The Real-Quaternionic Indicator for Finite-Dimensional Self-conjugate Representations of Real Reductive Lie Groups

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Abstract

The real-quaternionic indicator, also called the δ indicator, indicates if a self-conjugate representation is of real or quaternionic type. It is closely connected to the Frobenius-Schur indicator, which we call the ε indicator. It is interesting to compute the ε and δ indicators. The computation of the ε indicator is relatively straightforward. In fact, it has been proven in large generality that $\varepsilon(\pi)$ is given by a particular value of the central character. We would like a similar result for the δ indicator.

When G is compact, $\delta(\pi)$ and $\varepsilon(\pi)$ coincide. In general, they are not necessarily the same. In this paper, we will give a relation between the two indicators when G is real reductive. We will also give a formula for $\delta(\pi)$ in terms of the central character when π is finite dimensional.

1 Introduction

Let G be a real reductive Lie group, meaning a group of real points for a complex connected reductive algebraic group, and (π, V) an irreducible finite dimensional representation of G. We view π as a $(\mathfrak{g}, K)$ module where $\mathfrak{g}$ is the real Lie algebra of G and K is a maximal compact subgroup of G.

The Real-Quaternionic indicator, referred to as the δ-indicator, of π is non-trivially defined when π is isomorphic to its conjugate $\overline{\pi}$. The δ-indicator is similar to the ε-indicator, sometimes called the Orthogonal-Symplectic indicator. The ε-indicator is non-trivially defined when $\pi \cong \pi^*$. When the group G is compact, $\delta = \epsilon$. However in general, $\delta \neq \varepsilon$. Around 1960s, Iwahori, J. Tits, etc. have done related work on calculating both indicators. In 1980s, Onishchik and Vinberg have listed formulas for the Real-Quaternionic indicators for different types of Lie groups in terms of the fundamental representations.

We give relation between the two indicators. The Orthogonal-Symplectic indicator is much easier to compute in general. So with this relation, we are able to write down an uniform formula for the Real-Quaternionic indicator.

2 Basic Definitions

In this section, we are going to precisely define all the important notions in this paper.

Throughout this section, (π, V) will denote a finite dimensional representation of the pair $(\mathfrak{g}, K)$ unless otherwise specified.

Definition 2.1. Let

$$V^* = \{f : V \to \mathbb{C} | f \text{ is linear }\}$$

be the set of linear functionals on V. We call (π^*, V^*) the *dual* or the *contragredient* representation of (π, V).

Definition 2.2. Let

$$V^h = \{\xi : V \to \mathbb{C} | \xi \text{ is conjugate linear }\}$$

be the set of conjugate linear functionals on V. We call (π^h, V^h) the *Hermitian dual* of (π, V) where $\pi^h(g) \cdot \xi(v) = \xi(\pi(g^{-1}) \cdot v)$.

Definition 2.3. Let

$$\overline{V} = \{F : V^h \to \mathbb{C} | F \text{ is linear }\}$$

be the set of linear functionals on V^h. We call $(\bar{\pi}, \overline{V})$ the *conjugate* representation of (π, V).

Remark. It is easy to see that $\bar{\pi} := (\pi^h)^* \cong (\pi^*)^h$. We claim that any two will compose into the third.

Let's show $\pi^* \cong \overline{\pi^h}$: suppose $(\pi^h)^* \cong (\pi^*)^h$. Take the Hermitian dual of both sides: $((\pi^h)^*)^h \cong ((\pi^*)^h)^h \cong \pi^*$, i.e., $(\bar{\pi})^h \cong \pi^*$ or $\overline{(\pi^h)} \cong \pi^*$. We leave the elementary proof of the rest of the claim to the reader.

Definition 2.4 (Alternative definition for conjugate representation)**.** Set

$$\overline{V} = \mathbb{C} \otimes_{\mathbb{C}} V$$

where $\mathbb{C}$ acts on $\mathbb{C}$ by: $w \cdot z = \bar{w}z$. The representation $(\bar{\pi}, \overline{V})$ is called the *conjugate* representation of (π, V); the actions are given by:

$$\bar{\pi}(X) \cdot (z \otimes v) = z \otimes \pi(X)v, \quad \forall X \in \mathfrak{g}$$
$$\bar{\pi}(k) \cdot (z \otimes v) = z \otimes \pi(k)v, \quad \forall k \in K$$

The proof that the two definitions are equivalent is elementary, mainly because we can identify the vector spaces V and $(V^*)^h$.

2.1 Self-dual representations

Assuming π is self-dual, there exists an intertwining operator

$$\eta : V \to V^*$$

such that the following diagram commute:

$$\begin{tikzcd} V \arrow[r, "\eta"] \arrow[d, "\pi(y)"'] & V^* \arrow[d, "\pi^*(y)"] \\ V \arrow[r, "\eta"] & V^* \end{tikzcd}$$

i.e.,

$$\eta(\pi(y)v) = \pi^*(y)(\eta(v)) \quad \forall y \in \mathfrak{g} \text{ or } K$$

Define a bilinear form B on V by:

$$B(v, w) := \eta(v)(w)$$

It's easy to see that B is invariant under the action of both $\mathfrak{g}$ and K:

$$\begin{aligned} B(\pi(X)v, w) &= \eta(\pi(X)v)(w) = [\pi^*(X)\eta(v)](w) \\ &= \eta(v)(-\pi(X)w) = -B(v, \pi(X)w \quad \forall X \in \mathfrak{g} \\ B(\pi(k)v, w) &= \eta(\pi(k)v)(w) = [\pi^*(k)\eta(v)](w) \\ &= \eta(v)(\pi(k^{-1})w) = B(v, \pi(k^{-1})w) \quad \forall k \in K \end{aligned}$$

The following discussion will show that this bilinear form is either symmetric or skew-symmetric.

The map η induces an isomorphism:

$$\begin{aligned} \eta^* : V^{**} &\to V^* \\ F &\mapsto F \circ \eta \end{aligned}$$

which then induces another isomorphism

$$\begin{aligned} \xi(\eta) = \eta^* \circ \iota : V &\to V^* \\ v &\mapsto F_v \circ \eta \end{aligned}$$

where $\iota : V \to V^{**}$ is the canonical isomorphism sends v to F_v such that $F_v(f) = f(v)$, $\forall f \in V^*$. It is not hard to show that $\xi(\eta)$ is an isomorphism between π and π^*:

$$\pi^*(k)[\xi(\eta)(v)](w) = [\xi(\eta)(\pi(k)v)](w)$$

$$\begin{aligned} LHS &= \xi(\eta)(v)(\pi(k^{-1})w) = [\eta^* \circ \iota(v)](\pi(k^{-1})w) = F_v \circ \eta(\pi(k^{-1})w) \\ &= \eta(\pi(k^{-1})w)(v) \\ RHS &= [F_{\pi(k)v} \circ \eta](w) = \eta(w)(\pi(k)v) = [\pi^*(k^{-1})\eta(w)](v) = \eta(\pi(k^{-1})w)(v) \\ &= LHS \end{aligned}$$

the verification for $X \in \mathfrak{g}$ is the same. By Schur's lemma, $\xi(\eta) = c\eta$ for some constant c. Moreover, we can show $\xi(\xi(\eta)) = \eta$:

$$\begin{aligned} [\xi(\xi(\eta))(v)](w) &= [\xi(\eta)^*(F_v)](w) = [F_v \circ \xi(\eta)](w) = [\xi(\eta)(w)](v) \\ &= [\eta^*(F_w)](v) = F_w(\eta(v)) = [\eta(v)](w) \end{aligned}$$

therefore $\xi(\xi(\eta)) = \xi(c\eta) = c^2\eta \Rightarrow c^2 = 1$. $\Rightarrow \xi(\eta)(v)(w) = \eta^*(F_v)(w) = F_v \circ \eta(w) = F_v(\eta(w)) = \eta(w)(v) \Rightarrow \eta(w)(v) = \pm\eta(v)(w) \Rightarrow B(w, v) = \pm B(v, w)$. It is easy to see that if we start with any other isomorphism $z\eta$ between π and π^*, we will get the same multiplier c. Therefore c is an invariant.

Definition 2.5. Suppose (π, V) is an irreducible self-dual representation of the pair $(\mathfrak{g}, K)$, then we can associate to V an invariant bilinear form $B : V \times V \to \mathbb{C}$. We define $\varepsilon(\pi)$, the *ε-indicator* of π, as

$$\varepsilon(\pi) = \begin{cases} 1 & B \text{ is symmetric} \\ -1 & B \text{ is skew-symmetric} \end{cases}$$

2.2 Self-conjugate representations

Assume representation (π, V) is self-conjugate. We will use Definition 2.4. Let $\eta : V \to \overline{V}$ be an isomorphism, η satisfies

$$\bar{\pi}(k) \cdot \eta(v) = \eta(\pi(k) \cdot v)$$

Let $\iota : \overline{V} \to V$ be the canonical identification:

$$\iota(z \otimes v) = zv$$

so ι is a conjugate linear isomorphism between vector spaces. Let

$$\mathcal{J} = \iota \circ \eta : V \to V$$

This map is $(\mathfrak{g}, K)$-invariant and conjugate linear. Both properties are easy to verify, we omit the proof. Let's consider $\mathcal{J}^2$, a linear $(\mathfrak{g}, K)$-invariant map between (π, V) and (π, V). By Schur's lemma we have $\mathcal{J}^2 = cI$ for $c \in \mathbb{C}^*$. It turns out $c \in \mathbb{R}^*$ because of the following:

$$\begin{aligned}\mathcal{J}^3(v) &= \mathcal{J}(\mathcal{J}^2(v)) = \mathcal{J}(cv) = \bar{c}\mathcal{J}(v) \\ &= \mathcal{J}^2(\mathcal{J}(v)) = c\mathcal{J}(v)\end{aligned}$$

therefore $c = \bar{c} \Rightarrow c \in \mathbb{R}$. We can now define the δ-indicator of self-conjugate representations.

Definition 2.6. Let (π, V) be an irreducible self-conjugate representation of the pair $(\mathfrak{g}, K)$. Then their exists a non-zero conjugate linear $(\mathfrak{g}, K)$ invariant map $\mathcal{J}$ from V to V such that $\mathcal{J}^2 = cI$, $c \in \mathbb{R}^*$. We define the δ-indicator for π as follows:

$$\delta(\pi) = \begin{cases} 1 & \text{if } c > 0 \\ -1 & \text{if } c < 0 \end{cases}$$

Remark. There are many more equivalent definitions for the δ indicator. In this paper, we will only work with this one.

Theorem 2.1. *Suppose (π, V) is irreducible, unitary and self-dual. The δ-indicator exists and $\delta(\pi) = \varepsilon(\pi)$.*

Proof. There exists a positive definite invariant Hermitian form on V, denoted $\langle , \rangle$. Because π is self-dual, there also exists an invariant bilinear form on V, denoted $(,)$.

Define map $\mathcal{J} : V \to V$ by $(v, w) = \langle v, \mathcal{J}(w) \rangle$. It is easy to see that $\mathcal{J}$ is conjugate linear, $(\mathfrak{g}, K)$-invariant, we leave the proof to the reader. From Section 2.2, we know that $\mathcal{J}^2 = cI$, some $c \in \mathbb{R}^*$. The constant c also satisfies:

$$\varepsilon(\pi)c = \frac{\langle \mathcal{J}(v), \mathcal{J}(w) \rangle}{\overline{\langle v, w \rangle}} \quad \text{if } \langle v, w \rangle \neq 0 \tag{1}$$

this is because:

$$\begin{aligned}\langle \mathcal{J}(v), \mathcal{J}(w) \rangle &= (\mathcal{J}(v), w) = \epsilon(\pi)(w, \mathcal{J}(v)) \\ &= \epsilon(\pi)\langle w, \mathcal{J}^2(v) \rangle \\ &= \epsilon(\pi)c\langle w, v \rangle \\ &= \epsilon(\pi)c\overline{\langle v, w \rangle}\end{aligned}$$

set $w = v$, we have

$$\varepsilon(\pi)c = \frac{\langle \mathcal{J}(v), \mathcal{J}(v)\rangle}{\langle v, v\rangle}$$

Because $\langle , \rangle$ is positive definite, $sign(\varepsilon(\pi)c) = 1$. Therefore $\delta(\pi) = sign(c) = \varepsilon(\pi)$. $\square$

In general, we won't have a unitary representation nor the positive definite invariant Hermitian form it carries. Instead, we have the c-Hermitian form which can be chosen to be positive definite. The c-Hermitian form and the ordinary Hermitian form are related by Equation 3.3 which we are going to talk about in the next section. We will have a c-invariant form version of Equation 1 in Lemma 4.1. Using the basic idea of the proof above, we will produce a relation between ε-indicator and δ-indicator in general. Essentially, the difference between ε-indicator and δ-indicator is reflected in Equation 3.3.

3 Hermitian Forms

The key of finding the relation between the two indicators is the study of Hermitian forms associated to different real forms.

3.1 Complexifications

In order to work on different real forms, we need to consider $(\mathfrak{g}(\mathbb{C}), K(\mathbb{C}))$-modules, where $\mathfrak{g}(\mathbb{C})$ is the complexification of $\mathfrak{g}$ and $K(\mathbb{C})$ is the complexification of compact group K.

As an attempt to be self-contained, we will have a short discussion about these concepts. For more detail, see [3], [4], and [2].

It is well known that a complex representation of a real Lie algebra $\mathfrak{g}$ amounts to a complex representation of the complexification

$$\mathfrak{g}(\mathbb{C}) := \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$$

It is also true that a representation (π, V) of a compact group K can be complexified to be a representation of $K(\mathbb{C})$ on the same vector space.

Definition 3.1. Set

$$R(K, \mathbb{C}) = \mathrm{Span}_{\mathbb{C}}\{\text{cx valued matrix coefficients of F.D. representations of } K\}$$

Let $K(\mathbb{C})$ be the set of all $\mathbb{C}$-algebra homomorphisms:

$$R(K, \mathbb{C}) \to \mathbb{C}$$

We call this algebraic group the *complexification* of K.

Definition 3.2. Let (π, V) be a representation of K. After picking basis of V, we can write is as:

$$\pi : k \mapsto (\pi_{ij}(k))$$

Define representation $(\pi(\mathbb{C}), V)$ of $K_{\mathbb{C}}$ as:

$$\pi(C) : s \mapsto (s(\pi_{ij}))$$

We call this the complexification of (π, V).

It is clear that the following diagram commutes:

$$\begin{tikzcd} K \arrow[rr, "\iota"] \arrow[dr, "\pi"'] & & K(\mathbb{C}) \arrow[dl, "\pi(\mathbb{C})"] \\ & GL(n,\mathbb{C}) & \end{tikzcd}$$

Theorem 3.1. *[2, Corollary 2.18] Every locally finite continuous representation π of K extends uniquely to an algebraic representation $\pi(\mathbb{C})$ of $K(\mathbb{C})$ on the same space; and every algebraic representation of $K(\mathbb{C})$ restricts to a locally finite continuous representation of K. This extension defines an equivalence of categories from $(\mathfrak{g}, K)$-modules to $(\mathfrak{g}(\mathbb{C}), K(\mathbb{C}))$-modules.*

3.2 σ-invariant Hermitian forms

In this subsection, we will define different Hermitian forms on a $(\mathfrak{g}(\mathbb{C}), K(\mathbb{C}))$-module.

Definition 3.3. [2, P44] A *real structure* on the pair $(\mathfrak{g}(\mathbb{C}), K(\mathbb{C}))$ consists two maps:

$$\sigma : \mathfrak{g}(\mathbb{C}) \to \mathfrak{g}(\mathbb{C}), \sigma : K(\mathbb{C}) \to K(\mathbb{C})$$

where $\sigma^2 = Id$, σ on $\mathfrak{g}(\mathbb{C})$ is a conjugate linear Lie algebra homomorphism; and on $K(\mathbb{C})$ is antiholomorphic preserving regular functions on $K(\mathbb{C})$. The two maps are required to be compatible.

Definition 3.4. [2, P47] The *σ-Hermitian dual of* π is the representation $(\pi^{h,\sigma}, V^{h,\sigma})$ on vector space $V^{h,\sigma} := V^h$ of the pair $(\mathfrak{g}(\mathbb{C}), K(\mathbb{C}))$ where the action is as follows:

$$\pi^{h,\sigma}(X) \cdot f(v) = f(-\sigma(X) \cdot v) \quad \forall f \in V^h, v \in V$$
$$\pi^{h,\sigma}(k) \cdot f(v) = f(\sigma(k^{-1}) \cdot v) \quad \forall f \in V^h, v \in V$$

Definition 3.5. [2, Definition 8.6] Suppose $(\mathfrak{g}(\mathbb{C}), K(\mathbb{C}))$ is a pair with a real structure σ, and V is an $(\mathfrak{g}(\mathbb{C}), K(\mathbb{C}))$-module. A *$\sigma$-invariant Hermitian form* on V is a Hermitian pairing $\langle , \rangle$ on V, satisfying

$$\langle X \cdot v, w\rangle = \langle v, -\sigma(X) \cdot w\rangle \ \ \forall X \in \mathfrak{g}(\mathbb{C})$$

and

$$\langle k \cdot v, w\rangle = \langle v, \sigma(k^{-1}) \cdot w\rangle \ \ \forall k \in K(\mathbb{C})$$

Remark. [2, P48] Such a form may be identified with an intertwining operator $T \in Hom_{\mathfrak{g}(\mathbb{C}),K(\mathbb{C})}(V, V^{h,\sigma})$. Satisfying the Hermitian condition: $T^h = T$.

Let σ_0 be a real structure such that

$$\mathfrak{g}(\mathbb{C})^{\sigma_0} = \mathfrak{g}, K(\mathbb{C})^{\sigma_0} = K$$

It is a well known fact that there exists a compact real structure σ_c on the pair $(\mathfrak{g}(\mathbb{C}), K(\mathbb{C}))$ such that σ_c and σ_0 commute and

$$\mathfrak{g}(\mathbb{C})^{\sigma_c} = \mathfrak{g}_c, K(\mathbb{C})^{\sigma_c} = K$$

Moreover, the composition $\sigma_0 \circ \sigma_c =: \theta$ is the associated Cartan involution of σ_0. Note that θ is holomorphic on $K(\mathbb{C})$.

Theorem 3.2. *1. For (π, V) a finite dimensional representation of the pair $(\mathfrak{g}(\mathbb{C}), K(\mathbb{C}))$, there always exists a non-degenerate σ_c-Hermitian form.*

2. There is a non-degenerate positive definite σ_c invariant form on finite dimensional representation (π, V).

Remark. The 1st part of the Theorem is true because the infinitesimal character of a finite dimensional representation is real and [2, Proposition 10.7] The 2nd part is essentially based on the fact that every finite dimensional representations of a compact group is unitary.

From now on, we fix real structure σ_0 and compact real structure σ_c (thus the Cartan involution θ). We are going to establish the relation between a σ_c invariant form and a σ_0 invariant form. Note that all these can be found in [2] with more generality, here we only take what we need.

Definition 3.6. Let (π, V) be a finite dimensional representation of the pair $(\mathfrak{g}(\mathbb{C}), K(\mathbb{C}))$. Then the *twist of V by θ* is the representation (π^θ, V^θ) on the same vector space $V^\theta := V$ defined by

$$\pi^\theta(X) \cdot v = \pi(\theta(X)) \cdot v, \quad \pi^\theta(k) \cdot v = \pi(\theta(k)) \cdot v \quad \forall X \in \mathfrak{g}(\mathbb{C}), k \in K(\mathbb{C})$$

Intuitively, the equation $\sigma_c \circ \sigma_0 = \theta$ indicates the difference between σ_0 and σ_c should be captured by θ. It turns out this can be made precise:

Theorem 3.3. *Suppose $D : V \to V^\theta$ is an isomorphism of the two representations, and $D^2(v) = \pi(\lambda) \cdot v$ for $\lambda \in K$. Let $\langle , \rangle^{\sigma_c}$ be a non-degenerate positive definite σ_c invariant Hermitian form on V. We define $\langle , \rangle^{\sigma_0} : V \times V \to \mathbb{C}$ as follows:*

$$\langle v, w \rangle^{\sigma_0} := \mu^{-1} \langle Dv, w \rangle^{\sigma_c} = \mu \langle v, Dw \rangle^{\sigma_c} \tag{2}$$

where μ is a square root of $\omega(D, D)\overline{\xi(D, D)}$, ω is defined in the following equation:

$$\langle Dv, Dw \rangle^{\sigma_c} = \omega(D, D) \langle v, w \rangle^{\sigma_c} \tag{3}$$

ξ is defined as in:

$$D^{-1} = \xi(D, D) D$$

The form $\langle , \rangle^{\sigma_0}$ is a non-degenerate σ_0-invariant Hermitian form on V.

Proof. The intertwiner D makes the following diagram commute:

$$\begin{tikzcd}
(\pi, V) \arrow[r, "D"] \arrow[d, "\pi(x)"'] & (\pi^\theta, V^\theta) \arrow[d, "\pi^\theta(x)"] \\
(\pi, V) \arrow[r, "D"] & (\pi^\theta, V^\theta)
\end{tikzcd}$$

i.e.

$$x \cdot_\theta D(v) = D(x \cdot v) \quad \forall x \in K(\mathbb{C}) \text{ or } \mathfrak{g}(\mathbb{C})$$

The map D^{-1} is also an intertwiner: apply D^{-1} to both sides and change v into $D^{-1}(v)$ and x into $\theta(x)$ we have:

$$D^{-1}(x \cdot v) = \theta(x) \cdot D^{-1}(v) = x \cdot_\theta D^{-1}(v) \quad \forall x \in K(\mathbb{C}) \text{ or } \mathfrak{g}(\mathbb{C})$$

First, let's verify that

$$\langle Dv, Dw\rangle^{\sigma_c} = \omega(D,D)\langle v,w\rangle^{\sigma_c}$$

is a valid definition of ω. It's enough to show that the form $\langle,\rangle_D^{\sigma_c} := \langle Dv, Dw\rangle^{\sigma_c}$ is a σ_c-invariant sesquilinear pairing. The sesquilinear claim is obvious, the σ_c-invariant is not hard to show either:

$$\begin{aligned}\langle k\cdot v, w\rangle_D^{\sigma_c} &= \langle D(k\cdot v), Dw\rangle^{\sigma_c} = \langle \theta(k)\cdot D(v), Dw\rangle^{\sigma_c}\\ &= \langle D(v), \sigma_c(\theta(k^{-1}))\cdot Dw\rangle^{\sigma_c} = \langle D(v), \theta(\sigma_c(k^{-1}))\cdot D(w)\rangle^{\sigma_c}\\ &= \langle D(v), D(\sigma_c(k^{-1})\cdot w)\rangle^{\sigma_c}\\ &= \langle v, \sigma_c(k^{-1})\cdot w\rangle_D^{\sigma_c} \quad \forall k\in K(\mathbb{C})\end{aligned}$$

similarly for $X\in\mathfrak{g}(\mathbb{C})$.

By [2, Proposition 8.5(j)], we know that $\langle,\rangle_D^{\sigma_c}$ and $\langle,\rangle^{\sigma_c}$ differ by a complex number, call it $\omega(D,D)$.

Now let's analyze the form $\langle v,w\rangle^{\sigma_0} := \mu^{-1}\langle D(v),w\rangle^{\sigma_c}$. It is obvious that this form is sesquilinear. It is also Hermitian:

$$\begin{aligned}\langle v,w\rangle^{\sigma_0} &:= \mu^{-1}\langle D(v),w\rangle^{\sigma_c} = \mu^{-1}\omega(D,D^{-1})\langle v, D(w)\rangle^{\sigma_c}\\ &= \mu\overline{\langle D(w),v\rangle^{\sigma_c}} = \overline{\mu^{-1}\langle D(w),v\rangle^{\sigma_c}} = \overline{\langle w,v\rangle^{\sigma_0}}\end{aligned}$$

Here we used the fact that $\overline{\mu} = \mu^{-1}$ which is a corollary of Lemma 3.4.

$\langle,\rangle^{\sigma_0}$ is σ_0-invariant:

$$\begin{aligned}\langle k\cdot v,w\rangle^{\sigma_0} &= \mu^{-1}\langle D(k\cdot v),w\rangle^{\sigma_c} = \mu^{-1}\langle\theta(k)\cdot D(v),w\rangle^{\sigma_c}\\ &= \mu^{-1}\langle D(v),\sigma_c(\theta(k^{-1}))\cdot w\rangle^{\sigma_c}\\ &= \langle v,\sigma_0(k^{-1})\cdot w\rangle^{\sigma_0} \quad \forall k\in K(\mathbb{C})\end{aligned}$$

similarly for $X\in\mathfrak{g}(\mathbb{C})$ □

Lemma 3.4. *Let D to be as in Theorem 3.3*

(1) $\omega(D,D)^2 = 1$

(2) $\overline{\xi} = \xi^{-1}$

Proof. (1) By the definition of $\omega(D,D)$ we have $\langle D^2(v), D^2(w)\rangle^{\sigma_c} = \omega^2\langle v,w\rangle^{\sigma_c}$. By the assumption in Theorem 3.3, the left hand side equals to $\langle\lambda\cdot v,\lambda\cdot w\rangle^{\sigma_c} = \langle v,\sigma_c(\lambda^{-1})\lambda\cdot w\rangle^{\sigma_c}$ Because we assumed $\lambda\in K$, $\sigma_c(\lambda)=\lambda$. Therefore $\langle v,\sigma_c(\lambda^{-1})\lambda\cdot w\rangle^{\sigma_c} = \langle v,w\rangle^{\sigma_c} = \omega^2\langle v,w\rangle$, therefore $\omega^2=1$.

(2) By the definition of $\xi(D,D)$, now simply denoted as only ξ, we have $D^{-1}=\xi D$. Raise it to the second power we have $D^{-2}=\xi^2D^2$, thus we have $\langle v, D^{-2}(w)\rangle^{\sigma_c} = \langle v,\xi^2D^2(w)\rangle^{\sigma_c} \Rightarrow \langle v,\lambda^{-1}\cdot w\rangle^{\sigma_c} = \overline{\xi}^2\langle v,\lambda\cdot w\rangle^{\sigma_c} \Rightarrow \langle\lambda\cdot v,w\rangle^{\sigma_c} = \overline{\xi}^2\langle\lambda^{-1}\cdot v,w\rangle^{\sigma_c}$ because λ is σ_c invariant. Therefore $\overline{\xi}^2D^{-2}=D^2$. Combining the two relations between D^2 and D^{-2}, we have $\overline{\xi}^2\xi^2=1$. The conclusion follows. □

4 An Invariant Conjugate Linear Map

In this section, we are going to assume (π, V) to be a self-conjugate and self-dual representation of pair $(\mathfrak{g}(\mathbb{C}), K(\mathbb{C}))$. Note that (π, V) is automatically Hermitian because dual compose with conjugate is Hermitian. This implies that $V \cong V^{\theta}$ [2, Proposition 8.16]

We are going to define a conjugate linear, $(\mathfrak{g}, K)$ invariant (it's actually going to be $(\mathfrak{g}(\mathbb{C}), K(\mathbb{C}))$, σ_0 twisted, invariant) map $\mathcal{J}$ from V to V using the invariant bilinear form and the σ_0-invariant Hermitian form on V. Then we are going to compute the sign of $\mathcal{J}^2$, i.e. the δ-indicator, using the relation between σ_0 and σ_c invariant Hermitian forms.

Here we present the key lemma in calculating the δ indicator:

Lemma 4.1. *Let $\langle,\rangle^{\sigma_c}$ be a non-degenerate positive-definite σ_c-invariant Hermitian form on V, and $\langle,\rangle^{\sigma_0}$ be as defined in Theorem 3.3, B is a non-degenerate invariant bilinear form. Define $f : V \to V$ to be such that*

$$B(v, w) = \langle v, f(w)\rangle^{\sigma_0} \tag{4}$$

Then f is conjugate linear, invariant under $(\mathfrak{g}, K)$ and f^2 acts by a scalar. Furthermore, if $D(f(D^{-1}(v))) = \alpha f(v)$ where $\alpha > 0$ for all $v \in V$, then

$$\omega(D, D)\overline{\xi(D, D)} \cdot \varepsilon(\pi) \cdot f^2 = c \cdot Id \quad \textit{with } c > 0$$

Proof. It is easy to see that f is conjugate linear and invariant under $(\mathfrak{g}, K)$:

$$\langle v, f(k\cdot w)\rangle^{\sigma_0} = B(v, k\cdot w) = B(k^{-1}\cdot v, w) = \langle k^{-1}\cdot v, f(w)\rangle^{\sigma_0} = \langle v, \sigma_0(k)\cdot f(w)\rangle^{\sigma_0}$$

for all $v, w \in V$ and $k \in K(\mathbb{C})$. By Claim 1 below we have:

$$f(k \cdot w) = \sigma_0(k) \cdot f(w) \quad \forall k \in K(\mathbb{C})$$

and similarly:

$$f(X \cdot w) = \sigma_0(X) \cdot f(w) \quad \forall X \in \mathfrak{g}(\mathbb{C})$$

It is then obvious that f intertwines the actions of $(\mathfrak{g}, K)$.

Though f is conjugate linear, $f^2 : V \to V$ is linear and f^2 intertwines the action of $(\mathfrak{g}(\mathbb{C}), K(\mathbb{C}))$. Therefore by Schur's lemma, f^2 is a scalar multiple of the identity, i.e., $f^2 = c \cdot I$.

Equation 1 can be re-written in terms of $\langle,\rangle^{\sigma_c}$:

$$\varepsilon(\pi)c = \frac{\mu^{-1}\langle D(f(v)), f(w)\rangle^{\sigma_c}}{\overline{\mu^{-1}\langle Dv, w\rangle^{\sigma_c}}}$$

set $w = Dv$,

$$\begin{aligned}
\varepsilon(\pi)f^2 &= \frac{\mu^{-1}\langle D(f(v)), f(Dv)\rangle^{\sigma_c}}{\overline{\mu^{-1}\langle Dv, Dv\rangle^{\sigma_c}}} \\
&= \frac{\mu^{-1}\langle \alpha f(Dv), f(Dv)\rangle^{\sigma_c}}{\overline{\mu^{-1}\langle Dv, Dv\rangle^{\sigma_c}}} \quad \text{by } D(f(D^{-1}(v))) = \alpha f(v) \\
&= \mu^{-2}\alpha\frac{\langle f(Dv), f(Dv)\rangle^{\sigma_c}}{\overline{\langle D(v), D(v)\rangle^{\sigma_c}}} \quad \text{because } |\mu|^2 = 1
\end{aligned}$$

Since $\langle,\rangle^{\sigma_c}$ is positive definite, $sign(\varepsilon f^2) = sign(\mu^{-2}\alpha) \Rightarrow \mu^2\varepsilon f^2 > 0$. Since $\mu^2 = \omega\overline{\xi}$, the conclusion follows.

□

Claim 1. Suppose V possess a non-degenerate σ invariant Hermitian form $\langle , \rangle^\sigma$. If $\langle v, u\rangle^\sigma = \langle w, u\rangle^\sigma \ \ \forall u \in V$ then $v = w$.

Proof. Suppose $v \neq w$, then $v - w \neq 0$ and

$$\langle v, u\rangle^\sigma - \langle w, u\rangle^\sigma = \langle v - w, u\rangle^\sigma = 0 \ \ \forall u \in V$$

This contradicts the fact that $\langle , \rangle^\sigma$ is non-degenerate. Therefore $v = w$. $\square$

The above Lemma already gives a formula for the δ indicator: it is

$$\delta(\pi) = \omega(D, D)\overline{\xi(D, D)}\varepsilon(\pi)$$

However, we are not satisfied with this formula for two reasons: 1. The existence of δ-indicator is independent of the existence of the ε-indicator; 2. The value of $\omega(D, D)\overline{\xi(D, D)}$ can be made more explicit. Therefore, we would like to improve our theory.

In the next section, we are going to give a description of the intertwiner D and of $\omega\bar{\xi}$.

5 Strong Involution

Most of the contents in this section can be found in [2].

Let Γ_θ be the distinguished involution in the inner class of θ, Γ_θ fixes the fundamental Cartan $H_f(\mathbb{C})$ [2, Section 12]. Define γ to be such that

$$\gamma g \gamma^{-1} = \Gamma_\theta(g), \quad \gamma^2 = 1$$

Remark. γ^2 doesn't have to be 1, it can be a central element fixed by Γ_θ. We choose 1 because it simplifies our proof and the invariant in question, the δ-indicator, does not depend on the choice of γ^2.

From the explicit formulas given in [2, Section 12], we see that

$$\Gamma_\theta^2 = Id, \quad \Gamma_\theta\theta = \theta\Gamma_\theta, \quad \Gamma_\theta\sigma_c = \sigma_c\Gamma_\theta \tag{5}$$

Definition 5.1. The *extended group* ${}^\gamma G(\mathbb{C})$ is the semidirect product

$${}^\gamma G(\mathbb{C}) = G(\mathbb{C}) \rtimes \{1, \gamma\}$$

Equation 5 implies that the automorphism Γ_θ preserves the subgroup G, K, and $K(\mathbb{C})$. We can therefore form all of the corresponding extended groups:

$${}^\gamma G = G \rtimes \{1, \gamma\}, \quad {}^\gamma K = K \rtimes \{1, \gamma\}, \quad {}^\gamma K(\mathbb{C}) = K(\mathbb{C}) \rtimes \{1, \gamma\}$$

Equation 5 implies $\sigma_0(\Gamma_\theta(g)) = \sigma_0(\gamma g \gamma^{-1}) = \sigma_0(\gamma)\sigma_0(g)\sigma_0(\gamma^{-1}) = \Gamma_\theta(\sigma_0(g)) = \gamma\sigma_0(g)\gamma^{-1}$, same thing happens to σ_c. This suggests that σ_0 and σ_c both *can* be extended to real forms of ${}^\gamma G(\mathbb{C})$ by acting trivially on γ. The reason we don't have to be very general on deciding how σ_0 and σ_c can act on γ is basically because our proof is constructive and the δ-indicator is an invariant.

Definition 5.2. [2] A *strong involution* for the real form G is an element $x = x_0\gamma \in {}^\gamma G(\mathbb{C}) \backslash G(\mathbb{C})$ with the property that $xgx^{-1} = \theta(g)$, $\forall g \in G(\mathbb{C})$.

The Definition implies $x \in {}^{\gamma}K(\mathbb{C})$ and $x^2 = z \in Z(G(\mathbb{C}))^{\theta}$. Furthermore, we can pick x_0 such that $x_0 \in (H_f^{\theta})_0$ has finite order [2]. Therefore σ_c fixes x_0 and thus σ_0 also fixes x_0, i.e. $x_0 \in K$. This implies that σ_0 and σ_c also fix x. This restriction helps us make sense of the action of x_0 on the $(\mathfrak{g}(\mathbb{C}), K(\mathbb{C}))$-module (π, V).

Proposition 5.1. *[2, Proposition 8.16] The following are equivalent:*

(1) $(\mathfrak{g}(\mathbb{C}), K(\mathbb{C}))$*-module* (π, V) *extends to a representation of* $(\mathfrak{g}(\mathbb{C}), {}^{\gamma}K(\mathbb{C}))$

(2) $V \cong V^{\gamma}$

(3) $V \cong V^{\theta}$

Proof. (1)⇒(2): Let $\varphi : V \to V^{\gamma}$ be $\varphi(v) = \gamma \cdot v$. It's obvious that φ intertwines π and π^{γ}. The map $\varphi^2 = Id$ since $\gamma^2 = 1$, therefore φ is an isomorphism.

(2)⇒(1): Suppose there is an isomorphism $\varphi : V \to V^{\gamma}$, then let the action of γ be $\gamma \cdot v = \varphi(v)$, $\forall v \in V$. This is a valid extension if and only if $\varphi^2 = Id$. This is done by modifying φ by a complex scalar.

(2)⇔(3): $V^{\gamma} \cong V^{\theta}$ by isomorphism $\varphi(v) = x_0 \cdot v$, where x_0 is the one mentioned right before this Proposition. The map φ is non-zero because x_0 is of finite order, i.e., $x_0^n = 1$, implies $x_0 \cdot v \neq 0$ when $v \neq 0$. □

Corollary. *Suppose the conditions hold in Proposition 5.1, then the action of* x *gives the isomorphism* $V \cong V^{\theta}$*. Moreover, set* $D = x$ *we have* $\omega(D, D)\overline{\xi(D, D)} = \chi_{\pi}(x^2)$*.*

Proof. From the proof of Proposition 5.1, it's clear that the map

$$V \xrightarrow{\gamma} V^{\gamma} \xrightarrow{x_0} V^{\theta}$$

gives an isomorphism $V \cong V^{\theta}$.

It is clear that $\omega = 1$ since $\langle x \cdot v, x \cdot w\rangle^{\sigma_c} = \langle v, w\rangle^{\sigma_c}$. On the other hand, $D^{-1}(v) = x^{-1} \cdot v = \chi_{\pi}(x^{-2})x \cdot v = \chi_{\pi}(x^{-2})D(v) \Rightarrow \chi_{\pi}(x^2) = \xi^{-1} = \overline{\xi}$. The conclusion follows. □

6 The δ Indicator

We take a look at the assumptions in Lemma 4.1:

$$D(f(D^{-1}(v))) = \alpha f(v), \quad \alpha > 0 \tag{6}$$

Equation 6 has some equivalent statements:

Lemma 6.1. *The following are equivalent:*

(1) $D(f(D^{-1}(v))) = \alpha f(v) \Leftrightarrow f(D^{-1}(v)) = \alpha D^{-1}(f(v))$

(2) $D^{-1}(f(D(v))) = \alpha^{-1} f(v) \Leftrightarrow f(D(v)) = \alpha^{-1} D(f(v))$

(3) $B(D(v), D(w)) = \omega^{-1} d \alpha^{-1} B(v, w)$

Proof. (1)⇔(2):

$$\begin{aligned} D(f(D^{-1}(v))) = \alpha f(v) &\Leftrightarrow D(f(D^{-1}(D(v)))) = \alpha f(D(v)) \\ &\Leftrightarrow D(f(v)) = \alpha f(D(v)) \\ &\Leftrightarrow D^{-1}(f(D(v))) = \alpha^{-1} f(v) \end{aligned}$$

(2)⇔(3):

$$\begin{aligned} D^{-1}(f(D(v))) = \alpha^{-1} f(v) &\Leftrightarrow \langle D(v), D^{-1}(f(D(w)))\rangle^{\sigma_c} = \langle D(v), \alpha^{-1} f(w)\rangle^{\sigma_c} \\ &\Leftrightarrow \omega \langle v, f(D(w))\rangle^{\sigma_c} = \langle D(v), \alpha^{-1} f(w)\rangle^{\sigma_c} \\ &\Leftrightarrow \omega d^{-1} \langle D^2(v), f(D(w))\rangle^{\sigma_c} = \langle D(v), \alpha^{-1} f(w)\rangle^{\sigma_c} \\ &\Leftrightarrow \omega d^{-1} \langle D(v), f(D(w))\rangle^{\sigma_0} = \langle v, \alpha^{-1} f(w)\rangle^{\sigma_0} \\ &\Leftrightarrow \omega d^{-1} B(D(v), D(w)) = B(v, \alpha^{-1} w) \\ &\Leftrightarrow B(D(v), D(w)) = \omega^{-1} d \alpha^{-1} B(v, w) \end{aligned}$$

□

Lemma 6.2.

(1) $\omega^2 = 1$

(2) $d \in \mathbb{R}$

(3) $\alpha^{-1} = \overline{\alpha}$

Proof. (1): The proof of (1) is in [2, Proposition 8.16]

(2): Let $D^2 = d \cdot Id$, it is a scalar because D^2 is an intertwiner of the irreducible representation (π, V). The fact that $\omega^2 = 1$ implies $d \in \mathbb{R}$:

$$\begin{aligned} \omega^2 \langle v, w\rangle^{\sigma_c} &= \omega \langle D(v), D^{-1}(w)\rangle^{\sigma_c} = \langle D^2(v), D^{-1}(w)\rangle^{\sigma_c} \\ &= \langle d \cdot v, d^{-1} \cdot w\rangle^{\sigma_c} = d\overline{d^{-1}} \langle v, w\rangle^{\sigma_c} \end{aligned}$$

implies that $\overline{d} = d$ i.e. $d \in \mathbb{R}$.

(3):

$$\begin{aligned} f(D^{-1}(v)) = \alpha D^{-1}(f(v)) &\Rightarrow f^2(D^{-1}(v)) = \overline{\alpha} f(D^{-1} f(v)) \\ &\Rightarrow D^{-1}(f^2(v)) = \overline{\alpha}\alpha D^{-1}(f^2(v)) \\ &\Rightarrow \overline{\alpha}\alpha = 1 \Rightarrow \alpha^{-1} = \overline{\alpha} \end{aligned}$$

□

The existence of $D : V \cong V^\theta$ relies on if (π, V) as a $(\mathfrak{g}(\mathbb{C}), K(\mathbb{C}))$-module is σ_0-Hermitian or not.

Lemma 6.3. *Irreducible representation (π, V) of $(\mathfrak{g}(\mathbb{C}), K(\mathbb{C}))$ is σ_0-Hermitian if and only if $V \cong V^\theta$.*

Proof. (π, V) of $(\mathfrak{g}(\mathbb{C}), K(\mathbb{C}))$ is σ_0-Hermitian if and only if $V \cong V^{h,\sigma_0}$. By [2, Proposition 8.16] we have:

$$V^{h,\sigma_0} \cong [V^{h,\sigma_c}]^\theta$$

Since (π, V) is always σ_c-Hermitian, we have $V \cong V^{h,\sigma_c}$. Therefore:

$$V^{h,\sigma_0} \cong V^\theta$$

The claim follows. □

Based on the existence of D and this Lemma, we set up the dichotomy of (π, V) being σ_0-Hermitian versus not σ_0-Hermitian.

6.1 Hermitian representations

In this subsection, we assume (π, V) as a $(\mathfrak{g}, K)$-module is Hermitian. When (π, V) is referred to as a $(\mathfrak{g}(\mathbb{C}), K(\mathbb{C}))$-module, we assume it is σ_0-Hermitian.

Corollary 5 and Lemma 6.1(3) implies Equation 6 holds if and only if $B(x \cdot v, x \cdot w) = \alpha^{-1} B(v, w)$.

Theorem 6.4. *Suppose G is equal rank, the formula for δ-indicator is:*

$$\delta(\pi) = \chi_\pi(x^2)\epsilon(\pi)$$

Proof. Real group G is equal rank implies $\gamma = 1$ and $x = x_0 \in K$. We then have

$$B(x \cdot v, x \cdot w) = B(v, w)$$

Directly apply Lemma 4.1 gives the formula. □

Theorem 6.5. *Suppose G is unequal rank, the formula for δ-indicator is:*

$$\delta(\pi) = \chi_\pi(x^2)\epsilon(\pi)$$

Proof. For unequal rank group G, $\gamma \notin G(\mathbb{C})$. So we need to make sense of the action $x \cdot v$. By Proposition 5.1 we can extend the representation (π, V) to a representation of $(\mathfrak{g}(\mathbb{C}), {}^\gamma K(\mathbb{C}))$ using the isomorphism $V \cong V^\gamma$, hence x acts by the isomorphism $V \cong V^\theta$. If the extended representation is still self-dual, then the bilinear form B is invariant under the action of γ, thus of x. By Lemma 6.6 below, B is invariant under the action of $x \in {}^\gamma G(\mathbb{C}) \backslash G(\mathbb{C})$. Therefore all conditions in Lemma 4.1 are satisfied, the formula follows. □

Lemma 6.6. *If (π, V) of $G(\mathbb{C})$ extends to representation of ${}^\gamma G(\mathbb{C})$, then the two possible extensions π_1 and π_2 are self dual, where $\pi_1(\gamma) = -\pi_2(\gamma)$.*

Proof. The fact that there are only two possible extensions is given in [2]. Claim that $(\pi_1)^*$ has to be either π_1 or π_2: by definition of dual representation:

$$(\pi_1)^*(\gamma) \cdot f(v) = f(\pi_1(\gamma^{-1}) \cdot v)$$

This action of γ extends π^* since it intertwines π^* and $(\pi^*)^{\Gamma_\theta}$:

$$\begin{tikzcd} V^* \arrow[r, "(\pi_1)^*(\gamma)"] \arrow[d, "\pi^*"'] & V^* \arrow[d, "(\pi^*)^{\Gamma_\theta}"] \\ V^* \arrow[r, "(\pi_1)^*(\gamma)"] & V^* \end{tikzcd}$$

It is elementary to check this:

$$\begin{aligned} [(\pi^*)^{\Gamma_\theta}(g) \cdot [(\pi_1)^*(\gamma) \cdot f]](v) &= [\pi^*(\Gamma_\theta(g)) \cdot [(\pi_1)^*(\gamma) \cdot f]](v) \\ &= (\pi_1)^*(\gamma) \cdot f(\pi_1(\Gamma_\theta(g^{-1})) \cdot v) \\ &= f(\pi_1(\gamma^{-1} \cdot \gamma \cdot g^{-1} \cdot \gamma^{-1}) \cdot v) = f(\pi_1(\gamma g^{-1}) \cdot v) \end{aligned}$$

$$\begin{aligned}[(\pi_1)^*(\gamma)\cdot[\pi^*(g)\cdot f]](v) &= [\pi^*(g)\cdot f](\pi_1(\gamma^{-1})\cdot v) = f(\pi(g^{-1})\pi_1(\gamma^{-1})\cdot v)\\ &= f(\pi_1(\gamma^{-1}g^{-1})\cdot v)\end{aligned}$$

So $(\pi_1)^*(\gamma)$ extends π^*, and since $\pi^* \cong \pi$, $(\pi_1)^*(\gamma)$ also extends π. It is either π_1 or π_2.

It turns out $(\pi_1)^* \cong \pi_1$, we will now prove this.

Since π is self-dual, we will instead prove: $(\pi_1)^* \cong (\pi^*)_1$. The latter is defined as:

$$(\pi^*)_1(g)\cdot f = \eta[\pi_1(g)\cdot\eta^{-1}(f)]$$

where η is the representation isomorphism $\eta : \pi \cong \pi^*$. It is clear that by this definition, we have:

$$\pi_1 \cong (\pi^*)_1$$

The idea of the proof is using the highest weight vector of π^* and analyse how $(\pi_1)^*(\gamma)$ and $(\pi^*)_1(\gamma)$ act on it. The highest weight vector is defined with respect to H_f and the pinning described in [2].

The reason for using the highest weight vector is that $(\pi^*)_1(\gamma)$ acts on it by a scalar

$$(\pi^*)_1(\gamma)\cdot\varphi = c\cdot\varphi$$

see Lemma 6.7 below for proof. Consequently, $(\pi^*)_2(\gamma)\cdot\varphi = -c\cdot\varphi$

It is convenient to use proof by contradiction, i.e. $(\pi_1)^* \ncong (\pi^*)_2$. Note that a necessary condition for $(\pi_1)^* \cong (\pi^*)_2$ is:

$$(\pi_1)^*(\gamma)\cdot\varphi = -c\cdot\varphi$$

because for any representation isomorphism $\varphi : (\pi_1)^* \cong (\pi^*)_2$, φ sends highest weights to highest weights and therefore preserves the scalar by which γ acts. To prove $(\pi_1)^* \ncong (\pi^*)_2$, we just need to show the action of $(\pi_1)^*$ and that of $(\pi^*)_2$ on the highest weight vector are different.

Let v be the highest weight vector of π with highest weight λ, claim that the dual basis $f_{\sigma_{\omega_0}\cdot v}$ in V^* is the highest weight vector for π^* with highest weight $-\omega_0\cdot\lambda = \lambda$. σ_{ω_0} is the element in the Tits group with respect to the pinning fixed in [2, Section 12]. The reason we use this element is:

$$\Gamma_\theta(\sigma_{\omega_0}) = \sigma_{\omega_0}$$

We first verify that $f_{\sigma_{\omega_0}\cdot v}$ is indeed the highest weight vector for π^*, for $H \in \mathfrak{h}(\mathbb{C})$:

$$\begin{aligned}\pi^*(H)\cdot f_{\sigma_{\omega_0}\cdot v}(\sigma_{\omega_0}\cdot v) &= f_{\sigma_{\omega_0}\cdot v}(-H\cdot\sigma_{\omega_0}\cdot v) \text{ by Definition of } \pi^*\\ &= f_{\sigma_{\omega_0}\cdot v}(-\sigma_{\omega_0}\sigma_{\omega_0}^{-1}H\sigma_{\omega_0}\cdot v) \text{ by } (\mathfrak{g}(\mathbb{C}),K(\mathbb{C}) \text{ module}\\ &= f_{\sigma_{\omega_0}\cdot v}(-\sigma_{\omega_0}\cdot\omega_0^{-1}(H)\cdot v) \text{ by Definition}\\ &= -\lambda(\omega_0(H))f_{\sigma_{\omega_0}\cdot v}(\sigma_{\omega_0}\cdot v)\\ &= -\omega_0(\lambda)(H)f_{\sigma_{\omega_0}\cdot v}(\sigma_{\omega_0}\cdot v) \text{ Weyl group acts on weights}\end{aligned}$$

Because the representation π is self-dual, $-\omega_0(\lambda) = \lambda$, this implies:

$$\pi^*(H)\cdot f_{\sigma_{\omega_0}\cdot v} = \lambda(H)f_{\sigma_{\omega_0}\cdot v}$$

Let's see how $(\pi_1)^*(\gamma)$ acts on $f_{\sigma_{\omega_0}\cdot v}$:

$$\begin{aligned}(\pi_1)^*(\gamma)\cdot f_{\sigma_{\omega_0}\cdot v}(\sigma_{\omega_0}\cdot v) &= f_{\sigma_{\omega_0}\cdot v}(\pi_1(\gamma^{-1})\cdot \sigma_{\omega_0}\cdot v)\\ &= f_{\sigma_{\omega_0}\cdot v}(\Gamma_\theta(\sigma_{\omega_0})\cdot \pi_1(\gamma)\cdot v)\\ &= f_{\sigma_{\omega_0}\cdot v}(\sigma_{\omega_0}\cdot c\cdot v) = c\cdot f_{\sigma_{\omega_0}\cdot v}(\sigma_{\omega_0}\cdot v)\end{aligned}$$

However, on the other hand:

$$(\pi^*)_2(\gamma)\cdot f_{\sigma_{\omega_0}\cdot v}(\sigma_{\omega_0}\cdot v) = -[\eta(\pi_1(\gamma)\cdot \eta^{-1}(f_{\sigma_{\omega_0}\cdot v}))](\sigma_{\omega_0}\cdot v)$$

Since $\eta^{-1}(f_{\sigma_{\omega_0}\cdot v})$ is going to be a highest weight vector of π and $\pi_1(\gamma)$ acts on that by a scalar c, we have:

$$(\pi^*)_2(\gamma)\cdot f_{\sigma_{\omega_0}\cdot v}(\sigma_{\omega_0}\cdot v) = -c\cdot f_{\sigma_{\omega_0}\cdot v}(\sigma_{\omega_0}\cdot v)$$

Therefore, we have a contraduction:

$$(\pi_1)^*(\gamma)\cdot f_{\sigma_{\omega_0}\cdot v} \neq (\pi^*)_2(\gamma)\cdot f_{\sigma_{\omega_0}\cdot v}$$

it has to be:

$$(\pi_1)^* \cong (\pi^*)_1 \cong \pi_1$$

□

Lemma 6.7. *Let v be a highest weight vector of π with highest weight λ,*

$$\pi_1(\gamma)\cdot v = c\cdot v, \quad c\in\mathbb{C}^*$$

Proof. Let γ denote this isomorphism $V\cong V^\gamma$

$$\begin{gathered}\gamma : V\to V\\ \pi^{\Gamma_\theta}(g)\cdot\gamma(v) = \gamma(\pi(g)\cdot v)\end{gathered}$$

Set the action of γ to be $\pi_1(\gamma)\cdot v := \gamma(v)$

Consider the weight space decomposition of π. Let $\{\lambda,\lambda_1,\lambda_2,\cdots,\lambda_n\}$ be the weights and $\{V,V_1,V_2,\cdots,V_n\}$ be the corresponding weight spaces, with λ the highest weight, V the highest weight space. The map γ sends V to the weight space of λ^{Γ_θ}:

$$\pi(H)\cdot\gamma(v) = \gamma(\Gamma_\theta(H)\cdot v) = \gamma(\lambda^{\Gamma_\theta}(H)\cdot v) = \lambda^{\Gamma_\theta}(H)\cdot\gamma(v)$$

the notation $\lambda^{\Gamma_\theta}(H)$ is defined as $\lambda(\Gamma_\theta(H))$ By the uniqueness of the weight space decomposition and the fact that γ is an isomorphism, we know that the action of γ on V permutes the weights:

$$\{\lambda^{\Gamma_\theta},\lambda_1^{\Gamma_\theta},\cdots\lambda_n^{\Gamma_\theta}\} = \{\lambda,\lambda_1,\cdots,\lambda_n\}$$

It turns out that twisting by Γ_θ does not change the order of the weights, thanks to the considerate choice of based root datum and γ in [2].

We give a brief proof of the fact that λ^{Γ_θ} is still the highest weight: Since λ is the highest weight, we have:

$$\lambda-\lambda_i = \sum_{j\in S}\alpha_j$$

where $\alpha_j \in R_1$ and R_1 is the set of positive roots chosen to be preserved by θ. S is some non-empty set, meaning we are suming over some subset of R_1.

Twisting by Γ_θ, we have:

$$\lambda^{\Gamma_\theta} - \lambda_i^{\Gamma_\theta} = \sum_{j \in S} \alpha_j^{\Gamma_\theta}$$

By the construction of γ from θ, we conclude γ also preserves R_1 and thus $\alpha_j^{\Gamma_\theta}$ are again positive roots, so

$$\lambda^{\Gamma_\theta} \succ \lambda_i^{\Gamma_\theta}, \quad \forall i$$

Therefore $\lambda^{\Gamma_\theta} = \lambda$ and

$$\gamma(v) = \pi_1(\gamma) \cdot v = c \cdot v, \quad c \in \mathbb{C}^*$$

□

6.2 Non-Hermitian representations

For irreducible representation (π, V) who is not σ_0-Hermitian, we cannot extend it to an $(\mathfrak{g}(\mathbb{C}), {}^\gamma K(\mathbb{C}))$-module. Also, all the theory about using isomorphism $D : V \to V^\theta$ to define σ_0-invariant Hermitian form from σ_c-Hermitian form cannot be used because there is no isomorphism D. So naturally, we would like to associate to π a σ_0-Hermitian representation.

Definition 6.1. Let $(\widetilde{\pi}, \widetilde{V})$ be the induced representation $\mathrm{Ind}_{(\mathfrak{g}(\mathbb{C}), K(\mathbb{C}))}^{(\mathfrak{g}(\mathbb{C}), {}^\gamma K(\mathbb{C}))}(V)$, we will simplify the notation and denote $\widetilde{V} = \mathrm{Ind}V$ or $\widetilde{\pi} = \mathrm{Ind}\pi$.

Remark. This induced representation is irreducible [2, Proposition 8.13] and σ_0-Hermitian. This is because $[\mathrm{Ind}V]^{h,\sigma_0}$=$\mathrm{Ind}V^{h,\sigma_0}$ and $V^{h,\sigma_0} \cong V^\gamma \cong V^\theta$. Moreover, it is self-dual because $V^* \cong [\overline{V}]^{h,\sigma_0} \cong V^{h,\sigma_0}$. Therefore it is self-conjugate

Corollary. *There exists a non-degenerate $(\mathfrak{g}(\mathbb{C}), {}^\gamma K(\mathbb{C}))$-invariant bilinear form $B : \widetilde{V} \times \widetilde{V} \to \mathbb{C}$; a non-degenerate $(\mathfrak{g}(\mathbb{C}), {}^\gamma K(\mathbb{C}))$-$\sigma_0$-invariant Hermitian form $\langle , \rangle^{\sigma_0}$ on $\widetilde{V}$.*

One more thing to check is that the two Hermitian forms still have the relation in Theorem 3.3.

Lemma 6.8. *Let $D : \widetilde{V} \to \widetilde{V}^\theta$ be an isomorphism of the two representations, satisfying the property that $D^2(v) = \lambda \cdot v$, where $\lambda \in {}^\gamma K$. Let $\langle , \rangle^{\sigma_c}$ be a non-degenerate positive definite σ_c invariant Hermitian form on V. Define*

$$\langle \tilde{v}, \tilde{w} \rangle^{\sigma_0} := \mu^{-1} \langle D(\tilde{v}), \tilde{w} \rangle^{\sigma_c} = \mu \langle \tilde{v}, D(\tilde{w}) \rangle^{\sigma_c}$$

with μ, ω and ξ are defined the same as in Theorem 3.3. The form $\langle , \rangle^{\sigma_0}$ is a σ_0-invariant Hermitian form on $\widetilde{V}$.

The proof of Theorem 3.3 applies here.

Theorem 6.9. *The δ-indicator of $(\widetilde{\pi}, \widetilde{V})$ is given by*

$$\delta(\widetilde{\pi}) = \varepsilon(\widetilde{\pi}) \chi_{\widetilde{\pi}}(x^2)$$

Proof. Let $f : \widetilde{V} \to \widetilde{V}$ be such that

$$B(\tilde{v}, \tilde{w}) = \langle \tilde{v}, f(\tilde{w}) \rangle^{\sigma_0} = \mu^{-1} \langle x \cdot \tilde{v}, f(\tilde{w}) \rangle^{\sigma_c}$$

The map f is conjugate linear and satisfies:

$$f(k \cdot w) = \sigma_0(k) \cdot f(w) \quad \forall k \in {}^{\gamma}K(\mathbb{C})$$

and similarly:

$$f(X \cdot w) = \sigma_0(X) \cdot f(w) \quad \forall X \in \mathfrak{g}(\mathbb{C})$$

Therefore $x \cdot f(x^{-1} \cdot v) = x \cdot \sigma_0(x^{-1}) \cdot f(v) = x \cdot x^{-1} \cdot f(v) = f(v)$ because x is fixed by σ. We can use the same arguments to show that $\omega(x,x)\overline{\xi(x,x)}\varepsilon(\widetilde{\pi})f^2 = c \cdot Id$ with $c > 0$, where $\omega(x,x)\overline{\xi(x,x)} = \chi_{\widetilde{\pi}}(x^2)$. This implies

$$\delta(\widetilde{\pi}) = \chi_{\widetilde{\pi}}(x^2) \cdot \varepsilon(\widetilde{\pi})$$

□

Lemma 6.10. *For representations (π, V) and $(\widetilde{\pi}, \widetilde{V})$, if $\delta(\pi)$ and $\delta(\widetilde{\pi})$ both exist, then $\delta(\pi) = \delta(\widetilde{\pi})$.*

Proof. Since π is self-conjugate, there exists $\mathcal{J} : V \to V$ conjugate linear and G invariant. By definition of induced representation $\widetilde{V} = V \oplus \gamma V$. Define $\widetilde{\mathcal{J}} : \widetilde{V} \to \widetilde{V}$ such that:

$$\widetilde{\mathcal{J}}(v + \gamma w) = \mathcal{J}(v) + \gamma \mathcal{J}(w), \quad \forall v \in V, \gamma w \in \gamma V$$

It is easy to see that $\widetilde{\mathcal{J}}$ is conjugate linear and ${}^{\gamma}G$-invariant. We will demonstrate the calculation for ${}^{\gamma}G$-invariance. For $g \in G$:

$$\begin{aligned}
\widetilde{\mathcal{J}}(g \cdot (v + \gamma w)) &= \widetilde{\mathcal{J}}(g \cdot v + \gamma \Gamma(g) \cdot w) = \mathcal{J}(g \cdot v) + \gamma \mathcal{J}(\Gamma(g) \cdot w) \\
&= g \cdot \mathcal{J}(v) + \gamma \Gamma(g) \cdot \mathcal{J}(w) = g \cdot \mathcal{J}(v) + g \cdot \gamma \mathcal{J}(w) \\
&= g \cdot \widetilde{\mathcal{J}}(v + \gamma w)
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{\mathcal{J}}(\gamma \cdot (v + \gamma w)) &= \widetilde{\mathcal{J}}(\gamma v + z w) = \gamma \mathcal{J}(v) + z \mathcal{J}(w) = \gamma(\mathcal{J}(v) + \gamma \mathcal{J}(w)) \\
&= \gamma \widetilde{\mathcal{J}}(v + \gamma w)
\end{aligned}$$

Then for $v \in V$:

$$\delta(\pi) v = \mathcal{J}^2(v) = \widetilde{\mathcal{J}}^2(v) = \delta(\widetilde{\pi}) v$$

□

Corollary. *Let (π, V) be irreducible self-conjugate representation of $(\mathfrak{g}, K)$ and π is not Hermitian, then the formula for δ-indicator is:*

$$\delta(\pi) = \chi_{\pi}(x^2) \varepsilon(\widetilde{\pi})$$

Proof. This is a direct consequence of Theorem 6.9 and Lemma 6.10. The change from $\chi_{\widetilde{\pi}}(x^2)$ to $\chi_{\pi}(x^2)$ is because x^2 is in the center of ${}^{\gamma}K(\mathbb{C})$ implies it is in the center of $K(\mathbb{C})$. □

Remark. The indicator $\varepsilon(\widetilde{\pi})$ is understood when G is simple, in that case the Chevalley involution C is either trivial or inner to γ. The formula for $\varepsilon(\widetilde{\pi})$ is given in [1]. The author predicts that in the case of G semi-simple and reductive, double extended group of G is needed for understanding the formula $\delta(\pi) = \chi_{\pi}(x^2)\varepsilon(\widetilde{\pi})$.

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