The Real-Quaternionic Indicator and it's relation with the Frobenius-Schur indicator

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The Real-Quaternionic Indicator

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Definitions

Definition (Frobenius-Schur indicator)

 (π, V) irrep of G, and $\pi \cong \pi^*$. Then $\exists B : V \times V \to \mathbb{C}$, which is G-inv and bilinear. Define:

$$\varepsilon(\pi) = \begin{cases} 1 & B \text{ symm} \\ -1 & B \text{ skew-symm} \end{cases}$$

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Definition (Real-Quaternionic indicator)

 (π, V) irrep of G, and $\pi \cong \overline{\pi}$. Then $\exists \mathcal{J} : V \to V$, which is G-inv, conj linear, and $\mathcal{J}^2 = c \cdot I$ for $c \in \mathbb{R}^*$. Define:

$$\delta(\pi) = sgn(c)$$

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$$\delta(\pi) = sgn(c)$$

 $\delta(\pi) = 1$ iff π is of real type; $\delta(\pi) = -1$ iff π is of quaternionic type.

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Theorem (Bourbaki)

 $G(\mathbb{C})$ conn red cx Lie group, π finite-dim'l and $\pi\cong\pi^*$. Then

$$\varepsilon(\pi) = \chi_{\pi}(z_{\rho})$$

 $\chi_{\pi} = \textit{central character, } z_{
ho} = \exp(2\pi i
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Results about δ :

▶ N. Iwahori 1958 "On Real Irreducible Reps of Lie Algebras"

Reduced the problem to fundamental representations.

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Results about δ :

- ▶ N. Iwahori 1958 "On Real Irreducible Reps of Lie Algebras" Reduced the problem to fundamental representations.
- ▶ Jacques Tits 1967 "Tabellen zu den einfachen Lie Gruppen und ihren Darstellungen" Listed values of $\delta(\lambda)$, where λ is a fundamental representations of simple Lie groups.

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Some differences between ε and δ :

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Some differences between ε and δ :

1. $\pi \cong \pi^* \leadsto \varepsilon \quad \pi \cong \overline{\pi} \leadsto \delta$

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Some differences between ε and δ :

- 1. $\pi \cong \pi^* \leadsto \varepsilon \quad \pi \cong \overline{\pi} \leadsto \delta$
- 2. $\varepsilon(\pi) = \chi_{\pi}(z_{\rho})$ independent of real form

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Some differences between ε and δ :

- 1. $\pi \cong \pi^* \leadsto \varepsilon \quad \pi \cong \overline{\pi} \leadsto \delta$
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Some possible connections:

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Some differences between ε and δ :

- 1. $\pi \cong \pi^* \leadsto \varepsilon \quad \pi \cong \overline{\pi} \leadsto \delta$
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Some possible connections:

1. $(\overline{\pi})^h \cong \pi^*$

Some differences between ε and δ :

- 1. $\pi \cong \pi^* \leadsto \varepsilon \quad \pi \cong \overline{\pi} \leadsto \delta$
- 2. $\varepsilon(\pi) = \chi_{\pi}(z_0)$ independent of real form δ is sensitive to real form. (Exercise: 2-dim'l rep of $SL(2,\mathbb{R})$ and SU(2))

Some possible connections:

- 1. $(\overline{\pi})^h \cong \pi^*$
- 2. ε and δ can relate by a factor given by the real form

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Theorem (C.)

Let $(\pi, V) \cong (\overline{\pi}, \overline{V})$ be finite-dim'l irrep a real reductive Lie group G, then

1. π Hermitian $\Rightarrow \delta(\pi) = \varepsilon(\pi)\chi_{\pi}(x^2)$

Theorem (C.)

Let $(\pi, V) \cong (\overline{\pi}, \overline{V})$ be finite-dim'l irrep a real reductive Lie group G, then

- 1. π Hermitian $\Rightarrow \delta(\pi) = \varepsilon(\pi)\chi_{\pi}(x^2)$
- 2. π non-Hermitian $\delta(\pi) = \varepsilon(\tilde{\pi})\chi_{\pi}(x^2)$, where $\widetilde{\pi} = Ind_G^{\gamma G} \pi$

x is the "strong real form", $Ad(x) = \theta$, $x^2 \in Z(G)$. ${}^{\gamma}G$ is a extended group of G.

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x is the "strong real form", $Ad(x) = \theta$, $x^2 \in Z(G)$. ${}^{\gamma}G$ is a extended group of G.

To Prove the Theorem

Construct $\mathcal{J}: V \to V$ then calculate $\delta(\pi) = sgn(\mathcal{J}^2)$

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Define \mathcal{J}

Setting:

G real reductive, π irred, finite-dim'l, $\pi\cong\overline{\pi}$, and $\pi\cong\pi^h$

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Define \mathcal{J}

Setting:

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Define \mathcal{J} :

 π Hermitian $\rightsquigarrow \langle , \rangle$ invariant Hermitian form

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Define \mathcal{J} :

 π Hermitian $\rightsquigarrow \langle, \rangle$ invariant Hermitian form $\pi \cong (\overline{\pi})^h \cong \pi^* \Rightarrow \exists B$ inv bilinear form on V.

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Setting:

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Define \mathcal{I} :

 π Hermitian $\rightsquigarrow \langle , \rangle$ invariant Hermitian form $\pi \cong (\overline{\pi})^h \cong \pi^* \Rightarrow \exists B \text{ inv bilinear form on } V.$ Define $\mathcal{J}: V \to V$ such that:

$$B(v, w) = \langle v, \mathcal{J}(w) \rangle$$

Setting:

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Define \mathcal{J} :

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$$B(v, w) = \langle v, \mathcal{J}(w) \rangle$$

 ${\mathcal J}$ is conjugate linear and G-invariant. Calculate $\delta(\pi):=\operatorname{sgn}({\mathcal J}^2)$

$$B(v, w) = \langle v, \mathcal{J}(w) \rangle$$

 \exists pos-def *G*-inv Hermitian form \langle,\rangle

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$$B(v, w) = \langle v, \mathcal{J}(w) \rangle$$

 \exists pos-def *G*-inv Hermitian form \langle , \rangle Short calculation \Rightarrow

$$\varepsilon(\pi)\delta(\pi) = \operatorname{sgn}\left(\frac{\langle \mathcal{J}(v), \mathcal{J}(w)\rangle}{\langle w, v\rangle}\right) \quad \forall v, w \in V$$

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Let v = w, unitarity $\Rightarrow RHS = 1$

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Let v = w, unitarity $\Rightarrow RHS = 1 \Rightarrow \delta(\pi) = \varepsilon(\pi)$

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$$B(v, w) = \langle v, \mathcal{J}(w) \rangle$$

π unitary

 \exists pos-def G-inv Hermitian form \langle , \rangle Short calculation \Rightarrow

$$\varepsilon(\pi)\delta(\pi) = \operatorname{sgn}\left(\frac{\langle \mathcal{J}(v), \mathcal{J}(w)\rangle}{\langle w, v\rangle}\right) \quad \forall v, w \in V$$

Let
$$v = w$$
, unitarity $\Rightarrow RHS = 1 \Rightarrow \delta(\pi) = \varepsilon(\pi)$

Theorem

 π irred unitary, $\pi \cong \overline{\pi}$, we have $\varepsilon(\pi) = \delta(\pi)$

Corollaries

Corollary

If π irred self-conj and $\varepsilon(\pi) \neq \delta(\pi)$, then π is not unitary.

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Corollary

If π irred self-conj and $\varepsilon(\pi) \neq \delta(\pi)$, then π is not unitary.

Corollary

Let G be a compact group, π be an irred self-conj rep of G, then $\varepsilon(\pi) = \delta(\pi)$

Main Tool

The c-inv Hermitian form \langle , \rangle^c

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Main Tool

The c-inv Hermitian form \langle , \rangle^c

▶ It is pos-def on *V*

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Main Tool

The c-inv Hermitian form \langle , \rangle^c

- ▶ It is pos-def on *V*
- ▶ It exists for all finite-dim'l (π, V)

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 π is Hermitian

c-invariant Hermitian Form

Ordinary inv Hermitian form satisfies:

$$\langle \pi(g)v, w \rangle = \langle v, \pi(g^{-1})w \rangle$$

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c-invariant Hermitian Form

Ordinary inv Hermitian form satisfies:

$$\langle \pi(g)v, w \rangle = \langle v, \pi(g^{-1})w \rangle$$

 $\sigma = \text{real structure},$

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 $\sigma = \text{real structure,a } \sigma \text{-invariant Hermitian form satisfies:}$

$$\langle \pi(g)v, w \rangle^{\sigma} = \langle v, \sigma(\pi(g^{-1}))w \rangle^{\sigma}$$

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Ordinary inv Hermitian form satisfies:

$$\langle \pi(g)v, w \rangle = \langle v, \pi(g^{-1})w \rangle$$

 $\sigma = \text{real structure,a } \sigma \text{-invariant Hermitian form satisfies:}$

$$\langle \pi(g)v, w \rangle^{\sigma} = \langle v, \sigma(\pi(g^{-1}))w \rangle^{\sigma}$$

 σ_0 given by the real form $G \rightsquigarrow \langle , \rangle$ σ_c given by the compact real form of $G(\mathbb{C}) \rightsquigarrow \langle , \rangle^c$.

π Hermitian

$$\varepsilon(\pi)\delta(\pi) = \operatorname{sgn}\left(\frac{\langle \mathcal{J}(v), \mathcal{J}(w)\rangle}{\langle w, v\rangle}\right) \quad \forall v, w \in V$$

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$\varepsilon(\pi)\delta(\pi) = \operatorname{sgn}\left(\frac{\langle \mathcal{J}(v), \mathcal{J}(w) \rangle}{\langle w, v \rangle}\right) \quad \forall v, w \in V$

Proposition

Let G be equal rank, \langle , \rangle^c a pos-def c-inv Hermiatian form. Define \langle , \rangle such that:

$$\langle v, w \rangle := \mu^{-1} \langle x \cdot v, w \rangle^c$$

is an ordinary Hermitian form.

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x is the "strong real form", i.e., $Ad(x) = \theta$, Cartan involution, and $x^2 \in Z(G)$. μ is a square root of $\chi_{\pi}(x^2)$

G equal rank

Rewrite $\varepsilon(\pi)\delta(\pi)$ in terms of \langle , \rangle^c :

$$\varepsilon(\pi)\delta(\pi) = \operatorname{sgn}\left(\mu^{-2} \frac{\langle x \cdot \mathcal{J}(v), \mathcal{J}(w) \rangle^{c}}{\overline{\langle x \cdot v, w \rangle^{c}}}\right)$$

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G equal $\mathsf{rk} \Rightarrow x \in G$

Rewrite $\varepsilon(\pi)\delta(\pi)$ in terms of \langle , \rangle^c :

$$\varepsilon(\pi)\delta(\pi) = \operatorname{sgn}\left(\mu^{-2} \frac{\langle x \cdot \mathcal{J}(v), \mathcal{J}(w) \rangle^{c}}{\overline{\langle x \cdot v, w \rangle^{c}}}\right)$$

$$G$$
 equal $\mathsf{rk} \Rightarrow x \in G \Rightarrow x \cdot \mathcal{J}(v) = \mathcal{J}(x \cdot v)$.

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Rewrite $\varepsilon(\pi)\delta(\pi)$ in terms of \langle , \rangle^c :

$$\varepsilon(\pi)\delta(\pi) = \operatorname{sgn}\left(\mu^{-2} \frac{\langle x \cdot \mathcal{J}(v), \mathcal{J}(w) \rangle^{c}}{\overline{\langle x \cdot v, w \rangle^{c}}}\right)$$

$$G$$
 equal $\mathsf{rk} \Rightarrow x \in G \Rightarrow x \cdot \mathcal{J}(v) = \mathcal{J}(x \cdot v)$. Set $w = x \cdot v$ $\Rightarrow \mathit{sgn}(\mathsf{RHS}) = \mathit{sgn}(\mu^{-2}) \Rightarrow$
$$\delta(\pi) = \varepsilon(\pi)\mu^2 = \varepsilon(\pi)\chi_\pi(x^2)$$

Result for π Hermitian

 ${\it G}$ unequal rank, use similar arguments on extended representation of the extended group ${}^{\gamma}{\it G}$.

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 ${\it G}$ unequal rank, use similar arguments on extended representation of the extended group ${}^{\gamma}{\it G}$.

Theorem

Suppose π is a Hermitian, irreducible, finite-dim'l and self-conjugate rep of real red group G, then

$$\delta(\pi) = \varepsilon(\pi)\chi_{\pi}(x^2) = \chi_{\pi}(z_{\rho})\chi_{\pi}(x^2)$$

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 $\pi \ncong \pi^h \Rightarrow \pi \ncong \pi^\gamma$

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 $\pi\ncong\pi^h\Rightarrow\pi\ncong\pi^\gamma\Rightarrow\widetilde{\pi}=\mathit{Ind}_G^{\gamma_G}\pi$ is irreducible.

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 $\pi \ncong \pi^h \Rightarrow \pi \ncong \pi^\gamma \Rightarrow \widetilde{\pi} = Ind_G^{\gamma} \pi$ is irreducible.

Lemma

 $\widetilde{\pi}$ is Hermitian, self-dual and self-conjugate.

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Lemma

 $\widetilde{\pi}$ is Hermitian, self-dual and self-conjugate. $\delta(\pi) = \delta(\widetilde{\pi})$

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Theorem

$$\delta(\pi) = \delta(\widetilde{\pi}) = \varepsilon(\widetilde{\pi})\chi_{\pi}(x^2)$$

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Similar arguments, using \langle,\rangle^c pos-def on Lowest K-types.

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Similar arguments, using \langle,\rangle^c pos-def on Lowest K-types.

Theorem

Let G be equal rank, (π, V) is an infinite-dim'l (\mathfrak{g}, K) -module. Suppose irrep $\pi \cong \pi^* \cong \overline{\pi} \cong \pi^h$, then

$$\delta(\pi) = \varepsilon(\pi)\chi_{\pi}(x^2)$$

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$$\delta(\pi) = \varepsilon(\pi)\chi_{\pi}(x^2)$$

Future work?

- ightharpoonup arepsilon and δ for p-adic groups
- Interaction between the indicators and Langlands correspondence
- ► C* algebra?