THU-70250403, Convex Optimization (Fall 2021) **Homework: 3**

Convex Functions (continue)

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Problem 1

For each of the following functions determine whether it is convex, concave, quasiconvex, or quasiconcave [1].

1. $f(x) = e^x - 1$ on $\mathbb{R}$.
2. $f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_{++}$.
3. $f(x_1, x_2) = 1/(x_1 x_2)$ on $\mathbb{R}^2_{++}$.
4. $f(x_1, x_2) = x_1/x_2$ on $\mathbb{R}^2_{++}$.
5. $f(x_1, x_2) = x_1^2/x_2$ on $\mathbb{R} \times \mathbb{R}_{++}$.
6. $f(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$, where $0 \leqslant \alpha \leqslant 1$, on $\mathbb{R}^2_{++}$.

Problem 2

Composition rules. Show that the following functions are convex [1].

1. $f(x) = -\log(-\log(\sum_{i=1}^{m} e^{\boldsymbol{a}_i^T \boldsymbol{x} + b_i}))$ on $\mathrm{dom} f = \{\boldsymbol{x} \mid \sum_{i=1}^{m} e^{\boldsymbol{a}_i^T \boldsymbol{x} + b_i} < 1\}$. You can use the fact that $\log(\sum_{i=1}^{n} e^{y_i})$ is convex.
2. $f(x, u, v) = -\sqrt{uv - \boldsymbol{x}^T \boldsymbol{x}}$ on $\mathrm{dom} f = \{(\boldsymbol{x}, u, v) \mid uv > \boldsymbol{x}^T \boldsymbol{x},\ u,\ v > 0\}$. Use the fact that $\boldsymbol{x}^T \boldsymbol{x}/u$ is convex in $(\boldsymbol{x}, u)$ for $u > 0$, and that $-\sqrt{x_1 x_2}$ is convex on $\mathbb{R}^2_{++}$.
3. $f(x, t) = -(t^p - |\boldsymbol{x}|_p^p)^{1/p}$ where $p > 1$ and $\mathrm{dom} f = \{(\boldsymbol{x}, t) \mid t \geqslant |\boldsymbol{x}|_p\}$. You can use the fact that $|\boldsymbol{x}|_p^p / u^{p-1}$ is convex in $(\boldsymbol{x}, u)$ for $u > 0$ (see Problem 1) , and that $-x^{1/p} y^{1-1/p}$ is convex on $\mathbb{R}^2_+$

Problem 3

(This problem is Optional)

Let $f_0, \ldots, f_n : \mathbb{R} \to \mathbb{R}$ be given continuous functions. We consider the problem of approximating f_0 as a linear combination of $f_1, \ldots, f_n$. For $\boldsymbol{x} \in \mathbb{R}^n$, we say that $f = x_1 f_1 + \cdots + x_n f_n$ approximates f_0 with tolerance $\epsilon > 0$

over the interval $[0, T]$ if $|f(t) - f_0(t)| \leqslant \epsilon$ for $0 \leqslant t \leqslant T$. Now we choose a fixed tolerance $\epsilon > 0$ and define the *approximation width* as the largest T such that f approximates f_0 over the interval $[0, T]$:

$$W(\boldsymbol{x}) = \sup\{T \mid |x_1 f_1(t) + \cdots + x_n f_n(t) - f_0(t)| \leqslant \epsilon \text{ for } 0 \leqslant t \leqslant T\}.$$

Show that W is quasiconcave [1].

References

[1] S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, 2004. http://www.stanford.edu/~boyd/cvxbook/