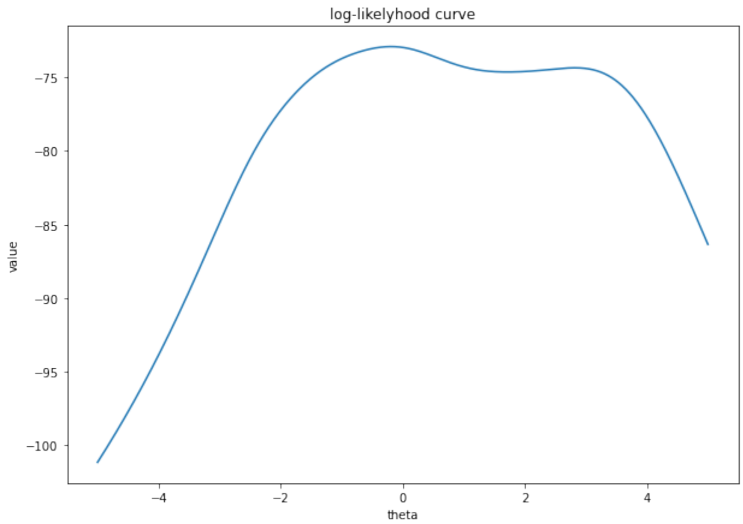
高等统计计算第 1 次作业

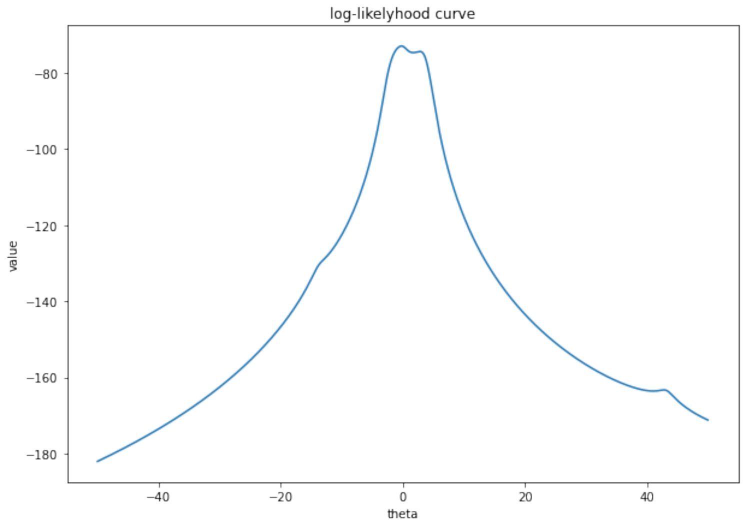
自硕 21 崔晏菲 2021210976

注：因为我不会 R，所以代码都是用 python 写的。代码文件见 homework1-code.ipynb

2.1

a.





初始值	收敛值
-11	-4.30537E+11
-1	-0.192286613
0	-0.192286613
1.5	1.713586835
4	2.817472166
4.7	-0.192286613
7	41.04084782
8	-6.00407E+11
38	42.79537747

可见，初始值的选取会影响最优值得收敛，这是显然的，因为这个对数似然函数并不是凸函数。数据点的平均值不是好选择，因为柯西分布的数学期望是发散的。

b.

得到最优解为-0.5，最优值为-73.050692。显然这个方法大部分情况下并不能获得全局最优值，因为这要求函数的导数在区间内是单调的，但是我们的对数似然函数的导数并不是单调的。

c.

```
for alpha in [1, 0.64, 0.25]:
    point3 = log_likely.fixed_point_method(-1, alpha, delta = 1e-10)
    if(point3 is not None):
        print("最优解为%f, 最优值为%f"%(point3, log_likely.value(point2)))
```

```
not lipschitz continous at scale factor = 1.000000
not lipschitz continous at scale factor = 0.640000
not lipschitz continous at scale factor = 0.250000
```

不动点法很难使用，因为很难保证 lipchitz 连续。

d.

```
: log_likely.secant_method(x0=-2, x1=-1, delta = 1e-10)
```

```
: -0.1922866132319395
```

```
: log_likely.secant_method(x0=-3, x1=3, delta = 1e-10)
```

```
: 2.8174721655730948
```

可以看到，算法收敛到不同的极值点，这是很正常的，毕竟这个函数不是凸函数。

e.

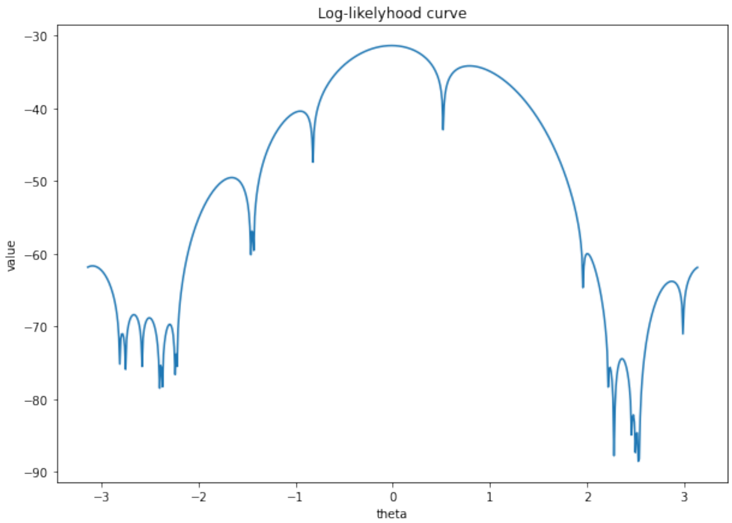
速度由大到小：二分法、牛顿法、弦截法、不动点法

稳定性由大到小：牛顿法、弦截法、二分法、不动点法

结论会在样本数据量改变时改变。

2.2

a.



b.

$$\int_0^{2\pi} xf(x)dx = \frac{1}{2\pi}\int_0^{2\pi} x(1-\cos(x-\theta))dx$$

$$= \pi + \sin\theta$$

故

$$\pi + \sin\theta = \frac{1}{n}\sum_{i=1}^{n} x_n$$

解得

$$\theta \approx 0.05844060614042408$$

c.

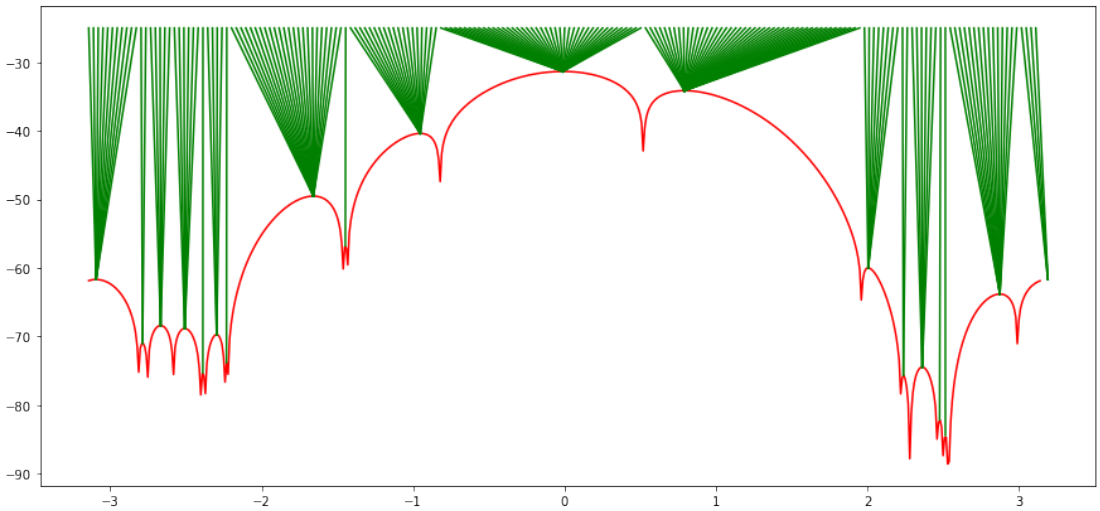
用上一问初始点，算出$MLE = -0.011972002283305973$

用-2.7，算出$MLE = -2.666699926100948$

用 2.7，算出$MLE = 2.8730945142450826$

可见，牛顿法会快速收敛到距离最近的极值点。

d.



可见，牛顿法会快速收敛到距离最近的极值点。

e.

2.4

解：概率密度函数为

$$f(x)=\frac{1}{\Gamma(2)}xe^{-x}=xe^{-x}$$

令$f'(x)=(1+x)e^{-x}=0$，得$x=1$是极值点，此时$f(x)=\frac{1}{e}$。设 95%概率密度区间里的最小概率密度为M，则有$0\le M\le\frac{1}{e}$。

设$f(x)=xe^{-x}-M=0$的解为x_1,x_2，则有

$$\int_{x_1}^{x_2}xe^{-x}dx=0.95$$

$$e^{-x_1}(1+x_1)-e^{-x_2}(1+x_2)=0.95$$

得

$$\begin{cases}e^{-x_1}-e^{-x_2}=0.95\\ x_1e^{-x_1}=x_2e^{-x_2}\end{cases}$$

使用数值算法解得

$$x_1=0.042364,x_2=4.76524323$$

$[0.042364,4.76524323]$就是$Gamma(2,1)$的最窄 95%概率区间.

2.5

解：

a. 假设我们已经知道了$N_1,\cdots,N_i,N_n\ (i=1,\cdots,n)$，对于$N_i$的概率密度函数为

$$f(N_i)=\frac{(\alpha_1b_{i1}+\alpha_2b_{i2})^{N_i}}{N_i!}e^{-(\alpha_1b_{i1}+\alpha_2b_{i2})}$$

故，对数似然函数为

$$\begin{aligned}l(\alpha_1,\alpha_2|\,b_1,b_2,N)&=\ln L(\alpha_1,\alpha_2|\,b_1,b_2,N)\\&=-\alpha_1\sum_{i=1}^n b_{i1}-\alpha_2\sum_{i=1}^n b_{i2}-\sum_{i=i}^n\ln(N_i!)\\&+\sum_{i=i}^n \mathrm{N_i}\cdot\ln(\alpha_1b_{i1}+\alpha_2b_{i2})\end{aligned}$$

求导，得

$$\nabla l(\alpha_1,\alpha_2)=\begin{pmatrix}-\sum_{i=1}^n b_{i1}+\sum_{i=i}^n\frac{N_ib_{i1}}{\alpha_1b_{i1}+\alpha_2b_{i2}}\\-\sum_{i=1}^n b_{i2}+\sum_{i=i}^n\frac{N_ib_{i2}}{\alpha_1b_{i1}+\alpha_2b_{i2}}\end{pmatrix}$$

Hessian 矩阵为

$$H(\alpha_1,\alpha_2)=-\sum_{i=i}^{n}\begin{pmatrix}\dfrac{N_i b_{i1}^2}{(\alpha_1 b_{i1}+\alpha_2 b_{i2})^2} & \dfrac{N_i b_{i1} b_{i2}}{(\alpha_1 b_{i1}+\alpha_2 b_{i2})^2}\\ \dfrac{N_i b_{i1} b_{i2}}{(\alpha_1 b_{i1}+\alpha_2 b_{i2})^2} & \dfrac{N_i b_{i2}^2}{(\alpha_1 b_{i1}+\alpha_2 b_{i2})^2}\end{pmatrix}$$

$$=-\sum_{i=i}^{n}\frac{N_i}{(\alpha_1 b_{i1}+\alpha_2 b_{i2})^2}\begin{pmatrix} b_{i1}^2 & b_{i1}b_{i2}\\ b_{i1}b_{i2} & b_{i2}^2\end{pmatrix}$$

因此，得到牛顿迭代公式为

$$\alpha^{(t+1)}=\alpha^{(t)}-H^{-1}\left(\alpha^{(t)}\right)\cdot\nabla l\left(\alpha^{(t)}\right)$$

b. Fisher 信息量矩阵为

$$I(\alpha_1,\alpha_2)=-E\left(H(\alpha_1,\alpha_2)\right)$$

$$=\sum_{i=i}^{n}\int_0^{+\infty}\frac{N_i}{(\alpha_1 b_{i1}+\alpha_2 b_{i2})^2}\begin{pmatrix} b_{i1}^2 & b_{i1}b_{i2}\\ b_{i1}b_{i2} & b_{i2}^2\end{pmatrix}\frac{(\alpha_1 b_{i1}+\alpha_2 b_{i2}\,)^{N_i}}{N_i!}e^{-(\alpha_1 b_{i1}+\alpha_2 b_{i2})}\,dN_i$$

$$=\sum_{i=i}^{n}\int_0^{+\infty}\frac{1}{(\alpha_1 b_{i1}+\alpha_2 b_{i2})}\begin{pmatrix} b_{i1}^2 & b_{i1}b_{i2}\\ b_{i1}b_{i2} & b_{i2}^2\end{pmatrix}\frac{(\alpha_1 b_{i1}+\alpha_2 b_{i2}\,)^{N_i-1}}{(N_i-1)!}e^{-(\alpha_1 b_{i1}+\alpha_2 b_{i2})}\,dN_i$$

$$=\sum_{i=i}^{n}\frac{1}{(\alpha_1 b_{i1}+\alpha_2 b_{i2})}\begin{pmatrix} b_{i1}^2 & b_{i1}b_{i2}\\ b_{i1}b_{i2} & b_{i2}^2\end{pmatrix}\int_0^{+\infty}\frac{(\alpha_1 b_{i1}+\alpha_2 b_{i2}\,)^{N_i-1}}{(N_i-1)!}e^{-(\alpha_1 b_{i1}+\alpha_2 b_{i2})}\,dN_i$$

$$=\sum_{i=i}^{n}\frac{1}{(\alpha_1 b_{i1}+\alpha_2 b_{i2})}\begin{pmatrix} b_{i1}^2 & b_{i1}b_{i2}\\ b_{i1}b_{i2} & b_{i2}^2\end{pmatrix}$$

故 Fisher Scoring 迭代公式为

$$\alpha^{(t+1)}=\alpha^{(t)}+I^{-1}\left(\alpha^{(t)}\right)\cdot\nabla l\left(\alpha^{(t)}\right)$$

c. 牛顿法和 Fisher Scoring 方法得到的结果一样，$\alpha_1=1.097153,\alpha_2=0.937555$，但是牛顿法的迭代次数明显少于 Fisher Scoring 方法。这两个方法的实现难度差不多。

当设置的误差$\epsilon=1\times10^{-10}$时，结果如下：

```
牛顿法搜索得到的最优点为：(1.097153, 0.937555)
迭代次数为6，迭代过程为：
(1.000000, 1.000000)
(1.090831, 0.942676)
(1.097128, 0.937575)
(1.097153, 0.937555)
(1.097153, 0.937555)
(1.097153, 0.937555)
```

```
Fisher Scoring方法搜索得到的最优点为: (1.097153, 0.937555)
迭代次数为21, 迭代过程为:
(1.000000, 1.000000)
(1.117785, 0.906284)
(1.090702, 0.947331)
(1.099195, 0.934459)
(1.096508, 0.938531)
(1.097356, 0.937246)
(1.097088, 0.937652)
(1.097173, 0.937524)
(1.097146, 0.937564)
(1.097155, 0.937552)
(1.097152, 0.937556)
(1.097153, 0.937554)
(1.097152, 0.937555)
(1.097153, 0.937555)
(1.097153, 0.937555)
(1.097153, 0.937555)
(1.097153, 0.937555)
(1.097153, 0.937555)
(1.097153, 0.937555)
(1.097153, 0.937555)
(1.097153, 0.937555)
```

d. 得到在 MLE 处的 Fisher 信息量为

```
np.sqrt(logPoisson.fisherI(max_point_Newton))
✓ 0.0s
```

```
array([[4.05118009, 3.06428095],
       [3.06428095, 2.80713521]])
```

故$std(\alpha_1) = 4.051180009, std(\alpha_2) = 2.80713521$

e. Sdada

```
Steepest Ascent方法搜索得到的最优点为: (1.097153, 0.937555)
迭代次数为65, 迭代过程为:
(1.000000, 1.000000)
(1.069551, 1.024199)
(1.049867, 0.996378)
(1.082775, 0.990198)
(1.068151, 0.971739)
(1.090170, 0.969765)
(1.079268, 0.957362)
(1.086722, 0.957386)
(1.085973, 0.948943)
(1.091176, 0.949461)
(1.090110, 0.944052)
(1.093764, 0.944723)
(1.092678, 0.941222)
(1.095259, 0.941884)
(1.094770, 0.940739)
(1.096113, 0.940178)
(1.095704, 0.939422)
(1.096597, 0.939151)
(1.096268, 0.938647)
(1.096868, 0.938531)
(1.096609, 0.938191)
(1.097017, 0.938155)
(1.096817, 0.937924)
...
(1.097153, 0.937555)
(1.097153, 0.937555)
(1.097153, 0.937555)
(1.097153, 0.937555)
```

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f.

```
Quasi-Newton方法使用StepHalvingBacktracking搜索得到的最优点为: (1.097153, 0.937555)
迭代次数为10, 迭代过程为:
(1.000000, 1.000000)
(1.069551, 1.024199)
(1.116833, 0.923460)
(1.098062, 0.937130)
(1.097143, 0.937555)
(1.097152, 0.937555)
(1.097153, 0.937555)
(1.097153, 0.937555)
(1.097153, 0.937555)
(1.097153, 0.937555)
```

Quasi-Newton方法不使用StepHalvingBacktracking搜索得到的最优点为：(1.097153, 0.937555)
迭代次数为11，迭代过程为：
(1.000000, 1.000000)
(2.112823, 1.387177)
(1.218099, 0.819302)
(0.894957, 1.183313)
(1.109377, 0.917175)
(1.098494, 0.941024)
(1.097771, 0.937359)
(1.097155, 0.937550)
(1.097153, 0.937555)
(1.097153, 0.937555)
(1.097153, 0.937555)

g.