RESEARCH STATEMENT

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My research is in the area of representation theory, particularly the Frobenius-Schur indicator and the Real-Quaternionic indicator for representations of real reductive groups as well as p-adic groups. I have also done research on the local Langlands correspondence of $GL(2, F)$, see Section 4 for more detail.

The Frobenius-Schur indicator (ε-indicator) is defined for those irreducible representations who are self-dual, and the Real-Quaternionic indicator (δ-indicator) for those which are self-conjugate. These indicators are natural invariants for irreducible representations. Their definitions are straightforward, but to compute them is another story. In fact, we don't know how to compute ε and δ in general. My research is driven by the belief that there should be a simple formula for the two indicators in general. I'm also motivated by the numerous connections that the δ-indicator has to quantum physics, see for example: [10], [3].

Much research has been done about how to obtain these two indicators, for example [7], [11], [2], [6], [9]. In [2], it is proved that the Frobenius-Schur indicator of finite-dimensional representation π has an expression in terms of the central character of π. My main contribution is that I found a relation between ε and δ. They are related by the strong real form x:

$$\delta(\pi) = \varepsilon(\pi)\chi_\pi(x^2)$$

in the case where G is real reductive, π is finite-dimensional and Hermitian. A similar formula holds when π is non-Hermitian. I am currently working on proving a relation between ε and δ when π is an infinite dimensional $(\mathfrak{g}, K)$-module.

1. Background

Definition 1.1 (Frobenius-Schur indicator)**.** Let (π, V) be a self-dual irreducible representation of a group G. Then there exists $B : V \times V \to \mathbb{C}$, a G invariant bilinear form. Define

$$\varepsilon(\pi) = \begin{cases} 1 & B \text{ symmetric} \\ -1 & B \text{ skew-symmetric} \end{cases}$$

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Theorem 1.1. [2] *Let $G(\mathbb{C})$ be a connected reductive complex Lie group, and let π be an irreducible self-dual finite-dimensional representation of $G(\mathbb{C})$. Then*

$$\varepsilon(\pi) = \chi_\pi(z_\rho)$$

where χ_π is the central character and $z_\rho = \exp(2\pi i\rho^\vee)$.

Definition 1.2 (Real-Quaternionic indicator)**.** Let (π, V) be a self-conjugate irreducible representation of a real reductive group G. Define:

$$\delta(\pi) = \begin{cases} 1 & \pi \cong \pi_0 \otimes_{\mathbb{R}} \mathbb{C}, \quad \pi_0 \text{ is an irreducible real representation} \\ -1 & \pi \text{ has a } G \text{ invariant } \mathbb{H} \text{ structure} \end{cases}$$

2. Relating ε and δ

In order to relate ε to δ, it is helpful to identify the differences between the two:

- ε is defined when $\pi \cong \pi^*$, δ is defined when $\pi \cong \overline{\pi}$.
- The formula for ε is independent of the real form. In other words, let G be any real form of $G(\mathbb{C})$, then the same theorem holds if replace $G(\mathbb{C})$ by G. On the other hand, δ is sensitive to real form.

The differences suggest the following possible connections:

- π^* and $\overline{\pi}$ are related by the Hermitian dual π^h of π. I.e., $\overline{\pi} \cong (\pi^*)^h$ (1)
- ε and δ can be related by a factor given by the real form.

The following is the main theorem:

Theorem 2.1 (R. Cui)**.** *Let G be a real reductive Lie group, (π, V) be an irreducible finite-dimensional representation of G, then*

(1) *$\delta(\pi) = \varepsilon(\pi)\chi_\pi(x^2)$ if π is Hermitian*
(2) *$\delta(\pi) = \varepsilon(\widetilde{\pi})\chi_\pi(x^2)$ if π is non-Hermitian.*

where $\widetilde{\pi} = Ind_G^{\gamma G}\pi$, ${}^\gamma G$ is an extended group of G. x is the "strong real form", described in [1]*, $x^2 \in Z(G)$, and $Ad(x) = \theta$, which is the Cartan involution.*

3. Sketch of Proof

The main tool I used in the proof this theorem is the c-invariant Hermitian form developed by Adams, van Leeuwen, Trapa, and Vogan in [1].

I will now sketch the proof where G is equal rank, and π is self-conjugate and finite-dimensional.

The working definition for δ is the following:

Definition 3.1. Let $(\pi, V) \cong (\overline{\pi}, \overline{V})$ be an irreducible representation of a real reductive group G. Then there exists $\mathcal{J} : V \to V$ a conjugate linear, G-invariant map. Such maps satisfy $\mathcal{J}^2 = c \cdot I$ for some $c \in \mathbb{R}^*$. Define

$$\delta(\pi) = sgn(c)$$

This definition is equivalent to Definition 1.2.

To prove Theorem 2.1, we will construct such a map $\mathcal{J}$ and compute the sign of c.

The group G being equal rank implies π is Hermitian, i.e., $\pi \cong \pi^h$. By Equation (1), $\pi \cong \pi^*$. Therefore we have an invariant Hermitian form $\langle , \rangle^0$ and an invariant bilinear form B. Define $\mathcal{J} : V \to V$ such that

$$B(v, w) = \langle v, \mathcal{J}(w)\rangle^0$$

The c-invariant Hermitian form is a Hermitian form $\langle , \rangle^c$ which satisfies:

$$\langle g \cdot v, w\rangle^c = \langle v, \sigma_c(g) \cdot w\rangle^c \quad \forall g \in G$$

where σ_c is the conjugate linear involution given by the compact real form. The forms $\langle , \rangle^0$ and $\langle , \rangle^c$ have the following relation:

Proposition 3.1. [1]

$$\langle v, w\rangle^0 := \mu^{-1}\langle x \cdot v, w\rangle^c = \mu\langle v, x \cdot w\rangle^c$$

where x is the "strong real form", i.e., $Ad(x) = \theta$, which is the Cartan involution, $x^2 \in Z(G)$, and μ is a square root of $\chi_\pi(x^2)$.

By the definition of $\mathcal{J}$ and a short calculation we have:

$$\varepsilon(\pi)\delta(\pi) = sgn\left(\frac{\langle \mathcal{J}(v), \mathcal{J}(w)\rangle^0}{\overline{\langle v, w\rangle^0}}\right) = sgn\left(\mu^{-2}\frac{\langle x \cdot \mathcal{J}(v), \mathcal{J}(w)\rangle^c}{\overline{\langle x \cdot v, w\rangle^c}}\right)$$

Set $w = x \cdot v$, then the right hand side equals to $sgn(\mu^{-2}) = \chi_\pi(x^2) \Rightarrow \varepsilon(\pi)\delta(\pi) = \chi_\pi(x^2)$.

Remark. The formula and the proof very clearly illustrated the difference/connection between the two indicators. The dual and conjugate are related by the Hermitian dual, and the ordinary Hermitian dual relates to the c-Hermitian dual by a factor from the real form. It is not surprising that in the end, ε and δ are related by the real form.

4. Local Langlands Correspondence of $GL(2, F)$

For $GL(2, F)$ where F is a non-Archimedean local field with characteristics 0, the Langlands conjecture has been proven. The construction of the correspondence used the famous theta correspondence, see [4] for the proof. My contribution is providing new proofs to several properties of the correspondence, in particular that the Langlands correspondence is independent of the additive character introduced by the theta correspondence. See the proof of Theorem 6 in [5].

5. Future Research

I am currently working on calculating the δ indicator for infinite-dimensional $(\mathfrak{g}, K)$ modules π of real reductive groups. I conjecture that the same or a similar formula will hold for $\delta(\pi)$.

It is also my interest to investigate the ε-indicator for self-dual representations of p-adic groups. This subject has been studied by Prasad [7] and other researchers. Formula for $\varepsilon(\pi)$ has been given in [7] for self-dual generic representations of quasi-split reductive group over p-adic fields.

I would like to find an analog to the Real-Quaternionic indicator in the p-adic setting and investigate the relation between that and the ε-indicator.

I would also like to study how the indicators behave under different correspondences. For example, suppose the Langlands correspondence gives $\rho \leftrightarrow \pi$ where ρ is a Weil-Deligne representation, and π is a representation of GL_n on a local field. If π is self-dual (or self-conjugate), what does it say about the self-duality (or self-conjugacy) of ρ? How does $\varepsilon(\pi)$ (or $\delta(\pi)$) relate to $\varepsilon(\rho)$ (or $\delta(\rho)$)?

There is the notion of real type and quaternionic type for irreducible representations of real C^* algebras defined by Rosenberg [8]. I am interested in the determining of the types for such representations.

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