Convex Optimization Theory and Applications

Topic 5 - Introduction to Convex Optimization Toolbox

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5.0. Outline

5.1. CVX

5.1.1 Basic Information

CVX is a modeling modeling system for disciplined convex programming developed by Michael Grant and Stephen Boyd. It was built on the ideas from earlier work by Löfberg (YALMIP), Dahl and Vandenberghe (CVXOPT).

It provides a proper assembly of convex atoms under a given ruleset with optimization problems that are provably convex.

CVX can solve standard problems such as linear programs (LPs), quadratic programs (QPs), second-order cone programs (SOCPs), and semidefinite programs (SDPs).

5.1. CVX

5.1.1 Basic Information

What is CVX not

CVX is not meant to be a tool for checking if your problem is convex. But if CVX accepts your problem, you can be sure it is convex.

CVX is not meant for very large problems. It is a great tool for experimenting with, and prototyping, convex optimization problems. But you can use CVX to solve scaled-down or simplified versions of the problem.

5.1. CVX

5.1.1 Basic Information

CVX does not solve any optimization problem itself. It only reformulates problems into a form (SDP and SOCP) that can be fed into a numerical convex optimization package, e.g.

SDPT3, http://www.math.nus.edu.sg/~mattohkc/sdpt5.html
SeDuMi, http://sedumi.ie.lehigh.edu/

These packages are distributed with cvx.

CVX removes the obstacle that keeps you from freely using such packages

5.1. CVX

5.1.1 Basic Information

You can choose the numerical solver, i.e. SeDuMi, as

```
cvx_solver sedumi
cvx_save_prefs
```

The second command makes your cvx toolbox remember your favorite choice

Since 2012, CVX supports some commencial software, e.g. Gurobi, MOSEK, please check the following link for details

http://cvxr.com/cvx/doc/solver.html

5.1. CVX

5.1.2 Installation

CVX can be downloaded at http://cvxr.com/cvx/download

Installation Steps

1. Unpack the .zip file wherever you like (do not install it into Matlab’s own toolbox directory);

2. At the Matlab prompt, type cd <cvxroot>;

5. Run command cvx_setup to configure the software.

5.1. CVX

5.1.2 Installation

```
Command Window

  To get started, select MATLAB Help or Demos from the Help menu.

>> cvx_setup

Testing the cvx distribution. If this script aborts with
an error, please report the error to the authors.
-------------------------------------------------------------
No errors! cvx has been successfully installed.

NOTE: The MATLAB path has been updated to point to the cvx distribution.
In order to use cvx regularly, you must save this new path definition.

>>
```

5.1. CVX

5.1.3 How to Use

To separate cvx specifications from surrounding Matlab codes, the CVX codes are preceded with the statement cvx_begin and followed with the statement cvx_end.

A specification can include any ordinary Matlab statements, as well as special cvx-specific commands for declaring primal and dual optimization variables, specifying constraints and objective functions.

cvx provide a number of options and control statements

cvx_quiet: turns on and off solver output

cvx_precision: control solver accuracy

5.1. CVX

5.1.4 Unconstraint Example

In a least-square problem, we seek $x \in R^n$ that minimizes $\|Ax - b\|_2$, where $A \in R^{m \times n}$ is skinny and full rank.

Let us create some test problem data for m, n, A and b in Matlab.

```
% Input data
m = 16; n = 8;
A = randn(m,n);
b = randn(m,1);
```

5.1. CVX

5.1.4 Unconstraint Example

Then the least-squares solution $x = (A^T A)^{-1} A^T b$ is easily computed using the backslash operator:

```
x_ls = A \ b; % Matlab version
```

Using cvx, the same problem can be solved as follows:

```
% cvx version
cvx_begin
    variable x(n);
    minimize( norm(A*x-b) );
cvx_end
```

5.1. CVX

5.1.4 Unconstraint Example

variable x(n) specifies that x is an optimization variable (in the form of a vector of length n)

minimize(norm(A*x-b)) gives the objective function and specifies that we wish to minimize it

following cvx_end, x is overwritten with the optimal point. That is, it will not be a symbolic variable since then.

5.1. CVX

5.1.4 Unconstraint Example

Command Window

```
Results:
---------
norm(A*x_ls-b): 2.0354
norm(A*x-b):    2.0354
cvx_optval:     2.0354
cvx_status:     Solved

Verify that x_ls == x:
   x_ls  = [ -0.2628  0.8828 -0.0734 -1.0844  0.3249 -0.3330  0.0603  0.3802 ]
   x     = [ -0.2628  0.8828 -0.0734 -1.0844  0.3249 -0.3330  0.0603  0.3802 ]
Residual vector:
   A*x-b = [ -0.3262 -0.0070 -0.9543  0.2447 -0.6418 -0.3426 -0.1870  0.2960  0.6024 -0.

Press Enter/Return for the next example...
```

5.1. CVX

5.1.5 Constraint Examples

Suppose we wish to add some simple upper and lower bounds to the least-squares problem above: i.e., we wish to solve

$$\begin{aligned} &\text{minimize } \|Ax - b\|_2 \\ &\text{subject to } l \le x \le u \end{aligned}$$

```
cvx_begin % cvx version
    variable x(n);
    minimize( norm(A*x-b) );
    subject to
        x >= l;
        x <= u;
cvx_end
```

5.1. CVX

5.1.5 Constraint Examples

The constraint is added after the subject to line

In CVX, the objective function must be convex, the constraint functions myst be convex or affine.

diag and trace are affine matrix functions.

CVX uses extended-valued extensions (i.e. convex functions are evaluated to ∞ outside of their domain, and concave functions are evaluated to $-\infty$).

5.1. CVX

5.1.6 Other Cases

Assume that we wish to ensure that $X \in R^{n \times n}$ is a positive semidefinite matrix. The cvx expression for this is

X == semidefinite(n)

```
cvx_begin
    variable X(n+1,n+1) symmetric;
    minimize( trace(L*X) );
    subject to
        diag(X) == ones(n+1,1);
        X == semidefinite(n+1);
cvx_end
```

5.1. CVX

5.1.6 Other Cases

CVX provides means to define new functions (atoms). The easiest way of adding functions to CVX is to simply write them (abiding to the CVX ruleset) as in MATLAB. For example, we can create the file deadzone.m containing

```
function y = deadzone( x )
y = max( abs( x ) - 1, 0 )
```

CVX will automatically recognize that it is a convex function. This function may be used as an objective or in a constraint just like any other function. It may also be used in MATLAB for numerical evaluation.

5.1. CVX

5.1.6 Other Cases

CVX simultaneously solved the primal and dual problem, and provide a means to obtain the dual variables. For example, we can solve the dual variables in the constrained least squares problem as

```
cvx_begin
    variable x(n);
    dual variable y;
    minimize( norm(A*x-b) );
    subject to
        y : x >= 0;
cvx_end
```

5.1. CVX

5.1.6 Other Cases

A good way to start is by

reading cvx_usrguide.pdf

reading http://cvxr.com/cvx/doc/

looking at examples distributed with cvx (includes all examples in the course book) http://cvxr.com/cvx/examples/

5.1. CVX

5.1.6 Other Cases

```
for k = 1:length(gamma),
    fprintf( 1, '%8.4e', gamma(k) );
    cvx_begin
        variable x(n);
        minimize( norm(A*x-b)+gamma(k)*norm(x,1) );
    cvx_end
    l1norm(k) = norm(x,1);
    l2norm(k) = norm(A*x-b);
    fprintf( 1, '   %8.4e   %8.4e\n', l1norm(k), l2norm(k) );
end
```

```
n = 2;
N=50;
X = randn(2,N); Y = randn(2,N);
X = X*diag(0.99*rand(1,N)./sqrt(sum(X.^2)));
Y = Y*diag((1.02+rand(1,N))./sqrt(sum(Y.^2)));
T = [1 -1; 2 1];    X = T*X;    Y = T*Y;

cvx_begin sdp
    variable P(n,n) symmetric
    variables q(n) r(1)
    P <= -eye(n);
    sum((X'*P).*X',2) + X'*q + r >= +1;
    sum((Y'*P).*Y',2) + Y'*q + r <= -1;
cvx_end
```

5.1. CVX

5.1.6 Other Cases

YALMIP, Yet Another LMI Parser

https://yalmip.github.io/

JuMP: A Modeling Language for Mathematical Optimization

https://github.com/JuliaOpt/JuMP.jl

5.2. CVXOPT

5.2.1 Basic Information

CVXOPT (Python Software for Convex Optimization) is a free software package for convex optimization based on the Python programming language.

It can be used with the interactive Python interpreter, on the command line by executing Python scripts, or integrated in other software via Python extension modules.

Its main purpose is to make the development of software for convex optimization applications straightforward by building on Python's extensive standard library and on the strengths of Python as a high-level programming language.

5.2. CVXOPT

5.2.2 Installation

The entire package is available as a zip file, containing the source, documentation, installation instructions, and examples. The installation of CVXOPT from source requires Python version 2.7 or higher and ATLAS or BLAS/LAPACK.

Installing via conda: The conda-forge channel provides pre-built CVXOPT packages for Linux, macOS, and Windows that can be installed using conda:

```
conda install -c conda-forge cvxopt
```

5.2. CVXOPT

5.2.2 Installation

Installing via pip: A pre-built binary wheel package can be installed using pip:

```
pip install cvxopt
```

These pre-built packages are linked against OpenBLAS and include all the optional extensions (DSDP, GLPK, GSL, and FFTW,).

DSDP provides free open source implementation of an interior-point method for semidefinite programming. https://www.mcs.anl.gov/hs/software/DSDP/DSDP5-Matlab-UserGuide.pdf

5.2. CVXOPT

5.2.2 Installation

The GLPK (GNU Linear Programming Kit) package is intended for solving large-scale linear programming (LP), mixed integer programming (MIP), and other related problems. It is a set of routines written in ANSI C and organized in the form of a callable library. http://www.gnu.org/software/glpk/

The Fastest Fourier Transform in the West (FFTW) is a software library for computing discrete Fourier transforms (DFTs). http://fftw.org/

The GNU Scientific Library (GSL) is a numerical library for C and C++ programmers. http://www.gnu.org/software/gsl/

5.2. CVXOPT

5.2.2 Installation

CVXOPT can be installed globally (for all users on a UNIX/Linux system) using the command:

```
python setup.py install
```

It can also be installed locally (for a single user) using the command:

```
python setup.py install -user
```

5.2. CVXOPT

5.2.2 Installation

To test that the installation was successful, run the included tests using:

```
python -m unittest discover -s tests
```

If Python does not issue an error message, the installation was successful. Please see the installation instruction for more installation options.

5.2. CVXOPT

5.2.3 Examples

Solving a linear program

```
>>> from cvxopt import matrix, solvers
>>> A = matrix([[-1.0, -1.0, 0.0, 1.0], [1.0, -1.0, -1.0, -2
.0]])
>>> b = matrix([ 1.0, -2.0, 0.0, 4.0 ])
>>> c = matrix([ 2.0, 1.0 ])
>>> sol=solvers.lp(c,A,b)
     pcost       dcost       gap    pres   dres   k/t
 0:  2.6471e+00 -7.0588e-01  2e+01  8e-01  2e+00  1e+00
 1:  3.0726e+00  2.8437e+00  1e+00  1e-01  2e-01  3e-01
 2:  2.4891e+00  2.4808e+00  1e-01  1e-02  2e-02  5e-02
 3:  2.4999e+00  2.4998e+00  1e-03  1e-04  2e-04  5e-04
 4:  2.5000e+00  2.5000e+00  1e-05  1e-06  2e-06  5e-06
 5:  2.5000e+00  2.5000e+00  1e-07  1e-08  2e-08  5e-08
Optimal solution found.
>>> print(sol['x'])
[ 5.00e-01]
[ 1.50e+00]
```

5.2. CVXOPT

5.2.3 Examples

Solving a quadratic program

```
>>> from cvxopt import matrix, solvers
>>> Q = 2*matrix([[2, .5], [.5, 1]])
>>> p = matrix([1.0, 1.0])
>>> G = matrix([[-1.0,0.0],[0.0,-1.0]])
>>> h = matrix([0.0,0.0])
>>> A = matrix([1.0, 1.0], (1,2))
>>> b = matrix(1.0)
>>> sol=solvers.qp(Q, p, G, h, A, b)
     pcost       dcost       gap    pres   dres
 0:  1.8889e+00  7.7778e-01  1e+00  2e-16  2e+00
 1:  1.8769e+00  1.8320e+00  4e-02  1e-16  6e-02
 2:  1.8750e+00  1.8739e+00  1e-03  1e-16  5e-04
 3:  1.8750e+00  1.8750e+00  1e-05  0e+00  5e-06
 4:  1.8750e+00  1.8750e+00  1e-07  2e-16  5e-08
Optimal solution found.
>>> print(sol['x'])
[ 2.50e-01]
[ 7.50e-01]
```

5.3. CVXPY and CVXGEN

CVXPY is a Python-embedded modeling language for convex optimization problems. It allows you to express your problem in a natural way that follows the math, rather than in the restrictive standard form required by solvers.

In addition to convex programming, CVXPY also supports a generalization of geometric programming.

CVXPY supports Python 3 on Linux, macOS, and Windows. You can use pip or conda for installation. You may want to isolate your installation in a virtualenv, or a conda environment. Please refer to the installation instruction.

5.3. CVXPY and CVXGEN

Solve a least-square problem

```
# Import packages .
import cvxpy as cp
import numpy as np
# Generate data .
m = 20 , n = 15
np. random.seed (1)
A = np.random.randn (m, n)
b = np.random.randn (m)
# Define and solve the CVXPY problem .
x = cp. Variable (n)
cost = cp. sum_squares (A @ x - b)
```

5.3. CVXPY and CVXGEN

```
prob = cp. Problem (cp. Minimize ( cost ))
prob . solve ()
# Print result .
print ("\n The optimal value is", prob . value )
print (" The optimal x is")
print (x. value )
print (" The norm of the residual is ", cp. norm (A @ x - b,
p=2).value )
```

5.3. CVXPY and CVXGEN

CVXGEN (Code Generation for Convex Optimization) generates fast custom code for small, QP-representable convex optimization problems, using an online interface with no software installation. With minimal effort, turn a mathematical problem description into a high speed solver.

CVXGEN is for convex, QP-representable problems only.

It works best for small problems, where the final system has around 2000 total coefficients in the constraints and objective. CVXGEN does not work well for larger problems.

5.4. Cplex

The CPLEX Optimizer was named for the simplex method as implemented in the C programming language, although today it also supports other types of mathematical optimization and offers interfaces other than C. It was originally developed by Robert E. Bixby and sold commercially from 1988 by CPLEX Optimization Inc. This was acquired by ILOG in 1997 and ILOG was subsequently acquired by IBM in January 2009.

目前 Cplex 可在 IBM 官网下载，在 support 下的 knowledge center 中，找到适合你自己版本的。然后是从 get started 开始读，找你想要的信息，根据 get started 里的教程学习建议，再配之搜索引擎，解决问题。

5.4. Cplex

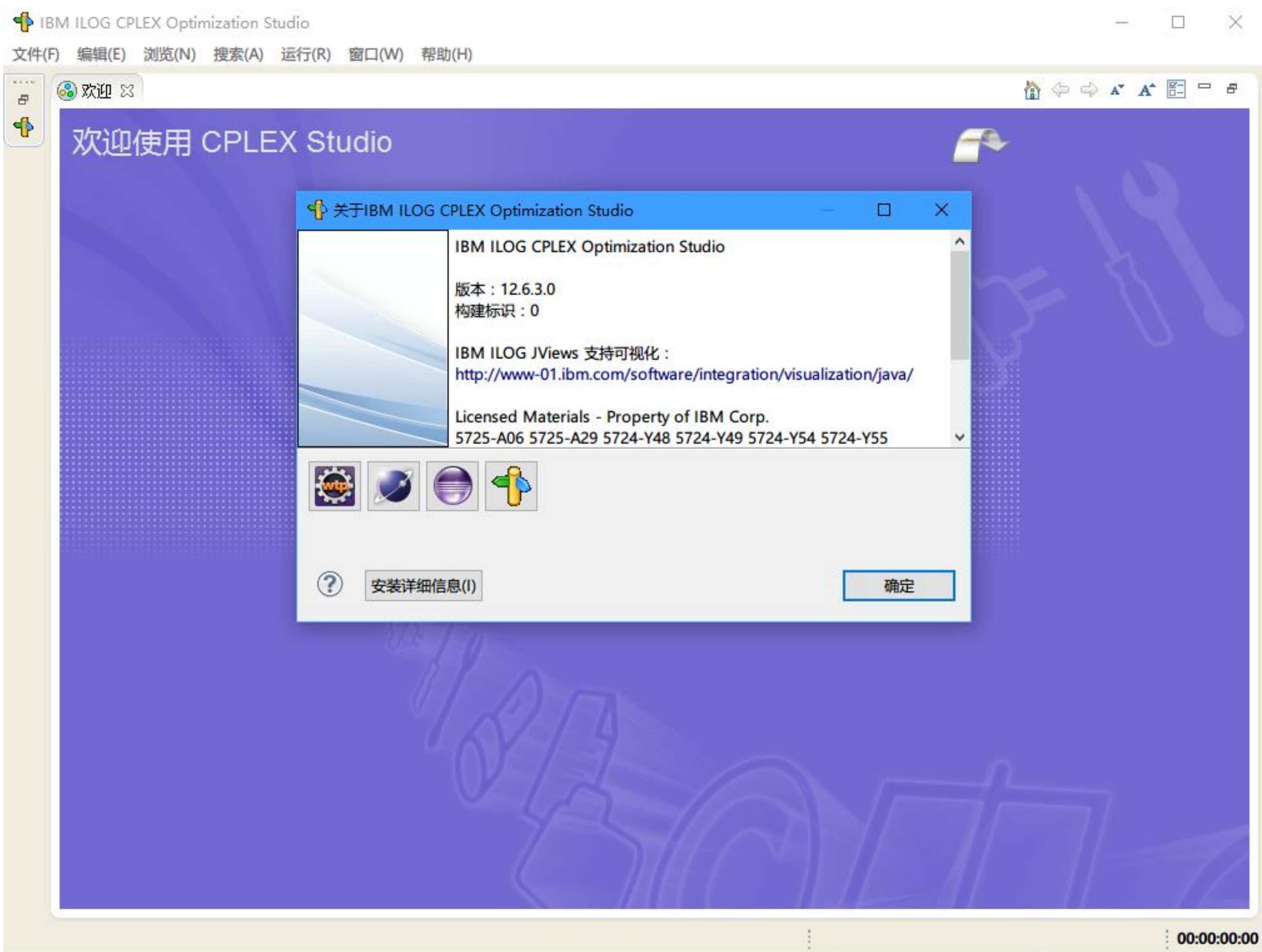
Cplex 专门用于求解大规模的线性规划（LP）、二次规划（QP）、带约束的二次规划（QCQP）、二阶锥规划（SOCP）等四类基本问题，以及相应的混合整数规划（MIP）问题。

目前，Cplex 学术版免费，同时对于求解变量是 unlimited 的，需要自己用学校的邮箱去申请。IBM 为 Cplex 提供了中文网页

https://www.ibm.com/cn-zh/analytics/cplex-optimizer

OR 学术圈，Cplex 是事实的计算基准软件。

5.4. Cplex



IBM ILOG CPLEX Optimization Studio
文件(F) 编辑(E) 浏览(N) 搜索(A) 运行(R) 窗口(W) 帮助(H)
欢迎
欢迎使用 CPLEX Studio
关于IBM ILOG CPLEX Optimization Studio
IBM ILOG CPLEX Optimization Studio
版本：12.6.3.0
构建标识：0
IBM ILOG JViews 支持可视化：
http://www-01.ibm.com/software/integration/visualization/java/
Licensed Materials - Property of IBM Corp.
5725-A06 5725-A29 5724-Y48 5724-Y49 5724-Y54 5724-Y55
安装详细信息(I)
确定
00:00:00:00

5.4. Cplex

C++风格的测试代码

```
#include "stdafx.h"
#include <ilcplex\ilocplex.h>
ILOSTLBEGIN

int main(int argc, char **argv)
{
   IloEnv env;                          %定义环境
   try
   {
      IloModel model(env);              %实例化
      IloNumVarArray vars(env);         %建立变量
```

5.4. Cplex

```
vars.add(IloNumVar(env, 0.0, 40.0));
vars.add(IloNumVar(env));
vars.add(IloNumVar(env));
model.add(IloMaximize(env, vars[0] + 1 * vars[1] + 3 * vars[2]));    %加入目标函数和约束
model.add(-vars[0] + vars[1] + vars[2] <= 20);
model.add(vars[0] - 3 * vars[1] + vars[2] <= 30);
IloCplex cplex(model);
if (!cplex.solve())    %求解
{
  env.error() << "Failed to optimize LP." << endl;
  throw(-1);
}
```

5.4. Cplex

```
    IloNumArray vals(env);            %输出结果
    env.out() << "Solution status = " << cplex.getStatus() << endl;
    env.out() << "Solution status = " << cplex.getObjValue() << endl;
    cplex.getValues(vals, vars);
    env.out() << "Values = " << vals << endl;
  }
  catch (IloException & e)
  {
    cerr << "Concert exception caught: " << e << endl;
  }
  catch (...)
```

5.4. Cplex

```
  {
    cerr << "Unknown exception caught" << endl;
  }
  env.end();

  return 0;
}
```

5.4. Cplex

The Netlib repository contains freely available software, documents, and databases of interest to the numerical, scientific computing, and other communities. The repository is maintained by AT&T Bell Laboratories, the University of Tennessee and Oak Ridge National Laboratory, and by colleagues world-wide. The collection is replicated at several sites around the world, automatically synchronized, to provide reliable and network efficient service to the global community.

大家有兴趣可以测试一下 Netlib 提供的一些线性规划标准问题，有趣的是，CVX+SDPT3 对于有些问题不能找到正确解。

http://www.netlib.org/lp/data/

5.4. Cplex

Cplex 的两个国产对手

杉数求解器（Cardinal Optimizer，COPT）是杉数科技自主研发的一款针对大规模优化问题的高效数学规划求解器套件。COPT 高效地实现了单纯形法和内点法，目前支持所有主流 64 位操作系统），包括：Windows、Linux 和 MacOS，并提供以下接口：Python、PuLP、Pyomo、C、C++、C#、Java、AMPL 和 GAMS。也支持 ARM64 平台

https://www.shanshu.ai/copt/

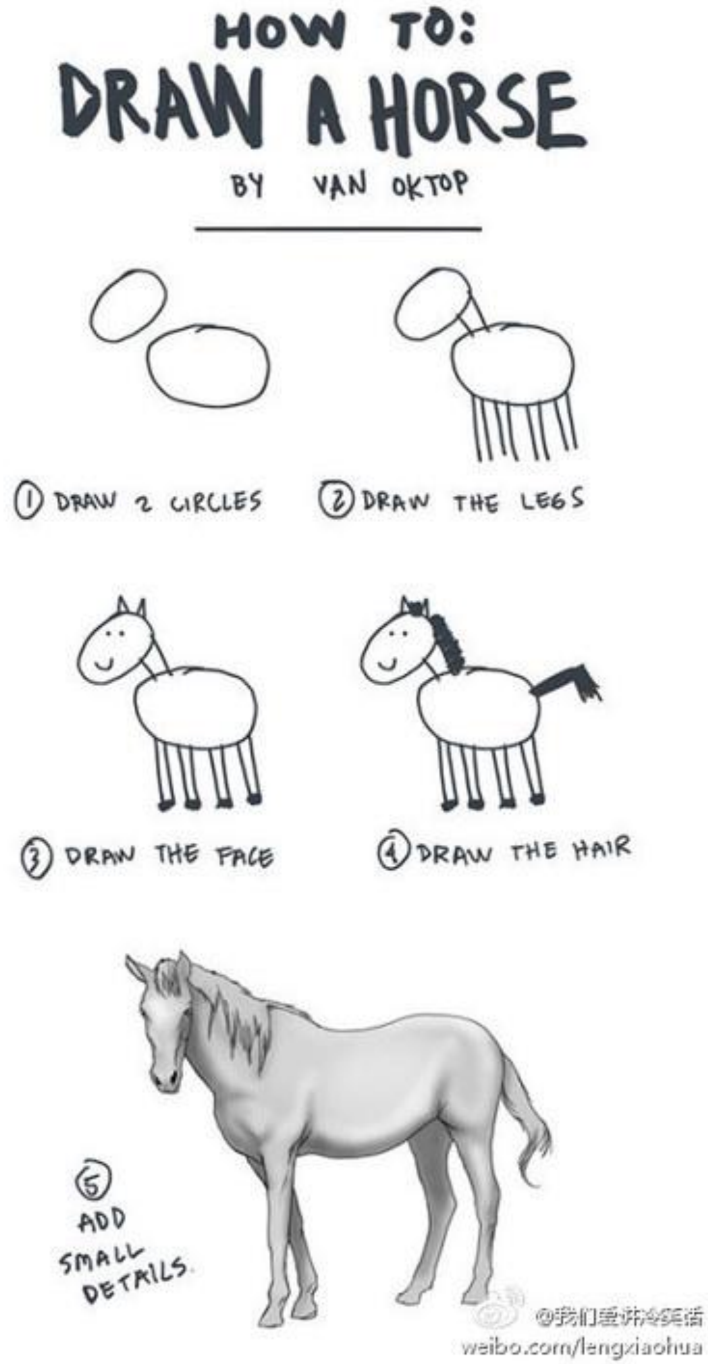
MindOpt 是达摩院决策智能实验室自主研发的数学规划求解器套件。目前支持单纯形法、内点法、并发法求解线性规划，有 C++、python 版本的 API。

https://solver.damo.alibaba.com/htmlpages/page#/

5.4. Cplex

Arizona State University 的 Hans D. Mittelmann 提供了测试平台，方便各种求解软件打榜

http://plato.asu.edu/
http://plato.asu.edu/bench.html


HOW TO:
DRAW A HORSE
BY VAN OKTOP
① DRAW 2 CIRCLES
② DRAW THE LEGS
③ DRAW THE FACE
④ DRAW THE HAIR
⑤ ADD SMALL DETAILS.
@我们爱讲冷笑话
weibo.com/lengxiaohua


I'LL protect you until you're ready
HARVARD
I'LL keep you warm until you're ready
CORNELL UNIVERSITY FOUNDED A.D. 1865
I'LL carry you until you're ready
TSINGHUA UNIVERSITY
~1911~
fly,
THESQUARECOMICS

5.5. References

[1] S. Boyd, L. Vandenberghe, *Convex Optimization*, Cambridge University Press, 2004. http://www.stanford.edu/~boyd/cvxbook/ http://www.ee.ucla.edu/~vandenbe/cvxbook

[2] CVX http://cvxr.com/cvx/

[3] http://web.cvxr.com/cvx/examples/cvxbook/Ch08_geometric_probs/html/quad_discr.html

[4] YALMIP, Yet Another LMI Parser, https://yalmip.github.io/

[5] I. Dunning, J. Huchette, M. Lubin, "JuMP: A modeling language for mathematical optimization," *SIAM Review*, vol. 59, no. 2, pp. 295-320, 2017.

[6] J.Mattingley, S. Boyd, "'CVXGEN' A code generator for embedded convex optimization," *Optimization and Engineering*, vol. 13, no. 1, pp. 1-27, 2012.

[7] http://www.netlib.org/lp/data/