Homework1

Hao Cui

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1 Prove $2n + \Theta(n^2) = \Theta(n^2)$

Let $f(n) \in \Theta(n^2)$, we can find c_1, c_2, n_0 such that $\forall n \geq n_0, 0 \leq c_1n^2 \leq f(n) \leq c_2n^2$, that is $c_1n^2 + 2n \leq f(n) + 2n \leq c_2n^2 + 2n$.
So we only need to prove $f(n) + 2n \in \Theta(n^2)$, that means we need to find c_3, c_4 so that $\forall n \geq n_0, 0 \leq c_3n^2 \leq f(n) + 2n \leq c_4n^2$.
Solve inequations:$\begin{cases} c_3n^2 \leq c_1n^2 + 2n \\ c_4n^2 \geq c_2n^2 + 2n \end{cases}$, We can find $\begin{cases} c_3 < c_1 \\ c_4 > 2/n_0 + c_2 \end{cases}$ such that $\forall n \geq n_0, 0 \leq c_3n^2 \leq f(n) + 2n \leq c_4n^2$.
Thus $f(n) + 2n \in \Theta(n^2)$, and $2n + \Theta(n^2) = \Theta(n^2)$.

2 Prove $\Theta(g(n)) \cap o(g(n)) = \emptyset$

Assume $\Theta(g(n)) \cap o(g(n)) \neq \emptyset$, then exists $f(n) \in \Theta(g(n))$ and $f(n) \in o(g(n))$, so $\exists c_1 > 0, c_2 > 0, n_0 > 0$ for that $\forall n \geq n_0, \forall c > 0,\begin{cases} 0 \leq c_1g(n) \leq f(n) \leq c_2g(n) \\ 0 \leq f(n) < cg(n) \end{cases}$, let c = $\frac{c_1}{2}$, so $c_1g(n) \leq f(n) < cg(n) = \frac{c_1}{2}g(n), c_1 < 0$.
This is a contradiction, thus $\Theta(g(n)) \cap o(g(n)) = \emptyset$.

3 Prove $\Theta(g(n)) \cup o(g(n)) \neq O(g(n))$

Let $f(n) = \begin{cases} g(n) \ when \ n \ is \ even \\ 0 \qquad otherwise \end{cases}$, $\exists c = 1, n_0 > 0, s.t. \forall n \geq n_0, 0 \leq f(n) \leq cg(n)$,so $f(n) \in O(g(n))$.
If $0 < c < 1$ and n is even number, $f(n) > cg(n)$, so $f(n) \notin o(g(n))$.
If n is not even number, f(n)=0, there is not a $c_1 > 0$ so that $0 \leq c_1g(n) \leq f(n)$, so $f(n) \notin \Theta(g(n))$.
Thus $f(n) \notin \Theta(g(n)) \cup o(g(n))$. Thus $\Theta(g(n)) \cup o(g(n)) \neq O(g(n))$.

4 Prove $max(f(n), g(n)) = \Theta(f(n) + g(n))$

To show that $max(f(n), g(n)) = \Theta(f(n) + g(n))$, we want to find constants $c_1, c_2, n_0 > 0$ such that $0 \leq c_1(f(n) + g(n)) \leq max(f(n), g(n)) \leq c_2(f(n) + g(n))$ for all $n \geq n_0$.
Note that $max(f(n), g(n)) \leq f(n) + g(n)$, we can find $c_2 = 1$ such that $max(f(n), g(n)) \leq c_2(f(n) + g(n))$.
Assume $f(n) \leq g(n)$, and let $c_1 = \frac{1}{2}$, we can find $\frac{1}{2}(f(n) + g(n)) \leq f(n) \leq g(n) = max(f(n), g(n))$.
So when $c_1 = \frac{1}{2}, c_2 = 1$, we can find $n_0 > 0$ such that $0 \leq c_1(f(n)+g(n)) \leq max(f(n), g(n)) \leq c_2(f(n)+g(n))$ for all $n \geq n_0$.
Thus $max(f(n), g(n)) = \Theta(f(n) + g(n))$.

5 Solve the recurrence $T(n) = 2T(\sqrt{n}) + 1$

Let $m = lgn$, then $T(2^m) = 2T(2^{\frac{m}{2}}) + 1$.
Let $S(m) = T(2^m)$, then $S(m) = 2S(\frac{m}{2}) + 1$.

According to the master method, $f(m) = 1 = O(m^{\log_2 2-1})$, so that $S(m) = \Theta(m)$.
Thus $T(n) = T(2^m) = S(m) = \Theta(m) = \Theta(lgn)$.

6 Solve the recurrence $nT(n) = (n-2)T(n-1) + 2$

$T(n) = 1$ when $n \geq 2$.
Prove:
1. n=2,2T(2)=2, T(2)=1.
2. Assume T(k)=1,$k > 2$, then (k+1)T(k+1)=(k-1)+2, then T(k+1)=1.

Thus $T(n) = 1 = \Theta(1)$.

7 CLRS,pp61,3-3

7.1 Rank the following functions by order of growth

Functions on the same line are in the same equivalence class.

$$
\begin{gathered}
2^{2^{n+1}} \\
2^{2^n} \\
(n+1)! \\
n! \\
e^n \\
n \cdot 2^n \\
2^n \\
(\tfrac{3}{2})^n \\
(lgn)^{lgn}, n^{lglgn} \\
(lgn)! \\
n^3 \\
n^2, 4^{lgn} \\
nlgn, lg(n!) \\
n, 2^{lgn} \\
\sqrt{2}^{lgn} \\
2^{\sqrt{2lgn}} \\
lg^2 n \\
lnn \\
\sqrt{lgn} \\
lnlnn \\
2^{lg^* n} \\
lg^* n, lg^*(lgn) \\
lg(lg^*)n \\
n^{\frac{1}{lgn}}, 1
\end{gathered}
$$

7.2 F(n) is neither $O(g_i(n))$ nor $\Theta(g_i(n))$

$$
f(n) = \begin{cases} 2^{2^{2^{n+1}}} & if\ n\ is\ even \\ 0 & if\ n\ is\ odd \end{cases}
$$