$ and (i) holds when $\varphi \in [\hat{\varphi}^2,\varphi^1]$.

If (i) holds for both φ and $\varphi' > \varphi$ when φ' is sufficiently close to φ, the values of q_L^* and q_H^* are fully determined by the indifferent conditions and thus independent of φ, which implies q_L^* and q_H^* are locally flat at π. If (ii) holds for φ and φ', the equilibrium is described by $f_{hN}(q_L^*(\varphi),q_H^*(\varphi)) = 0$ and the following relationship,

$$q_H^*(\varphi) = \frac{\lambda}{1-\lambda}q_L^*(\varphi) - \frac{(1-\pi)(2\lambda-1)}{1-\lambda}\varphi.$$

It follows that q_L^* increases and q_H^* decreases locally at φ.

In sum, if $\varphi^1 \leq \varphi^2$, q_L^* strictly increases and q_H^* strictly decreases when $\varphi < \varphi^1$; when $\varphi^2 < \varphi^1$, then the same pattern holds when φ is small, but later q_L^* and q_H^* are flat when φ is sufficiently close to φ^1.

Scenario 2. Suppose $\varphi^1 \leq \varphi^2$, consider $\varphi \in [\varphi^1,\varphi^2]$. Then we have $f_{hN}(q_L^\dagger(\varphi),q_H^\dagger(\varphi)) \leq 0$ and $f_{hN}(q_L^\dagger(\varphi),q_H^\dagger(\varphi)) \geq 0$. By Claim 3 in the proof of Proposition 2, the influencer plays a pure equilibrium under φ. It follows that $q_L^*(\varphi) = q_L^\dagger(\varphi)$ and $q_L^*(\varphi) = q_L^\dagger(\varphi)$, both of which are increasing in φ. Therefore, if the influencer still plays a pure equilibrium after φ increases slightly, we have q_L^* and q_H^* both increase locally at π. The borderline case in which the influencer randomizes after φ increases slightly could be absorbed into Scenario 1 or 3.

Scenario 3. Now consider $\varphi \geq \varphi^1$ and $\varphi \geq \varphi^2$ (which requires $\varphi^2 \leq 1$). Then $f_{hN}(q_L^\dagger(\varphi),q_H^\dagger(\varphi)) \leq 0$ and $f_{hN}(q_L^\dagger(\varphi),q_H^\dagger(\varphi)) < 0$. By Claim 2 in the proof of Proposition 2, in equilibrium we have $f_{\ell Y}(q_L^*(\varphi),q_H^*(\varphi)) = 0$. In addition, there exists $y' \in [0,1]$ (vary with φ continuously) such that $f_{hN}(q_L(0,y'),q_H(0,y')) = 0$. Since $q_H(0,y') = \frac{1-\lambda}{\lambda}q_L(0,y') + \frac{(2\lambda-1)(\pi+\varphi(1-\pi))}{\pi}$, as φ increases, $q_L(0,y')$ decreases and $q_L(0,y')$ increases. When $\varphi = 1$, we have $q_L(0,y') = 1-\lambda$ and $q_H(0,y') = \lambda$; when $\varphi = \varphi^1$, we instead have $q_L(0,y') = q_L^\dagger(\varphi^1) > 1-\lambda$ and $q_H(0,y') = q_H^\dagger(\varphi^1) < \lambda$. In addition, (iii) if $f_{\ell Y}(q_L(0,y'),q_H(0,y')) \leq 0$, the influencer is indifferent between E and $\varnothing$ at (h, N) as well; (iv) if $f_{\ell Y}(q_L(0, y'), q_H(0, y')) > 0$ the influencer plays $\varnothing$ with probability 1 at (h, N). Note that with $f_{hN}(q_L(0, y'), q_H(0, y')) = 0$, we can again equivalently write $f_{\ell Y}(q_L(0, y'), q_H(0, y'))$ as a function increasing in $q_H(0, y')$ and decreasing in $q_L(0, y')$; thus, $f_{\ell Y}(q_L(0, y'), q_H(0, y'))$ increases in φ. Again, $f_{\ell Y}(q_L(0, y'), q_H(0, y'))$ is positive when $\varphi = 1$. Suppose $\varphi^1 \leq \varphi^2$. When $\varphi = \varphi^2 \geq \varphi^1$, we know that $f_{\ell Y}(q_L^\dagger(\varphi), q_H^\dagger(\varphi)) = 0$ and $f_{hN}(q_L^\dagger(\varphi), q_H^\dagger(\varphi)) \leq 0$ (since it's decreasing in φ). Since both $(q_L^\dagger(\varphi), q_H^\dagger(\varphi))$ and $(q_L(0, y'), q_H(0, y'))$ are on the line $q_H = \frac{1-\lambda}{\lambda} q_L + \frac{(2\lambda-1)(\pi+\varphi(1-\pi))}{\pi}$, it must be that $(q_L(0, y'), q_H(0, y')) \leq (q_L^\dagger(\varphi), q_H^\dagger(\varphi))$. It follows that when $\varphi = \varphi^2$, $f_{\ell Y}(q_L(0, y'), q_H(0, y')) \geq 0$. Therefore, condition (iv) holds for all $\varphi \in [\varphi^2, 1]$. On the other hand, if $\varphi^2 < \varphi^1$, then $f_{\ell Y}(q_L(0, y'), q_H(0, y'))$ is negative when $\varphi = \varphi^1$. Therefore, there exists a cutoff $\hat{\varphi}^1 \in (\varphi^1, 1)$ such that condition (iii) holds when $\varphi \in [\varphi^1, \hat{\varphi}^1]$ and condition (iv) holds when $\varphi \in [\hat{\varphi}^1, 1]$.

If (iii) holds for φ and $\varphi' > \varphi$ when they are sufficiently close, analogously we have q_L^* and q_H^* are locally flat at π. If instead (iv) holds for φ and φ', then the equilibrium is described by $f_{\ell Y}(q_L^*(\varphi), q_H^*(\varphi)) = 0$ and the following relationship,

$$q_H^*(\varphi) = \frac{1-\lambda}{\lambda} q_L^*(\varphi) + \frac{(2\lambda-1)(\pi+\varphi(1-\pi))}{\lambda}. \tag{15}$$

Same for the equilibrium under φ'. With this relationship, we can equivalently write

$$\begin{aligned} f_{\ell Y}(q_L^*, q_H^*) =& \frac{1}{\delta} N(\pi) G\left(\frac{1-\lambda}{\lambda} + \frac{(2\lambda-1)(\pi+\varphi(1-\pi))}{\lambda q_L^*}\right) + \lambda W\left(\frac{\pi\,(1-\lambda)}{q_L^*}\right) \\ &+ (1-\lambda)\, W\left(\frac{\pi\lambda}{q_H^*}\right) - W\left(\frac{\pi}{2-q_H^*-q_L^*}\right), \end{aligned}$$

with the right-hand side increasing in φ and increasing in q_H^* and q_L^*. It follows that $q_H^*(\varphi') > q_H^*(\varphi)$, because otherwise $f_{\ell Y}(q_L^*(\varphi), q_H^*(\varphi)) = 0$ implies that $q_L^*(\varphi') > q_L^*(\varphi)$, contradicting the equality Eq. (15). In addition, we must have $q_H^*(\varphi')/q_L^*(\varphi') > q_H^*(\varphi)/q_L^*(\varphi)$, because otherwise $q_H^*(\varphi')/q_L^*(\varphi') \leq q_H^*(\varphi)/q_L^*(\varphi)$ implies $q_L^*(\varphi') > q_L^*(\varphi)$, while $f_{\ell Y}(q_L^*(\varphi), q_H^*(\varphi)) = 0$ implies that $q_L^*(\varphi') < q_L^*(\varphi)$—a contradiction. However, the comparison between $q_L^*(\varphi')$ and $q_L^*(\varphi)$ depends on the comparison between the increase in endorsement credibility and the decrease in the continuation payoff, which remains undetermined.

In sum, if $\varphi^1 \leq \varphi^2$, then q_H^* strictly increases when $\varphi > \varphi^2$ while the change of q_L^* is ambiguous; if $\varphi^1 > \varphi^2$, then q_H^* and q_L^* are flat when φ is close to φ^1, but as φ grows close to 1, q_H^* strictly increases while the change of q_L^* is ambiguous.

A.4 Proof of Lemma 1

First notice that the influencer's preference ranking over E_y and E_n is independent from her sponsorship status s because the flow payoffs associated with E_y and E_n are always equal to each other for all $s \in \{S, N\}$. Suppose the influencer strictly prefers E_y to E_n when receiving signal h at any s. This happens if and only if

$$\lambda W(\pi^{E_y+}) + (1-\lambda)W(\pi^{E_y-}) > \lambda W(\pi^{EN+}) + (1-\lambda)W(\pi^{EN-}).$$

But this means that we must have $\pi(E_n|\ell s) > 0$ for some $s \in S$ because otherwise Eq. (12) implies that $q_H^{E_y} = q_L^{E_n} = 1$, which contradicts the above inequality since W is strictly increasing. Hence, E_n should be weakly optimal at ℓ, and we have

$$\lambda W(\pi^{E_y-}) + (1-\lambda)W(\pi^{E_y+}) \leq \lambda W(\pi^{EN-}) + (1-\lambda)W(\pi^{EN+}).$$

These two inequalities together imply that

$$\pi^{E_y+} > \pi^{E_n+} \text{ and } \pi^{E_y-} < \pi^{E_n-},$$

which further reduce to

$$q_H^{E_n} > \frac{1-\varphi}{\varphi} q_H^{E_y} \text{ and } q_L^{E_n} < \frac{1-\varphi}{\varphi} q_L^{E_y}. \tag{16}$$

By assumption, $\sigma(E_n|hs) = 0$ for all $s \in S$, so

$$\begin{aligned} q_H^{E_n} &= \pi\lambda\,(1-\varphi) + (1-\pi)\,(1-\lambda)\,[\varphi\sigma\,(E_n|\ell Y) + (1-\varphi)\sigma\,(E_n|\ell N)]\,, \\ q_L^{E_n} &= \pi(1-\lambda)\,(1-\varphi) + (1-\pi)\,\lambda\,[\varphi\sigma\,(E_n|\ell Y) + (1-\varphi)\sigma\,(E_n|\ell N)]\,. \end{aligned}$$

We can write the inequalities in Eq. (16) as

$$\begin{aligned} (1-\lambda)\,[\varphi\sigma\,(E_n|\ell Y) + (1-\varphi)\sigma\,(E_n|\ell N)] &> \frac{1-\varphi}{\varphi}\sum_{Z,S}\sigma\,(E_y|zs)\,Pr\,(zs|H) \\ \lambda\,[\varphi\sigma\,(E_n|\ell Y) + (1-\varphi)\sigma\,(E_n|\ell N)] &< \frac{1-\varphi}{\varphi}\sum_{Z,S}\sigma\,(E_y|zs)\,Pr\,(zs|L) \end{aligned}$$

which implies that

$$\sum_{Z,S} \sigma\left(E_y|zs\right) Pr\left(zs|H\right) < \frac{1-\lambda}{\lambda} \sum_{Z,S} \sigma\left(E_y|zs\right) Pr\left(zs|L\right)$$

But this inequality cannot hold for any feasible σ since $\frac{Pr(zs|H)}{Pr(zs|L)} \geq \frac{1-\lambda}{\lambda}$. We could analogously prove that the influencer cannot strictly prefer E_n over E_y when receiving signal h. A similar argument also holds when the influencer receives signal ℓ. Therefore, it can be shown that the influencer must be indifferent betwee E_y and E_n at all times. This is only possible if and only if

$$\pi^{E_y+} = \pi^{E_n+} \text{ and } \pi^{E_y-} = \pi^{E_n-},$$

and thus by definition, $\frac{q_H^{E_y}}{q_L^{E_y}} = \frac{q_H^{E_n}}{q_L^{E_n}}$.

A.5 Proof of Proposition 4

This proof consists of two parts, the first proving the four properties of the static equilibrium and the second establishing its existence and uniqueness.

[Part A.] Parallel to the proof of Proposition 1, I first show that fixing $W \in C_{++}$, $\pi \in (0,1)$, and a positive rate $R^T > 0$, the four properties hold in the static equilibrium.

Lemma 3. *Fix $W \in C_{++}$ and $\pi \in (0,1)$. In any static equilibrium under transparency with $R^T > 0$, we have $\pi^{E_n-} = \pi^{E_n+}$ and $\pi^{E_y-} < \pi^{E_y+}$.*

Proof. Using the same logic as in Lemma 2, we have $\pi^{E_n-} \leq \pi^{E_n+}$ and $\pi^{E_y-} \leq \pi^{E_y+}$, where equality holds only if that message is never sent by an influencer who receives a low signal. Suppose for contradiction that $\pi^{E_n-} < \pi^{E_n+}$ and $\pi^{E_y-} = \pi^{E_y+}$, then an immediately observation is that $\sigma^T(E_y|\ell Y) = \sigma^T(E_y|\ell N) = 0$ and $\sigma^T(E_n|\ell N) > 0$. It then follows from $R^T > 0$ and $\pi^{E_y-} = \pi^{E_y+}$ that $\sigma^T(E_y|hY) = 1$, and from $\pi^{E_n-} < \pi^{E_n+}$ that $\sigma^T(E_n|hN) = 1$. However, this means $\pi^{E_n+} < \pi < \pi^{\varnothing}$, contradicting the optimality of E_n at (ℓ, N).

Next suppose that $\pi^{E_n-} = \pi^{E_n+}$ and $\pi^{E_y-} = \pi^{E_y+}$, then we know $\sigma^T(\varnothing|\ell Y) = \sigma^T(\varnothing|\ell N) = 1$. It follows that $\sigma^T(\varnothing|hN) = 0$, because otherwise $\pi^{\varnothing} \leq \pi < \pi^{E_n-}$, contradicting the optimality of $\varnothing$ at hN in the first place. If $\sigma^T(E_y|hN) > 0$, then since $R^T > 0$, we must have $\sigma^T(E_y|hY) = 1$, implying that $\pi^{E_y+} \leq \pi < \pi^{E_n-}$ and contradicting the optimality of E_y at (h, Y). If $\sigma^T(E_n|hN) = 1$, then $\pi^{E_y+} \leq \pi = \pi^{E_n-}$; since $R^T > 0$, the strategic influencer at (ℓ, Y) should strictly prefer E_y over $\varnothing$, contradiction.

Finally, suppose that $\pi^{E_n-} < \pi^{E_n+}$ and $\pi^{E_y-} < \pi^{E_y+}$, then $\sigma^T(E_y|\ell Y) > 0$ and $\sigma^T(E_n|\ell N) > 0$. Since $R^T > 0$, it immediately follows that $\sigma^T(E_y|hY) = 1$ and $\sigma^T(\varnothing|hN) = 0$. Since then we must have $\pi^{E_y+} < \pi < \pi^{\varnothing}$, we infer that $\sigma^T(E_n|hN) = 1$, but this implies $\pi^{E_n+} < \pi < \pi^{\varnothing}$ and contradicts the optimality of E_n at (h, N). □

Suppose that $\sigma^T(\varnothing|\ell N) < 1$ in equilibrium. By Lemma 3, this implies $\sigma^T(E_y|\ell N) > 0$. But since $\pi^{E_y-} < \pi^{E_y+}$ and $R^T > 0$, we infer that the strategic influencer must strictly prefers E_y over $\varnothing$ at all other circumstances $(h, Y), (\ell, Y), (h, N)$. This then implies $\pi^{E_y+} < \pi < \pi^{\varnothing}$ and contradicts the optimality of E_y at (ℓ, N). Therefore, in equilibrium we must have $\sigma^T(\varnothing|\ell N) = 1$. The second and third observations that in equilibrium $\sigma^T(E_y|hY) = 1$ and $\sigma^T(E_y|\ell Y) > 0$ can be shown using the same arguments as in the proof of Proposition 1. Finally, note that $\sigma^T(E_y|hY) = 1$ and $\sigma^T(E_y|\ell Y) > 0$ imply that $\pi^{E_y+} < \pi$. By Lemma 3, $\pi^{E_n+} = \pi^{E_n-}$. Since the belief about the influencer's type is a martingale, at least one of π^{E_n+} and $\pi^{\varnothing}$ has to be higher than π, so $\sigma^T(E_y|hN) = 0$. The rest of the argument is the same as in the proof of Proposition 1.

[Part B.] I now show that there exists a unique equilibrium and it features a positive rate $R^T > 0$. Suppose there exists an equilibrium with a negative rate $R^T < 0$, then analogously as in the proof of Proposition 2 we can show that $\sigma^T(\varnothing|\ell Y) = 1$ and $\sigma^T(E_n|hN) = 1$. But then upon receiving E_y, the expected endorsement credibility q_H^y/q_L^y is positive, contradicting $R^T < 0$. Now suppose there exists an equilibrium where the rate is zero, $R^T = 0$. Then the strategic influencer behaves truthfully, resulting in a positive rate $\overline{R} > 0$. Therefore, the rate R^T must be positive in any static equilibrium.

I first show an additional property that must hold in equilibrium, $\sigma^T(E_n|hN) > 0$.[4] Suppose not, then since the message E_n is sent by the strategic influencer with probability 0, we have $\pi^{E_n+} = \pi^{E_n-} = 1 > \pi^{\varnothing}$, contradicting the optimality of $\varnothing$ at (h, N). With the properties shown in Part A, we can write

$$q_H^T = \tilde{q}_H^{E_y} + \tilde{q}_H^{E_n}, q_L^T = \tilde{q}_L^{E_y} + \tilde{q}_L^{E_n},$$

where

$$\tilde{q}_H^{E_n} := Pr(E_n|H) = \pi\lambda(1-\varphi) + (1-\pi)\lambda(1-\varphi)\sigma^T(E_n|hN)$$
$$\tilde{q}_L^{E_n} := Pr(E_n|L) = \pi(1-\lambda)(1-\varphi) + (1-\pi)(1-\lambda)(1-\varphi)\sigma^T(E_n|hN)$$

[4]This contrasts with the equilibrium without the transparency regulation, in which the strategic influencer might play $\varnothing$ with probability 1 at (h, N).

$$\tilde{q}_H^{E_y} := Pr(E_y|H) = \pi\lambda\varphi + (1-\pi)\varphi[\lambda + (1-\lambda)\sigma^T(E_y|\ell Y)]$$
$$\tilde{q}_L^{E_y} := Pr(E_y|L) = \pi(1-\lambda)\varphi + (1-\pi)\varphi[1-\lambda + \lambda\sigma^T(E_y|\ell Y)]$$

We can analogously define $f_{hN} : [0,1] \to \mathbb{R}$ and $f_{\ell Y} : [0,1] \to \mathbb{R}$,

$$\begin{aligned} f_{hN}^T(\tilde{q}_H^{E_y}, \tilde{q}_L^{E_y}, \tilde{q}_H^{E_n}, \tilde{q}_L^{E_n}) &:= \lambda W\left(\frac{\pi\lambda(1-\varphi)}{\tilde{q}_H^{E_n}}\right) + (1-\lambda)W\left(\frac{\pi(1-\lambda)(1-\varphi)}{\tilde{q}_L^{E_n}}\right) \\ &\quad - W\left(\frac{\pi}{2-\tilde{q}_H^{E_y}-\tilde{q}_L^{E_y}-\tilde{q}_H^{E_n}-\tilde{q}_L^{E_n}}\right), \\ f_{\ell Y}^T(\tilde{q}_H^{E_y}, \tilde{q}_L^{E_y}, \tilde{q}_H^{E_n}, \tilde{q}_L^{E_n}) &:= \frac{1}{\delta}G\left(\frac{\tilde{q}_H^{E_y}}{\tilde{q}_L^{E_y}}\right) + \lambda W\left(\frac{\pi(1-\lambda)(1-\varphi)}{\tilde{q}_L^{E_y}}\right) \\ &\quad + (1-\lambda)W\left(\frac{\pi\lambda(1-\varphi)}{\tilde{q}_H^{E_y}}\right) - W\left(\frac{\pi}{2-\tilde{q}_H^{E_y}-\tilde{q}_L^{E_y}-\tilde{q}_H^{E_n}-\tilde{q}_L^{E_n}}\right). \end{aligned}$$

Since the influencer randomizes at (h,N), in equilibrium it must be true that $f_{hN}^T(\tilde{q}_H^{E_y}, \tilde{q}_L^{E_y}, \tilde{q}_H^{E_n}, \tilde{q}_L^{E_n}) = 0$. Fixing $\sigma^T(E_y|\ell Y)$ (thus also fixing $\tilde{q}_H^{E_y}$ and $\tilde{q}_L^{E_y}$), f_{hN} is decreasing in both $\tilde{q}_H^{E_n}$ and $\tilde{q}_L^{E_n}$. So f_{hN} reaches its infimum value $W(\pi) - W\left(\frac{\pi}{1+\varphi-\tilde{q}_H^{E_y}-\tilde{q}_L^{E_y}}\right)$ when $\sigma^T(E_n|hN) = 1$, and this value is negative since $\tilde{q}_H^{E_y} + \tilde{q}_L^{E_y} > \varphi$; meanwhile, f_{hN} reaches its maximum value $W(1) - W\left(\frac{\pi}{2-\pi(1-\varphi)-\tilde{q}_H^{E_y}-\tilde{q}_L^{E_y}}\right) > 0$ when $\sigma^T(E_n|hN) = 0$ since $\tilde{q}_H^{E_y} + \tilde{q}_L^{E_y} \le \varphi(2-\pi)$. Therefore, for each possible profile $(\tilde{q}_H^{E_y}, \tilde{q}_L^{E_y})$, there exists a unique interior solution $\sigma^T(E_n|hN)$ to $f_{hN}^T = 0$. Moreover, as $\sigma^T(E_y|\ell Y)$ increases, while the solution $\sigma^T(E_n|hN)$ decreases, the implied continuation payoff from not endorsing $W(\pi^\varnothing)$ must also increase. So $f_{\ell Y}^T(\tilde{q}_H^{E_y}, \tilde{q}_L^{E_y}, \tilde{q}_H^{E_n}, \tilde{q}_L^{E_n})$ could be rewritten as a strictly decreasing function of $\sigma^T(E_y|\ell Y)$. Therefore, either there exists an interior solution $\sigma^T(E_y|\ell Y) \in (0,1)$ such that $f_{\ell Y}^T = 0$, or when $\sigma^T(E_y|\ell Y) = 1$, we have $f_{\ell Y}^T \ge 0$. In the former case, we have an equilibrium in which the strategic influencer randomizes at (ℓ, Y); in the latter case, we have an equilibrium in which the strategic influencer plays E_y with probability 1 at (ℓ, Y).

A.6 Proof of Proposition 5

I first prove $\sigma^T(E_y|\ell Y) \le \sigma^*(E|\ell Y)$ and $\sigma^T(E_n|hN) \ge \sigma^*(E|hN)$. Let $\mathit{\Delta} := \sigma^T(E_y|\ell Y) - \sigma^*(E|\ell Y)$. Suppose $\mathit{\Delta} > 0$, then $\sigma^*(E|\ell Y) < 1$. First observe that $\tilde{q}_H^{E_y} > \varphi q_H^*$ and $\tilde{q}_L^{E_y} > \varphi q_L^*$, and thus $\tilde{\pi}^{E_y+} < \pi^{+*}$ and $\tilde{\pi}^{E_y-} < \pi^{-*}$. We can write

$$\begin{aligned}\frac{\tilde{q}_H^{E_y}}{\tilde{q}_L^{E_y}} &= \frac{\pi\lambda\varphi + (1-\pi)\varphi[\lambda + (1-\lambda)\sigma^T(E_y|\ell Y)]}{\pi(1-\lambda)\varphi + (1-\pi)\varphi[1-\lambda+\lambda\sigma^T(E_y|\ell Y)]} \\ &= \frac{q_H^* - \pi\lambda(1-\varphi) - (1-\pi)\lambda(1-\varphi)\sigma^*(E|hN) + (1-\pi)(1-\lambda)\varphi\Delta}{q_L^* - \pi(1-\lambda)(1-\varphi) - (1-\pi)(1-\lambda)(1-\varphi)\sigma^*(E|hN) + (1-\pi)\lambda\varphi\Delta} < \frac{q_H^*}{q_L^*},\end{aligned}$$

where the inequality follows from $\Delta > 0$ and the fact that $\frac{q_H^*}{q_L^*} \in \left(\frac{1-\lambda}{\lambda}, \frac{\lambda}{1-\lambda}\right)$. Therefore, the equilibrium payoff from sending E_y under transparency, denoted by $\tilde{V}_{\ell Y}$, is strictly lower than the the equilibrium payoff from sending E without transparency, $V_{\ell Y}^*$.

Now suppose $0 \leq \sigma^T(E|hN) < \sigma^*(E|hN) \leq 1$, then it immediately follows that $\tilde{\pi}^{E_n+} > \pi^{+*}$ and $\tilde{\pi}^{E_n-} > \pi^{-*}$, and thus $\tilde{V}_{hN} > V_{hN}^*$. This cannot arise because in this case $\tilde{V}_{\ell Y} \geq \tilde{U} \geq \tilde{V}_{hN} > V_{hN}^* = U^* = V_{\ell Y}^*$. Suppose instead that $1 \geq \sigma^T(E|hN) > \sigma^*(E|hN) \geq 0$, then it follows that $\tilde{\pi}^{\varnothing} > \pi^{\varnothing *}$, and thus $\tilde{U} > U^*$. This cannot arise because in this case $U^* = V_{\ell Y}^* > \tilde{V}_{\ell Y} \geq \tilde{U}$. So we must have $\Delta \leq 0$. Suppose that $\Delta \leq 0$ and $0 \leq \sigma^T(E|hN) < \sigma^*(E|hN) \leq 1$. Then it follows that $\tilde{\pi}^{\varnothing} < \pi^{\varnothing *}$, $\tilde{\pi}^{E_n+} > \pi^{+*}$, and $\tilde{\pi}^{E_n-} > \pi^{-*}$. Hence, $\tilde{U} < U^*$ and $\tilde{V}_{hN} > V_{hN}^*$. But this also cannot arise in equilibrium because $\tilde{V}_{hN} \leq \tilde{U}$ and $V_{hN}^* = U^*$. The desired result then follows.

Now let's show that the consumer welfare is higher under transparency. Without regulation, the total expected payoff of customers is simply

$$CS^* = \frac{1}{2}\int_{\underline{v}}^{q_H^*/q_L^*} (q_H^* - vq_L^*)dG(v).$$

Under regulation, the total payoff is given by

$$CS^T = \frac{1}{2}\int_{\underline{v}}^{\tilde{q}_H^{E_n}/\tilde{q}_L^{E_n}} (\tilde{q}_H^{E_n} - v\tilde{q}_L^{E_n})dG(v) + \frac{1}{2}\int_{\underline{v}}^{\tilde{q}_H^{E_y}/\tilde{q}_L^{E_y}} (\tilde{q}_H^{E_y} - v\tilde{q}_L^{E_y})dG(v).$$

To facilitate the comparison, first define

$$\begin{aligned}\hat{q}_H^{E_n} &:= \pi\lambda(1-\varphi) + (1-\pi)\lambda(1-\varphi)\sigma^*(E_n|hN), \\ \hat{q}_L^{E_n} &:= \pi(1-\lambda)(1-\varphi) + (1-\pi)(1-\lambda)(1-\varphi)\sigma^*(E_n|hN), \\ \hat{q}_H^{E_y} &:= \pi\lambda\varphi + (1-\pi)\varphi[\lambda + (1-\lambda)\sigma^*(E_y|\ell Y)], \\ \hat{q}_L^{E_y} &:= \pi(1-\lambda)\varphi + (1-\pi)\varphi[1-\lambda+\lambda\sigma^*(E_y|\ell Y)].\end{aligned}$$

Then these quantities satisfy $\hat{q}_H^{E_n} + \hat{q}_H^{E_y} = q_H^*$ and $\hat{q}_L^{E_n} + \hat{q}_L^{E_y} = q_L^*$. In addition, notice that

$\hat{q}_H^{E_n}/\hat{q}_L^{E_n} = \lambda/(1-\lambda) > q_H^*/q_L^*$, so it must be that $\hat{q}_H^{E_y}/\hat{q}_L^{E_y} < q_H^*/q_L^*$. Therefore,

$$\begin{aligned}\hat{CS} :=& \frac{1}{2}\int_{\underline{v}}^{\hat{q}_H^{E_n}/\hat{q}_L^{E_n}} (\hat{q}_H^{E_n} - v\hat{q}_L^{E_n})dG(v) + \frac{1}{2}\int_{\underline{v}}^{\hat{q}_H^{E_y}/\hat{q}_L^{E_y}} (\hat{q}_H^{E_y} - v\hat{q}_L^{E_y})dG(v) \\ >& \frac{1}{2}\int_{\underline{v}}^{q_H^*/q_L^*} (\hat{q}_H^{E_n} - v\hat{q}_L^{E_n})dG(v) + \frac{1}{2}\int_{\underline{v}}^{q_H^*/q_L^*} (\hat{q}_H^{E_y} - v\hat{q}_L^{E_y})dG(v) \\ =& \frac{1}{2}\int_{\underline{v}}^{q_H^*/q_L^*} (\hat{q}_H^{E_n} - v\hat{q}_L^{E_n} + \hat{q}_H^{E_y} - v\hat{q}_L^{E_y})dG(v) = CS^*.\end{aligned}$$

Since $\sigma^T(E_y|\ell Y) \leq \sigma^*(E|\ell Y)$ and $\sigma^T(E_n|hN) \geq \sigma^*(E|hN)$, we know that $\tilde{q}_H^{E_n}/\tilde{q}_L^{E_n} \geq \hat{q}_H^{E_n}/\hat{q}_L^{E_n}$ and $\tilde{q}_H^{E_n}/\tilde{q}_L^{E_n} \geq \hat{q}_H^{E_n}/\hat{q}_L^{E_n}$. Moreover, for any $v \in [\underline{v}, \lambda/(1-\lambda)]$, we have $\tilde{q}_H^{E_n} - v\tilde{q}_L^{E_n} \geq \hat{q}_H^{E_n} - v\hat{q}_L^{E_n}$ and $\tilde{q}_H^{E_y} - v\tilde{q}_L^{E_y} \geq \hat{q}_H^{E_y} - v\hat{q}_L^{E_y}$. Therefore, $CS^* < \hat{CS} \leq CS^T$.

A.7 Proof of Theorem 1

Let $\overline{W} = \frac{f(1)+N(1)G(\lambda/(1-\lambda))}{1-\delta}$, which bounds the influencer's possible equilibrium payoff from above. For any constant $\underline{\kappa} > 0$, define

$$\Omega_{\underline{\kappa}} = \left\{W \in \mathcal{C}([0,1],[0,\overline{W}]) : W(\pi^2) - W(\pi^1) \geq \underline{\kappa}(\pi^2 - \pi^1) \text{ for all } \pi^2 \geq \pi^1.\right\}$$

That is, the set $\Omega_{\underline{\kappa}}$ contains continuous and strictly increasing functions with its slope weakly larger than $\underline{\kappa}$. It is a convex, compact subset of the space $\mathcal{C}([0,1],[0,\overline{W}])$ with the sup-norm denoted by d.[5]

Given a continuation value function $W \in \Omega_{\underline{\kappa}}$, by Proposition 1, the influencer's equilibrium is given by

$$\begin{aligned} T(W)(\pi) =& \frac{1}{2}\Big[\varphi V_{hY}^*(\pi,W) + \varphi \max\{V_{\ell Y}^*(\pi,W), U^*(\pi,W)\} && (17)\\ &+ (1-\varphi)\max\{V_{hN}^*(\pi,W), U^*(\pi,W)\} + (1-\varphi)U^*(\pi,W)\Big]. && (18)\end{aligned}$$

Below, I show that there exists $\overline{\lambda} \in (\frac{1}{2},1)$ such that when the signal accuracy $\lambda \in (\frac{1}{2},\overline{\lambda})$, the mapping T is a continuous contraction on the metric space $(\Omega_{\underline{\kappa}}, d)$, and so has a unique fixed point.

I prove the statement in two parts. Part A shows that if λ is close enough to $1/2$, we have $T(\Omega_{\underline{\kappa}}) \subset \Omega_{\underline{\kappa}}$. Part B further shows that if λ is close enough to $1/2$, there exists $\xi < 1$

[5] Formally, $d(W^1,W^2) = \sup_\pi |W^1(\pi) - W^2(\pi)|$.

such that $d(T(W^1), T(W^2)) \leq \xi \cdot d(W^1, W^2)$ for all $W^1, W^2 \in \Omega_{\underline{\kappa}}$.

[Part A.] Fix $W \in \Omega_{\underline{\kappa}}$. It directly follows from the proof of Proposition 2 that $\sigma^*_{\pi W}$ is continuous in π, which implies the continuity of $T(W)$ in π. Pick $\pi^1, \pi^2 \in [0,1]$ and let $q^*_L(\pi^i, W) = q^i_L$ and $q^*_H(\pi^i, W) = q^i_H$, $\forall i \in \{1,2\}$. To show that $T(W) \in \Omega_{\underline{\kappa}}$ in π, it suffices to show that for any π^1, when π^2 is larger than and sufficiently close to π^1, we have

$$T(W)(\pi^2) - T(W)(\pi^1) \geq F(\pi^2) - F(\pi^1). \tag{19}$$

I only discuss three cases and leave the other cases to the Supplemental Appendix.

Case 1. Suppose in the static equilibrium associated with (π^1, W), the influencer randomizes between E and $\varnothing$ at both (h, N) and (ℓ, Y). Then by continuity, the influencer also randomizes at both (h, N) and (ℓ, Y) in the equilibrium associated with (π^2, W) if π^1 and π^2 are close. The following indifference condition holds for $\pi^i \in \{\pi^1, \pi^2\}$.

$$V^*_{\ell Y}(\pi^i, W) = F(\pi^i) + N(\pi^i) G\left(\frac{q^i_H}{q^i_L}\right) + \delta\left[(1-\lambda) W\left(\frac{\pi^i \lambda}{q^i_H}\right) + \lambda W\left(\frac{\pi^i(1-\lambda)}{q^i_L}\right)\right] \tag{20}$$

$$= V^*_{hN}(\pi^i, W) = F(\pi^i) + \delta\left[\lambda W\left(\frac{\pi^i \lambda}{q^i_H}\right) + (1-\lambda) W\left(\frac{\pi^i(1-\lambda)}{q^i_L}\right)\right] \tag{21}$$

$$= U^*(\pi^i, W) = F(\pi^i) + \delta W\left(\frac{\pi^i}{2 - q^i_H - q^i_L}\right) \tag{22}$$

I first show the following inequalities hold:

$$V^*_{\ell Y}(\pi^2, W) - V^*_{\ell Y}(\pi^1, W) > F(\pi^2) - F(\pi^1), \tag{23}$$

$$V^*_{hN}(\pi^2, W) - V^*_{hN}(\pi^1, W) > F(\pi^2) - F(\pi^1), \tag{24}$$

$$U^*(\pi^2, W) - U^*(\pi^1, W) > F(\pi^2) - F(\pi^1), \tag{25}$$

If $q^1_L \leq q^2_L$ and $q^1_H \leq q^2_H$, then the inequalities Eqs. (23) to (25) follow from the third expression Eq. (22). If $q^1_L > q^2_L$ and $q^1_H > q^2_H$, then the inequalities follow from the second expression Eq. (21). Suppose instead $q^1_L > q^2_L$ and $q^1_H \leq q^2_H$. Then we can arrange Eqs. (20) and (21) to obtain

$$V^*_{hN}(\pi^i, W) = F(\pi^i) + \delta W\left(\frac{\pi^i(1-\lambda)}{q^i_L}\right) + \frac{\lambda}{2\lambda - 1} N(\pi^i) G\left(\frac{q^i_H}{q^i_L}\right) \tag{26}$$

and so the inequalities follow as well. Finally, suppose $q^1_L \leq q^2_L$ and $q^1_H > q^2_H$. Note that we can arrange Eqs. (20) and (21) to obtain the equivalent expression

$$V^*_{hN}(\pi^i, W) = F(\pi^i) + \delta W\left(\frac{\pi^i\lambda}{q^i_H}\right) - \frac{1-\lambda}{2\lambda-1}N(\pi^i)G\left(\frac{q^i_H}{q^i_L}\right). \tag{27}$$

If $N(\pi^1)G\left(\frac{q^1_H}{q^1_L}\right) \geq N(\pi^2)G\left(\frac{q^2_H}{q^2_L}\right)$, then the three inequalities Eqs. (23) to (25) follow analogously. Suppose instead that $N(\pi^1)G\left(\frac{q^1_H}{q^1_L}\right) < N(\pi^2)G\left(\frac{q^2_H}{q^2_L}\right)$. If $\frac{\pi^1}{2-q^1_H-q^1_L} \leq \frac{\pi^2}{2-q^2_H-q^2_L}$ then the three inequalities follow from the expression Eq. (22). If instead we have $\frac{\pi^1}{2-q^1_H-q^1_L} > \frac{\pi^2}{2-q^2_H-q^2_L}$, then the indifference condition at (h, N) and the observation that $\frac{\pi^1\lambda}{q^1_H} < \frac{\pi^2\lambda}{q^2_H}$ imply that $\frac{\pi^1(1-\lambda)}{q^1_L} > \frac{\pi^2(1-\lambda)}{q^2_L}$. However, we then have

$$N(\pi^2)G\left(\frac{q^2_H}{q^2_L}\right) < N(\pi^2)G\left(\frac{q^1_H}{q^1_L}\frac{\pi^1}{\pi^2}\right) \leq N(\pi^1)G\left(\frac{q^1_H}{q^1_L}\right),$$

where the inequality follows from our assumption on N and G. This contradicts the assumption and thus this possibility cannot arise in equilibrium.

It remains to verify Eq. (19). Suppose $N(\pi^1)G\left(\frac{q^1_H}{q^1_L}\right) \leq N(\pi^2)G\left(\frac{q^2_H}{q^2_L}\right)$, then since

$$V^*_{hY}(\pi^i, W) = N(\pi^i)G\left(\frac{q^i_H}{q^i_L}\right) + V^*_{hN}(\pi^i, W),$$

we have $V^*_{hY}(\pi^2, W) - V^*_{hY}(\pi^1, W) \geq V^*_{hN}(\pi^2, W) - V^*_{hN}(\pi^1, W) > f(\pi^2) - f(\pi^1)$. Combined with the inequalities Eqs. (23) to (25), it follows that Eq. (19) holds. Suppose instead that $N(\pi^1)G\left(\frac{q^1_H}{q^1_L}\right) > N(\pi^2)G\left(\frac{q^2_H}{q^2_L}\right)$. Then since $V^*_{\ell Y}(\pi^i, W) = V^*_{hN}(\pi^i, W)$, we have

$$N(\pi^i)G\left(\frac{q^i_H}{q^i_L}\right) = \delta(2\lambda-1)\left[W\left(\frac{\pi^i\lambda}{q^i_H}\right) - W\left(\frac{\pi^i(1-\lambda)}{q^i_L}\right)\right]$$

and thus

$$W\left(\frac{\pi^1\lambda}{q^1_H}\right) - W\left(\frac{\pi^1(1-\lambda)}{q^1_L}\right) > W\left(\frac{\pi^2\lambda}{q^2_H}\right) - W\left(\frac{\pi^2(1-\lambda)}{q^2_L}\right).$$

On the other hand, Eqs. (23) and (24) imply that

$$\begin{aligned}(1-\lambda)W\left(\frac{\pi^1\lambda}{q^1_H}\right) + \lambda W\left(\frac{\pi^1(1-\lambda)}{q^1_L}\right) &\leq (1-\lambda)W\left(\frac{\pi^2\lambda}{q^2_H}\right) + \lambda W\left(\frac{\pi^2(1-\lambda)}{q^2_L}\right), \\ \lambda W\left(\frac{\pi^1\lambda}{q^1_H}\right) + (1-\lambda)W\left(\frac{\pi^1(1-\lambda)}{q^1_L}\right) &\leq \lambda W\left(\frac{\pi^2\lambda}{q^2_H}\right) + (1-\lambda)W\left(\frac{\pi^2(1-\lambda)}{q^2_L}\right).\end{aligned}$$

Taken together, these observations imply that

$$W\left(\frac{\pi^1\lambda}{q_H^1}\right) \leq W\left(\frac{\pi^2\lambda}{q_H^2}\right),$$
$$W\left(\frac{\pi^1(1-\lambda)}{q_L^1}\right) \leq W\left(\frac{\pi^2(1-\lambda)}{q_L^2}\right).$$

Note that

$$\begin{aligned}
&T(W)(\pi^i) - f(\pi^i)\\
=&\frac{1}{2}\left[\varphi V_{\ell Y}^*(\pi^i, W) + (1-\varphi)V_{hN}^*(\pi^i, W) + \varphi V_{hY}^*(\pi^i, W) + (1-\varphi)U^*(\pi^i, W)\right]\\
=&\frac{1}{2}\varphi\delta(2\lambda-1)\left[W\left(\frac{\pi^i\lambda}{q_H^i}\right) - W\left(\frac{\pi^i(1-\lambda)}{q_L^i}\right)\right] + V_{hN}^*(\pi^i, W)\\
=&\frac{1}{2}\varphi\delta(2\lambda-1)\left[W\left(\frac{\pi^i\lambda}{q_H^i}\right) - W\left(\frac{\pi^i(1-\lambda)}{q_L^i}\right)\right] + \delta\left[\lambda W\left(\frac{\pi^i\lambda}{q_H^i}\right) + (1-\lambda)W\left(\frac{\pi^i(1-\lambda)}{q_L^i}\right)\right]\\
=&\delta\left[\left(\frac{1}{2}\varphi(2\lambda-1)+\lambda\right)W\left(\frac{\pi^i\lambda}{q_H^i}\right) + \left((1-\lambda) - \frac{1}{2}\varphi(2\lambda-1)\right)W\left(\frac{\pi^i(1-\lambda)}{q_L^i}\right)\right],
\end{aligned}$$

so Eq. (19) follows if the constants before both continuation values are positive, i.e. $\varphi < \frac{2(1-\lambda)}{2\lambda-1}$. Since $\varphi < 1$, a sufficient condition is $\lambda < \frac{3}{4}$.

To show contraction: Use Blackwell's contraction mapping theorem to prove monotonicity and thus contraction.

Case 2. Suppose in the static equilibrium associated with (π^1, W), the influencer randomizes at (h, N) but plays E with probability 1 at (ℓ, Y). So we have

$$V_{\ell Y}^*(\pi^1, W) \geq V_{hN}^*(\pi^1, W) = U^*(\pi^1, W) \text{ and } \sigma_{\pi^1 W}^*(E|\ell Y) = 1.$$

Then in the static equilibrium associated with (π^1, W), by continuity and Proposition 1, we either have

$$V_{\ell Y}^*(\pi^2, W) \geq V_{hN}^*(\pi^2, W) = U^*(\pi^2, W) \text{ and } \sigma_{\pi^2 W}^*(E|\ell Y) = 1 \tag{28}$$

or

$$\begin{aligned}
V_{\ell Y}^*(\pi^1, W) =& V_{hN}^*(\pi^1, W) = U^*(\pi^1, W) \text{ and } \sigma_{\pi^1 W}^*(E|\ell Y) = 1,\\
V_{\ell Y}^*(\pi^2, W) =& V_{hN}^*(\pi^2, W) = U^*(\pi^1, W) \text{ and } \sigma_{\pi^2 W}^*(E|\ell Y) \in (0, 1).
\end{aligned} \tag{29}$$

If Condition (29) holds, then we can use a similar argument as in Case 1 to obtain the three inequalities Eqs. (23) to (25). Now suppose instead the situation is as described by Condition (28). Note that since $\sigma^*_{\pi^i W}(E|\ell Y) = 1$, we have

$$q_L^i = \frac{1-\lambda}{\lambda} q_H^i + \frac{(1-\pi^i)(2\lambda - 1)\varphi}{\lambda}.$$

Plug this relationship into the indifference condition,

$$\lambda W\left(\frac{\pi^i \lambda}{q_H^i}\right) + (1-\lambda) W\left(\frac{\pi^i(1-\lambda)}{q_L^i}\right) = W\left(\frac{\pi^i}{2 - q_L^i - q_H^i}\right),$$

we obtain an equation with the left-hand side continuously decreasing in q_H^i and the right-hand side continuously increasing in q_H^i. Since $\pi^1 < \pi^2$ and both sides increase with π^i, we must have

$$V^*_{hN}(\pi^1, W) = U^*(\pi^1, W) \leq V^*_{hN}(\pi^2, W) = U^*(\pi^2, W).$$

It remains to verify Eq. (19). Note that $q_H^1/q_L^1 \geq q_H^2/q_L^2$ iff

$$\frac{1-\pi^1}{q_H^1} \leq \frac{1-\pi^2}{q_H^2}.$$

Meanwhile, the posteriors satisfy $\pi^1\lambda/q_H^1 < \pi^2\lambda/q_H^2$ iff

$$\frac{\pi^1}{q_H^1} < \frac{\pi^2}{q_H^2},$$

and $\pi^1(1-\lambda)/q_L^1 < \pi^2(1-\lambda)/q_L^2$ iff

$$\frac{\pi^1}{(1-\lambda)q_H^1 + (1-\pi^1)(2\lambda-1)\varphi} < \frac{\pi^2}{(1-\lambda)q_H^2 + (1-\pi^2)(2\lambda-1)\varphi}.$$

Note that the first inequality implies the second, which in turn implies the third. Let's now analyze all possibilities. Suppose $\pi^1\lambda/q_H^1 \geq \pi^2\lambda/q_H^2$, then the comparison above implies that $q_H^1/q_L^1 < q_H^2/q_L^2$, and the observation that $V^*_{hN}(\pi^1, W) \leq V^*_{hN}(\pi^2, W)$ implies that $\pi^1(1-\lambda)/q_L^1 \leq \pi^2(1-\lambda)/q_L^2$. Since

$$V^*_{hY}(\pi^i, W) = N(\pi^i) G\left(\frac{q_H^i}{q_L^i}\right) + V^*_{hN}(\pi^i, W),$$

$$V_{\ell Y}^*(\pi^i, W) = N(\pi^i)G\left(\frac{q_H^i}{q_L^i}\right) - \delta(2\lambda-1)\left[W\left(\frac{\pi^i\lambda}{q_H^i}\right) - W\left(\frac{\pi^i(1-\lambda)}{q_L^i}\right)\right] + V_{hN}^*(\pi^i, W),$$

it follows that $V_{hY}^*(\pi^1, W) \leq V_{hY}^*(\pi^2, W)$ and $V_{\ell Y}^*(\pi^1, W) \leq V_{\ell Y}^*(\pi^2, W)$ and thus Eq. (19) holds. Suppose instead that $\pi^1\lambda/q_H^1 < \pi^2\lambda/q_H^2$, then we immediately have $\pi^1(1-\lambda)/q_L^1 < \pi^2(1-\lambda)/q_L^2$. If we further have $q_H^1/q_L^1 < q_H^2/q_L^2$, then similarly the influencer obtains higher payoff at both (h, Y) and (ℓ, Y), and so Eq. (19) holds. The most involved case is when $q_H^1/q_L^1 \geq q_H^2/q_L^2$. Here, $q_H^1 > q_H^2$ and $q_L^1 > q_L^2$. As I will show below, Eq. (19) holds in this case when λ is close enough to $1/2$.

First observe that

$$\begin{aligned}
&N(\pi^2)G\left(\frac{q_H^2}{q_L^2}\right) - N(\pi^1)G\left(\frac{q_H^1}{q_L^1}\right)\\
\geq &N(\pi^2)G\left(\frac{1}{\frac{1-\lambda}{\lambda} + \frac{(1-\pi^2)(2\lambda-1)\varphi}{\lambda q_H^2}}\right) - N(\pi^1)G\left(\frac{1}{\frac{1-\lambda}{\lambda} + \frac{(1-\pi^1)(2\lambda-1)\varphi}{\lambda q_H^1}}\right)\\
\geq &- N(\pi^1)\overline{g}\left(\frac{\lambda}{1-\lambda}\right)^2 \frac{(1-\pi^1)(2\lambda-1)\varphi}{\lambda}\left(\frac{1}{q_H^2} - \frac{1}{q_H^1}\right)\\
= &- c_1 N(\pi^1)(2\lambda-1)\left(\frac{1}{q_H^2} - \frac{1}{q_H^1}\right),
\end{aligned}$$

where $\overline{g} = \max_{R\in[\frac{1-\lambda}{\lambda}, \frac{\lambda}{1-\lambda}]} G'(R)$ and $c_1 > 0$. On the other hand,

$$W\left(\frac{\pi^2\lambda}{q_H^2}\right) - W\left(\frac{\pi^1\lambda}{q_H^1}\right) \geq \underline{\kappa}\lambda\pi^1\left(\frac{1}{q_H^2} - \frac{1}{q_H^1}\right) = c_2\pi^1\left(\frac{1}{q_H^2} - \frac{1}{q_H^1}\right)$$

Note that we can rewrite the influencer's equilibrium payoff as

$$\begin{aligned}
T(W)(\pi) = &F(\pi) + \varphi N(\pi)G\left(\frac{q_H}{q_L}\right)\\
&+ \delta\left[\left(\frac{1}{2}\varphi + \lambda(1-\varphi)\right)W\left(\frac{\pi^i\lambda}{q_H^i}\right) + \left(\frac{1}{2}\varphi + (1-\lambda)(1-\varphi)\right)W\left(\frac{\pi^i(1-\lambda)}{q_L^i}\right)\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&T(W)(\pi^2) - T(W)(\pi^1) - (F(\pi^2) - F(\pi^1))\\
\geq &\left[\delta\left(\frac{1}{2}\varphi + \lambda(1-\varphi)\right)c_2\pi^1 - \varphi c_1 N(\pi^1)(2\lambda-1)\right]\left(\frac{1}{q_H^2} - \frac{1}{q_H^1}\right).
\end{aligned}$$

Notice that since $\lim_{\pi\to 0}\frac{N(\pi)}{\pi} \geq n$ for some $n > 0$ and N is strictly increasing over the interval $[0,1]$, there exists $n' > 0$ such that $\frac{N(\pi)}{\pi} \geq n$ for all $\pi \in [0,1]$. Therefore, there exists $\overline{\lambda}_1$ such that when $\lambda \in (\frac{1}{2}, \overline{\lambda}_1)$, we have

$$\frac{\delta\left(\frac{1}{2}\varphi + \lambda(1-\varphi)\right)c_2\pi^1}{\varphi c_1 N(\pi^1)(2\lambda-1)} \geq \frac{\delta\left(\frac{1}{2}\varphi + \lambda(1-\varphi)\right)c_2}{\varphi c_1(2\lambda-1)}n' > 1$$

and thus Eq. (19) holds.

Case 3. Suppose in the static equilibrium associated with (π^1, W), the influencer randomizes at (ℓ, S) but plays $\varnothing$ with probability 1 at (h, N). So,

$$V^*_{\ell Y}(\pi^1, W) = U^*(\pi^1, W) \geq V^*_{hN}(\pi^1, W).$$

Then in the static equilibrium associated with (π^1, W), by continuity and Proposition 1, when (π^1, W) is sufficiently close to (π^2, W), we either have

$$V^*_{\ell Y}(\pi^2, W) = U^*(\pi^2, W) \geq V^*_{hN}(\pi^2, W) \text{ and } \sigma^*_{\pi^2 W}(\varnothing|hN) = 1, \tag{30}$$

or

$$\begin{aligned} V^*_{\ell Y}(\pi^1, W) =& U^*(\pi^1, W) = V^*_{hN}(\pi^1, W) \text{ and } \sigma^*_{\pi^1 W}(\varnothing|hN) = 1, \\ V^*_{\ell Y}(\pi^2, W) =& U^*(\pi^2, W) = V^*_{hN}(\pi^2, W) \text{ and } \sigma^*_{\pi^2 W}(\varnothing|hN) \in (0,1). \end{aligned} \tag{31}$$

If the latter situation comes true, then we can use a similar argument as in Case 1 to obtain Eq. (19). Now consider the first possibility. Note that since $\sigma^*_{\pi^i W^i}(\varnothing|hN) = 1$, we have

$$q^i_H = \frac{1-\lambda}{\lambda}q^i_L + \frac{(2\lambda-1)\left(\pi^i + \varphi\left(1-\pi^i\right)\right)}{\lambda}, \tag{32}$$

The influencer is indifferent at (ℓ, Y), so

$$N(\pi^i)G\left(\frac{q^i_H}{q^i_L}\right) + \delta\left[(1-\lambda)W\left(\frac{\pi^i\lambda}{q^i_H}\right) + \lambda W\left(\frac{\pi^i(1-\lambda)}{q^i_L}\right)\right] = \delta W\left(\frac{\pi}{2-q^i_H-q^i_L}\right). \tag{33}$$

Plugging in the expression for q^i_H in Eq. (32), we obtain an equation with the right-hand side decreasing in q^i_L and the left-hand side increasing in q^i_L, which, given (π^i, W), pins down a unique q^i_L. Since both sides increase with π^i, it follows that

$$V^*_{\ell Y}(\pi^1, W) = V^*_{hN}(\pi^1, W) = U^*(\pi^1, W) \leq V^*_{\ell Y}(\pi^2, W) = V^*_{hN}(\pi^2, W) = U^*(\pi^2, W)$$

.

Note that $q_H^1/q_L^1 \leq q_H^2/q_L^2$ iff

$$\frac{\pi^1 + \varphi(1-\pi^1)}{q_L^1} < \frac{\pi^2 + \varphi(1-\pi^2)}{q_L^2}.$$

The posteriors $\frac{\pi^1(1-\lambda)}{q_L^1} < \frac{\pi^2(1-\lambda)}{q_L^2}$ iff

$$\frac{\pi^1}{q_L^1} < \frac{\pi^2}{q_L^2}.$$

The posterior $\frac{\pi^1\lambda}{q_H^1} < \frac{\pi^2\lambda}{q_H^2}$ iff

$$\frac{\pi^1}{(1-\lambda)q_L^1 + (2\lambda-1)\left(\pi^1 + \varphi\left(1-\pi^1\right)\right)} < \frac{\pi^2}{(1-\lambda)q_L^2 + (2\lambda-1)\left(\pi^2 + \varphi\left(1-\pi^2\right)\right)}.$$

Note that the first inequality is stronger than the second inequality, and the second is stronger than the third. As in Case 2, let's analyze all possibilities. If $\frac{\pi^1(1-\lambda)}{q_L^1} \geq \frac{\pi^2(1-\lambda)}{q_L^2}$ but $\frac{\pi^1\lambda}{q_H^1} < \frac{\pi^2\lambda}{q_H^2}$, then since $V_{hY}^*(\pi^i, W) = V_{\ell Y}^*(\pi^i, W) + \delta(2\lambda-1)\left(W\left(\frac{\pi^i\lambda}{q_H^i}\right) - W\left(\frac{\pi^i(1-\lambda)}{q_L^i}\right)\right)$, we infer that $V_{hY}^*(\pi^1, W) \leq V_{hY}^*(\pi^1, W)$ and so Eq. (19) holds. If $\frac{\pi^1(1-\lambda)}{q_L^1} < \frac{\pi^2(1-\lambda)}{q_L^2}$, $\frac{\pi^1\lambda}{q_H^1} < \frac{\pi^2\lambda}{q_H^2}$, and in addition $N(\pi^1)G\left(\frac{q_H^1}{q_L^1}\right) \leq N(\pi^2)G\left(\frac{q_H^2}{q_L^2}\right)$, we reach the same conclusion that $V_{hY}^*(\pi^1, W) \leq V_{hY}^*(\pi^1, W)$. Now instead the posteriors increase while the endorsement price decreases, i.e. $N(\pi^1)G\left(\frac{q_H^1}{q_L^1}\right) > N(\pi^2)G\left(\frac{q_H^2}{q_L^2}\right)$. Then the problem again boils down to a comparison between the decrease in the price and the increase in the continuation payoff. Similar as before,

$$\begin{aligned}
&N(\pi^2)G\left(\frac{q_H^2}{q_L^2}\right) - N(\pi^1)G\left(\frac{q_H^1}{q_L^1}\right) \\
&\geq -N(\pi^1)\overline{g}\frac{(\pi_1 + \varphi(1-\pi^1))(2\lambda-1)}{\lambda}\left(\frac{1}{q_L^2} - \frac{1}{q_L^1}\right),
\end{aligned}$$

while

$$W\left(\frac{\pi^2(1-\lambda)}{q_L^2}\right) - W\left(\frac{\pi^1(1-\lambda)}{q_L^1}\right) \geq \underline{\kappa}(1-\lambda)\pi^1\left(\frac{1}{q_L^2} - \frac{1}{q_L^1}\right).$$

Since

$$T(W)(\pi^2) - T(W)(\pi^1) - (F(\pi^2) - F(\pi^2))$$

$$\geq \frac{1}{2}\varphi\left[N(\pi^2)G\left(\frac{q_H^2}{q_L^2}\right) - N(\pi^1)G\left(\frac{q_H^1}{q_L^1}\right) + \delta W\left(\frac{\pi^2(1-\lambda)}{q_L^2}\right) - \delta W\left(\frac{\pi^1(1-\lambda)}{q_L^1}\right)\right],$$

it follows from similar arguments as in Case 2 that there exists $\overline{\lambda}_2$ such that when $\lambda \in (\frac{1}{2}, \overline{\lambda}_2)$, the above payoff difference is positive and thus Eq. (19) holds.

Finally, suppose $\frac{\pi^1(1-\lambda)}{q_L^1} \geq \frac{\pi^2(1-\lambda)}{q_L^2}$, $\frac{\pi^1\lambda}{q_H^1} \geq \frac{\pi^2\lambda}{q_H^2}$, then the observation that $V_{\ell Y}^*(\pi^1, W) \leq V_{\ell Y}^*(\pi^2, W)$ implies we must have $N(\pi^1)G\left(\frac{q_H^1}{q_L^1}\right) \leq N(\pi^2)G\left(\frac{q_H^2}{q_L^2}\right)$. It also implies that

$$\begin{aligned}
&\delta\lambda\left[W\left(\frac{\pi^2(1-\lambda)}{q_L^2}\right) - W\left(\frac{\pi^1(1-\lambda)}{q_L^1}\right)\right] + \delta(1-\lambda)\left[W\left(\frac{\pi^2\lambda}{q_H^2}\right) - W\left(\frac{\pi^1\lambda}{q_H^1}\right)\right] \\
\geq& - N(\pi^2)G\left(\frac{q_H^2}{q_L^2}\right) + N(\pi^1)G\left(\frac{q_H^1}{q_L^1}\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&T(W)(\pi^2) - T(W)(\pi^1) - (F(\pi^2) - F(\pi^2)) \\
\geq& - \frac{1}{2}\varphi\frac{2\lambda-1}{1-\lambda}\left[N(\pi^2)G\left(\frac{q_H^2}{q_L^2}\right) - N(\pi^1)G\left(\frac{q_H^1}{q_L^1}\right)\right] \\
&+ (1-\frac{1}{2}\varphi)\left[W\left(\frac{\pi^2}{2-q_H^2-q_L^2}\right) - W\left(\frac{\pi^1}{2-q_H^1-q_L^1}\right)\right]
\end{aligned}$$

But notice that

$$N(\pi^2)G\left(\frac{q_H^2}{q_L^2}\right) - N(\pi^1)G\left(\frac{q_H^1}{q_L^1}\right) \leq G\left(\frac{q_H^1}{q_L^1}\right)\sup_{\pi} N'(\pi)(\pi^2-\pi^1),$$

and

$$W\left(\frac{\pi^2}{2-q_H^2-q_L^2}\right) - W\left(\frac{\pi^1}{2-q_H^1-q_L^1}\right) \geq \frac{\kappa}{2}(\pi^2-\pi^1).$$

it follows from similar arguments that there exists $\overline{\lambda}_3$ such that when $\lambda \in (\frac{1}{2}, \overline{\lambda}_3)$, the difference in the price is smaller than the difference in the continuation value, and thus Eq. (19) holds.

Case 4. Suppose at both π^1 and π^2, the equilibrium prescribes the influencer to play pure strategies, i.e. to play E at (ℓ, Y) and $\varnothing$ at (h, N). Then the increase in reputation directly translate into an increase in all posteriors and the endorsement quality and thus the price. So Eq. (19) holds.

[Part B.] Fix $\pi \in (0,1)$ and $W^1, W^2 \in \Omega_{\overline{\kappa}}$. Let $q_H^i = q_H^*(\pi, W^i)$ and $q_L^i = q_L^*(\pi, W^i)$.

Define a new value function $\hat{W}^2 : [0,1] \to \mathbb{R}$ such that $\hat{W}^2(\pi) = W^2(\pi) + d(W^1, W^2)$ for all π, then $\hat{W}^2(\pi) \geq W^1(\pi)$ for all π. Since W^2 and $\hat{W}^2$ only differ by a constant, they lead to the same equilibrium behavior. Analogously define $\hat{W}^1(\pi) = W^1(\pi) + d(W^1, W^2), \forall \pi$. I discuss ten different cases and show that there exists $\overline{\delta} < 1$ such that in each case,

$$|T(W^1)(\pi) - T(W^2)(\pi)| \leq \overline{\delta} d(W^1, W^2). \tag{34}$$

This then implies that

$$\delta d(T(W^1), T(W^2)) \leq \overline{\delta} d(W^1, W^2)$$

Case 1. Suppose both W^1 and W^2 induce an interior equilibrium at π. Using similar arguments as in Case 1 of Part A (and even simpler arguments for the case in which $q_L^1 \leq q_L^2, q_H^1 > q_H^2$ as π stays the same), it can be shown that $T(W^1)(\pi) \leq T(\hat{W}^2)(\pi)$ when λ is close enough to $\frac{1}{2}$. Therefore,

$$T(W^1)(\pi) \leq T(W^2)(\pi) + \delta d(W^1, W^2).$$

and similarly,

$$T(W^2)(\pi) \leq T(W^1)(\pi) + \delta d(W^1, W^2).$$

Therefore, it follows that

$$|T(W^1)(\pi) - T(W^2)(\pi)| \leq \delta d(W^1, W^2).$$

and so there exists $\overline{\delta} < 1$ such that Eq. (34) holds. This argument essentially replicates the proof of Blackwell's Contraction Mapping Theorem.

Case 2. Suppose both W^1 and W^2 induce an equilibrium in which the influencer only randomizes at (h, N) at π. Since $\hat{W}^2 \geq W^1$, we similarly infer that the influencer obtains a higher payoff under $\hat{W}^2$ at (h, N) and (ℓ, N) than under W^1; same for $\hat{W}^1$ and W^2. Since

$$q_L^i = \frac{1-\lambda}{\lambda} q_H^i + \frac{(1-\pi)(2\lambda - 1)\varphi}{\lambda},$$

we either have $(q_L^1, q_H^1) \geq (q_L^2, q_H^2)$ or $(q_L^1, q_H^1) < (q_L^2, q_H^2)$. Since the choice of W^1 and W^2 are arbitrary we can without loss of generality assume that $(q_L^1, q_H^1) \geq (q_L^2, q_H^2)$. Notice then

$\frac{\pi\lambda}{q_H^2} \geq \frac{\pi\lambda}{q_H^1}$, $\frac{\pi(1-\lambda)}{q_L^2} \geq \frac{\pi(1-\lambda)}{q_L^1}$ and $q_H^1/q_L^1 \geq q_H^2/q_L^2$.

First consider W^1 and $\hat{W}^2$. Similar to Case 2 of Part A, it can be shown that

$$N(\pi)G\left(\frac{q_H^2}{q_L^2}\right) - N(\pi)G\left(\frac{q_H^1}{q_L^1}\right) \geq -c_1 N(\pi)(2\lambda-1)\left(\frac{1}{q_H^2} - \frac{1}{q_H^1}\right),$$

and

$$\hat{W}^2\left(\frac{\pi\lambda}{q_H^2}\right) - W^1\left(\frac{\pi\lambda}{q_H^1}\right) \geq \ c_2\pi^1\left(\frac{1}{q_H^2} - \frac{1}{q_H^1}\right).$$

So when λ is close enough to $\frac{1}{2}$, we have $T(W^1)(\pi) \leq T(\hat{W}^2)(\pi)$.

Next consider $\hat{W}^1$ and W^2. Since $q_H^1/q_L^1 > q_H^2/q_L^2$, it follows that $V_{hY}^*(\pi, \hat{W}^1) \geq V_{hY}^*(\pi, W^2)$. We also know that at least one of the following holds: (i) $\hat{W}^1\left(\frac{\pi\lambda}{q_H^1}\right) \geq W^2\left(\frac{\pi\lambda}{q_H^2}\right)$, (ii) $\hat{W}^1\left(\frac{\pi(1-\lambda)}{q_L^1}\right) \geq W^2\left(\frac{\pi(1-\lambda)}{q_L^2}\right)$. If both hold, then $V_{\ell Y}^*(\pi, \hat{W}^1) \geq V_{\ell Y}^*(\pi, W^2)$; if (i) doesn't hold while (ii) holds, then we have the same observation since $V_{\ell Y}^*(\pi, W) = V_{hY}^*(\pi, W) - \delta(2\lambda-1)\left[W\left(\frac{\pi\lambda}{q_H}\right) - W\left(\frac{\pi(1-\lambda)}{q_L}\right)\right]$. In either case, $T(W^2)(\pi) \leq T(\hat{W}^1)(\pi)$. Now suppose (i) holds while (ii) doesn't. Either we still end up having $T(W^2)(\pi) \leq T(\hat{W}^1)(\pi)$, or $T(\hat{W}^1)(\pi) < T(W^2)(\pi)$. In the latter case, since $V_{hN}^*(\pi, \hat{W}^1) \geq V_{hN}^*(\pi, W^2)$, we know that

$$\begin{aligned}
&-(1-\lambda)\left[\hat{W}^1\left(\frac{\pi\lambda}{q_H^1}\right) - \hat{W}^1\left(\frac{\pi(1-\lambda)}{q_L^1}\right) - W^2\left(\frac{\pi\lambda}{q_H^2}\right) + W^2\left(\frac{\pi(1-\lambda)}{q_L^2}\right)\right] \\
&\geq W^2\left(\frac{\pi\lambda}{q_H^2}\right) - \hat{W}^1\left(\frac{\pi\lambda}{q_H^1}\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
0 > T(\hat{W}^1)(\pi) - T(W^2)(\pi) \geq & \frac{1}{2}\varphi\delta\frac{2\lambda-1}{1-\lambda}\left[W^2\left(\frac{\pi\lambda}{q_H^2}\right) - \hat{W}^1\left(\frac{\pi\lambda}{q_H^1}\right)\right] \\
\geq & -\frac{1}{2}\varphi\delta\frac{2\lambda-1}{1-\lambda}d(W^2, \hat{W}^1) \geq -\varphi\delta\frac{2\lambda-1}{1-\lambda}d(W^1, W^2)
\end{aligned}$$

where the third inequality follows from $W^2\left(\frac{\pi\lambda}{q_H^2}\right) - \hat{W}^1\left(\frac{\pi\lambda}{q_H^1}\right) \geq W^2\left(\frac{\pi\lambda}{q_H^2}\right) - \hat{W}^1\left(\frac{\pi\lambda}{q_H^2}\right)$ and the fourth quality follows from $d(W^2, \hat{W}^1) = d(W^2, W^1 + d(W^1, W^2)) \geq 2d(W^1, W^2)$. When λ is close to $\frac{1}{2}$, we then have

$$|T(W^1)(\pi) - T(W^2)(\pi)| \leq |\delta\left(1 + \varphi\frac{2\lambda-1}{1-\lambda}\right)d(W^1, W^2)| \leq \tilde{\delta}_1 d(W^1, W^2)$$

for some $\tilde{\delta}_1 < 1$. Therefore, there exists $\overline{\delta} < 1$ such that Eq. (34) holds.

Case 3. Suppose at π, both W^1 and W^2 induce an equilibrium in which the influencer only randomizes at (ℓ, Y). Since $\hat{W}^2 \geq W^1$, we infer that the influencer obtains a higher payoff under $\hat{W}^2$ at $(\ell, Y), (h, N)$ and (ℓ, N) than under W^1; same for $\hat{W}^1$ and W^2. Since

$$q_H^i = \frac{1-\lambda}{\lambda} q_L^i + \frac{(2\lambda - 1)(\pi + \varphi(1-\pi))}{\lambda},$$

we either have $(q_L^1, q_H^1) \geq (q_L^2, q_H^2)$ or $(q_L^1, q_H^1) < (q_L^2, q_H^2)$. Suppose without loss of generality that $(q_L^1, q_H^1) < (q_L^2, q_H^2)$. Then we have $\frac{\pi\lambda}{q_H^2} < \frac{\pi\lambda}{q_H^1}, \frac{\pi(1-\lambda)}{q_L^2} < \frac{\pi(1-\lambda)}{q_L^1}$ and $q_H^1/q_L^1 > q_H^2/q_L^2$.

Consider $\hat{W}^1$ and W^2 first. Since $\hat{W}^1 \geq W^2$, We immediately have $V_{hY}^*(\pi, \hat{W}^1) \geq V_{hY}^*(\pi, W^2)$. The same monotonicity arguments show that $T(W^2)(\pi) \leq T(\hat{W}^1)(\pi)$.

Next consider $\hat{W}^2$ and W^1. Since $V_{\ell Y}^*(\pi, \hat{W}^2) \geq V_{\ell Y}^*(\pi, W^1)$, at least one of the following holds: (i) $\hat{W}^2\left(\frac{\pi\lambda}{q_H^2}\right) \geq W^1\left(\frac{\pi\lambda}{q_H^1}\right)$, (ii) $\hat{W}^2\left(\frac{\pi(1-\lambda)}{q_L^2}\right) \geq W^1\left(\frac{\pi(1-\lambda)}{q_L^1}\right)$. If (i) holds but (ii) doesn't, then $V_{hY}^*(\pi, \hat{W}^2) \geq V_{hY}^*(\pi, W^1)$ and thus $T(W^1)(\pi) \leq T(\hat{W}^2)(\pi)$. Suppose instead that (ii) holds, then either we still have $T(W^1)(\pi) \leq T(\hat{W}^2)(\pi)$, or $T(W^1)(\pi) > T(\hat{W}^2)(\pi)$. In the latter case, $V_{\ell Y}^*(\pi, \hat{W}^2) \geq V_{\ell Y}^*(\pi, W^1)$ implies that

$$\begin{aligned}
&(1-\lambda)\left[\hat{W}^2\left(\frac{\pi\lambda}{q_H^2}\right) - W^1\left(\frac{\pi\lambda}{q_H^1}\right) - \left[\hat{W}^2\left(\frac{\pi(1-\lambda)}{q_L^2}\right) - W^1\left(\frac{\pi(1-\lambda)}{q_L^1}\right)\right]\right] \\
\geq &-\left[\hat{W}^2\left(\frac{\pi(1-\lambda)}{q_L^2}\right) - W^1\left(\frac{\pi(1-\lambda)}{q_L^1}\right)\right] \\
\geq &-\left[\hat{W}^2\left(\frac{\pi(1-\lambda)}{q_1^2}\right) - W^1\left(\frac{\pi(1-\lambda)}{q_L^1}\right)\right] \\
\geq &-(1+\delta) d(W^1, W^2).
\end{aligned}$$

Therefore, when λ is close enough to $\frac{1}{2}$, for some $\tilde{\delta}_2 < 1$,

$$0 \geq T(W^2)(\pi) - T(W^1)(\pi) \geq -\left[\delta + \frac{1}{2}\varphi\frac{2\lambda-1}{1-\lambda}(1+\delta)\right] d(W^1, W^2) \geq \tilde{\delta}_2 d(W^1, W^2).$$

Therefore, there exists $\overline{\delta} < 1$ such that Eq. (34) holds.