A DIFFERENT CUSTOM BOOK

TABLE OF CONTENTS

HABITS

look for patterns: to look for patterns amongst a set of numbers or figures

tinker: to play around with numbers, figures, or other mathematical expressions in order to learn something more about them or the situation; experiment

describe: to describe clearly a problem, a process, a series of steps to a solution; modulate the language (its complexity or formalness) depending on the audience

visualize: to draw, or represent in some fashion, a diagram in order to help understand a problem; to interpret or vary a given diagram

represent symbolically: to use algebra to solve problems efficiently and to have more confidence in one's answer, and also so as to communicate solutions more persuasively, to acquire deeper understanding of problems, and to investigate the possibility of multiple solutions

prove: to desire that a statement be proved to you or by you; to engage in dialogue aimed at clarifying an argument; to establish a deductive proof; to use indirect reasoning or a counterexample as a way of constructing an argument

check for plausibility: to routinely check the reasonableness of any statement in a problem or its proposed solution, regardless of whether it seems true or false on initial impression; to be particularly skeptical of results that seem contradictory or implausible, whether the source be peer, teacher, evening news, book, newspaper, internet or some other; and to look at special and limiting cases to see if a formula or an argument makes sense in some easily examined specific situations

take things apart: to break a large or complex problem into smaller chunks or cases, achieve some understanding of these parts or cases, and rebuild the original problem; to focus on one part of a problem (or definition or concept) in order to understand the larger problem

conjecture: to generalize from specific examples; to extend or combine ideas in order to form new ones

change or simplify the problem: to change some variables or unknowns to numbers; to change the value of a constant to make the problem easier; change one of the conditions of the problem; to reduce or increase the number of conditions; to specialize the problem; make the problem more general

work backwards: to reverse a process as a way of trying to understand it or as a way of learning something new; to work a problem backwards as a way of solving

re-examine the problem: to look at a problem slowly and carefully, closely examining it and thinking about the meaning and implications of each term, phrase, number and piece of information given before trying to answer the question posed

change representations: to look at a problem from a different perspective by representing it using mathematical concepts that are not directly suggested by the problem; to invent an equivalent problem, about a seemingly different situation, to which the present problem can be reduced; to use a different field (mathematics or other) from the present problem's field in order to learn more about its structure

create: to invent mathematics both for utilitarian purposes (such as in constructing an algorithm) and for fun (such as in a mathematical game); to posit a series of premises (axioms) and see what can be logically derived from them

HABITS

LESSON 1: DEFINING NEW SYMBOLS

Introduction

You have a pocket calculator that can only do one thing — when you type in two whole numbers, it takes the first number, adds the second number to it, adds the first number to the sum, then takes that whole answer and multiplies it by the second number. The number you see on the screen is its final answer after this series of steps.

1 If you type in 7 and 2, what will your calculator show?

2 What if you type in 2 then 7?

3 Your friend uses the calculator. You can't see the first number she types in, but the second number is 3. The answer that the calculator gives is 39. What was the first number?

Development

The calculator's rule (from the problems above) has a symbol to represent it: "$\triangle$". For example, "$7\triangle 2$" refers to what you did in question #1.

4 What is $2\triangle 10$?

5 What is $2\triangle y$, in terms of y? Simplify as much as possible (no parentheses in your answer).

6 What is $x \triangle 8$?

7 What is $x \triangle y$, in terms of x and y? Write this as an equation.

If you invented the symbol "$\triangle$" for a math problem you wrote, and wanted to explain it in an equation rather than in words, you would say:

"Let $x \triangle y$ ______________________" (your answer to question 7).

This is called the algebraic rule for $\triangle$.

Here the word "Let" is used the same way as when a problem says "Let x be the number of boxes you purchase."

8 After trying out a few different whole numbers as inputs to the $\triangle$ rule, John makes the following claim: "To get an odd number for your answer from $\triangle$, you need to input odd numbers for a and b."

a. Does John's claim seem reasonable to you? If it doesn't, find a counterexample — a specific example which proves that his statement is not always true.

b. Is John's claim true?

These two questions are quite different — in many situations, there can be a variety of different predictions that seem reasonable, but only one of them may be actually true! The habit of seeking proof is not only about learning how to prove a statement to be true — it's also about learning to ask "Why would that be true?" when you are presented with a reasonable statement.

9 The symbol “&”, applied to two whole numbers, means that you take the first number and then add the product of the numbers.

a. Find 4& 7.

b. Find 7 & 4.

c. Write an algebraic rule for &.

d. Here is another of John’s claims. “To get an odd number for your answer from a & b, you need to input an odd number for a and an even number for b.” Is his new claim true? If you think it is true, carefully explain why. If you believe it’s not true, give a counterexample.

Practice

10 Just like any symbol regularly used in mathematics ($+, -, \cdot, \div$), symbols that we create can be used inside an equation. Let $m☼n = 3n - m$. Calculate the following:

a. $5☼2$

b. $6 \cdot (5☼2)$

c. $(1☼3.5)-(3.5☼1)$

d. $2☼(3☼5)$

e. $(2☼3)☼5$

f. $(a☼b)☼c$

11 In problem 10, is $m☼n = n☼m$ true in general, for ANY input numbers m and n? Explain.

Problems

Problems

12

When you give ◎two numbers, it gives you the third number in the addition/ subtraction pattern. For example, $◎\{21, 23\} = 25$ (going up by 2's), and $◎\{95, 90\} = 85$ (going down by 5's).

a. Find $◎\{6, 11\}$ and $◎\{11, 6\}$.

b. Find $◎\{11, ◎\{6, 11\}\}$.

c. Write an equation for $◎\{m, n\}$.

13

When you give $ two numbers, it gives you the third number in the multiplication/ division pattern. For example, $\$\{3, 15\} = 75$, and $\$\{48, 24\} = 12$.

a. Find $\$\{2, 3\}$, $\$\{12, 13\}$, and $\$\{102, 103\}$.

b. What is $\$\{2009, 1\}$?

c. If $\$\{x, 2\} = \$\{3, 1\}$, then what is x?

14

The symbol ¤ takes a single number, squares it, and then subtracts 4.

a. What is ¤(6)?

b. Can you ever get a negative answer for ¤(x)? Why or why not?

c. Find an x so that ¤(x) is divisible by 5.

15

Let the symbol ¥ mean: Add up the two numbers, then take that answer and subtract it from the product of the two numbers. What is 5¥8? 8¥5?

16

Look back at the problem above. Do you think the same thing would happen for any pair of numbers, if you used the same symbol? Explain your answer.

When switching the order of the input numbers never has an effect on the answer, the symbol you are working with is said to have the commutative property. For example, the symbol ¥ (from questions 15 and 16 above) had the commutative property, but the symbol © (from question 12) did not.

One important thing to note is that there's no such thing as "sometimes" having the commutative property. For example, ¥ has the commutative property because a¥b and b¥a are equal for ANY input numbers a and b, not just because it worked for 5 and 8.

Proving a statement false is as easy as finding one counterexample, but it is sometimes difficult to prove that a statement that appears to be true is indeed true. In the following problems (17-21), you will need to decide whether statements are true or false, and to also clearly support your position.

17

Fergie claims that each of the following symbols has the commutative property. Examine each of his claims.

a. $x \, \$ y$ means add 1 to y, multiply that answer by x, and then subtract x.

b. Take two whole numbers x and y. To do $x \, \% y$, you divide x by 2, round down if it's not a whole number, and then multiply by y.

c. To calculate $x£y$, imagine that you walk x miles east and then y miles northeast. $x£y$ is how far away you end up from your starting point.

d. ¿ works by adding up the two numbers, multiplying that by the first number, and then adding the square of the second number.

e. ¢ takes two numbers. You reverse the first number (for instance, 513 becomes 315); one-digit numbers stay the same), then add the reversed number to the second number, and finally add up the digits of your answer.

18

Let $x \bigstar y = x^2 - y^2$. For example, $4 \bigstar 3 = 16 - 9 = 7$. True or false: $x \bigstar y$ always equals the sum of the two numbers — for example, $4 \bigstar 3 = 7$ which equals $4 + 3$. If it's true, justify your claim. If it's false, try to find out what kinds of numbers do make the claim work.

19

When you give ® two numbers, it finds the sum of the two numbers, then multiplies the result by the first number. Finally it subtracts the square of the first number. Flinch claims that ® is commutative. Is he correct?

20

Here's how you might prove Flinch's claim in problem 19 for *any* two starting numbers.

a. Let's call your first number m and your second number n. Write down and simplify as much as you can the expression for $m®n$.

b. Write down and simplify as much as you can the expression for $n®m$.

You should be able to convince anyone, using your work in problem 20 that Flinch's statement is always true, no matter which numbers we start with.

21

Is the rule below commutative? The rule $\sim$ adds up the two numbers, doubles the answer, multiplies the answer by the first number, then adds the square of the second number and subtracts the square of the first number.

22

Let $x \clubsuit y = \frac{1}{1{,}000{,}000{,}000} x^y$.

a. Is there a value of y such that $10 \clubsuit y > 1$?

b. Is there a value of y such that $1.001 \clubsuit y > 1$?

23

The symbol "&", applied to *any* two numbers (not only whole numbers), means that you take the first number and then add the product of the numbers.

a. When you calculate x & y, you get 120. What could x and y be? Give several different answers.

b. When you calculate x & y, you get 120. Write an equation that expresses this fact, then solve for y in terms of x (meaning, write an equation $y =$.... with only x's in the equation).

24

The command "CircleArea" is a rule that finds the area of the circle with the given radius. For example, $\text{CircleArea}(3) = 9\pi$.

a. Find CircleArea(4).

b. Write the equation for CircleArea(x).

c. Can you find CircleArea(-4)? Why or why not?

25

It's January 1st, and you are counting the days until your birthday. Let m be the month (as a number between 1 and 12) and d the day (between 1 and 31) of your birthday, and pretend that there are exactly 31 days in each month of the year.

a. How many days are there until January 25? Until April 10?

b. For January 25th, $m = 1$ and $d = 25$, and for April 10th, $m = 4$ and $d = 10$. By looking at what you did in part a, explain how you can use the numbers m and d to count the days from January first to until any day of the year.

c. Let the symbol ♥ represent this count. Write an algebraic rule for calculating m♥d. Test your rule with an example.

26

Now, count how many days are from your birthday to New Year's Eve (December 31st). Represent this with the symbol ?. Again, pretend there are 31 days in each month.

Write an equation for in terms of m and d, and use an example to show that your equation works. (You might want to try explaining it in words or in an example first.)

27

"max(a,b)" takes any two numbers and gives you the larger of the two. The symbol ∂ is defined by $\partial(a, b) = \max(a,b) - \min(a,b)$. Is ∂ commutative?

28 We say that the counting numbers (i.e., 1, 2, 3, ...) are "closed under addition" because any time you add two counting numbers, you get another counting number. Decide whether or not the counting numbers are closed under each of the following operations. In each case where the answer is no, try to find a group of numbers that *is* closed under that operation.

a. * (multiplication)

b. - (subtraction)

c. / (division)

d. The symbol ∂, from problem 27

29 The rule ← adds twelve to a number and divides the sum by four. What number x can you input into $\leftarrow(x)$ to get an answer of 5? An answer of -3?

30 To do the rule $, add 5 to the first number and add 1 to the second number, then multiply those two answers.

a. What's 3$1?

b. What's $x\$y$?

c. Your friend tells you that she needs to find numbers x and y so that $x\$y$ gets her an answer of A— a whole number that she does not reveal. In terms of A, tell her what to plug in for x and y. Make sure that your strategy would always get her the answer she wants.

d. What values of x and y give you an odd answer? Prove that your description is true and complete. (Make sure you know what it means to prove it's complete!)

e. What values of x and y would give you an answer of zero?

31 Create 3 different rules that give an answer of 21 when you plug in 2 and 8.

32 Let ? be a rule that acts on a single number.

a. Create a rule for ? so that $?(2) = 12$ and $?(11) = 75$.

b. Now, create a new rule g such that g(5)=26 and g(10)=46.

33 Let $x \triangle y = x^2 + 2xy$. ($x$ and y have to be integers.)

a. If you plug in 6 for x, find a number you could plug in for y to get an answer of zero.

b. Using part a as an example, describe a general strategy for choosing x and y to get an answer of zero, without making x zero. Explain why your strategy works.

c. Suppose you use the same number for x and y— call this number N. (So, you're doing $N \triangle N$.) What is your answer, in terms of N? Simplify as much as possible.

d. Suppose $x \triangle y = 20$. Solve for y in terms of x.

e. Describe a strategy to get any odd number that you want. Give an example, and also show why your strategy will always work (either give a thorough explanation, or use algebra to prove that it works).

34 Tinker to find a rule for $x \# y$ that gets the following answer: $5 \# 1 = 24$. Then try to write a rule that gives $5 \# 1 = 24$ <u>and</u> $4 \# 2 = 14$.

35 Create a rule α that works with two numbers, so that $\alpha(1, 1) = 5$ <u>and</u> $\alpha(2, 3) = 10$. Give your answer as an equation in terms of a and b. $\alpha(a, b) =$ ________________________.

36 Let $\otimes\{a, b\}$ be the two digit number where the tens digit equals a and the units digit equals b. For example, $\otimes\{9, 3\} = 93$.

a. In terms of f, what do you get when you do $\otimes\{5, f\}$? Write your answer as an equation. (Remember, writing $5f$ doesn't work, because it means $5 \cdot f$).

b. What do you get when you do $\otimes\{a, 4\}$?

c. When you calculate $\otimes\{a, 4\} - \otimes\{4, a\}$, what do you get in terms of a? Simplify as much as possible.

d. $\otimes\{6, x\} - \otimes\{x, 3\} = 12$. Find x. Show your work algebraically.

37 Now, let's redefine $\otimes\{a, b\}$. Let $\otimes\{a, b\}$ be the three-digit number where the hundreds and units digits are a, and the tens digit is b.

a. In terms of b, what is $\otimes\{b, 2\}$? (Again, $b2b$ won't work).

b. In terms of a and b, what do you get when you calculate $\otimes\{a, b\} - \otimes\{b, a\}$? Simplify as much as possible and check your answer with an example.

38 For any two positive numbers a and b, let $a ? b$ be the perimeter of the rectangle with length a and width b.

a. Is $?$ commutative?

b. Is $?$ associative? In other words, is $(a ? b) ? c = a ? (b ? c)$?

Exploring in Depth

39 A positive whole number is called a "staircase number" if the digits of the number go up from left to right. For example, 1389 works but not 1549.

The rule $x \nabla y$ works by gluing x and y together. For example, $63 \nabla 998$ gives an answer of 63998. If x and y are both staircase numbers, and x is bigger than y, is it always true that $x \nabla y$ is bigger than $y \nabla x$? If you think it's always true, explain why. If not, explain what kind of input would make it true.

40 Suppose that, for any two positive numbers a and b, $a \blacksquare b$ represents the area (ignoring units) of the rectangle with length a and width b.

Is $\blacksquare$ associative?

41 In problem 23, you were asked to find some pairs of numbers x and y so that $x \,\&\, y = 120$. (Recall that $x \,\&\, y$ takes the first number and adds the product of the numbers).

a. Do you think you could find numbers x and y to get any number you wanted? Try a few possibilities and explain what you find.

b. Develop a rule for choosing x and y to get the number that you want, which we will call N. (Hint: Try starting out by picking a number for x, and then figuring out what y would have to be. You might have to try different numbers for x to find a solid strategy.)

42 Don't use a calculator for this problem.

a. Find $\frac{3}{4} + \frac{5}{6}$

b. Find $2 \div \frac{1}{2}$

c. Simplify $3x - 2(x + 1)$

d. Write $.\overline{6}$ as a fraction.

e. Find $2\frac{1}{4} + 5\frac{7}{8}$

43 Let $x \square y = xy - x - y$. Develop a strategy to pick x and y so that you can get any number you want.

44 For a number x, $f(x)$ subtracts twice x from 100.

a. What is $f(10)$?

b. What is $f(-4)$?

c. Write an expression for $f(x)$.

d. Write an expression for $f(3x)$.

45 Let Let $x \square y = xy - x - y$. Prove or disprove: if x and y are both larger than 2, then $x \square y$ gives a positive answer. Make sure your explanation is thorough.

46 Write an equation for a rule $a \diamondsuit b$, so that the answer is odd only when both a and b are even.

47 Let $a \ \% \ b = a^2 b - b^2 a$. What would have to be true about a and b for $a \ \% \ b$ to be positive?

48 Let $a☺b = ab + b^2$. What would have to be true about a and b for $a☺b$ to be negative?

49 Let $x \& y = xy + y/2$, where x and y are whole numbers.

a. Describe what values of x and y would get you a negative answer.

b. Describe what values of x and y would get you an even whole number answer. (Careful! Your answer won't be just in terms of odds and evens. You'll have to tinker to see what type of numbers work. Think through it step by step and explain your reasoning).

c. What could x and y be to get 21? Give at least 3 different options.

d. Find a strategy to get any whole number you want, called N. You can explain your strategy in algebraic terms ("To get an answer of N let x equal ... and let y equal ...") or in words. Show an example to demonstrate that your strategy works.

50 For a number x that might not be a whole number, $\Phi\,(x)$ represents the integral part of x— for example $\Phi\,(6.51)$ is 6 and $\Phi\,(10)$ is 10. Using the Φ notation, write a rule that gives an answer of 0 if x is a whole number, and gives a non-zero answer if x is not a whole number. For example should equal 0 and should give an answer not equal to zero.

51 The rule $[b]$ is called the "greatest integer function" — it outputs the largest integer that is not above b. For example, $[9.21] = 9$, $[12] = 12$, $[.92] = 0$, and so on.

Write an equation for the rule $\{b\}$, which rounds b down to the hundreds — for example $\{302\} = 300$, $\{599\} = 500$, and $\{2\} = 0$. For your equation to calculate $\{b\}$, you can use any standard operations, and you should use the operation $[b]$.

52 To understand the rule behind the symbol $\diamond$, you need to draw a picture (graph paper will help). To draw the picture for $5\diamond3$, pick a point to start and then draw a line stretching 5 units to the right. From there, draw a line stretching 3 units up. Then draw a line stretching 3 units to the right, and then a line stretching 5 units up. Finally, draw a line back to your starting point.

a. Let $5\diamond3$ be the area of the shape you just drew. Calculate this number exactly.

b. Pick 2 new numbers (the 1st number should still be bigger). Draw the picture and calculate the answer that $\diamond$ would give for your numbers.

c. Draw a diagram to help you find an algebraic rule for $x \diamond y$ (again, you can assume that x is a bigger number than y). Check your rule by making sure it would give you the right answer for $5\diamond3$ and for your numbers in part b.

d. Will your rule still work if the two numbers are the same, such as $4\diamond4$? Explain why it will or will not always work.

e. Will your rule still work if the first number is smaller, such as $3\diamond7$? Explain why it will or will not always work.

LESSON 1: GRAPH THEORY

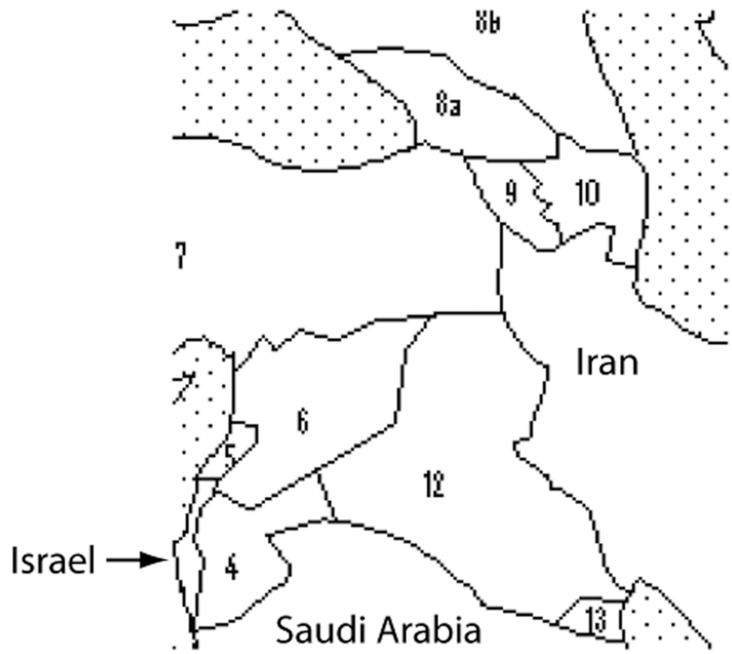
Introduction: Three Challenges

Felix will fly into Birmingham tomorrow to visit some family members. Today, he has his Alabama map spread out over a table in the local coffee shop to determine the route he will drive with his pay-per-mile rental car. He has a cousin in Birmingham, grandparents in Huntsville, an aunt in Cottondale, and a brother in Tuscaloosa. When Felix realized that he'll soon be served some of his grandma's chicken liver-sardine-cranberry casserole, he gagged on his cappuccino and accidentally spilled the whole cup on his map. The only usable part of the map remaining is the mileage chart between cities. Here is the mileage between the relevant cities:

	Birmingham	Huntsville	Cottondale	Tuscaloosa
Birmingham	X	95	50	56
Huntsville	95	X	148	143
Cottondale	50	148	X	9
Tuscaloosa	56	143	9	X

1 In what order should Felix visit his relatives to minimize the number of miles he drives with the rental car? Remember that he has to return to Birmingham to return the rental car.

Here is a map of part of the Middle East. You have different colors you can use to color in the different countries. If two countries are touching, then you should make them different colors so someone can easily tell they're different countries.



(For your reference and edification, the numbered countries are 4-Jordan, 5-Lebanon, 6-Syria, 7-Turkey, 8a-Georgia, 8b-Russia, 9-Armenia, 10-Azerbaijan, 12-Iraq, 13-Kuwait)

(image from http://catholic-resources.org/Courses/SCTR19-Spring2007-Worksheets.htm)

2 Color the map using as few colors as possible.
(If you don't have markers, just write the name of the color or use shading.)

Kim is figuring out the schedule for next year. It's tough, because out of thirteen electives, several students have expressed interest in taking more than one of them. She doesn't want to schedule two classes in the same block if there are students who want to take both of those classes.

In the grid below, an "x" indicates that at least one student has expressed an interest in taking both classes in that row and column of the grid.

	Poetry	Adv calc	Crim law	Foren.	Pract.	Stat	Prod.	Animal behavior	Civ. Lib.	Java	Shakespeare	Astro	Discrete
Poetry		x	x	x									
Adv Calc	x			x									
Crim Law	x			x	x	x							
Forensics	x	x	x		x						x		
Practicum			x	x		x	x	x			x		
Stat			x		x		x						
Production					x	x							
Animal					x				x	x	x		
Civ Lib								x		x		x	x
Java								x	x		x	x	
Shakespeare				x	x			x		x		x	
Astro									x	x	x		x
Discrete									x			x	

3

a. Assign each class a block (A through F) so that you never put two classes in the same block if there's anyone who wants to take both classes.

b. Now, suppose that all of these electives are to be scheduled on an ABC day. Can Kim create a successful, conflict-free schedule with only three blocks? If not, what is the fewest number of blocks that would accommodate these electives?

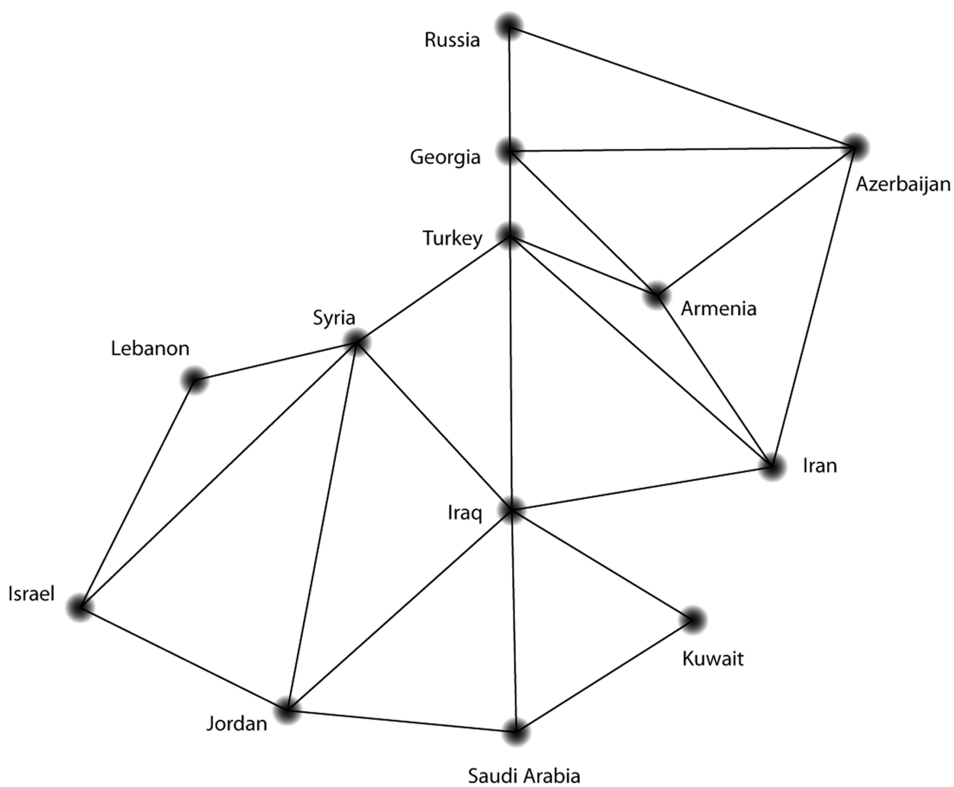
Development

To determine Felix's driving route, you probably recognized a way to represent the situation visually — whether you sketched out a map or drew a diagram.

The second and third problems, however, didn't have an obvious new representation to apply. Some people probably applied labels; others may have made lists. It turns out, though, if we use the method of the first problem with the second and third, we can learn much more from the data!

Most likely, the diagram you used in the first problem was actually a graph. A graph, in its most general sense, is a collection of dots (called "nodes") and lines connecting them (called "edges"). In your graph of Felix's problem, the nodes represented cities, and the edges related the nodes by showing the distances between.

We can represent the second situation with a graph, too, if we let each country be represented by a node. We can draw an edge between nodes when those countries border each other. Our graph comes out to look like this:

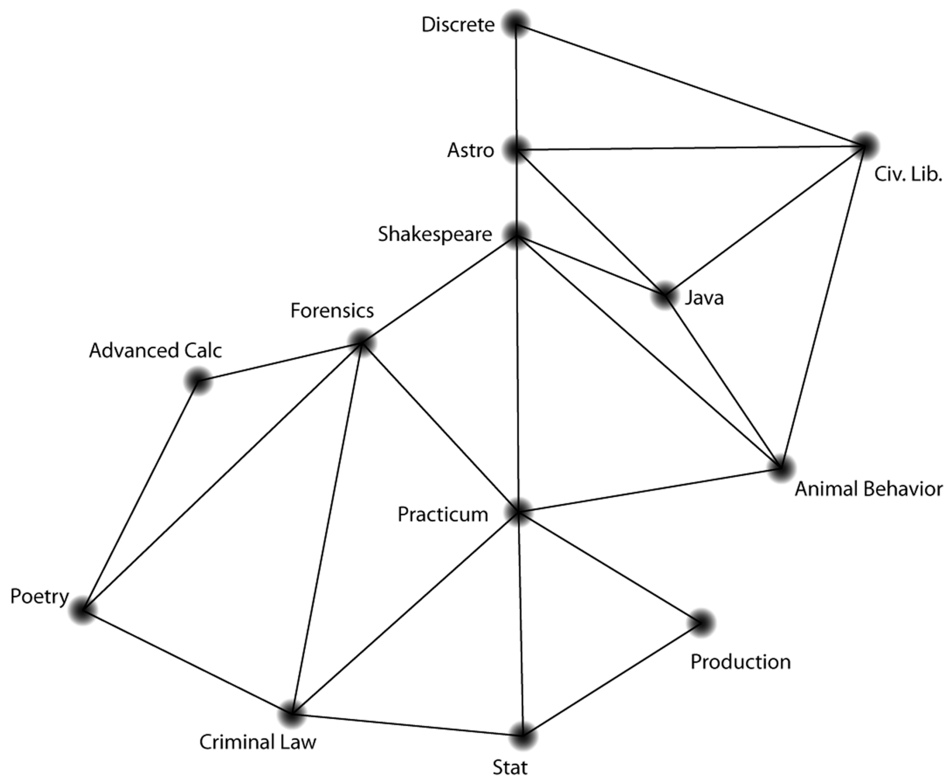


The problem now becomes to color each dot in such a way that any two dots connected by lines are always different colors.

4 What is the minimum number of colors you need now?

5 Did you think this version of the "coloring" problem was easier than in the introduction? Why or why not?

Similarly, we can represent the scheduling problem as a graph. The nodes can represent classes, and an edge can connect the nodes in the cases where some student wants to take both classes, as in the figure below.



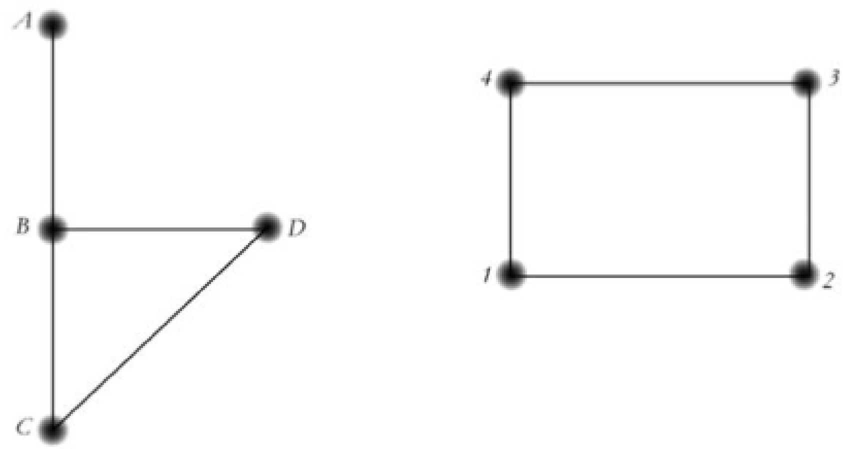
6 Solve the scheduling problem using this representation. What minimum number of blocks can you identify now?

7 Was this version of the schedule problem easier than in the introduction? Why or why not?

The tools you have been using so far in this lesson are the tools of graph theory. Already, graph theory has been über-useful in helping us represent relationships in a very streamlined way. What's more, we can now see that the map-coloring problem and the scheduling problem — initially very different — are actually remarkably similar.

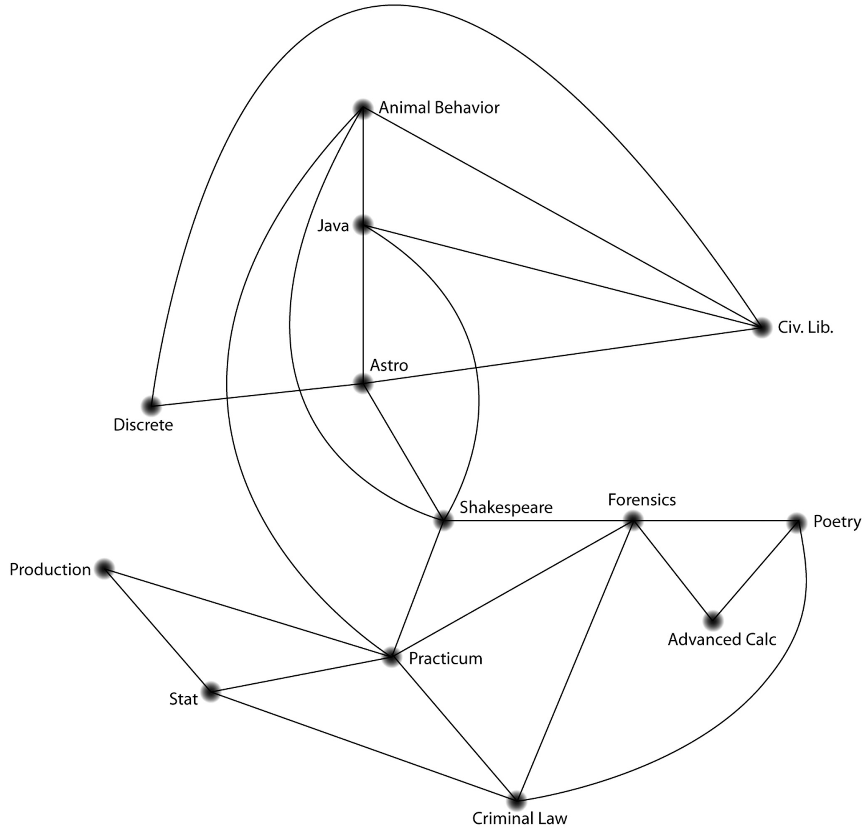
In fact, the identically structured graphs of the map-coloring problem and the scheduling problem are isomorphic. Two graphs are isomorphic when each node in one graph has a "partner" node in the other — a "partner" with the same connections. For example, Georgia in the first graph and Astronomy in the second correspond to one another. So do Lebanon and Advanced Calc. This correspondence comes from their identically structured connections, not their actual position on the page.

The following two graphs are not isomorphic.



An easy way to see that the graphs are not isomorphic is to see that node B has no "partner." In the left hand graph, B has three edges coming out of it. On the right, no node has three edges coming out of it, so B has no possible "partner." The number of edges connected to a node is a very important characteristic. The degree of a node (abbreviated deg(n)) is the number of edges coming out of it. So in the left-hand graph deg(A) = 1 and deg(B) = 3. What is deg(C)?

Isomorphisms can be trickier to spot than we've seen so far. For example, someone else with the task of drawing the scheduling graph might have come up with the following:



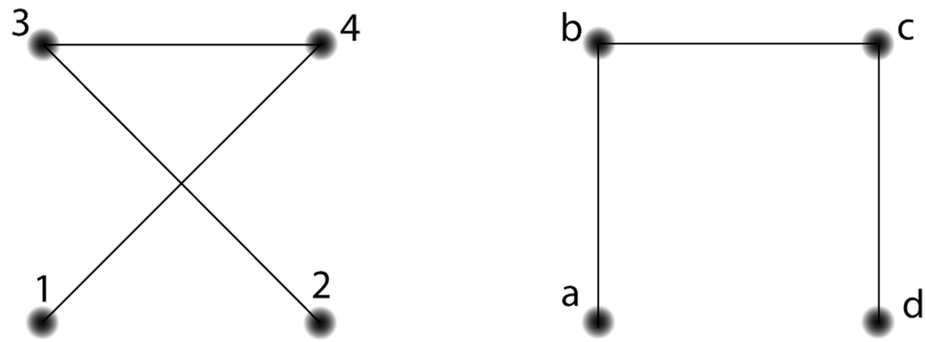
8 Check to see if this graph is still an accurate representation of the scheduling problem. Explain, in language a sixth-grader could understand, what you need to do to check if two graphs are isomorphic.

One final important concept that we will use is planarity. In the second version of the schedule graph, some edges crossed at places other than nodes. (For example, the edge connecting Animal Behavior to Shakespeare crosses the edge connecting Astro to Discrete.) In these cases, you can think of the edges passing under or over each other.

Graphs where you don't have to do this — where there is a way you can draw them without crossing the edges — are called planar.

9 Why do you think the word "planar" might be used?

Note that a graph might be drawn with the edges crossing, and yet be isomorphic to a graph that is clearly planar, as in the following two graphs (with pairing 1-d, 2-a, 3-b, 4-c):



In this case, we still say the first graph is planar because it can be drawn with no edges crossing and still represent the same information. Imagine picking up nodes 1 and 2, with their edges still attached, and "uncrossing" them to get something with the same shape as the second graph.

10 Draw a nonplanar graph.

11 Find some degrees of nodes in the Middle East graph.

a. What is deg(Saudi Arabia)?

b. What is deg(Syria)?

c. What country's node has the highest degree?

Practice

12 A shelter needs to find homes for nine cats. The table below indicates with an "x" which pairs of cats do not get along. Assuming that cats who do not get along cannot go with the same family, what is the minimum number of families needed to adopt all of these cats?

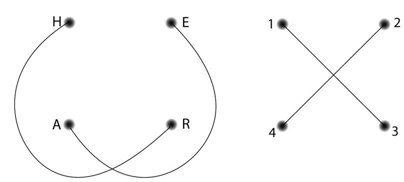
	A	B	C	D	E	F	G	H	I
A		x							
B	x		x		x				
C		x				x		x	
D					x		x	x	x
E		x		x		x	x		

F			x		x		x	x	
G				x	x	x		x	x
H			x	x		x	x		x
I				x			x	x	

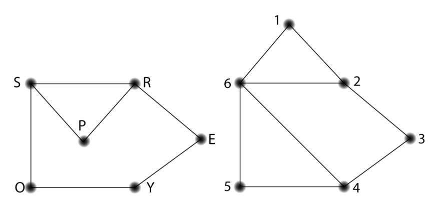
13 Which of these pairs of graphs are isomorphic? For graphs that are isomorphic, give the pairing of the nodes.

Now is a good time to think about visualizing the edges stretching and bending to see if you can make the shape on the left look like the shape on the right.

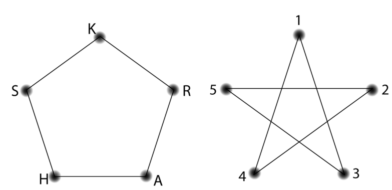
a.



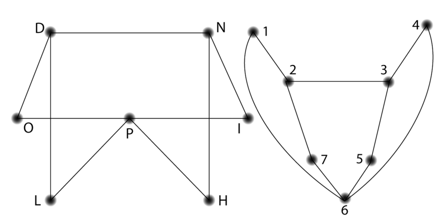
b.



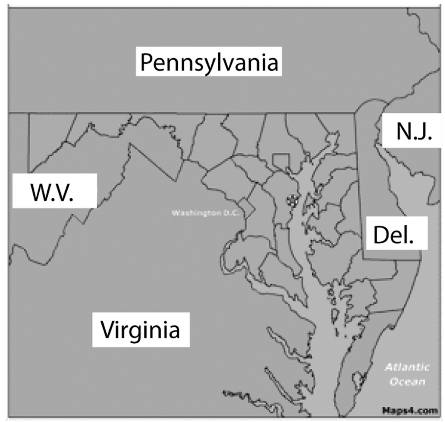
c.



d.

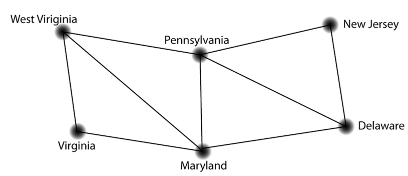


14 Below is a map of the region surrounding Maryland.
(image from http://www.maps4.com/maryland_map.gif)

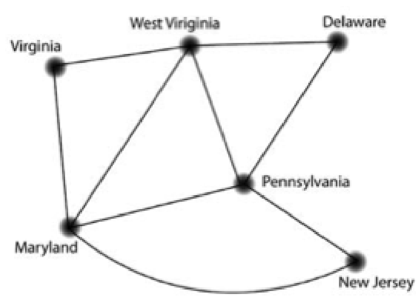


Which of the next three graphs is an accurate representation of the borders shared by the states on the map?

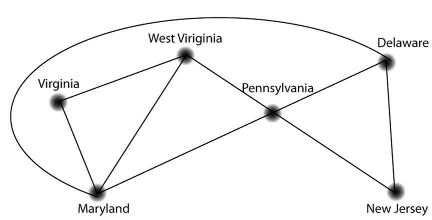
a.



b.

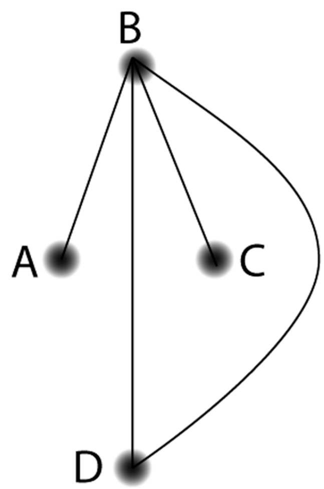


c.

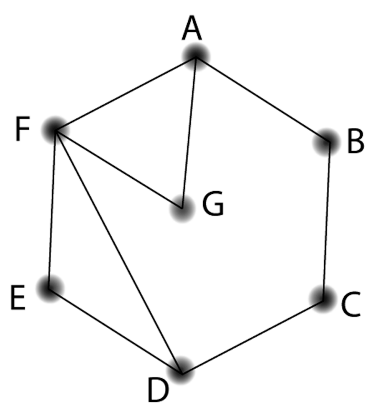


15 What's the degree of each node in each of these graphs?

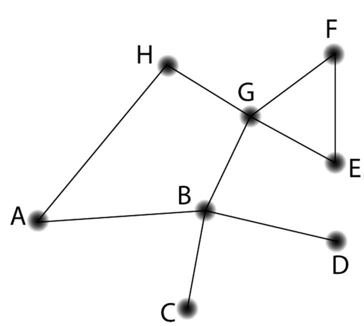
a.



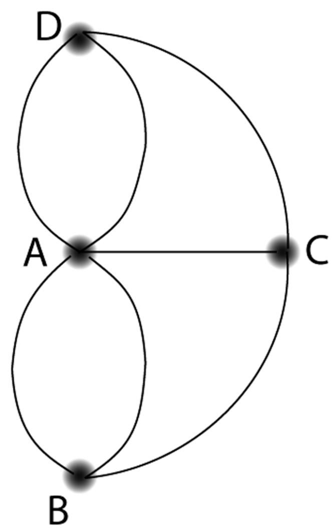
b.



c.

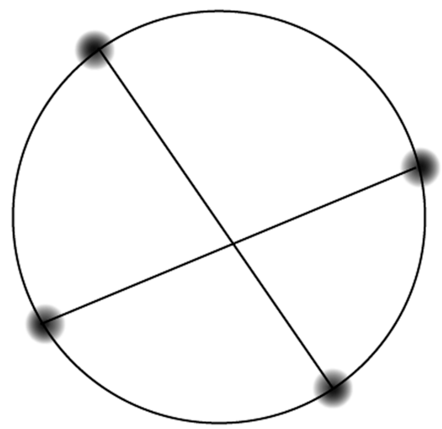


d.

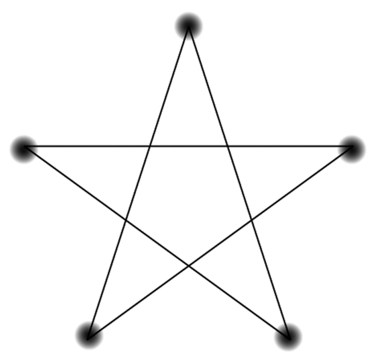

D
A
C
B

16 Which of these graphs are planar? For the ones that are planar, show how you can draw them with no edges crossing.

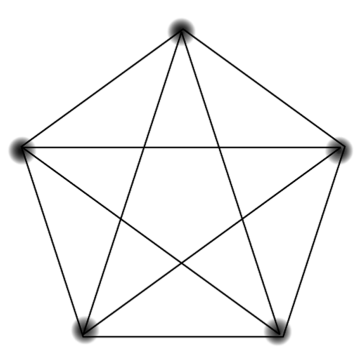
a.



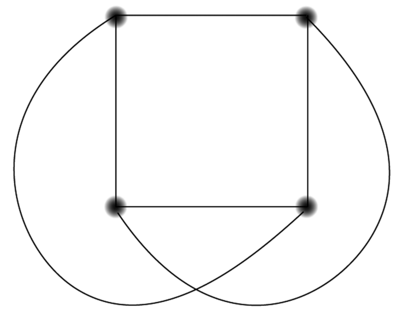
b.



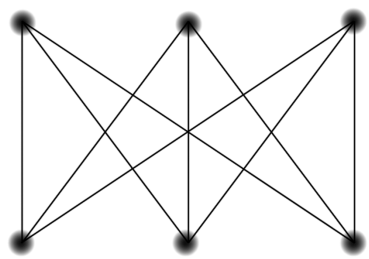
c.



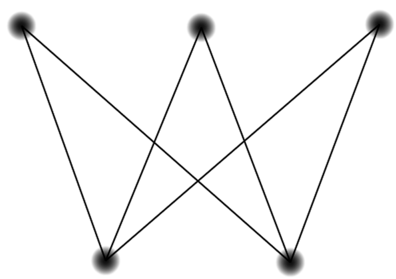
d.



e.



f.



17 A school has four exam periods. If one student is enrolled in two different courses, the exams for those courses need to be scheduled in different periods so that the student may take both exams. The following chart shows the pairs of courses having at least one student enrolled in both.

	Graphic Design	Philosophy	Architecture	Russian History	French	Statistics	Chemistry	20th Century Lit
Graphic Design		x		x		x		x
Philosophy	x			x		x		x
Architecture				x	x		x	
Russian History	x	x	x			x		x
French			x				x	
Statistics	x	x		x			x	x
Chemistry			x		x	x		
20th Century Lit	x	x		x		x		

a. Represent this situation as a graph.

b. Is it possible to create an exam schedule so that no student has to be in two places at once?

c. Find an exam schedule that uses four different exam periods.

Problems

18

(From Finkbeiner and Lindstrom) The Board of Directors of XYZ Corporation has 16 members and 6 committees of 5 persons each as indicated by the following chart.

Committee	Members
1	A, D, G, K, N
2	A, B, G, L, M
3	H, J, L, P, R
4	C, E, F, M, Q
5	C, J, K, N, R
6	D, E, H, P, Q

How many distinct meeting times are needed to conduct the committee meetings of the board?

19

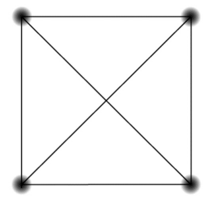
Draw a graph in which the total degree — that is, the sum of the degrees of each node — is odd.

20

The graph we used to represent bordering countries in problem 4 was planar. Create a map of your own countries for which the corresponding graph is nonplanar.

21

A graph is complete when every node is connected to every other node by an edge. Here's a complete graph with four nodes:



Draw:

a. A complete graph with five nodes.

b. A complete graph with six nodes.

c. A complete graph with three nodes.

22

How many edges will a complete graph with each of the following number of nodes have?

a. Five nodes

b. Six nodes

c. Three nodes

23

How would you calculate the number of edges of a complete graph with...

a. A hundred nodes?

b. n nodes?

24 Now that you know how the number of nodes and edges in a complete graph are related, investigate this relationship further.

a. Are the number of nodes and the number of edges in a complete graph related directly? That is, if you double the number of nodes, does the number of edges double, too?

b. Are the number of nodes and edges in a complete graph related linearly? Or does the number of edges go up "faster" or "slower" than in a linear relationship?

25 What's the minimum number of nodes needed for a graph with 1000 edges, if at most one edge can connect two nodes, and no edge can connect a node to itself?

26 If a complete graph has 500 nodes, can you estimate the number of edges quickly? Is it more likely to be roughly 12500, 125000, or 1250000 edges?

27 In the complete graph with four nodes, pictured earlier, the graph is drawn so that some of the edges crossed. However, just because it is drawn that way in that particular diagram does not mean the graph is not planar.

a. Is the complete graph with four nodes planar?

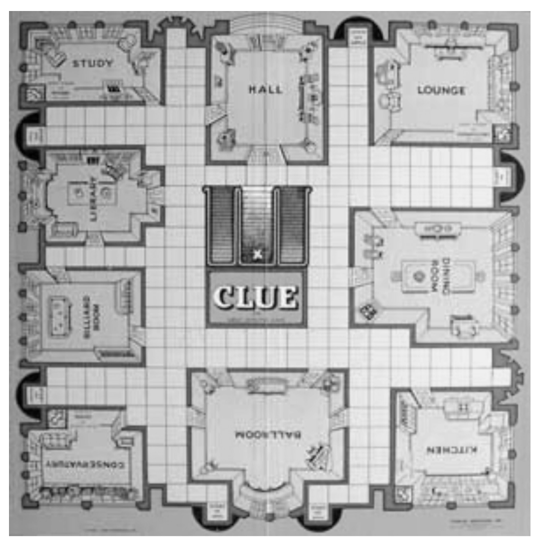
b. Is the complete graph with five nodes planar?

c. How about complete graphs with more than five nodes?

28 The symbol "Deg," as in the degree of a node, represents a function. What kinds of objects does Deg take as input? What kinds of objects does it output? By the way, the set of allowable inputs to a function is called its domain, and the set of allowable outputs is called its range.

For the next seven problems, it will be useful to change the representation of the problem using the tools of graph theory. The first question to ask yourself is always, "what should the nodes represent, and what should the edges represent?"

29 In this "Clue" game board, draw a graph in which you show which rooms can be reached from the others by the roll of a 6 on a die. Include the secret passages that you can take no matter what you roll from the Lounge to the Conservatory and from the Kitchen to the Study. If you roll a number higher than the number of spaces you need, you can still go in the room. By the way, is the Clue graph planar?



30

A group of six sophomores is going down to the Lower School to read to a group of six first-graders. The teachers asked for a list of preferences, both from the first-graders and the sophomores, as to who would read to whom.

(First grader)	**(would like to be read to by)**
Algernon	Xenophon, Yvonne, Ulysses
Bartleby	Ulysses, Vlad, Yvonne, Zaphod
Curio	Vlad, Winnifred, Zaphod
Duncan	Winnifred, Zaphod, Xenophon
Evangeline	Ulysses, Xenophon
Francine	Ulysses ,Vlad, Winnifred, Zaphod

(Sophmore)	**(would like to read to)**
Ulysses	Algernon, Bartleby, Evangeline, Francine
Vlad	Bartleby, Curio, Francine
Winnifred	Bartleby, Curio, Duncan, Evangeline, Francine
Xenophon	Algernon, Evangeline
Yvonne	Algernon, Duncan, Francine
Zaphod	Bartleby, Curio, Francine

Is there a way to form partners that will make everybody happy?

31

(From Epp) A traveler in Europe wants to visit each of six cities shown on this map exactly once, starting and ending in Brussels.
The distance (in kilometers) between each pair of cities is given in the table. Suggest a sequence of cities she can visit that minimizes the distance traveled.

	Berlin	**Brussels**	**Dusseldorf**	**Luxembou**
Brussels	783	--		
Dusseldorf	564	223	--	
Luxembourg	764	219	224	--
Munich	585	771	613	517
Paris	1,057	308	497	375

32 Wiki-Racing! The game of wiki-racing involves starting on a randomly generated Wikipedia page and taking Wikipedia links to navigate through until you reach a designated ending page. For example, two players on two different computers might agree to end on the page for "Fruit Bat." The first player to arrive at the Fruit Bat Wikipedia page using only the links on Wikipedia pages would win.

Below is a list of webpages in our miniature version of wiki-racing. Each webpage links to some, but not necessarily all, of the others.

Wikipedia Page	Links on that page
Toy Story	Walt Disney Pictures Pixar Animation Studios
Toy Story 3	Buzz Lightyear Toy Story
Pixar Animation Studios	Meet the Robinsons
Buzz Lightyear	Pixar Animation Studios Walt Disney Pictures Toy Story
Meet the Robinsons	Walt Disney Pictures Toy Story 3
Walt Disney Pictures	Pixar Animation Studios

a. How could you represent this situation using a graph?

b. What is the fewest number of clicks it will take to go from the Toy Story page to the Toy Story 3 page? From the Toy Story page to the Buzz Lightyear page?

33 Oh, no — your parents just announced that they're dropping by your apartment for a surprise visit. But your place is a mess. Fortunately, you've got some elves — as many as you need — to help you clean the place, but even an elf can't wash lights with darks. Here's what you and the elves have got to do, and the time it takes to do each thing:

- Wash your load of light-colored laundry (30 mins)
- Wash your load of dark-colored laundry (30 mins, and remember you've only got one washer)
- Dry your lights (45 mins, and you have to wash the lights first)
- Dry your darks (45 mins, you have to wash the darks first, and you've only got one dryer)
- Scrape off the dishes in your sink (10 mins)
- Load and run the dishwasher (60 mins, and you have to scrape first)
- Pick your stuff up off of the floor (20 minutes)
- Vacuum (10 minutes, and you have to pick up your stuff first)

a. Find a good graph-theory way to represent this problem. How long will it take you to put your apartment in order?

b. What is the minimum number of elves you need to do everything in that amount of time?

34

You have a 3 × 3 chessboard. The black knights are in the top two corners, and the white knights are in the bottom two corners. No other pieces are on the board. (Recall that the only moves knights are allowed to make is "2 up or down, 1 left or right," or "1 up or down, 2 left or right," but that they can "jump" pieces.)

a. How many moves will it take to switch the position of the knights, so that the black knights are now in the bottom two corners and the white knights are in the top two corners?

b. How many moves will it take just to switch the two rightmost knights?

35

Below is a map of the town of Königsberg, Prussia, as it existed in the 18th Century. A popular puzzle at the time was to cross all seven bridges without ever going back over a bridge you've crossed, and wind up at your starting point. Solve the puzzle yourself. (The mathematician Leonhard Euler invented graph theory by analyzing this problem.)

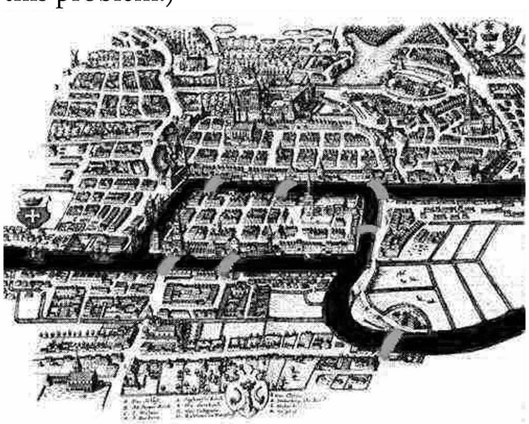
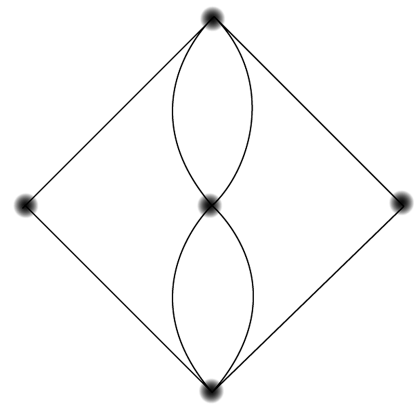


Image from solipsys.com.uk/new/KoenigsbergImages.html

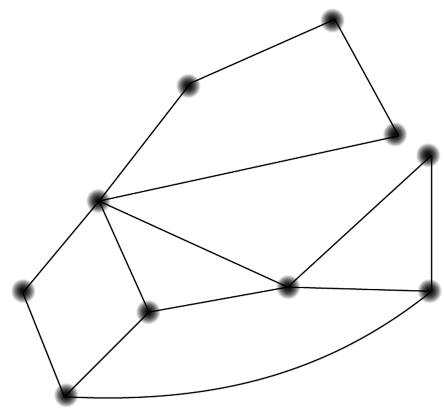
36

For each figure, say whether you can trace it with your pencil without going over any segment twice, ending at your starting point.

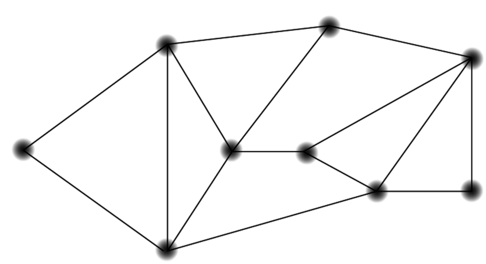
a.



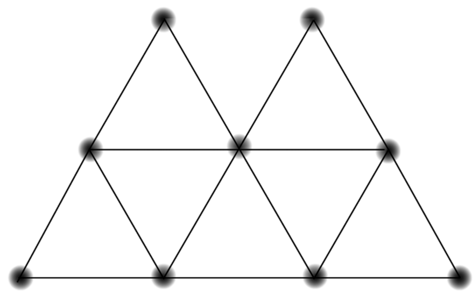
b.



c.



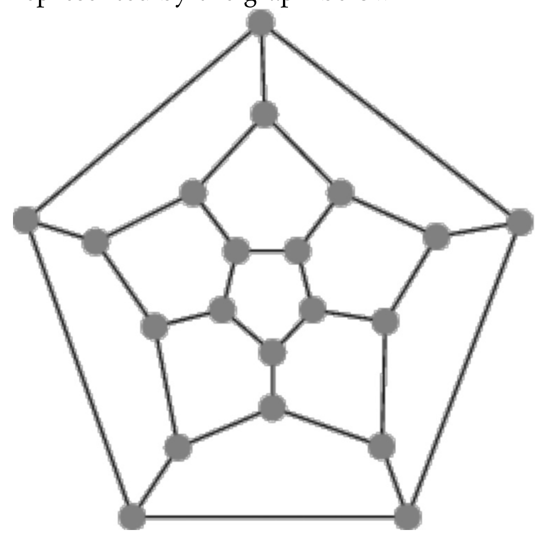
d.



37 A graph that can be traced in the manner of the previous problem is said to have an **Euler Circuit**, named after the mathematician who invented graph theory. Propose a method for determining if a given graph has an Euler circuit or not. (Are any of the concepts you've learned in this lesson relevant?) Make up new examples to test, if necessary.

38 Which complete graphs have Euler circuits?

39 The Irish mathematician William Rowan Hamilton invented a game in the 1850s called the Icosian Game. In it, you had to visit a list of cities connected by roads, as represented by the graph below.

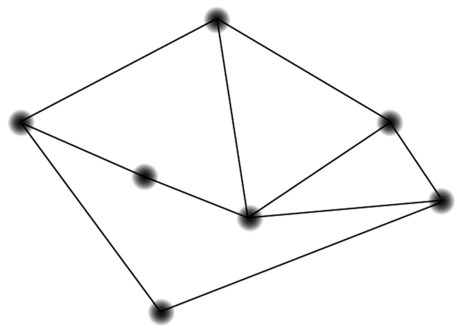


(image from mathworld.wolfram.com/IcosianGame.html)

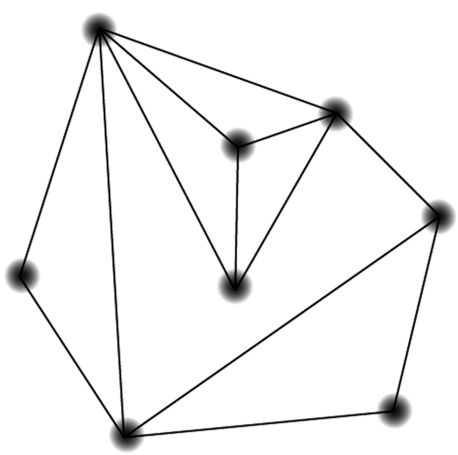
The challenge was to start in one city and follow the roads to visit all the cities, winding up at your starting city without ever having visited a city twice. Solve the Icosian Game.

40 If you read the previous problem carefully, you won't be surprised to learn that a circuit of the type you found — one where you need to visit each node exactly once and return to your starting node — is called a Hamiltonian circuit. Find Hamiltonian circuits for each graph below.

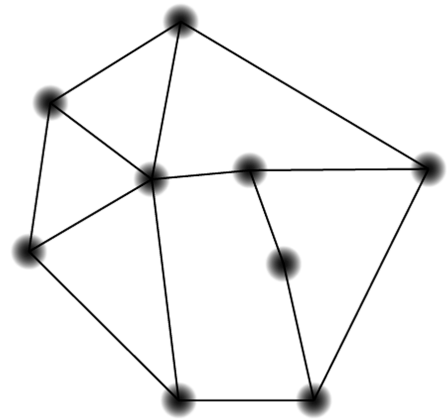
a.



b.



c.



41 Do you suppose that all graphs have Hamiltonian circuits?

42 Don't use a calculator for this problem.

a. Factor $x^2 + 8x + 16$

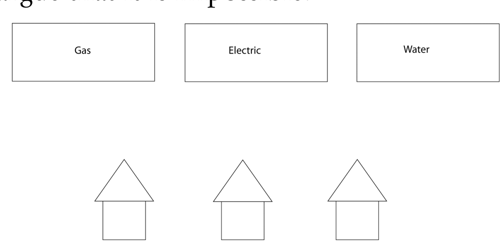
b. Factor $x^2 - 16$

c. Solve for x: $x^2 - 5x + 6 = 0$

d. Subtract $4\frac{7}{5} - 2\frac{1}{3}$

e. If $x^2 + y^2 = 30$ and $(x + y)^2 = 484$, what is xy ?

43 The diagram below shows a blueprint of three different utility companies and three houses. You want to add to the blueprint lines running from each utility company to each house, but the lines cannot cross one another. Suggest a way to do this or argue that it is impossible.



You may not have needed a graph to fully analyze the previous problem — but clearly, you could have drawn a graph to represent the situation. Graphs like the one in problem 43 that have two different groupings of nodes are called bipartite. In a bipartite graph, edges may only be drawn between nodes in two different groups (like house and utility); they never connect nodes in the same group. When you solved Problem 30, you also would have used a bipartite graph; however that graph was not a complete bipartite graph, whereas the graph in 43 is complete bipartite. Do you see why?

44

Draw a complete bipartite graph between two groups of two nodes. This graph is abbreviated $K_{2,2}$. Now draw $K_{2,4}$. Using this notation, what graph were you working with in Problem 43?

45

How many colors are needed to color the graph $K_{2,4}$? How about $K_{m,n}$?

46

Find a formula for the total degree of $K_{m,n}$.

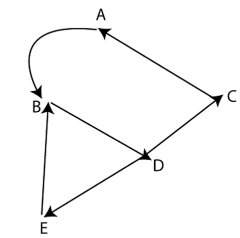
47

In the three-utilities puzzle (see problem 43), suppose now that lines are allowed to cross, and so the lines are actually run that way. If you accidentally cut one of the pipes without knowing which one it is, what's the probability that the gas in the leftmost house will go out? How about if there are m houses and n utilities, instead?

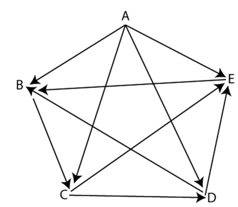
48

Any graph that could represent "who beat whom" in a situation where each team plays every other team once is called a tournament. Which of the graphs below are tournaments?

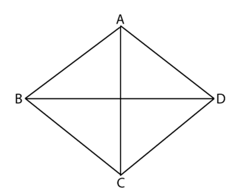
a.



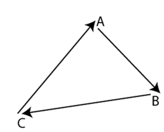
b.



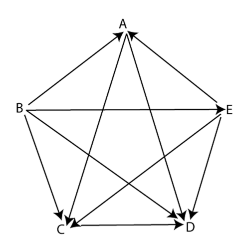
c.



d.



e.



49 The Game of Sprouts: Mark a few dots on a piece of paper. A move consists of two actions: connect two different dots with a single edge then place a dot at around the middle of this edge. You can only connect two dots if each has a degree of less than 3, and you may not repeat an edge or cross edges. The person who has no move is the loser.

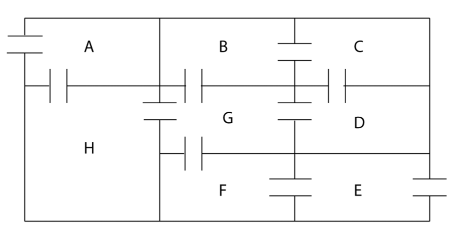
a. If you start with only two dots, then how many moves can the game last?

b. At the end of the two-dot sprouts game, what will be the total degree of the diagram?

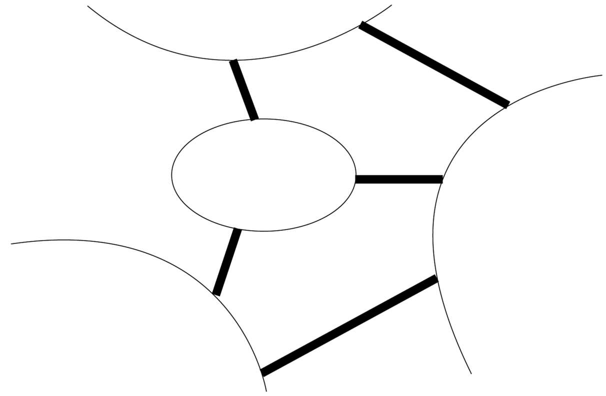
c. At the end of the *n*-dot sprouts game, what will be the total degree of the diagram?

d. How many moves will the *n*-dot sprouts game take?

50 The following is a floor plan of a house. Is it possible to enter the house in room A, travel through every interior doorway of the house exactly once, and exit out of room E?



51 The following diagram represents land connected by bridges. Can you walk these bridges without retracing your steps? (Though not necessarily winding up in the same place you started.)



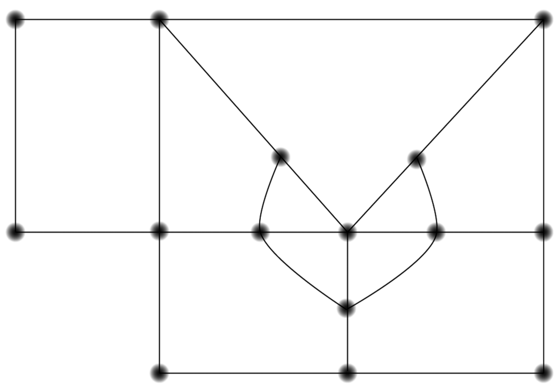
52 Propose a criterion by which you'd be able to tell if a graph has an Euler path. Does it matter which nodes are your starting and ending points? Remember to examine a similar problem by making up specific cases for yourself.

53 In problem 32 and the tournament problem (see problem 48), you drew a directed graph — a graph where the edges are one-way. The question of finding Euler circuits is more complicated on directed graphs. Do some experiments and propose criteria for whether a directed graph has an Euler circuit.

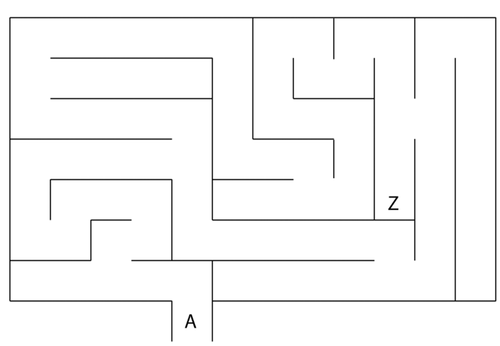
54 Does every tournament have a Hamiltonian path? What does it mean for the team rankings if there is more than one path through a tournament?

55 Come up with your own system to determine the winner of a tournament. It may help to create some of your own examples of tournaments.

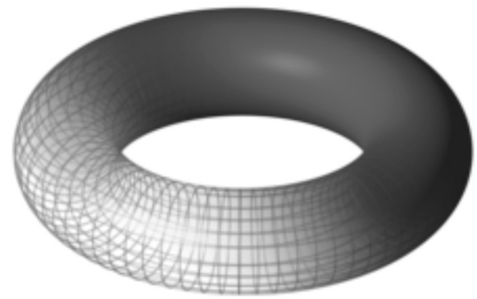
56 The following graph represents streets and corners; the portion of a street between two corners is a "block". The neighborhood association decides to place streetlights at some of the various corners so that the neighborhood never gets too dark at night. They want there to be enough streetlights so that every home is on a block that has a streetlight on at least one of its corners. Assume that there are homes between each pair of adjacent corners. Find the smallest number of street lights needed.



57 Use a graph to represent and solve the following maze.



58 The "doughnut" shape below is called a torus. You know you can't draw a complete graph with five nodes on a flat surface. But can you draw it on the surface of a torus?



59 If the blueprint for the houses and utilities (see problem 43) were printed on paper shaped like a torus, could you run the wiring with no overlaps?

LESSON 2: PRIMES PRODUCTS AND EXPONENTS

Development

As you learned last year, factoring a number means expressing it as the product of two or more other numbers. For example, 40 can be factored as $2 \cdot 20$. Of course, 40 can be factored in many other ways as well.

1 What are all the ways that 40 can be factored into two whole numbers?

Since 20 can be factored as $4 \cdot 5$, 40 can also be factored into 3 numbers as $2 \cdot 4 \cdot 5$; and since 4 can also be factored into $2 \cdot 2$, 40 can be factored into 4 numbers as $2 \cdot 2 \cdot 2 \cdot 5$. You might also think of saying it could be factored into 5 or 6 numbers as $1 \cdot 2 \cdot 2 \cdot 2 \cdot 5$ or $1 \cdot 1 \cdot 2 \cdot 2 \cdot 2 \cdot 5$, but since those aren't really different from $2 \cdot 2 \cdot 2 \cdot 5$, we will agree that 1 won't "count" as a factor in the following problems, even though technically 1 is a factor of every number. Thus, at this point, 40 can't be "broken down" any more than $2 \cdot 2 \cdot 2 \cdot 5$ (we are restricting ourselves to positive integers here, so we wouldn't consider turning 5 into $2.5 \cdot 2$ or into $-5 \cdot -1$).

2 Find two different ways to factor 72 into two positive integers. For each of your answers, continue factoring them until they can't be "broken down" any more. Finally, rearrange each of your final factorings so that the factors are in order from smallest to largest.

3 Factor 210 into the product of as many positive integers as possible. Compare your answer with a classmate's. Based on this question and on question 2, what generalization might you now conjecture?

A number has been factored as much as possible when each of its individual factors cannot be broken down any more. Numbers greater than 1 that cannot be broken down any more (i.e., cannot be factored into smaller positive integers) are called **primes**. We say that the **prime factorization** of 40 is $2 \cdot 2 \cdot 2 \cdot 5$, or $2^3 \cdot 5$ for short; we know it is a prime factorization because all of the factors (2, 2, 2, and 5) are primes. As suggested by questions 2 and 3, every positive integer greater than 1 has a unique prime factorization. As we will see, this uniqueness means that rewriting a number in its prime factored form will often be extremely useful, even though all we have done is written the same number in a different way.

Sometimes it can be tricky to know what the factors of a number are: that is, what divides into it evenly, leaving no remainder. For example, what numbers is 153 divisible by? It would clearly be helpful if one knew easy procedures (known as divisibility tests) that allowed one to see if smaller numbers divided arbitrary larger ones evenly. As a matter of fact, the divisibility tests for 2 and 5 are easy (what are they?), but it helps to know the tests for 3 and 11 as well (sadly, there is no easy test for 7).

If, when you add up the digits of a number, its sum is divisible by 3, then the number is divisible by 3 as well; on the other hand, if the sum is not divisible by 3, the number is not divisible by 3. So 14353 is not divisible by 3 (because $1+4+3+5+3=16$, which is not divisible by 3), while 55521 is. We will learn why the divisibility test for 3 is true later this chapter, but you might find it fun to ponder why it works on your own first!

Some of you may remember learning in 9th grade that a number is divisible by 11 only when the "alternating sum" is divisible by 11. So 4565 is divisible by 11 because is divisible by 11, as is 527494 because $+5-2+7-4+9-4=11$, but 6587 is not because $+6-5+8-7=2$ is not divisible by 11.

And now, a few problems to check your understanding of the above:

4 What is the largest number (other than 495 itself) that divides evenly into 495? Why do you know it's the largest number that can do so?

5 Is 4806 prime? How about 4807? Quickly name 5 numbers between 4800 and 4900 that are divisible by 11.

6 Determine the prime factorization of 2310.

7 Is 91 prime? How could you be sure of your answer? What about 221? Or 223?

8 Come up with a procedure that can determine, with reasonable efficiency, whether a number is prime or not.

Determining in general if a positive integer is prime (i.e., has no factors other than itself and 1) turns out to be an increasingly difficult task the larger the number becomes. In fact, one of the most powerful ways of encrypting data on the internet — the so-called "RSA algorithm" — is based on the difficulty of finding the prime factorization of very large (over 100 digits!) integers.

Prime factorization also allows us to determine the answers to questions about the relationships between two (or more) numbers. Remember that it can be helpful to write the final prime factorization using powers; for example, the prime factorization of 484 is $2^2 \cdot 11^2$.

9 What is the largest number that divides evenly into both 216 and 270? This number is called the greatest common divisor of the two numbers and is written as gcd(180, 156). (For example, gcd(24, 36) = 12, as 12 is the largest integer that "goes into" both 24 and 36.)

10 If your answer to question 9 involved testing out a lot of different numbers, try writing out the prime factorization of each original number (i.e. 216 and 270) first and see if you can use them to be more efficient in finding their gcd.

11 Find gcd(168, 448) efficiently. How can you be sure you have found the largest possible number that "works"?

12 Find gcd($2^{16} \cdot 3^4$, $2^{13} \cdot 3^8$), or put another way, gcd(5308416, 53747712). Looking at the original prime factorizations of each number, what do you notice about the prime factorization of your answer?

13 Create a procedure that allows one to efficiently determine the gcd of two positive integers. Check it with 3 examples that you make up, where at least one of the examples is similar to problem 12 (i.e., where the two numbers are large but have simple prime factorizations).

Crucially, then, a prime factorization is important because all the factors of a number derive from the number's prime factors. 24, for example, is $2^3 \cdot 3$, and all of its factors — 2, 3, 4, 6, 8, 12, and 24 — are made up of combinations of the 2's and 3's found in the prime factorization.

14

In the following exercises, learn how powerful prime factorization is by solving them without your calculator!

a. How can you check to see if $32 \cdot 35$ equals $28 \cdot 40$ without determining either product?

b. Juniper says that when she multiplies $6 \cdot 24 \cdot 55$ it is equal to $21 \cdot x$, where x is an integer. Explain if she is telling the truth or not.

c. Simplify $\frac{64 \cdot 25 \cdot 14 \cdot 33}{28 \cdot 44 \cdot 15}$ as much as possible.

d. What is the positive integer N for which $22^2 \times 55^2 = 10^2 \times N^2$? Copyright *www.mathleague.com.*

e. Is $6^4 \cdot 10^3$ greater than, less than, or equal to $8^2 \cdot 15^3 \cdot 6$?

The last few problems also are a reminder of the laws of exponents that you learned in middle school. Since 5^6 is just $5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5$ and 5^4 is just $5 \cdot 5 \cdot 5 \cdot 5$, it makes sense that

$$\begin{aligned} 5^6 \cdot 5^4 &= (5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5) \cdot (5 \cdot 5 \cdot 5 \cdot 5) \\ &= 5^{10} \end{aligned}$$

Similarly, because $\frac{5^6}{5^4}$ is just $\frac{5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5}{5 \cdot 5 \cdot 5 \cdot 5}$, it makes sense that some pairs of 5's in the numerator and denominator would "cancel" (because $\frac{5}{5} = 1$) and thus

$$\frac{5^6}{5^4} = \frac{5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5}{5 \cdot 5 \cdot 5 \cdot 5}$$
$$= \frac{5 \cdot \cancel{5} \cdot \cancel{5} \cdot \cancel{5} \cdot \cancel{5} \cdot 5}{\cancel{5} \cdot \cancel{5} \cdot \cancel{5} \cdot \cancel{5}}$$
$$= 5^2.$$

Lastly, $(5^4)^3$ is just $(5 \cdot 5 \cdot 5 \cdot 5)(5 \cdot 5 \cdot 5 \cdot 5)(5 \cdot 5 \cdot 5 \cdot 5)$, so $(5^4)^3 = 5^{12}$.

15

By using the reasoning of the previous paragraph, finish the following equations, where x is the base, and a and b are exponents: $x^a \cdot x^b =$ $(x^a)^b =$

16

For the second law in question 15, what are you assuming about a and b ? Try different values of a and b to clarify your answer.

17 Simplify the following:

a. $x^{13} \cdot x^7$

b. $\frac{3^{28}}{3^{24}}$

c. $\left(x^5\right)^6$

d. $(\frac{x^8 \cdot x^9}{x^6})^2$

18 What do you think 3^0 should be equal to? Why? Come up with a specific argument to defend your view (and don't use your calculator!).

19 Let's explore the expression a bit more.

a. If $3^4 = \frac{3^p}{3^q}$, then what are possible pairs of values for p and q? Give at least 3 pairs.

b. Using similar reasoning as in part a, what would it appear 3^0 should be equal to? What about 497^0?

c. Which answer do you trust more, your answer to question 18, or your answer to part b above? Explain.

d. Finally, make a chart of the powers of 3, starting at 3^5 and going down to 3^1. Looking at this chart, what do you think should be?

e. Based on part d and your own conclusions, can you revise your answer a little to question 18?

20 What do you think 6^{-3} should be equal to? Is your answer the same as $-(6^3)$? Again, come up with a specific argument to justify your answer.

21

Let's look a bit deeper into the idea of negative powers.

a. Using similar reasoning as in question 19a, what would it appear 6^{-3} should be equal to? (Write your answer as a reduced fraction.) Does your answer equal what you thought in question 20?

b. Make a chart of the powers of 6, starting at 6^5 and going down to 6^0. Following this idea, what do you think 6^{-1} would be? How about 6^{-2} and 6^{-3}? Again, write your answers as fractions, rather than using decimals.

c. Based on parts a and b and your own conclusions, can you revise your answer to question 16 even more than you did in question 19e?

22

What should x^6 be multiplied by to equal x^2? What should x^6 be divided by to equal x^2? Test that your answers work with a specific value of x, and then explain how the answers are related to each other.

23

Rewrite $\frac{3^{-5}}{3^6}$, $\frac{2^{-8}}{2^{-3}}$, $\frac{x^{-6}}{x^{-4}}$ and $\frac{x^{-7}}{x^{-11}}$ so that your final expressions have only positive exponents.

24

Simplify $\frac{x^8x^5}{x^3x^7}$, $\left(\frac{x^8x^5}{x^3x^7}\right)^4$, and $\left(\frac{x^8x^5}{x^3x^7}\right)^{-4}$. Then write each of your "simplified" answers with only positive exponents, if they aren't already.

Practice

25

Without using a calculator, find all the prime factors of 99792 that you can. Then, using a calculator, find the complete prime factorization of 99792.

26

Find gcd(1240, 4400). If you wish, you can express your answer not as a number, but as the prime factorization of that number.

27 Determine if $\frac{60480}{2268}$ is an integer without using a calculator!

28 What values of x and y satisfy $36 \times 5^x = 225 \times 4^y$?
Copyright *www.mathleague.com.*

29 Arrange from biggest to smallest: 3^{-3}, $(-3)^3$, 2^{-2}, $(-2)^2$, $(-3)^{-3}$, $(-2)^{-2}$.

30 Simplify $\frac{2^7 \cdot 2^5}{2^4 \cdot 2^6}$, $\left(\frac{2^7 \cdot 2^5}{2^4 \cdot 2^6}\right)^3$, and $\left(\frac{2^7 \cdot 2^5}{2^4 \cdot 2^6}\right)^{-4}$ so that each answer is extremely simple: 2 raised to an integer power. Don't use a calculator.

31 Can $3^4 \cdot 2^5$ be written in the simpler form a^b (where a and b are integers and $b \neq 1$)? If yes, check your answer by trying it out with specific numbers a and b just to be sure.

Going Further

32 Do the laws of exponents continue to hold when dealing with two variables? Let's check this out.

a. Does $(xy)^5 = x^5y^5$? Try testing this "rule" with specific numbers, and, if it seems to work, see if you can prove the rule using algebra.

b. Generalizing from part a, what would $(xy)^a$ be equal to?

c. Using similar reasoning, how could one rewrite $\left(\frac{x}{y}\right)^a$?

d. What would $(xy)^8(xy)^4$ equal? How about $(xy)^3(xy)^{-7}$?

e. Following the pattern in part d, how could you rewrite $(xy)^a(xy)^b$?

f. Can you rewrite $\frac{(xy)^a}{(xy)^b}$ as well?

g. How about $(x+y)^a$— does it simplify easily? How can you check your answer?

33 Now let's apply what we've learned in question 32.

a. . Can $\frac{x^5y^7}{y^3x^2}$ be simplified? How about $\left(\frac{x^5y^7}{y^3x^2}\right)^3$? Finally, howzabout $\left(\frac{x^5y^7}{y^3x^2}\right)^{-6}$?

b. Simplify $\frac{2^3 \cdot 3^4}{2^5 \cdot 3}$ and $\left(\frac{2^3 \cdot 3^4}{2^5 \cdot 3}\right)^{-2}$ so that your answers are in the form 2^x3^y, where x and y are integers.

c. Simplify $\left(\frac{4x^{-3}y^7}{x^8y^3}\right)^{-2}$ and $\left(\frac{x^8y^3}{4x^{-3}y^7}\right)^2$. Are you surprised by the two answers?

d. Rewrite $\left(\frac{kx^ay^b}{x^cy^d}\right)^e$ so that your answer is in the form $k^Px^Qy^R$, where P, Q, and R are integers that you determine.

34

Here are a couple more doozies based on what you learned in question 33. Don't use a calculator!

a. What simple number is equivalent to $\left(3^2 \cdot 7^4\right)^4 \left(5^8 \cdot 7^{-5}\right)^3 \left(3 \cdot 5^4\right)^{-6}$?

b. What is $\left(\frac{16g^6}{9^6 t^{-5}}\right)^{-2} \cdot \left(\frac{3^6 t^{-3}}{2^2 g^3}\right)^4$, simplified?

35

You know from the laws of exponents that , or that $x^a x^b = x^{a+b}$. We can use this identity to help us think of square roots in a different way — as an exponent.

a. What does $(\sqrt{x})(\sqrt{x})$ equal?

b. So if $\sqrt{x} = x^N$, what must N be?

c. If $p \cdot p \cdot p = x$, what must p equal, in terms of x? Put another way, if we say that $p = x^N$, what does N equal? (This number is called a "cube root" of x, i.e., since $2 \cdot 2 \cdot 2 = 8$, 2 is a cube root of 8. This can also be written as $2 = \sqrt[3]{8}$, where the $\sqrt[3]{}$ symbol indicates "cube root".)

d. If $p^5 = x$, what would p equal in terms of x? That is, if we say that $p = x^N$, what does N equal?

e. If $p = \left(\frac{1}{32}\right)^{\left(\frac{1}{5}\right)}$, what would p be? If $q = \sqrt[5]{\frac{1}{32}}$, what is q ?

f. In general, then, if $\sqrt[n]{x} = x^N$, how are N and n related? Give a specific example or two to clarify.

36 Using what you learned in question 35, simplify each expression as far as possible. No calculator necessary, although feel free to check!

a. $\left(16^{\frac{1}{2}}\right) + \left(16^{\frac{1}{4}}\right)$

b. $64^0 + 64^{\frac{1}{3}} + 64^{\frac{1}{2}} + 64^1$

c. $\left(\sqrt[3]{241}\right)\left(241^{\frac{1}{3}}\right)\left(\sqrt[3]{241}\right)$

d. $\left(81^{\frac{1}{4}}\right)^{-2}$

e. $125^{\frac{1}{3}} \cdot 36^{-\frac{1}{2}}$

37 You now know how to calculate 64^0, $64^{\frac{1}{3}}$, $64^{\frac{1}{2}}$, 64^1 and 64^2. Let's broaden our horizons even more!

a. Suppose a friend told you that he had just figured out how to calculate $64^{1.5}$. Based on what you already know about powers of 64, about how big a number do you think this is? Explain.

b. $64^{1.5}$ can also be written as 64^N, where N is a simple fraction. What is N?

c. Now further rewrite 64^N by using the property of exponents that tells us we can rewrite x^{12} as $\left(x^3\right)^4$.

d. Using your result from part c, you should now be able to compute a precise answer to $64^{\frac{3}{2}}$ $(= 64^{1.5})$. What is it, and does it agree with your answer in part a?

e. Similarly, what would $64^{\frac{2}{3}}$ $(= 64^{.\overline{666}})$ equal? How about $4^{2.5}$?

f. Finally, what is $7^{1.3}$? Explain what it means to raise a number to the 1.3 power. Once you've done that, do the same for $7^{1.29}$.

38 One can also solve equations quite easily when they contain powers. For example, $x^3 = 11$ can be solved by taking advantage of the laws of exponents and raising each side to the 1/3 power: $\left(x^3\right)^{\frac{1}{3}} = 11^{\frac{1}{3}}$, the idea being that because the exponents on the left side multiply, the exponent becomes "1", and so it simplifies to $x = 11^{\frac{1}{3}}$ (or $\sqrt[3]{11}$). Solve the following equations for x, keeping in mind the strategies in the paragraph above:

a. $x^5 = 32$

b. $x^{\frac{1}{5}} = 32$

c. $3x^3 = 81$

d. $\frac{16}{64}x^5 = 8$

e. $x^{\frac{2}{3}} = 9$

f. $x^{2.71} = 126$

g. $5x^{3.87} = 1000$

Practice (No Calculators!!)

39 Simplify $\left(x^{-17}x^8\right)^{-3}$ in two different ways:

a. By simplifying inside the parentheses first, and then applying the outside exponent.

b. By applying the outside exponent first, and then simplifying.

c. Confirm that your answers in parts a and b are the same.

40 Simplify $\left(\frac{p^9q^{-5}}{q^7p^4}\right)^3$ and $\left(\frac{p^9q^{-5}}{q^7q^4}\right)^{-3}$. What do you notice?

41 Does $\left(\frac{w^6t^7}{t^{-8}}\right)^4$ equal $\left(\frac{w^3t^{20}}{w^{-5}}\right)^3$? Explain.

42 Don't use a calculator for this problem.

a. Divide $\frac{10}{3} \div \frac{5}{2}$.

b. Factor $4x^3 - 12x^2$.

c. Factor $x^2 - 25$.

d. Solve for x: $x(x+3) - 1 = -3$.

e. Solve for x, looking up the quadratic formula if necessary:
$x^2 - 4x - 6 = 0$.

43 What is $\frac{(m^{-4}n^{12})^{-5}}{(n^{-6}m^3)}$ simplified?

44 Simplify $\left(\frac{12x^{-2}y^5}{8x^{-6}y^9}\right)^3$.

45 The volume of a cube is 2197 cm^3. Find the total surface area of the cube. (Calculators allowed.)

46 Write $5^5 \cdot 5^8 \cdot 8^5 \cdot 8^8$ in the form a^b.

47 What is $\sqrt{17^2}$? What is $\sqrt[3]{5^3}$?

48 Which is bigger: $\sqrt[3]{37^2}$ or $(\sqrt[3]{37})^2$? Why?

49 What does $8^{\left(\frac{4}{3}\right)} \cdot 4^{\left(\frac{3}{2}\right)}$ equal?

50 Simplify $\sqrt[3]{\sqrt[4]{x^{24}}}$.

51 Write $25^4 \cdot 125^{-2}$ as a power of 5.

52 Solve $2r^4 = 162$.

Problems (Still No Calculators!)

53 How many prime numbers under 10000 have digits that add up to 9?

54 If the sum of two prime numbers is 999, what is their product?
Copyright *www.mathleague.com.*

55 Explain why the product of any three consecutive integers has to be divisible by 6.

56 The dimensions of a rectangular box (in cm) are all positive integers, and the volume of the box is 2002 cm^3. What is the least possible sum of the three dimensions?

57 (Calculators allowed.) What is the smallest positive integer that is NOT a factor of $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times \ldots \times 18 \times 19 \times 20$?
Copyright *www.mathleague.com.*

58 Does $6^6 + 6^6 + 6^6 + 6^6 + 6^6 + 6^6$ equal 36^6? 6^{36}? 6^7? 36^{36}? Or something else in the form a^b? Explain.
(Appeared on AMC-12 competition, 1992)

59 Express $\frac{2^1+2^0+2^{-1}}{2^{-2}+2^{-3}+2^{-4}}$ as a single, simple fraction. (Appeared on AMC-12 competition, 1987)

60 When 270 is divided by the odd number X, the answer is a prime number. What is X?
Copyright *www.mathleague.com.*

61 What is the simplest expression for $\frac{2^{40}}{4^{20}}$?
Copyright *www.mathleague.com.*

62 What is the simplified value of $\frac{4444^4}{2222^4}$?
Copyright *www.mathleague.com.*

63 If you double x, by what factor does the expression $\frac{x^7 x^{-2}}{x^3}$ grow?

64 A composite number is a whole number that is greater than 1 and not prime — that is, it has factors other than itself and 1.

a. What is the smallest composite number that is not divisible by 2 or 3?

b. What is the smallest composite number that is not divisible by 2, 3, or 4?

c. What is the smallest composite number that is not divisible by 2, 3, 4, or 5?

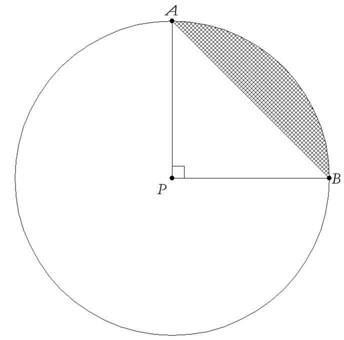
d. (Calculators allowed.) Now determine the smallest composite number that is not divisible by 2, 3, 4, 5, 6, 7, or any of the numbers up to and including 100.

65 Order these, from smallest to largest, without using a calculator: $1000^{\frac{1}{1000}}$, $\left(\frac{1}{1000}\right)^{1000}$, 1000^{-1000}, $\left(\frac{1}{1000}\right)^{\frac{1}{1000}}$, $\left(\frac{1}{1000}\right)^{-1000}$

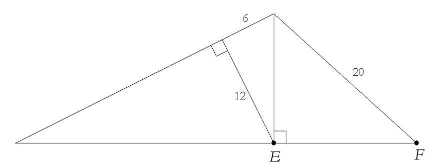
66 You are going to randomly choose three integers (repetition allowed) from 0 to 5, and call them k, m, and n. What's the probability that $2^k3^m5^n$ is NOT divisible by 5?

67 What is the only number x which satisfies $\sqrt{1992} = 1992\sqrt{x}$?

68 In circle P, the length of $\overline{AB}$ is $\sqrt{50}$. Find the area of the shaded region.



69 Find EF exactly.



70 Triangle ABC has a right angle at C. If sin B = $\frac{2}{3}$ what is tan B, expressed as a fraction?

71 Why must $\sqrt{17}$ be a non-terminating decimal? Put another way, why can't a terminating decimal multiplied by itself = 17?

72 What is the positive number x for which and $\sqrt{y} = 8$?
Copyright *www.mathleague.com.*

73 If $x^{64} = 64$, what is the exact value of x^{32}?
Copyright *www.mathleague.com.*

74 In simplest form, what is the numerical value of $\sqrt{1985}\ \sqrt[3]{1985}\ \sqrt[6]{1985}$?
Copyright *www.mathleague.com.*

75 Which kind of number does not appear to have a 10th root? Which kind of number has an integer as its 10th root?

76 If $\sqrt[5]{x} = 4$, what is the value of $\sqrt{x}$?
Copyright *www.mathleague.com.*

77 What does $\sqrt{7}\sqrt{3}\sqrt{2}\sqrt{5}\sqrt{10}\sqrt{21}$ equal? Think before calculating!!

78 Rewrite $\left(\frac{1}{4}\right)^{-\frac{1}{4}}$ as the root (i.e. square root, cube root, and so on—you choose !) of an integer that is NOT 4.

79 What is the value of x which satisfies $\sqrt[3]{x\sqrt{x}} = 4$?
Copyright *www.mathleague.com.*

80 If $x \geq 0$, then $\sqrt{x\sqrt{x\sqrt{x}}}$ is equivalent to x^N. What is N?:

81 The function s takes a number and outputs the sum of its “proper” divisors, meaning that 1 is considered to be a divisor but the number itself is not.

a. Find $s(8)$.

b. A perfect number is a number n for which $s(n) = n$. How many perfect numbers are between 2 and 10?

c. An abundant number is a number n for which $s(n) > n$. Find the first abundant number.

d. A pair of numbers is said to be amicable when the proper divisors of each number add up to the other number.

e. Verify that 220 and 284 are an amicable pair.

f. Write the definition of amicable numbers using the $s(n)$ notation.

Exploring in Depth (No Calculators Still!)

82 Which is bigger, 2^{3000} or 3^{2000}? Prove your answer.
Copyright *www.mathleague.com.*

83 Write $\frac{15^{30}}{45^{15}}$ in the form a^b, if possible. If it is not possible, explain why not.
(Appeared on AMC-12 competition, 1993)

84 If $3^x = 5$, what is the value of $3^{(2x+3)}$?
Copyright *www.mathleague.com.*

85 What is the integer n for which $5^n + 5^n + 5^n + 5^n + 5^n = 5^{25}$?
Copyright *www.mathleague.com.*

86 Take any 6-digit number that repeats 3 digits twice in the same order, like 596596. It will always be divisible by 7, 11, and 13. Why? (Calculators allowed.)

87 How many distinct pairs of positive integers (m, n) satisfy $m^n = 2^{20}$?
Copyright *www.mathleague.com.*

88 How many positive integers less than 50 have an odd number of positive integer divisors?
(Appeared on AHSME 41 competition)

89 If $x > y > 0$, then express $\frac{x^y y^x}{y^y x^x}$ using only a single exponent.
(Appeared on AMC-12 competition, 1992)

90 Earlier, you learned how to calculate a number such as $8^{\frac{2}{3}}$.

a. In fact, there are two different plausible ways of evaluating a number like $8^{\frac{2}{3}}$: as $\left(8^{\frac{1}{3}}\right)^2$ or as $\left(8^2\right)^{\frac{1}{3}}$. Are these two numbers in fact equal? Explain why or why not.

b. Try calculating both $27^{\frac{4}{3}}$ and $32^{\frac{6}{5}}$ in each of the ways described in part a. Which is easier, and why?

91 Multiplying and dividing numbers and/or expressions that have roots and/or exponents in them becomes considerably easier if all the roots are converted to exponents. Also often helpful in harder problems is to try to give all the exponents the same base. For example, the easiest way to simplify $2^5 \cdot 8^4$ is to think of "8" as "2^3":

$$\begin{aligned} 2^5 \cdot 8^4 &= 2^5 \cdot \left(2^3\right)^4 \\ &= 2^5 \cdot 2^{12} \\ &= 2^{17}. \end{aligned}$$

Simplify the following:

a. $4^3 \cdot 64^5$

b. $2^{-4} \cdot 32^2$

c. $\frac{3^5 \cdot 18^4}{16}$

d. $7^{\frac{5}{6}} \cdot 49^3$

e. . $\frac{81^{\frac{3}{8}}}{27^{\frac{1}{2}}}$

92 (Calculators allowed.) In addition to the greatest common divisor, another interesting numerical concept is called the least common multiple.

a. What is the smallest positive integer that is a multiple of 30 and also a multiple of 42? This number is called the "least common multiple" and is written as lcm(30, 42). (For example, lcm(6, 8) = 24, as 24 is the smallest number that is both a multiple of 6 and a multiple of 8.)

b. If your answer to part a involved testing a large amount of numbers, try using the prime factorizations of the numbers to help you see how to find the lcm efficiently.

c. Find lcm(84, 126) efficiently. How can you be sure you have found the smallest possible number that works?

d. Find lcm($2^{16} \cdot 3^4$, $2^{13} \cdot 3^8$). Looking at the original prime factorizations of each number, what do you notice about the prime factorization of your answer?

e. Create a procedure that allows one to efficiently determine the lcm of two numbers. Once again, check with 3 examples you devise, with at least one of the examples similar to part d.

93 Take any two positive integers x and y. (Calculators allowed.)

a. Find their gcd and lcm.

b. Multiply the gcd and lcm together.

c. Now multiply the original numbers x and y together. What do you notice?

d. Try parts a through c with 5 different pairs of positive integers x and y.

e. What is going on here? Explain carefully in terms of the prime factorizations of x and y. (Hint: Go back to questions 13 and 92e and look at how you were able to find the gcd and the lcm efficiently, and compare.)

94 Sometime around 5th grade, you were taught to add fractions $\frac{a}{b}$ and $\frac{c}{d}$ by finding a common denominator. Assume b and d are positive.

a. Explain whether you should be finding the gcd or the lcm of b and d if you want to find their smallest possible common denominator.

b. By the way, why do you need a "common" denominator to add fractions anyway? Why not just add them using different denominators?

95 If $x = \sqrt{2000}$ and $y = \sqrt{2001}$, what is the simplified numerical value of $(x + y)^2 + (x - y)^2$?
Copyright *www.mathleague.com.*

96 What on earth could $4^{\sqrt{2}}$ mean? Could you determine approximately how big it would have to be?

97 Let's learn about how simplest radical form directly relates to prime factorization. (Calculators allowed.)

a. Put $\sqrt{99}$ and $\sqrt{884}$ into simplest radical form.

b. Explain, for an arbitrary integer n, how you would go about putting $\sqrt{n}$ into simplest radical form.

c. Now try simplifying $\sqrt{907}$. What makes you confident that you are definitely in simplest radical form?

d. Explain why you don't have to test any numbers above 31 when checking to see if 907 is prime.

e. Now try simplifying $\sqrt{529}$, $\sqrt{551}$, and $\sqrt{557}$.

f. Given what you learned in parts c through e, can your answer in part b be improved or clarified? Are you confident your method will efficiently put $\sqrt{n}$ into simplest radical form? Explain.

g. Try your method on $\sqrt{343}$, $\sqrt{379}$, $\sqrt{403}$, $\sqrt{765}$, $\sqrt{1517}$, $\sqrt{1373}$, and $\sqrt{1763}$.

98 Let's take a look at a proof the Greek mathematician Euclid came up with over 2000 years ago to show that there are an infinite number of primes. Euclid did this by assuming that there were a finite number of primes, and then showed that assuming that is the case turns out to be self-contradictory. Follow along to see his "moves"!

a. If there were a finite number of primes, we could call the largest one "P". Euclid then asked us to consider "Q", the number that has a prime factorization that includes each of the primes exactly once. Explain clearly how one could calculate "Q".

b. Euclid then made his biggest "move". Remembering that all the primes divide evenly into "Q", Euclid asks us to consider the number $Q+1$. Which of the primes do you think would go evenly into$Q+1$ as well? Why?

c. What can you conclude about $Q+1$, now that you have established that it has no prime factors?

d. Why does your conclusion contradict what you assumed at the start of the problem?

LESSON 4: COMBINATORICS INDISTINGUISHABLE ORDERINGS

Introduction: Hey batta! Hey batta!

Jan bats .500 in softball — i.e., on average, for each 10 at-bats she gets five hits. We wish to determine the probability that she will get no hits if she comes to bat 5 times during a game. In order to calculate this probability we must assume that she stays a ".500 hitter" from one at-bat to the next, which is to say that success or failure at each at-bat is not affected by the previous at-bats. Given this assumption we can perform the following calculation:

1. What is the probability that Jan will get exactly one hit in 5 at-bats?

2. What's the probability that Jan will get exactly 4 hits in 5 at-bats?

3. What's the probability that Jan will get exactly 2 hits?

The last question probably took some time to calculate since there are 10 different ways for Jan to get exactly 2 hits. For example, one string of at-bats might look like HNHNN, where H = "hit" and N = "no hit". Imagine trying to calculate the probability that Jan would get exactly 3 hits in 10 at-bats. Don't do this by listing all the strings of 10 at-bats with 3 hits. You will be cursing by the end of this process, if not earlier. Fortunately, there is a more elegant way of counting how many strings have exactly 3 hits in 10 at-bats. Let's look at one approach.

Development

This approach will require us, first, to review two basic counting principles that you learned last year. Suppose you're setting up a password for a new account.

4 If the password must be 3 letters followed by 3 digits, then how many possible passwords are there? The password is not case-sensitive so there's no distinction between "a" and "A".

5 If the password must be 3 letters followed by 3 digits, but no letter or digit can be repeated, then how many passwords are there? Again, the passwords are not case-sensitive.

6 If the password must be 10 digits, no letters, with no repetition, then how many passwords are there? There's a nice shorthand way of writing this answer. Do you remember it?

The problems above involve a basic principle of counting called the Multiplication Principle of Counting. You also needed to pay attention to whether repetition mattered. Lastly, Problem 6 alludes to some shorthand notation that comes in handy when the final count gets very large. Keep in mind that this is only notation and doesn't solve problems for you. As you progress through this lesson, you'll need to remember that the formulas you might derive are not all-purpose ones that solve counting problems. They tend to work in highly specialized situations. You will find more success if you read each problem carefully, think about what is being counted (visualize, even), and then apply basic principles like the Multiplication Principle of Counting. Now, let's see how to tackle the batting problem by looking at a similar problem.

7 The word error has 5 letters in it, two letters that appear once and one that appears three times. We're going to look at different 5-letter "words" that can be formed by rearranging these 5 letters. These words need not make any sense, so "rroer" is a word.

a. How many 5-letter words can be formed, assuming that you can tell the difference between the r's? In other words, think of the r's as being a red r, a blue r, and a green r (or as r_1, r_2, and r_3). By the way, each of the letters can only be used once per word.

b. Write down two 5-letter words that differ from each other only if the r's are different colors. How many other 5-letter words differ from these two only if the r's are different colors?

c. Think of another 5-letter word that is different from the ones in Part b. How many versions are there of this 5-letter word that are distinguishable from each other only if the r's are different colors? Would all of these 5-letter words be part of the count you made in Part a?

d. So, how many 5-letter words can be formed by rearranging the letters in *error*, if each letter can only be used once per word (and the r's are indistinguishable from each other)?

e. Lastly, you should be able to write your answer to Part d in the form $\frac{5!}{n!}$. What is n?

8 How many distinguishable 9-letter words can be formed from the letters in freestone, where each letter can only be used once per word?

9 How many distinguishable 9-letter words can be formed from the letters in freewheel, where each letter can only be used once per word?

10 How many distinguishable 9-letter words can be formed from the letters in appraisal, where each letter can only be used once per word?

11 Using the letters H, H, H, H, H, N, N, N, N, N, N, N only, how many 12-letter words can be formed? (Note: though the word "distinguishable" is not appearing you should assume you are just counting words that are different from each other.)

In the previous problems you were always counting letters. If you think about what you were doing, however, you should realize that the method of counting wasn't dependent on letters. Problem 11, for example, could have initially been a problem about how many ways can you step right five times and left seven times in a sequence of twelve steps.

12 Bill is flipping a coin 8 times, keeping track of the sequence of heads and tails. What is the probability that he'll get exactly 5 heads in total? Exactly 4 heads? 3 heads?

13 Alix's mathematics class wants to form a committee of 4 students to investigate whether AP classes should be taught in the school. If there are 16 students in the class then how many different committees can be formed?

The last problem may have seemed quite different from the previous ones, but there is a way of seeing it as no different. Imagine that you're working with all 16 students and that you have 16 Velcro labels. Twelve of these labels are a large N and four of them are a large Y. You line the 16 students up and then try to count how many different ways you can assign the labels to the students. In this sense, you are trying to determine how many different ways you can arrange 12 N's and 4 Y's in a sequence.

14 The answer to the last sentence (and to Problem 13) can be written in the form $\frac{a!}{b!\,c!}$, where a, b, and c have specific meanings in the context of the problem. What are a, b, and c and what are their meanings?

15 The fraction in Problem 14 can be simplified so that there is only one factorial expression in the denominator. Do this.

Look back at your answer to Problem 15 and remember that this fraction is an answer to Problem 13. Now, use your understanding of the relationship between this fraction and Alix's committee problem to answer the following question.

16 You're going to order a triple-scoop bowl of ice cream. If you plan on choosing three different types of ice cream from 8 possible choices, then how many different triple-scoop combinations are there?

In this lesson you have seen some problems where two different orderings of the same things are to be thought of as being different — e.g., the passwords 3857210964 and 6710892534. In other problems, you've had to discount these differences — e.g., the triple-scoop combination of chocolate-vanilla-pistachio does not differ from the combination pistachio-chocolate-vanilla. As you work through the remaining problems in these lessons, bear in mind these distinctions. Some problems will fall into one category and not the other, while some will fall into both. And, of course, some will fall into neither.

Practice

17 John and Mary are ordering hamburgers that come with 3 types of toppings: cheese (cheddar, blue, swiss), vegetable (lettuce, onion, tomato, mushroom), and sauce (mustard, ketchup). They will each order one of each type of topping.

a. John doesn't care about the order of his toppings, figuring that the whole mess ends up in the same place. How many different burgers does John think he can order?

b. Mary believes very strongly that the order in which the toppings are on her burger does subtly change the taste of the hamburger. How many different burgers does Mary think she can order?

18 There are 8 books on a shelf — five novels and three biographies. How many different ways can you line the books up on the shelf if

a. you naturally think of the books as being different from each other?

b. you don't distinguish the novels from each other or the biographies from each other, but you do distinguish novels as different from biographies?

19 You shuffle a standard 52-card deck several times.

a. How many different sequences of 5 cards are there at the top of the deck?

b. You deal a hand of 5 cards to yourself. How many possible 5-card hands are there?

c. Estimate the amount of time it would take to create all the possible 5-card hands of part b, assuming you can create 1 new hand every second. Would it take a day, a week, a month, a year, or a decade, or longer? What if you were trying to create all the possible 7 card hands? What about all the possible 52 card hands? Check your 3 estimates with a calculator and see if you were right in each case within a factor of 10 (also known as an "order of magnitude").

20 A true-false test has 25 questions on it. If you take the test by randomly deciding that 10 of the questions are true and the rest are false, then how many ways can you write down the answers to the test?

21 How many different ways can you arrange the letters in Mississippi?

22 Horatio has 3 tulips, 4 irises, and 3 daffodils growing in his yard. For each of ten days, he cuts one flower and brings it to his math teacher. How many ten-day sequences of flowers are there? (An example of one such sequence is iris-tulip-iris-iris-daffodil-tulip-daffodil-daffodil-tulip-iris.)

23 Using the seven digits 9, 9, 9, 3, 3, 3, and 3, once each, how many 7-digit numbers can be formed?

24 Each of the 240 students attending an Upper School dance has a ticket with a number for a door prize. If three different numbers are randomly selected, how many ways are there to award the prizes if the prizes are different from each other? if the prizes are identical?

25 In how many ways can you choose 3 letters from the word problems, if the order in which you choose the letters is important? is unimportant?

Problems

26 There are 20 cities in a certain country, and every pair of them is connected by an air route. How many air routes are there?

27 Remember Jan who bats .500 in softball. What's the probability that she will get exactly 4 hits in 10 at-bats? What's the probability that she will get no more than 4 hits in 10 at-bats?

28 How many "words" can you form from 5 A's and no more than 3 B's?

29 There are 6 books on a shelf. How many ways are there to arrange some (or all) of the books in a stack? The stack may consist of one book.

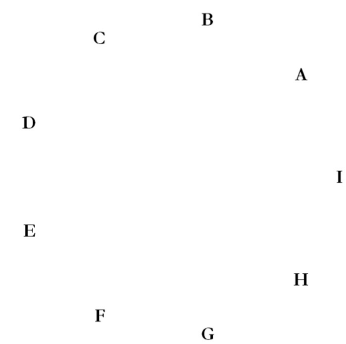
30 A teacher has a collection of 40 true-false questions. She wishes to create a test using 5 of these questions. How many different tests can she create if

a. the order of the questions doesn't matter?

b. the order of the questions does matter?

31 How many three-letter words can be formed from the letters A, B, C, D, E, F, G, H, and I, where each letter can only be used once per word?

32 The points A, B, C, D, E, F, G, H, and I are vertices of a regular polygon. If three of the points are chosen, and connected by line segments, a triangle is determined. How many triangles can be formed in this way? If four points were chosen to form a quadrilateral, then how many such quadrilaterals will there be?



33 You have six sticks of lengths 1, 2, 3, 4, 5, and 6 inches. How many non-congruent triangles can be formed by using three of these sticks as sides?

34 There are three rooms in a dormitory: one single, one double, and one for four students (called a "quad"). How many ways are there to house 7 students in these rooms?

35 Creamy Heaven serves 30 different ice cream flavors. A customer can order a single-, double-, or triple-scoop cone.

a. How many double-scoop cones are there if both scoops can be the same flavor and the order of the scoops doesn't matter?

b. How many possible cones are there if flavors can be repeated and the order of the scoops doesn't matter?

36 Three couples go to the movies and sit together in a row of seats. How many ways can they arrange themselves if each couple sits together?

37 Eight people are at Chen's house for a party (Chen is one of the eight). Chen has several options for sitting people for dinner.

a. If Chen gets a long table she can sit people on one side of the table so that they can watch a movie while they're eating. How many ways can she sit the eight people in this situation?

b. Chen can get a shorter, but rectangular table, and sit four people on one side and the other four on the opposite side. How many ways can she sit the eight in this situation? Assume that two arrangements that are reflections of each other over the long axis of the table are indistinguishable from each other.

c. Chen can sit the eight around a circular table. How many ways, now? Assume that two arrangements that are rotations of each other are indistinguishable from each other?

d. If Chen sits n people around a circular table, then how many ways can this be done if two arrangements that are rotations of each other are indistinguishable?

38 A bug jumps from lattice point to lattice point on a piece of graph paper according to the following pattern: from (a, b), the bug can jump only to $(a+1, b)$ or to $(a, b+1)$. How many different paths can the bug take if it wishes to start at $(0, 0)$ and end up at $(6, 4)$? Hint: tinker with this problem a bit so that you can get a feel for what the bug is doing.

39 Homer starts at the origin and performs a five-step random walk along a number line. Each second he steps either one unit to the left or one unit to the right.

a. Describe all the places that Homer can wind up at the end of this walk.

b. How likely is Homer to end up 5 steps to the right? 3 steps to the right?

c. If Homer were to perform this 5-step random walk 32000 times, what is your prediction of the average of all the final positions? What is your prediction of the average of all the distances from the origin to Homer's final position?

40 How many diagonals are there in a pentagon? a hexagon? a decagon? an n -gon? A diagonal is a straight line segment that connects any two non-consecutive vertices.

41 The rules of a checkers tournament indicate that each contestant must play every other contestant exactly once. How many games will be played if there are 12 participants? How many will be played if there are n participants?

42 Don't use a calculator for this problem.

a. Reduce the fraction $\frac{6a^2}{4a^2-6a}$.

b. Solve for x: $x^2 - 5x = 14$.

c. Solve for x: $x^2 - x - 1 = 0$.

d. Divide $\frac{2}{x} \div \frac{4}{x^2}$.

e. Write as one fraction: $\left(\frac{b}{2a}\right)^2 - \frac{c}{a}$.

43 How many ways can you split up 12 people into six pairs?

44 You have learned that any positive integer has a prime factorization of the form $2^a \cdot 3^b \cdot 5^c \cdot 7^d \cdot 11^e \cdot 13^f \ldots$, where the "..." represents the rest of the primes, and a, b, c, d, e, f, and so on can each equal 0, 1, 2, 3, ... So for example, 360 is equal to $2^3 \cdot 3^2 \cdot 5^1$. Remembering that every possible factor of 360 comes from one of the possible combinations of these prime factors (e.g. $20 = 2^2 \cdot 3^0 \cdot 5^1$ and $90 = 2^1 \cdot 3^2 \cdot 5^1$), and using the principles of counting that you learned studying combinatorics, determine the number of factors that 360 has. Note that 1 is considered a factor of 360 in this problem, i.e. $1 = 2^0 \cdot 3^0 \cdot 5^0$. (You might try answering for 84 instead of 360 if you want an easier example to start with.)

45 Generalizing Problem 44, find a way to determine the number of factors that any positive integer has. Try out your method with at least 3 other positive integers to check to see if it works.

46 Let $\binom{n}{r}$ stand for $\frac{n!}{r!(n-r)!}$, where n and r are nonnegative integers, and $r \leq n$.

a. Calculate $\binom{6}{2}$.

b. Calculate $\binom{7}{1}$.

c. Is $\binom{n}{r}$ ever not an integer? Explain.

d. Prove: $\frac{n(n-1)(n-2)\cdots(n-r+1)}{r(r-1)(r-2)\cdots 2\cdot 1} = \binom{n}{r}$.

e. Determine what should replace the question mark so that the following statement is true, then prove that the statement is true:
$\binom{n+1}{r} = \binom{?}{r-1} + \binom{?}{r}$

Exploring in Depth

47 How many different 5-person basketball teams can be chosen from a group of 10 people? How many ways can you split up 10 people into two different 5-person basketball teams?

48 The points A, B, C, D, E, and F are vertices of a regular hexagon. What is the probability that three randomly chosen vertices from this hexagon will form an equilateral triangle? What's the probability that four randomly chosen vertices from this hexagon will form a rectangle?

49 This is a continuation of Problem 38. In this problem you will see a connection between algebra and counting.

a. Expand $(r+u)^5$ and explain how this can help you determine how many paths there are from $(0,0)$ to $(2,3)$. Recall that when simplifying and collecting expressions such as rru, rur, and urr, we can write the final result as $3r^2u$.

b. What expression can you expand and simplify in order to answer Problem 38?

c. What is the coefficient of the x^3y^5 term in the expansion of $(x+y)^8$?

d. What is the coefficient of the x^5y^7 term in the expansion of $(2x+y)^{12}$?

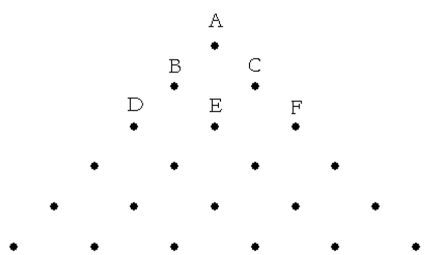
50 What is the sum of all the 3-digit numbers that can be formed using three different odd digits?

51 *Box Problems* You probably didn't realize that there are combinatorial problems that involve boxes, but there are.

a. Four boxes are numbered 1 through 4. How many ways are there to put 8 identical balls into these boxes (some of the boxes can be empty)? Hint: Imagine that you have 12 slots to work with.

b. Four boxes are numbered 1 through 4. How many ways are there to put 8 identical balls into these boxes so that none of them is empty?

52 Below is shown a triangle of dots. Suppose you start at dot A and move down through the triangle in the following manner: from any dot you can only move to the next dot that is down left or down right. So, if you are at dot A you can move to either dots B or C, but not directly to D or E. Below each dot in the triangle write the number of distinct paths from dot A to that dot.



53 There is a well-known triangle called Pascal's Triangle. What is this triangle and how does it relate to counting?

Park School Mathematics

This curriculum was written by members (past and present) of the upper school math department at the Park School of Baltimore. This was possible thanks to Park School's F. Parvin Sharpless Faculty and Curricular Advancement (FACA) program, which supports faculty every summer in major curricular projects. In addition to the support of FACA by the Nathan L. Cohen Faculty Enhancement fund and the Joseph Meyerhoff FACA fund, this project was also funded by grants from the E. E. Ford Foundation, the Benedict Foundation, Josh and Genine Fidler, and an anonymous donor.