

1 Normal operation analysis

Definitions:

Definition. A state graph consists of $\langle V, S, T \rangle$ where V is the ordered set of finite signals where $1 \leq k \leq n$ indexes into V and gives the signal v_k . S is the set of all states in the state graph $V \mapsto B$ $B = 0, 1$ (note that this is a subset of 2^n) and $s(k)$ denotes the (binary) value of the state at signal v_k . T is the set of all transitions in the state graph $T \subseteq S \times S$ and $(s_i, s_j) \in T$ denotes a transition from the state s_i to s_j . A transition pair s_i and s_j are under additional constraint $\exists k$ st $s_i(k) \neq s_j(k)$ and $s_i(l) = s_j(l)$ for all other $l \neq k$.

State graphs can be implemented as an asynchronous circuit where the signals of a state graph s maps to wires in the circuit w . We can also map the set S to a set S_w of state of wires. The circuit has a set of transitions, for all $w_i \in S_w$, (w_i, w_j) is a transition of the circuit if and only if $(s_i, s_j) \in T$.

Definition. A wire k in the circuit is **excited** in state w_i if (w_i, w_j) is a transition and $w_i(k) \neq w_j(k)$. Alternatively if one treats the output of wire k as a function of the current state w_i , then $f(w_i) \neq w_i(k)$

Definition. An asynchronous circuit is **semi-modular** if $\forall w_i \in S_w$ and $\forall (w_i, w_j)$ transitions and $w_i(k) \neq w_j(k)$, and $l \neq k$ then if $w_i(l)$ is excited, $w_j(l)$ is also excited

Semi-modularity means a wire stays excited until it transitions to the excited value. Circuits that are semi-modular are also speed independent.

Definition. *Trace* of a circuit is a sequence of states of length n . $[0..n-1] \mapsto S$ and $(s_i, s_j) \in T$ for $0 \leq i \leq n-2$

Definition. Two circuits A and B are **similar** if all traces from A can occur in B and vice versa

An additional property of the asynchronous circuit is that one can view each wire as separate black boxes. Inputs from the rest of the circuit (other w_k) feed into a logic block and outputs the wire value w_l . We call the instantaneous value of the logic block l of a state s as $f_l(s)$.

We transform the circuit by making a duplicated copy of the logic blocks to each wire, and interconnecting the outputs of the logic blocks to two c-elements. We label the duplicated circuit as circuit A and circuit B. Then we add in intermediate wires $w^{A'}(k)$ and $w^{B'}(k)$ that is the output of the logic blocks k of circuits A and B respectively. $w^{A'}(k)$ and $w^{B'}(k)$ are inputs to two C-elements. We can (arbitrarily) label one of the C-element outputs as $w^A(k)$ and the other output as $w^B(k)$. These then connect to other circuit A and circuit B logic blocks respectively. We want to show this circuit behaves similarly to the

original circuit under normal operation. Of course this depends on the initial configuration of the transformed circuit. But if we set up the initial correspondence correctly, we can show certain properties in all of the traces that can occur.

To show that the transformed circuit behaves similar to the original circuit, we have the following properties:

1. the trace that occurs in A and B is a trace in original circuit. If a transition happens from s_i , then the next state s_j is either $s_i = s_j$ or $(s_i, s_j) \in T$
2. Given a state w in the original circuit, if wire k is excited in this state, $f_k(w) \neq w(k)$, then if half of the duplicated circuit A or B is also in this state then $f_k(w) \neq w'(k)$ or $w'(k) \neq w(k)$ (exactly one of these is true). In addition, if wire k is not excited in the original circuit, $f_k(w) = w(k)$, then in the duplicated circuit half $f_k(w) = w'(k) = w(k)$.
3. For some k , if $w^A(k) \neq w^B(k)$ then $w^{A'}(k) = w^{B'}(k)$ (if the same output wire in A and B are different, this indicates that one of the c-elements to A or B is excited.)

We prove these properties through induction. Assume we are in a state $\{w_i^A, w_j^B\}$ where these 3 properties are true. Then

1. Due to property 2) we know that only wires excited in the original circuit (in state w_i^A or w_j^B) may have excited components. We focus on w_i^A first, for all wires k that are excited in the original, if $f_k(w_i^A) \neq w_i^{A'}(k)$ then a transition on $w_i^{A'}(k)$ may occur. Otherwise if $w_i^{A'}(k) \neq w_i^A(k)$ then if $w_i^{A'}(k) = w_j^{B'}(k)$ the c-element is excited and $w_i^A(k)$ can transition to the new value $w_{i+1}^A(k) = w_i^{A'}(k)$. Since k is excited in the original circuit, w_{i+1}^A is also a state in the original circuit and $(w_i^A, w_{i+1}^A) \in T$. Otherwise the c-element is not excited and no transition can occur. The same is true for w_j^B
2. Using the same reasoning as above a transition may only take place in circuit A if the wire k is excited in the original. if $f_k(w_i^A) \neq w_i^{A'}(k)$ and a transition on $w_i^{A'}(k)$ occurs then $f_k(w_{i+1}^A) = w_{i+1}^{A'}(k)$ and $w_{i+1}^{A'}(k) \neq w_{i+1}^A(k)$. If $w_i^{A'}(k) \neq w_i^A(k)$ and $w_i^{A'}(k) = w_j^{B'}(k)$ and $w_i^A(k)$ transitions then $w_{i+1}^{A'}(k) = w_{i+1}^A(k)$, in addition since w_{i+1}^A is also a state in the original circuit (from above) then the set of newly excited wires are the same in the original circuit and circuit A. For all wires k in the newly excited set, since k was previously not excited, then $f_k(w_{i+1}^A) \neq w_{i+1}^{A'}(k)$ and $w_{i+1}^{A'}(k) = w_{i+1}^A(k)$. Due to semi-modularity the wires excited at w_{i+1}^A is still excited at $w_{i+1}^{A'}$ with the exception of the wire that transitioned and the same is true in circuit A. And the wires not excited at w_{i+1}^A is not excited at $w_{i+1}^{A'}$ and the same is true in circuit A. Thus property 2 holds in any next state w_{i+1}^A
3. Suppose this property is not true in a possible next state $(w_{i+1}^A(k) \neq w_{j+1}^B(k) \text{ and } w_{i+1}^{A'}(k) \neq w_{j+1}^{B'}(k))$. This means that in the current state

if $w_i^A(k) \neq w_j^B(k)$ $w_i^{A'}(k) = w_j^{B'}(k)$ then a transition on $w_i^{A'}(k)$ or $w_j^{B'}(k)$ occurs. Without loss of generality, assume $w_j^B(k) = w_j^{B'}(k)$, then $w_i^{A'}(k)$ cannot transition or it would violate property 2 above. Then $w_j^{B'}(k)$ transitions. Because of assumption of 3 in state w_i^A, w_j^B there is a possible sequence of transitions on the inputs of wire k in circuits A and B so that they are equal. This means following a trace of h steps in the original circuit $w_{i+h}^A(k)$ can be excited which again violates property 2. Thus the initial assumption is false and this property is true in all possible next states

Finally there is always a transition available since if $w_i^A(k) \neq w_j^B(k)$ we can always use the trick above to produce a sequence of transitions so that $w_i^A(k) = w_j^B(k)$. And if the circuit is in this state there is a next available transition following the original circuit