On the Fixed-Parameter Tractability of Some Matching Problems Under the Color-Spanning Model

Sergey Bereg[1], Feifei Ma[2], Wencheng Wang[2], Jian Zhang[2], and Binhai Zhu[3](✉)

[1] Department of Computer Science, University of Texas-Dallas, Richardson, TX 75080, USA
besp@utdallas.edu

[2] State Key Laboratory of Computer Science, Institute of Software, Chinese Academy of Sciences, Beijing 100080, China
{maff,whn,zj}@ios.ac.cn

[3] Gianforte School of Computing, Montana State University, Bozeman, MT 59717, USA
bhz@montana.edu

Abstract. Given a set of n points P in the plane, each colored with one of the t given colors, a color-spanning set $S \subset P$ is a subset of t points with distinct colors. The minimum diameter color-spanning set (MDCS) is a color-spanning set whose diameter is minimum (among all color-spanning sets of P). Somehow symmetrically, the largest closest pair color-spanning set (LCPCS) is a color-spanning set whose closest pair is the largest (among all color-spanning sets of P). Both MDCS and LCPCS have been shown to be NP-complete, but whether they are fixed-parameter tractable (FPT) when t is a parameter are still open. (Formally, the problem whether MDCS is FPT was posed by Fleischer and Xu in 2010.) Motivated by this question, we consider the FPT tractability of some matching problems under this color-spanning model, where $t = 2k$ is the parameter. The results are summarized as follows: (1) Min-Sum Matching Color-Spanning Set, namely, computing a matching of $2k$ points with distinct colors such that their total edge length is minimized, is FPT; (2) MaxMin Matching Color-Spanning Set, namely, computing a matching of $2k$ points with distinct colors such that the minimum edge length is maximized, is FPT; (3) MinMax Matching Color-Spanning Set, namely, computing a matching of $2k$ points with distinct colors such that the maximum edge length is minimized, is FPT; and (4) k-Multicolored Independent Matching, namely, computing a matching of $2k$ vertices in a graph such that the vertices of the edges in the matching do not share common edges in the graph, is W[1]-hard. With (2), we show that LCPCS is in fact FPT.

1 Introduction

Given a set of n points Q with all points colored in one of the t given colors, a *color-spanning* set (sometimes also called a *rainbow* set) is a subset of t points

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with distinct colors. (In this paper, as we focus on matching problems, we set $t = 2k$. Of course, in general t does not always have to be even.) In practice, many problems require us to find a specific color-spanning set with certain property due to the large size of the color-spanning sets. For instance, in data mining a problem arises where one wants to find a color-spanning set whose diameter is minimized (over all color-spanning sets), which can be solved in $O(n^t)$ time using a brute-force method [2,14]. (Unfortunately, this is still the best bound to this date.)

Since the color-spanning set problems were initiated in 2001 [1], quite some related problems have been investigated. Many of the traditional problems which are polynomially solvable, like Minimum Spanning Tree, Diameter, Closest Pair, Convex Hull, etc., become NP-hard under the color-spanning model [6,7,9]. Note that for the hardness results the objective functions are usually slightly changed. For instance, in the color-spanning model, we would like to maximize the closest pair and minimize the diameter (among all color-spanning sets). On the other hand, some problems, like the Maximum Diameter Color-Spanning Set, remain to be polynomially solvable [4].

In [6,7], an interesting question was raised. Namely, if t is a parameter, is the NP-complete Minimum Diameter Color-Spanning Set (MDCS) problem fixed-parameter tractable? This question is still open. In this paper, we try to investigate some related questions along this line. The base problem we target at is the matching problem, both under the geometric model and the graph model. We show that an important graph version is W[1]-hard while all other versions in consideration are fixed-parameter tractable (FPT). With that, we show that the symmetric version of MDCS, the Largest Closest Pair Color-Spanning Set is in fact FPT.

This paper is organized as follows. In Sect. 2, we define the basics regarding FPT algorithms and the problems we will investigate. In Sect. 3, we illustrate the positive FPT results on the geometric version MinSum Matching (and a related graph version). In Sect. 4, we show the positive results on the MaxMin Matching and MinMax Matching under the color-spanning model. In Sect. 5, we show that a special graph version is W[1]-hard. In Sect. 5, we conclude the paper.

2 Preliminaries

We make the following definitions regarding this paper. An Fixed-Parameter Tractable (FPT) algorithm is an algorithm for a decision problem with input size n and parameter k whose running time is $O(f(t)n^c) = O^*(f(t))$, where $f(-)$ is any computable function on t and c is a constant. FPT algorithms are efficient tools for handling some NP-complete problems as they introduce an extra dimension t. If an NP-complete problem, like Vertex Cover, admits an FPT algorithm, then it is basically polynomially solvable when the parameter t is a small constant [3,8].

Of course, it is well conceived that not all NP-hard problems admit FPT algorithms. It has been established that

$$FPT \subseteq W[1] \subseteq W[2] \subseteq \cdots W[z] \subseteq XP,$$

where XP represents the set of problem which must take $O(n^t)$ time to solve (i.e., not FPT), with t being the parameter. Typical problems in W[1] include Independent Set and Clique, etc. For the formal definition and foundation, readers are referred to [3,8].

Given a set Q of n points in the plane with t colors, a *color-spanning* set $S \subset Q$ is a subset of t points with distinct colors. If S satisfies a property Π among all color-spanning sets of Q, we call the corresponding problem of computing S the Property-Π Color-Spanning Set. For instance, the Minimum Diameter Color-Spanning Set (MDCS) is one where the diameter of S is minimized (among all color-spanning sets of Q) and the Largest Close Pair Color-Spanning Set (LCPCS) is one where the closest pair of S is maximized (among all color-spanning sets of Q). All the distances between two points in the plane are Euclidean (or L_2). We next define the matching problems we will investigate in this paper.

Given a set P of n points in the plane with $2k$ colors, a *color-spanning* set $S \subset P$ is a subset of $2k$ points with distinct colors. The points in S always form a perfect matching, i.e., a set M of k edges connecting the $2k$ points in S. Among all these matchings (over all color-spanning sets), if a matching M satisfies a property Π, we call the problem the *Property-Π Matching Color-Spanning Set* or *Property-Π Color-Spanning Matching*. The three properties we focus on are MinSum, MinMax and MaxMin.

MinSum means that the sum of edge lengths in M is minimized, MinMax means that the maximum edge length in M is minimized, and MaxMin means that the minimum edge length in M is maximized. The main purpose of this paper is to investigate the FPT tractability of the three problems: MinSum Matching Color-Spanning Set, MinMax Matching Color-Spanning Set, and MaxMin Matching Color-Spanning Set. We show that all these problems are in fact FPT.

We also briefly mention some of the related problems on graphs, where we are given a general graph G whose vertices are colored with $2k$ colors, the problem is to determine whether a perfect matching M exists such that M contains exactly $2k$ vertices of distinct colors. We call this problem k-Multicolored Matching, and we will show that this problem is FPT.

Finally, we will study a special version on graphs where the (vertices of the) edges in M cannot share edges in G. We call the problem k-Multicolored Independent Matching, and we will show that this problem is W[1]-hard.

3 MinSum Matching Color-Spanning Set Is FPT

In this section, we consider the MinSum Matching Color-Spanning Set (MSMCS) problem, namely, given a set P of n points in the plane, each colored with one of the $2k$ colors, identify $2k$ points with distinct colors such that they induce a matching with the minimum total weight (among all feasible color-spanning matchings). Recall that the weight of an edge (p_i, p_j) is the Euclidean distance between p_i and p_j. For a point p_i, let $color(p_i)$ be the color of p_i. For this problem, we in fact have a good property of the optimal solution which is stated as follows.

Lemma 1. *In an optimal solution of MSMCS, let p_i and p_j be a matched edge in the optimal matching, then (p_i, p_j) must be the closest pair between points of $color(p_i)$ and $color(p_j)$.*

With this property, among $\binom{2k}{2}$ pairs of colors, we need to select k disjoint pairs and for each feasible solution compute the matching (by computing the bichromatic closest pair between points in each of the k paired colors) in $O(kn \log n)$ time [13]. This gives an FPT algorithm running in $k^{O(k)}O(n \log n)$ time. We hence have

Theorem 1. *MinSum Matching Color-Spanning Set (MSMCS) is FPT.*

We next consider the graph version of the MSMCS problem, or, the k-Multicolored Matching problem, which is formally defined as follows.

INSTANCE: An undirected graph $G = (V, E)$ with each vertex colored with one of the $2k$ given colors.

QUESTION: Is there a matching $E' \subseteq E$ including all the $2k$ colors? That is, are there k disjoint edges in E', and all the vertices of the edges in E' have different colors.

The following theorem shows that k-Multicolored Matching is also FPT.

Theorem 2. *k-Multicolored Matching is FPT.*

Proof. We could simulate the method for Theorem 1 as follows. First, make G into a complete weighted graph (K_n, w). For an edge $e \in E(K_n)$, if $e \in E(G)$ then set $w(e) = 1$; if $e \notin E(G)$, then set $w(e) = 2$. Then, once the k pairs of colors are given, we could compute the minimum weight matching (by computing the bichromatic closest pair between points in each of the k paired colors) in $O(kn^2)$ time. Then, there is a solution for k-Multicolored Matching if and only if the minimum color-spanning matching, over all possible pairs of $2k$ colors, has a weight k. Similar to Theorem 1, the running time is $k^{O(k)}O(n^2)$. The theorem is hence proved. $\square$

In the next section, we investigate the MaxMin and MinMax Matching Color-Spanning Sets problems.

4 MaxMin and MinMax Matching Color-Spanning Sets Are FPT

We first study the MaxMin Matching Color-Spanning Set problem. The first attempt is to try to see whether a property similar to Lemma 1 holds or not.

In Fig. 1, we show an example where such a property does not hold for MaxMin Matching Color-Spanning Set, i.e., the minimum edge length is maximized among all feasible color-spanning matchings. In Fig. 1, the MinSum Color-Spanning Matching is $\{(a,c),(b,d)\}$, with a total weight of $3-2\epsilon$. The Max-Sum Color-Spanning Matching is $\{(c,d),(e,f)\}$, with a total weight of $3+6\epsilon$. The optimal solution for MaxMin Matching Color-Spanning Set is $\{(c,e),(d,f)\}$, with a solution value of 1.5 (while the total weight is 3). Again, note that (c,e) and (d,f) do not form the closest pairs among the subsets of respective colors.

Nonetheless, it is easy to see that MaxMin Matching Color-Spanning Set is NP-complete. The reduction is from LCPCS: just compute an optimal solution for the MaxMin Color-Spanning Matching on set P, the edge with the minimum weight in the matching must be a solution for LCPCS. We next show that Maxmin Matching Color-Spanning Set is FPT.

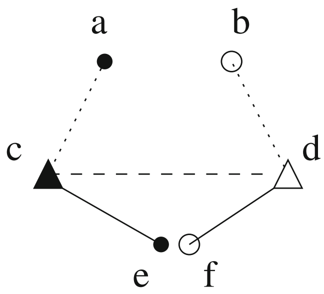

Fig. 1. A simple multicolored point set, the dotted, dashed and solid segments have lengths $1.5-\epsilon$, 3 and 1.5 respectively; moreover, $d(e,f)=6\epsilon$.

Theorem 3. *MaxMin Matching Color-Spanning Set is NP-complete and is FPT.*

Proof. We first enumerate all possible ways to pair colors. Following Theorem 1, there are an $k^{O(k)}$ number of such valid pairs. Let C_i and C_j be the set of points with $color_i$ and $color_j$ respectively. If in the optimal solution for MaxMin Matching Color-Spanning Set $color_i$ and $color_j$ are paired together, then the edge connecting two points $p \in C_i$ and $q \in C_j$ must be the ℓ-th closest pair between C_i, C_j, with $\ell \leq k$. (The reason is that there are only k edges in the matching, the other $k-1$ edges in the matching could make at most the first $k-1$ closest pairs between C_i and C_j infeasible — or, are longer than these closest pairs.) Then our FPT algorithm for MaxMin Matching Color-Spanning set is straightforward: given each valid pairing of $2k$ colors, for each pair of colors $color_i$ and $color_j$, compute the $\leq k$ closest pairs between the corresponding point sets with $color_i$ and $color_j$ respectively, then enumerate k^k possible solutions for the matching and return the one whose minimum edge length is maximized. For a fixed pairing of $2k$ colors, this takes $O^*(k^k)$ time. As we have $k^{O(k)}$ number of such pairings, the total running time is still $O^*(k^{O(k)})$. $\square$

Corollary 1. *The parameterized version of LCPCS, where the number of colors k is the parameter, is FPT.*

Now we consider the MinMax Matching Color-Spanning Set, namely, the maximum edge length is minimized among all feasible color-spanning matchings. Not surprisingly, the property illustrated in Lemma 1 also does not hold. In fact, such a matching might have nothing to do with the MinSum Color-Spanning Matching or the MaxSum Color-Spanning Matching. In Fig. 2, the MinSum Color-Spanning Matching is $\{(a, b), (c, d)\}$, with a total weight of 3. The MaxSum Color-Spanning Matching is $\{(a, c), (b, d)\}$ or $\{(c, d), (e, f)\}$, with a total weight of $4 + 2\epsilon$. For the MinMax Color-Spanning Matching problem, all of the above matchings give a solution value of $2 + \epsilon$. The optimal solution is $\{(c, e), (d, f)\}$, with a solution value of $1.5 + \epsilon$ (while the total weight is $3 + 2\epsilon$). Also, note that (c, e) and (d, f) do not form the farthest pairs among the subsets of respective colors.

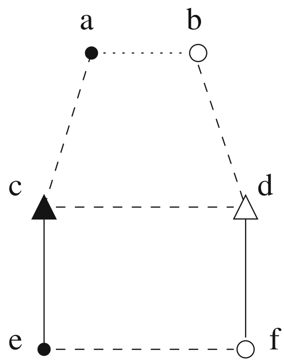

Fig. 2. A simple multicolored point set, the dotted, dashed and solid segments have lengths $1 - \epsilon$, $2 + \epsilon$ and $1.5 + \epsilon$ respectively.

It is unknown whether MinMax Color-Spanning Matching is NP-complete. Nonetheless, we show that it is FPT. The algorithm is almost identical to that for MaxMin Color-Spanning Matching in Theorem 4. The only difference is that when a pairing is fixed for point sets C_i and C_j, with colors $color_i$ and $color_j$ respectively, we enumerate the $\leq k$ farthest pairs between C_i and C_j.

Corollary 2. *MinMax Color-Spanning Matching is FPT.*

In the next section, we show that a special version on graphs is in fact W[1]-hard.

5 *k*-Multicolored Independent Matching Is W[1]-Hard

The k-Multicolored Independent Matching problem is defined as follows.

INSTANCE: An undirected graph $G = (V, E)$ with each vertex colored with one of the $2k$ given colors.

QUESTION: Is there an independent matching $E' \subseteq E$ including all the k colors? That is, are there k edges in E' such that any two edges in E' are not incident to any vertex in V, and all the vertices of the edges in E' have different colors.

The problem originates from an application in shortwave radio broadcast, where the matched nodes represent the shortwave channels which should not directly affect each other [12]. (We also comment that this problem seems to be related to the uncolored version of 'Induced Matching' which is known to be W[1]-hard as well [10,11].) We will show that this problem is not only NP-complete but also W[1]-hard. The problem to reduce from is the k-Multicolored Independent Set, which is defined as follows.

INSTANCE: An undirected graph $G = (V, E)$ with each vertex colored with one of the k given colors.

QUESTION: Is there an independent set $V' \subseteq V$ including all the k colors? That is, are there k vertices in V' incurring no edge in E, and all the vertices in V' have different colors.

When $U \subseteq V$ contains exactly k vertices of different colors, we also say that U is *colorful*.

For completeness, we first prove the following lemma, similar to what was done by Fellows *et al.* on k-Multicolored Clique problem [5].

Lemma 2. *k-Multicolored Independent Set is W[1]-complete.*

Proof. The proof can be done through a reduction from k-Independent Set. Given an instance $(G = (V, E), k)$ for k-Independent Set, we first make k copies of G, G_i's, such that the vertices in each G_i are all colored with color i, for $i = 1..k$. For any $u \in V$, let u_i be the corresponding mirror vertex in G_i. Then, for each $(u, v) \in E$ and for each pair of i, j, with $1 \leq i \neq j \leq k$, we add four edges (u_i, u_j), (v_i, v_j), (u_i, v_j) and (u_j, v_i). Let the resulting graph be G'. It is easy to verify that G has a k-independent set if and only if G' has a k-multicolored independent set. As k-Independent Set is W[1]-complete [3], the lemma follows. □

The following theorem shows that k-Multicolored Independent Matching is not only NP-complete but also W[1]-hard.

Theorem 4. *k-Multicolored Independent Matching is W[1]-hard, i.e., it does not admit any FPT algorithm unless FPT=W[1].*

Proof. We reduce k-Multicolored Independent Set (IS) to the k-Multicolored Independent Matching problem.

Given an instance of k-Multicolored IS problem, i.e., a graph $G = (V, E)$ with each vertex in $V = \{v_1, v_2, \ldots, v_n\}$ colored with one of the k colors $\{1, 2, \ldots, k\}$, the question is whether one could compute an IS of size k, each with a distinct color.

We construct an instance for the k-Multicolored Independent Matching as follows. First, make a copy of G (with the given coloring of k colors). Then, construct a set $U = \{u_1, u_2, \ldots, u_k\}$ such that u_i has color $k + i$. Finally, we connect each $u_i \in U$ to each $v_j \in V$, i.e., we construct a set $E' = \{(u_i, v_j)|u_i \in U, v_j \in V, 1 \leq i \leq k, 1 \leq j \leq n\}$. Let the resulting graph be $G' = (V \cup U, E \cup E')$, with each vertex in G' colored with one of the $2k$ colors. We claim that G has a colorful independent set of size k if and only if G' has a colorful independent matching of size k. The details are given as follows.

If G has a colorful independent set $V' \subseteq V$ of size k, we select the k vertices in V' and match them up with the k vertices in U. As the vertices in V' are independent and no two vertices in U share an edge (i.e., vertices in U are also independent), we have a colorful independent matching for G'.

If G' has a colorful independent matching of size k, then exactly k vertices of V must match up with the k vertices in U. (Otherwise, if two vertices v_i and v_j in V form an edge in the optimal colorful matching then we cannot have k edges in the matching. This is because at least two vertices in U cannot match up with the vertices in V of the same color as v_i and v_j. Then the colorful matching contains at most $k-1$ edges, a contradiction.) By the definition of colorful independent matching, the k vertices from V cannot share any edge hence form a independent set for G.

As the reduction takes polynomial time, the theorem is proved. □

We have the following corollary.

Corollary 3. *The optimization version of k-Multicolored Independent Matching (called Multicolored Maximum Independent Matching) does not admit a factor $n^{1-\epsilon}$ polynomial-time approximation, for some $\epsilon > 0$, unless P = NP.*

Proof. As the reductions in Lemma 2 and Theorem 4 are both L-reductions, the Multicolored Maximum Independent Matching problem is as hard to approximate as the Independent Set problem, which does not admit a factor $n^{1-\epsilon}$ polynomial-time approximation, for some $\epsilon > 0$, unless P = NP [15]. □

6 Closing Remarks

Motivated by the open question of Fleischer and Xu, we studied the FPT tractability of LCPCS and some related matching problems under the color-spanning model. We show in this paper that most of these problems are FPT, except one version on graphs which can be considered as a generalization of the multicolored independent set problem. The original question on the FPT tractability of Minimum Diameter Coloring-Spanning Set (MDCS), is, unfortunately, still open.

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