The Real-Quaternionic Indicator

and it's relation with the Frobenius-Schur indicator

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Outline

Definitions

Definition (Frobenius-Schur indicator)

(π, V) irrep of G, and $\pi \cong \pi^*$. Then $\exists B : V \times V \to \mathbb{C}$, which is G-inv and bilinear. Define:

$$\varepsilon(\pi) = \begin{cases} 1 & B \text{ symm} \\ -1 & B \text{ skew-symm} \end{cases}$$

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$$\delta(\pi) = sgn(c)$$

$\delta(\pi) = 1$ iff π is of real type;
$\delta(\pi) = -1$ iff π is of quaternionic type.

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Existing Results

Theorem (Bourbaki)

$G(\mathbb{C})$ conn red cx Lie group, π finite-dim'l and $\pi \cong \pi^*$. Then

$$\varepsilon(\pi) = \chi_\pi(z_\rho)$$

$\chi_\pi =$ *central character,* $z_\rho = \exp(2\pi i \rho^\vee)$

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- N. Iwahori 1958 "On Real Irreducible Reps of Lie Algebras"
  Reduced the problem to fundamental representations.
- Jacques Tits 1967 "Tabellen zu den einfachen Lie Gruppen und ihren Darstellungen"
  Listed values of $\delta(\lambda)$, where λ is a fundamental representations of simple Lie groups.

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ε and δ

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1. $(\overline{\pi})^h \cong \pi^*$
2. ε and δ can relate by a factor given by the real form

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A Brief History
Definitions
Existing Results
Main Theorem
ε and δ
Main Theorem
Outline of Proof
π is Hermitian
π is not Hermitian
π Infinite-Dimensional

Main Theorem

Theorem (C.)

Let $(\pi, V) \cong (\overline{\pi}, \overline{V})$ *be finite-dim'l irrep a real reductive Lie group* G*, then*

1. π *Hermitian* $\Rightarrow \delta(\pi) = \varepsilon(\pi)\chi_\pi(x^2)$

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1. π *Hermitian* $\Rightarrow \delta(\pi) = \varepsilon(\pi)\chi_\pi(x^2)$
2. π *non-Hermitian* $\delta(\pi) = \varepsilon(\widetilde{\pi})\chi_\pi(x^2)$, *where* $\widetilde{\pi} = Ind_G^{\gamma G}\pi$

x is the "strong real form", $Ad(x) = \theta$, $x^2 \in Z(G)$. ${}^\gamma G$ is a extended group of G.

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To Prove the Theorem

Construct $\mathcal{J} : V \to V$ then calculate $\delta(\pi) = sgn(\mathcal{J}^2)$

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$$B(v, w) = \langle v, \mathcal{J}(w) \rangle$$

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$\mathcal{J}$ is conjugate linear and G-invariant.
Calculate $\delta(\pi) := sgn(\mathcal{J}^2)$

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Short calculation $\Rightarrow$

$$\varepsilon(\pi)\delta(\pi) = sgn\left(\frac{\langle \mathcal{J}(v), \mathcal{J}(w) \rangle}{\langle w, v \rangle}\right) \quad \forall v, w \in V$$

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Theorem

π irred unitary, $\pi \cong \overline{\pi}$, we have $\varepsilon(\pi) = \delta(\pi)$

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Corollary

Let G be a compact group, π be an irred self-conj rep of G, then $\varepsilon(\pi) = \delta(\pi)$

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The c-inv Hermitian form $\langle , \rangle^c$

- It is pos-def on V
- It exists for all finite-dim'l (π, V)

c-invariant Hermitian Form

Ordinary inv Hermitian form satisfies:

$$\langle \pi(g)v, w\rangle = \langle v, \pi(g^{-1})w\rangle$$

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σ_0 given by the real form $G \rightsquigarrow \langle , \rangle$
σ_c given by the compact real form of $G(\mathbb{C}) \rightsquigarrow \langle , \rangle^c$.

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Proposition

Let G be equal rank, $\langle , \rangle^c$ a pos-def c-inv Hermiatian form. Define $\langle , \rangle$ such that:

$$\langle v, w\rangle := \mu^{-1}\langle x \cdot v, w\rangle^c$$

is an ordinary Hermitian form.

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is an ordinary Hermitian form.
x is the "strong real form", i.e., $Ad(x) = \theta$, Cartan involution, and $x^2 \in Z(G)$. μ is a square root of $\chi_\pi(x^2)$

G equal rank

Rewrite $\varepsilon(\pi)\delta(\pi)$ in terms of $\langle , \rangle^c$:

$$\varepsilon(\pi)\delta(\pi) = sgn\left(\mu^{-2}\frac{\langle x\cdot\mathcal{J}(v),\mathcal{J}(w)\rangle^c}{\overline{\langle x\cdot v, w\rangle^c}}\right)$$

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G equal rk $\Rightarrow x \in G \Rightarrow x\cdot\mathcal{J}(v) = \mathcal{J}(x\cdot v)$. Set $w = x\cdot v$
$\Rightarrow sgn(RHS) = sgn(\mu^{-2}) \Rightarrow$

$$\delta(\pi) = \varepsilon(\pi)\mu^2 = \varepsilon(\pi)\chi_\pi(x^2)$$

Result for π Hermitian

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Theorem

Suppose π is a Hermitian, irreducible, finite-dim'l and self-conjugate rep of real red group G, then

$$\delta(\pi) = \varepsilon(\pi)\chi_\pi(x^2) = \chi_\pi(z_\rho)\chi_\pi(x^2)$$

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Lemma

$\widetilde{\pi}$ is Hermitian, self-dual and self-conjugate.

π not Hermitian

$\pi \not\cong \pi^h \Rightarrow \pi \not\cong \pi^\gamma \Rightarrow \widetilde{\pi} = Ind_G^{\gamma G} \pi$ is irreducible.

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Theorem

Let G be equal rank, (π, V) is an infinite-dim'l $(\mathfrak{g}, K)$-module. Suppose irrep $\pi \cong \pi^* \cong \overline{\pi} \cong \pi^h$, then

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Future work?

- ε and δ for p-adic groups
- Interaction between the indicators and Langlands correspondence
- C^* algebra?