THU-70250403, Convex Optimization (Fall 2021) **Homework: 11**

Unconstrained Minimization

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Student:

Problem 1

i) Suppose $f(x)$ and $g(x)$ are two convex functions on $\mathbb{R}$, please show how many intersection points they can have at most.

ii) Suppose $f(x)$ and $g(x)$ are two convex functions on $\mathbb{R}$ which have only countable intersection points, please show how many intersection points they can have at most.

Problem 2

Let us consider a unconstrained minimization constant step size $t \in \mathbb{R}_{++}$.

Suppose

i) $f : \mathbb{R}^{\ltimes} \to \mathbb{R}$ is convex on $\mathbb{R}^n$.

ii) The gradient of f is Lipschitz continuous, such that

$$\left| \nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y}) \right|_2 \leq L \left| \boldsymbol{x} - \boldsymbol{y} \right|_2 \tag{1}$$

iii) The optimal value of f is finite and can be reached at a certain $\boldsymbol{x}^*$.

A unconstrained minimization constant step size can be formulated as

$$\boldsymbol{x}^{k+1} = \boldsymbol{x}^k - t \nabla f(\boldsymbol{x}^k) \tag{2}$$

Please prove

1)

$$f\left(\boldsymbol{x}^k - t\nabla f(\boldsymbol{x}^k)\right) \leq f(\boldsymbol{x}^k) - t\left(1 - \frac{Lt}{2}\right) \left|\nabla f(\boldsymbol{x}^k)\right|_2^2 \tag{3}$$

So, if we choose $0 < t \leq \frac{1}{L}$, we have

$$f(\boldsymbol{x}^{k+1}) \leq f(\boldsymbol{x}^k) - \frac{t}{2} \left|\nabla f(\boldsymbol{x}^k)\right|_2^2 \tag{4}$$

In other words, the value of $f(\boldsymbol{x}^k)$ is always decreasing when k increases.

2)

$$0 \leq f(\boldsymbol{x}^{k+1}) - f(\boldsymbol{x}^*) \leq \frac{1}{2t}\left(\left|\boldsymbol{x}^k - \boldsymbol{x}^*\right|_2^2 - \left|\boldsymbol{x}^{k+1} - \boldsymbol{x}^*\right|_2^2\right) \tag{5}$$

This indicates that our distance to the optimal point $\boldsymbol{x}^*$ is decreasing, when k increases.

3)

$$f(\boldsymbol{x}^k) - f(\boldsymbol{x}^*) \leq \frac{1}{2kt}\left|\boldsymbol{x}^0 - \boldsymbol{x}^*\right|_2^2 \tag{6}$$

So, the number of iterations required to satisfy $f(\boldsymbol{x}^k) - f(\boldsymbol{x}^*) \leq \epsilon$ should be $O(1/\epsilon)$.

4) If f is strong convex with $f(\boldsymbol{x}) - \frac{m}{2}\left|\boldsymbol{x}\right|_2^2$ is a convex function, we can choose $0 < t < \frac{2}{m+L}$, such that

$$\left|\boldsymbol{x}^{k+1} - \boldsymbol{x}^*\right|_2^2 \leq \left(1 - t\frac{2mL}{m+L}\right)\left|\boldsymbol{x}^k - \boldsymbol{x}^*\right|_2^2 \tag{7}$$

In other words, we have

$$f(\boldsymbol{x}^k) - f(\boldsymbol{x}^*) \leq \frac{\left(1 - t\frac{2mL}{m+L}\right)^k L}{2}\left|\boldsymbol{x}^0 - \boldsymbol{x}^*\right|_2^2 \tag{8}$$

So, the number of iterations required to satisfy $f(\boldsymbol{x}^k) - f(\boldsymbol{x}^*) \leq \epsilon$ should be $O(\log(1/\epsilon))$.

Problem 3

Please solve the following unconstrained optimization problems using the normalized steepest descent directions separately with respect to the l_1, l_2 and l_∞ norm.

The exact linear search (0.618 method) should be used for all linear searches. The initial point is taken to be $\boldsymbol{x}^0 = (0,0)^T$, and the stopping criterion is $|\nabla f(\boldsymbol{x})|_2 \leq 10^{-8}$.

It is required to draw the trajectory of the iteration points in the 2-dimensional plane (by connecting each point) and the change of the objective function value with respect to the number of iterations.

$$\min_{\boldsymbol{x}\in\mathbb{R}^2} f(\boldsymbol{x}) = \left(1 - x_1\right)^2 + 3\left(x_2 - x_1^2\right)^2 \tag{9}$$

References