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HABITS

look for patterns: to look for patterns amongst a set of numbers or figures

tinker: to play around with numbers, figures, or other mathematical expressions in order to learn something more about them or the situation; experiment

describe: to describe clearly a problem, a process, a series of steps to a solution; modulate the language (its complexity or formalness) depending on the audience

visualize: to draw, or represent in some fashion, a diagram in order to help understand a problem; to interpret or vary a given diagram

represent symbolically: to use algebra to solve problems efficiently and to have more confidence in one's answer, and also so as to communicate solutions more persuasively, to acquire deeper understanding of problems, and to investigate the possibility of multiple solutions

prove: to desire that a statement be proved to you or by you; to engage in dialogue aimed at clarifying an argument; to establish a deductive proof; to use indirect reasoning or a counterexample as a way of constructing an argument

check for plausibility: to routinely check the reasonableness of any statement in a problem or its proposed solution, regardless of whether it seems true or false on initial impression; to be particularly skeptical of results that seem contradictory or implausible, whether the source be peer, teacher, evening news, book, newspaper, internet or some other; and to look at special and limiting cases to see if a formula or an argument makes sense in some easily examined specific situations

take things apart: to break a large or complex problem into smaller chunks or cases, achieve some understanding of these parts or cases, and rebuild the original problem; to focus on one part of a problem (or definition or concept) in order to understand the larger problem

conjecture: to generalize from specific examples; to extend or combine ideas in order to form new ones

change or simplify the problem: to change some variables or unknowns to numbers; to change the value of a constant to make the problem easier; change one of the conditions of the problem; to reduce or increase the number of conditions; to specialize the problem; make the problem more general

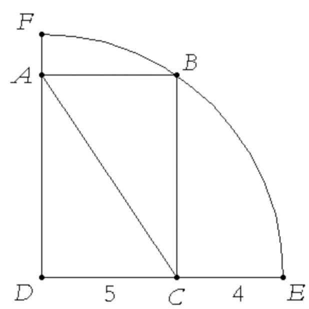
work backwards: to reverse a process as a way of trying to understand it or as a way of learning something new; to work a problem backwards as a way of solving

re-examine the problem: to look at a problem slowly and carefully, closely examining it and thinking about the meaning and implications of each term, phrase, number and piece of information given before trying to answer the question posed

change representations: to look at a problem from a different perspective by representing it using mathematical concepts that are not directly suggested by the problem; to invent an equivalent problem, about a seemingly different situation, to which the present problem can be reduced; to use a different field (mathematics or other) from the present problem's field in order to learn more about its structure

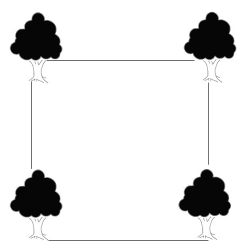
create: to invent mathematics both for utilitarian purposes (such as in constructing an algorithm) and for fun (such as in a mathematical game); to posit a series of premises (axioms) and see what can be logically derived from them

1 In the following picture, rectangle ABCD is inscribed in a quarter-circle. $DC = 5$, and $CE = 4$. Can you figure out the length of diagonal AC?

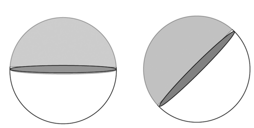


Often a problem that can seem particularly perplexing can be solved by looking at it in a different way. Sometimes the best way to keep track of the different information and variables in a problem is to draw a picture of some sort, to visualize the information so that it is in a form that is easier to understand. You saw this last year as you solved a variety of problems, at times by constructing models, and at times by finding clever ways to visually represent things that at first seemed quite nonvisual.

2 Four trees are planted at each corner of a square park. The city wants to expand the park to twice its current area, but in such a way that the park is still a square, and none of the four trees is in the interior of the park. (The trees cannot be transplanted.) Draw a plan for a new park that meets these criteria.



3 The shaded parts of the spheres below are hemispheres.



You throw three darts onto the surface of a globe, each from a randomly chosen direction. What is the probability that all three darts lie in one hemisphere?

4 The circles in the diagram each have radius 1 cm, are tangent to each other and also to the square PARK. Their centers are on the line PR. Find the area of the square PARK.

5 Triangles of many different sizes and shapes can be created, two of whose sides have lengths 12 and 13 cm respectively. Which of these triangles has the greatest area?

6 A rectangle and a square are inscribed in congruent circles. The rectangle has a width of 6 and a length of 8. What is the area of the square?

7 Three identical, spherical oranges are placed in a bin as part of a supermarket display. The bin is exactly long and wide enough to have two oranges fit snugly in the bottom, but there's plenty of room to layer oranges above these two. If the radius of an orange is 2 inches, find the height of this small stack.

8 A street has parallel curbs 40 ft apart. A crosswalk bounded by two parallel stripes crosses the street at an angle. The length of the curb between the stripes is 15 feet, and each stripe is 50 feet long. Find the distance between the stripes.

9 Let A and B be any two points in a plane.

a. How many different circles can you draw that go through points A and B? Can you give the radius of the smallest possible circle? Of the biggest?

b. How many different rectangles can you draw with opposite vertices on points A and B?

10 PQ and QR are diagonals of two faces of a cube. Find the measure of $\angle PQR$.

11 A trapezoid is inscribed in a circle of radius 5 cm so that one base is a diameter of the circle, and the other base has length 5 cm. What is the perimeter of the trapezoid?

12 If you start with $\frac{1}{2}$ then add $\frac{1}{4}$, then $\frac{1}{8}$, then $\frac{1}{16}$, and so on, ad infinitum, what do you suppose the answer would be? Draw a diagram that would justify your response.

13 How many sides does a cube have? How about a pyramid?

Can you build a closed 3-dimensional shape out of 4 flat sides? How about out of 3 flat sides? Give examples, or explain why not.

14 Suppose you have a box that has a base of 1 inch by 5 inches and that stands 8 inches tall. How many ½ inch radius spherical balls can you get into this box if you can't let any ball protrude above the top of the box?

15 Craziola, the wacky pizza guy, has decided he wants to cut a pizza into as many pieces as possible, with as few straight cuts as possible. He doesn't care at all if the pieces are of equal size, he just wants to make the most number of distinct pieces. With 1 cut, he produces 2 pieces. With 2 cuts, he creates a maximum of 4 pieces, no matter how crazy the 2 cuts he makes are. How many pieces can he possibly make with 3 cuts? 4 cuts? What about n cuts? Can you find the pattern?

16 Is it possible to arrange six pencils so that they all touch each other?

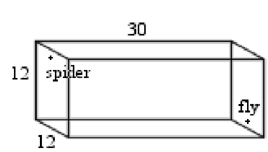
17 Mr. Shimano gives an extremely difficult Japanese test. The highest score was 74% and the lowest 31%. Rather than give a retest, Mr. Shimano decides to raise the 74% to 93% and the 31% to 61%. For a student who scored 57 on the original test, what score do you think Mr. Shimano should give him after the adjustment?

18 A circle is sitting inside an equilateral triangle so that it's tangent to the sides of the triangle in three places. Another equilateral triangle is inside the circle with its three vertices on the circle. If the length of a side of the smaller triangle is 1 cm, find the length of a side of the bigger triangle.

19 Two sides of a triangle measure 6 and 11 cm respectively. If the length of the third side is also an integer, what possible lengths can the third side have?

20 What is the maximum number of acute angles a convex polygon can have?

21 A hungry spider and a fly are in a room 30 feet long, 12 feet wide, and 12 feet high. The spider is on one of the smaller walls, 6 feet from each side and 1 foot from the ceiling. The fly is on the opposite wall, 6 feet from each side and 1 foot from the floor. Assume that the fly does not move (it is paralyzed by fear!) and find the shortest path that the spider can take to eat the fly.



If you were told by a classmate that taking a cup of brackish lukewarm water three times a day for a month would heal your broken leg, you would probably laugh out loud. And even though your laughter might be a bit more subdued were the source of that statement the health segment of the evening news, I bet that you would nonetheless be quite skeptical. On many levels this would seem to you to be highly implausible, and so you might either absolutely dismiss it or yield to that nagging curiosity and check out some other sources.

On the other hand there are a host of other more reasonable sounding claims that you might be prepared to let slide. Examples might include the claim that an increase in oil prices pushes a decline in the Dow Industrial averages, or that there is a strong correlation between wealth and SAT scores. It doesn't seem unreasonable to hold a healthy skepticism for these as well.

In the context of mathematics, to **check for plausibility** is to routinely check the reasonableness of any statement in a problem or its proposed solution, regardless of whether it seems true or false on initial impression; to be particularly skeptical of results that seem contradictory or implausible, whether the source be peer, teacher, evening news, book, newspaper, internet or some other; and to look at special and limiting cases to see if a formula or an argument makes sense in some easily examined specific situations.

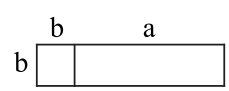
1

Your classmate proposed traveling 240 miles to the beach at an average speed of 80 mph, and the next day traveling home at 40 mph. He pointed out that since you have an average speed of 60 mph, the total 480 miles there and back takes 8 hours. Yet when you went to the beach, it didn't take 8 hours round trip, even though you two drove at precisely the speeds he had proposed. Why?

It certainly seems at first glance that your classmate is correct, but just because a result seems superficially correct doesn't make it so. When there is an apparent contradiction in a problem, it often pays to make a quick check or two to see if what is being said is plausible. For example, how long does each half of the beach trip take?

2

Martin says, looking at the figure below, that the entire rectangle's area divided by the area of the square inside is equal to $\frac{a}{b} + 1$. Melissa thinks that such a peculiarly simple answer is unlikely to be correct. Martin insists he's right, and that in fact his formula would work for any values of a and b one might choose. How could Melissa check to see if Martin's formula is at least plausibly correct?



One particular way that we check the reasonableness of a solution to a problem is to examine special and limiting cases. In the problem above, what would those cases be? Well, one special case would surely be when $a = b$, because you could quickly determine the right answer and check if Martin's formula is correct there. What about if we examined when "a" was small and "b" was large—say $a = 1$ and $b = 100$, or even $a = 1$ and $b = 10000$? Does Martin's formula make sense in those cases as well? Can you come up with another limiting case, and see if his formula is plausible for that case as well?

Once you have checked a solution for all the limiting cases you can think of, if it still seems like a reasonable solution, you might think about how to prove it always works yourself from first principles. Can you come up with Martin's formula?

check for plausibility

3 If one looks at $y = x^2$ on the calculator in "ZOOM SQR" mode (so that the scale in the x and y directions are the same and the graph looks most accurate), it appears that the graph is getting so steep, so quickly, that it will eventually become a vertical line. Does it? If so, estimate for what x value it becomes vertical; if not, explain why that can never happen.

4 While teaching in Brazil, Tony was approached by a fellow math teacher who said the following: "Hey Tony, do you know how to prove that all right triangles are similar? I was trying to show my students in class today and I couldn't quite do it." Could you have helped him out?

5 On a 3-D blueprint for an Olympic swimming pool, 1 ft. represents 16 actual feet. In order to determine how much water would be needed to fill the pool, Tim computes the volume from the blueprint, which is 4 ft^3, and then he multiplies by the scale factor of 16 to get 64 ft^3 of water to fill the pool. Tim is a little unsure if that is the correct amount, but it seems right. What do you think?

6 If Jorge told you that 3.16227765 was an exact solution to $x^2 = 10$, how could you determine without a calculator if he is correct, or slightly off?

7 Hero of Alexandria came up with a formula to determine the area of any triangle based solely on the lengths of its 3 sides. Below are 4 formulas, all of which purport to be Hero's formula. In all 4 formulas, a, b, and c are the lengths of the sides, and s is the semi-perimeter, which is equal to half the perimeter.

$$Area = (s-a)\,(s-b)\,(s-c)$$

$$Area = \sqrt{\frac{(s)(a)(b)(c)}{10}}$$

$$Area = \sqrt{(s)(s-a)(s-b)(s-c)}$$

$$Area = \sqrt{3(a+b+c)}$$

a. Which of these formulas do you think is the right one? Why?

b. Try seeing if the formulas give the kind of answers you would expect for various "common" triangles you have experience with.

c. Test to see if "extreme" triangles (ones with very large or small values of some of the sides) also give reasonable results.

d. What units do you typically measure area in? Does that also help you in deciding which formulas are most plausible?

e. Now pick the formula you are most confident is the right one. Does the fact that it has passed all your "tests" prove that it is correct? If so, explain why. If not, explain how you could become convinced that Hero was in fact correct.

8 Look at these 6 numbers: 1, 3, 6, 9, 11 and 12. Their mean is 7. Subtracting the mean from each of the numbers and adding those together gives us $-6 + -4 + -1 + 2 + 4 + 5$, which equals 0. Juniper is unimpressed, and says that you would always get 0, regardless of the 6 numbers you chose. Sassafras disagrees, and says it is highly dependent on choosing the right 6 numbers; for example, in this case, exactly 3 were above the mean and exactly 3 were below, and also there were no decimals to complicate matters. Who is right?

9 Zargo says that instead of doing lots of intricate calculations, he can find the area of a rhombus by just multiplying the diagonal lengths together. See if you can determine in a minute if his method is plausible.

10 If you draw a line from the vertex of any triangle to the midpoint of the opposite side (i.e. the median), will it be perpendicular to that side, or would it bisect the vertex angle from which it was drawn?

11 A pollster interviewed 100 families, and reported that the mean number of children was 2.037 and the median was 1.8. I do not believe either of these figures. Do you? Why?

12 Brian thinks he remembers that the area of a parallelogram is equal to the product of consecutive sides, but he isn't quite sure. You can't remember whether he's right either, but you know you can check his formula to see if it is plausible. Is it?

13 Show that the formula for the area of a triangle can be viewed as just a special case of the area of a trapezoid.

14 Which is bigger, $4^{\frac{1}{4}}$ or $10^{\frac{1}{10}}$? No calculators allowed! (Hint: try thinking about limiting cases.)

15 Hero's formula gives us a formula for the area of a triangle based only on the lengths of its 3 sides (see problem 7 in this lesson). No one has yet come up with a formula for the area of a quadrilateral based only on the lengths of its 4 sides. Why do you think that is?

check for plausibility

16

a. Veneeta graphs $y = x^2$, $y = x^2 + 4$ and $y = x^2 - 10$ on the same calculator screen using ZOOM STD. What does she notice about how their shapes compare to each other?

b. Grunchik changes the viewing window on his calculator so that it graphs x values from -4 to 4, and y values from -10 to 20. He then graphs the same three equations that Veneeta graphed. What does he notice about how their shapes compare to each other?

c. Veneeta and Grunchik compared their different answers, but aren't sure what to make of them. What do you think?

17

Prashad says that in a quadrilateral ABCD he has examined, AB + BC + CD + DA is equal to 1.8 times the diagonal AC. Evaluate whether what he says is possible or not.

18

a. When an object falls under gravity, its speed increases by a constant amount each second. Two stones are dropped at the same time from a cliff, but one of them is 10 feet higher up than the other at the time of dropping. As they fall, will the distance between them always be the same?

b. Later on, two stones are dropped at the same height from a cliff, but one stone is released one second before the other. As they fall, will the distance between them always be the same?

19

On the same screen as $y = x^2$, graph $y = x$ in ZOOM STANDARD. Then "ZOOM IN" once.

Note that the graph of $y = x$ is "above" the graph of $y = x^2$ for some values of x.

Hermione thinks that those values of x will be the only ones where $y = x$ is above $y = x^2$. What do you think?

20 Bart is feeling a little sick. Having recently read about simpsonitis, a very rare and debilitating disease, and being somewhat hypochondriacal, he goes to see his physician, Dr. Kalvakian. The doctor checks him out and decides to administer a special blood test for detecting the disease.

This diagnostic test is 98% accurate (returns a positive result) for people who have simpsonitis and 95% accurate (returns a negative result) for people who do not have simpsonitis. Approximately 0.3% of people in the country actually have this disease.

Unfortunately, several days after taking the blood test, Bart receives a phone call from Dr. Kalvakian. The doctor tells him that he tested positive for the disease. Bart asks, "What's the chance that I actually have simpsonitis? I mean, you said that the test was not 100% accurate." Dr. Kalvakian replies, "Well, there's a 98% chance that you have the disease, Bart."

Bart initially has a cow, but then he decides to tell the brainy Lisa what Dr. Kalvakian said. What do you suppose Lisa told Bart in response?

21 Zollywog the crazy Geometry student has come up with a formula for the length of a median in a triangle! He claims that the length of the median m that bisects side a of a triangle (with other sides b and c, of course) is:

$$m = \sqrt{\frac{2b^2 + 2c^2 - a^2}{4}}.$$

Could this possibly be true?

22 Felipe tells Bradley that he has just come up with a cool fact: a regular polygon of n sides, with distance r from the middle of the polygon to any one of the vertices, will always have an area less than $4r^2$. Bradley is skeptical of Felipe, since as n increases the area keeps getting bigger. Can you resolve their dispute?

23 Jillian says that, for any positive integer x, $\dfrac{420\,(x+1)!}{x}$ will always be an integer. Explain why she is correct.

LESSON 1: CIRCULAR FUNCTIONS

Introduction

You've seen sine, cosine, and tangent before. In fact, you've even used them to find missing sides in triangles.

1 How would you define the sine of θ ? Cosine of θ ?

2 Use careful construction of a triangle to find $\sin 37^\circ$. Do not use the sine function on your calculator!

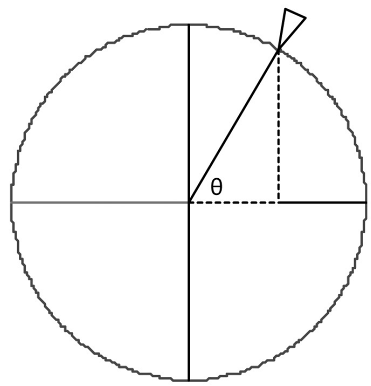
3 Why can't you use a construction to find $\sin 102^\circ$?

Even though we can't construct a triangle to find the sine of 102°, your calculator can tell you an approximate value! In this lesson, we will discover a new way to think about sine and cosine—a way that makes calculating the values of $\sin 102^\circ$, $\cos 1002^\circ$, and $\tan(-10002^\circ)$ possible.

Development

Most of the problems in this section require carefully drawn diagrams.

4 A wheel of radius one foot is placed so that its center is at the origin, and a pin on the rim is at (1, 0). The diagram below shows the wheel after it has been spun an angle θ in a counterclockwise direction.



Now consider the function $P(\theta)$, which outputs the coordinates of the pin after the wheel has been spun an angle θ in a counterclockwise direction. So, for example, $P(0^\circ) = (1, 0)$ and $P(270^\circ) = (0, -1)$. Find $P(\theta)$ when:

a. $\theta = 90^\circ$

b. $\theta = 45^\circ$. Give an exact answer.

c. $\theta = 30^\circ$. Give an exact answer.

d. $\theta = 57^\circ$

e. θ is some measure between 0° and 90°. (Your answer should be expressed in terms of θ.)

f. For values of θ between 0° and 90°, how are $\cos\theta$ and $P(\theta)$ related?

5 Now, let's consider angles greater than 90°.

a. Calculate $P(150^\circ)$, with P being the same position function as in question 4.

b. What is $P(253^\circ)$?

c. Compare these values to the sine and cosine of 150° and 253°.

d. Using the diagram and your calculations, what do you think the tangent of 253° is? The tangent of 150°? Check them on your calculator.

It is the convention for rotations that motion in a counterclockwise direction is considered **positive**, while motion in a clockwise direction is considered **negative**. So if our wheel is spun 57° counterclockwise we would input 57° in our function $P(\theta)$, as we did in problem 1, but if our wheel is spun 57° clockwise we would input -57° in our function $P(\theta)$. So the output for would be the coordinates of the pin after the wheel is spun 57° clockwise.

6 Find:

a. $P(-240^\circ)$. Give an exact answer.

b. $P(-2640^\circ)$. Give an exact answer.

c. $P(-237^\circ)$.

d. $\tan(-237^\circ)$.

e. $P(-\theta)$, where $0^\circ < \theta < 90^\circ$. (Your answer should be expressed in terms of θ.)

7 Now define sine, cosine, and tangent of θ for every value of θ.

8 For angles between 0 and 90 degrees, is using your definition in problem 7 equivalent to using your definition in problem 1? Why or why not?

9 $P(46280^\circ) = (-0.94, -0.34)$. Without using the cosine button on your calculator, find the cosine of 44480°.

10 Let $\cos\theta = -0.4$.

a. For how many angles is that true?

b. How many of these angles are between -180° and 360°?

c. With the help of your calculator find all the angles in part b.

11 Assuming that $\cos 80^\circ = 0.17$, use the symmetry of the circle to find $\cos 100^\circ$, $\cos(-260^\circ)$, $\cos 260^\circ$, $\cos 280^\circ$, $\sin 190^\circ$, and $\sin(-10^\circ)$.

12 Let's revisit question 4, but with a wheel of radius 7 feet instead of 1 foot. The wheel is still centered at $(0, 0)$, and still with a pin at $(7, 0)$. Let $Q(\theta)$ be the function which outputs the coordinates of the pin after this larger wheel has been spun an angle θ in a counterclockwise direction.

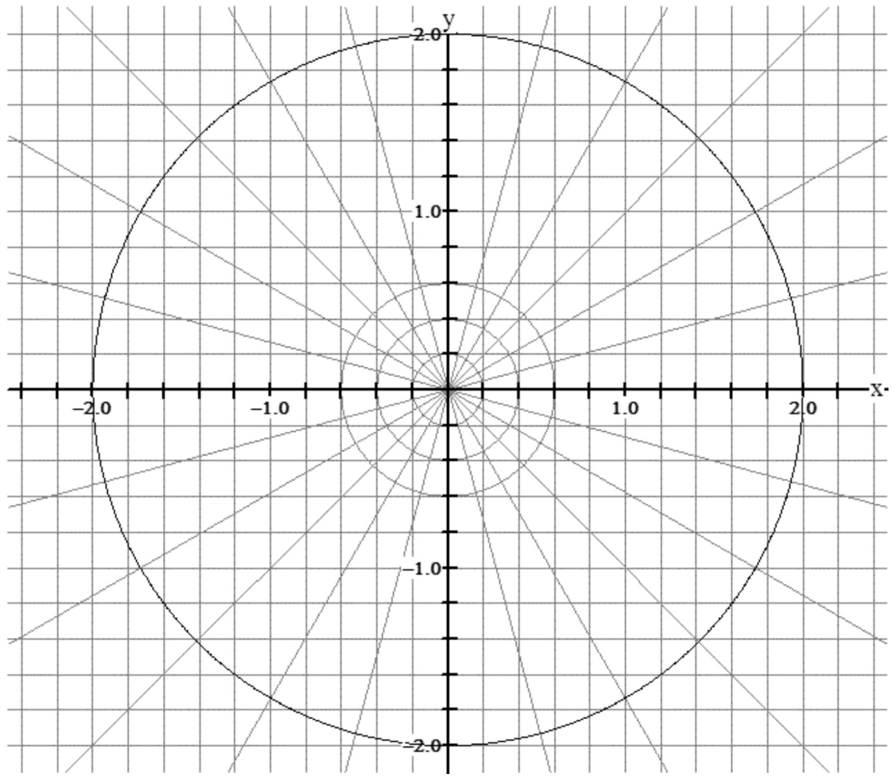
a. What is $Q(48^\circ)$?

b. What is $Q(109^\circ)$?

c. How would you define sine and cosine of θ using this $Q(\theta)$ function? How would you define tangent?

d. How would you define sine, cosine, and tangent using a circle of radius r?

13 In the diagram below, there is a circle of radius 2 inches, with radii drawn at 15° intervals.



a. Use careful estimates and the conclusion of problem 12 to calculate the sines of 30°, 60°, 90°, 120°, ..., 360°.

b. Using your calculations in part a, sketch a graph with θ on the horizontal axis and $\sin\theta$ on the vertical axis. (Use values of θ from $-360°$ to $360°$.)

Surprise! There are actually three more trigonometric ratios (functions) in addition to the three you already know. Here are their names and definitions:

Cosecant of angle θ, written $\csc\theta$ is defined thus: $\csc\theta = \frac{r}{y}$

Secant of angle θ, written $\sec\theta$ is defined thus: $\sec\theta = \frac{r}{x}$.

Cotangent of angle θ, written $\cot\theta$ is defined thus: $\cot\theta = \frac{x}{y}$

14 Use the unit circle to find $\csc\theta$, $\sec\theta$ and $\cot\theta$ for $\theta = 240°$.

Practice

15 Use the symmetry of the circle to complete the following chart. Give exact answers. Copy the chart into your notebooks.

θ	30°	45°	60°	315°	–210°	210°	–315°	150°	240°
$\sin\theta$									
$\cos\theta$									
$\tan\theta$									

16 (For this problem use the circle of problem 13.)

a. Use careful measurements and the conclusion of problem 7 to calculate the cosines of 30°, 60°, 90°, etc.

b. Using your calculations in part a, sketch a graph with θ on the horizontal axis and $\cos\theta$ on the vertical axis. (Use values of θ from -360° to 360°.)

17 Determine, without using your calculator, which of the following expressions are the same as $\sin 27^\circ$.

$\sin(180^\circ - 27^\circ)$, $\sin(180^\circ + 27^\circ)$, $\sin(-27^\circ)$, $\sin(360^\circ + 27^\circ)$, $\sin(-207^\circ)$

18 Find, without using your calculator, two of the following expressions which are the same.

$\sin 27^\circ$, $\cos 27^\circ$, $\sin(-153^\circ)$, $\cos(-153^\circ)$, $\cos 63^\circ$

19 Find at least two values for θ that fit the equation $\sin\theta^\circ = \frac{\sqrt{3}}{2}$. How many such values are there?

Problems

20 At constant speed, a wheel rotates once counterclockwise every 8 seconds. The center of the wheel is $(0,0)$ and its radius is 1 foot. A pin is initially at $(1,0)$. Where is it 69 seconds later?

21 A wheel whose radius is 1 is placed so that its center is at $(3,2)$. A pin on the rim is located at $(4,2)$. The wheel is spun θ degrees in the counterclockwise direction. Now what are the coordinates of that pin? Does your answer work for 90 degrees? 180 degrees?

22 For the following equations use a circle of radius 2 (and your calculator for part b only) to find all solutions θ between 0° and 360°:

a. $\cos\theta = -\dfrac{\sqrt{3}}{2}$

b. $\tan\theta = 6.3138$

c. $\sin\theta = -\dfrac{\sqrt{2}}{2}$

d. $\cos\theta = \cos(251^\circ)$

e. $\sin\theta = \sin 580^\circ$

f. $(\tan\theta)^2 = 3$

23 Find all solutions t between 360° and 720° :

a. $\cos t = \sin t$ (no calculator)

b. $\sin t = -0.9397$

c. $\cos t < \dfrac{\sqrt{3}}{2}$ (no calculator)

In the next few problems you are asked to come up with conjectures or to examine the validity of various statements. You might want to see if the conclusion reached is even plausible, by looking at specific, easy-to-check cases. A simple check will either give the conclusion credence (and thus makes it worth trying to prove) or disprove it instantly.

24 Asked to simplify the expression $\sin(180^\circ - \theta)$, Alex volunteered the following solution: $\sin(180^\circ - \theta) = \sin 180^\circ - \sin\theta$, and, because $\sin 180^\circ$ is zero, it follows that $\sin(180^\circ - \theta)$ is the same as $-\sin\theta$.

a. Is this conclusion plausible?

b. If it is plausible, try to prove the result. If it isn't, can you come up with a correct way to express $\sin(180^\circ - \theta)$ in simpler form?

c. Answer the same questions for $\cos(180^\circ - \theta)$.

25 Find simpler, equivalent expressions for the following: $\sin(180^\circ + \theta)$, $\cos(180^\circ + \theta)$, $\tan(180^\circ + \theta)$, $\cos(360^\circ - \theta)$, $\sin(360^\circ - \theta)$, $\tan(360^\circ - \theta)$, $\cos(360^\circ + \theta)$, $\sin(360^\circ + \theta)$, $\tan(180^\circ - \theta)$, $\tan(360^\circ + \theta)$

26 Is it the case that $\sin(90^\circ - \theta) = \cos\theta$? Explain. Then write simpler, equivalent expressions for $\cos(90^\circ - \theta),\ \sin(90^\circ + \theta),\ \cos(90^\circ + \theta)$.

27 Is $\sin(-\theta)$ always the same as $-\sin(\theta)$? What can be said about $\cos(-\theta)$?

28 Do the following problems without using a calculator. Explain your reasoning.

a. Which is larger: $\cos 311^\circ$ or $\cos 312^\circ$?

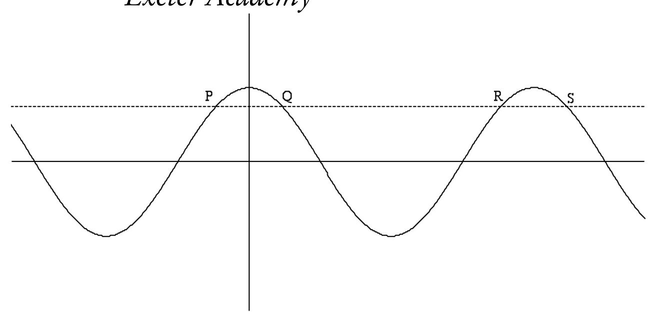
b. Which is larger: $\sin 311^\circ$ or $\sin 312^\circ$?

29 If $\sin A$ is known to be 0.96, then what is $\cos A$? What if it is also known that A is an obtuse angle?

30 Hendrickson is thinking of an angle θ where $\tan\theta > \sin\theta$, and also where $\sin\theta < \cos\theta$. Give two possible values for θ that are in different quadrants.

31 Rodney is running around the circular track $x^2 + y^2 = 10000$, whose radius is 100 meters, at 4 meters per second. Rodney starts at the point $(100, 0)$ and runs in the counterclockwise direction. After 30 minutes of running, what are Rodney's coordinates? *Copyright Phillips Exeter Academy*

32 Below are the graphs of $y = \cos x$ and y = 0.7431 (dotted). Given that $Q = (42, 0.7431)$, find coordinates for the intersection points P, R, and S without using a calculator. Use a calculator to check your answers. *Copyright Phillips Exeter Academy*



33 For how many angles in the first quadrant will $\sin\theta$ be rational?

34 You use a circle of radius 2 to find the sine of a 60 degree angle. Find the length of the arc of the circle that is intercepted by this 60 degree angle.

35 Choose an angle θ and calculate $(\cos\theta)^2 + (\sin\theta)^2$. Repeat with several other values of θ. Explain the results. (Note that it is customary to write $\cos^2\theta + \sin^2\theta$ instead of $(\cos\theta)^2 + (\sin\theta)^2$.)

36 For any angle θ, is it true that $1 + \tan^2\theta = \sec^2\theta$? If so, prove it. If not, produce a counterexample.

37 Asked to find an expression that is equivalent to $\cos(\alpha + \beta)$, Laura responded $\cos(\alpha) + \cos(\beta)$. What do you think of Laura's answer, and why?

38 On your calculator, draw a graph of the function $y = \tan x$ with $0^\circ < x < 360^\circ$. Something appears to be wrong here. Say what you think is wrong and explain what's going on.

39 Solve the following without finding the angle θ.

a. Given that $\sin\theta = \frac{12}{13}$, with $0^\circ < \theta < 90^\circ$, find the values of $\cos\theta$ and $\tan\theta$.

b. Given that $\cos\theta = \frac{7}{25}$, with $270^\circ < \theta < 360^\circ$, find $\sin\theta$ and $\tan\theta$.

c. Suppose that $\csc\theta = \dfrac{-25}{7}$ and $\tan\theta < 0$. Evaluate, in fractional form, the remaining five trigonometric functions of θ.

d. The point $P(-12, 16)$ is on a circle whose center is the origin. Find the cosine of angle θ, the angle between the positive x -axis and the radius to P.

e. The point P is on a circle whose center is the origin, and the x-coordinate of P is five times its y-coordinate. Find the cosine of angle θ, the angle between the positive x -axis and the radius to P.

40 Is it possible for $\sin\theta$ to be exactly twice the size of $\cos\theta$? If so, find such an angle θ . If not, explain why not.

41 Find the three smallest positive solutions to $2\sin\theta = -1.364$.

42 Don't use a calculator for this problem.

a. Reduce: $\dfrac{5ab-10bc}{ab}$

b. Reduce: $\dfrac{3x^2-24}{3x}$

c. Solve for x: $2x^2-3x-2=0$

d. Add: $\dfrac{7}{4}+\dfrac{3a}{b}$

e. Subtract: $\dfrac{x}{4}-\dfrac{x-1}{2}$

43 Find all the solutions x, given $0^\circ \leq x \leq 360^\circ$.

a. $(\cos x-.5)(\sin x+\dfrac{\sqrt{3}}{2})=0$

b. $(\cos x)^2-5\cos x+4=0$

c. $3(\tan x)^2+5\tan x=0$

d. $2(\sin x)^2+\sin x=3$

Exploring in Depth

44 Paul rides a Ferris wheel for five minutes. The diameter of the wheel is 10 meters, and its center is 6 meters above the ground. Each revolution of the wheel takes 30 seconds. Being more than 9 meters above the ground causes Paul to suffer an anxiety attack. For how many seconds does Paul feel uncomfortable?

45 Jasper's bike has wheels that are 27 inches in diameter. After the front wheel picks up a tack, Jasper rolls another 100 feet and stops. How far above the ground is the tack?

46 You have a circular dartboard. Its target area is defined in an unusual way. Take any point within the dartboard. Draw a straight line that goes through this point and the center of the board. Measure the angle this line makes with the line going from the center to the "easternmost" point on the dart board. If the tangent of this angle is between -0.2 and 0.6, then the point is shaded. Otherwise, it is not shaded. What are your chances of hitting this target area?

47 Given that $\sin\theta=k$, and that $0^\circ<\theta<90^\circ$, find expressions for $\cos\theta$ and $\tan\theta$.

48 What do the graphs of $y=\sin x$ and $y=\sin 2x$ have in common, and how do they differ? How about the graphs of $y=\cos x$ and $y=\cos mx$, where m is any positive integer?

49 Starting at the same spot on a circular track that is 80 meters in diameter, Andy and Brandon run in opposite directions, at 300 meters per minute and 240 meters per minute, respectively. They run for 50 minutes. What distance separates Andy and Brandon when they finish? Interpret the word *distance* any way you wish.

LESSON 2: LAW OF SINES AND LAW OF COSINES

Introduction

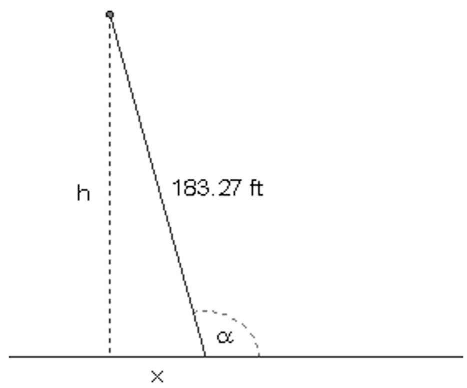
The Leaning Tower of Pisa

The Leaning Tower of Pisa has been leaning almost since the onset of construction in 1173. From ground to top, the tower is 183.27 ft along the lowest side and 186.02 ft along the highest side. The angle of slant of the tower—formed by its shorter slant height and the ground—is 95.5°. An apocryphal tale states that Galileo Galilei (1564-1642), an Italian physicist, mathematician, and philosopher, dropped two cannon balls of different weights from the top of this leaning tower in trying to demonstrate that the descending speed of a falling body is independent of its weight.

1 What is the distance traveled by an object dropped from the top of the Tower of Pisa, on its lowest side, when it hits the ground?

2 What is the horizontal distance from the point where the object hits the ground to the base of the tower?

The Tower of Pisa first acquired a slant after the third floor was built in 1178. More recently, in 1990, it was closed to the public because of safety fears. In fact, the tower was on the verge of collapse, and it was projected that it would have collapsed between 2030 and 2040. However, it has been straightened a bit, but still remains slanted (the tower has been reopened to the public). Thus, the angle of slant of the Tower of Pisa, α, has been changing for centuries. The following is a sketch of the situation.



3 In trying to determine a general expression for the distance, h, traveled by an object dropped from the top of the Tower of Pisa's lowest side when it hits the ground (assuming that the shorter length of the tower has not changed), Gerald found the following expression.

$$h = 183.27 \sin \alpha$$

For the horizontal distance from the point where the object hits the ground to the base of the tower, Gerald found the following expression.

$$x = -183.27 \cos \alpha$$

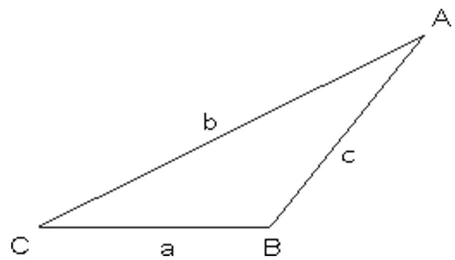
Do you agree with these statements? Explain your answer.

4 You have no more than two seconds after reading the statement of this problem to solve it.

What is the value of $\sin^2\left(\dfrac{\sqrt[7]{123456}}{\pi+13}\right) + \cos^2\left(\dfrac{\sqrt[7]{123456}}{\pi+13}\right)$?

Development

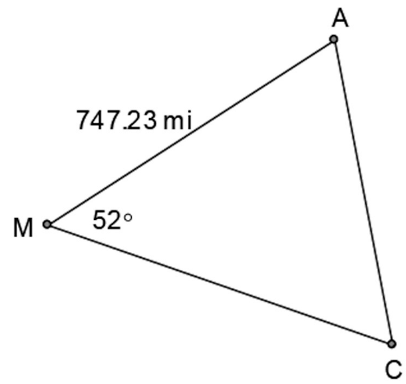
We follow the convention of labeling the angles of a triangle using capital letters, and the lengths of the corresponding opposite sides with the corresponding lower-case letters. For example, we may label A, B, C the angles of the triangle, and a, b, c the corresponding opposite sides, as in the figure below.



The Bermuda Triangle

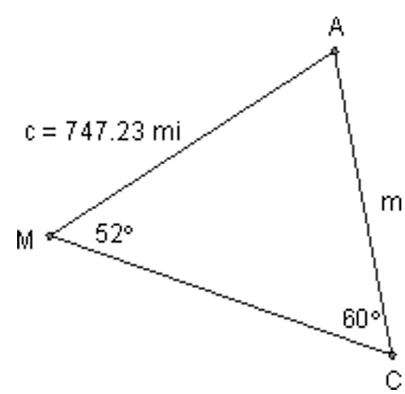
The "Bermuda Triangle" or "Devil's Triangle" is an imaginary area located off the southeastern Atlantic coast of the United States of America, which is noted for a supposedly high incidence of unexplained disappearances of ships and aircraft. The vertices of the triangle are generally believed to be Bermuda, Miami (Florida), and San Juan (Puerto Rico.)

One of the amazing stories from this triangle tells that an aircraft (A in the figure below) was 747.23 miles from the airport of Miami (M in the figure), on the line joining Miami and Bermuda, when its crew received an SOS signal from Cyclops (C in the figure), a ship located at a point on the line joining Miami airport and San Juan, which was sinking. Flight controllers at the airport were able to estimate the measure of $\angle AMC$ to be 52°.

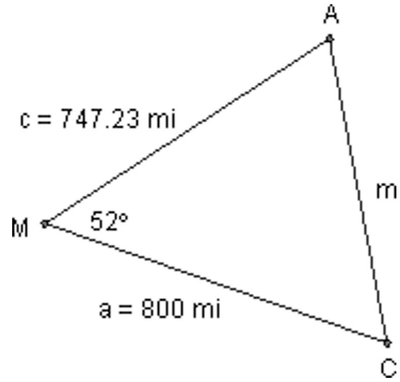


Based on the previous story, Mr. Thomas Howard designed a problem for his mathematics class. The initial situation described in the story, however, contained more than the minimal information required to solve the problem; therefore, he divided the class into two groups, Group I and Group II, and gave each group a problem with different pieces of information, but the same goal to determine how many miles the aircraft had to travel through the purportedly dangerous Bermuda Triangle before reaching Cyclops.

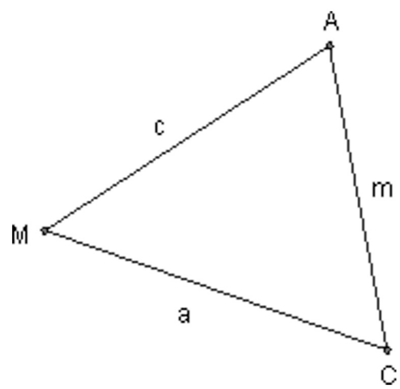
5 Group I: Besides the general information described above, it is known that $m\angle C = 60^{\circ}$. What is the distance between the aircraft A and the sinking ship Cyclops C? Find a solution to this problem and explain how you arrived at your answer.



6 Group II: Besides the original and general information, it is known that the distance from Cyclops to the airport of Miami is 800 miles, as shown in the figure below. What is the distance between the aircraft A and the sinking ship Cyclops C? Find a solution to this problem and explain how you arrived at your answer.



7 In Problem 5, if the distance given were m and the one to be found were c, the problem could be solved in a similar way. Consider the more general triangle below.



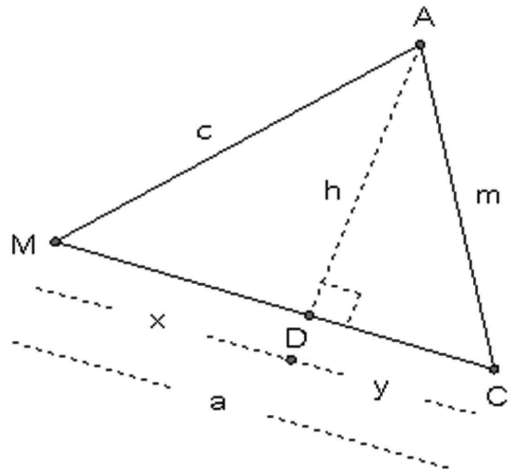
a. Draw the altitude from A to side $\overline{MC}$, and prove that $\dfrac{c}{\sin C} = \dfrac{m}{\sin M}$.

Hint: If h is the length of the altitude from A to $\overline{MC}$, find h in ΔADM in terms of c and $\angle M$. Then find $\sin C$ in ΔADC.

b. Now, prove that $\dfrac{m}{\sin M} = \dfrac{a}{\sin A}$

In group II, Rebecca noticed that if the measure of $\angle M$ were 90° rather than 52°, the Pythagorean Theorem would guarantee that $m^2 = a^2 + c^2$. “However,” she said, “since $\angle M$ is less than 90°, the Pythagorean relationship among m, a, and c must be adjusted.” This adjustment is the result that you are about to find in Part c of the following problem.

8 Consider the triangle in Mr. Howard’s problem in a more general form, as shown below, and prove the following.



a. $h = c \sin M$

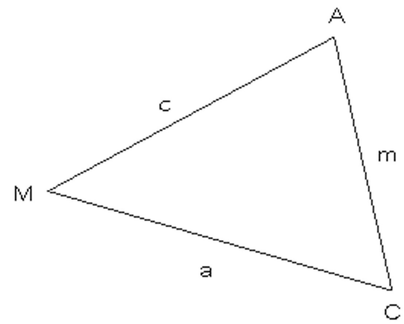
b. $y = a - c \cos M$

c. $m^2 = c^2 \sin^2 M + (a - c \cos M)^2$

9 In part c of the previous problem, expand the square on the right side of the equality. Then use the trigonometric identities that you have learned thus far to find an expression as simple as possible relating m to c, a, and $\angle M$.

As announced above, the expression relating m with c, a, and $\angle M$ that you may have found in the previous problem is the adjustment to the Pythagorean relationship $m^2 = a^2 + c^2$ required when $m\angle M < 90°$. In a similar way, adjustments to the Pythagorean relationship relating a to m and c as well as to that relating c to a and m may be needed when either $m\angle A$ or $m\angle C$ is not 90°, as in the case illustrated above.

10 Consider the general triangle shown below.



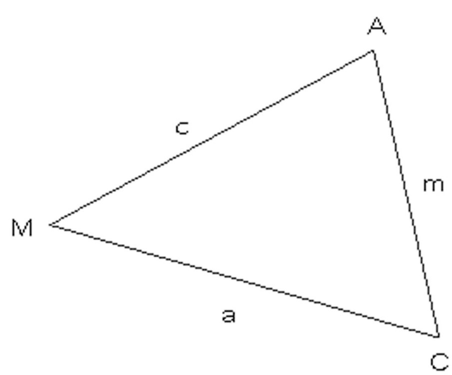
a. If $m\angle A = 90^\circ$, how would a be related to m and c?

b. Conjecture an adjustment to the Pythagorean relationship in part a that may be required when $m\angle A \neq 90^\circ$. Would the proof of your conjecture in this case be quite different from that developed in Problems 8 and 9? Explain.

c. If we had that $m\angle C = 90^\circ$, how would c be related to a and m?

d. Conjecture an adjustment to the Pythagorean relationship in part c that may be required when $m\angle C \neq 90^\circ$. Would the proof of your conjecture in this case be quite different from that required in Part b? Explain.

Summarizing, given a triangle as the one below,



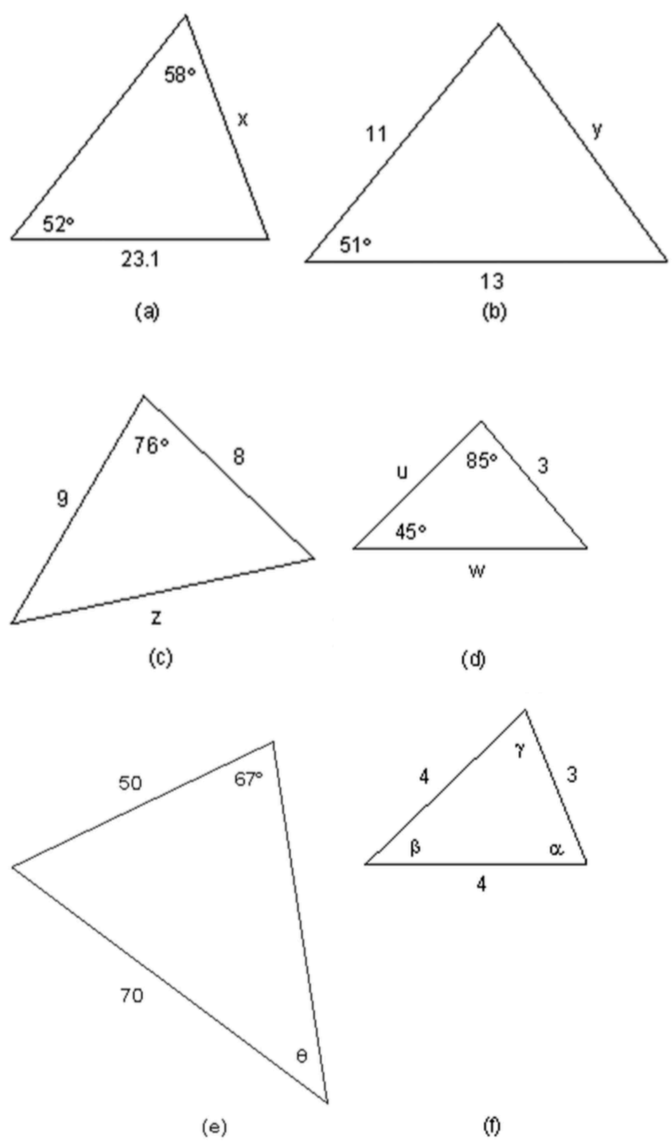
two sets of equalities have been found.

On the one hand, the set of equalities that you may have proven in Problem 7 is known as the **Law of Sines**.

On the other hand, the set of equalities that you may have found in Problems 9, 10b, and 10d is known as the **Law of Cosines**.

Practice

11 Find the indicated side lengths or angle measures in the following figures.

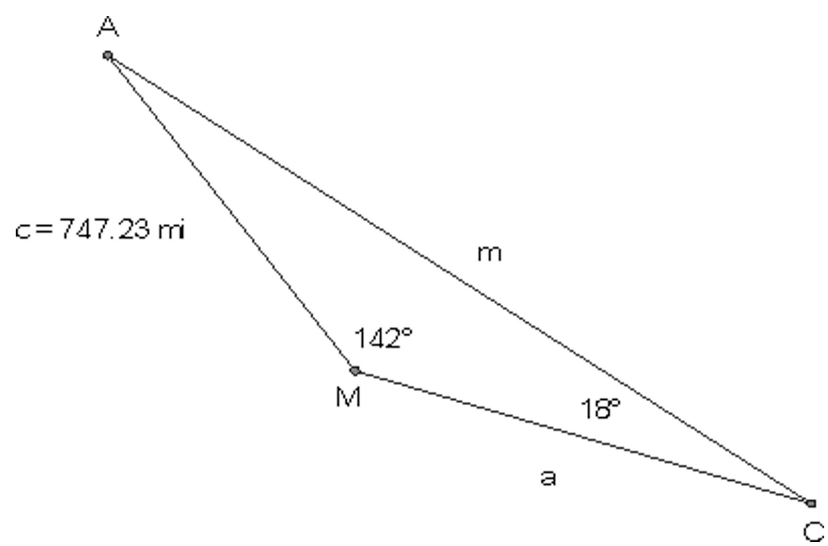


Further Development

When the students in Mr. Howard's class shared their findings (the Law of Sines and the Law of Cosines) the information contained in these laws was considered "the key" to find missing side lengths or angles in any triangle, when basic information about the triangle is known. However, in reality in all the proofs only acute triangles—triangles with each of their three angles being less than 90°—were used.

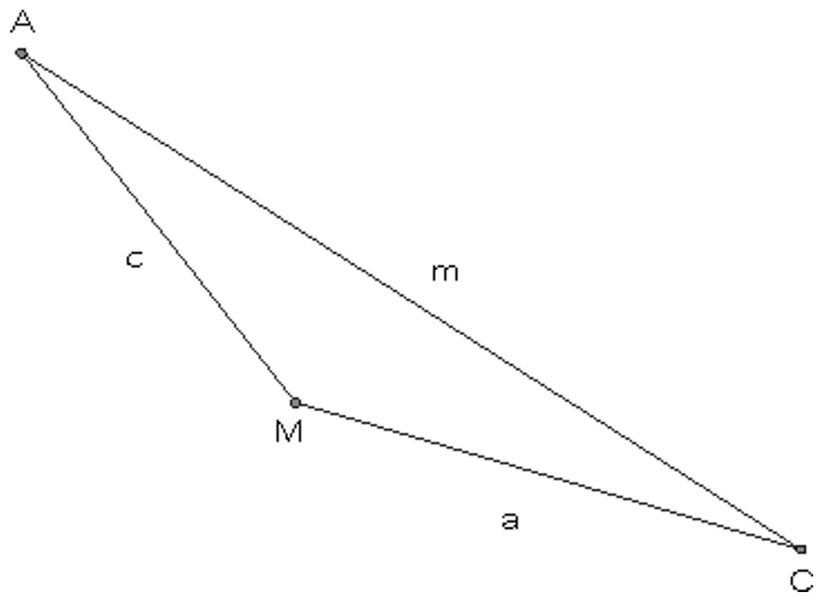
12

Regarding Mr. Howard's problem, consider the following situation where $m\angle M = 142°$.



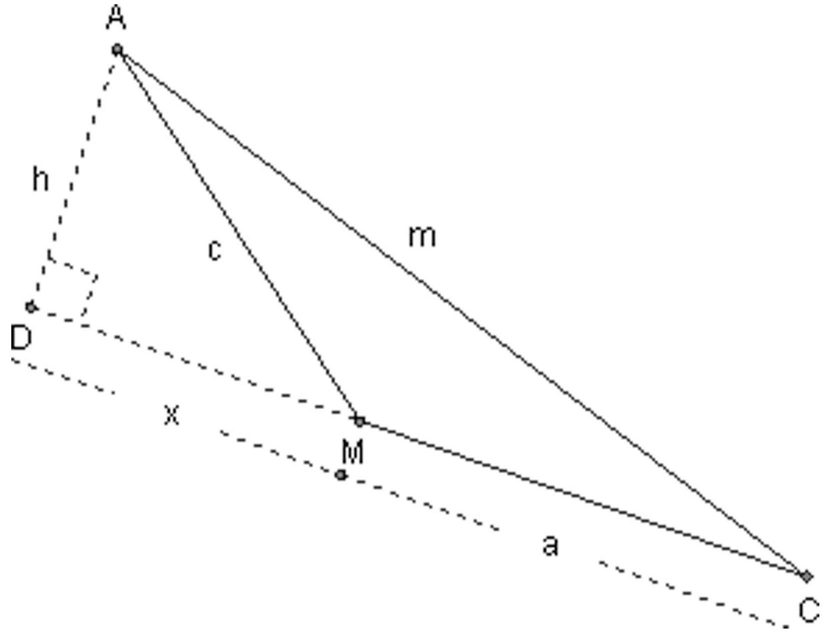
Does the proof of the Law of Sines (in Problem 7), or that of the Law of Cosines (in Problems 8 and 9), support the use of either of these laws to find m in this triangle, which is not an acute triangle? Explain.

13 Consider the following obtuse ΔAMC (that is, a triangle containing an angle greater than $90°$).



Prove that $\dfrac{c}{\sin C} = \dfrac{m}{\sin M} = \dfrac{a}{\sin A}$.

14 Consider again the obtuse ΔAMC. Dashed lines have been added to help you prove the equality stated in Part (a).



a. Prove that $m^2 = a^2 + c^2 - 2ac\cos M$.

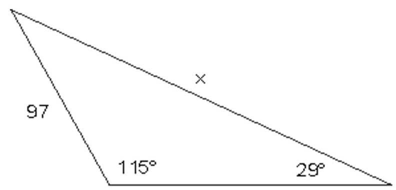
b. Prove that $a^2 = m^2 + c^2 - 2mc\cos A$.

c. Prove that $c^2 = m^2 + a^2 - 2ma\cos C$.

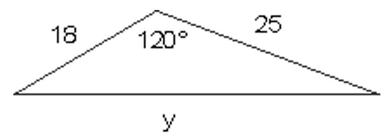
15 Is the Law of Sines or the Law of Cosines worth remembering? Would it be easier always to construct appropriate perpendicular lines and use only trigonometric ratios to solve the problems that may require them?

Practice

16 Find the indicated side lengths or angle measures in the following figures.

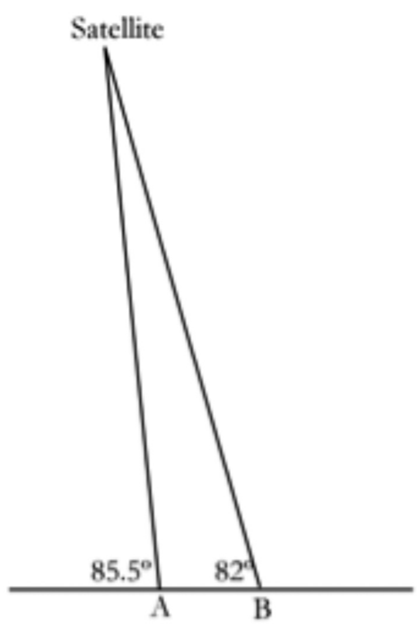


(a)



(b)

17 Two tracking stations are monitoring the path of a satellite, which has passed to the west of both stations. From station A, the angle of elevation to the satellite is 85.5 degrees. From station B, the angle of elevation to the satellite is 82 degrees. Stations A and B are 65 miles apart.



a. Find the distance from the satellite to tracking station A.

b. Find the height of the satellite above the ground.

18 The distance from Chicago to St. Louis is 440 km, from St. Louis to Atlanta 795 km, and from Atlanta to Chicago 950 km. What are the angles in the triangle with these three cities as vertices?

19 In the figure below, find the measure of angle A

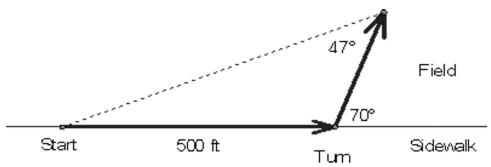
After having done these problems in the Practice section, you may be better prepared to answer question 15, repeated here:

20 Is the Law of Sines or the Law of Cosines worth remembering? Would it be easier always to construct appropriate perpendicular lines and use only trigonometric ratios to solve the problems that may require them?

Problems

21

Melissa walks along the path shown below: She goes 500 ft along a sidewalk adjacent to a field, then turns 70 degrees, walks a way across the field, and stops. Looking back, she measures a 47 degree angle between her path across the field and her line of sight to her starting point.

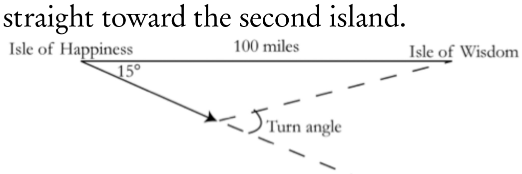


a. Find the distance that Melissa walked across the field.

b. How far away is Melissa from her starting point?

22

A sailboat is attempting to sail between two islands, 100 miles apart. From the very beginning, a wind blows the boat 15 degrees off its course. After the sailboat has been sailing for an hour and a half at 25 mph, it corrects its course so it is sailing straight toward the second island.

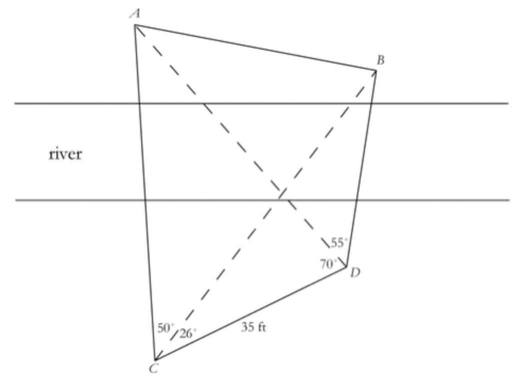


a. By how much does the sailboat need to increase its speed if it wants to arrive at its destination at the same time it would have going 25 mph along the straight path?

b. Find the "turn angle" – the number of degrees the sailboat needed to turn in order to correct its course.

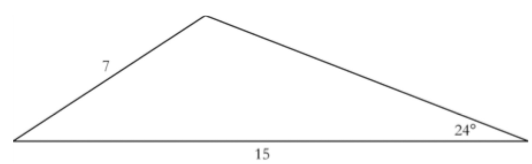
23

Julie needs to find the distance between two trees A and B on the opposite side of a river. On her side of the river, she chooses two points C and D, 35 feet apart. Then she measures the angles shown in the diagram. What distance between the trees will she calculate?



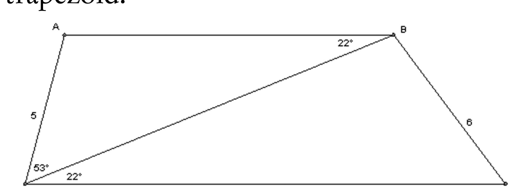
24

Find all three angles of this triangle. Check to be sure that your answer is plausible.

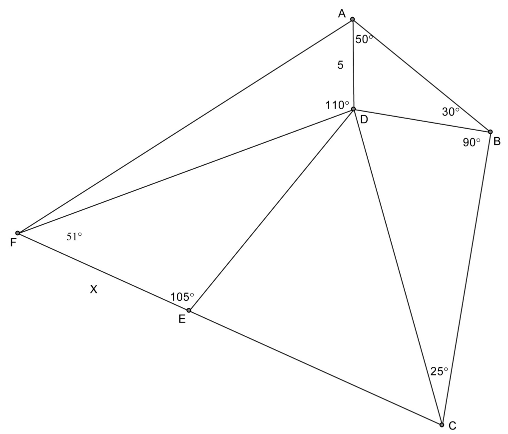


25

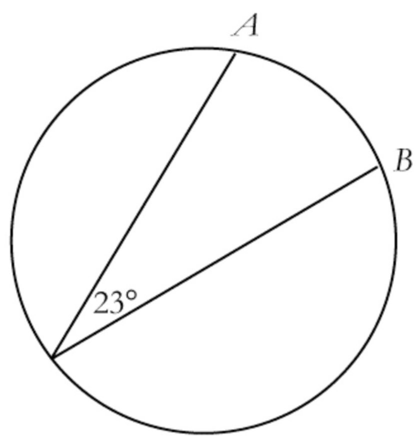
Find the measure of all the angles in the trapezoid.



26 In the figure below, find X.



27 In the figure below, the length of the chord drawn from A is 8, and the length of the chord drawn from B is 10. Find the length of arc AB. (Hint: you will need to know the radius of the circle.)



28 The function ThirdSide takes an angle, theta, and outputs the third side of a triangle with sides 3 and 4 and included angle theta.

a. Find $Thirdside(80^\circ)$.

b. What are the minimum and maximum possible values for $Thirdside(\theta)$? Justify your answer in two different ways:

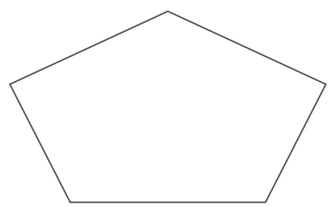
i. by visualizing what different triangles would look like for different values of θ.
ii. by looking at the Law of Cosines formula and seeing how the value of theta affects the value of each term.

Often times, a problem that seems to be hard may be simplified a lot by just drawing a diagram or adding a couple of lines—or even just points—to a diagram already in place. These additions (to the concrete situation given) are implemented to **visualize** and better understand problems which may have been initially confusing. In the following problems, 29 through 33, you will have the chance to use this mathematical habit of mind repeatedly.

29 A triangle has a 13-inch, a 14-inch, and a 15-inch side. To the nearest tenth of an inch, how long is the median drawn to the 14-inch side? (Recall that a median is a line segment drawn from a vertex of a triangle to the midpoint of the opposite side.)

30

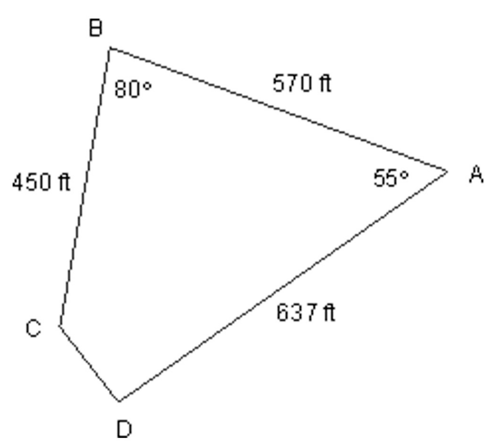
A "half-regular" pentagon isn't perfectly regular, but it does fold perfectly in half (the left half is the same as the right half – it's symmetric across a vertical axis). The half-regular pentagon below has a top angle of 160°, a side length of 8 for the top two sides, a side length of 9 for the base, and a total height of 14 (from the top point down to the base).



Find the area and perimeter of the pentagon.

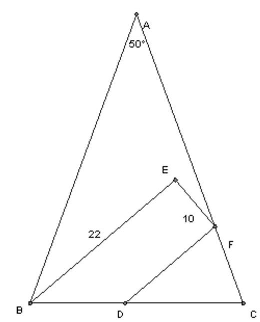
31

The diagram below represents a plot of a piece of land. Find the area of the plot.



32

In this diagram, ΔABC and ΔCDF are both isosceles. $AB = AC$ and $DF = DC$. $\angle E$ and $\angle EFD$ are right angles.

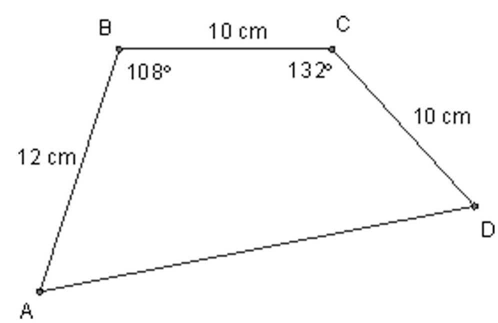


a. Find all the angles in the diagram.

b. Find AF.

33

The lengths of three sides and the measure of two angles of a quadrilateral are given, as shown in the figure below.

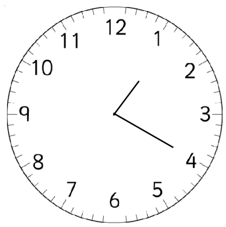


a. Determine the length of the diagonals of this quadrilateral. Round answers to two decimal places.

b. Determine the perimeter of this quadrilateral. Round answer to two decimal places.

c. Determine the area of this quadrilateral. Round answer to two decimal places.

34 You're looking at the hour hand and the minute hand on a clock at exactly 1:20. The tips of the hands are 3 inches apart. The hour hand is 2.15 inches long. (Remember – the hour hand is not pointing directly at the 1, since it's after 1:00!)

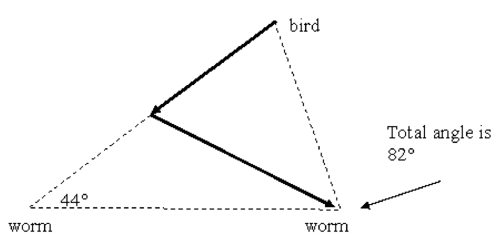


a. What's the angle between the hands?

b. Draw the triangle that this forms, and find the other angles in the triangle.

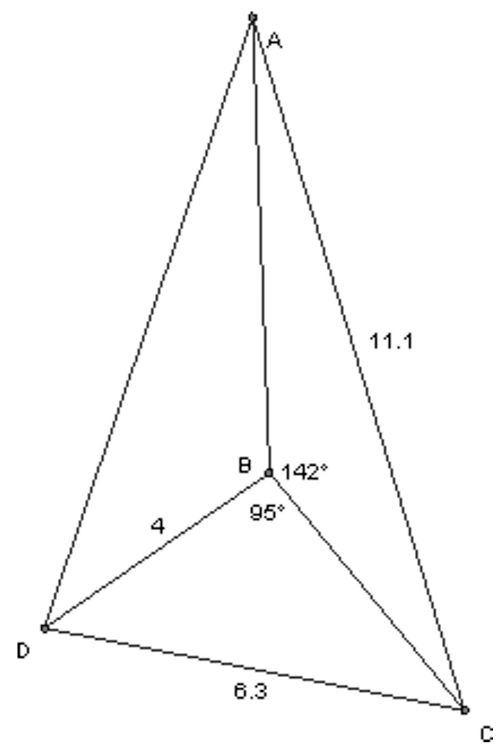
c. Find the length of the minute hand.

35 A bird sees two worms on the ground. The worms are 23 inches apart. The bird flies at the worm on the left, but when it's exactly halfway to the worm, it turns and flies to the worm on the right. The angles are as marked, and the angle marked 82° refers to the entire angle on the right side. Find all the other lengths and angles in the diagram below.



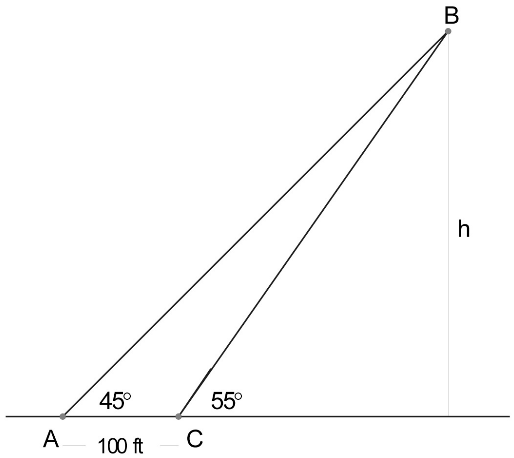
36 Use the law of cosines to determine the angles in a triangle that has sides of lengths 7.3, 23.1, and 15.7 . Why would this problem be easier if the sides were 8.1, 23.9, and 16.2?

37 In the figure below, find AB.



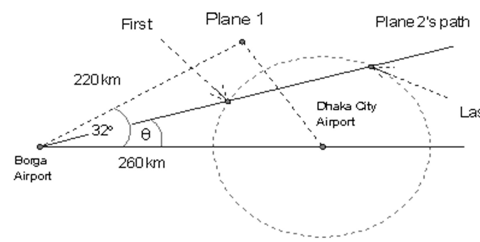
Exploring in Depth

38 A balloon, B, is tethered to the ground by wires $\overline{AB}$ and $\overline{CB}$ as shown in the figure below. How high, h, is the balloon above the ground?



39 In any triangle ABC, prove that $m\angle C = \cos^{-1}\left(\dfrac{a^2+b^2-c^2}{2ab}\right)$.

40 Borga and Dhaka City are two of the main cities in Bangladesh. Borga Airport and Dhaka City Airport are 260 km apart. The ground controllers at Dhaka City monitor planes within a 100-km radius of the airport.



a. Plane 1 is 220 km from Borga Airport at an angle of 32° to the straight line between the airports. Is it within the range of Dhaka City Ground Control?

b. Plane 2 takes off from Borga Airport toward Dhaka City Airport at an angle θ with the line between the airports. If θ is small enough, there is a point when Plane 2 first comes within range of Dhaka City Ground Control, and another point when it is last within range. Is there a value of θ for which Plane 2 is within range of Dhaka City Ground Control at just one point? If so, what is the magnitude of this angle?

c. If $\theta = 15°$, how far will Plane 2 be from Borga Airport when it first comes within range of Dhaka City Ground Control? How far from Borga Airport is it when it is last within range?

41

A triangle has six parts: three sides and three angles.

a. If we know only two out of the six parts of a triangle, is it enough information to precisely describe what triangle it is? Explain.

b. What if we know its three angles? Is it enough information to precisely describe that triangle? Explain.

c. What is the minimal information about the six parts of a triangle needed to precisely describe a triangle?

42

Don't use a calculator for this problem.

a. Reduce: $\dfrac{x^2y + y^2x}{xy}$

b. Reduce: $\dfrac{4x - 20y}{16x + 20y}$

c. Simplify: $\left(\dfrac{84x^5y^2}{14x^{-2}y^4}\right)^{-2}$

d. Rewrite using fractional exponents: $\sqrt{\sqrt[3]{\sqrt{x}}}$

e. If $a = \sqrt{b}$, find a^3 in terms of b.

As you may have explained in Problem 41, in general a triangle is determined by three of its six parts, where at least one of these parts is a side. These are the possibilities. Case SAA: when one side and two angles are known. Case SSA: when two sides and the angle opposite one of those sides are known. Case SAS: when two sides and the included angle are known. Case SSS: when the three sides are known.

43

In the Case SSA described above, what can we say about a triangle for which two sides and the angle opposite one of those sides are known?

Note: Case SSA is known as the **ambiguous case**. Are there good reasons for this name?

44

Find the side lengths and measures of the angles of if $m\angle A = 43.1^\circ$, $a = 186.2$, and $b = 248.6$.

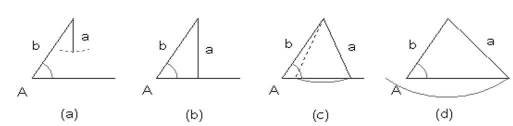
45

Find the side lengths and measures of the angles of the angles of ΔABC if $m\angle A = 42^\circ$, $a = 70$, and $b = 122$.

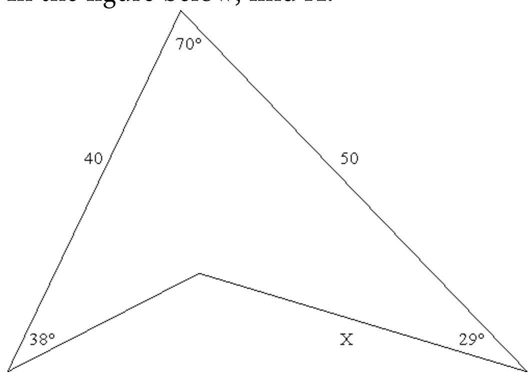
46

The Ambiguous Case

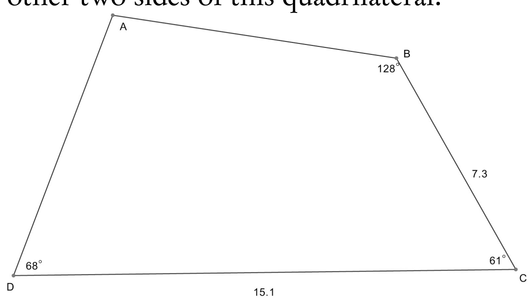
In Case SSA, when two sides and an angle opposite one of those sides are given, it is possible that none, one, or two triangles may exist satisfying the given information. These possibilities are illustrated in the figure below, where $\angle A$, a, and b are the angle and two sides given. For each case, (a) through (d), explain how $\angle A$, a, and b are related.



47 In the figure below, find X.



48 In the figure below, find the lengths of the other two sides of this quadrilateral.



49 **Heron's Formula**

Suppose you have ΔABC .

a. Prove that its area, A, is given by
$A = \frac{1}{2}ab\sin C$

b. Prove that
$A^2 = \frac{1}{4}a^2b^2\left(1-\cos C\right)\left(1+\cos C\right)$

c. Use the Law of Cosines to express $\cos C$ in terms of a, b, and c, and from Part b prove Heron's Formula:
$A = \sqrt{s(s-a)(s-b(s-c)}$,

where $s = \frac{1}{2}(a+b+c)$ is the semiperimeter of the triangle.

LESSON 1: GETTING COMFORTABLE WITH TRIG

Introduction

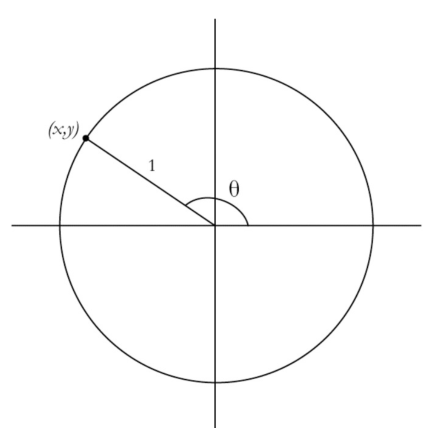
In an earlier lesson, you learned how the trigonometric functions sine, cosine, and tangent were defined for angles other than those between 0 and 90 degrees. Here is a quick refresher.

Imagine sitting at the edge of a merry-go-round with radius 1 meter. You've set up a coordinate system with (0,0) at the center of the merry-go-round, and the point (1,0) is due east from (0,0). When the wheel starts to turn counterclockwise, your position is (1,0).

1. If the merry-go-round has rotated 42° from its starting position, find the coordinates of your position.

2. If the merry-go-round has rotated 200° from its starting position, find the coordinates of your position.

This may be enough to remind you of the generalized definitions of sine and cosine. In the figure below, a "spoke" of a circle of radius 1 is drawn so that it makes a central angle of θ with the positive x-axis. The spoke intercepts the circle at the point (x, y). The sine of an angle θ is the y-coordinate of this point. The **cosine** of θ is the x-coordinate of this point.



3 Using the figure above, how would you define the tangent of θ?

4 In middle school, you learned that $\sin\theta$ is defined as the ratio of the length of the side opposite θ in a right triangle to the length of the hypotenuse. Where is the right triangle in the figure above, and what are the opposite and hypotenuse?

5 How would you determine the sine, cosine, and tangent of an angle using a circle with a different radius than 1?

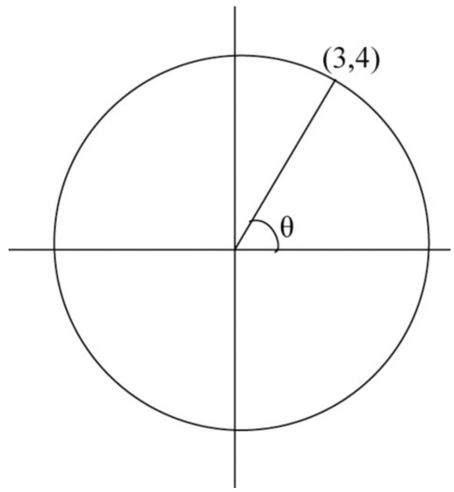
The circle with radius one is better known in mathematical circles as the "unit circle." You'll want to draw it — or a circle with a well-chosen radius — almost every time you solve a trig problem.

Development

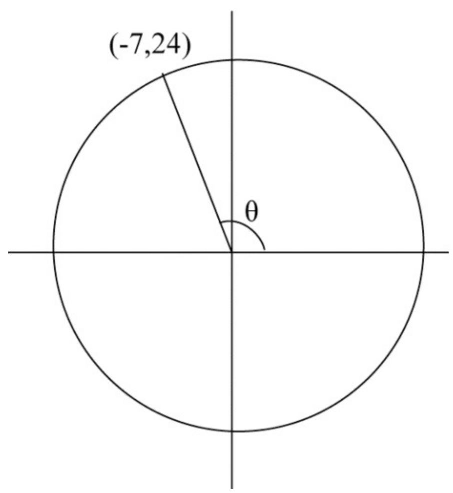
6 Cecilia is on a Ferris wheel and notices that, when she has rotated about 143 degrees from due east, she is 24 meters west and 18 meters above the center of the wheel. Use this information to approximate the sine, cosine, and tangent of 143 degrees. Then see how you did by asking your calculator for these values.

7 For each diagram, find the sine and cosine of the angle θ.

a.



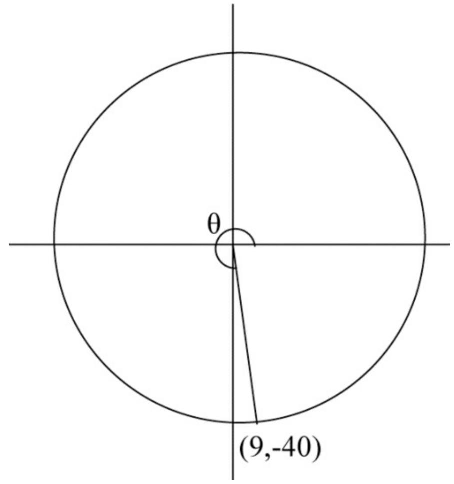
b.



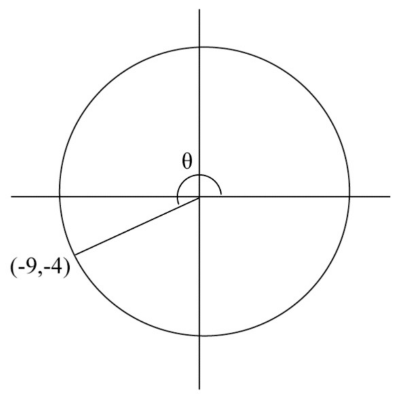
c.

8 Copy the following 30-60-90 triangle and 45-45-90 triangle into your notebook. Then write in the remaining angles and find the remaining sides in terms of x.

Problem 2 may have reminded you that whether the sine or cosine of an angle is positive or negative depends on which **quadrant** the angle is in. This allows you to answer questions like the following.



d.



9 Go back to your 30-60-90 and 45-45-90 triangles and think about positioning them on the unit circle.

a. Name all angles between 0 and 360 degrees that have a sine of $\frac{1}{2}$.

b. Name all angles between 0 and 360 degrees that have a cosine of $\frac{1}{2}$.

c. Name all angles between 0 and 360 degrees that have a tangent of 1.

d. Now explain how to find *all* the angles that have a sine of $\frac{1}{2}$.

10 Solve the equation $\sin\theta = \frac{1}{2}$. How many solutions are there? How many solutions are interestingly different from one another?

Of course, you might also have to solve equations that don't involve sines or cosines of angles in special triangles. For instance, if you had to solve $\sin\theta = .2531$, you would be forced to resort to the inverse sine function on your calculator.

11 Make sure your calculator is in degree mode, then take the inverse sine of .2531 to solve this equation.

12 Draw the "wheel" again, and draw a spoke at the angle that has sine .2531. Remembering that the sine of an angle is the height of a point above the x-axis, indicate the other point on the wheel that would have the same sine.

13 How would you calculate the size of this second angle with sine .2531? Do so.

14 Give *all* the solutions to the equation $\sin\theta = .2531$.

15 Now find *all* the solutions to the equation $\cos\theta = .2531$.

Practice

16 A wheel of radius 2 ft spins around the point $(0, 0)$. A ladybug sticker is initially at the point $(2, 0)$. Find the coordinates of the sticker once the wheel has spun

a. 115°

b. 220°

c. 300°

d. 660°

17 Calculate the exact value of each expression. Do not use a calculator.

a. $\sin 120^\circ$

b. $\cos(-60^\circ)$

c. $\cos 315^\circ$

d. $\tan 150^\circ$

e. $\sin 270^\circ$

f. $\cos 0^\circ$

18 Given that the sine of 138 degrees is about .6691,

a. Find all angles between 0 and 360 degrees that have a sine of .6691.

b. Describe all angles that have a sine of .6691, including those not between 0 and 360 degrees. Can you find a way to write your answer using symbols?

19 Find all solutions to the equation $\sin\theta = .6691$.

20 Find all solutions to the equation $3\sin\theta - 1 = 1.0074$.

21 Find all solutions between 0° and 360° to the equation $\tan\theta = \frac{-1}{\sqrt{3}}$.

22 Find all solutions between 0° and 360° to the equation $\cos\theta = -.8834$.

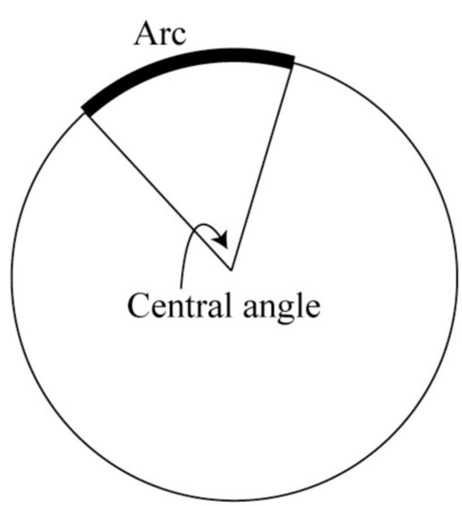
23 In which quadrants is the tangent negative? In which quadrants is the tangent positive?

Further Development

You're used to measuring temperature in degrees Fahrenheit, but you also know that much of the world measures temperature using degrees Celsius. Similarly, the world uses miles and kilometers, ounces and grams, gallons and liters.

When the Babylonians invented degree measure, they chose a 360-degree circle for two reasons. One was that, since there are 365 days in a year, a degree would be very close to one day's progress around the circular path they believed the sun made every year. Another reason is that 360 is divisible by lots of numbers, so we can describe a fourth, a sixth, an eighth, and a tenth of a circle as an integer number of degrees. (Try this with a 100 degree circle and you'll see it doesn't work as well.)

As time went on, though, mathematicians realized that the number 360 didn't really have anything to do with circles, and in fact some of their calculations would be easier if they used a different system. This system was inspired by the close relationship between a central angle of a circle and the arc it intercepts. It was also inspired by the importance of the unit circle.



Specifically, we'd like the measure of a central angle in the unit circle to be literally the same as the length of the arc it intercepts.

24

What is the circumference of the unit circle?

Since a 360 degree "central angle" intercepts the entire circle your answer to Problem 24 will be the equivalent of 360 degrees in our new system.

25 How much of a circle does a 180 degree central angle intercept? So, in this new system, how should we represent a 180 degree angle?

26 Under this new system, what should we call the measure of an angle that intercepts a quarter-circle?

27 What should we call the measure of an angle that intercepts a sixth of a circle?

When you measure angles this way, you are measuring in **radians**. (Try pressing the MODE button on your calculator and notice that there is a radian vs. degree option.) The next problems suggest a reason for the name.

28 How many times does the radius of a circle "fit" around the circumference? (Hint: remember that the circumference of a circle is $C = 2\pi r$.)

29 Say you have an angle that intercepts an arc on the unit circle equal to two of its radii. How many radians is that angle?

If you are about to solve an equation like $\sin\theta = \frac{\sqrt{3}}{2}$, you might not know whether to give your answer in degrees or radians. Your class should agree on which system to use.

Practice

30 If 60 degrees is equivalent to $\frac{\pi}{3}$ radians, how many radians are in

a. 120 degrees?

b. 240 degrees?

c. 30 degrees?

d. 15 degrees?

31 Use your answers to the previous problem to help convert each degree measure to radians.

a. 75 degrees

b. 150 degrees

c. 330 degrees

32 If 45 degrees is equivalent to $\frac{\pi}{4}$ radians, how many degrees are in

a. $\frac{3\pi}{4}$ radians?

b. $\frac{7\pi}{4}$ radians?

c. π radians?

33 Copy this unit circle into your notebook and write the radian measure of each angle marked.

34 Give the exact value of each expression.

a. $\sin \frac{\pi}{3}$

b. $\cos \frac{\pi}{3}$

c. $\sin \frac{2\pi}{3}$

d. $\cos \frac{2\pi}{3}$

e. $\tan \frac{7\pi}{4}$

f. $\cos \frac{5\pi}{4}$

Problems

35 Write each set of values in order of smallest to biggest. Do not use a calculator.

a. $\sin 60^\circ, \sin 70^\circ, \sin 80^\circ$

b. $\cos 60^\circ, \cos 70^\circ, \cos 80^\circ$

c. $\tan 60^\circ, \tan 70^\circ, \tan 80^\circ$

36 Write each set of values in order of magnitude (that is, you would write 2 before −10) and then say which are positive and which are negative.

a. $\sin 89^\circ, \cos 89^\circ, \tan 89^\circ$

b. $\sin 269^\circ, \cos 269^\circ, \tan 269^\circ$

c. $\sin 359^\circ, \cos 359^\circ, \tan 359^\circ$

37 Solve each equation for all values of θ. You may use your calculator, but keep in mind that it will not give you all the answers you need.

a. $\cos\theta = .4642$

b. $\frac{1}{3}\sin\theta = .2862$

c. $2\tan\theta - 4 = -3.5174$

d. $\sin(\theta - 15^\circ) = .7327$

38 Solve for all values of x. Give answers to Part a using degrees and answers to Part b using radians.

a. $\sqrt{2}\sin(x + 45^\circ) = 1$

b. $\sqrt{3}\tan(x + 4) = 3$

39 Solve for all values of θ.

a. $\tan^2\theta = 3$

b. $\left(\sqrt{3}\sin\theta + 1\right)(2\sin\theta - 1) = 0$

40 Does $\sin^{-1}x$ refer to an angle, or just a number that does not represent an angle? How about $\sin x$?

41 A wheel on a buggy has a radius of one foot. How many degrees has it spun counterclockwise if a chalk mark originally on the wheel's rightmost point is for the first time .8 feet off the ground?

42 Don't use a calculator for this problem.

a. Solve: $x^4 - 9x^2 = 0$

b. Solve: $\frac{3}{27}x^2 = 1$

c. Find the base-ten logs of a million, a billion, and a million times a billion.

d. Solve: $\frac{x+1}{x+3} = 14$

e. . Simplify: $\frac{\left(x^2\right)^6}{x^3}$

43 A Ferris wheel has a radius of 132 feet. Its bottom seat is 6 feet off the ground. You are sitting in the seat at the wheel's rightmost point when the wheel begins to spin counterclockwise.

a. What is your height off the ground when the Ferris wheel has rotated $\frac{5\pi}{4}$ radians?

b. What is your height off the ground when the Ferris wheel has rotated $\frac{9\pi}{4}$ radians?

c. What is your height off the ground when the Ferris wheel has rotated $\frac{13\pi}{4}$ radians?

d. You have fallen asleep on the Ferris wheel. When you wake up, you realize that you are level with a tower that you know to be 100 feet off the ground. What are the possibilities for how many radians the wheel has rotated since the beginning of the ride?

44 How many degrees is one radian? How many radians is one degree?

45 Come up with a conversion formula between degrees and radians.

46 Another way of thinking of radian measure is as a ratio. In a circle, it is the ratio of the arc an angle intercepts to the radius of the circle.

a. Draw a circle of radius 5 inches, with an angle inside that intercepts an arc that is $\frac{1}{4}$ the length of the circle. Find the ratio of the arc length to the radius.

b. Repeat part a, but using a circle of radius 7 inches. Your angle will still intercept $\frac{1}{4}$ of the circle.

c. What is the radian measure of an angle that intercepts a quarter-circle?

47 In 10th grade, you found a formula for the length of an arc intercepted by an angle θ: $\text{arclength} = 2\pi r \cdot \frac{\theta}{360^\circ}$. Find a similar formula that works when the central angle is measured in radians instead of degrees.

48 In 10th grade, you also found a formula for the area of the sector the angle intercepts: $\text{sector area} = \pi r^2 \cdot \frac{\theta}{360^\circ}$. Find a similar formula that works when the central angle is measured in radians instead of degrees.

49 From what we've seen in this lesson, there are often two solutions to the equation $\cos\theta = A$ between 0° and 360°.

a. Are there any values of A for which there would only be one solution? How many values of A?

b. Are there any values of A for which there would be no solutions?

50 Suppose an angle has a sine of .4221.

a. How many angles have this sine?

b. Your calculator tells you that $\sin^{-1}(.4221) \approx 24.97^\circ$. Why do you suppose it only gives you one answer?

c. Try to figure out how the calculator "chooses" the answers it gives for the inverse sine button. What's the biggest answer it will give? What's the smallest?

51 Repeat part c of the previous problem for inverse cosine and inverse tangent. Why do you suppose the range of answers is not the same for the sine and cosine?

52 If $2\theta = 26^\circ + 360^\circ n$, where n stands for any integer,

a. Find all possible values for θ.

b. Did your possible values include $\theta = 553^\circ$? If not, revise your answer to part a.

53 If $3\theta = 150^\circ + 360^\circ n$, where n stands for any integer, find all possible values of θ.

54 Solve each equation.

a. $\sin 2\theta = .1647$

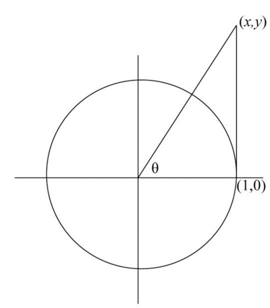
b. $\tan 2\theta + 5 = 42$

c. $2\sin 3\theta = \sqrt{3}$ (exact answers)

d. $\cos 3\theta = \dfrac{1}{\sqrt{2}}$ (exact answers)

Exploring in Depth

55 Find the coordinates of the point labeled (x, y) in terms of θ. Does the diagram suggest a certain word origin?



56 How many solutions does $\sin n\theta = \dfrac{\sqrt{3}}{2}$ have between 0° and 360°? Answer in terms of n.

57 Refer back to Problem 46, in which you thought of the radian measure of an angle as a ratio. Explain why, unlike degrees, miles, or kilograms, radian measure doesn't require .

Park School Mathematics

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