THU-70250403, Convex Optimization (Fall 2021) **Homework: 8**

Parameter Estimation and Norm Approximation (continue)

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Student:

Problem 1

Please explain the geometric meaning of the following optimization problems and derive its dual problem.

$$\min_{\boldsymbol{x}\in\mathbb{R}^n} \quad \boldsymbol{x}^T Q \boldsymbol{x} \tag{1}$$

$$\text{s.t.} \quad A\boldsymbol{x} = \boldsymbol{b} \tag{2}$$

where $Q \in \mathbb{S}^n_{++}$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$.

Problem 2

Show that the following three convex problems are equivalent [1]. Carefully explain how the solution of each problem is obtained from the solution of the other problems. The problem data are the matrix $A \in \mathbb{R}^{m\times n}$ (with rows $\boldsymbol{a}_i^T$), the vector $\boldsymbol{b} \in \mathbb{R}^m$, and the constant $M > 0$.

(1) The *robust least-squares problem*

$$\text{minimize} \quad \textstyle\sum_{i=1}^m \phi(\boldsymbol{a}_i^T\boldsymbol{x} - b_i),$$

with variable $\boldsymbol{x} \in \mathbb{R}^n$, where $\phi : \mathbb{R} \to \mathbb{R}$ is defined as

$$\phi(u) = \begin{cases} u^2 & |u| \leqslant M \\ M(2|u| - M) & |u| > M. \end{cases}$$

(This function is known as the *Huber penalty function* [2].)

(2) The *least-squares problem with variable weights*

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^m (\boldsymbol{a}_i^T\boldsymbol{x} - b_i)^2/(w_i + 1) + M^2\mathbf{1}^T\boldsymbol{w} \\ \text{subject to} & \boldsymbol{w} \geq 0, \end{array}$$

with variables $x \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$, and domain $\mathcal{D} = \{(x, w) \in \mathbb{R}^n \times \mathbb{R}^m \mid \boldsymbol{w} \geq -\mathbf{1}\}$.

Hint. Optimize over $\boldsymbol{w}$ assuming $\boldsymbol{x}$ is fixed, to establish a relation with the problem (1).

(This problem can be interpreted as a weighted least-squares problem in which we are allowed to adjust the weight of the ith residual. The weight is one if $w_i = 0$, and decreases if we increase w_i. The second term in the objective penalizes large values of w, *i.e.*, large adjustments of the weights.)

(3) The *quadratic program*

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{m}(u_i^2 + 2Mv_i) \\ \text{subject to} & -\boldsymbol{u} - \boldsymbol{v} \preceq A\boldsymbol{x} - \boldsymbol{b} \preceq \boldsymbol{u} + \boldsymbol{v} \\ & 0 \preceq \boldsymbol{u} \preceq M\mathbf{1} \\ & \boldsymbol{v} \succeq 0. \end{array}$$

Problem 3

Please establish a maximum likelihood estimation for Huber loss function [2]-[3].

References

[1] S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, 2004. http://www.stanford.edu/~boyd/cvxbook/

[2] P. J. Huber, "Robust estimation of a location parameter," *The Annals of Mathematical Statistics*, vol. 35, no. 1, pp. 73-101, 1964.

[3] G. P. Meyer, "An alternative probabilistic interpretation of the Huber loss," *CVPR*, 2021. https://openaccess.thecvf.com/content/CVPR2021/papers/Meyer_An_Alternative_Probabilistic_Interpretation_of_the_Huber_Loss_CVPR_2021_paper.pdf