THU-70250403, Convex Optimization (Fall 2020) **Homework: 2**

Convex Functions

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Student:

Problem 1

Suppose $f(x) : \mathbb{R} \mapsto \mathbb{R}$ is a convex function. Please prove or disprove that

$$f(x_1) + f(x_2) + f(x_3) + 3f((x_1 + x_2 + x_3)/3) \geq 2f((x_1 + x_2)/2) + 2f((x_1 + x_3)/2) + 2f((x_2 + x_3)/2) \tag{1}$$

Please also discuss whether the above inequality holds for convex function $f(x) : \mathbb{R}^n \mapsto \mathbb{R}$.

Solution

Case 1: The above inequality holds for convex function $f(x) : \mathbb{R} \mapsto \mathbb{R}$

The inequality that we need to prove is also called as "Popoviciu's inequality" found in 1965 by Tiberiu Popoviciu [1]. Its generalize extensions can be found in [2].

The proof process is shown as below. In general, we assume that $x_1 \leq x_2 \leq x_3$.

If $x_2 \leq (x_1 + x_2 + x_3)/3$, then $(x_1 + x_2 + x_3)/3 \leq (x_1 + x_3)/2 \leq x_3$ and $(x_1 + x_2 + x_3)/3 \leq (x_2 + x_3)/2 \leq x_3$. There exist α and β that satisfy the following equations.

$$\frac{x_1 + x_3}{2} = \alpha \frac{x_1 + x_2 + x_3}{3} + (1 - \alpha)x_3 \tag{2}$$

$$\frac{x_2 + x_3}{2} = \beta \frac{x_1 + x_2 + x_3}{3} + (1 - \beta)x_3 \tag{3}$$

Eliminating x_3, we get $(\alpha + \beta - 3/2)(x_1 - x_2) = 0$.

Therefore, $\alpha + \beta = 3/2$ (If $x_1 = x_2$, then $\alpha = \beta = 3/4$.).

Summing up the inequalities

$$f(\frac{x_1 + x_3}{2}) \leq \alpha f(\frac{x_1 + x_2 + x_3}{3}) + (1 - \alpha)f(x_3) \tag{4}$$

$$f(\frac{x_2 + x_3}{2}) \leq \beta f(\frac{x_1 + x_2 + x_3}{3}) + (1 - \beta)f(x_3) \tag{5}$$

$$f(\frac{x_1 + x_2}{2}) \leq \frac{1}{2}f(x_1) + \frac{1}{2}f(x_2) \tag{6}$$

to conclude that

$$f(x_1) + f(x_2) + f(x_3) + 3f((x_1 + x_2 + x_3)/3) \geq 2f((x_1 + x_2)/2) + 2f((x_1 + x_3)/2) + 2f((x_2 + x_3)/2) \tag{7}$$

If $x_2 \geq (x_1 + x_2 + x_3)/3$, then $(x_1 + x_2 + x_3)/3 \leq (x_1 + x_2)/2 \leq x_3$ and $(x_1 + x_2 + x_3)/3 \leq (x_2 + x_3)/2 \leq x_3$. The proof process is similar to the process above.

Case 2: The above inequality does not generally hold for convex function $f(x) : \mathbb{R}^n \mapsto \mathbb{R}$ if $n \geq 2$

1)We first prove the above inequality does not hold for convex function $f(x) : \mathbb{R}^n \mapsto \mathbb{R}$ if $n = 2$.

Example 1

There exists a function $f_2(\boldsymbol{x}) : \mathbb{R}^2 \mapsto \mathbb{R}$ satisfying the following constraints. First, its domain of definition is a triangle Δ_{ABC} as shown in Fig.1. Second, its epigraph $\{(\boldsymbol{x}, t) | \boldsymbol{x} \in \Delta_{ABC}, f_2(\boldsymbol{x}) \leq t\}$ is a upside-down pyramid as shown in Fig.1. According the two constraints, $f_2(\boldsymbol{x})$ is a convex function. From Fig.1, we can see $f_2(A) = f_2(B) = f_2(C) = 0$, $f_2(\frac{A+B}{2}) = f_2(\frac{A+C}{2}) = f_2(\frac{B+C}{2}) = 0$, and $f_2(\frac{A+B+C}{3}) < 0$. Then $f_2(A) + f_2(B) + f_2(C) + 3f_2(\frac{A+B+C}{3}) < 2f_2(\frac{A+B}{2}) + 2f_2(\frac{A+C}{2}) + 2f_2(\frac{B+C}{2})$. Therefore, the above inequality does not hold for convex function $f(x) : \mathbb{R}^2 \mapsto \mathbb{R}$.

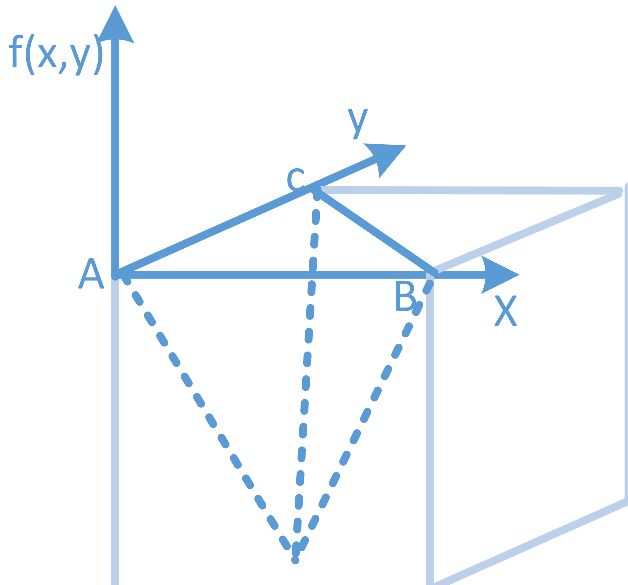

Figure 1: The domain of definition and the epigraph of $f_2(x)$

Example 2 Take this function for example: $f_2(x_1, x_2) = max\{x - y, y, -x - y\}$.

If $A = (0, -1), B = (-2, 1), C = (2, 1)$, then $f_2(A) = f_2(B) = f_2(C) = 1$, $f_2(\frac{A+B}{2}) = f_2(\frac{A+C}{2}) = f_2(\frac{B+C}{2}) = 1$, and $f_2(\frac{A+B+C}{3}) < \frac{1}{3}$. Then $f_2(A) + f_2(B) + f_2(C) + 3f_2(\frac{A+B+C}{3}) < 2f_2(\frac{A+B}{2}) + 2f_2(\frac{A+C}{2}) + 2f_2(\frac{B+C}{2})$. Therefore, the above inequality does not generally hold for convex function $f(x) : \mathbb{R}^2 \mapsto \mathbb{R}$.

2)We then prove the above inequality does not hold for convex function $f(x) : \mathbb{R}^n \mapsto \mathbb{R}$ if $n > 2$.

There exists a function $f_n(\boldsymbol{x}) = f_2(\boldsymbol{x}[1:2])\mathbb{R}^n \mapsto \mathbb{R}$ where $\boldsymbol{x}[1:2]$ represents the vector formed by the first two dimensions of $\boldsymbol{x}$. Apparently, the above inequality does not generally hold for convex function $f_n(\boldsymbol{x})$.

Problem 2

Given $a, b > 0$, please solve the following optimization problem.

$$\min_{x \in (0, \frac{\pi}{2})} \frac{a}{\sin x} + \frac{b}{\cos x} \tag{8}$$

Solution 1

We define a function $f(x)$ as following

$$f(x) = \frac{a}{\sin x} + \frac{b}{\cos x} \tag{9}$$

where $x \in (0, \frac{\pi}{2})$. Then,

$$f'(x) = -\frac{a\cos x}{\sin^2 x} + \frac{b\sin x}{\cos^2 x} \tag{10}$$

$$f''(x) = a\frac{1+\cos^2 x}{\sin^3 x} + b\frac{1+\sin^2 x}{\cos^3 x} \tag{11}$$

As $x \in (0, \frac{\pi}{2})$, $f''(x) > 0$. Therefore, $f(x)$ is convex.

Let's denote the optimal x as x_0, then $f'(x_0) = 0$, i.e

$$\frac{\sin x_0}{\cos x_0} = \frac{a^{1/3}}{b^{1/3}} \tag{12}$$

Therefore,

$$\sin x_0 = \frac{a^{1/3}}{(a^{2/3} + b^{2/3})^{1/2}} \tag{13}$$

$$\cos x_0 = \frac{b^{1/3}}{(a^{2/3} + b^{2/3})^{1/2}} \tag{14}$$

It is easy to verify that $x_0 = (0, \frac{\pi}{2})$

Therefore,

$$\min_{x\in(0,\frac{\pi}{2})} \frac{a}{\sin x} + \frac{b}{\cos x} = \frac{a}{\sin x_0} + \frac{b}{\cos x_0} = (a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}} \tag{15}$$

Solution 2

Given any $\boldsymbol{y} \in \mathbb{R}^n$ and $\boldsymbol{z} \in \mathbb{R}^n$, the Cauchy's inequality indicates

$$\prod_{i=1}^{i=n}(y_i + z_i) \geq ((\prod_{i=1}^{i=n} y_i)^{\frac{1}{n}} + (\prod_{i=1}^{i=n} z_i)^{\frac{1}{n}})^n \tag{16}$$

The two sides are equal if and only if $\boldsymbol{y}$ and $\boldsymbol{z}$ are linearly independent (meaning they are parallel: one of the vector's magnitudes is zero, or one is a scalar multiple of the other).

Let's take $k = n-2$, $y_1 = y_2 = ... = y_k = \sin^2 x$, $z_1 = z_2 = ... = z_k = \cos^2 x$, $y_{k+1} = y_{k+2} = \frac{a}{\sin^k x}$, $z_{k+1} = z_{k+2} = \frac{b}{\cos^k x}$. Then, we can get the following inequality

$$(\sin^2 x + \cos^2 x)^k(\frac{a}{\sin^k x} + \frac{b}{\cos^k x})^2 \geq (a^{\frac{2}{k+2}} + b^{\frac{2}{k+2}})^{k+2} \tag{17}$$

The two sides are equal if and only if $\frac{a}{b} = (\frac{\sin x}{\cos x})^{k+2}$.

Therefore,

$$\frac{a}{\sin x} + \frac{b}{\cos x} \geq (a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}} \tag{18}$$

The two sides are equal if and only if $\frac{a}{b} = (\frac{\sin x}{\cos x})^3$.

Problem 3

Suppose $n \in \mathbb{N}$, $0 \leq x_1 \leq ... \leq x_n \leq \frac{\pi}{2}$ satisfy that

$$\sum_{k=1}^{n} \sin x_k = 1 \tag{19}$$

Please apply Jensen's Inequality [3] to prove or disprove that

$$n \arcsin \frac{1}{n} \leq \sum_{k=1}^{n} x_k \leq \frac{\pi}{2} \tag{20}$$

Solution

We first to prove the left part of the inequality.

Since $\sin(x)$ is concave at $[0, \pi]$, applying Jesen's Inequality we obtain

$$\sin\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) \geq \frac{1}{n}\sum_{i=1}^{n} \sin(x_i) = \frac{1}{n}$$

Note $0 \leq x_1 \leq ... \leq x_n \leq \frac{\pi}{2}$, we have $0 \leq \frac{\sum_{i=1}^{n} x_i}{n} \leq \frac{\pi}{2}$. Since $\sin(x)$ is monotonically increasing at $[0, \frac{\pi}{2}]$, we obtain

$$\frac{1}{n}\sum_{i=1}^{n} x_i \geq \arcsin\left(\frac{1}{n}\right)$$

We then prove the right part of the inequality.

Note $\frac{2}{\pi}x \leq \sin(x)$ will be hold at $[0, \frac{\pi}{2}]$, we then have

$$\frac{2}{\pi}\sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} \sin(x_i) = 1$$

which shows the right part of the inequality is true.

Problem 4

(This problem is Optional)

Suppose $f(x) : \mathbb{R} \mapsto \mathbb{R}$ is a convex function. Please prove or disprove that

(a) if $f(x)$ is bounded, it must be a constant value function.

(b) if $f(x)$ satisfies

$$\lim_{x \to -\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} \frac{f(x)}{x} = 0 \tag{21}$$

it must be a constant value function.

Solution

(a) If $f(x)$ has a upperbound, say if $f(x) \leq m$ and $f(x)$ is not a constant value function, there must exist $x_1 < x_2$, which satisfies,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \neq 0.$$

Suppose $\frac{f(x_2)-f(x_1)}{x_2-x_1} > 0$, $\forall x \geq x_2$ we have:

$$\frac{f(x) - f(x_1)}{x - x_1} \geq \frac{f(x_2) - f(x_1)}{x_2 - x_1},$$

then

$$f(x) \geq f(x_1) + (x - x_1) \cdot \left(\frac{x_2 - x_1}{f(x_2) - f(x_1)} \right).$$

If we set

$$x = (m + 1 - f(x_1)) \cdot \left(\frac{x_2 - x_1}{f(x_2) - f(x_1)} \right) + x_1,$$

then $f(x) \geq 1 + m$, which is contradicted with $f(x) \leq m$. So $f(x)$ is a constant value function.

If $f(x)$ has a lowerbound, the proof is similiar as above.

(b) We can prove it by contradiction. If $f(x)$ is not a constant value function, $\exists x_1 \neq x_2, f(x_1) \neq f(x_2)$.

Suppose $\exists x_1 < x_2, f(x_1) < f(x_2)$, from (a) we know that, $\forall x \geq x_2$,

$$\frac{f(x) - f(x_1)}{x - x_1} \geq \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Set $\{a_n\}$, where $a_n = \frac{f(x_n)-f(x_1)}{x_n-x_1}$ and $0 < a_n \leq a_{n+1}$. Then we have

$$\lim_{n\to+\infty} a_n = \lim_{x\to+\infty} \frac{f(x) - f(x_1)}{x - x_1} = \lim_{x\to+\infty} \frac{f(x)}{x} = 0,$$

which is contradicted.

Similiarly, when $\exists x_1 < x_2, f(x_1) > f(x_2)$, the result is contradicted with

$$\lim_{x\to-\infty} \frac{f(x)}{x} = 0.$$

From above, we know that $f(x)$ is a constant value function.

References

[1] Popoviciu's inequality. https://en.wikipedia.org/wiki/Popoviciu'sinequality

[2] Mihai, M. V., Mitroi-Symeonidis, F. C. (2016). New Extensions of Popoviciu's Inequality. *Mediterranean Journal of Mathematics*, 13(5), 3121-3133.

[3] http://en.wikipedia.org/wiki/Jensen's_inequality