Convex Optimization Theory and Applications

Topic 11 - Geometric Problem

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11.0. Outline

11.1. Geometric Problems

- 11.1.1 Projection and Distance for Convex Sets
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11.2. Location Optimization

11.1. Geometric Problems

11.1.1 Projection and Distance for Convex Sets

Projection of point x on set C is defined as

$$P_C(x) = \arg\min_{z \in C} \| x - z \| \tag{11.1}$$

That is, point in C that is closest to x

Suppose C has form

$$C = \{x \mid Ax = b,\ f_i(x) \le 0, i = 1, \cdots, m\} \tag{11.2}$$

where $f_i : R^n \to R$ is convex

11.1. Geometric Problems

11.1.1 Projection and Distance for Convex Sets

We have the following convex programming problem

$$P_C(x) = \arg\min \| x - z \| \qquad (11.3)$$

subject to $Az = b$, $f_i(z) \le 0$, $i = 1, \cdots, m$

11.1. Geometric Problems

11.1.1 Projection and Distance for Convex Sets

Distance between sets C and $\widetilde{C}$ is defined as

$$dist(C,\widetilde{C}) = \min_{z \in C, \widetilde{z} \in \widetilde{C}} \| z - \widetilde{z} \| \qquad (11.4)$$

suppose sets C and $\widetilde{C}$ are convex, with form

$$C = \{x \mid Ax = b,\ f_i(x) \le 0, i = 1, \cdots, m\} \qquad (11.5)$$

$$\widetilde{C} = \{z \mid \widetilde{A}z = \widetilde{b},\ \widetilde{f}_i(z) \le 0, i = 1, \cdots, \widetilde{m}\} \qquad (11.6)$$

where f_i, $\widetilde{f}_i : R^n \to R$ are convex

11.1. Geometric Problems

11.1.1 Projection and Distance for Convex Sets

$dist(C, \widetilde{C})$ is then a convex programming problem

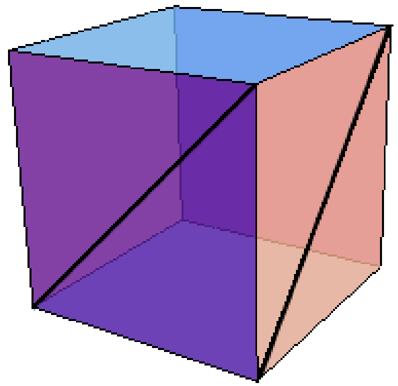
$$\min \ \| z - \widetilde{z} \| \qquad (11.7)$$

subject to $Az = b$, $\widetilde{A}\widetilde{z} = \widetilde{b}$, $f_i(z) \le 0$, $i = 1, \cdots, m$, $\widetilde{f}_i(\widetilde{z}) \le 0$, $i = 1, \cdots, \widetilde{m}$

11.1. Geometric Problems

11.1.1 Projection and Distance for Convex Sets

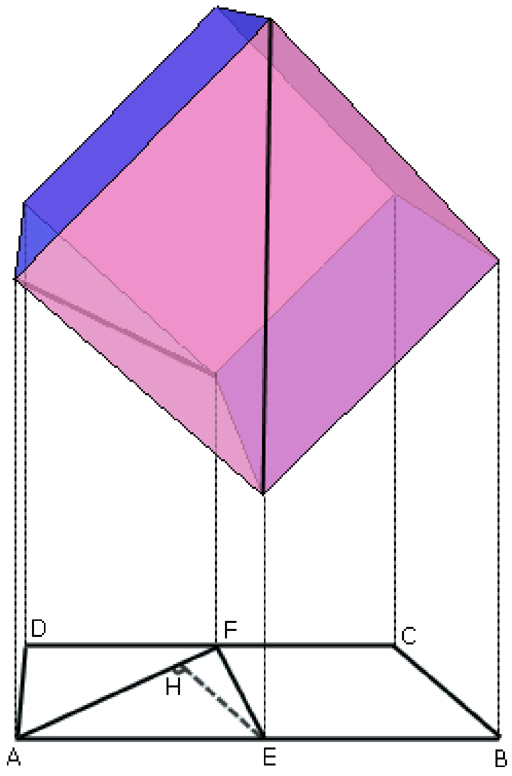
Suppose this is a unit cube.



Question: what is the minimum Euclidean distance between the two bold black lines?

11.1. Geometric Problems

11.1.1 Projection and Distance for Convex Sets

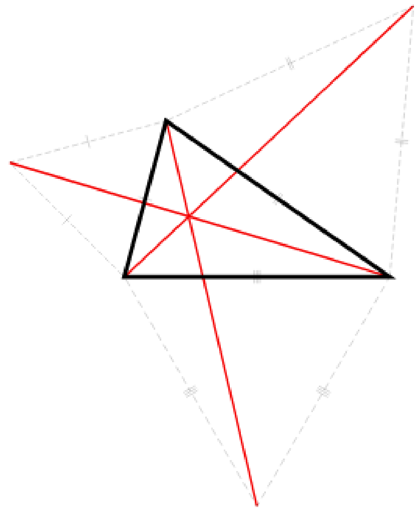


Answer: $1/\sqrt{3}$

11.1. Geometric Problems

11.1.1 Projection and Distance for Convex Sets

Fermat Point Problem: the Fermat point of a triangle, also called the Torricelli point or Fermat–Torricelli point, is a point such that the total distance from the three vertices of the triangle to the point is the minimum possible. It is so named because this problem is first raised by Fermat in a private letter to Evangelista Torricelli, who solved it.



11.1. Geometric Problems

11.1.1 Projection and Distance for Convex Sets

Another approach to find a point within the triangle, from where sum of the distances to the vertices of triangle is minimum, is to use one of the optimization (mathematics) methods. In particular, method of the Lagrange multipliers and the law of cosines.

We draw lines from the point within the triangle to its vertices and call them **X**, **Y** and **Z**. Also, let the lengths of these lines be x, y, and z, respectively. Let the angle between **X** and **Y** be α, **Y** and **Z** be β. Then the angle between **X** and **Z** is (2π − α − β). Using the method of Lagrange multipliers we have to find the minimum of the Lagrangian L, which is expressed as:

$$L = x + y + z + \lambda_1 (x^2 + y^2 - 2xy\cos(\alpha) - a^2) + \lambda_2 (y^2 + z^2 - 2yz\cos(\beta) - b^2) + \lambda_3 (z^2 + x^2 - 2zx \cos(\alpha + \beta) - c^2)$$

where a, b and c are the lengths of the sides of the triangle.

Equating each of the five partial derivatives $\delta L/\delta x$, $\delta L/\delta y$, $\delta L/\delta z$, $\delta L/\delta\alpha$, $\delta L/\delta\beta$ to zero and eliminating λ_1, λ_2, λ_3 eventually gives sin(α) = sin(β) and sin(α + β) = − sin(β) so α = β = 120°. However the elimination is a long tedious business and the end result only covers Case 2.

11.1. Geometric Problems

11.1.2 Intersection and Containment of Polyhedra

Give two polyhedra $P_1 = \{x \mid a_i^T x \le b_i, i = 1, \cdots, m\} = \{x \mid Ax \le b\}$, $P_2 = \{x \mid f_i^T x \le g_i, i = 1, \cdots, l\} = \{x \mid Fx \le g\}$, the checking problem $P_1 \cap P_2 = \phi$? can be solved by solving the feasibility problem

$$Ax \le b, \quad Fx \le g \tag{11.8}$$

And the checking problem $P_1 \subseteq P_2$? for $k = 1, \cdots, l$ is

$$\sup\{f_k^T x \mid Ax \le b\} \le g_k \tag{11.9}$$

11.1. Geometric Problems

11.1.2 Intersection and Containment of Polyhedra

We can solve a number of LPs

$$\text{maximize} \quad f_k^T x \qquad (11.10)$$

subject to $Ax \le b$

11.1. Geometric Problems

11.1.2 Intersection and Containment of Polyhedra

If we denote the polyhedra via convex hull $P_1 = Co\{v_1, \cdots, v_K\}$, $P_2 = Co\{w_1, \cdots, w_L\}$

To check $P_1 \cap P_2 = \phi$? is to find $\lambda_1, \cdots, \lambda_K, \mu_1, \cdots, \mu_L \geq 0$ such that

$$\lambda_1 + \cdots + \lambda_K = 1$$

$$\mu_1 + \cdots + \mu_L = 1$$

$$\lambda_1 v_1 + \cdots + \lambda_K v_k = \mu_1 w_1 + \cdots + \mu_L w_L \tag{11.11}$$

11.1. Geometric Problems

11.1.2 Intersection and Containment of Polyhedra

To check $P_1 \subseteq P_2$?, $k = 1, \cdots, K$ is to find $\mu_i \geq 0$ such that

$$v_k = \mu_1 w_1 + \cdots + \mu_L w_L$$

$$1 = \mu_1 + \cdots + \mu_L \qquad (11.12)$$

11.1. Geometric Problems

11.1.3 Some Theorems of Polyhedra

Given a nonempty convex set C, a vector $x \in C$ is said to be an extreme point of C, if it does not lie strictly between the endpoints of any line segment contained in the set, i.e., if there do not exist vectors $y \in C$ and $z \in C$, with $y \neq x$ and $z \neq x$, and a scalar $\alpha \in (0,1)$ such that $x = \alpha y + (1-\alpha)z$.

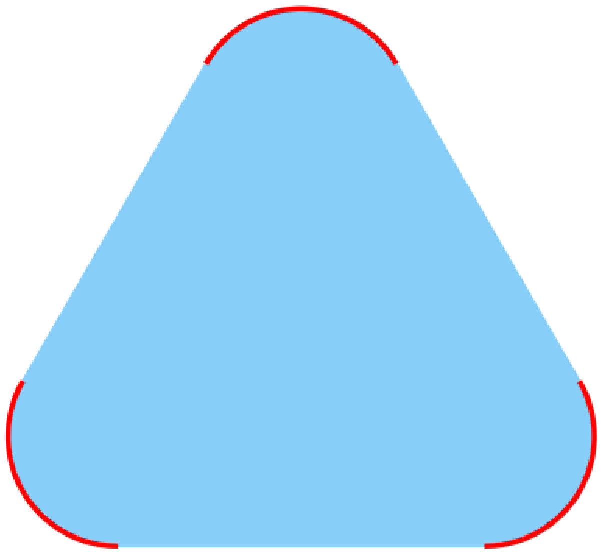
It can be seen that an equivalent definition is that x cannot be expressed as a convex combination of some vectors of C, all of which are different from x.

A geomatrically apparent property of extreme points is the so called Krein-Milam Theorem.

11.1. Geometric Problems

11.1.3 Some Theorems of Polyhedra

Krein-Milam Theorem: if C is a nonempty compact convex subset of R^n, then C is equal to the convex hull of its extreme points.



11.1. Geometric Problems

11.1.3 Some Theorems of Polyhedra

Lemma 1: if a hyperplane H contains C in one of its closed halfspaces, then every extreme point of $C \cap H$ is also an extreme point of C.

Proof: Let $\bar{x}$ be an extreme point of $C \cap H$ and to arrive at a contradiction, it is not an extreme point of C. Then we have certain $\bar{x} = \alpha y + (1-\alpha)z$.

Since $\bar{x} \in H$, the closed halfspace containing C is of the form $\{x \mid a^T x \geq a^T \bar{x}\}$, where $a \neq 0$. And the hyperplane H should be $\{x \mid a^T x = a^T \bar{x}\}$.

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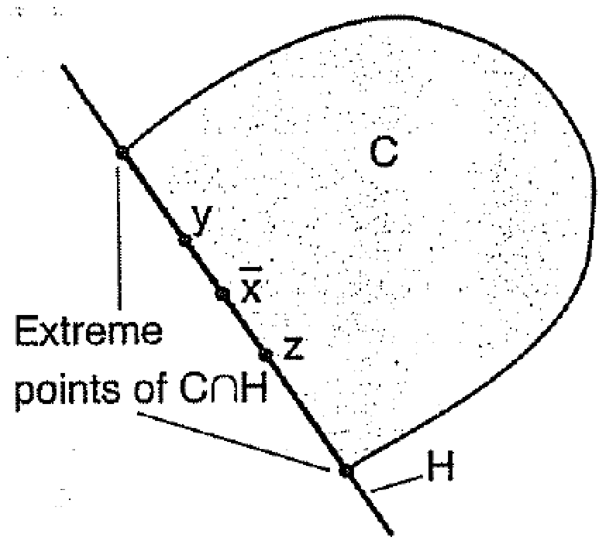
11.1.3 Some Theorems of Polyhedra

Thus, we have $\begin{cases} a^T y \geq a^T \overline{x} \\ a^T z \geq a^T \overline{x} \\ \overline{x} = \alpha y + (1-\alpha) z \end{cases}$, which gives $\begin{cases} a^T y = a^T \overline{x} \\ a^T z = a^T \overline{x} \end{cases}$.

So $y \in C \bigcap H$ and $z \in C \bigcap H$, which contradicts $\overline{x} \in C \bigcap H$.



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11.1.3 Some Theorems of Polyhedra

By convexity, C contains the convex hull of its extreme points. To show the reverse inclusion, we use induction on the dimension of the space. On the real line, a compact convex set C is a line segment whose endpoints are the extreme points of C, so every point in C is a convex combination of the two endpoints. Suppose now that every vector in a compact and convex subset of R^{n-1} can be represented as a convex combination of extreme points of the set. We will show that the same is true for compact and convex subsets of R^n.

Arbitrarily choose any $x \in C$, if x is the only point in C, it is an extreme point and we done.

11.1. Geometric Problems

11.1.3 Some Theorems of Polyhedra

Assume $\bar{x}$ is another point in C and consider the line that passes over x and $\bar{x}$. Since C is compact, the intersection of this line and C is a compact line segment whose endpoints, say x_1 and x_2, belong to the relative boundary of C. Let H_1 be a hyperplane that passes through x_1 and contains C in one of its closed halfspaces. Similarly, let H_2 be a hyperplane that passes through x_2 and contains C in one of its closed halfspaces. The intersections $H_1 \cap C$ and $H_2 \cap C$ are compact convex sets in the hyperplanes H_1 and H_2, respectively. By viewing as H_1 and H_2 as $(n-1)$ dimensional spaces, and by using the induction hypothesis, we see that each of the sets $H_1 \cap C$ and $H_2 \cap C$ is the convex hull of its extreme points.

11.1. Geometric Problems

11.1.3 Some Theorems of Polyhedra

Thus, x_1 is a convex combination of some extreme points of $H_1 \cap C$, and x_2 is a convex combination of some extreme points of $H_2 \cap C$.

According to Lemma 1, all the extreme points of $H_1 \cap C$ and all the extreme points of $H_2 \cap C$ are also the extreme points of C. So, x_1 and x_2 are the extreme points of C.

Since x lies in the line segment connecting x_1 and x_2, it is a convex combination of some extreme points of C, showing that C is contained in the convex hull of the extreme points of C. Hence, we prove Krein-Milman Theorem.

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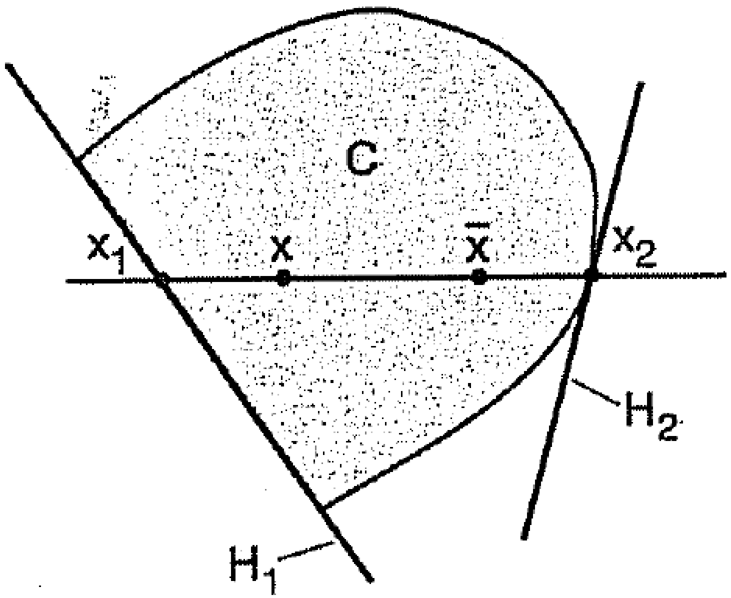
11.1.3 Some Theorems of Polyhedra



A more general discussion can be found in A. Barvinok, *A Course in Convexity*, American Mathematical Society, 2002, page 57.

11.1. Geometric Problems

11.1.3 Some Theorems of Polyhedra

Carathéodory's Theorem: if a point x of R^n lies in the convex hull of a set P, there is a subset P' of P consisting of $n+1$ or fewer points such that x lies in the convex hull of P'.

Proof: x must be a convex combination of a finite number of points in P, let us denote this fact as

$$x = \sum_{j=1}^{k} \lambda_j x_j \tag{11.13}$$

where every x_j is in P, every λ_j is positive, $\sum_{j=1}^{k} \lambda_j = 1$.

11.1. Geometric Problems

11.1.3 Some Theorems of Polyhedra

Suppose $k > n+1$ (otherwise, there is nothing to prove). Then, the points $x_2 - x_1$, ..., $x_k - x_1$ are linearly dependent. So there are real scalars μ2, ..., μk, not all zero, such that

$$\sum_{j=2}^{k} \mu_j (x_j - x_1) = 0 \tag{11.14}$$

If μ_1 is defined as $\mu_1 = -\sum_{j=2}^{k} \mu_j$, we have

$$\sum_{j=1}^{k} \mu_j x_j = 0, \quad \sum_{j=1}^{k} \mu_j = 0 \tag{11.15}$$

11.1. Geometric Problems

11.1.3 Some Theorems of Polyhedra

and not all of the μ_j are equal to zero. Therefore, at least one $\mu_j > 0$. Then,

$$x = \sum_{j=1}^{k} \lambda_j x_j - \alpha \sum_{j=1}^{k} \mu_j x_j = \sum_{j=1}^{k} (\lambda_j - \alpha \mu_j) x_j \tag{11.16}$$

for any real α. In particular, the equality will hold if α is defined as

$$\alpha := \min_{1 \le j \le k} \left\{ \frac{\lambda_j}{\mu_j} : \mu_j > 0 \right\} = \frac{\lambda_i}{\mu_i} \tag{11.17}$$

11.1. Geometric Problems

11.1.3 Some Theorems of Polyhedra

Note that $\alpha > 0$, and for every j between 1 and k,

$$\lambda_j - \alpha\mu_j \geq 0 \tag{11.18}$$

In particular, $\lambda_i - \alpha\mu_j = 0$ by definition of α. Therefore,

$$x = \sum_{j=1}^{k} \lambda_j x_j - \alpha \sum_{j=1}^{k} \mu_j x_j = \sum_{j=1}^{k} (\lambda_j - \alpha\mu_j) x_j \tag{11.19}$$

where every $\lambda_i - \alpha\mu_j$ is nonnegative. Particularly, $\lambda_i - \alpha\mu_i = 0$, their sum is 1.

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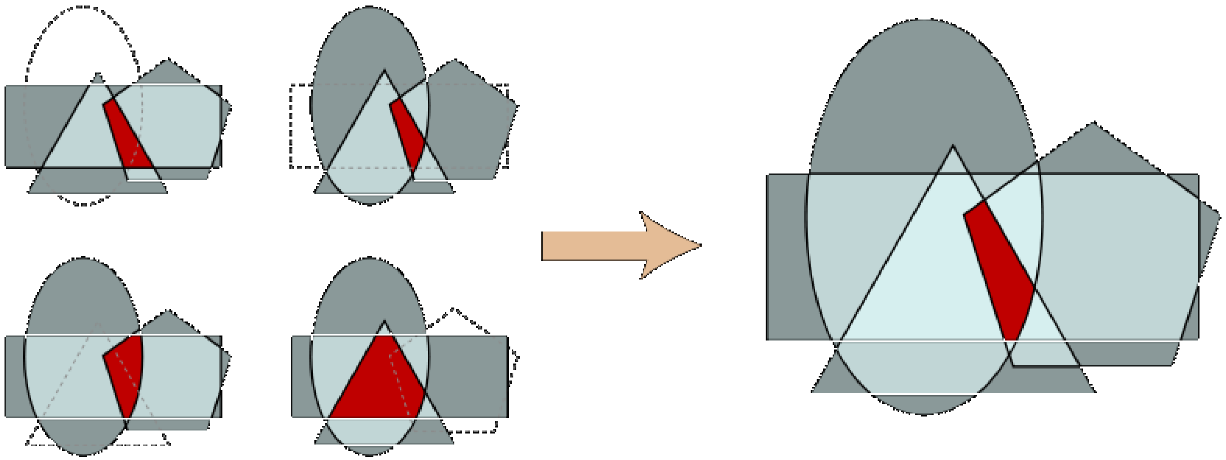
11.1.3 Some Theorems of Polyhedra

In other words, x is represented as a convex combination of at most $k-1$ points of P. This process can be repeated until x is represented as a convex combination of at most $n+1$ points in P. (why?)

11.1. Geometric Problems

11.1.3 Some Theorems of Polyhedra

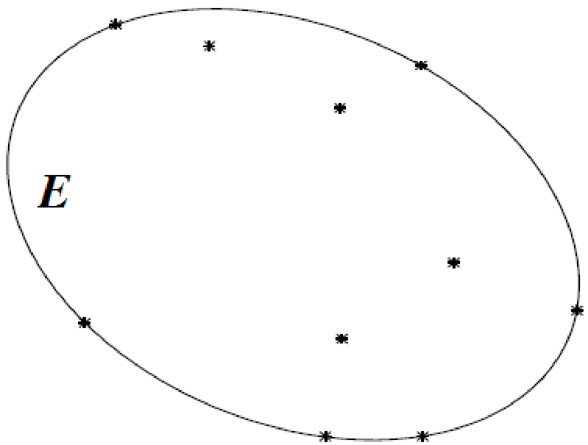
Helly's Theorem (Radon's Theorem): assume $C_1, \ldots, C_n$ is a finite collection of convex subsets of R^d, where $n > d$. If the intersection of every $(d+1)$ of these sets is nonempty, then the whole collection has a nonempty intersection $\bigcap_{j=1}^{n} C_j \neq \Phi$.



11.1. Geometric Problems

11.1.4 Ellipsoid Problems

Find ellipsoid $E = \{x \mid \| Ax - b \| \le 1\}$ with the minimum volume containing points $v_1, \cdots, v_k \in R^n$



The center of the ellipsoid is $A^{-1}b$ with $A = A^T > 0$, and the volume is proportional to $\det A^{-1}$.

11.1. Geometric Problems

11.1.4 Ellipsoid Problems

Thus, we can formulate it into the following problem

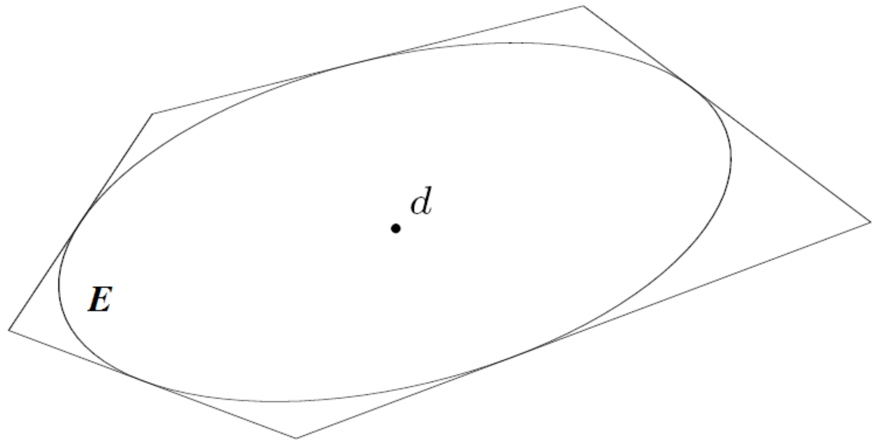
$$\min \quad \log\det A^{-1} \tag{11.20}$$

subject to $A = A^T > 0$, $\| Av_i - b \| \le 1$, $i = 1, \cdots, K$

This is not a convex optimization problem, but we can consider a psd matrix $M = A^{-1/2}$ as an alternative decision variable. Please check the textbook for detailed explanations.

11.1. Geometric Problems

11.1.4 Ellipsoid Problems



Find ellipsoid $E = \{By + d \mid \|y\| \le 1\}$ with maximum volume in a polyhedra $P = \{x \mid a_i^T x \le b_i, i = 1, \cdots, L\}$

The center of the ellipsoid is d, $B = B^T$, and the volume is proportional to $\det B$. Based on the above discussions, we see

11.1. Geometric Problems

11.1.4 Ellipsoid Problems

$$E \subseteq P \Leftrightarrow a_i^T(By + d) \le b_i \text{ for all } \|y\| \le 1$$

$$\Leftrightarrow \sup_{\|y\|\le 1}(a_i^T By + a_i^T d) \le b_i$$

$$\Leftrightarrow \|Ba_i\| + a_i^T d \le b_i, \quad i = 1, \cdots, L$$

Thus, to find the maximum volume $E \subseteq P$ is a convex proprammming problem on variables

$$\min \quad \log\det B \tag{11.21}$$

subject to $B = B^T > 0$, $\|Ba_i\| + a_i^T d \le b_i$, $i = 1, \cdots, L$

11.1. Geometric Problems

11.1.5 Cutting Plane Method and Center Problems

Goal: find a point in convex set $X \subseteq R^n$, or prove $X = \phi$.

Approach: our description of X is through the following cutting-plane oracle: when a cutting-plane oracle is queried at $X \subseteq R^n$, it either

1) asserts that $x \in X$, or
2) returns a separating hyperplane between x and X

$$a^T z = b$$

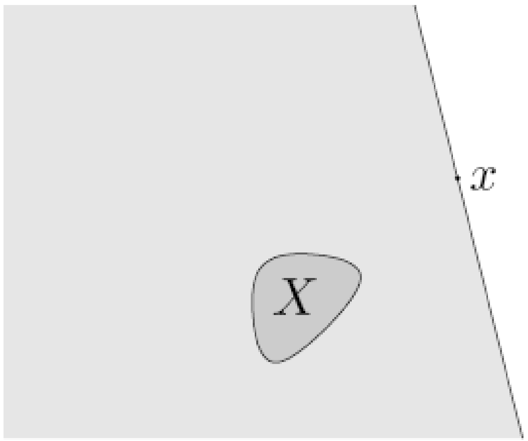
where (a,b) is called a cutting-plane, or a cut, because it eliminates the halfspace $\{z \mid a^T z > b\}$ from our search for a point in X

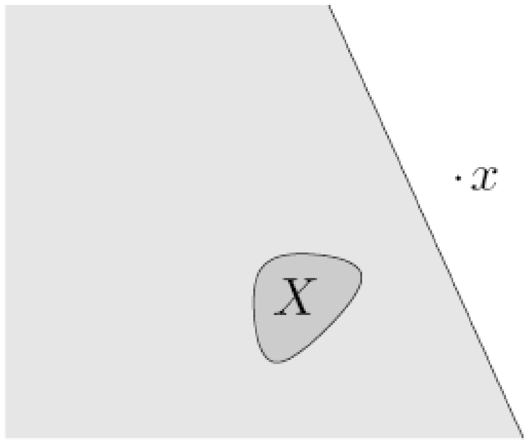
11.1. Geometric Problems

11.1.5 Cutting Plane Method and Center Problems

if $a^T x = b$ (x is on boundary of halfspace that is cut) this cutting-plane is called a neutral cut

if $a^T x > b$ (x lies in interior of halfspace that is cut), cutting-plane is called deep cut





11.1. Geometric Problems

11.1.5 Cutting Plane Method and Center Problems

Goal: minimize smooth convex function $f : R^n \to R$

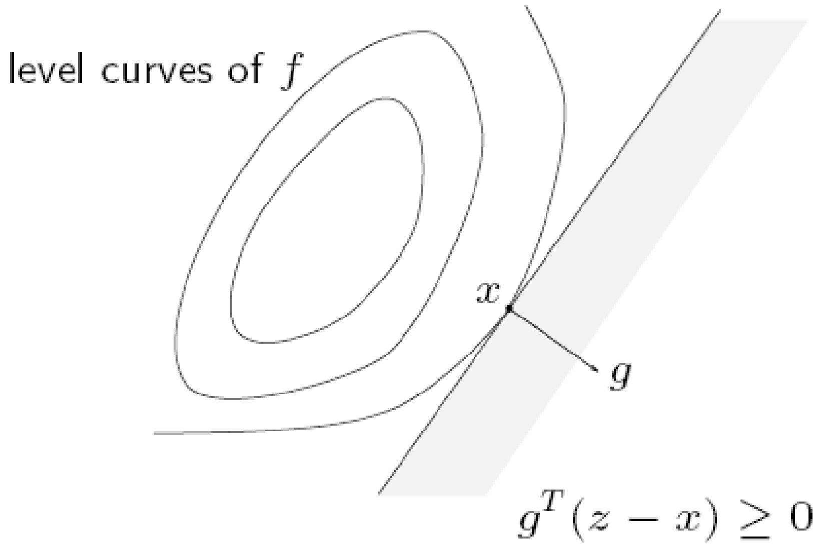
Suppose X is the set of optimal points (minimizers), given x, we can find $g = \nabla f(x)$. From $f(z) \geq f(x) + g^T(z-x)$, we can conclude

$$g^T(z-x) > 0 \Rightarrow f(z) > f(x) \qquad (11.22)$$

That is, all points in the halfspace $g^T(z-x) \geq 0$ are worse than x, and in particular not optimal. so $g^T(z-x) \leq 0$ is (neutral) cutting-plane at x ($a = g$, $b = g^T x$)

11.1. Geometric Problems

11.1.5 Cutting Plane Method and Center Problems



by evaluating $g = \nabla f(x)$, we rule out a halfspace in our search for x^*. Thus, we can get one bit of information (on location of x^*) by evaluating g.

Actually, we can similarly attack non-smooth convex function $f: R^n \to R$ by choose $g \in \partial f(x)$

11.1. Geometric Problems

11.1.5 Cutting Plane Method and Center Problems

Suppose we know a number $\bar{f}$ with $f(x) > \bar{f} \geq f^*$ (e.g., the smallest value of f that is found so far in an algorithm). From $f(z) \geq f(x) + g^T(z-x)$, we have

$$f(x) + g^T(z-x) > \bar{f} \Rightarrow f(z) > \bar{f} \geq f^* \qquad (11.23)$$

so we have a deep cut

$$g^T(z-x) + f(x) - \bar{f} \leq 0 \qquad (11.24)$$

What is a and b here?

11.1. Geometric Problems

11.1.5 Cutting Plane Method and Center Problems

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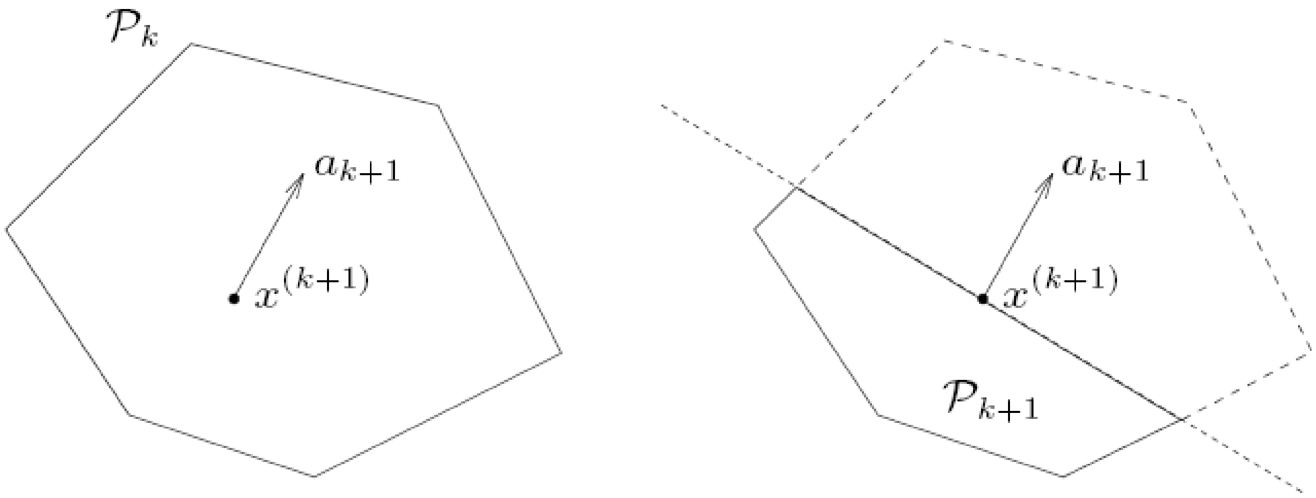
11.1.5 Cutting Plane Method and Center Problems



P_k gives our uncertainty of x^* at iteration k, intuitively we want to pick $x(k+1)$ so that P_{k+1} is as small as possible, no matter what cut is made. Therefore, we want $x(k+1)$ to be near center of P_k

11.1. Geometric Problems

11.1.5 Cutting Plane Method and Center Problems

We have different choices of the center of a polyhedra

1) center of gravity (CG) algorithm: $x(k+1)$ is center of gravity of P_k

2) maximum volume ellipsoid (MVE) cutting-plane method: $x(k+1)$ is center of maximum volume ellipsoid contained in P_k

3) Chebyshev center cutting-plane method: $x(k+1)$ is Chebyshev center of P_k

4) analytic center cutting-plane method (ACCPM): $x(k+1)$ is analytic center of P_k

11.1. Geometric Problems

11.1.5 Cutting Plane Method and Center Problems

1) Center of gravity algorithm $CG(P_k) == \int_{P_k} x dx \Big/ \int_{P_k} dx$

Star at the center of CG and make a line, we can split the polyhedral into two parts. Moreover, we have

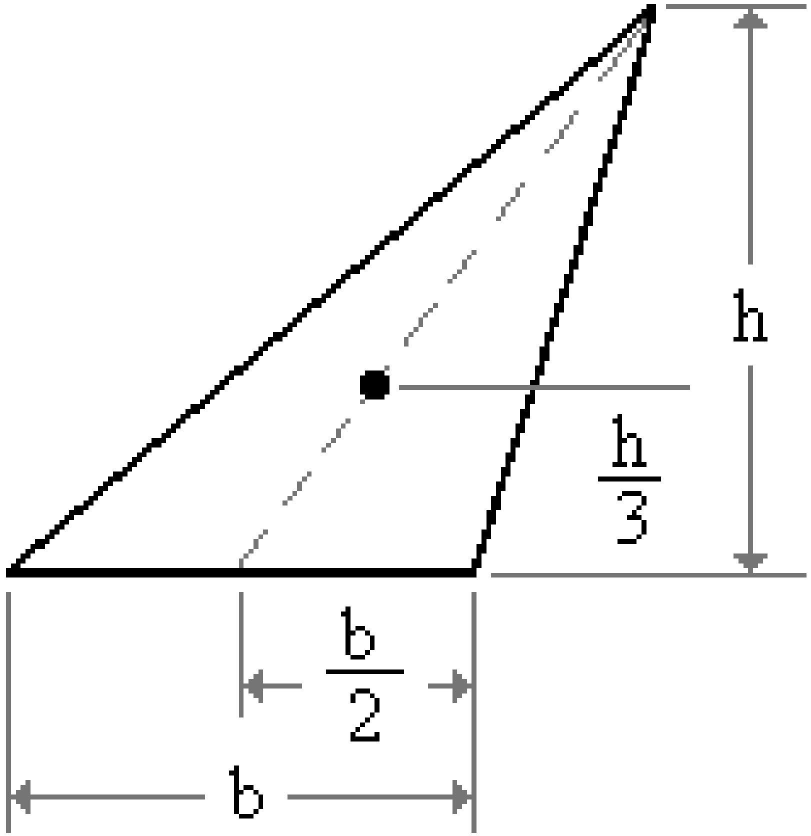
Theorem : if $C \subseteq R^n$ convex, $x_{cg} = CG(C)$, $g \neq 0$, we have

$$vol(C \cap \{g^T (x - x_{cg}) \leq 0\}) \leq (1 - 1/e) vol(C) \approx 0.63 vol(C) \tag{11.25}$$

which is independent of dimension n.

11.1. Geometric Problems

11.1.5 Cutting Plane Method and Center Problems



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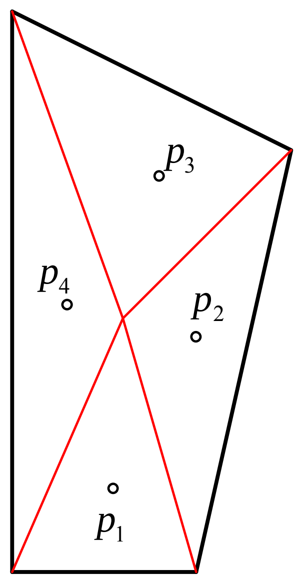
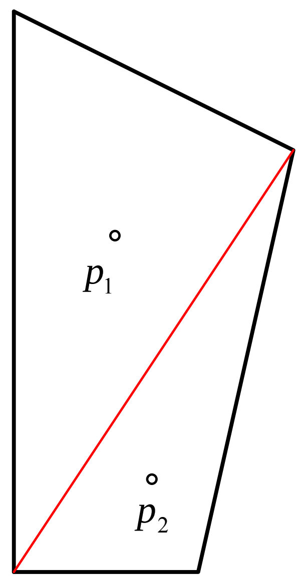
11.1.5 Cutting Plane Method and Center Problems

Exact solution of center of gravity of 2D convex:

a)Take a random interior point or a vertex, and divide the convex polygon into triangles (Why CG is always inside?)





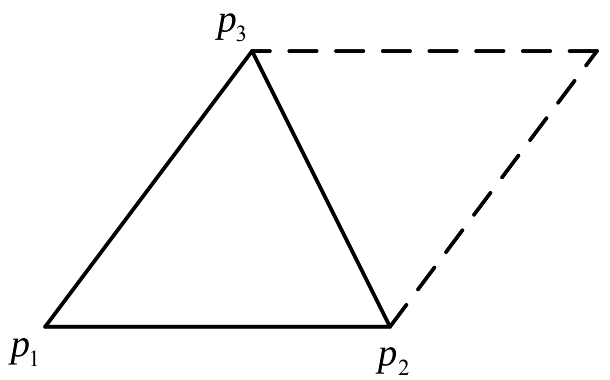
11.1. Geometric Problems

11.1.5 Cutting Plane Method and Center Problems

b) compute the area (mass) of the j th triange as m_j

$$m_j = \frac{1}{2}\vec{u} \times \vec{v} \tag{11.26}$$

where $\vec{u} = p_2 - p_1$, $\vec{v} = p_3 - p_1$. p_1, p_2, p_3 are the vertices.



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11.1.5 Cutting Plane Method and Center Problems

Or we can use Hero's formula: suppose the length of the three sides of a triangle is a, b and c, we have the rea of this triangle is

$$A = \sqrt{s(s-a)(s-b)(s-c)} \qquad (11.27)$$

where $s = \dfrac{a+b+c}{2}$.

You can prove it using the Cosine Theorem. But why we do not prefer this method?

11.1. Geometric Problems

11.1.5 Cutting Plane Method and Center Problems

c) finally get the center of gravity as

$$x_c = \frac{\sum_{j=1}^{M} x_j m_j}{\sum_{j=1}^{M} m_j}, \quad y_c = \frac{\sum_{j=1}^{M} y_j m_j}{\sum_{j=1}^{M} m_j} \qquad (11.28)$$

http://www.cosc.brocku.ca/Offerings/3P98/course/lectures/comp_geom/

How to extend to n D cases? How to divide a convex n D polyhedra? How to calculate the mass?

11.1. Geometric Problems

11.1.5 Cutting Plane Method and Center Problems

The area (mass) of a N vertice convex hull is

$$m = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & \ldots & x_N \\ y_1 & y_2 & \ldots & y_N \end{vmatrix} \qquad (11.29)$$

11.1. Geometric Problems

11.1.5 Cutting Plane Method and Center Problems

2) Maximum volume ellipsoid method

We had solved in above

3) The Chebyshev center of a set $P = \{x \mid a_i^T x \le b_i, i = 1, \cdots m\}$ having non-empty interior is the center of the minimal radius ball $B = \{x_c + u \mid \|u\|_2 \le r\}$ enclosing the entire set P.

$$\text{maximize} \quad r \tag{11.30}$$

subject to $a_i^T x_c + r \| a_i \|_2 \le b_i$, $i = 1, \cdots m$

11.1. Geometric Problems

11.1.5 Cutting Plane Method and Center Problems

Suppose P_0 lies in a ball of radius R and simultaneously includes a ball of radius r.

Given $x^{(1)},\cdots x^{(k)} \notin X$, so $P_k \supseteq X$, we have

$$\alpha_n r^n \le vol(P_k) \le (0.63)^k vol(P_0) \le (0.63)^k \alpha_n R^n \tag{11.31}$$

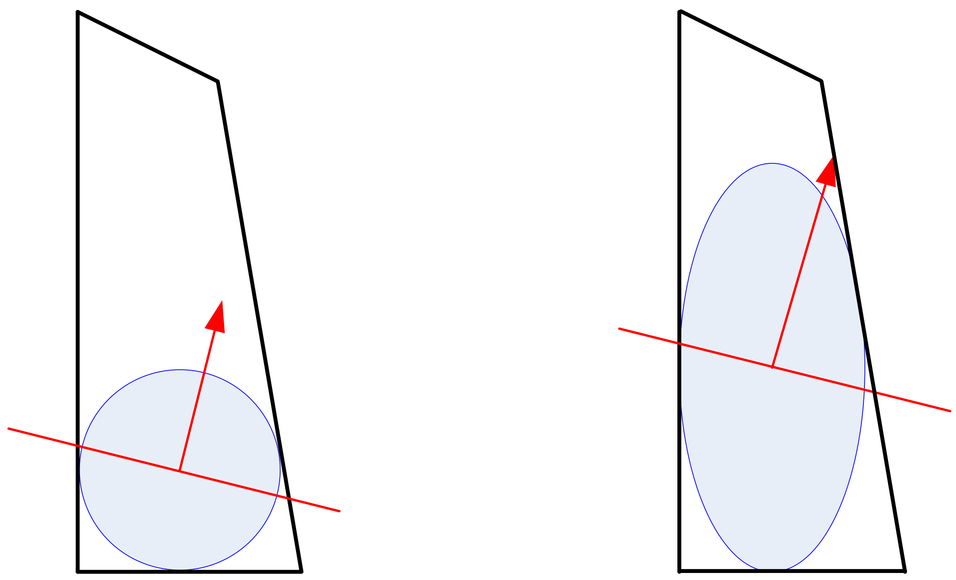
where α_n is volume of unit ball in R^n. So $k \le 1.5n\log_2(R/r)$.

Related to Löwner-John ellipsoid (John, Kiefer-Wolfowitz, Khachiyan, Nesterov-Nemirovski)

11.1. Geometric Problems

11.1.5 Cutting Plane Method and Center Problems

A simple example to illustrate why the maximum volume ellipsoid (MVE) cutting-plane method is usually better than the Chebyshev center cutting-plane method (we can cut off much larger area during one step)



11.1. Geometric Problems

11.1.5 Cutting Plane Method and Center Problems

4) Analytic center cutting-plane method

$x^{(k+1)}$ is the analytic center of $P_k = \{z \mid a_i^T z \le b_i, i = 1, \cdots, q\}$

$$x^{(k+1)} = \arg\min_x - \sum_{i=1}^{q} \log(b_i - a_i^T x) \qquad (11.32)$$

$x^{(k+1)}$ can be computed using infeasible start Newton method, which works quite well in practice. So, analytic center can be more easily computed than MVE or Chebyshev center.

11.1. Geometric Problems

11.1.5 Cutting Plane Method and Center Problems

Given a polyhedral $C = \{x \mid a_i^T x \le b_i\}$, the analytic center is the solution of the following optimization problem

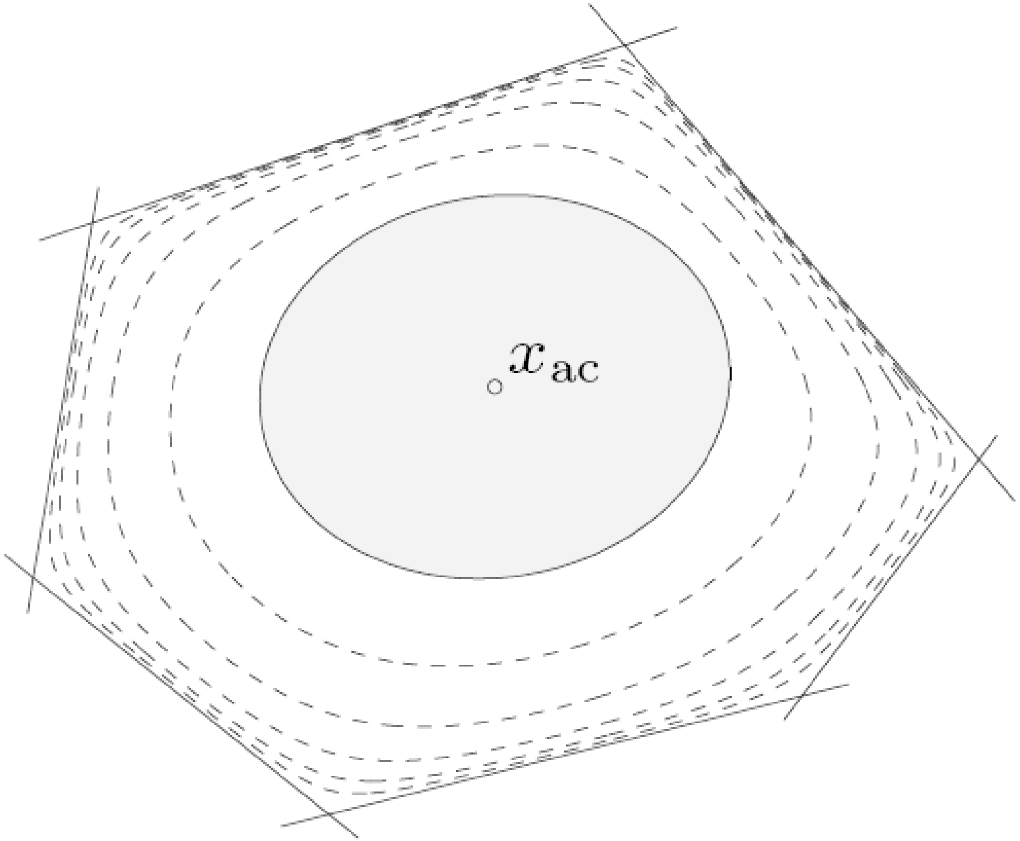
$$\text{maximize} \quad \prod_{i=1}^{m} s_i \qquad (11.33)$$

subject to $a_i^T x + s = b$, which is easily seen to be the same as

$$\text{minimize} \quad -\sum_{i=1}^{m} \ln(s_i) = -\sum_{i=1}^{m} \ln(b_i - a_i^T x) \qquad (11.34)$$

11.1. Geometric Problems

11.1.5 Cutting Plane Method and Center Problems



Interior point method?

11.1. Geometric Problems

11.1.6 Angles Problems

Suppose we have n vectors $x \in R^n$, please find the largest possible value of the minimum intersection angle among every two of these vectors.

11.1. Geometric Problems

11.1.6 Angles Problems

The largest possible value of the minimum intersection angle among every two of these vectors is $\arccos(-\frac{1}{n-1})$.

We can transform the original problem into finding the smallest possible value of the maximum inner product of every two of these vectors:

$$\min_{\boldsymbol{x}_i,\boldsymbol{x}_j} \max_{i,j,i\neq j} \boldsymbol{x}_i^T\boldsymbol{x}_j \tag{2}$$

Assuming $|\boldsymbol{x}_i|_2 = 1$ for all $i = 1, 2, \ldots, n$, we have

$$\begin{aligned}
\max_{i,j,i\neq j} \boldsymbol{x}_i^T\boldsymbol{x}_j \geq & \frac{1}{n(n-1)}\sum_{i\neq j}\boldsymbol{x}_i^T\boldsymbol{x}_j \\
= & \frac{1}{n(n-1)}\left(\left|\sum_i \boldsymbol{x}_i\right|_2^2 - \sum_i |\boldsymbol{x}_i|_2^2\right) \\
= & \frac{1}{n(n-1)}\left(\left|\sum_i \boldsymbol{x}_i\right|_2^2 - n\right) \\
\geq & \frac{-1}{n-1}
\end{aligned}$$

The equality sign in the first inequality holds if all products of every two vectors are the same, and the equality sign in the second inequality holds if the sum of all vectors equals to zero. Now the remaining job is to prove that we can find such a matrix X that satisfies the above conditions.

11.1. Geometric Problems

11.1.6 Angles Problems

We do it by rewrite the two conditions into the following form:

$$X^T X = \begin{bmatrix} 1 & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} \\ \frac{-1}{n-1} & 1 & \cdots & \frac{-1}{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-1}{n-1} & \frac{-1}{n-1} & \cdots & 1 \end{bmatrix}$$

The matrix in the above equation is a positive semidefinite matrix (easy to prove) and its eigenvalues are:

$$\underbrace{\frac{n}{n-1}, \frac{n}{n-1}, \ldots, \frac{n}{n-1}}_{n-1}, 0$$

Then we have the following eigenvalue decomposition of matrix $X^T X$:

$$X^T X = Q^T \begin{bmatrix} \frac{n}{n-1} & & & & \\ & \frac{n}{n-1} & & & \\ & & \ddots & & \\ & & & \frac{n}{n-1} & \\ & & & & 0 \end{bmatrix} Q$$

11.1.6 Angles Problems

Thus one possible solution is

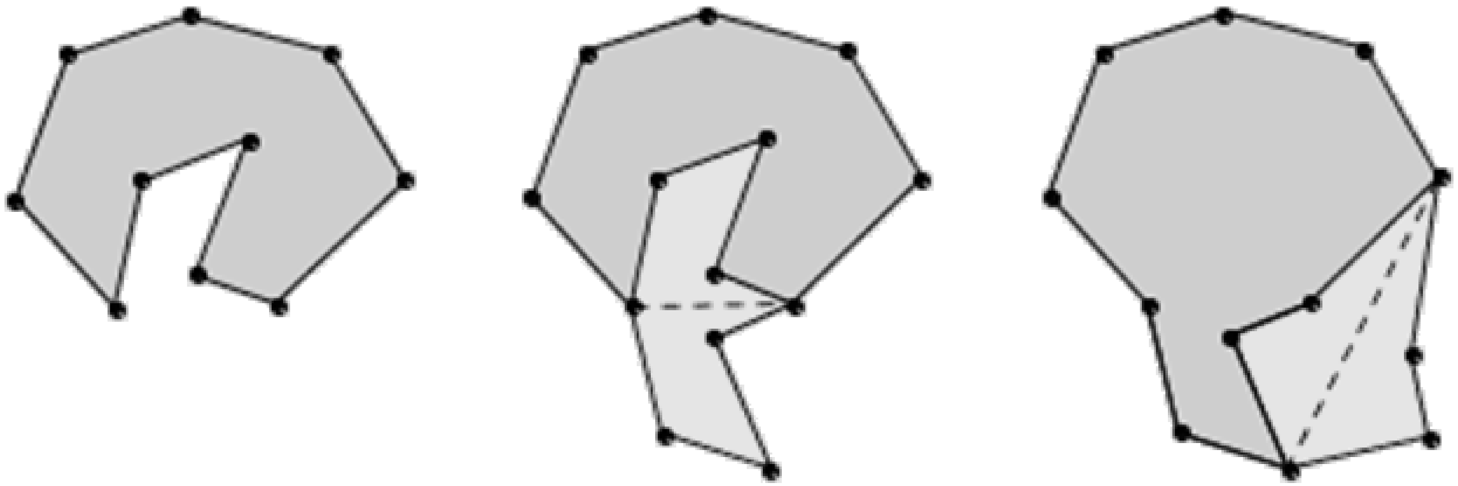
$$X = \begin{bmatrix} \sqrt{\frac{n}{n-1}} & & & & \\ & \sqrt{\frac{n}{n-1}} & & & \\ & & \ddots & & \\ & & & \sqrt{\frac{n}{n-1}} & \\ & & & & 0 \end{bmatrix} Q$$

which concludes the proof.

11.1. Geometric Problems

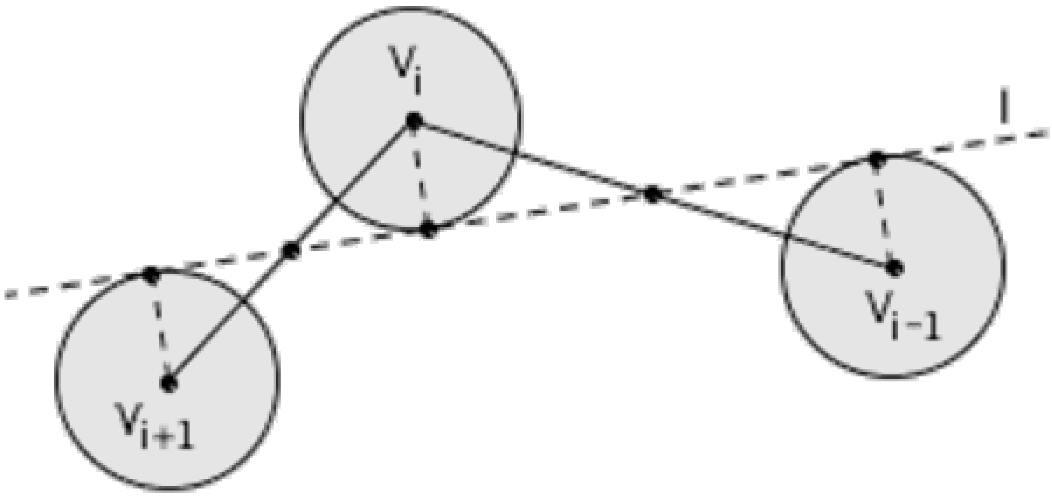
11.1.7 Puzzles

The Erdős-Nagy theorem states that a non-convex simple polygon can be transfered into a convex polygon by a finite sequence of flips. The flips are defined by taking a convex hull of a polygon and reflecting a pocket with respect to the boundary edge.



11.1. Geometric Problems

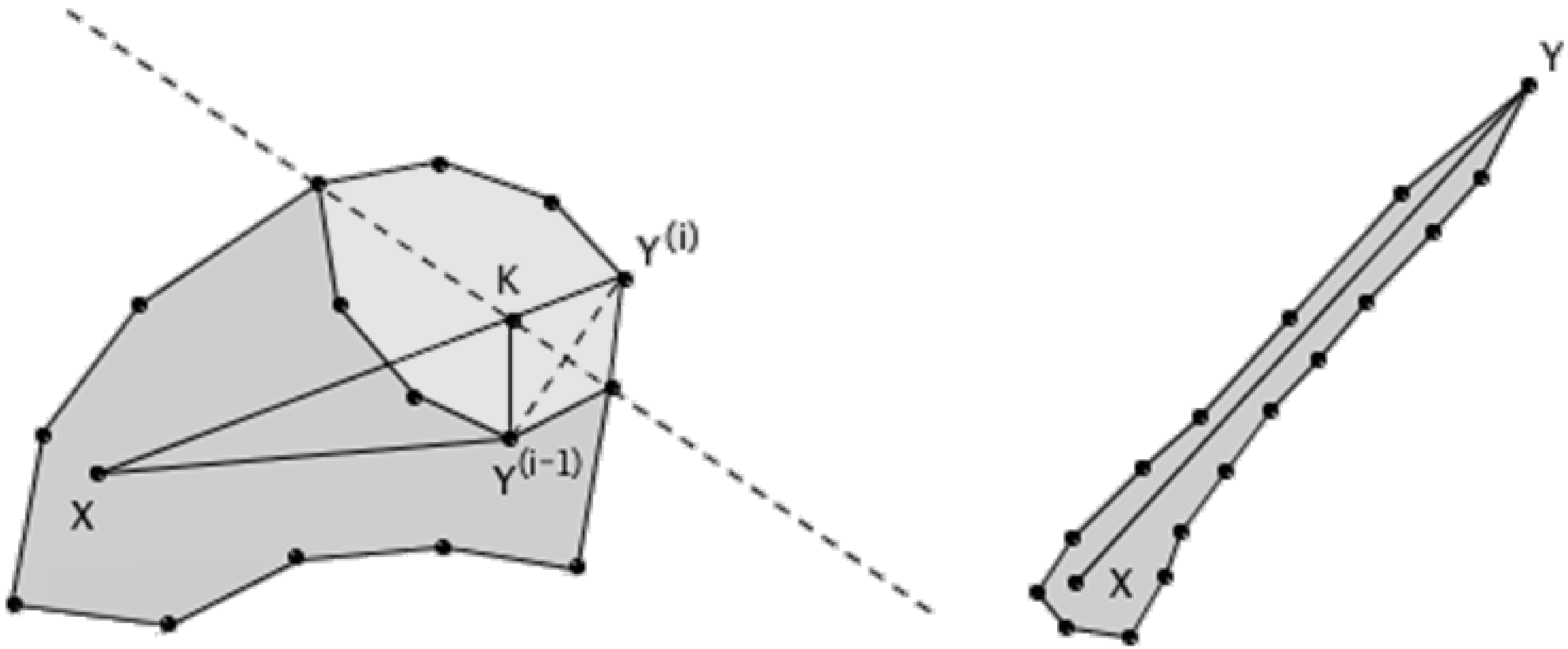
11.1.7 Puzzles



Lemma 1: Given a convex polygon, there exists a positive real number ε such that if some or all of the vertices are each moved by a distance less than ε, then the polygon remains convex.

11.1. Geometric Problems

11.1.7 Puzzles



$$XY^{(i)} = XK + KY^{(i)} = XK + KY^{(i-1)} \geq XY^{(i-1)}$$

11.1. Geometric Problems

11.1.7 Puzzles

A flip either maintains or increases the distance from a fixed point and a vertex. However, the distance is bounded from the half of the perimeter of the polygon. One fact: the flips do not change the perimeter of a polygon.

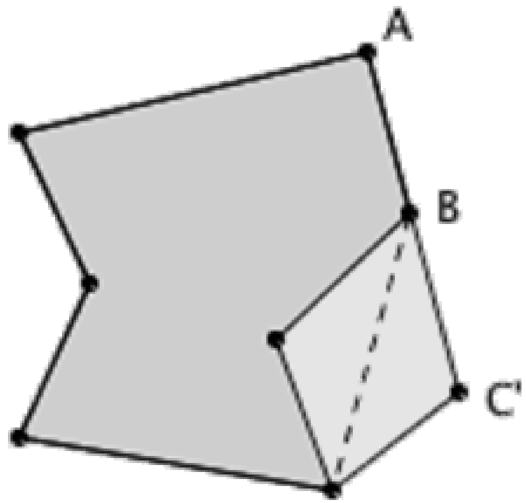
First, the limit polygon must be a simple polygon. That is, different vertices cannot converge to one and the same limit vertex.

Secondly, the limit polygon must be convex; otherwise, being a simple polygon, another new flip would alter its shape contradicting that it is the limit polygon.

11.1. Geometric Problems

11.1.7 Puzzles

Thirdly, some vertices of will have interior angles equal to π and others less than π. Note also that whenever a vertex becomes straight, it remains straight for all descendants of this vertx. Therefore we may ignore straight vertices in the analysis.



11.1. Geometric Problems

11.1.7 Puzzles

So, after a finite sequence of flips, every vertex has entered its limit disk and since the vertices must then remain in their respective disks, the final simple polygon must be convex.

The theorem is named after mathematicians Paul Erdős and Béla Szőkefalvi-Nagy. Paul Erdős conjectured the result in 1935 as a problem in the American Mathematical Monthly, and Szőkefalvi-Nagy published a proof in 1939. The problem has a curious history and had been repeatedly rediscovered, until Branko Grünbaum surveyed the results in 1995. As it turns out, the original proof had a delicate mistake, which has been since corrected.

11.1. Geometric Problems

11.1.7 Puzzles

Mikhalev's Octahedrons: Two octahedrons are composed of exactly same set of faces; one of them convex, the other concave. Which one has a greater volume?

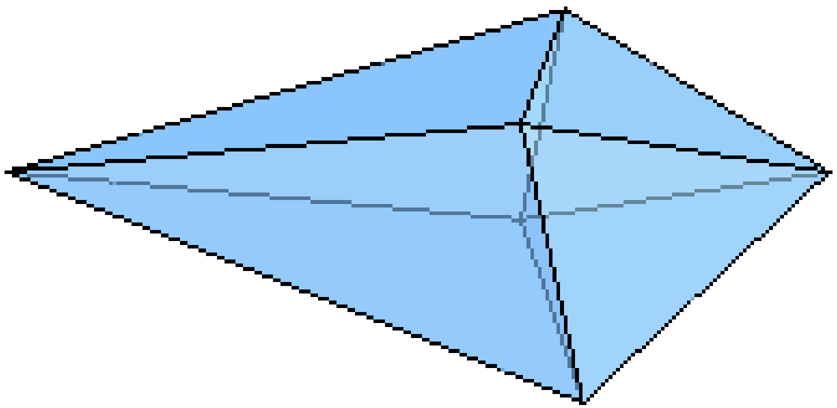
Surprisingly, there is a pair of such octahedron in which the concave one has a greater volume. The pair has been discovered in 2002 by S. N. Mikhalev. The ratio of the volumes is about 1.163.

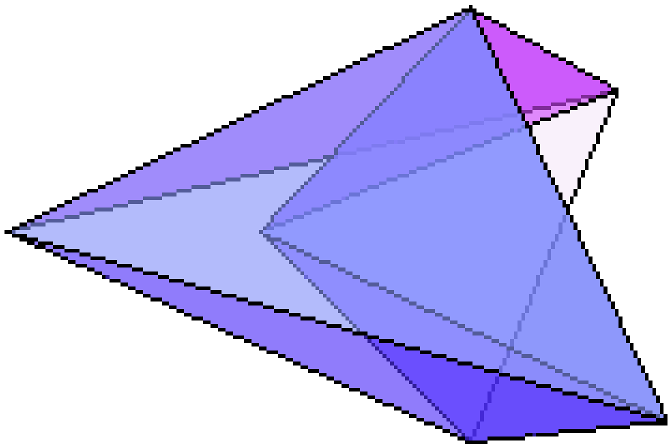
Convex octahedron: N(0, 0, 1)，A(10, 1, 0)，B(0, 6, 0)，C(-10, 1, 0)，D(0, -10, 0)，S(0, 0, -1)

11.1. Geometric Problems

11.1.7 Puzzles

Concave octahedron: N(0, 0, $\sqrt{61/3}$), A($\sqrt{71}$, $4\sqrt{2/3}$, 0), B(0, $-5\sqrt{2/3}$, 0), C($-\sqrt{71}$, $4\sqrt{2/3}$, 0), D(0, $-11\sqrt{2/3}$, 0), S(0, 0, $-\sqrt{61/3}$)





How to calculate the volume?

11.2. Location Optimization

placement problem

N points with coordinates $x_i \in R^2$ (or R^3)

some positions x_i are given; the other x_i's are variables

for each pair of points, a cost function $f_{ij}(x_i, x_j)$

minimize $\sum_{i \neq j} f_{ij}(x_i, x_j)$

variables are positions of free points

interpretations

points for plants/warehouses, $f_{ij}(x_i, x_j)$ is transportation cost between facilities i and j

points represent cells on an IC; $f_{ij}(x_i, x_j)$ represents wirelength

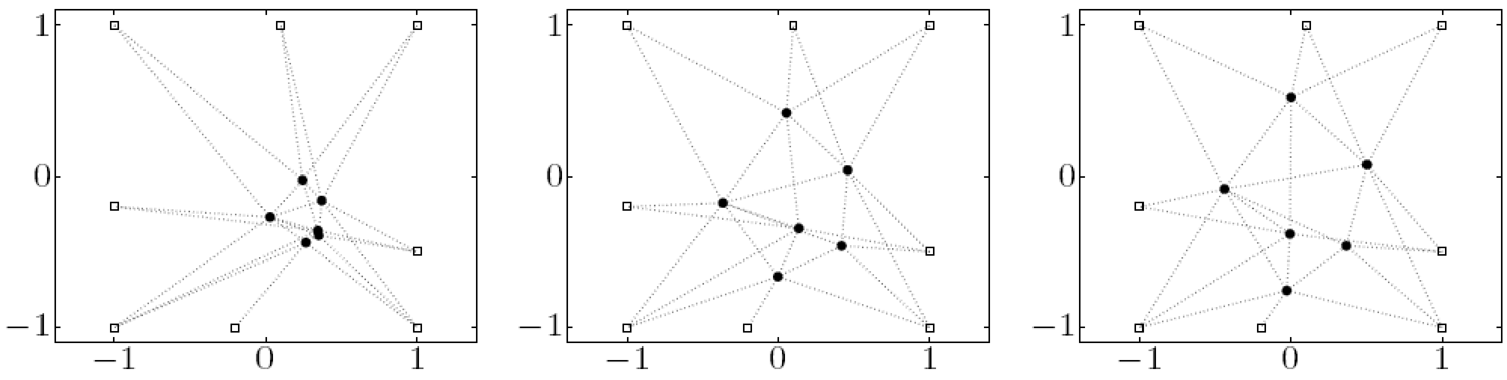
11.2. Location Optimization

For l_1 norm $\sum_{i=1}^{k}\left[\left|u-u_i\right|+\left|v-v_i\right|\right]$, an optimal point is any median of the fixed points. The centre-of-gravity method?

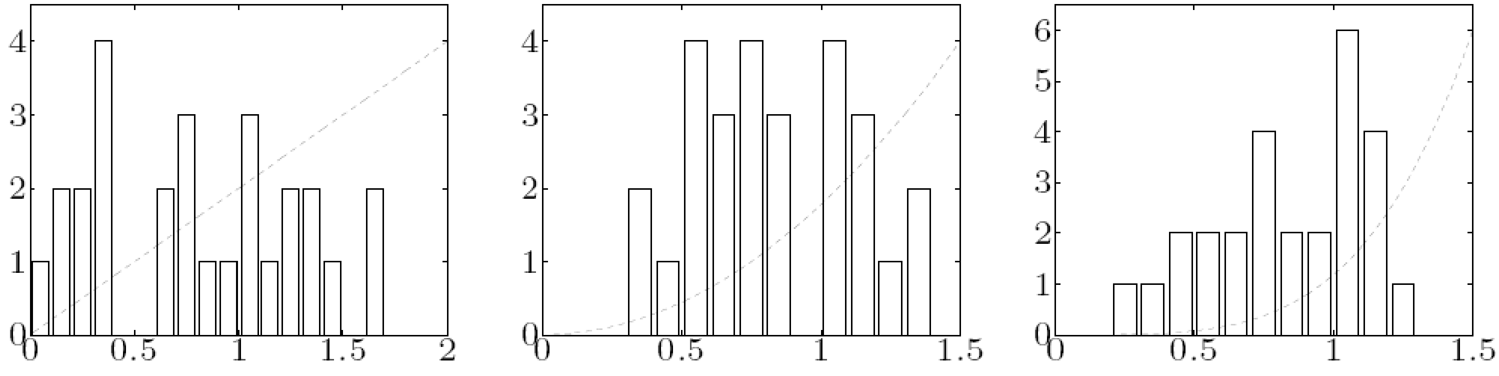
11.2. Location Optimization

example: minimize $\sum_{(i,j)\in\mathcal{A}} h(\|x_i - x_j\|_2)$, with 6 free points, 27 links

optimal placement for $h(z) = z$, $h(z) = z^2$, $h(z) = z^4$



histograms of connection lengths $\|x_i - x_j\|_2$



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