Chapter #5 Exercises

5.1

5.1

The Taylor expandion at $x=x_i$ for f is:

$$f(x)=f(x_i)+f'(x_i)(x-x_i)+\frac{1}{2}f''(x_i)(x-x_i)^2+O(|x-x_i|^3)$$

Value at $x=x_{i+1}$ is

$$f(x_{i+1})=f(x_i)+f'(x_i)(x_{i+1}-x_i)+\frac{1}{2}f''(x_i)(x_{i+1}-x_i)^2+O(|x_{i+1}-x_i|^3)$$

$P_i(x)=f(x_i)+(x-x_i)\frac{f(x_{i+1})-f(x_i)}{x_{i+1}-x_i}$, take $f(x_{i+1})$ into $P_i(x)$.

$$P_i(x)=f(x_i)+(x-x_i)f'(x_i)+\frac{1}{2}f''(x_i)(x-x_i)(x_{i+1}-x_i)+(x-x_i)\frac{O(|x_{i+1}-x_i|^3)}{x_{i+1}-x_i}$$

$$=f(x_i)+(x-x_i)f'(x_i)+\frac{1}{2}f''(x_i)(x-x_i)(x_{i+1}-x_i)+O(n^{-3})$$, which is (5.14).

5.2

5.2

For Simpson rule, $m=2$.

$p_{i0}(x)=\frac{x-x_{i1}^*}{x_{i0}^*-x_{i1}^*}\cdot\frac{x-x_{i2}^*}{x_{i0}^*-x_{i2}^*}$, $p_{i1}(x)=\frac{x-x_{i0}^*}{x_{i1}^*-x_{i0}^*}\cdot\frac{x-x_{i2}^*}{x_{i1}^*-x_{i2}^*}$, $p_{i2}(x)=\frac{x-x_{i0}^*}{x_{i2}^*-x_{i0}^*}\cdot\frac{x-x_{i1}^*}{x_{i2}^*-x_{i1}^*}$, $x_{i0}^*=x_i$, $x_{i1}^*=\frac{x_i+x_{i+1}}{2}$, $x_{i2}^*=x_{i+1}$

$$A_{i0}=\int_{x_i}^{x_{i+1}}p_{i0}(x)dx=\int_{x_i}^{x_{i+1}}\frac{x-x_{i1}^*}{x_{i0}^*-x_{i1}^*}\cdot\frac{x-x_{i2}^*}{x_{i0}^*-x_{i2}^*}dx=\frac{1}{(x_{i0}^*-x_{i1}^*)(x_{i0}^*-x_{i2}^*)}\int_{x_i}^{x_{i+1}}x^2-x_{i1}^*x-x_{i2}^*x+x_{i1}^*x_{i2}^*\,dx$$

$$=\frac{2}{(x_i-x_{i+1})^2}\left(\frac{1}{3}(x_{i+1}^3-x_i^3)-\frac{1}{2}(x_{i+1}^2-x_i^2)\cdot\left(\frac{x_i+3x_{i+1}}{2}\right)+\frac{x_ix_{i+1}+x_{i+1}^2}{2}\cdot(x_{i+1}-x_i)\right)$$

$$=\frac{2}{(x_i-x_{i+1})^2}\left(\frac{1}{3}x_{i+1}^3-\frac{1}{3}x_i^3-\frac{1}{4}x_{i+1}^2x_i-\frac{3}{4}x_{i+1}^3+\frac{1}{4}x_i^3+\frac{3}{4}x_i^2x_{i+1}+\cancel{\frac{1}{2}x_ix_{i+1}^2}+\frac{1}{2}x_{i+1}^3-\frac{1}{2}x_i^2x_{i+1}-\cancel{\frac{1}{2}x_ix_{i+1}^2}\right)$$

$$=\frac{2}{(x_i-x_{i+1})^2}\left(\frac{1}{12}x_{i+1}^3-\frac{1}{12}x_i^3-\frac{1}{4}x_i^2x_{i+1}+\frac{1}{4}x_i^2x_{i+1}\right)$$

$$=\frac{1}{6}(x_{i+1}-x_i)$$

$$A_{i1}=\int_{x_i}^{x_{i+1}}p_{i1}(x)dx=\int_{x_i}^{x_{i+1}}\frac{x-x_{i0}^*}{x_{i1}^*-x_{i0}^*}\cdot\frac{x-x_{i2}^*}{x_{i1}^*-x_{i2}^*}dx=\frac{-4}{(x_{i+1}-x_i)^2}\int_{x_i}^{x_{i+1}}x^2-(x_{i0}^*+x_{i2}^*)x+x_{i0}^*\cdot x_{i2}^*\,dx$$

$$=\frac{-4}{(x_{i+1}-x_i)^2}\left(\frac{1}{3}x_{i+1}^3-\frac{1}{3}x_i^3-\frac{1}{2}(x_{i+1}^2-x_i^2)(x_{i+1}+x_i)+(x_i\cdot x_{i+1})(x_{i+1}-x_i)\right)$$

$$=\frac{-4}{(x_{i+1}-x_i)^2}\left(\frac{1}{3}x_{i+1}^3-\frac{1}{3}x_i^3-\frac{1}{2}x_{i+1}^3-\frac{1}{2}x_{i+1}^2x_i+\frac{1}{2}x_i^2x_{i+1}+\frac{1}{2}x_i^3+x_i\cdot x_{i+1}^2-x_i^2\cdot x_{i+1}\right)$$

$$=\frac{-4}{(x_{i+1}-x_i)^2}\left(\frac{1}{6}x_i^3-\frac{1}{6}x_{i+1}^3+\frac{1}{2}x_{i+1}^2x_i-\frac{1}{2}x_i^2x_{i+1}\right)$$

$$=\frac{2}{3}(x_{i+1}-x_i)$$

$$A_{i2}=\int_{x_i}^{x_{i+1}}p_{i2}\,dx=\int_{x_i}^{x_{i+1}}\frac{x-x_{i0}^*}{x_{i2}^*-x_{i0}^*}\cdot\frac{x-x_{i1}^*}{x_{i2}^*-x_{i1}^*}dx=\frac{2}{(x_{i+1}-x_i)^2}\int_{x_i}^{x_{i+1}}x^2-(x_{i0}^*+x_{i1}^*)x+x_{i0}^*\cdot x_{i1}^*\,dx$$

$$=\frac{2}{(x_{i+1}-x_i)^2}\left(\frac{1}{3}x_{i+1}^3-\frac{1}{3}x_i^3-\frac{1}{2}(x_{i+1}^2-x_i^2)\left(\frac{3x_i+x_{i+1}}{2}\right)+\frac{x_ix_{i+1}+x_i^2}{2}(x_{i+1}-x_i)\right)$$

$$=\frac{2}{(x_{i+1}-x_i)^2}\left(\frac{1}{3}x_{i+1}^3-\frac{1}{3}x_i^3-\frac{3}{4}x_ix_{i+1}^2-\frac{1}{4}x_{i+1}^3+\frac{3}{4}x_i^3+\frac{1}{4}x_i^2x_{i+1}+\frac{1}{2}x_ix_{i+1}^2-\frac{1}{2}x_i\cancel{x_{i+1}}-\frac{1}{2}x_i^3+\frac{1}{2}x_i^2\cancel{x_{i+1}}\right)$$

$$=\frac{2}{(x_{i+1}-x_i)^2}\left(\frac{1}{12}x_{i+1}^3-\frac{1}{12}x_i^3-\frac{1}{4}x_ix_{i+1}^2+\frac{1}{4}x_i^2x_{i+1}\right)$$

$$=\frac{1}{6}(x_{i+1}-x_i)$$

Hence, $A_{i0}=A_{i2}=\frac{1}{6}(x_{i+1}-x_i)$, $A_{i1}=\frac{2}{3}(x_{i+1}-x_i)$.

5.3

a.

The function we want to integrate is p.d.f of $N(\mu, 9/7)$ times p.d.f of $Cauchy(5, 2)$, where we only need to consider the sufficient statistics of μ.

```
x <- c(6.52,8.32,0.31,2.82,9.96,0.14,9.64)
xbar <- mean(x)
f <- function(mu){
  (1/sqrt(2*pi*9/7))*exp(-7/18*(mu-xbar)^2)*(1/(2*pi))*(4/((mu-5)^2+4))
}


riemann <- function(interval,n,f){
      h   <- (interval[2]-interval[1])/n
      x   <- interval[1] + (0:(n-1))*h
      out <- h*sum(f(x))
      return(out)
}

1/riemann(c(-10000,10000),1e7,f)
```

```
## [1] 7.846538
```

b.

```
trapezoidal <- function(interval,n,f){
      h   <- (interval[2]-interval[1])/n
      x   <- interval[1] + (1:(n-1))*h
      out <- h*sum(f(x)) + h/2*sum(f(interval))
      return(out)
}

simpsons <- function(interval,n,f){
      h   <- (interval[2]-interval[1])/n
      x   <- interval[1] + (0:n)*h
      out <- 0
      for(i in 1:(n/2)){
        out <- h/3*(f(x[2*i-1])+4*f(x[2*i])+f(x[2*i+1]))+out
      }
      return(out)
}

n <- 2
k <- 7.84654
int01 = int02 = int03 <- 1
eps <- 1
inte <- c(2,8)
while (eps > 1e-4) {
  int1 <- k*riemann(inte,n,f)
  eps1 <- (int1-int01)/int01
  int01 <- int1

  int2 <- k*trapezoidal(inte,n,f)
  eps2 <- (int2-int02)/int02
  int02 <- int2

  int3 <- k*simpsons(inte,n,f)
  eps3 <- (int3-int03)/int3
  int03 <- int3

  eps <- min(eps1,eps2,eps3)
  n <- 2*n
}

c(int01,int02,int03)
```

```
## [1] 0.9902608 0.9962186 0.9082608
```

```
0.99605 - c(int01,int02,int03)
```

```
## [1]  0.0057891769 -0.0001686053  0.0877891948
```

C.

5.3

$$f(\mu)=k\cdot\frac{1}{2\pi}\cdot\frac{4}{4+(\mu-3)^2}\cdot\frac{\sqrt{7}}{\sqrt{2\pi}\,3}\exp\left\{-\frac{7}{18}(\bar{x}-\mu)^2\right\}$$

Let $u=\frac{\exp\{\mu\}}{1+\exp\{\mu\}}$, $\mu=\log\frac{u}{1-u}$, $d\mu=\frac{1-u}{u}\cdot\frac{1-u+u}{(1-u)^2}=\frac{1}{u(1-u)}$

Hence,

$$\int_3^{+\infty}f(\mu)\,d\mu=\int_{\frac{e^3}{1+e^3}}^{1}f\left(\log\frac{u}{1-u}\right)\cdot\frac{1}{u(1-u)}\,du$$

```
ff <- function(u){
  f(log(u/(1-u)))/(u*(1-u))
}

res1 <- k*riemann(c(exp(3)/(1+exp(3)),0.9999),1000,ff)
res2 <- k*trapezoidal(c(exp(3)/(1+exp(3)),0.9999),1000,ff)
res3 <- k*simpsons(c(exp(3)/(1+exp(3)),0.9999),1000,ff)

c(res1,res2,res3)
```

```
## [1] 0.9907174 0.9907699 0.9907823
```

```
0.99086 - c(res1,res2,res3)
```

```
## [1] 1.425985e-04 9.005999e-05 7.774885e-05
```

```
res11 <- k*riemann(c(3,1e6),10000,f)
res21 <- k*trapezoidal(c(3,1e6),10000,f)
res31 <- k*simpsons(c(3,1e6),10000,f)

c(res11,res21,res31)
```

```
## [1] 2.3954128 1.1977064 0.7984709
```

```
0.99086 - c(res11,res21,res31)
```

```
## [1] -1.4045528 -0.2068464  0.1923891
```

d.

```
ff2=function(x){
  f(1/x)/x^2
}

res1 <- k*riemann(c(1e-6,1/3),1000,ff2)
res2 <- k*trapezoidal(c(1e-6,1/3),1000,ff2)
res3 <- k*simpsons(c(1e-6,1/3),1000,ff2)

c(res1,res2,res3)
```

```
## [1] 0.9908235 0.9908594 0.9908595
```

```
0.99086 - c(res1,res2,res3)
```

```
## [1] 3.650891e-05 5.777153e-07 5.233999e-07
```

5.4

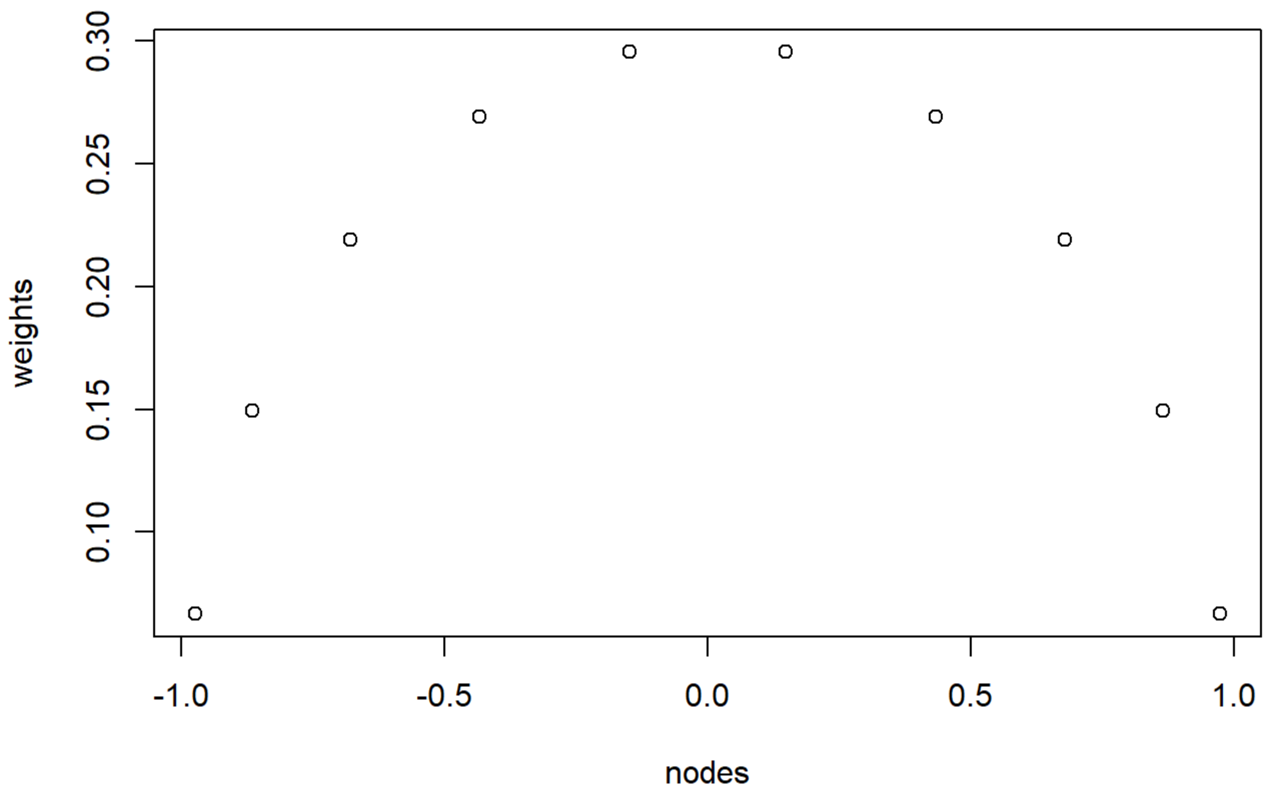
```
a <- exp(1)
m <- 6
Romberg <- matrix(0,7,7)
f  <- function(x){
  1/x
}

for(i in 0:6){
  Romberg[i+1,1]=trapezoidal(c(1,a),2^i,f)
}

for(j in 1:6){
  for(i in j:6){
    Romberg[i+1,j+1]=(4^j*Romberg[i+1,j]-Romberg[i,j])/(4^j-1)
  }
}

Romberg
```

```
##          [,1]      [,2]     [,3]      [,4]     [,5] [,6] [,7]
## [1,] 3.525604 0.0000000 0.000000 0.0000000 0.000000    0    0
## [2,] 1.049718 0.2244225 0.000000 0.0000000 0.000000    0    0
## [3,] 1.013039 1.0008128 1.052572 0.0000000 0.000000    0    0
## [4,] 1.003307 1.0000631 1.000013 0.9991788 0.000000    0    0
## [5,] 1.000830 1.0000042 1.000000 1.0000001 1.000003    0    0
## [6,] 1.000208 1.0000003 1.000000 1.0000000 1.000000    1    0
## [7,] 1.000052 1.0000000 1.000000 1.0000000 1.000000    1    1
```

5.5

a.

```
nodes <- c(0.148874338981631, 0.433395394129247, 0.679409568299024, 0.865063366688985, 0.973906528
517172)
nodes <- c(nodes,-nodes)
weights <- c(0.295524224714753, 0.269266719309996, 0.219086362515982, 0.149451394150581, 0.0666713
44308688)
weights <- c(weights,weights)
plot(nodes,weights)
```



b.

```
sum(nodes^2*weights)
```

```
## [1] 0.6666667
```

```
2/3 - sum(nodes^2*weights)
```

```
## [1] -6.735012e-08
```

5.6

5.6

(a)

For Gauss-Hermite quadrature rule,

$H_0(x)=1,\ H_1(x)=x,\ H_k(x)=x\cdot H_{k-1}(x)-(k-1)P_{k-2}(x).$

Hence, $H_2(x)=x^2-1,\ H_3(x)=x^3-x-2x=x^3-3x,\ H_4(x)=x^4-3x^2-3x^2+3=x^4-6x^2+3.$

$H_5(x)=x^5-6x^3+3x-4x^3+12x=x^5-10x^3+15x.$

Hence it relies on $H_5(x)=c(x^5-10x^3+15x)$

```
f <- function(x) {
  gamma(x+1)/(gamma(x/2+1)*2^(x/2))
}

f(10) - 20*f(8) + 130*f(6) - 300*f(4) + 225*f(2)
```

```
## [1] 120
```

b.

(b)

$$\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}}\cdot H_5^2(x)\,dx = c^2\cdot\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}}(x^5-10x^3+15x)^2\,dx = c^2\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}}(x^{10}+100x^6+225x^2-20x^8+30x^6-300x^4)\,dx$$

$$= c^2\sqrt{2\pi}\,E(x^{10}-20x^8+130x^6-300x^4+225x^2)$$

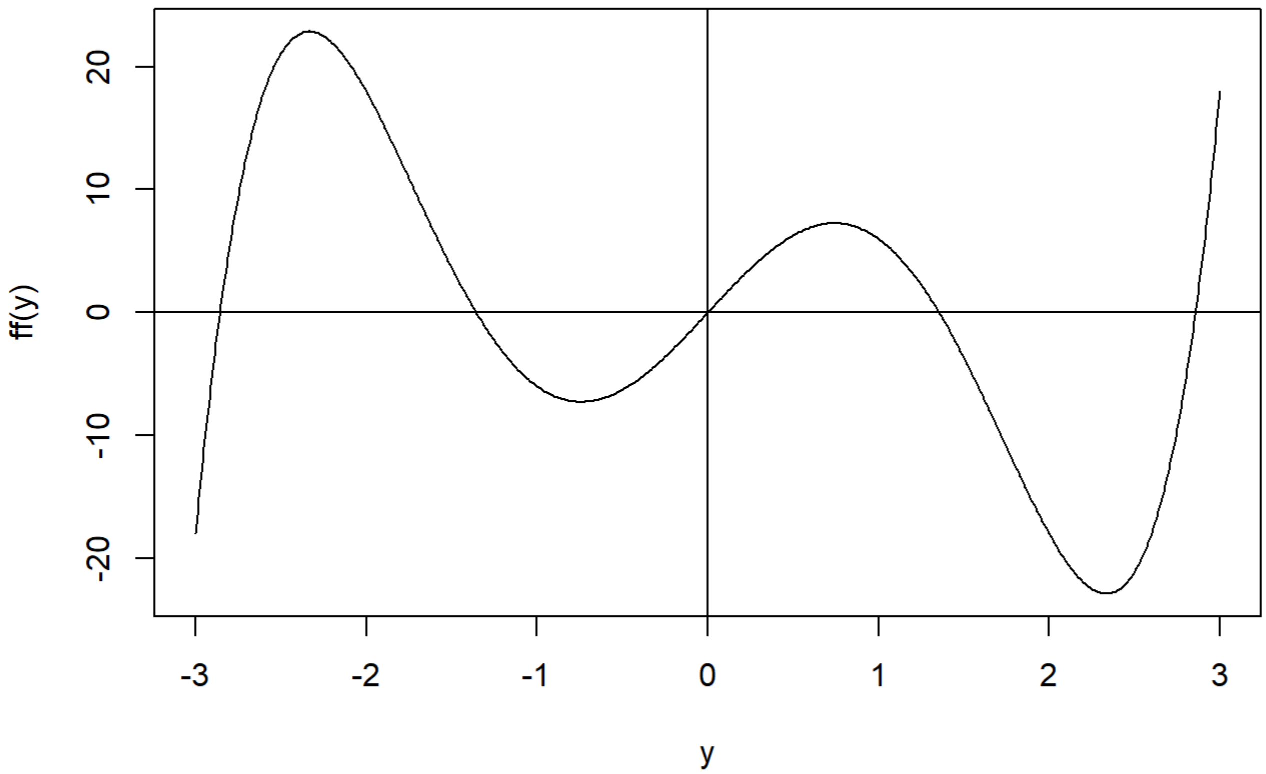
$$= c^2\cdot\sqrt{2\pi}\left(\frac{10!}{5!\cdot 2^5}-20\cdot\frac{8!}{4!\cdot 2^4}+130\cdot\frac{6!}{3!\cdot 2^3}-300\cdot\frac{4!}{2!\cdot 2^2}+225\cdot\frac{2!}{1!\cdot 2^1}\right)$$

$$= c^2\cdot\sqrt{2\pi}\cdot 120 = 1$$

Hence, $c=\frac{1}{\sqrt{120\cdot\sqrt{2\pi}}}$

c.

```
ff <- function(x){
  x^5 - 10*x^3 + 15*x
}

y <- seq(-3,3,0.01)

plot(y,ff(y),type = 'l')
abline(h = 0,v = 0)
```




```
#The five roots are
roots <- Re(polyroot(c(0,15,0,-10,0,1)))
roots
```

```
## [1]  0.000000  1.355626 -1.355626 -2.856970  2.856970
```

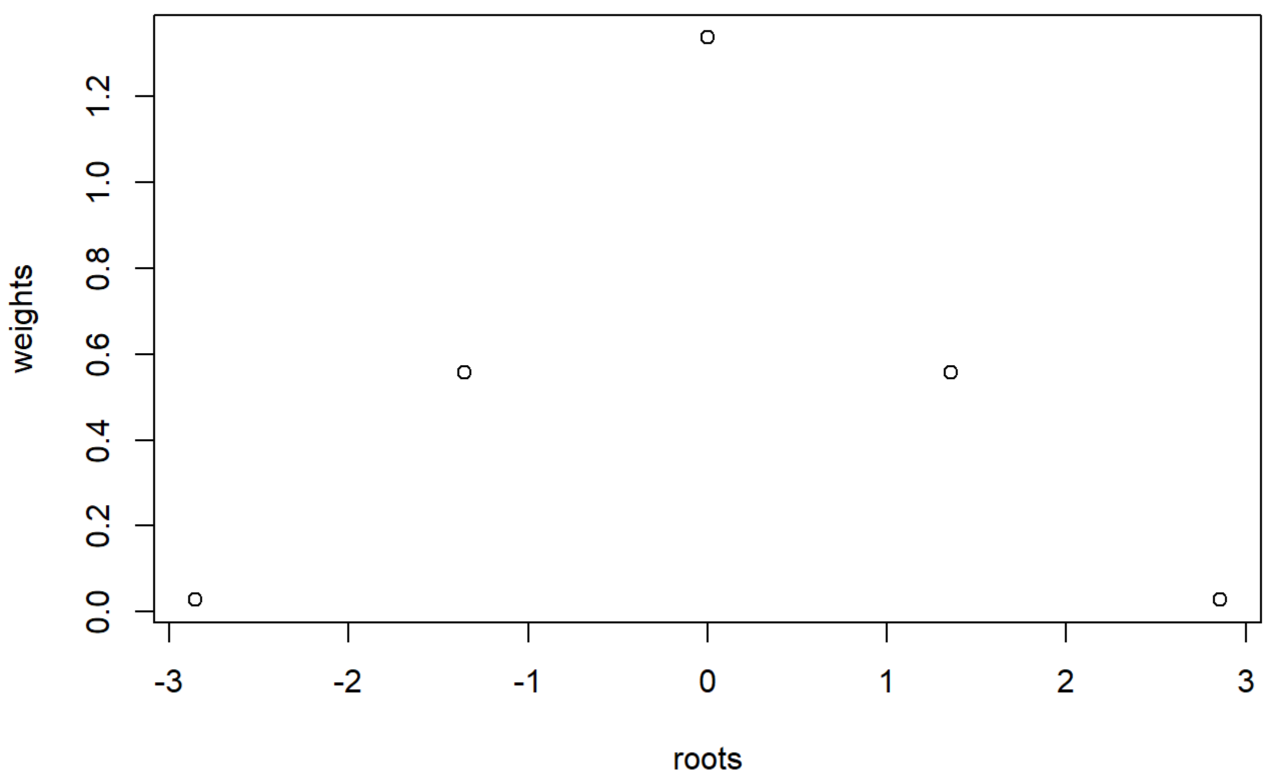
d.

$$\begin{aligned} p_6(x) &= xp_5(x) - 5p_4(x) \\ &= x(x^5 - 10x^3 + 15x) - 5(x^4 - 6x^2 + 3) \\ &= x^6 - 15x^4 + 45x^2 - 15 \end{aligned}$$

```
c6 <- 1/(sqrt(720*sqrt(2*pi)))
h6 <- function(x){
  c6*(x^6-15*x^4+45*x^2-15)
}

c5 <- 1/(sqrt(120*sqrt(2*pi)))
h51 <- function(x){
  c5*(5*x^4 -30*x^2 + 15)
}

weights <- -c6/(c5*h6(roots)*h51(roots))
weights
```

```
## [1] 1.33686841 0.55666179 0.55666179 0.02821815 0.02821815
```

```
plot(roots, weights)
```



e.

e.

Since $\frac{\mu-50}{8} \sim t_1$, $F(\mu)=F_{t_1}(\frac{\mu-50}{8})$, hence $f(\mu) \propto (1+(\frac{\mu-50}{8})^2)^{-1}$

Hence $P_{\theta|x} \propto \exp\{-\frac{(47-\mu)^2}{10}\} \cdot (1+(\frac{\mu-50}{8})^2)^{-1}$

Hence we could calculate $\int_{-\infty}^{+\infty} \exp\{-\frac{(47-\mu)^2}{10}\} \cdot (1+(\frac{\mu-50}{8})^2)^{-1} d\mu$ to get normalizing constant.

Consider $f^* = \frac{P_{\theta|x}}{W(x)}$, we could use Gauss-Hermite on $\int_{-\infty}^{+\infty} f^*(x) \cdot W(x)\, dx$

Then the posterior expectations of μ is

$$\int \mu^2 \cdot P_{\theta|x}\, dx - \left(\int \mu \cdot P_{\theta|x}\, dx\right)^2$$

```
f <- function(mu){
  exp(-(47-mu)^2/10)*(1+((mu-50)/8)^2)^(-1)/exp(-mu^2/2)
}

gh <- function(f){
  sum(weights*f(roots))
}

c <- 1/gh(f)

c
```

```
## [1] 9.056712e+85
```

```
E1 <- function(mu){
  mu*c*f(mu)
}

E2 <- function(mu){
  mu^2*c*f(mu)
}

gh(E2) - gh(E1)^2
```

```
## [1] 2.476341e-06
```

The result seems a little bit wired, but I didn't figure out what is wrong……

Here is another method, using method of substitution.

Let $X = \frac{47-\mu}{\sqrt{5}}$, $\mu = 47+\sqrt{5}$.

Hence $\int_{-\infty}^{+\infty} p_{\theta|x}\, d\theta = \int_{-\infty}^{+\infty} \underbrace{\sqrt{5}\left(1+\left(\frac{\sqrt{5}X-3}{8}\right)^2\right)^{-1}}_{f(x)} \exp\left\{-\frac{x^2}{2}\right\} dx$

```r
f1 <- function(x){
  sqrt(5)/(1+((sqrt(5)*x-3)/8)^2)
}


gh <- function(f){
  sum(weights*f(roots))
}

c <- 1/gh(f1)

c
```

```
## [1] 0.2110069
```

```r
E1 <- function(x){
  (47+sqrt(5)*x)*c*f1(x)
}

E2 <- function(x){
  (47+sqrt(5)*x)^2*c*f1(x)
}

gh(E2) - gh(E1)^2
```

```
## [1] 4.53585
```