BOOK 4: QUADRATICS

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HABITS

look for patterns: to look for patterns amongst a set of numbers or figures

tinker: to play around with numbers, figures, or other mathematical expressions in order to learn something more about them or the situation; experiment

describe: to describe clearly a problem, a process, a series of steps to a solution; modulate the language (its complexity or formalness) depending on the audience

visualize: to draw, or represent in some fashion, a diagram in order to help understand a problem; to interpret or vary a given diagram

represent symbolically: to use algebra to solve problems efficiently and to have more confidence in one's answer, and also so as to communicate solutions more persuasively, to acquire deeper understanding of problems, and to investigate the possibility of multiple solutions

prove: to desire that a statement be proved to you or by you; to engage in dialogue aimed at clarifying an argument; to establish a deductive proof; to use indirect reasoning or a counterexample as a way of constructing an argument

check for plausibility: to routinely check the reasonableness of any statement in a problem or its proposed solution, regardless of whether it seems true or false on initial impression; to be particularly skeptical of results that seem contradictory or implausible, whether the source be peer, teacher, evening news, book, newspaper, internet or some other; and to look at special and limiting cases to see if a formula or an argument makes sense in some easily examined specific situations

take things apart: to break a large or complex problem into smaller chunks or cases, achieve some understanding of these parts or cases, and rebuild the original problem; to focus on one part of a problem (or definition or concept) in order to understand the larger problem

conjecture: to generalize from specific examples; to extend or combine ideas in order to form new ones

change or simplify the problem: to change some variables or unknowns to numbers; to change the value of a constant to make the problem easier; change one of the conditions of the problem; to reduce or increase the number of conditions; to specialize the problem; make the problem more general

work backwards: to reverse a process as a way of trying to understand it or as a way of learning something new; to work a problem backwards as a way of solving

re-examine the problem: to look at a problem slowly and carefully, closely examining it and thinking about the meaning and implications of each term, phrase, number and piece of information given before trying to answer the question posed

change representations: to look at a problem from a different perspective by representing it using mathematical concepts that are not directly suggested by the problem; to invent an equivalent problem, about a seemingly different situation, to which the present problem can be reduced; to use a different field (mathematics or other) from the present problem's field in order to learn more about its structure

create: to invent mathematics both for utilitarian purposes (such as in constructing an algorithm) and for fun (such as in a mathematical game); to posit a series of premises (axioms) and see what can be logically derived from them

1

In a roomful of businesspeople, everyone wants to network with the others in the room. To achieve this, each person shakes hands with every other person in the room. If there are n people in the room, then, in terms of n, how many handshakes will occur?

When you solved this problem, you may have used the strategy of change or simplify the problem, trying the problem with four or five businesspeople and noticing the pattern. However you solved it, in order to generalize to n people, you had to describe, if only to yourself, a process by which you could count the number of handshakes. If you discussed this problem in class, you probably found yourself having to put this process into words very clearly, so that someone else following your process would also arrive at the correct answer: $n + (n-1) + (n-2) + \ldots + 1$.

In mathematics, taking the time to describe something — a procedure, a set of data, an image, or a definition — can give you a deeper insight into the nature of the problem. The focus in this lesson will first be on writing thorough descriptions. In later problems, you will find that coming up with a complete description of a method for tackling the problem will be of use in solving it and other problems like it — as was the case with Problem 1.

2 Without looking up what a pentagon is, explain to a 5th grader how to tell if a shape is a pentagon. Remember that it might not even be a polygon. You can assume, however, that the shape is something that can be drawn with pencil and paper.

When you describe something, in order for your description to be as insightful as possible, you should have the following goals in mind:

Your description should be *complete*, addressing all possibilities that could arise. In the problem above, did you take into account the fact that the shape might have curved lines? That it might have lines intersecting each other? Or even multiple lines intersecting at the same point as in the letter "K".

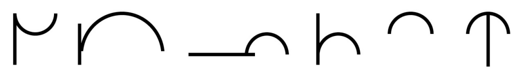
Your description should be *understandable* (to the intended reader). For example, you would be correct if you gave Wikipedia's definition of a pentagon, "a five-sided polygon," but this doesn't do any good if the fifth grader doesn't know what a polygon is.

Your description should be *unambiguous*. This means that your description should be completely clear, and not open to any different interpretations. You should ask yourself, "Would it be possible for the reader to read my description carefully, but misunderstand what I mean?" For example, if you said "a pentagon must have five sides," would someone reading the description think that this meant "exactly five sides" or "at least five sides"? If you think there is any possibility that your language might be misinterpreted, you should be more specific.

Try this next exercise to practice writing a thorough description:

3 Describe how to draw a lowercase letter "r", to someone who's never seen our alphabet. Once your description is complete, go on to the next problem.

4 The shapes below are some possible ways that the reader could draw an "r" incorrectly – make sure your description from the previous problem is clear enough



5 How would you explain what speed is to someone who doesn't understand the phrase "how fast you're going"?

If you thought that the task of describing how to draw an r was a little bit silly, you're right. Everyone in your class knows how to draw an r, and children who don't know how to make an r are taught by example, not by description. Likewise, the chances are slim of your running into someone without a layperson's understanding of speed.

On the other hand, the task of providing a basic description can be illuminating in its own right. It's easy to believe that you understand what speed is, but when encouraged to define it for someone who doesn't understand certain phrases, you're likely to think more carefully about what speed actually is and therefore be able to use this understanding in future problems dealing with speed. Providing a careful description of something often helps you to understand it, bringing assumptions to light and revealing questions that you may not have known you had.

With a few exceptions, the next batch of problems does not require you to pretend you have an audience with gaping holes in their knowledge.

Instead, the problems are opportunities for you to practice writing complete, understandable, unambiguous descriptions targeted at people with a mathematics background similar to your own. In Problems 9 and 10 in particular, working towards a description that covers all the cases will guide your investigation.

Targeting your explanations toward an audience of your peers will give you the chance to practice another virtue: making your descriptions *concise*. Charged with writing a complete description, people sometimes respond by using as many words as possible. But a beautiful description is thorough while at the same time using as few words as possible. Keep that in mind while working on the following problems.

6

Last year you learned to solve a system of equations such as

$2x - 4y = 15$

$-2x + 8y = -1$

by adding the equations to each other in order to create a simpler equation that we could solve. Explain why we can solve the original system of equations by using this new, simpler equation in place of this system.

7

In the standard window of your calculator to graph $y = \frac{(x+2)(x+3)}{x+3}$. In the ZOOM menu choose the ZDecimal option. With this option, your calculator will set up a window where Xmin $= -4.7$, Xmax $= 4.7$, Ymin $= -3.1$, and Ymax $= 3.1$. Now, describe the graph of the equation and explain why it is this way.

8

You learned in the Combinatorics lesson that the number of ways four people can stand in a line is 4x3x2x1=24. Describe a procedure they could use for systematically standing in all 24 configurations, without repeating any configuration.

9

You may remember that the absolute value function $|x|$ takes an number x and makes it positive... so $|3| = 3$ and $|-3| = 3$.

a. Make a table of values, then use it to graph the equation $y = |x|$.

b. Make a table of values, then use it to graph the equation $y = |x - 3|$.

c. Describe the graph of $y = |x - h|$ clearly enough that, once you tell someone a specific value of h, they could draw the graph accurately.

10

An equation of the form $y = x^n$ is called a power function. $y = x^2$ and $y = x^3$ are two simple examples, with $n = 2$ and $n = 3$. Use your calculator to graph some different power functions, making sure you have chosen a sensible window. Describe the effect of the different values of n on the graph. Make sure you try values of n that are negative, fractional, etc. Try to categorize all the different types of graphs you can get depending on what n is.

11 Describe a circle to someone who's never seen anything circular. You can either explain how to draw one, or define what it is.

12 Do this problem with a partner. First, draw a meaningless shape on a piece of paper — just doodle for a few seconds, and make it look fairly random, but not too complicated. Don't let your partner see the shape.

Then, your partner should get a pen and paper, and you should describe what your shape looks like and have them try to draw it. Don't look at what they're drawing or make any gestures — just tell them what to do in words. At the end, look to see how accurately it turned out.

You should each be the "describer" twice. Try to make your second try more accurate based on what you learned from your first try.

There are even more benefits to providing a clear description! Not only can it help you convince others that your solution is correct, a clear, complete description can help you be sure that you haven't overlooked anything in your solution to the problem. In addition, by doing problems such as the average speed or the four-people-in-a-line problem, you might have learned something deep about the nature of the concepts involved. Finally, you may find that, in providing a complete description of your solution to one problem, you may see a way to apply the solution to another problem.

In the final set of problems, the description will no longer be the task itself. Each of these problems requires you to come up with a clear solution method. The trick will be to come up with a plan for solving each problem, after which the solution becomes much easier. As you work the problems, imagine explaining your solution method to your classmates. This will not only help you be ready to convince your classmates that your solution works, it will make sure that you are really honest about following your method.

Don't forget: after initially tinkering to get a feel for the problem, ask yourself, "what's my plan?"

13 A merchant visited three state fairs. At the first, she doubled her money and spent \$150. At the second, she tripled her money and spent \$270. And, at the third, she quadrupled her money and spent \$360. Alexi found that she had \\$240 at the end. How much money did she start with? Is there another solution?

14 Rihanna decides to give her three friends — Alicia, Jason, and Sali — some of her nickels that she brought to school. She gives Alicia half of her nickels plus one half of a nickel. Then she gives Jason half of her remaining nickels, plus half a nickel. Finally, she gives her last remaining nickel to Sali. How many nickels did Rihanna bring to school?

15 A book has been opened at random. What pages has it been opened to if the product of the facing pages is 298662?

16 A store was selling toothbrushes for 50¢ each. When the price was reduced, though not too much, the remaining stock sold for \$31.93. What was the reduced price?

17 What is the next term in the sequence $1, 4, 6, 26, 158, 4110$? Write a formula for the sequence using the T_n notation you learned when studying sequences.

18 Examine the following sequence: 3, 6, 12, 24...

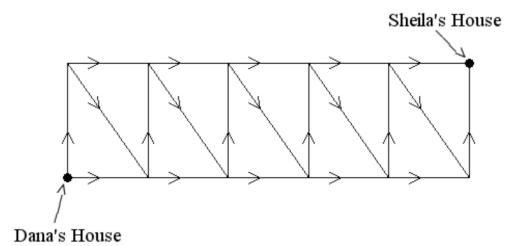
a. Find the next term, then write a formula for the nth term of the sequence.

b. Sequences that can be written in the form $T_n = T_1 \cdot r^{n-1}$ are called geometric sequences. Make up three more geometric sequences, with an eye toward teaching yourself what the effect of both T_1 and are on the sequence.

c. Do geometric sequences always get bigger and bigger as n increases? If not, under what conditions do they get bigger and bigger, and what are the other possibilities?

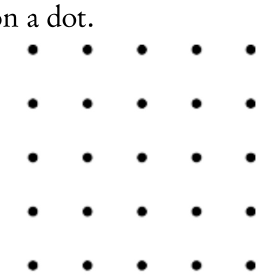
19 Students in Mr. Willard's P.E. class are spaced evenly around a circle and then told to count off beginning with 1. If student number 16 is directly opposite student number 47, then how many students are in Mr. Willard's class?

20 How many different routes are there from Dana's house to Sheila's if their town was laid out as shown below. Dana can only move in the direction of the arrows.



21 Find a way to cut any parallelogram into pieces that can be arranged into a rectangle.

22 These dots are spaced one unit apart, measured horizontally and vertically. Find a way to draw a square with area exactly 10 square units, so that each vertex of the square is on a dot.



HABITS

LESSON 1: QUADRATIC MODELS AND ALGEBRA

Flu Season, Again.

One of the most unpleasant aspects of going to school is that when a few people get sick, they can transmit their illness to others very easily, and as a consequence most of the school can end up “catching” it.

Epidemiologists, when faced with the spread of a disease, try to predict how quickly people will be infected by coming up with equations that seem to fit the data well. One reasonably accurate equation is called the logistic model, and is based on the following idea. Let I stand for the number of people already infected and N stand for the number who are not infected. Then the rate at which the disease is spreading is proportional to the product of I and N. For the sake of simplicity let’s call this product the “disease quotient” of DQ. Let’s assume, also, that the DQ is exactly equal to the product of I and N. The DQ is, therefore, a measure of how quickly the disease is spreading: the higher it is, the more people get infected that day.

How would you expect DQ to change while the disease is spreading? At what point during the spread of the disease would the disease spread most quickly?

Development

Let’s look at a specific case — a local high school with 400 students.

1 What is the DQ when no students are infected? When 1 student is infected? When 70 students are? When 300 are infected?

2 Write an expression describing the DQ if one knows the number x of students that are infected.

3 Rewrite the equation in Problem 2 so that it's in the form $DQ = ax^2 + bx + c$. What are a, b, and c in your equation?

Any equation that can be written in the form $y = ax^2 + bx + c$, where $a \neq 0$, is called a quadratic equation. In this lesson we will be looking at some algebraic tools that can be used to manipulate these equations so as to learn more about how the solutions of such an equation behave.

4 Determine the number of people who are infected if $DQ = 0$. You should get two answers.

5 Pick one of your answers from problem 4 and explain why, without talking about the disease quotient or the number infected or not infected, it must be a solution to the equation $x(400 - x) = 0$.

6 Think carefully about your answers to Problem 4 and how they solve the equation given in Problem 5. Now, explain why 2 and -3/4 are solutions to the equation $(4x + 3)(x - 2) = 0$.

7 Solve each of the following equations.

a. $x(x + 10) = 0$

b. $(x - 3)(x + 2) = 0$

c. $(x + 5)(x + 7) = 0$

d. $-2(x - 4)(x + 9) = 0$

e. $(x + 1)^2(3x - 1) = 0$

f. $(4x + 5)(x - 3)(x + 1) = 0$

8 What must be true about A or B if $A \cdot B = 0$? Can the same thing be said if $A \cdot B = 8$?

In Problem 8 you're being asked to conjecture a very important theorem of mathematics that is often referred to as the **Zero Product Property** — anytime you have a product, then it can equal zero only if at least one of its factors is zero. This is a very powerful theorem in situations in which we can write an equation in the form BLAH $= 0$, where BLAH can be written as a product. So, it seems useful here to focus on how to write expressions as products, don't you think?

There is a way to rewrite expressions $x^2 + 7x + 10$ as a product of two expressions, so that it looks like $(x + \ldots)(x + \ldots)$. You may have seen this technique before: it is called **factoring**.

9 Working on your own or with your peers, factor $x^2 + 7x + 10$. When you have an answer, look carefully at your factored form of the expression and how it relates to the original expression.

10 Solve the equation $x^2 + 7x + 6 = 0$.

11 Solve each of the following equations by factoring.

a. $x^2 + 3x + 2 = 0$

b. $x^2 + 8x + 12 = 0$

c. $x^2 + 20x + 64 = 0$

d. $x^2 + 6x + 9 = 0$

e. $x^2 + 5x - 14 = 0$

In Problem 11 Part e, you needed to resort to negative numbers in order to factor the expression. Trying to diagram this expression as a way of factoring it would certainly be tricky, but using the number patterns that you've noticed about the original expression and its factored form lets you get around the difficulty of constructing the diagram.

12 Factor each of the following expressions.

a. $x^2 + 4x - 45$

b. $x^2 - 5x - 36$

c. $x^2 - x - 42$

Another hopeful outcome from the Algebra Blocks lesson was that you would realize that factoring is not a process reserved for expressions like $x^2 + 6x + 8$, but can also be done on expressions like $3x^2 + 7x$. In essence, factoring is a process devoted to reversing distribution: can we rewrite $3x^2 + 7x$ so that its factored form, when multiplied out, would give us back $3x^2 + 7x$?

13 Factor $3x^2 + 7x$. Check your work by doing the distribution.

14 Solve each of the following equations by factoring.

a. $2x - 5x^2 = 0$

b. $6x^2 + 4x = 0$

c. $5x^3 - 12x^2 = 0$

d. $x^3 - 8x^2 + 16x = 0$

Practice

15 Kyle is managing an upcoming concert at school and is concerned about the income generated by the concert just through ticket sales. Over the years he has come to realize that when the price of the ticket increases, the number of people willing to attend goes down. He knows that all 360 students will attend if the concert is free, but for each $1 increase in the price of the ticket, 20 fewer students will attend.

a. How many students will attend if Kyle charges $2 per ticket? How many if he charges \$5?

b. What income will Kyle generate if he charges $2 per ticket? What if he charges \$8?

c. Write an expression for the income I generated by the concert when the price of the ticket is p. Is this a quadratic equation?

d. Set the expression in Part c equal to zero and solve this equation. Explain the meaning of the solution in the context of the problem situation.

16 Factor each of the following expressions.

a. $x^2 + 16x + 15$

b. $6x - 10x^3$

c. $x^2 + 3x - 28$

d. $x^2 - 7x - 18$

17 Solve each of the following equations by factoring.

a. $x^2 + x - 20 = 0$

b. $5x^4 + x^3 = 0$

c. $x^3 + 5x^2 - 14x = 0$

Going Further

18 Graph the equation you came up with in Problem 2 on a calculator. (If you have trouble seeing the graph, take a look at the values you computed in Problem 1.)

19 Use the graph in Problem 18 to determine when the DQ is 25600. Explain why both of your answers make sense in terms of the spread of a disease.

20 How many people have been infected if $DQ = 20000$?

21 Determine the maximum DQ. How many people are infected at this point?

22 Explain how you can use your answers to Problem 19 to solve Problem 21. Could you have used the answers from Problem 20 to solve problem 21?

The two numbers that you calculated in Problem 21 serve as the coordinates of a specific point on the graph of the DQ equation. The graph of the equation is called a **parabola**, and the point that you calculated in Problem 21 is called the **vertex** of the parabola. The vertex can be thought of as the point where the parabola turns.

23 The graph of $DQ = x(400 - x)$ is symmetric over a vertical line. Write an equation for this line.

24 An answer to Problem 15 Part c could be $I = p\,(360 - 20p)$. Graph this equation on your calculator, then write an equation for its **line (or axis) of symmetry.** You might look at Problem 15 Part d to help you solve this problem.

25 The graph of $I = p\,(360 - 20p)$ intersects the p-axis. What are the coordinates of the intersection(s)?

You may recall that the points you found in Problem 25 are referred to as the p -intercepts of the graph of $I = p\,(360 - 20p)$. They would be called the x -intercepts, if the horizontal axis had been labeled with x. Notice that if you determine the intercepts on the horizontal axis of a parabola then you can determine the axis of symmetry and the vertex of the parabola.

26 Determine the vertex of the graph of $y = (x + 5)\,(x - 4)$ without using the graphing or table features of your calculator.

27 Determine the vertex of the graph of $y = (x + 5)(\mathrm{x}{-}4)$ without using the graphing or table features of your calculator.

28 Describe a process for determining the exact coordinates of the vertex of the graph of $y = x^2 - 5x - 14$. At no time can the process rely on a graph of the equation.

Practice

29 Let $y = x^2 - 3x - 54$. Solve the following problems without using the table or graphing features of your calculator.

a. Determine the exact coordinates of the x-intercepts of the graph of the equation.

b. Determine the exact coordinates of the vertex of the graph.

c. Write an equation for the axis of symmetry of the graph.

30 The points $(4, 10)$, $(8, 25)$, and $(26, 10)$ are on the graph of a parabola. Write an equation for the parabola's line of symmetry.

31 If the x-coordinate of the vertex of a parabola is $\frac{3}{7}$ and one of its roots is $\frac{5}{6}$, what is its other root? (Answer as a fraction.)

32 For each equation below, determine the exact coordinates of the vertex of the graph.

a. $y = x^2 + 6x + 8$

b. $y = 2x^2 - 5x$

c. $y = -2\left(x^2 - x - 12\right)$

Problems

33 The King of Prussia wishes to make a magnificent rectangular temple. He has just enough gold to adorn a perimeter of 98 meters. The king wishes his temple to have the greatest possible area, so that as many people can stand in it as possible.

a. How can one express the relationship between the length, width and perimeter of the temple in an equation?

b. How can one express the relationship between the length, width and area of the temple in an equation?

c. By using your answers to Parts a and b, write an expression that allows one to compute the area in terms of the length of the temple alone.

d. What length of the temple allows the most people to fit inside?

e. For religious reasons, the King of Prussia built a temple that was 18 meters long instead; it only let in P of the people that the “ideal” temple you computed in Part d would have. What is P?

34 The 9th grade is selling tickets to a Rock-Paper-Scissors tournament. Top-secret research has shown that the number (N) of high school students who are willing to enter the tournament is highly dependent on the entry fee (P). If there were no entry fee, 200 people would enter the tournament; for every dollar the entry fee is above \$0, 50 less people will enter.

a. Write an expression for the number of students (N) entering the tournament in terms of the entry fee (P).

b. If the entry fee is \
$3.50, how many tickets will be sold? Also, how much mone$
I\$)?

c. Write an equation for I in terms of N and P.

d. Using part a, now write an equation for I in terms of P alone.

e. What entry fees would produce zero income for the 9th grade?

f. If the 9th grade wants to make the most money, what should the entry fee be? How much income would the 9th grade then make?

35

So far you haven't been asked to factor an expression of the form $ax^2 + bx + c$, where $a \neq 1$ or 0.

a. For example, how would you factor $2x^2 + 3x + 1$? Try doing this without a diagram.

b. Now try factoring $2x^2 + 7x + 6$.

c. Use your understanding of Parts a and b to factor each of the following expressions.

i. $3x^2 - 2x - 8$
ii. $2x^2 - 10x + 12$
iii. $-2x^2 + 10x - 12$
iv. $4x^2 + 4x - 15$

36

Come up with values for a and b so that $(x + a)(x + b) = x^2 - c$, where c is some unknown constant.

37

(Continuation of Problem 36) Factor each of the following expressions, if possible. Note that each of these expressions could be called a **difference of squares**.

a. $y^2 - 9$

b. $x^2 - 100$

c. $x^2 - b^2$

d. $n^4 - 16$

e. $x^2 + 25$

f. $w^2 - 49x^2$

38

Solve each of the following equations.

a. $x^2 - 2x = 8$

b. $3x^3 + 11x^2 - 4x = 0$

c. $x^2 = 144$

d. $r^2 - 9 = 8r$

e. $(x + 6)(x - 2) = 9$

f. $x^2 + 4.5x + 3.5 = 0$

g. $x + \frac{x}{x-2} = \frac{2}{x-2}$

39

Let $x^2 + 15x - 324 = 0$.

a. Find the sum of the solutions and the product of the solutions.

b. A quadratic equation $x^2 + bx + c = 0$ has two solutions that have a sum of 2 and a product of -143. Find b and c.

40

f is a function that takes a number, subtracts the square of the number, and then adds 30.

a. Write a formula $f(x) = \ldots$

b. What is the largest value that f can output?

41

25 is a perfect square since $25 = 5 \cdot 5$. Find d if $4x^2 + 12x + d$ is a perfect square.

42

Don't use a calculator for this problem.

a. Solve for x: $\frac{6}{7} = \frac{33}{x}$

b. Evaluate $-(3)^2 + (-3)^2$

c. Find an equation for the line that goes through the points $(-5, 4)$ and $(5, -2)$.

d. Solve the inequality $-4x + 13 < x + 3$

e. Try to make all the numbers from 1-10 by using basic mathematical operations on four 4's. For example, to make the number 1 you could do $(4 - 4) + (4 \div 4)$.

43

π is approximately 3.141593. Without using a calculator determine whether $\pi^2 - 7\pi + 12$ is negative, positive, or zero.

44

Graph $y = x^2 + 4x + 7$ on your calculator. Make sure to choose a WINDOW for your calculator that will show the basic shape of the graph.

a. Explain why the algebraic methods you know don't seem to help in determining the exact coordinates of the vertex of the graph.

b. Calculate the y-intercept of the graph of the equation.

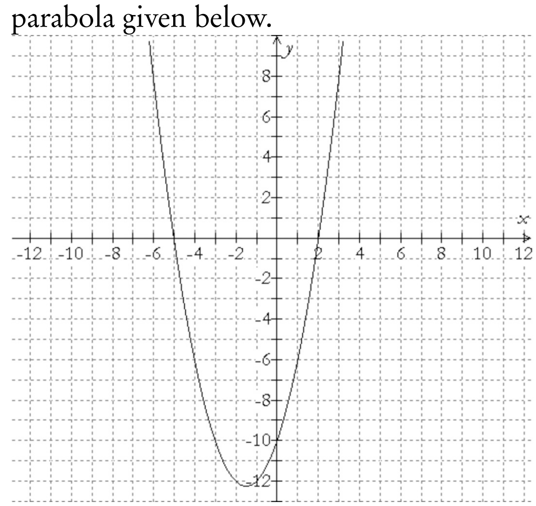
c. Now, use your understanding of the symmetry of the graph and the value of the y-intercept to determine another point on the graph.

d. Calculate the exact coordinates of the vertex of the parabola.

e. Calculate the exact coordinates of the vertex of the graph of $y = -x^2 - 9x + 12$.

45

Write an equation for the graph of the parabola given below.



46 You are selling candy for a price of p dollars per candy bar. 100 people want candy if it's free, but every time the price goes up by a dollar, 3 fewer people want candy.

a. Write an equation for n (the number of people who want candy) in terms of p.

b. The government charges a candy tax of \
1.50*percandybar*. *Sowhensomeonebuysacandybar*, \$1.50*ofwhattheypaygoestothetax*. *Writeanequationfor*
p\$.

c. Write an equation for M, the total amount of money you make, if you charge price p. Make sure M is just in terms of p.

d. Without graphing on your calculator, find the maximum amount of money you can make.

e. Determine a value for $p > 0$ that would make no sense in the context of this problem. Explain why.

f. Determine the values of $p > 0$ that would lead you to earning zero money.

47 When Alyssa kicks a football, its height (in feet) as a function of time (in seconds)can be written $h(t) = -16t^2 + 40t$.

a. When will Alyssa's football hit the ground?

b. Find the maximum height it will reach.

48 Hickory, Hunk, and Zeke are building a rectangular pigpen. One side of the pen will be the wall of a barn, and the other three will be made of the 1000 feet of fencing they have.

a. Draw a picture of the situation.

b. Write a formula for a function, $A(x) = \ldots$ that outputs the area of the pen when one of the sides perpendicular to the barn has length x.

c. What value of x should the farmhands choose if they want the pigs to be able to roam over the largest area possible?

49 Without using the graphing or table features of your calculator, determine the exact coordinates of three solutions to the equation $y = x^2 - 5x - 24$ that would allow you to sketch a reasonable graph of the equation. Why did you pick these three solutions?

Exploring in Depth

50 For what values of n is
$\left(n^2 - 3n - 4\right)\left(n^2 + 8n + 2\right) = (5n + 12)\left(n^2 - 3n\right.$

51 You learned how to factor $x^2 - 1$, but is there a way of factoring $x^3 - 1$, $x^4 - 1$, or $x^{99} - 1$, for that matter?

a. We multiply $x - 1$ by $x + 1$ in order to get $x^2 - 1$. What should we multiply $x - 1$ by in order to get $x^3 - 1$?

b. What should we multiply $x - 1$ by in order to get $x^4 - 1$? How about in order to get $x^7 - 1$?

c. $x^4 - 1$ can also be seen as a variation on the expressions in Problem 37. Use this fact to figure out how to factor $x^3 + x^2 + x + 1$.

52 The expression $x - 9$ can be factored in the same way that you factored a difference of squares. Come up with this factoring. Is it true that $x - 9$ equals this factoring for all values of x? Explain.

53 What, if any, integer values of b are there such that $x^2 + bx + 71$ is factorable? And, more generally, if $x^2 + bx + c$ is factorable, what can you say for sure about c? about b ?

54 Solve the following inequalities by using algebra and some number sense.

a. $(x - 3)(x + 5) \leq 0$

b. $x^2 + 9x + 14 > 0$

c. $-x^2 \leq x - 42$

d. $x(x + 3)(x - 2) \geq 0$

LESSON 2: TRANSFORMATIONS

Introduction

In Lesson 1 of this chapter you saw several different quadratic models. Two of these models were $y = -x^2 + 400x$ and $y = -20x^2 + 360x$. These models are different in the sense that the equations look different, but are their graphs different, too?

1 Set the window dimensions of your calculator to Xmin $= -50$, Xmax $= 400$, Xscl $= 50$, Ymin $= -5000$, Ymax $= 40000$, and Yscl $= 5000$. Now, graph $y = -20x^2 + 360x$ and $y = -x^2 + 400x$. What do you see? Are the graphs identical to each other?

While the graphs were not identical, they were both parabolas. When we studied transformations we learned that we can take a triangle, for example, and transform it into a triangle of the same shape but a different size and in a different place. We also learned that we can stretch a triangle in just one direction and change its shape while still preserving the fact that it's a triangle. A natural question at this point might be the following: Can the graphs of the two parabolas seen in Problem 1 be transformed into each other? Put another way: Is it possible to take the graph of $y = -x^2 + 400x$ and shift it, stretch it, or flip it over some line in order to get it to be identical to the graph of $y = -20x^2 + 360x$? Rather than tackle this actual question now, we will examine a similar problem by looking at simpler equations.

Development

2 Use the standard window to graph the quadratic function $y = x^2$ on your calculator. The standard window has Xmin $= -10$, Xmax $= 10$, Xscl $= 1$, Ymin $= -10$, Ymax $= 10$, and Yscl $= 1$.

a. For what values of x does the graph, as you move from left to right, go down? go up?

b. Why does the graph not appear in the 3rd or 4th quadrants?

The previous questions were designed to help you get a feel for the shape of the graph of $y = x^2$. Often it helps to have this kind of feel before you start to investigate how you can transform this shape. So, now that you have a feel for this graph, let's look at some transformations of it.

3 In the same window you graphed $y = x^2$, graph $y = x^2 + 3$.

a. Do the graphs appear congruent — the exact same size and shape?

b. Complete the table below.

x	-2	-1	0	1	2	3	4
$y = x^2$	4						
$y = x^2 + 3$	7						

The completed table in Problem 3 Part b should lead you to think that the graphs of $y = x^2$ and $y = x^2 + 3$ really are congruent to each other, differing only in their location. It's important to note that we haven't proven this to be true; we've only gathered evidence that supports this notion.

4 In what direction should you shift the graph of $y = x^2$ so that it would exactly coincide with the graph of $y = x^2 + 3$? By how much should the graph be shifted? You should be able to give a precise answer to this.

One way of answering Problem 4 is by saying that the graph of $y = x^2 + 3$ is a vertical translation up 3 units of the graph of $y = x^2$. There is synonymous language — vertical shift up 3 units, for example. Don't worry about using the exact same terms that everyone else is using when trying to describe the transformation that has occurred, but do worry about whether your language is accurate.

5 In the standard window graph $y = x^2$, $y = x^2 + 6$, $y = x^2 - 2$, and $y = x^2 - 10$. Thinking about what you wrote in Problem 4, and what you read in the above paragraph, what conjecture can you make regarding the graphs of $y = x^2$ and $y = x^2 + k$? Be specific.

6 Graphical transformations allow us to produce new quadratic functions that as a consequence have transformed coordinate pairs.

a. The point $(5, 25)$ is on the graph of $y = x^2$. How could you shift the graph of $y = x^2$ to produce a new quadratic function where $(5, 32)$ is on its graph? What would be the equation of this new graph?

b. The point $(4, 16)$ is on the graph of $y = x^2$. How could you shift the graph of $y = x^2$ to produce a new quadratic function where $(4, -77)$ is on its graph? What would be the equation of this new graph?

7 Graph $y = x^2$, $y = 3x^2$, and $y = \frac{1}{2}x^2$ in the standard window. Make sure you can see all of the important features of each of the graphs. Now, describe how you would transform the graph of $y = x^2$ into each of the other graphs. Be as precise as possible.

There are many possible answers to Problem 7. A very common one is to say that you would horizontally compress (make narrower) toward the y-axis the graph of $y = x^2$ in order to get the graph of $y = 3x^2$ and horizontally stretch (make wider) away from the y-axis the graph of $y = x^2$ in order to get the graph of $y = \frac{1}{2}x^2$. Sometimes a person who describes the changes in this way will go on to write that the graph of $y = x^2$ would be compressed by a factor of 3 (made 1/3 as small) in the first case and stretched by a factor of 2 in the second case. It's not easy checking the accuracy of these statements, but as it turns out, these statements about the factors are not quite right. Let's look at another way of transforming the graph of $y = x^2$ into the graph of $y = 3x^2$ or the graph of $y = \frac{1}{2}x^2$.

In Problems 3-6 we saw that if you add a quantity (positive or negative) to x^2 the graph of the new equation will shift vertically. Since 3 is multiplying x^2 in the equation $y = 3x^2$, let's see whether the graph of $y = x^2$ changes vertically. In order to do this it's best that we plug the same x-values into each equation and compare corresponding y-values.

8 Complete the table below.

x	-2	-1	0	1	2	3	4
$y = x^2$	4						
$y = 3x^2$	12						

9 Use the table in Problem 8 to answer the following questions.

a. What is the height of the graph of $y = x^2$ at $x = -2$— i.e., what is the vertical distance between the x-axis and the point on the graph where $x = -2$? What is the height of $y = 3x^2$ at this same value of x?

b. How do the two answers you arrived at in Part a relate to each other? Would this be the same relationship if instead of looking at $x = -2$ you looked at $x = -1$? $x = 4$? If your answer is “no”, then come up with a relationship that would be true for the y-values at each of these x-values.

c. Using what you have learned in Part b describe precisely — using the language of transformations that you learned back in Chapter 1 — how you can transform the graph of $y = x^2$ into the graph of $y = 3x^2$.

10 Now, think about what you determined in Problem 9 and describe precisely how you can transform the graph of $y = x^2$ in order to get the graph of $\frac{1}{2}x^2$. If you need to, make a table similar to the one in Problem 8 as a way of checking your description.

In Problems 8-10 you only looked at how the graph of $y = x^2$ can be transformed into the graph of $y = ax^2$ when $a > 0$. What happens when $a < 0$?

11 Describe precisely how you can transform the graph of $y = x^2$ into the graph of $y = -x^2$.

12 Now describe precisely how you can transform the graph of $y = x^2$ into the graph of $y = -2x^2$. Be prepared to justify your description with a table.

13 There are two transformations that occur in Problem 12: a reflection of the graph over the x-axis and a vertical stretch away from the x-axis by a factor of 2. Does the order of the transformations matter — in other words, if you reflect over the x-axis first and then vertically stretch will that result in the same graph if you had vertically stretched first and then reflected over the x-axis?

14 What can you say in general about the relationship between the graphs of $y = x^2$ and $y = ax^2$ when $a \neq 0$?

15 The points $(10, 100)$ and $(-4, 16)$ are on the graph of $y = x^2$.

a. How can you vertically stretch or shrink the graph $y = x^2$ to produce a new quadratic function where $(10, 400)$ and $(-4, 64)$ are on its graph? What would be the equation of this new graph?

b. How can you vertically stretch or shrink the graph of $y = x^2$ to produce a new quadratic function where $(10, 20)$ and $(-4, 3.2)$ are on its graph? What would be the equation of this new graph?

16 The point $(6, -18)$ is on the graph of $y = ax^2$. Determine the value of a. Can you do this by using transformations?

You may have noticed that it's quite tedious to keep saying that the graph has been vertically stretched "away from the x-axis" or compressed "toward the x-axis". In Chapter 1 you learned that there is a simple formula that describes dilations or compressions if their done around $(0, 0)$. Throughout this lesson, we will be seeking simple, formulaic ways of describing vertical stretching or shrinking, so we will always take the x-axis as the line from which we stretch or toward which we compress. Hence, we will say, "the graph of $y = 3x^2$ is a vertical stretch by a factor of 3 of the graph of $y = x^2$"; there's no need to mention the x-axis.

Practice

17 Describe how to transform the graph of $y = x^2$ into the graph of the given equation.

a. $y = x^2 + 0.5$

b. $y = -10x^2$

c. $y = x^2 - 10$

18 Without using your calculator, graph each of the following equations on the same axis system.

a. $y = x^2$

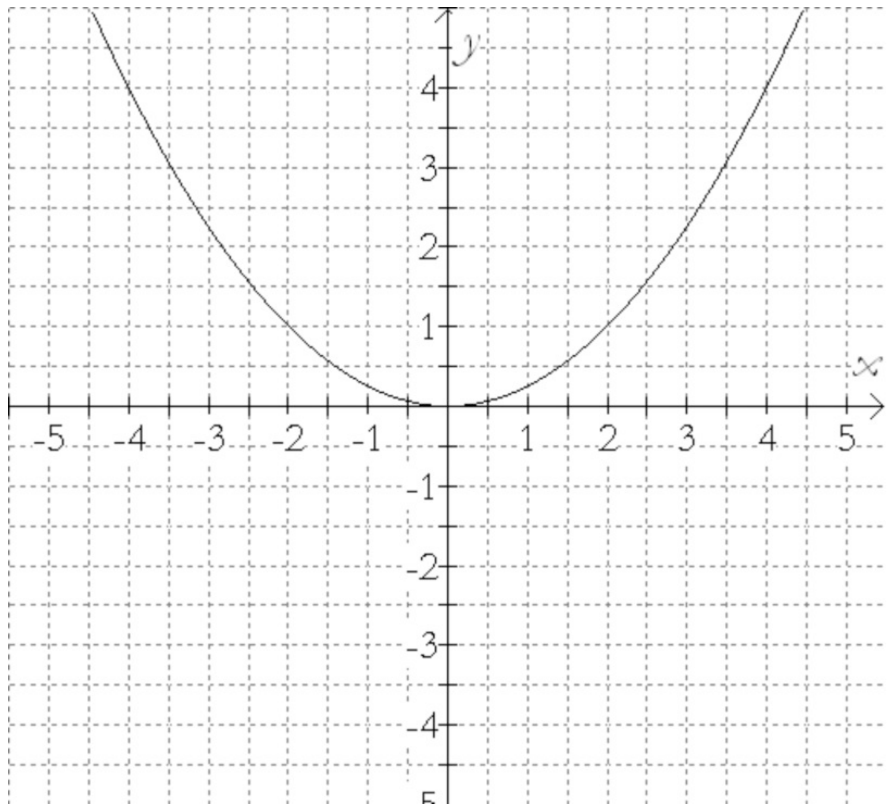
b. $y = x^2 - 2$

19 Determine the value of a such that $(2, 10)$ is on the graph of $y = ax^2$.

20 Determine the value of k such that $(2, 10)$ is on the graph of $y = x^2 + k$.

21 Write an equation for the given graph.

a.

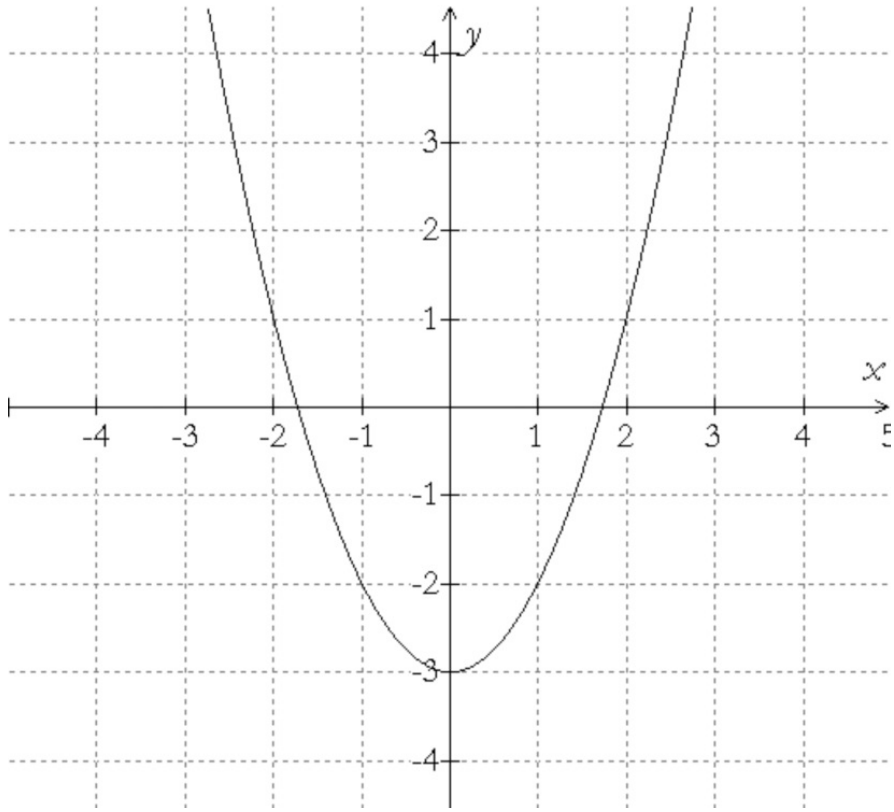


b.

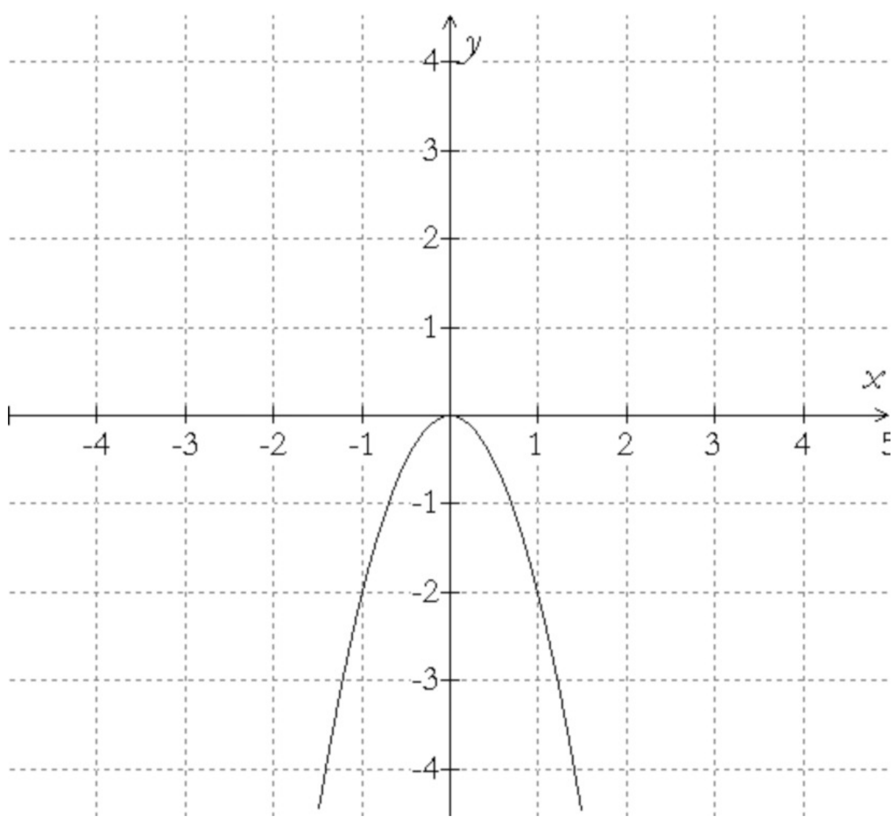
Going Further

22 Graph $y = x^2$ and $y = (x - 4)^2$ in the same window on your calculator. Make sure you can see complete graphs of each of the functions. Describe as precisely as possible how you can transform the graph of $y = x^2$ into the graph of $y = (x - 4)^2$.

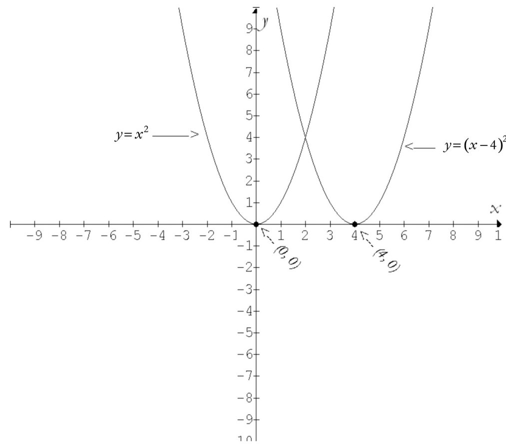
You may have noticed in Problem 22 that the two graphs appear to be the same size and shape but are in different places along the x-axis. In this sense the graphs look like they're horizontal translations of each other. One way of determining the precise magnitude and direction (left or right) of this translation is to focus on where the vertex is on each graph. Since these points correspond to each other — in the same way that the corner points of two triangles would correspond to each other — you can probably figure out what the horizontal translation is by looking



c.



at these points. The diagram below shows the two graphs, along with their indicated vertices.



From this diagram we can see that the graph of $y = (x - 4)^2$ appears to be 4 units to the right of the graph of $y = x^2$. So, if we horizontally translate the graph of $y = x^2$ to the right 4 units we can get the graph of $y = (x - 4)^2$. Notice that though the equation indicates that 4 is being subtracted from x, the graph moves *right* 4 units, not left 4 units. Let's look at a few more graphs before we investigate this apparently strange behavior.

23 Graph $y = x^2$ and $y = (x + 3)^2$ in the same window. Describe how you would transform the graph of $y = x^2$ in order to get the graph of $y = (x + 3)^2$.

24 Without using your calculator describe how you would transform the graph of $y = x^2$ in order to get the graph of $y = (x + 1)^2$. How would you transform it into the graph of $y = (x - 7)^2$? Check your work by using your calculator.

Problems 23 and 24 probably confirmed your sense of what was going on in Problem 22, not only in that the transformation is horizontal but also that it is "opposite" of what you might expect just from looking at the equation. One way of trying to understand what's going on here is to isolate the variable that appears to be affected by the transformation — the "x" variable — by controlling the values of the other variable — the "y" variable. We can do this by creating a table in which the y-values are the same for each equation.

25 Complete the table below. For some y-values there are two values of x that could go into the box. Write only one of these x-values. Also, you'll need to think about which x-value you want to write if you want to uncover a pattern.

y-values	0	1	4	9	16	25	36
x-values for $y = x^2$	0						
x-values for $y = (x-3)^2$	3						

26 What is the relationship between the x-values for $y = x^2$ and the corresponding x-values for $y = (x-3)^2$? Is this relationship consistent with your answers to Problems 22-24?

27 What can you say in general about the relationship between the graphs of $y = x^2$ and $y = (x-h)^2$?

28 Three points on the graph of $y = x^2$ are $P = (3, 9)$, $Q = (7, 49)$, and $R = (-1.5, 2.25)$. Just as the graph of $y = x^2$ gets trans two graphs of Problem 24, points P, Q, and R are transformed as well and end up with "new" coordinates.

a. Give the "new" coordinates for P, Q, and R on the graph of $y = (x+5)^2$.

b. Give the "new" coordinates for P, Q, and R on the graph of $y = (x-1.5)^2$.

c. Give the "new" coordinates for P, Q, and R on the graph of $y = (x+b)^2$.

29 There are two transformations done on the graph of $y = x^2$ in order to get the graph of $y = (x-3)^2 - 4$. What are they? Does the order in which they are applied matter?

Practice

30 Describe how to transform the graph of $y = x^2$ into the graph of the given equation.

a. $y = (x+3)^2 - 2$

b. $y = -(x-5)^2$

c. $y = 110(x-35)^2$

31 Without using your calculator, graph each of the following equations. Identify the exact coordinates of two points on the graph.

a. $y = (x-1)^2 + 2$

b. $y = \frac{1}{2}(x+6)^2$

32 Determine the values of h and k such that the given point will be the vertex of the graph of $y = (x-h)^2 + k$.

a. $(3, -6)$

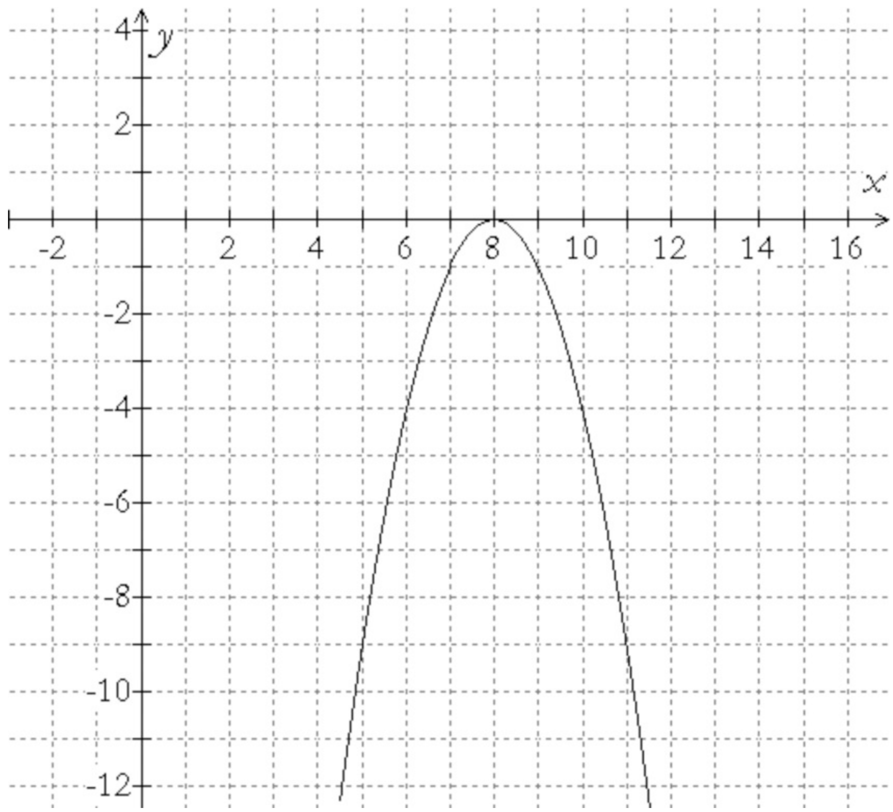
b. $(-2, 5)$

c. $(-7, 0)$

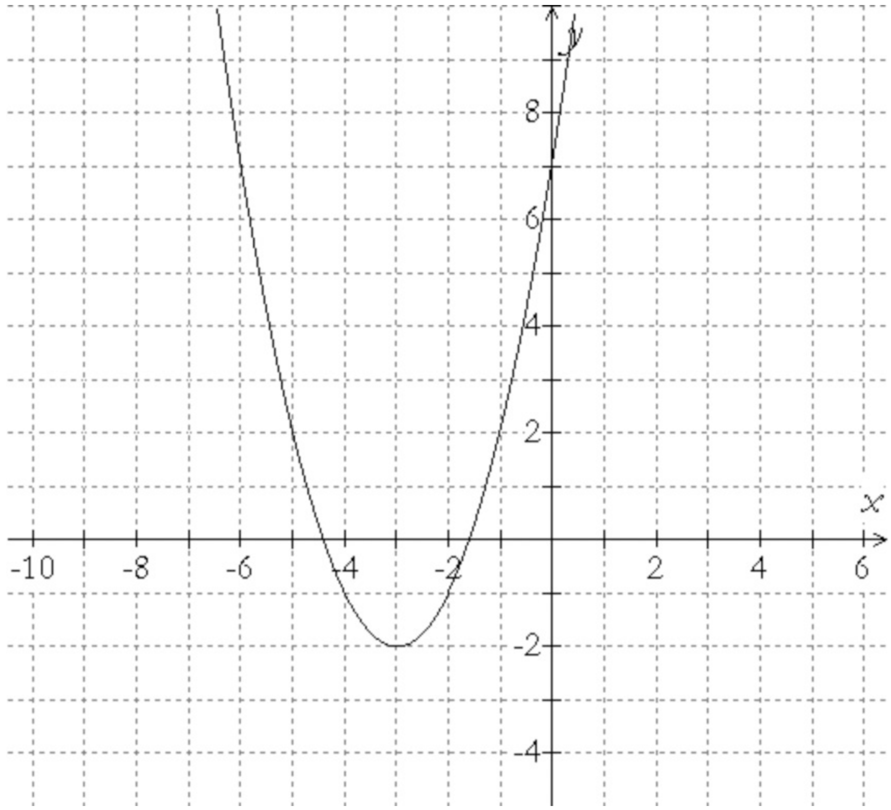
33

Write an equation for the given graph.

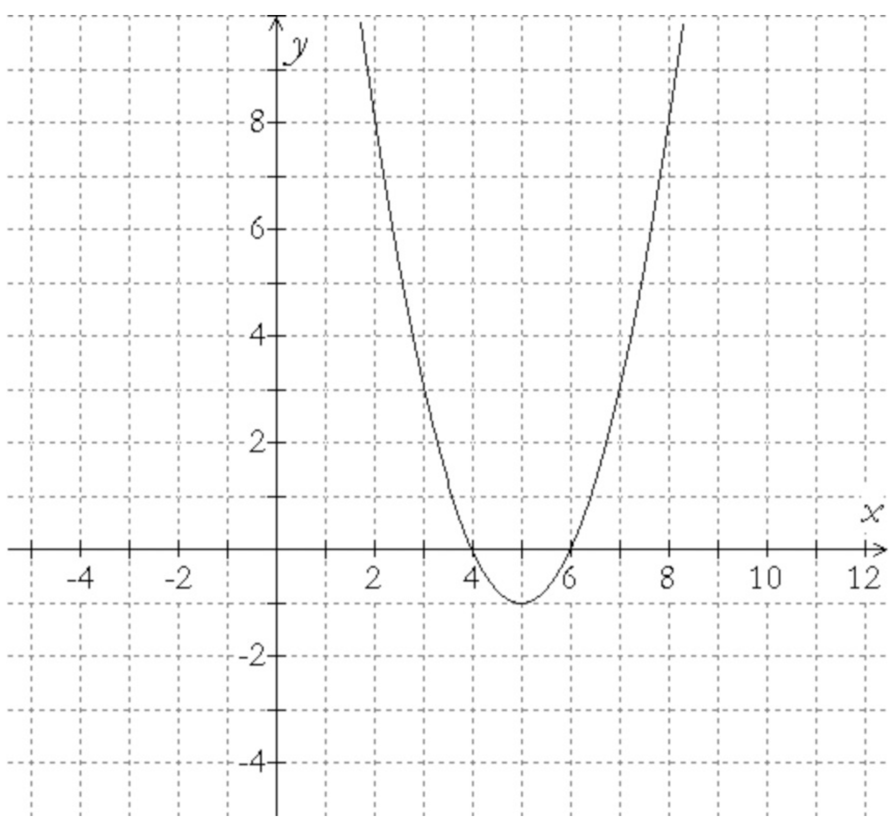
a.



b. .



c. .



Problems

34 Describe three different transformations of the graph of $y = x^2$ that would transform the point $(2, 4)$ to $(2, 20)$. How many transformations do you think there could be? Justify your answer.

35 The point $(5, 4)$ is on the graph of $y = (x - b)^2$: it is a transformation of the point $(2, 4)$ on $y = x^2$. Find a value of b that makes sense. Can you find a second value for b that works as well? Why would there be two values?

36 Write two tables that clearly illustrate what type of transformation is used on the graph of $y = -x^2$ to get the graph of $y = -(x + 2)^2$.

37 Describe a way that you can prove to someone that the graph of $y = 5x^2$ is not a horizontal compression by a factor of 5 of the graph of $y = x^2$. Your explanation cannot be akin to saying, "Well...because it's a vertical stretch by a factor of 5."

38 Mickey was told to vertically stretch the graph in Figure 1 by a factor of 3 from $y = 0$. His work is shown in Figure 2.

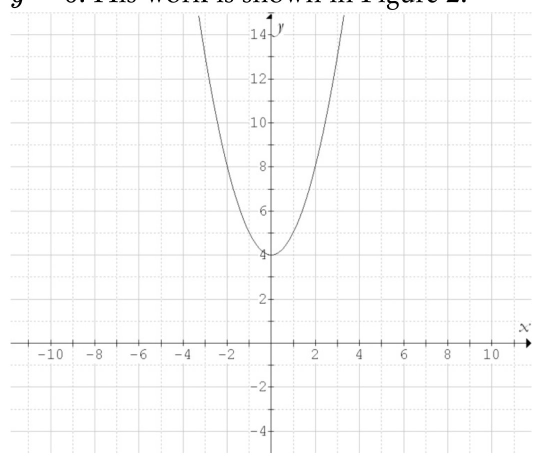


Figure 1

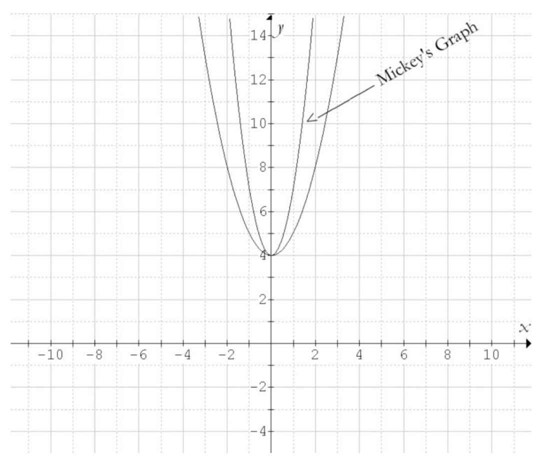


Figure 2

a. Explain why Mickey's graph is incorrect.

b. On Figure 1, draw the correct graph.

c. Write an equation for Mickey's graph. Now, write an equation for the graph you drew in Part b.

39 The graph of $y = 3(x+2)^2$ can be transformed into the graph of $y = \frac{1}{2}(x-3)^2$.

a. Describe one such transformation.

b. Describe how to transform the graph of $y = \frac{1}{2}(x-3)^2$ into the graph of $y = 3(x+2)^2$.

c. What relation, if any, does your answer in Part a have to your answer in Part b?

40 The graph of a quadratic equation has x -intercepts of -4 and 3 and a y-intercept of -7. Suppose you transform this graph by translating it 3 right and then reflecting it over the x-axis. Identify the exact coordinates of 3 points on this new graph. Are any of these points a y-intercept? Why?

41 Let $f(x) = (x-1)^2 + 2$.

a. Make a graph of the function f, with x on the horizontal axis and values of $f(x)$ on the vertical axis.

b. On separate axes, graph the equation $y = f(x) + 3$– that is, to calculate the y-value for a certain x, take and then add 3. Any surprises?

c. On separate axes, graph the equation $y = f(x+3)$– that is, to calculate the y -values for a certain x, add 3 to x and then take f of the result. Any surprises here?

d. Use what you have learned in parts b and c to quickly graph the equation $y = f(x+1) - 5$.

42 Don't use a calculator for this problem.

a. Divide $3 \div \frac{2}{3}$

b. Simplify $[2(-5+3) - (-4-1)](342 - 142)$

c. Subtract $1\frac{4}{5} - \frac{7}{10}$

d. Solve for x if $|x-7| = 3$. Then solve for x if $|x| - 7 = 3$.

e. Solve for x and y: $\begin{aligned} 2x + 5y &= -10 \\ 6x - 10y &= 45 \end{aligned}$

43 At what x-value(s) does the graph of $y = ax^2 - b$ cross the x-axis? Is this true for all possible values of a and b?

44 Graph $y = x^2$ and $y = x^2 + 6x$ in the same window. It's difficult to see whether the graphs are the same size and shape since they are not in the same place.

a. How would you transform the graph of $y = x^2$ so that it is in the same place as the graph of $y = x^2 + 6x$?

b. Once you have transformed the graph of $y = x^2$ you should be able to write an equation in the form $y = (x-h)^2 + k$ for this transformed graph. What will be the values of h and k?

c. Are the graphs of $y = x^2$ and $y = x^2 + 6x$ the same size and shape? How do you know that your answer must be correct?

45

Given what you have learned in Problem 44, see what you can do with the problems below.

a. Are the graphs of $y = x^2$ and $y = x^2 - 10x$ the same size and shape? How do you know that your answer must be correct?

b. Are the graphs of $y = x^2$ and $y = x^2 + 8x - 1$ the same size and shape? Justify your response.

c. Show that $y = x^2 + 4x - 3$ and are the same size and shape.

46

Let $y = (x + 3)(x - 5)$.

a. Without using your calculator, draw a graph of the function. Label the exact coordinates of the vertex, the two x -intercepts, and the y-intercept.

b. An equation in the form $y = (x - h)^2 + k$ can be written for the graph in Part a. Write this equation.

c. Use algebra to prove that the original factored-form equation and the equation that you wrote in Part b are equivalent.

47

Suppose you were trying to explain to someone why the graph of $y = (x - 2)^2$ is a horizontal translation to the right by 2 units of the graph of $y = x^2$. Construct two tables, each having 5 ordered pairs (5 points), that you could use to show why the translation occurs the way it does. The tables should make this plainly obvious. Write a short explanation indicating how the tables show the correct translation.

48

Let $y = (x + 4)(x + 6)$.

a. What are the x-intercepts of the graph of the function?

b. Suppose the graph was translated 3 units to the left. Write, in factored form, an equation for this new graph.

c. Demonstrate that your answer in part b is correct by showing algebraically that the graph of your answer is a translation of the graph of the original equation.

d. In thinking about what she learned about horizontal translations Jessica decided to solve part b by replacing each " x" in the original equation with "$x + 3$": literally erase each x (not $x + 4$ or $x + 6$) and write $x + 3$ in its place. Do this and show that Jessica's thinking is on the mark.

e. Suppose the original graph was translated 5 units to the right. Use Jessica's thinking to write an equation for this new graph.

49

When you learned transformations in the context of coordinate geometry you learned that translating a figure does not change its size or shape: in a sense you may have thought that translations don't change any aspect of the figure. Is this true for graphs of quadratic functions? Are there aspects of the graph (x-intercepts, y -intercept, vertex, etc.) that don't change? Are there aspects that do? Are these changes predictable? Are there aspects that change but there is essentially no way of predicting what the outcome will be?

Exploring

in Depth

50 The graph of the equation $y = 2x^2 + 4$ involves two transformations of the graph of $y = x^2$: a vertical translation up 4 units and a vertical stretch by a factor of 2.

a. If you were going to graph $y = 2x^2 + 4$ by actually transforming the graph $y = x^2$ should you apply the translation first and then the vertical stretch or should the transformations be done in the other order? Explain how you know that your answer is correct without referring to the graphing or table features of your calculator.

b. There is an algebraically equivalent form of the equation $y = 2x^2 + 4$ that will allow you to apply a vertical translation first and then a vertical stretch, though the magnitudes of the translation and/or the stretch will not necessarily be 4 and 2, respectively. Find this equivalent form.

c. For each equation, determine the transformations needed in order to transform $y = x^2$ into the graph of the given equation.

i. $y = -3x^2 - 2$

ii. $y = \frac{1}{2}(x^2 + 4)$

iii. $y = 3(x + 4)^2 - 1$

iv. $2y + 1 = (x - 3)^2$

51 Describe how to transform the graph of $y = -x^2 + 400x$ into the graph of $y = -20x^2 + 360x$.

52 Let $y = (x + 2) + 3$. Do not simplify this equation. And, “yes”, the equation has been written correctly.

a. The graph of the given equation is a transformation of the graph of what much simpler equation?

b. What are the transformations involved of the graph of the simpler equation?

c. On a coordinate axis system, draw a graph of the simple equation you identified in Part a. Now, apply the transformations that you identified in Part b to this graph. Is the final graph the correct graph for $y = (x + 2) + 3$? Check your work.

d. Graph $y = -3x$ by transforming the graph of a simpler equation.

53 Compare the graphs of $y = x^2$ and $y = (3x)^2$. What type of horizontal transformation can you do on the graph of $y = x^2$ in order to get the graph of $y = (3x)^2$? Be precise in your description and justify your description with two tables that clearly show why the transformation is what it is.

54 Determine the value(s) of p such that $(p, 5p)$ is on the graph of $y = 3x^2 - 2$.

55 Recall that a circle consists of the set of points in a plane that are equidistant (the same distance) from a fixed point.

a. Let (x, y) be a point that is 5 units from the fixed point $(0, 0)$. There are many possible points that (x, y) could represent. Sketch all of them.

b. Write an equation that describes the fact that (x, y) is 5 units from $(0, 0)$. Write your final answer to this question so that it has no square roots in it.

c. Now, sketch all of the points (x, y) that are 5 units from the fixed point $(2, 0)$. Use the Pythagorean Theorem to write an equation that describes the fact that (x, y) is 5 units from the fixed point $(2, 0)$.

d. Explain how you can use transformations to write the equation you derived in Part c.

e. Use transformations to write the equation that describes all the points (x, y) that are 5 units from the fixed point $(4, 1)$. Show that your answer is correct by deriving your answer through the Pythagorean Theorem.

LESSON 3: COMPETING THE SQUARE

Introduction

In Lesson 1 you investigated how to solve quadratic equations like $5\left(x+3\right)\left(x-7\right)=0$ or $x^2-3x-28=0$. You can use the Zero Product Property to solve the first equation, but you'll need to factor (reverse distribution) the second equation before you can use that property to solve this equation. It's really nice, in fact, when you can use factoring as a way of transforming an equation into another form that is easier to analyze. Unfortunately, you also saw a few equations that did not readily yield to factoring. Below are some equations of this type:

$x^2+4x+2=0$,
$x^2-6x+6=0$, and
$x^2+10x-20=0$

Notice that had these equations initially been derived from the equations shown below —

$y=x^2+4x+2$,
$y=x^2-6x+6$, and
$y=x^2+10x-20$

—you would not be able to use factoring as a way of determining the exact values of the x-intercepts. You would be able to approximate them with your calculator but never get the exact values for them. You actually do know a way — a process — of solving this problem, however. Let's take a look at a process that will get you the exact values.

Development

1 There are two values of x that satisfy the equation $x^2 = 9$. What are they? These are called the **solutions** of the equation.

The two numbers you found in Problem 1 are each called a **square root** of 9.

2 What are the solutions to $x^2 = 49$? What are the square roots of 49?

3 What are the solutions to $x^2 = 50$? What are the square roots of 50?

If your answers to problem 3 were 7.071067812 and -7.071067812 (or shorter versions of one or both of these numbers) then you did not find the exact solutions. These numbers are *approximations* of the square roots of 50.

4 Explain why 7.07106781^2 cannot equal 50.

Given your answer to Problem 4, then, you may wonder why your calculator indicates that the square of 7.071067812 is 50. Your calculator does this because it cannot hold onto enough digits to show you what this squared number actually is. A more powerful calculator could show you that $7.071067812^2 = 50.000000001902467344$. Pretty close to 50, but not quite! The fact is that there is no way to write down the exact value of the positive square root of 50 by using decimal notation.

When we want to indicate the exact solutions to the equation $x^2 = 50$, or the exact square roots of 50, we just write $\sqrt{50}$ to indicate the positive square root of 50 and $-\sqrt{50}$ to indicate the negative square root of 50. Keep this in mind when you're being asked to determine the exact solutions to an equation and you need to find square roots in order to do this.

5

Find the exact solutions to each equation.

a. $x^2 = 16$

b. $x^2 - 20 = 0$

c. $(x - 2)^2 = 9$

If you weren't sure what to do in Problem 5 Part c think about replacing the "$x - 2$" with A, solve for A, and then figure out what x would be from there. Try Problem 5 Part c again if you didn't correctly solve it the first time.

6

Find the exact solution to each equation.

a. $(x + 3)^2 = 10$

b. $(x - 6)^2 - 25 = 0$

c. $(x + 1)^2 - 35 = 0$

d. $(x - 10)^2 + 45 = 56$

At this point you should have a pretty good idea about how to solve quadratic equations that are in the form $(x - h)^2 - k = 0$. So, if you can take an equation like

$$x^2 + 4x + 2 = 0$$

and rewrite it in the form

$$(x - h)^2 - k = 0$$

Then you would be able to solve the original equation since you know how to solve an equation in this last form. So, let's figure out how to get a quadratic equation of the form $x^2 + bx + c = 0$ into the form $(x - h)^2 - k = 0$.

7

Complete the table below. The first row has been filled in for you so that you have an idea of what to do.

Form V	Expanded and simplified version of Form V
$(x - 4)^2 - 22 = 0$	$x^2 - 8x - 6 = 0$
$(x - 7)^2 - 39 = 0$	

$(x-5)^2-17=0$	
$(x+2)^2-2=0$	
$(x+3)^2-5=0\vert$	
$(x+9)^2-20=0$	

8 Look carefully at each equation in the right column of the table in Problem 7 and compare it to its corresponding equation in the left column of the table. Look at one row at a time and see if you can pick up on any patterns between the equation on the right and its partner on the left. Once you have come up with some patterns try to use one or more of them to rewrite the equation below so that it is in the form $(x-h)^2-k=0$ (Form V in the table above).

$x^2+8x+1=0.$

9 Once you have rewritten the equation in Problem 8 solve it the same way that you solved the equations in Problem 6. Check your answers in the original equation to make sure that they work.

The process of adding a constant to the terms x^2+bx+c so that the resulting expression can be factored as a perfect square is called **completing the square.** As you can see in Problems 8 and 9, completing the square can be used to solve quadratic equations that don't appear to factor.

10 Use completing the square to solve the following equations. Find exact solutions.

a. $x^2+12x-21=0.$

b. $x^2+2x-4=0$

c. $x^2-4x+1=0$

d. $x^2-20x-100=0$

In Problem 10 the solutions you were finding are called the **roots** of the equation since each equation is in the form BLAH $=0$. As you've seen before, these roots (these solutions) can be interpreted as x-intercepts of a corresponding quadratic function. For example, the roots of the equation $x^2+12x-21=0$ are the x -intercepts of $y=x^2+12x-21$. As it turns out, the values of x that make y equal to zero are called the **zeros** of the function. And, as you learned in *Quadratic Models and Algebra*, these zeros can be used to determine the exact coordinates of the vertex.

11 Find the exact zeros of the following quadratic functions.

a. $y = x^2 + 6x - 1$

b. $y = x^2 - 10x + 4$

c. $y = x^2 - 2x - 15$

12 Find the exact coordinates of the vertex of each function presented in Problem 11 without using the graphing or table features of your calculator.

13 Use the information you have developed in Problems 11 and 12 to draw a graph of $y = x^2 - 10x + 4$. Do not use the graphing or table features of your calculator. Label, on your graph, the exact coordinates of the vertex, the x-intercepts, and the y-intercept.

You should pat yourself on the back after completing Problem 13. You not only can fully analyze a linear function ($y = ax + b$): solve it, graph it, find its slope and intercepts without the calculator; you now have some ideas about how to fully analyze a quadratic function of the form $y = x^2 + bx + c$. Later, we will get to analyzing the more general form $y = ax^2 + bx + c$, where a can be any real number other than zero, and we will even look at what happens when there don't appear to be any zeros (it turns out that there are zeros, but they are of a completely different nature).

Practice

14 What are the square roots of 400?

15 What are the square roots of 56?

16 Find the exact roots $x^2 - 61 = 0$.

17 In each part below, determine what c must be so that the resulting expression is a perfect square.

a. $x^2 + cx + 25$

b. $x^2 - cx + 1$

c. $x^2 + 24x + c$

d. $x^2 - 14x - 5 + c$

18 Find the exact solutions for each equation.

a. $x^2 + 18x - 100 = 0$

b. $x^2 - 4x - 1 = 0$

c. $x^2 = 14x - 20$

d. $x^2 + 6x + 8 = 0$

19 Find all the intercepts of each quadratic function.

a. $y = x^2 - 2x - 9$

b. $y + 5 = x^2 + 4x$

20 Find the exact coordinates of the vertex without using the graphing or table features of your calculator.

a. $y = x^2 + 30x - 100$

b. $y = x^2 - 18x + 80$

21 Without using the graph or table features of your calculator draw a graph of the equation. On the graph, label the exact coordinates of the vertex, the x-intercepts, and the y-intercept.

a. $y = x^2 + 30x - 100$

b. $y = x^2 - 6x + 6$

Problems

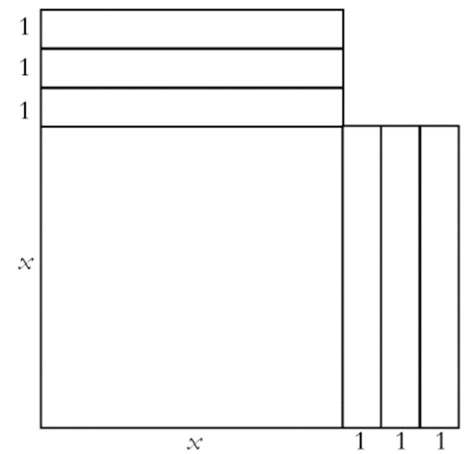
22 Find the exact solutions to $-x^2 = 40 - 22x$.

23 Find the exact solutions to $x^2 + 5x - 4 = 0$.

24 Find the exact solutions to $x^2 - 8x + 1 = 2x + 9$.

25 Find the exact solutions to $x^2 + bx + c = 0$. The solutions here will not be explicit numbers, per se, but will have b's and c's in them.

26 The diagram below consists of an x^2-block and six -blocks.



a. Use the diagram to figure out the right number to add to "complete the square" on the expression $x^2 + 6x$. Then write the identity your diagram illustrates.

b. Make a similar diagram to show how to complete the square on the expression $x^2 + 10x$.

27 Where precisely do the graphs of $y = x^2 - 4x + 1$ and $y = 2x + 4$ intersect?

28 Where precisely do the graphs of $y = -x^2$ and $y = x^2 - 4$ intersect?

29 Is there a number that has only one square root? No square roots?

30 If x is a real number then what is the smallest value of $x^2 + 8$? What about $x^2 + 8x + 8$?

31 There are two points $(x, 4)$ that are 6 units from $(-2, 7)$. Find the exact values of x.

32 Let $y = x^2 - 6x + 5$.

a. Rewrite the equation so that it's in the form $y = (x - h)^2 + k$.

b. Explain why $y = (x - h)^2 + k$ is called the **vertex form** of the quadratic equation.

33 In a right triangle, the hypotenuse is 14 and one leg is 3 longer than the other. Find the lengths.

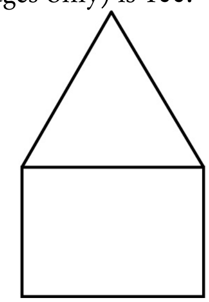
34 You take 3 consecutive numbers. You add the middle one to the product of the other two, and get 505. Find the three numbers.

35 A whiteboard is 8 feet wider than it is tall. It was damaged in a freak rainstorm and is missing a rectangle that is 3 feet by 2 feet in the lower right hand corner. The total area of useable whiteboard is 20 square feet.

a. To the nearest hundredth of a foot, find the dimensions of the whiteboard. What are these dimensions in inches?

b. Find the exact dimensions (in feet) of the whiteboard.

36 In the house below, the roof is an equilateral triangle of side length x and the bottom part is a rectangle of dimensions x by y. The total perimeter of the shape (outside edges only) is 100.



a. Write an equation for the perimeter in terms of x and y, and then write an equation for the area in terms of x and y.

b. Write the area in terms of x alone.

c. Find the values of x and y that would make the area the largest.

37 You jump off a 9-meter diving board. At the same time, your friend jumps off a 10-meter diving board next to you. However, you and your friend have a different style of jumping. You jump up off the board with an initial speed of 1.6 meters/second. Your friend just steps off the board, without jumping at all. The functions describing your heights off the surface of the water in terms of time (in seconds) are

You: $a(t) = -5t^2 + 1.6t + 9$

Friend: $b(t) = -5t^2 + 10$

a. Who hits the water first?

b. When you jump in the air, do you ever get as high as the 10-meter board?

38 The line segment below is divided into two parts by point A. The ratio of its right part to its left part is the same as the ratio of the entire line segment to its right part. Note that the left part has a length of 1. Let x stand for the length of its right part.

a. Write an equation saying the ratios are equal.

b. Determine the exact value of x.

Exploring in Depth

39 You've learned how to solve equations of the form $x^2 + bx + c = 0$. In this problem you'll tackle quadratic equations where the coefficient (the number in front) of x^2 is something other than 1. You might consider what you can do to each equation in order to turn it into an equation of the form $x^2 + bx + c = 0$.

a. $2x^2 + 8x - 12 = 0$

b. $3x^2 - 12x = 21$

c. $-4x^2 + 8x - 5 = 0$

d. $-2x^2 + 10x - 8 = 0$

e. $\frac{2}{3}x^2 = x + 1$

f. $ax^2 + bx + c = 0$, where $a \neq 0$.

40 (Continuation of Problem 25) Determine a formula for the x-coordinate of the vertex of the graph of $y = x^2 + bx + c$.

41 (Continuation of Problem 39 Part f) Determine a formula for the x-coordinate of the vertex of the graph of $y = ax^2 + bx + c$, where $a \neq 0$.

42 Don't use a calculator for this problem.

a. Evaluate
$1^2 - (-1)^3 + 0^4 - 1^5 + (-1)^6 - 0^7$

b. Add $\frac{a}{b} + \frac{c}{d}$

c. Solve for x: $\frac{7}{x+2} = \frac{2}{x-3}$

d. 16 hungry students come across 5 and a half leftover pizzas. If each pizza has 8 slices and the students split the pizzas evenly, how many slices does each student get?

e. Simplify $\frac{\frac{1}{2}}{2}$, then simplify $\frac{1}{\frac{2}{2}}$.

43 Show that the graph of $y = -3x^2 + 12x - 1$ is a transformation of the graph of $y = x^2$.

44 There are some equations that are "quadratic in form," meaning that they are not quadratic but they can be reduced to a quadratic equation. For example, $x^4 - 2x^2 - 8 = 0$ is such an equation if you let $y = x^2$ and then substitute into the equation, getting $y^2 - 2y - 8 = 0$. Use this kind of thinking to solve the following equations.

a. $x^4 - 2x^2 - 8 = 0$

b. $x^4 - 4x^2 + 1 = 0$

c. $x^4 - 6x^2 + 8 = 0$

d. $y^6 + 8y^3 = 20$

e. $y - 4\sqrt{y} - 12 = 0$

45 A ball is thrown from the top of a 65-feet tall building. The ball's height above the ground is modeled by the equation $h = -16t^2 + 100t + 65$.

a. Sketch a reasonable graph of the height of the ball above the ground.

b. Determine the approximate time (accurate to the nearest tenth of a second) at which the ball is first 150 feet above the ground. Now find the exact time.

c. Use algebra to explain why the ball never reaches a height above the ground of 225 feet.

d. What is the maximum height of the ball?

e. For how long is the ball more than 150 feet above the ground?

f. Determine the exact time when the ball hits the ground.

46 To lay wall-to-wall carpeting in a living room and dining room takes 612 square feet of carpet. The living room floor is 3 feet longer than it is wide. The width of the dining room floor is 6 feet more than the width of the living room, and its length is twice the width of the living room. What are the dimensions of each room?

47 At one point we found the coordinates of the vertex of a parabola by finding the exact coordinate of the x-intercepts and then averaging these as a way of getting the x-coordinate of the vertex. What happens, however, if there are no x-intercepts? Let's explore this issue by using the quadratic equation $y = x^2 + 4x + 5$.

a. Use algebra to try to determine the exact coordinates of the x-intercepts. What happens? What does this mean in regards to finding the x-intercepts?

b. . Let's suppose $\sqrt{-1}$ exists (and so does $-\sqrt{-1}$). Now, complete the algebra you started in Part a and use these zeros to find the exact x-coordinate of the vertex of the parabola. Check your work by inspecting the graph. Hmmm...

c. See if the kind of algebraic reasoning you used in Parts a and b will let you determine the coordinates of the vertex of $y = x^2 - 6x + 12$.

1 In a roomful of businesspeople, everyone wants to network with the others in the room. To achieve this, each person shakes hands with every other person in the room. If there are n people in the room, then, in terms of n, how many handshakes will occur?

When you solved this problem, you may have used the strategy of change or simplify the problem, trying the problem with four or five businesspeople and noticing the pattern. However you solved it, in order to generalize to n people, you had to describe, if only to yourself, a process by which you could count the number of handshakes. If you discussed this problem in class, you probably found yourself having to put this process into words very clearly, so that someone else following your process would also arrive at the correct answer: $n + (n-1) + (n-2) + \ldots + 1$.

In mathematics, taking the time to describe something — a procedure, a set of data, an image, or a definition — can give you a deeper insight into the nature of the problem. The focus in this lesson will first be on writing thorough descriptions. In later problems, you will find that coming up with a complete description of a method for tackling the problem will be of use in solving it and other problems like it — as was the case with Problem 1.

2 Without looking up what a pentagon is, explain to a 5th grader how to tell if a shape is a pentagon. Remember that it might not even be a polygon. You can assume, however, that the shape is something that can be drawn with pencil and paper.

When you describe something, in order for your description to be as insightful as possible, you should have the following goals in mind:

Your description should be *complete*, addressing all possibilities that could arise. In the problem above, did you take into account the fact that the shape might have curved lines? That it might have lines intersecting each other? Or even multiple lines intersecting at the same point as in the letter “K”.

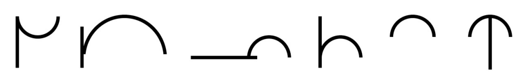
Your description should be *understandable* (to the intended reader). For example, you would be correct if you gave Wikipedia’s definition of a pentagon, “a five-sided polygon,” but this doesn’t do any good if the fifth grader doesn’t know what a polygon is.

Your description should be *unambiguous*. This means that your description should be completely clear, and not open to any different interpretations. You should ask yourself, “Would it be possible for the reader to read my description carefully, but misunderstand what I mean?” For example, if you said “a pentagon must have five sides,” would someone reading the description think that this meant “exactly five sides” or “at least five sides”? If you think there is any possibility that your language might be misinterpreted, you should be more specific.

Try this next exercise to practice writing a thorough description:

3 Describe how to draw a lowercase letter “r”, to someone who’s never seen our alphabet. Once your description is complete, go on to the next problem.

4 The shapes below are some possible ways that the reader could draw an “r” incorrectly – make sure your description from the previous problem is clear enough



5 How would you explain what speed is to someone who doesn’t understand the phrase “how fast you’re going”?

If you thought that the task of describing how to draw an r was a little bit silly, you’re right. Everyone in your class knows how to draw an r, and children who don’t know how to make an r are taught by example, not by description. Likewise, the chances are slim of your running into someone without a layperson’s understanding of speed.

On the other hand, the task of providing a basic description can be illuminating in its own right. It’s easy to believe that you understand what speed is, but when encouraged to define it for someone who doesn’t understand certain phrases, you’re likely to think more carefully about what speed actually is and therefore be able to use this understanding in future problems dealing with speed. Providing a careful description of something often helps you to understand it, bringing assumptions to light and revealing questions that you may not have known you had.

With a few exceptions, the next batch of problems does not require you to pretend you have an audience with gaping holes in their knowledge.

Instead, the problems are opportunities for you to practice writing complete, understandable, unambiguous descriptions targeted at people with a mathematics background similar to your own. In Problems 9 and 10 in particular, working towards a description that covers all the cases will guide your investigation.

Targeting your explanations toward an audience of your peers will give you the chance to practice another virtue: making your descriptions *concise*. Charged with writing a complete description, people sometimes respond by using as many words as possible. But a beautiful description is thorough while at the same time using as few words as possible. Keep that in mind while working on the following problems.

6 Last year you learned to solve a system of equations such as
$2x - 4y = 15$
$-2x + 8y = -1$
by adding the equations to each other in order to create a simpler equation that we could solve. Explain why we can solve the original system of equations by using this new, simpler equation in place of this system.

7 In the standard window of your calculator to graph $y = \frac{(x+2)(x+3)}{x+3}$. In the ZOOM menu choose the ZDecimal option. With this option, your calculator will set up a window where Xmin $= -4.7$, Xmax $= 4.7$, Ymin $= -3.1$, and Ymax $= 3.1$. Now, describe the graph of the equation and explain why it is this way.

8 You learned in the Combinatorics lesson that the number of ways four people can stand in a line is 4x3x2x1=24. Describe a procedure they could use for systematically standing in all 24 configurations, without repeating any configuration.

9 You may remember that the absolute value function $|x|$ takes an number x and makes it positive... so $|3| = 3$ and $|-3| = 3$.

a. Make a table of values, then use it to graph the equation $y = |x|$.

b. Make a table of values, then use it to graph the equation $y = |x - 3|$.

c. Describe the graph of $y = |x - h|$ clearly enough that, once you tell someone a specific value of h, they could draw the graph accurately.

10 An equation of the form $y = x^n$ is called a power function. $y = x^2$ and $y = x^3$ are two simple examples, with $n = 2$ and $n = 3$. Use your calculator to graph some different power functions, making sure you have chosen a sensible window. Describe the effect of the different values of n on the graph. Make sure you try values of n that are negative, fractional, etc. Try to categorize all the different types of graphs you can get depending on what n is.

11 Describe a circle to someone who's never seen anything circular. You can either explain how to draw one, or define what it is.

12 Do this problem with a partner. First, draw a meaningless shape on a piece of paper — just doodle for a few seconds, and make it look fairly random, but not too complicated. Don't let your partner see the shape.

Then, your partner should get a pen and paper, and you should describe what your shape looks like and have them try to draw it. Don't look at what they're drawing or make any gestures — just tell them what to do in words. At the end, look to see how accurately it turned out.

You should each be the "describer" twice. Try to make your second try more accurate based on what you learned from your first try.

There are even more benefits to providing a clear description! Not only can it help you convince others that your solution is correct, a clear, complete description can help you be sure that you haven't overlooked anything in your solution to the problem. In addition, by doing problems such as the average speed or the four-people-in-a-line problem, you might have learned something deep about the nature of the concepts involved. Finally, you may find that, in providing a complete description of your solution to one problem, you may see a way to apply the solution to another problem.

In the final set of problems, the description will no longer be the task itself. Each of these problems requires you to come up with a clear solution method. The trick will be to come up with a plan for solving each problem, after which the solution becomes much easier. As you work the problems, imagine explaining your solution method to your classmates. This will not only help you be ready to convince your classmates that your solution works, it will make sure that you are really honest about following your method.

Don't forget: after initially tinkering to get a feel for the problem, ask yourself, "what's my plan?"

13 A merchant visited three state fairs. At the first, she doubled her money and spent \$150. At the second, she tripled her money and spent \$270. And, at the third, she quadrupled her money and spent \$360. Alexi found that she had \\$240 at the end. How much money did she start with? Is there another solution?

14 Rihanna decides to give her three friends — Alicia, Jason, and Sali — some of her nickels that she brought to school. She gives Alicia half of her nickels plus one half of a nickel. Then she gives Jason half of her remaining nickels, plus half a nickel. Finally, she gives her last remaining nickel to Sali. How many nickels did Rihanna bring to school?

15 A book has been opened at random. What pages has it been opened to if the product of the facing pages is 298662?

16 A store was selling toothbrushes for 50¢ each. When the price was reduced, though not too much, the remaining stock sold for \$31.93. What was the reduced price?

17 What is the next term in the sequence $1, 4, 6, 26, 158, 4110$? Write a formula for the sequence using the T_n notation you learned when studying sequences.

18 Examine the following sequence: 3, 6, 12, 24...

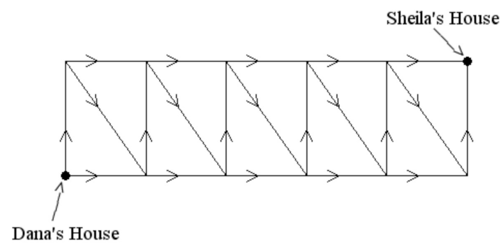
a. Find the next term, then write a formula for the nth term of the sequence.

b. Sequences that can be written in the form $T_n = T_1 \cdot r^{n-1}$ are called geometric sequences. Make up three more geometric sequences, with an eye toward teaching yourself what the effect of both T_1 and are on the sequence.

c. Do geometric sequences always get bigger and bigger as n increases? If not, under what conditions do they get bigger and bigger, and what are the other possibilities?

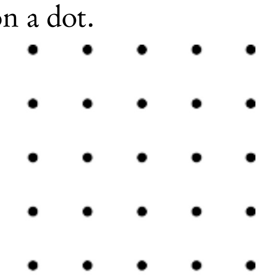
19 Students in Mr. Willard's P.E. class are spaced evenly around a circle and then told to count off beginning with 1. If student number 16 is directly opposite student number 47, then how many students are in Mr. Willard's class?

20 How many different routes are there from Dana's house to Sheila's if their town was laid out as shown below. Dana can only move in the direction of the arrows.



21 Find a way to cut any parallelogram into pieces that can be arranged into a rectangle.

22 These dots are spaced one unit apart, measured horizontally and vertically. Find a way to draw a square with area exactly 10 square units, so that each vertex of the square is on a dot.



Park School Mathematics

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