THU-70250403, Convex Optimization (Fall 2021) **Homework: 4**

Convex Programming Problems

Lecturer: Li Li `li-li@tsinghua.edu.cn`

Student:

Problem 1

Problems involving l_∞ and l_1-norms. Formulate the following problems as Linear programming problems (LPs). Explain in detail the relation between the optimal solution of each problem and the solution of its equivalent LP [1].

Problem (a)

$$\min_{\boldsymbol{x}} \quad |A\boldsymbol{x} - \boldsymbol{b}|_1 \tag{1}$$
$$\text{s.t.} \quad |\boldsymbol{x}|_\infty \leq 1 \tag{2}$$

Problem (b)

$$\min_{\boldsymbol{x}} \quad |\boldsymbol{x}|_1 \tag{3}$$
$$\text{s.t.} \quad |A\boldsymbol{x} - \boldsymbol{b}|_\infty \leq 1 \tag{4}$$

Problem (c)

$$\min_{\boldsymbol{x}} \; |A\boldsymbol{x} - \boldsymbol{b}|_1 + |\boldsymbol{x}|_\infty \tag{5}$$

The matrix $A \in \mathbb{R}^{m\times n}$ and the vector $\boldsymbol{b} \in \mathbb{R}^m$ are given.

Problem 2

The sum of the largest elements of a vector.

Define $f : \mathbb{R}^n \to \mathbb{R}$ as

$$f(\boldsymbol{x}) = \sum_{i=1}^{r} \boldsymbol{x}_{[i]},$$

where r is an integer between 1 and n , and $\boldsymbol{x}_{[1]} \geqslant \boldsymbol{x}_{[2]} \geqslant \cdots \geqslant \boldsymbol{x}_{[r]}$ are the components of x sorted in decreasing order. In other words, $f(x)$ is the sum of the r largest elements of x . In this problem we study the constraint

$$f(\boldsymbol{x}) \leqslant \alpha.$$

This is a convex constraint, and equivalent to a set of $n!/(r!(n-r)!)$ linear inequalities

$$\boldsymbol{x}_{i_1} + \cdots + \boldsymbol{x}_{i_r} \leqslant \alpha, \quad 1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n.$$

The purpose of this problem is to derive a more compact representation.

1. Given a vector $\boldsymbol{x} \in \mathbb{R}^n$, show that $f(\boldsymbol{x})$ is equal to the optimal value of the LP

$$\begin{array}{ll} \text{maximize} & \boldsymbol{x}^T \boldsymbol{y} \\ \text{subject to} & \mathbf{0} \preceq \boldsymbol{y} \preceq \mathbf{1} \\ & \mathbf{1}^T \boldsymbol{y} = r \end{array}$$

   with $\boldsymbol{y} \in \mathbb{R}^n$ as variable.

2. Derive the dual of the LP in part (a). Show that it can be written as

$$\begin{array}{ll} \text{minimize} & rt + \mathbf{1}^T \boldsymbol{u} \\ \text{subject to} & t\mathbf{1} + \boldsymbol{u} \succeq \boldsymbol{x} \\ & \boldsymbol{u} \succeq \mathbf{0}, \end{array}$$

   where the variables are $t \in \mathbb{R}$, $\boldsymbol{u} \in \mathbb{R}^n$. By duality this LP has the same optimal value as the LP in (a), *i.e.*, $f(x)$. We therefore have the following result: $\boldsymbol{x}$ satisfies $f(\boldsymbol{x}) \leqslant \alpha$ if and only if there exist $t \in \mathbb{R}$, $\boldsymbol{u} \in \mathbb{R}^n$ such that

$$rt + \mathbf{1}^T \boldsymbol{u} \leqslant \alpha, \qquad t\mathbf{1} + \boldsymbol{u} \succeq \boldsymbol{x}, \qquad \boldsymbol{u} \succeq \mathbf{0}.$$

   These conditions form a set of $2n+1$ linear inequalities in the $2n+1$ variables $\boldsymbol{x}, \boldsymbol{u}, t$.

References

[1] S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, 2004. http://www.stanford.edu/~boyd/cvxbook/