Homework 5

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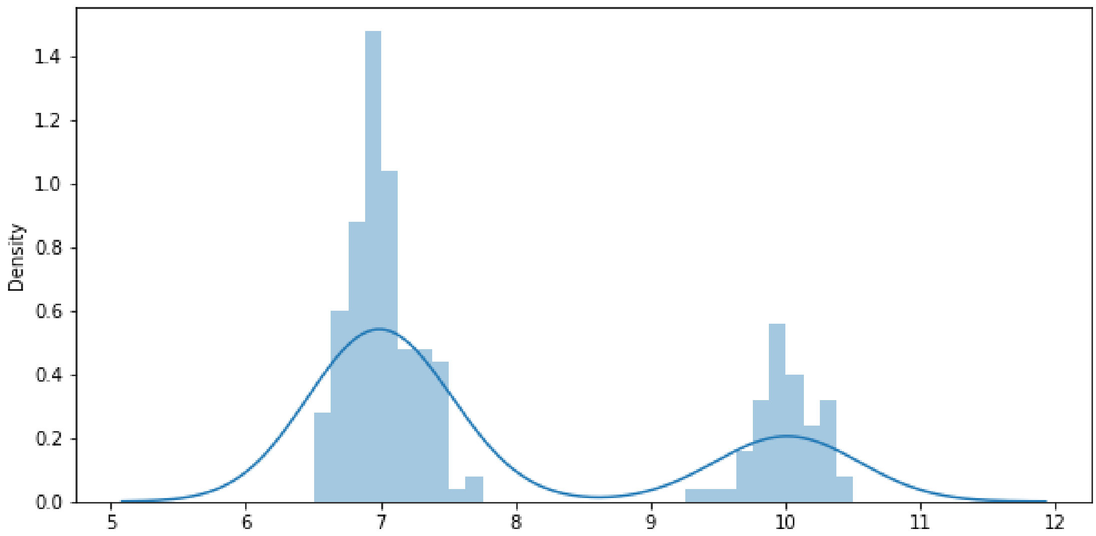
注：因为我不会 R 语言，所以代码都是用 Python 写的。代码文件见 homework5-code.ipynb

7.1

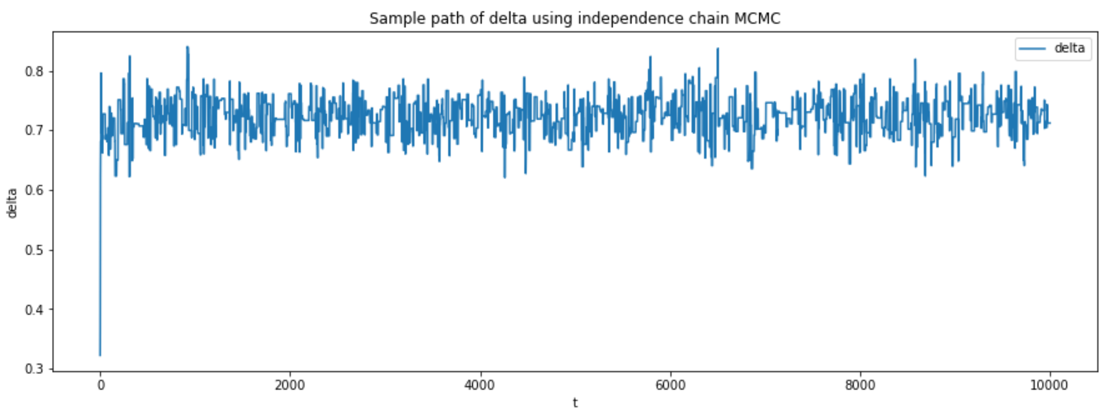
解：

a.

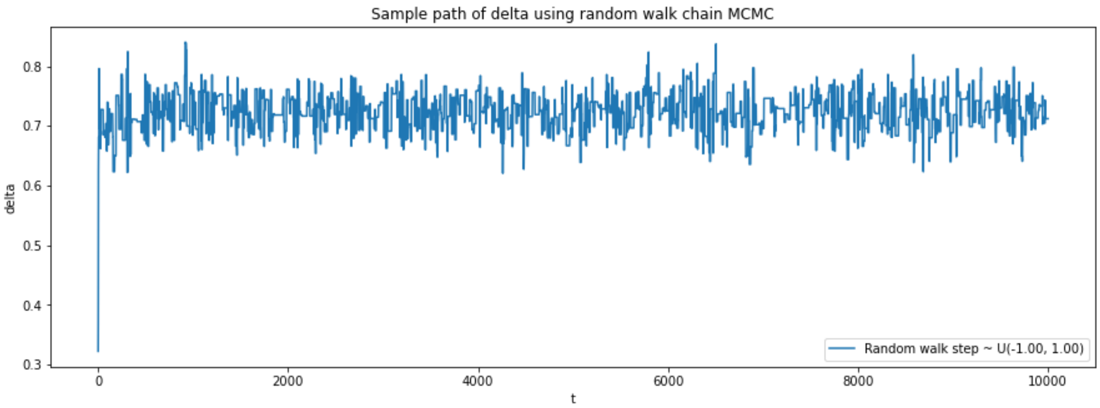


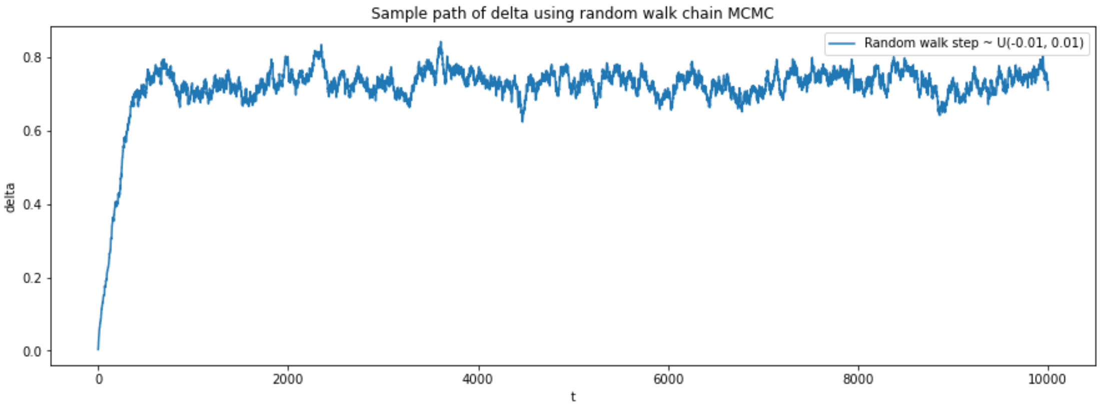
b.

$$P(X,\delta)=\prod_{i=1}^{n}\left(\delta\frac{1}{0.5\sqrt{2\pi}}e^{-\frac{1}{2}\frac{(x_i-7)^2}{0.5^2}}+(1-\delta)\frac{1}{0.5\sqrt{2\pi}}e^{-\frac{1}{2}\frac{(x_i-10)^2}{0.5^2}}\right)$$



c.





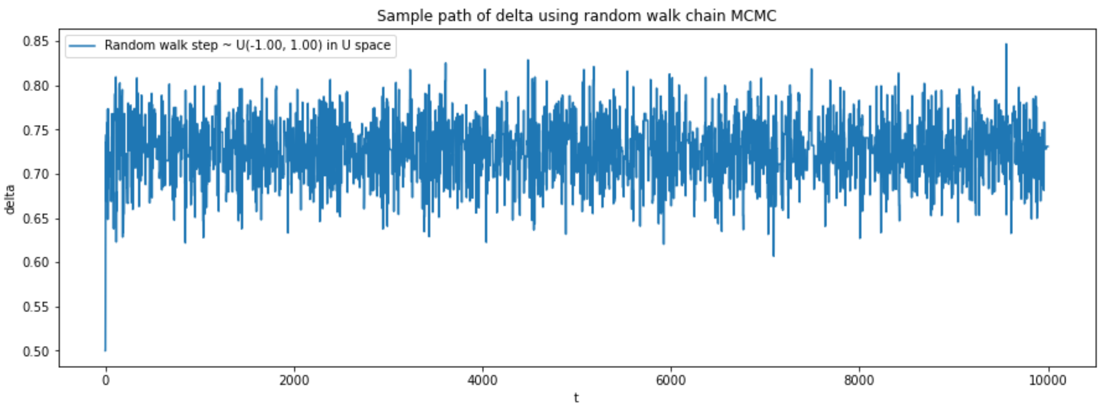
d.

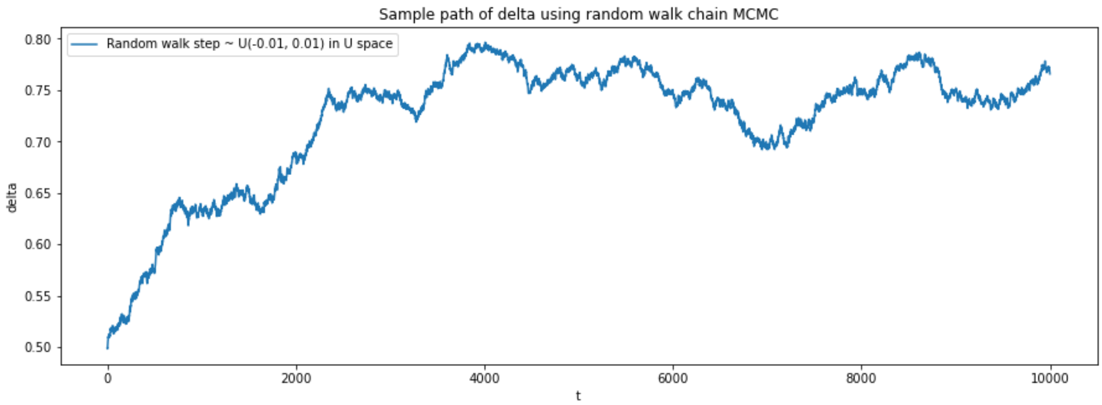
$$U = \ln\frac{\delta}{1-\delta}, \delta = \frac{1}{(e^{-U}+1)}$$

在 U 空间进行$\epsilon \sim \mathrm{U}(-b, b)$的随机游走，则

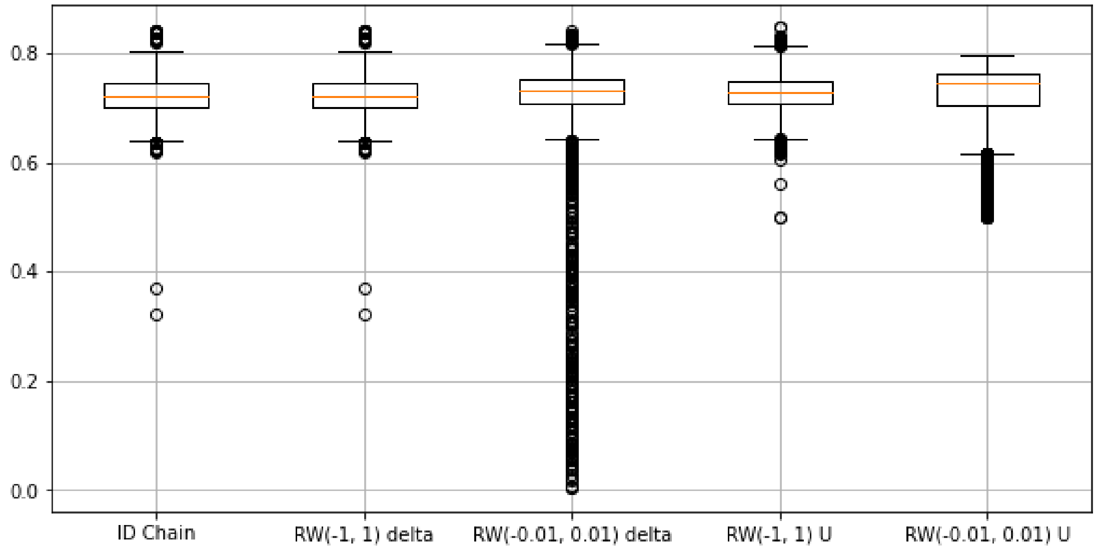
$$g(U^t|U^*) = g(U^*|U^t) = \frac{1}{2b}$$

$$|Jacob(U)| = \frac{e^{-U}}{(e^{-U}+1)^2} = e^{-U}\delta^2$$





e.

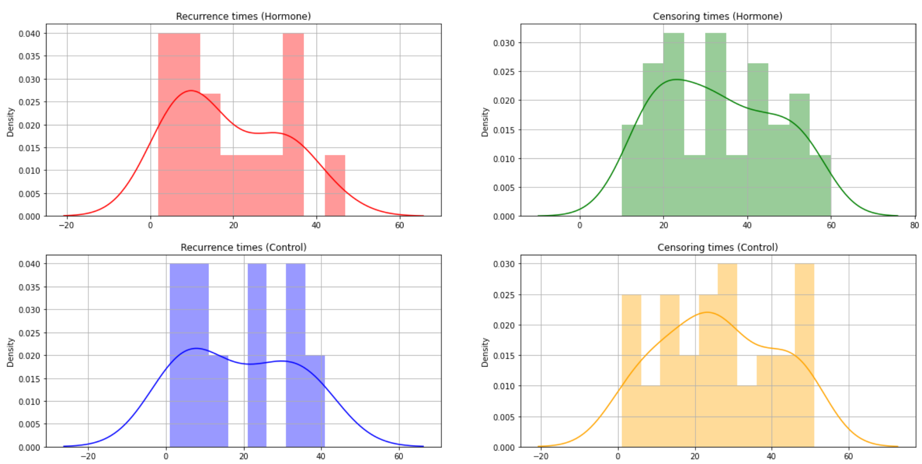


可见，当随机游走步数为$U(-1,1)$时，在δ空间的随机游走和独立链的结果是等价的。当随机游走的步长越小，收敛速度越慢。在 U 空间中的随机游走不论是方差还是收敛速度都要弱于在 delta 空间的随机游走。这是因为非线性变换造成的步长扭曲。

7.5

解：

a.



b.

$$P(\theta|\tau)$$

$$= \frac{\theta^{a}\tau^{b}e^{(-c\theta-d\theta\tau)}}{\int_{0}^{+\infty}\theta^{a}\tau^{b}e^{(-c\theta-d\theta\tau)}\mathrm{d}\theta}$$

$$= \frac{\theta^{a}e^{(-c\theta-d\theta\tau)}}{(d\tau+c)^{-a-1}\Gamma(a+1)}$$

$$\sim\Gamma(a+1, d\tau+c)$$

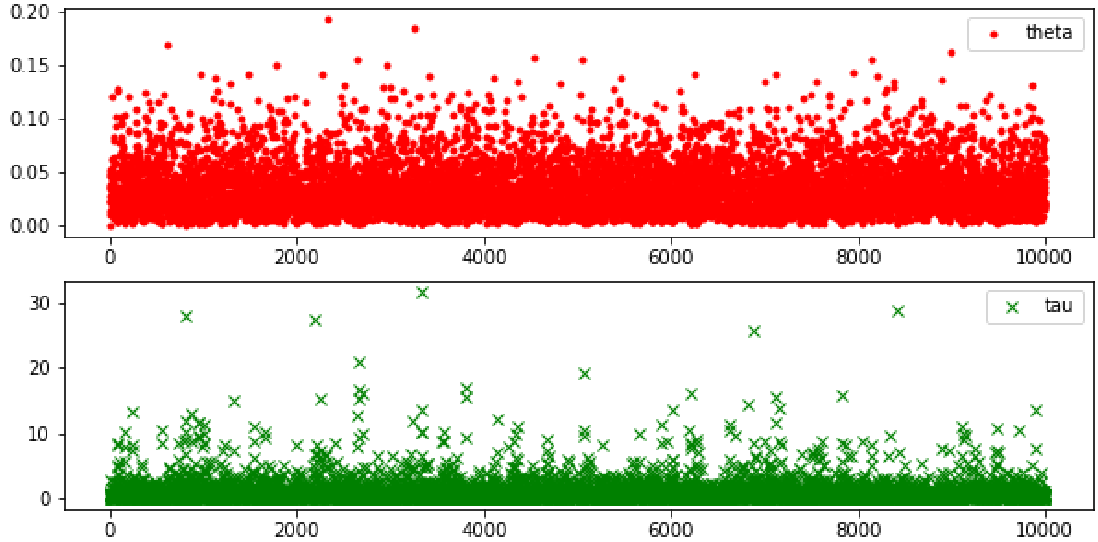
$$P(\tau|\theta)$$

$$= \frac{\theta^{a}\tau^{b}e^{(-c\theta-d\theta\tau)}}{\int_{0}^{+\infty}\theta^{a}\tau^{b}e^{(-c\theta-d\theta\tau)}\mathrm{d}\tau}$$

$$= \frac{\tau^{b}e^{-d\theta\tau}}{(d\theta)^{-b-1}\Gamma(b+1)}$$

$$\sim\Gamma(b+1, d\theta)$$

c. 采样结果为:

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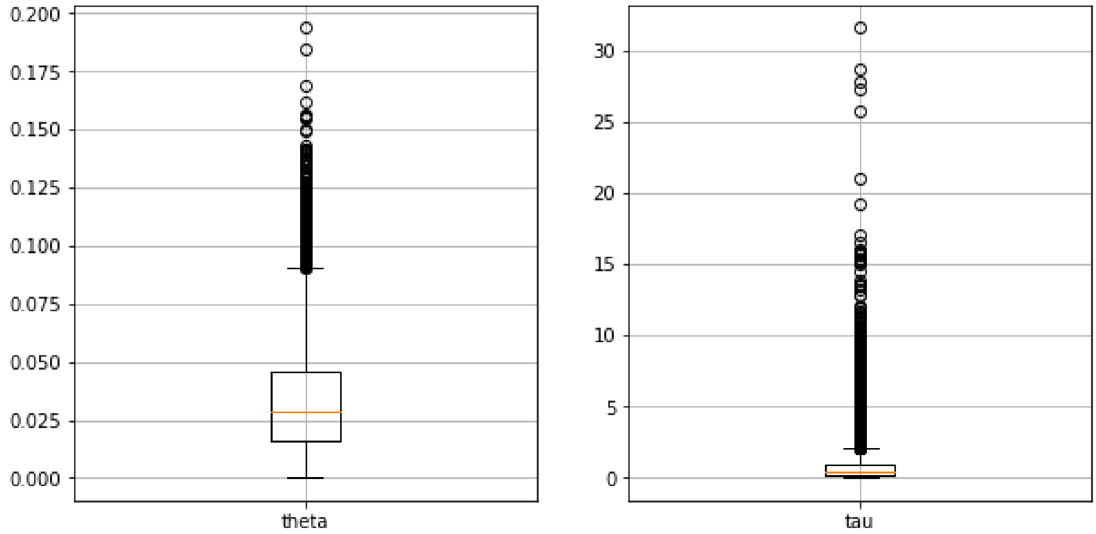
Using Burn-in and Run Length method, get R_theta = 1.000192, R_tau = 1.000344
```

可见，拟合得相当好。

d.

	marginal mean	std	95% interval LB	95% interval UB
theta	0.034032	0.023688	0.004239	0.094301
tau	0.916754	1.523507	0.052322	4.537479

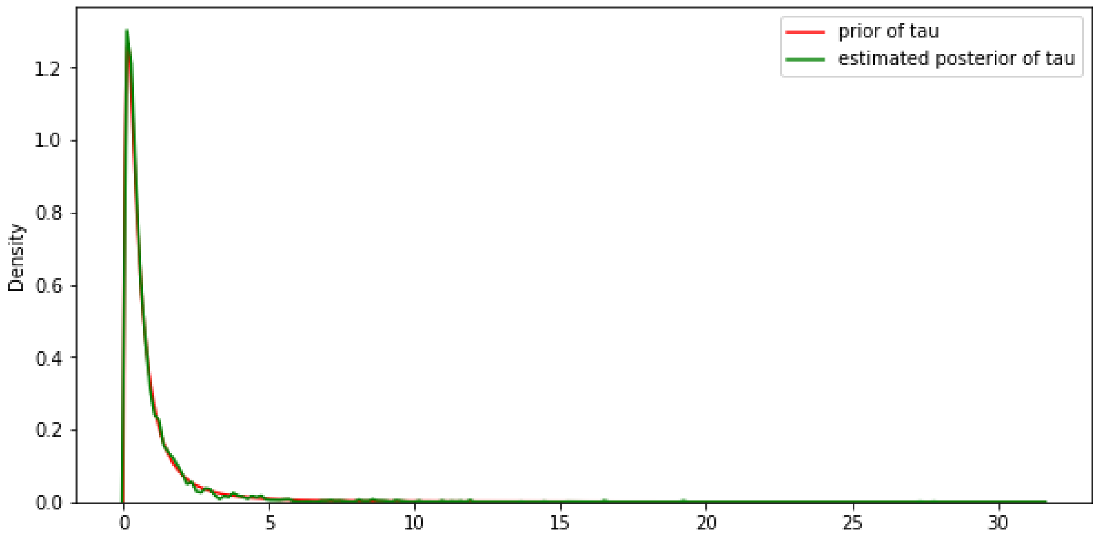


e. τ的先验分布为

$$f(\tau) \propto \int_0^{+\infty} \theta^a \tau^b e^{(-c\theta - d\theta\tau)}\,\mathrm{d}\theta = (d\tau + c)^{-a-1}\Gamma(a+1)\tau^b$$

归一化后，得

$$f(\tau) = \frac{c^{a-b} d^{b+1}\Gamma(a+1)}{\Gamma(a-b)\Gamma(b+1)}(d\tau + c)^{-a-1}\tau^b$$



f.

τ是一个比例系数，$\tau < 1$时才意味着该激素对病人有效。而根据吉布斯采样得到的均值和置信区间，我们倾向于认为该激素对病人无效。

g.

```
The statics of origin hyperparemeters:
       marginal mean       std  95% interval LB  95% interval UB
theta       0.034032  0.023688         0.004239         0.094301
tau         0.916754  1.523507         0.052322         4.537479

The statics of origin hyperparemeters divide by 2:
       marginal mean        std  95% interval LB  95% interval UB
theta       0.033225   0.033424         0.000729         0.122396
tau         5.244312  33.200553         0.046139        32.349776

The statics of origin hyperparemeters times 2:
       marginal mean       std  95% interval LB  95% interval UB
theta       0.033540  0.016717         0.008956         0.073503
tau         0.496958  0.478717         0.067289         1.824140
```

可见，τ对超参数非常敏感。建议先对超参数进行更准确的估计。

7.7

解：

a. 证明：

$$
\begin{aligned}
&\mu^{(t+1)}|\left(\alpha^{(t)},\beta^{(t)},y\right)\\
&=E\left[y_{ij}-\alpha_i^{(t)}-\beta_{ij}^{(t)}-\epsilon_{ij}\right]\\
&=\frac{1}{n}\sum_{i=1}^{I}\sum_{j=1}^{J_i}\left(y_{ij}-\alpha_i^{(t)}-\beta_{ij}^{(t)}-\epsilon_{ij}\right)\\
&=\frac{1}{n}y_{ij}-\frac{1}{n}\sum_{i=1}^{I}J_i\alpha_i^{(t)}-\frac{1}{n}\sum_{i=1}^{I}\sum_{j=1}^{J_i}\beta_{ij}^{(t)}-\frac{1}{n}\sum_{i=1}^{I}\sum_{j=1}^{J_i}\epsilon_{ij}
\end{aligned}
$$

而$y_{ij},\alpha_{ij}^{(t)},\beta_{ij}^{(t)}$都是已知量，且$\epsilon_{ij}\sim N(0,\sigma_\epsilon^2)$，故

$$
\mu^{(t+1)}|\left(\alpha^{(t)},\beta^{(t)},y\right)\sim N\left(y_{..}-\frac{1}{n}\sum_i J_i\alpha_i^{(t)}-\frac{1}{n}\sum_{j(i)}\beta_{j(i)}^{(t)},\frac{\sigma_\epsilon^2}{n}\right)
$$

$$
\begin{aligned}
&P\left(\alpha_i^{(t+1)}|\left(\mu^{(t+1)},\beta^{(t)},y\right)\right)\\
&\propto P\left(\alpha_i^{(t+1)}\right)\prod_{j=1}^{J_i}P\left(\epsilon_{ij}=y_i-\alpha_i^{(t+1)}-\mu^{(t+1)}-\beta_{ij}^{(t)}\right)\\
&\propto e^{-\frac{1}{2}\frac{\alpha_i^{(t+1)^2}}{\sigma_\alpha^2}}\prod_{j=1}^{J_i}e^{-\frac{1}{2}\sum_{j=1}^{J_i}\frac{\left(y_i-\alpha_i^{(t+1)}-\mu^{(t+1)}-\beta_{ij}^{(t)}\right)^2}{\sigma_\epsilon^2}}
\end{aligned}
$$

$$\propto e^{-\frac{1}{2}\left(\frac{\alpha_i^{(t+1)^2}}{\sigma_\alpha^2}+\sum_{j=1}^{J_i}\frac{\alpha_i^{(t+1)^2}-2\alpha_i^{(t+1)}y_{ij}+2\alpha_i^{(t+1)}\mu^{(t+1)}+2\alpha_i^{(t+1)}\beta_{ij}^{(t)}}{\sigma_\epsilon^2}\right)}$$

$$= e^{-\frac{1}{2}\left(\frac{\alpha_i^{(t+1)^2}\left(\frac{\sigma_\epsilon^2}{J_i}+\sigma_\alpha^2\right)-\alpha_i^{(t+1)}\frac{1}{J_i}\sum_{j=1}^{J_i}\left(y_{ij}-\mu^{(t+1)}-\beta_{ij}^{(t)}\right)}{\frac{\sigma_\alpha^2\sigma_\epsilon^2}{J_i}}\right)}$$

$$\propto \exp\left(-\frac{1}{2}\frac{\left(\alpha_i^{(t+1)}-\frac{1}{\left(\frac{\sigma_\epsilon^2}{J_i}+\sigma_\alpha^2\right)}\frac{1}{J_i}\sum_{j=1}^{J_i}\left(y_{ij}-\mu^{(t+1)}-\beta_{ij}^{(t)}\right)\right)^2}{\frac{\sigma_\alpha^2\sigma_\epsilon^2}{J_i}\cdot\frac{1}{\left(\frac{\sigma_\epsilon^2}{J_i}+\sigma_\alpha^2\right)}}\right)$$

故，$\alpha_i^{(t+1)} \sim N\left(\frac{J_iV_1}{\sigma_\epsilon^2}\left(y_{i\cdot}-\mu^{(t+1)}-\frac{1}{J_i}\beta_{j(i)}^{(t)}\right),V_1\right)$，其中，$V_1=\left(\frac{J_i}{\sigma_\epsilon^2}+\frac{1}{\sigma_\alpha^2}\right)^{-1}$

同理，$\beta_{j(i)}^{(t)} \sim N\left(\frac{V_2}{\sigma_\epsilon^2}\left(y_{ij}-\mu^{(t+1)}-\alpha_i^{(t+1)}\right),V_2\right)$，其中$V_2=\left(\frac{1}{\sigma_\epsilon^2}+\frac{1}{\sigma_\beta^2}\right)^{-1}$

证毕.

b. 证明：

$$P\left(\mu^{(t+1)}|\left(\gamma^{(t)},\eta^{(t)},y\right)\right)$$

$$=\prod_{i=1}^{I}P\left(\alpha_i=\gamma_i^{(t)}-\mu^{(t+1)}\right)$$

$$\propto e^{-\frac{1}{2}\left(\sum_{i=1}^{I}\frac{\left(\gamma_i^{(t)}-\mu^{(t+1)}\right)^2}{\sigma_\alpha^2}\right)}$$

$$\propto e^{-\frac{1}{2}\left(\frac{I\mu^{(t+1)^2}-2\mu^{(t+1)}\sum_{i=1}^{I}\gamma_i^{(t)}}{\sigma_\alpha^2}\right)}$$

$$\propto \exp\left(-\frac{1}{2}\frac{\left(\mu^{(t+1)}-\frac{1}{I}\sum_{i=1}^{I}\gamma_i^{(t)}\right)^2}{\frac{\sigma_\alpha^2}{I}}\right)$$

故$\mu^{(t+1)}|\left(\gamma^{(t)},\eta^{(t)},y\right)\sim N\left(\frac{1}{I}\sum_{i=1}^{I}\gamma_i^{(t)},\frac{\sigma_\alpha^2}{I}\right)$

同理，

$$P\left(\gamma_i^{(t+1)}|\left(\mu^{(t+1)},\eta^{(t)},y\right)\right)$$

$$=P\left(\alpha_i=\gamma_i^{(t+1)}-\mu^{(t+1)}\right)\cdot\prod_{j=1}^{J_i}P\left(\beta_{ij}=\eta_{ij}^{(t)}-\gamma_i^{(t+1)}\right)$$

$$\propto e^{-\frac{1}{2}\frac{\left(\gamma_i^{(t)}-\mu^{(t+1)}\right)^2}{\sigma_\alpha^2}} \cdot e^{-\frac{1}{2}\left(\sum_{j=1}^{J_i}\frac{\left(\eta_{ij}^{(t)}-\gamma_i^{(t+1)}\right)^2}{\sigma_\beta^2}\right)}$$

$$\propto e^{-\frac{1}{2}\left(\frac{\gamma_i^{(t)^2}-2\mu^{(t+1)}\gamma_i^{(t)}}{\sigma_\alpha^2}+\frac{\frac{1}{J_i}\sum_{j=1}^{J_i}\gamma_i^{(t+1)^2}}{\frac{\sigma_\beta^2}{J_i}}-2\frac{\gamma_i^{(t+1)}\sum_{j=1}^{J_i}\eta_{ij}^{(t)}}{\sigma_\beta^2}\right)}$$

$$= e^{-\frac{1}{2}\left(\left(\frac{J_i}{\sigma_\beta^2}+\frac{1}{\sigma_\alpha^2}\right)\gamma_i^{(t+1)^2}-2\left(\frac{1}{\sigma_\beta^2}\sum_{j=1}^{J_i}\eta_{ij}^{(t)}+\frac{\mu^{(t+1)}}{\sigma_\alpha^2}\right)\gamma_i^{(t+1)}\right)}$$

$$\propto \exp\left(-\frac{1}{2}\frac{\left(\gamma_i^{(t+1)}-\left(\frac{J_i}{\sigma_\beta^2}+\frac{1}{\sigma_\alpha^2}\right)^{-1}\left(\frac{1}{\sigma_\beta^2}\sum_{j=1}^{J_i}\eta_{ij}^{(t)}+\frac{\mu^{(t+1)}}{\sigma_\alpha^2}\right)\right)^2}{\left(\frac{J_i}{\sigma_\beta^2}+\frac{1}{\sigma_\alpha^2}\right)^{-1}}\right)$$

故 $\gamma_i^{(t+1)}|\left(\mu^{(t+1)},\eta^{(t)},y\right)\sim N\left(V_3\left(\frac{1}{\sigma_\beta^2}\sum_{j=1}^{J_i}\eta_{ij}^{(t)}+\frac{\mu^{(t+1)}}{\sigma_\alpha^2}\right),V_3\right)$，其中 $V_3=\left(\frac{J_i}{\sigma_\beta^2}+\frac{1}{\sigma_\alpha^2}\right)^{-1}$

同理，有$\eta_{ij}^{(t+1)}|\left(\mu^{(t+1)},\gamma^{(t+1)},y\right)\sim N\left(V_2\left(\frac{y_{ij}}{\sigma_\epsilon^2}+\frac{\gamma_i^{(t+1)}}{\sigma_\beta^2}\right),V_2\right)$