

Homework1

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1 Prove $2n + \Theta(n^2) = \Theta(n^2)$

Let $f(n) \in \Theta(n^2)$, we can find c_1, c_2, n_0 such that $\forall n \geq n_0, 0 \leq c_1 n^2 \leq f(n) \leq c_2 n^2$, that is $c_1 n^2 + 2n \leq f(n) + 2n \leq c_2 n^2 + 2n$.

So we only need to prove $f(n) + 2n \in \Theta(n^2)$, that means we need to find c_3, c_4 so that $\forall n \geq n_0, 0 \leq c_3 n^2 \leq f(n) + 2n \leq c_4 n^2$.

Solve inequations: $\begin{cases} c_3 n^2 \leq c_1 n^2 + 2n \\ c_4 n^2 \geq c_2 n^2 + 2n \end{cases}$, We can find $\begin{cases} c_3 < c_1 \\ c_4 > 2/n_0 + c_2 \end{cases}$ such that $\forall n \geq n_0, 0 \leq c_3 n^2 \leq f(n) + 2n \leq c_4 n^2$.

Thus $f(n) + 2n \in \Theta(n^2)$, and $2n + \Theta(n^2) = \Theta(n^2)$.

2 Prove $\Theta(g(n)) \cap o(g(n)) = \emptyset$

Assume $\Theta(g(n)) \cap o(g(n)) \neq \emptyset$, then exists $f(n) \in \Theta(g(n))$ and $f(n) \in o(g(n))$,

so $\exists c_1 > 0, c_2 > 0, n_0 > 0$ for that $\forall n \geq n_0, \forall c > 0, \begin{cases} 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \\ 0 \leq f(n) < c g(n) \end{cases}$,

let $c = \frac{c_1}{2}$, so $c_1 g(n) \leq f(n) < c g(n) = \frac{c_1}{2} g(n), c_1 < 0$.

This is a contradiction, thus $\Theta(g(n)) \cap o(g(n)) = \emptyset$.

3 Prove $\Theta(g(n)) \cup o(g(n)) \neq O(g(n))$

Let $f(n) = \begin{cases} g(n) & \text{when } n \text{ is even} \\ 0 & \text{otherwise} \end{cases}$, $\exists c = 1, n_0 > 0, s.t. \forall n \geq n_0, 0 \leq f(n) \leq c g(n)$, so $f(n) \in O(g(n))$.

If $0 < c < 1$ and n is even number, $f(n) > c g(n)$, so $f(n) \notin o(g(n))$.

If n is not even number, $f(n)=0$, there is not a $c_1 > 0$ so that $0 \leq c_1 g(n) \leq f(n)$, so $f(n) \notin \Theta(g(n))$.

Thus $f(n) \notin \Theta(g(n)) \cup o(g(n))$. Thus $\Theta(g(n)) \cup o(g(n)) \neq O(g(n))$.

4 Prove $\max(f(n), g(n)) = \Theta(f(n) + g(n))$

To show that $\max(f(n), g(n)) = \Theta(f(n) + g(n))$, we want to find constants $c_1, c_2, n_0 > 0$ such that $0 \leq c_1(f(n) + g(n)) \leq \max(f(n), g(n)) \leq c_2(f(n) + g(n))$ for all $n \geq n_0$.

Note that $\max(f(n), g(n)) \leq f(n) + g(n)$, we can find $c_2 = 1$ such that $\max(f(n), g(n)) \leq c_2(f(n) + g(n))$.

Assume $f(n) \leq g(n)$, and let $c_1 = \frac{1}{2}$, we can find $\frac{1}{2}(f(n) + g(n)) \leq f(n) \leq g(n) = \max(f(n), g(n))$.

So when $c_1 = \frac{1}{2}, c_2 = 1$, we can find $n_0 > 0$ such that $0 \leq c_1(f(n) + g(n)) \leq \max(f(n), g(n)) \leq c_2(f(n) + g(n))$ for all $n \geq n_0$.

Thus $\max(f(n), g(n)) = \Theta(f(n) + g(n))$.

5 Solve the recurrence $T(n) = 2T(\sqrt{n}) + 1$

Let $m = \lg n$, then $T(2^m) = 2T(2^{\frac{m}{2}}) + 1$.

Let $S(m) = T(2^m)$, then $S(m) = 2S(\frac{m}{2}) + 1$.

According to the master method, $f(m) = 1 = O(m^{\log_2 2 - 1})$, so that $S(m) = \Theta(m)$.

Thus $T(n) = T(2^m) = S(m) = \Theta(m) = \Theta(\lg n)$.

6 Solve the recurrence $nT(n) = (n-2)T(n-1) + 2$

$T(n) = 1$ when $n \geq 2$.

Prove:

1. $n=2, 2T(2)=2, T(2)=1$.

2. Assume $T(k)=1, k > 2$, then $(k+1)T(k+1)=(k-1)+2$, then $T(k+1)=1$.

Thus $T(n) = 1 = \Theta(1)$.

7 CLRS, pp61, 3-3