Collect Animal Crossing Amiibo Cards

Introduction

Animal Crossing is a social simulation video game developed by Nintendo. There are hundreds of cute animal characters in the game you can make friends with, and each of them is featured with a trading card called Amiibo Card. The front of the card displays the photo of the animal character with its name and birthday. The Amiibo Card is a Near-Field Communication (NFC) card, which players can tap to the NFC touchpoint on the Nintendo game console. If they own the Amiibo cards, players can interact with corresponding animal characters in the game.

Five different series of Animal Crossing Amiibo Cards have been released as of 2021. The main four series contain 100 cards each, representing a total of 400 different animal characters. The cards are available in random blind packs of 6 in North America, and 3 in Europe or Japan, with the suggested retail price of $6.00 / £3.49 / ￥300.

For players who are obsessed with card collection, collecting a complete set of cards is their goal. However, one issue that bothers the players is that if we buy the random blind pack, we will often collect duplicate cards. The most extreme case is that the players spend a large amount of money on the last few cards they have not collected. To avoid this problem, players turn to purchase particular cards from the resellers. We can buy one single card we want from the sources such as Amazon or eBay at a price higher than 6 dollars. In this way, we can get rid of collecting duplicate cards and save money.

Here comes the problems our team wants to figure out:

1. What's the expected value of cost if we draw cards from the blind pack of 1 random card?
2. What’s the expected value of the distinct cards we can collect given a certain budget?
3. What’s the expected value of cost if we collect cards from the blind pack of 6 random cards?
4. What’s the expected value of cost if we buy particular cards from a reseller?
5. If we buy blind packs at first, when should we stop collecting and turn to buying certain cards? (e.g., until how much money we have already spent or how many distinct cards we have already collected?)
6. How much money do we spend on the optimal strategy in total? Why is it optimal?

Analysis

Introduction of CCP

Our project is derived from the Coupon Collecting Problem (CCP), which is a practical theory in mathematical books(eg: Ross 2012, Ferrante 2014, etc). In the classical CCP, the collector collects one type of coupon at a time continuously until each type is collected. It is usually assumed that each type of coupon has a fixed probability, and each time the collection of coupons is independent of those previously obtained. One of the main issues in CCP is the computation of the expected value of coupons needed for the collector to make a complete set.

Applying CCP in our Project

Ⅰ. Problem 1: Draw 1 random card through the blind pack

In order to reach the final goals of our project, we will first simplify the situation: assuming we collect one Amiibo card through a blink pack at each draw.

The first problem is to calculate the expected value of cost if collectors want to collect a complete set of cards in this situation.

Mathematical Proof

We assume $\mathbf{X}_i$ *for* ***i=1, 2, …, N***, be the number of additional cards needed to obtain the *i*-th type after (*i*-1) distinct types have been collected. Thus, the total number of cards needed is:

$$X = X_1 + X_2 + \dots + X_N = \sum_{i=1}^{N} X_i \qquad (1)$$

For any *i*, *i*-1 distinct types of cards have already been collected. In our problem, the first draw will always contain a distinct type of cards.

$$p(i = 1) = 1$$

To collect second distinct type of cards, for each subsequent collection, there is

$$p(i = 2) = \frac{N-1}{N}$$

chance that the card we draw will be different than the first, thus the expected number of cards collector will draw to get a different type of card is:

$$E(X_2) = \frac{N}{N-1}$$

Once two distinct type of cards in the set are collected, there is:

$$p(i = 3) = \frac{N-2}{N}$$

chance that a subsequent draw will obtain a new card, which is different from the first two distinct types of cards we collected. Similarly, the expected value of the cards we draw to obtain a different card is:

$$E(X_3) = \frac{N}{N-2}$$

We can find out that the expected value is the reciprocal of probability for collecting a new card. In conclusion, given that *i*-1 distinct types of cards have already been collected, the probability of obtaining a new card of a different type is:

$$1 - \frac{i-1}{N} = \frac{N-i+1}{N} \qquad (2)$$

Essentially to obtain the *i*-th distinct type of card, the probability is:

$$p = \frac{N-i+1}{N} \qquad (3)$$

The expected value of additional cards needed for obtaining a new type of card is:

$$E(X_i) = \frac{N}{N-i+1} \qquad (4)$$

And the expected value of cards needed to collect all distinct types can be expressed as:

$$E(X) = \sum_{i=1}^{N} E[X_i] = \sum_{i=1}^{N} \frac{N}{N-i+1} = N(\frac{1}{N} + \frac{1}{N-1} + ... + \frac{1}{1})$$

(5)

Application

In this problem, we assume the cost for drawing a single card is 1 dollar. The total number of Amiibo cards is 400.

The probability of collecting the first type is 1 because we have no previous card.

$$p1 = \frac{N-i+1}{N} = 1$$

$$E(X_1) = \frac{N}{N-i+1} = \frac{400}{400}$$

The probability of collecting the second type is:

$$p2 = \frac{N-i+1}{N} = 0.9975$$

The expected value of extra cost for the second new card is:

$$E(2) = \frac{N}{N-i+1} = \frac{400}{399}$$

Thus, the expected value of total cost to collect a complete set is:

$$E(cost) = price \times E(X) = price \times \sum_{i=1}^{400} E[X_i] \tag{6}$$

$$E(cost) = \$1 \times 400 \times (\frac{1}{400} + \frac{1}{399} + ... + \frac{1}{1}) = \$2627.97$$

We can use the formula (6) to calculate the expected value of cost:

Table1: Values of expected value of cost

i	P(i)	E(X_i)	E(cost for i)
1	1	1.00	1.00
2	0.9975	1.00	1.00
3	0.995	1.01	1.01
4	0.9925	1.01	1.01
5	0.99	1.01	1.01
6	0.9875	1.01	1.01
7	0.985	1.02	1.02
8	0.9825	1.02	1.02
9	0.98	1.02	1.02
10	0.9775	1.02	1.02
…	…	…	…
398	0.0075	133.33	133.33
399	0.005	200	200
400	0.0025	400	400

i (***i***-type of card),
P(i) (Probability of collecting ***i***-th type)
E(X_i) (Expected value of extra draws needed for ***i***-th type)
E(cost) (Expected value of cost for collecting ***i***-th type)

Ⅱ. Problem 2: Collect cards within a given budget

Next, we will calculate how many distinct cards can collectors collect within a given budget. This problem will help us illustrate the result in the following parts.

We first compute the expected value of distinct cards for the simple case that ***n***, the given budget, is 1.

We introduce indicator random variables I_i for i=1,2,3,....,N, where

$$\begin{cases} Ii = 1, & \text{if at least one card of the i-th type in the set of n cards} \\ Ii = 0, & \text{otherwise} \end{cases}$$

If we buy one card with 1 dollar, we are not sure which type of card is the first purchase. Thus the probability that it is not the ***1*-st** card type is $\frac{399}{400}$. The probability that it is the ***1*-st** card type is $\frac{1}{400}$.

Then, if we have two dollars now and buy two Amiibo cards, the probability that they are not the ***2*-nd** card type is $(\frac{399}{400})^2$. The probability that the collector buys at least one the ***2*-nd** type of card is $1-(\frac{399}{400})^2$.

Next, we consider the general case that n＞2. Let ***Y*** be the number of distinct types of cards in the set of n cards (one dollar for one card. It also means the cost), so we have:

$$Y = I_1 + I_2 + ... + I_N = \sum_{i=1}^{N} I_i$$

For each collected card, the probability that it is not the i-th card type is

$$\frac{N-1}{N}$$

Since all n collected cards, the probability that none of the n card is the i-th card type is:

$$P(I_i = 0) = (\frac{N-1}{N})^n$$

and we have:

$$E[I_i] = P(I_i = 1) = 1 - (\frac{N-1}{N})^n$$

Thus, we have:

$$E(Y) = \sum_{i=1}^{N} E[I_i] = N - N(\frac{N-1}{N})^n$$

We can use the formula to calculate the expected value of the number of distinct cards with ***n*** dollars:

Table2: Expected value of distinct cards with given budget

n (dollars)	***E(Y) (cards)***
1	1.00
2	2.00
3	2.99
4	3.99
5	4.98
6	5.96
7	6.95
8	7.93
9	8.91
10	9.89
…	…
996	366.94
997	367.02

998	367.10
999	367.19
1000	367.27

n-the given cost

For example, if we have 1000 dollars, we can collect 367 distinct cards.

I II. Problem 3: Draw cards from the blind pack containing multiple random cards

In Problem 1, we proved that the expected value of draws needed to collect a complete set is expressed as:

$$E(X) = \sum_{i=1}^{N} E[X_i] = \sum_{i=1}^{N} \frac{N}{N-i+1} = N(\frac{1}{N} + \frac{1}{N-1} +...+ \frac{1}{1})$$

If there are multiple random cards in the blind pack, without duplicates, on average how many draws are needed to collect a complete set?

Mathematical Proof

A complete set consists of n cards, and a blind pack contains m random cards without duplicates. If we already have k distinct cards, let X_k be the expected value of additional packs needed to obtain all n cards.

The total number of possible draws is $\binom{n}{m}$. To obtain i new cards, we can choose i cards from the (n-k) types we have not seen and choose (m-i) cards from the k types we have already collected. Therefore, the probability to obtain i new cards is:

$$p_{k,i} = \frac{\binom{k}{m-i}\binom{n-k}{i}}{\binom{n}{m}}$$

We need to make X_{k+i} further draws to get i new cards, when $i \in \{0, 1, 2..., m\}$. Then we come up with the recursion relationship of X_k:

$$X_k = 1 + \sum_{i=0}^{m} p_{k,i} X_{k+i}$$

Simplify the formula, we get:

$$X_k \times (1 - p_{k,0}) = 1 + \sum_{i=1}^{m} p_{k,i} X_{k+i}$$

$$X_k = \frac{1}{(1-p_{k,0})} \times (1 + \sum_{i=1}^{m} p_{k,i} X_{k+i})$$

$$X_k = \frac{\binom{n}{m}}{\binom{n}{m}-\binom{k}{m}} \times (1 + \sum_{i=1}^{m} \frac{\binom{k}{m-i}\binom{n-k}{i}}{\binom{n}{m}} X_{k+i})$$

Each X_k depends on X_{k+1}, X_{k+2}, ..., X_{k+m}. The terminal condition of the recursion is $X_n = 0$, because 0 further cards are needed when we have already collected n cards. Then we can calculate X_{n-1}, X_{n-2}, ... backwards, until we get the final result X_0.

Application

We use a python program to calculate the expected number of draws.

```
# A function to calculate the expected number of draws needed.
def expected_draws(n, m):
   # calculate the binomial matrix
   B = calculate_binomial(n, m)

   # comb(n,r) can make use of the binomial matrix
   def comb(n, r):
       if(r > m):
           return 0
       else:
           return B[n][r]

   X = [0 for i in range(n+m)] # list to store the sequence
   # start from n-1, calculate backwards
   for k in range(n-1, -1, -1):
       q0 = comb(n, m)/(comb(n, m)-comb(k, m))
       q = [comb(k, m-i)*comb(n-k, i)/comb(n, m) for i in range(1, m+1)]
       X[k] = q0*(1 + sum([a * b for a, b in zip(q, X[k+1:k+1+m])]))
   return X
```

The cards are sold in the blind pack containing 6 random cards. When n=400, m = 6, the expected number of packs required is:

$$X_0 = \frac{\binom{400}{6}}{\binom{400}{6}-\binom{0}{6}} \times \left(1 + \sum_{i=1}^{6} \frac{\binom{0}{6-i}\binom{400}{i}}{\binom{400}{6}} X_i\right) \approx 435.67$$

Therefore, on average we need to buy 436 packs to collect a complete set. The price of one pack is about 6 dollars. The total cost will be:

$$E(cost) = price \times X_0 = \$6 \times 436 = \$2616$$

Ⅰ V. Problem 4: Expected value of cost if we buy certain cards from a reseller

Buying a particular card from the reseller on eBay, generally, we can get any single card we want for an average price of 6.5 dollars. As a result, the total cost would be:

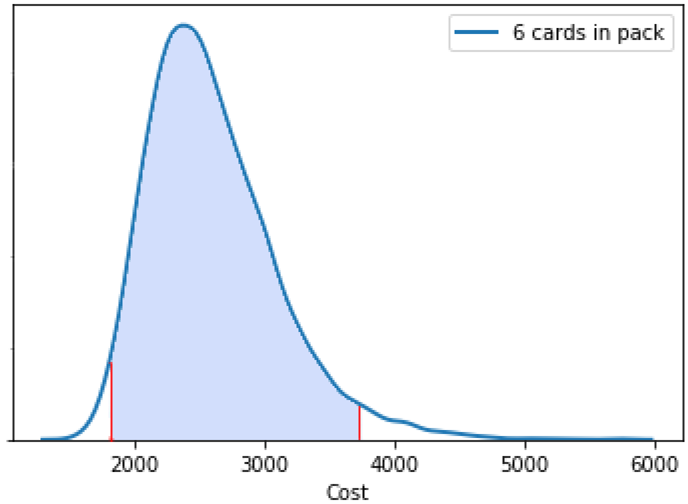
$$\$6.5 \times 400 = \$2600$$

V. Problem 5: When should we stop drawing blind packs and turn to buying certain cards

Mathematical Explain: Why do we need to stop drawing blind packs and turn to buy certain cards?

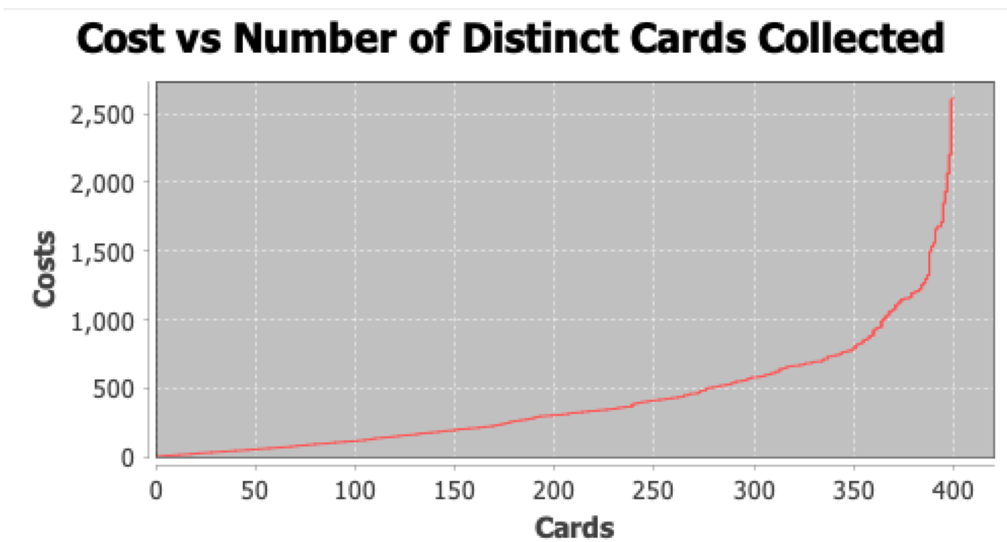
Firstly, let’s figure out what kind of event we are studying. We use a python program to simulate the process of collecting cards through purchasing blind packs.

We simulate the collecting process 10000 times, and come up with the distribution of the event, which is shown in the below picture.



Most people would spend approximately $1900 - 3800 on collecting a set of cards by drawing random cards.

We visualized the relations between costs and cards following the below plot:



When we have already collected around 350 cards, the costs would surge. Thus we should turn to buy after this tipping point. In summary, we should combine drawing and buying to collect a set of cards economically.

Application

When the expected spending for collecting a new-type card from blind packs surpasses the spending of buying a new card, we should stop collecting from blind packs and turn to buying the remaining cards we have not collected.

Because X_{i-1} is the expected number of drawing packs to collect the remaining (n-i+1) cards, and X_i is the expected number of drawing packs to collect the remaining (n-i) cards. If we are trying to obtain the i-th card, we need to estimate how many packs we need to purchase, which is about $(X_{i-1} - X_i)$.

$$E(packs\ to\ draw\ the\ i_{th}\ card) \approx X_{i-1} - X_i$$

Afterwards, we can multiply it with the price of each pack($6) to calculate the expected spending on the i-th card:

$$E(cost\ on\ drawing\ the\ i_{th}\ card) \approx (X_{i-1} - X_i) * \$6$$

Then we calculate the expected spending for 400 cards. In our case, as the below chart shows, the 340-th card is a break-even point.

Table3: Expected cost for collecting the i-th new card from blind packs

The i-th card	***E(cost on the i-th card)***
1	0.994
2	0.996
3	0.999
4	1.001
5	1.004
…	…
338	6.309
339	6.411
340	6.516
341	6.625
342	6.737
…	…
396	79.499

397	99.373
398	132.498
399	198.739
400	400

The expected value for the spending on the 340th card is around 6.516 dollars, which is almost the same as the spending of buying a certain card from reseller, 6.50 dollars. If we collect more than 339 cards, the expected cost for which would be larger than 6.50 dollars for each card. Since we already deduced the expected cost of drawing the i-th card before:

$$E(cost\ on\ \ drawing\ the\ i_{th}\ card) \approx (X_{i-1} - \ X_{i}) * \ \$6$$

Then we can calculate the expected total cost following the below steps:

$$E(Cost\ of\ drawing\ the\ first\ 339\ cards) \ = \sum_{i=1}^{339}(X_{i-1} - X_{i}) * \$6 \approx \$744.77$$

$$E(Cost\ of\ buying\ the\ remaining\ 61\ cards) \ \approx 61 \ * \$6.5 \approx \$396.50$$

$$E(Total\ cost) \ = E(Cost\ of\ drawing\ 339\ cards) \ + E(Cost\ of\ buying\ the\ remaining\ 61\ cards)$$

$$E(Total\ cost) \ = \$744.77 + \$396.50 = \$1141.27$$

As a result, we found that if we stop collecting when we leave 61 cards to buy, we can expect to spend the least money to collect a set of cards. **The total cost would be 1141.27 dollars.**

VI. Problem 6: How much money do we spend totally on the optimal strategy? Why is it optimal?

From problem 5, we deduced our optimal strategy: we should combine draw and buy to collect a full set of cards. We would draw cards until obtaining 339 distinct cards, and then purchase the remaining type of cards from second-hand markets.

By applying the optimal strategy, we predict to **spend $1141.27** on collecting a complete set of cards, which is way much lower than the expected cost of drawing cards ($2616) or the expected cost of buying cards ($2600). Our optimal strategy could significantly **reduce the overall cost by 57%**.

A Derived Problem: Cost of drawing cards in Japan vs North America

In Japan, the cards are sold in the blind pack containing 3 random cards. When n=400, m = 3, the expected number of packs required is:

$$X_0 = \frac{\binom{400}{3}}{\binom{400}{3}-\binom{0}{3}} \times \left(1 + \sum_{i=1}^{3} \frac{\binom{0}{3-i}\binom{400}{i}}{\binom{400}{3}} X_i\right) \approx 874.13$$

Therefore, on average we need to buy 875 packs to collect a complete set. The price of one pack is about 300 Yen in Japan. If converted into dollars based on the exchange rate of 113 JPY = 1 USD, The total Cost will be:

$$Cost = price \times X_0 = ¥300 \times 875 \div 113 = \$2323$$

Compared to North America, we can spend a little less money to collect a complete set in Japan.

Conclusion

From the above analysis, we can draw the conclusion that the optimal strategy of collecting the complete set of cards is combining draw and buy. We would buy blind packs to draw cards until we already obtain 339 distinct cards and then purchase the remaining 61 cards on the second-hand market. This strategy would cost us around $1141.27 to collect a set of cards, which is 57% lower than the cost of only drawing cards from blind packs or always buying cards on the second-hand market.

However, there are some weaknesses and limitations in our study.

Firstly, since we are dealing with probability, it’s still possible for some players to spend less money than our optimal strategy when collecting a set of cards. For instance, if they rarely draw duplicate cards, the cost would be only $402. However the probability of this situation is extremely low. Thus we only focus on the most players’ situation.

Secondly, we are assuming the cards have an equal chance of being in a pack. However, in the real game, some special cards have lower occurrence probability than regular cards. Because of this, the price of these special cards may skyrocket on the second-hand market. To simplify our study, we didn’t consider these special cards.

Therefore, our study still has many avenues to go in the future. These directions are containing but not limited to:

1. Cards can be exchanged between friends. If there are two people to collect cards together, how much money would they spend in total? How about three people to collect cards?
2. There are a certain number of regular cards (83%) and special cards (17%) in the total of 400 cards. If a collector wants to collect only all the regular cards or all the special cards, how much money would he/she spend?
3. If we can also sell the duplicate cards to other players for the price of 4 dollars per card, what's the total cost for us?
4. If there is a high-efficiency trade market that could let players trade cards freely, what's the expected price of each card should be?

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