BOOK 8: FUNCTIONS AND MODELS I

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HABITS

look for patterns: to look for patterns amongst a set of numbers or figures

tinker: to play around with numbers, figures, or other mathematical expressions in order to learn something more about them or the situation; experiment

describe: to describe clearly a problem, a process, a series of steps to a solution; modulate the language (its complexity or formalness) depending on the audience

visualize: to draw, or represent in some fashion, a diagram in order to help understand a problem; to interpret or vary a given diagram

represent symbolically: to use algebra to solve problems efficiently and to have more confidence in one's answer, and also so as to communicate solutions more persuasively, to acquire deeper understanding of problems, and to investigate the possibility of multiple solutions

prove: to desire that a statement be proved to you or by you; to engage in dialogue aimed at clarifying an argument; to establish a deductive proof; to use indirect reasoning or a counterexample as a way of constructing an argument

check for plausibility: to routinely check the reasonableness of any statement in a problem or its proposed solution, regardless of whether it seems true or false on initial impression; to be particularly skeptical of results that seem contradictory or implausible, whether the source be peer, teacher, evening news, book, newspaper, internet or some other; and to look at special and limiting cases to see if a formula or an argument makes sense in some easily examined specific situations

take things apart: to break a large or complex problem into smaller chunks or cases, achieve some understanding of these parts or cases, and rebuild the original problem; to focus on one part of a problem (or definition or concept) in order to understand the larger problem

conjecture: to generalize from specific examples; to extend or combine ideas in order to form new ones

change or simplify the problem: to change some variables or unknowns to numbers; to change the value of a constant to make the problem easier; change one of the conditions of the problem; to reduce or increase the number of conditions; to specialize the problem; make the problem more general

work backwards: to reverse a process as a way of trying to understand it or as a way of learning something new; to work a problem backwards as a way of solving

re-examine the problem: to look at a problem slowly and carefully, closely examining it and thinking about the meaning and implications of each term, phrase, number and piece of information given before trying to answer the question posed

change representations: to look at a problem from a different perspective by representing it using mathematical concepts that are not directly suggested by the problem; to invent an equivalent problem, about a seemingly different situation, to which the present problem can be reduced; to use a different field (mathematics or other) from the present problem's field in order to learn more about its structure

create: to invent mathematics both for utilitarian purposes (such as in constructing an algorithm) and for fun (such as in a mathematical game); to posit a series of premises (axioms) and see what can be logically derived from them

HABITS

LESSON 0: REEXAMINE THE PROBLEM

> to look at a problem slowly and carefully, closely examining it and thinking about the meaning and implications of each term, phrase, number and piece of information given before trying to answer the question posed.

Some mathematics problems are challenging because they just do not seem to provide enough information to the reader to be solved. Often, though, that is because we have not looked carefully enough at the question and thought about what each of the words being used implies. To Re-examine the Problem is to look at each and every word, term, and equation in the statement of the problem and to see if there is more information to be gleaned from them if you look more closely.

In the problem above, at first it seems that much of the information given is completely unhelpful and irrelevant. Although there seems to be insufficient information to determine the daughters' ages, that should prompt us to re-examine the problem to see if some of that "irrelevant" information can in fact be used productively. While knowing the product of the ages is 36 doesn't tell us the ages of the daughters, it does limit the possibilities of what the ages could be. What are the possibilities? And although we don't know the house number, the census taker does, and yet she still cannot say for sure what the ages are. So, which of the possibilities that you listed are consistent with that fact? And while blueberry pancakes are soon to be eaten by the man and his daughter, is that the only piece of additional information you can get from that sentence?

Problems

1 A census taker came to a house where a man lived with 3 daughters. "What are your daughters' ages?" she asked. The man replied, "The product of their ages is 36, and the sum of their ages is equal to our house number." "That's still not enough information" the census taker said, after looking at the number. "Sorry, I have to go", the man said, "my oldest daughter needs help making blueberry pancakes".

The census taker then promptly wrote down the three daughters' ages and moved on to the next house. How did she know the ages? What are the ages?

2 Three squares are placed next to each other as shown. The vertices A, B, and C are collinear. Find the dimensions of the largest square. Copyright Phillips Exeter Academy.

Sometimes, even when a problem does not seem to be all that complicated, we still find that there seems to be insufficient information. Often that is because we have not looked fully at what the words in the question imply. For example, although you likely took care to label the lengths of all 12 sides of the 3 squares, did you ask yourself what it means for A, B, and C to be collinear? What do line segments AB and BC thus have in common? It is easy for us to "read past" certain words without fully grasping what they are telling us. Carefully re-examining the problem gives us a chance to really slow down and look, as if with a microscope, at each individual part.

3 A triangle where each side is an integer has a perimeter of 8. What is its area?

4 A man was looking at a portrait. Someone asked him, "Whose picture are you looking at?" He replied: "Brothers and sisters have I none, but this man's father is my father's son."

Whose picture was the man looking at?

5 Which will save you the most amount of gas in a year: changing from a 10 mpg to a 20 mpg car, changing from a 20 mpg to a 40 mpg car, or changing from a 40 mpg to an 80 mpg car?

6 At 6 o'clock the wall clock struck 6 times. Checking with my watch, I noticed that the time between the first and last strokes was 30 seconds. How long will the clock take to strike 12 at midnight?

7 If 4 copiers can process 400 sheets of paper in 4 hours, how long would it take 8 copiers to process 800 sheets?

8 If your answer to #7 was "8 hours", read the question again, slowly, and think carefully about what the information you are given is actually telling you!

9

Smallville and Tinytown are 200 miles apart and are connected by a straight railroad track. At 2 o'clock a train leaves Smallville at 50 mph and another train leaves Tinytown at 40 mph. When they eventually meet, which train will be closer to Tinytown?

10

Take a look at the following proof:

$a = b$

$a^2 = ba$

$a^2 - b^2 = ba - b^2$

$a^2 - b^2 = b(a - b)$

$(a + b)(a - b) = b(a - b)$

$a + b = b$

$2b = b$

$2 = 1$

What went wrong?

11

In the figure below, can one determine $\angle A + \angle B + \angle C + \angle D$ in terms of angle n? If not, explain why. If so, what does it equal?

12

Amber bought several identical boxes of CrazyCreme cookies to serve at a big party. She noticed as she was leaving the store that the number of boxes she bought is the same as the number of cents in the cost of a single box. If she spent $37.44 total, what is the cost of a single box of cookies?

13

How many integers from 1 to 400, inclusive, are not the square of an integer?

14

If $0 \leq w \leq 6$ and $-2 \leq p \leq 4$, what is the largest and smallest value wp can have?

15

Three businessmen — Smith, Robinson, and Jones — all live in the Leeds-Sheffield district. Three railwaymen (a guard, a stoker, and an engineer) of similar names live in the same district. The businessman Robinson and the guard live at Sheffield, the businessman Jones and the stoker live at Leeds, while the businessman Smith and the railway engineer live halfway between Leeds and Sheffield. The guard's namesake earns $10,000 per annum, and the engineer earns exactly one-third of the salary of the businessman living nearest to him. Finally, the railwayman Smith beats the stoker at billiards. What is the engineer's name?

16

For what values of x does $2^x + x^4 + 3 = 0$?

17

If , is $b - a > 2$?

18

In a triangle where $\angle A$ is the largest angle, $(\sin A)^2 + (\tan A)^2 + (\cos A)^2 = 2$. What does $m\angle B + m\angle C$ equal?

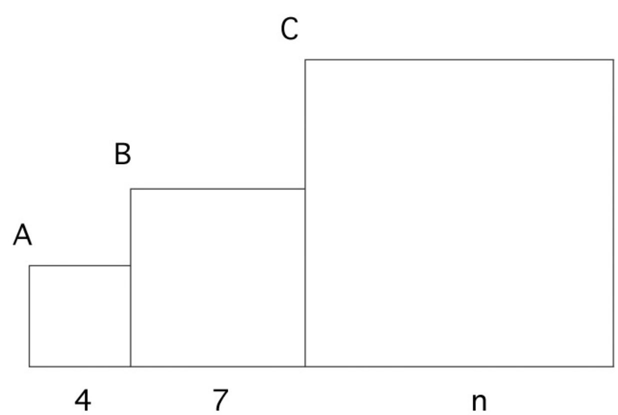
19

A standard deck of playing cards contains 26 black cards and 26 red cards. A deck is randomly divided into two unequal piles, such that the probability of drawing a red card from the smaller pile is 1/3. At the same time, the probability of drawing a black card from the larger pile is 5/14. How many cards are in the larger pile?

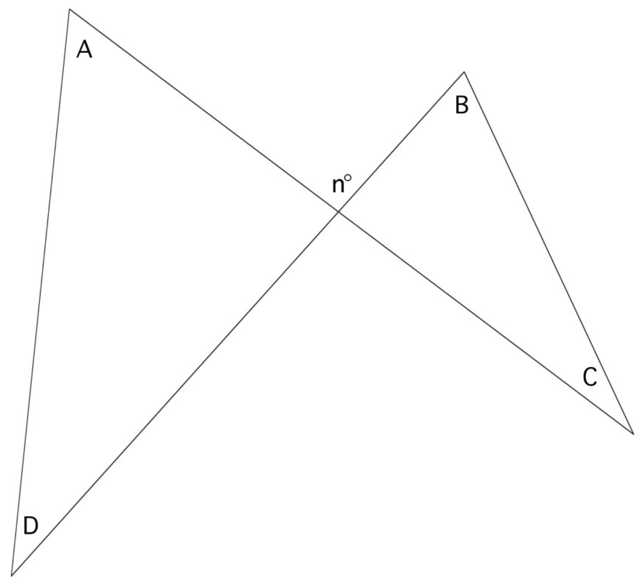
20 In isosceles triangle ABC, $\overline{BC}$ is not quite twice as long as either of the other two equal sides. Which is larger, sin A or cos A ?

21 A magician put four yellow, four green, and four red eggs in each of two hats. He called a person from the audience, blindfolded him, and asked him to transfer five eggs from hat 1 to hat 2. The magician then asked the audience to tell the blindfolded assistant how many eggs to return to hat 1 to ensure that in hat 1 there will be at least three eggs of each of the three colors. What would you have told the assistant?

22 Let X and Y be circles of diameter 2 that are tangent to each other in the plane. How many circles of diameter 6 are in this plane and tangent to X and to Y?



re-examine the problem



re-examine the problem

LESSON 1: GEOMETRIC SEQUENCES AND SERIES

Introduction

Designers, poets, mathematicians, and most everyone appreciates patterns. Our bodies and minds like patterns, such as sequences of behaviors and mental practices that are organized and efficient, like the pattern of actions you follow in coming to school every morning and the procedure you know to solve a particular class of problems.

You saw patterns of numbers that formed sequences when you worked with arithmetic sequences, such as: 3, 8, 13, 18, ... Below are three sequences. Which two of the three share a common characteristic? What is that special effect?

1. Putting a picture on a copier and hitting the 100% enlarge button five times; and then creating a sequence of numbers representing the widths of each of the five different images.

2. Taking a cherry from a bowl, then taking 3 more than the first pick, then taking 6 more than the second pick, then taking 12 more; and then writing down the number of cherries that you have taken one time after the other.

3. Starting with a cookie, take away half a cookie, then take away a half of the remaining portion, then another half of what remains, and another half; and then creating a sequence of how much of the original cookie remained after each removal.

Development: Geometric Sequences

Earlier you considered arithmetic sequences, which are sequences having a common difference, d, between terms. So we would be inclined to describe the sequence 6, 10, 14 as an arithmetic sequence, for we could see the sequence having a first term $T_1 = 6$ with .

4 The two sequences in problems 1-3 above that were similar are referred to as geometric sequences. How would you describe them in English?

The next term of the sequence 2, 6, 18, 54 could be 162, as the sequence begins with 2 and each succeeding term can be created by multiplying the prior term by 3. So we could say it was a geometric sequence where each term after the first is created by multiplying by 3, so $54 \cdot 3 = 162$. We will call 3 the multiplier, or common ratio.

5 In the sequence 2, 6, 18, 54, 162, ..., we can represent the first term, 2, as T_1, the second term, 6, as T_2, etc.

a. How would you represent the third term, T_3, in terms of T_1 and the multiplier?

b. How would you represent the fourth term, in terms of T_1?

c. How about the nth term, T_n?

6 If we had the geometric sequence 5, 10, 20, what would T_5 and T_7 be?

In the last problem if you weren't told that the sequence was geometric, you could think that the sequence 5, 10, 20, ... wasn't geometric. You might well have noticed that the successive differences between terms form a pattern: first 5, then 10; so this could lead you to think that the next term of the sequence would be 15 more than the prior term of 20; namely, 35. In general any sequence, no matter how many terms are given, is open to the next term being of infinite possibility — if one is clever enough to choose it so that it fits the form of the sequence terms.

7 If the sequence had 35 after 20, how could you argue that the sequence wasn't geometric?

8 Considering the geometric sequence: 4, 6, 9, ...,

a. What would you say would be the next two terms?

b. How would you write the general term, T_n, of the sequence?

c. Can you describe another pattern beginning with the terms 4, 6, and 9 that would not be geometric? What would be the 6th term?

9 Create values for a and b that establish patterns for each of the following sequences. See if you can generalize by expressing mathematically the general term, T_n. [Note: these need not be geometric sequences.]

a. 16, a, 24, b, ...

b. 17, a, 1, -7, ...

c. 4, 2, 8, a, b, ...

d. 1, 2, 4, a, b, ...

e. 6, 1, a, 1/36, b, ...

10 The following sequences are to be geometric.

a. Find x: 5, x, 125?

b. What about in the sequence 15, x, 375?

c. How about $\frac{5}{2}, x, \frac{125}{2}$? And $\frac{2}{5}, x, \frac{2}{125}$? What is r, the multiplier?

d. Can you create a proportion for each of these three-term sequences so that the common ratio, r, can be easily observed?

11 What would you think the graph of a geometric sequence would look like? Choose some geometric sequence and draw the graph with the number of the term, n, on the x-axis and the value of the term, T_n, on the y-axis. How would you compare this image to the one obtained by drawing an arithmetic sequence?

Practice

12 Suppose T_n is a geometric sequence with $T_1 = 2$ and multiplier $r = 1.5$. Find the 4th term and the 14th term. How many terms would you need before the sequence contains the first value beyond 100,000?

13 For a geometric sequence let $T_1 = 4$ and let $r = \frac{1}{2}$. What would the 4th term and the 14th term be of the < ="" span=""> sequence? What would you guess would be the 4th and 14th terms if you halved r? Check your guess.

Suppose in a geometric sequence $T_4 = 1$. Find T_1 if $r = \frac{1}{4}$.

Create three geometric sequences where $T_2 = 16$, such that one would increase, one would decrease, and one would oscillate between positive and negative numbers.

14 Choose a number between 0 and 1 and let this be the value of r for a geometric sequence. Now, let the first term of this sequence be 2. What will happen to the sequence values as you find more and more terms?

15 If $T_1 = 6$ and $T_6 = 6144$ in a geometric sequence, find T_7.

16 If $T_5 = 640$ in a geometric sequence, what value of r would make $T_8 = 40960$?

17 In a geometric sequence, if the 2nd term is 85 and the 7th term is -27, then:

a. .Would the 30th term be positive or negative?

b. What about the 1,259,776,901st term?

18 Is there a sequence that can be claimed to be both arithmetic and geometric?

19 Can you create an arithmetic sequence where the terms tend to get closer and closer to 0, without ever going past 0, as you continue to create more and more terms?

20 How would you describe the impact r has on the value of the terms in a geometric sequence? Be careful in choosing r— see what happens if it is positive or negative. Also, are the effects on the geometric sequence different if $0 < r < 1$ versus $r > 1$?

21 Write two geometric sequences that begin with the same value, but where in one the tenth term is positive, and in the other the tenth term is negative.

22 If $T_n = 3\left(\sqrt{5}\right)^n$, what are all the values of n that would make T_n rational?

23 The function $Next(a, b)$ takes any two consecutive terms of a geometric sequence and gives you the next term (assuming a is the earlier term and b is the later term).

a. .What is $Next(14, 42)$?

b. What is $Next(A, 1)$?

c. Write an equation for $Next(a, b)$.

d. What is $Next(3, Next(3, 12))$?

24 In a geometric sequence An with common multiplier r,

a. What is the ratio of $A8$ to $A5$?

b. What is the ratio of $A3$ to $A9$?

25 Draw a 3-sided figure, 4-sided figure, 5-sided figure, and a 6-sided figure. Now draw all the diagonals you can in each figure. How many do you find for each of the four shapes? Does the number of diagonals suggest an arithmetic, geometric, or some other type of sequence? If you put these values in a table, see if you can guess how many diagonals there would be for a 7-sided figure if the pattern continues. 10-sided? Any idea about the n-sided polygon?

26 Bowser and Lulu are playing chess and eating pizza. Lulu asks Bowser what would be the dimensions of a square board that had half the area of the original $8 \times 8\}$ board. And the dimensions of another square board that would have half the area of the prior one? And half again? Bowser said, “I don’t know. But to keep things in proportion, we should halve the area of our pizza with the 8” radius every time we halve the area of the board.” As Bowser and Lulu are dogs, what would you determine the four consecutive halvings of the board and the pizza to be?

As you saw with arithmetic sequences, we create an arithmetic series by summing the consecutive terms of such a sequence.

27 Do you recall for any arithmetic series how to express the sum of the first n terms? If not, make up an arithmetic sequence and see if that triggers any memories.

Because we clearly didn’t want to spend time adding all the terms for an arithmetic series when the number of terms is beyond a few, we made a commitment to find a formula. And to do that we had to look at the series in an imaginative way. Well, that holds also for coming up with a formula for a geometric series.

The clever technique that allows us to generate the formula for the sum of a geometric series is called “telescoping”. Let’s try the technique with a concrete instance. Suppose we are asked to find the sum of the geometric series: $1 + 2 + 4 + 8 + \ldots 512$. Let’s call the sum S and write:

$S = 1 + 2 + 4 + 8 + \ldots 512$ y .

28 Now multiply both sides by 2 (since $r = 2$), and write a new equation. Try to combine the two equations (by addition or subtraction) into one equation that gives you an "easy" way to calculate S. Can you make sense of why the method is called "telescoping"?

29 Use your understanding of what you did in Problem 30 to calculate the sum of the following geometric series: $S = \frac{3}{1} + \frac{3}{4} + \frac{3}{16} + \frac{3}{64} + \frac{3}{256}$.

30 Use your understanding of what you did in Problems 30 and 31 to describe how to sum up any geometric series.

Going Further: Geometric Series Practice

31 Find the sum of the first 11 terms of the following geometric series:

a. $2 - 10 + 50 - 250 + \ldots$

b. $6 + 1\frac{1}{2} + \frac{3}{8} + \frac{3}{32} + \ldots$

c. $28 - 14 + 7 - \ldots$

32 Find the sum, S, of the following geometric series:

a. .

b. $32 + 48 + \ldots + 162$

c. $1000 + 10 + \ldots + 10^{-17}$

33 Suppose you had an $8 \times 8\}$ checkerboard in front of you. Now imagine putting one grain of rice on the first square, 2 grains on the second square, 4 on the third, and continue doubling. (Ask your teacher or a website about the famous story that lies behind this problem.)

a. Give your best estimate (no calculators or paper, please!) of how many grains of rice there would be on the 20th and 64th squares.

b. Now calculate how many grains of rice there would be on the 20th square.

c. What would be the total of all the rice grains up to the 20th square?

d. What would be the power of 10 that would be closest to the sum of the rice grains on all 64 squares?

34 Write the sum of the following series in as compact a form as possible: $375(1.21)^0 + 375(1.21)^1 + 375(1.21)^2 + \ldots 375(1.21)^n$

35 Write the sum of in as compact a form as possible.

36 What is the sum of the first 23 terms of the geometric sequence that begins with $T_1 = 32$ and has $r =$ 1.1?

37 What would be the first term of a geometric series if the sum of the first 11 terms is 3069 and the common ratio is $r =$ 2?

38 If the first term of a geometric series is $T_1 = 2$ and $r =$ 3, is it possible that the series sum is 4920? Explain.

39

Look at the formula for the sum of a geometric series, from problem 37.

a. What do you imagine would happen to the formula if r was between 0 and 1, and the number of terms would be infinite — that is, the multiplication would continue indefinitely?

b. Using what you just found, what would be the sum of the infinite series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$?

c. What would be the sum of the infinite series $\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}\ldots$?

d. Make up your own infinite geometric series and find the sum. Now see what your classmates came up with.

40

Don't use a calculator for this problem.

a. Add .

b. Solve for x: $x^2+4x+1=2$.

c. Simplify $\sqrt{3000000}$.

d. Simplify $\frac{x^3y^8+x^2y^4}{x^5y^7}$.

e. Simplify $(-1)^2+(-1)^3-(-1)^4-(-1)^5$.

41 Draw a good-sized equilateral triangle filling about a third of an entire page. Let's say the area is 1 square unit.

a. Now divide each side of the triangle into thirds.

b. On the middle segment of each side, draw another equilateral triangle going out from that side of the triangle where the lengths of the sides are the same as the base.

c. What's the additional area that has been added by the three triangles?

d. Now on each of these three triangles, divide each side into thirds and again create triangles.

e. Can you determine the additional area created by the six new triangles?

f. This progression is known as a Koch snowflake. Can you find the area if the procedure was continued for three more divisions and sets of created triangles?

g. What about for *n* divisions? An infinite number of divisions?

42 Suppose you had the choice of getting paid $10,000 for your month's work or getting the amount determined by summing the amount of money you would receive for 30 days, starting with one penny on the first day, and doubling the amount for each succeeding day. Which salary structure would generate the most money? By how much?

43 Suppose you invested $50 in an ecology stock, which was doing really well — going up by 1% per month. How much money would your initial investment be worth after a year and a half?

44

Imagine dropping a ball from a height of 5 feet. Suppose the material it's made out of and the surface it's hitting combine to have the effect of the ball bouncing back up two-thirds of the way.

a. Considering just the falling distances, what would be the total distance the ball fell after 4 bounces? 8 bounces?

b. Now what would be the total distance for each of these numbers of bounces if you considered its full path — going up as well as down?

c. What do you think would be the total distance the ball would travel?

45

Find the sum of the series: $0.7 + 0.07 + 0.007 + 0.0007$. Can you convert this series to have a geometric form? What does your form say the next term would be? Is it correct decimal-wise?

46

When Jane met Tarzan, Tarzan naturally wanted to show off to Jane and also arrange to get a pair of decent sneakers because his feet were killing him after running on the ground barefoot all these years. To show off a bit, get the sneakers, and save wear and tear on his feet, Tarzan told Jane that he would swing from vine to vine without coming down for an hour. She agreed to give him money for each vine switch, starting with $1, and then doubling the amount for each subsequent switch (since the same activity gets harder as even Tarzan gets tired over time).

a. How many vines would Tarzan have to switch in order to get $60 to buy a pair of sneakers?

b. If each switch took 5 seconds, how long will he have to stay in the air in order to reach his goal of a pair of sneakers?

47

A geometric sequence has 4 for the first term, and the sum of the first three terms of the series equals 172. Find the possible values for the common ratio.

48

How many terms must be added in the geometric series $10 + 5 + 2.5 + \ldots$ in order to first pass the sum of 18? 19? 20?

49 Here's a very special sequence from history. It's called the Fibonacci sequence named after Leonardo of Pisa, whose surname was Fibonacci. He put together the following sequence: 1, 1, 2, 3, 5, 8, Perhaps you've seen it before.

a. What do you think the next two terms of the sequence are?

b. Do you notice anything about the 3rd, 6th, 9th, ... terms?

c. What about the 4th, 8th, 12th? And the 6th, 12th, 18th? Any conjectures come to mind?

50 Creating sequences can be great fun and sometimes the outcomes can be quite outstanding.

a. Consider the sequence 1/1, 3/2, 7/5, ... where each succeeding term is created by adding the top and bottom of the fraction to create the new denominator, and adding the top and twice the bottom to get the numerator. Try this for a number of terms. And if you can write or find someone who can write a calculator program, see if you can guess where this sequence is going.

b. Here is another special sequence: 4/1, 8/3, 32/9, ... where to get the nth term first determine if n, the number of the term, is even or odd. If it's even, multiply the previous term by $\frac{n}{n+1}$; if it's odd multiply by $\frac{n+1}{n}$. See if you get an idea where this sequence is going toward. (From Prime Obsession, by John Derbyshire.)

Exploring in Depth

LESSON 2: MODELING WITH EXPONENTIAL FUNCTIONS

Introduction: From the Discrete to the Continuous

1 The town of Tinyville has been growing at a steady rate for many years now. Its population at the end of the last three years is indicated in the table below. The population has been growing geometrically.

Year	2004	2005	2006
Population	212960	222640	232760

What will Tinyville's population be at the end of 2007? at the end of 2008? n years after 2006?

What will Tinyville's population be 4.3 years after the end of 2006?

Your answer to Problem 1 Part b is based on an assumption — that is, that the formula for describing Tinyville's population isn't restricted to inputting whole numbers. In other words, it seems perfectly natural to assume that Tinyville's population doesn't wait until the end of the year to grow; population change is happening on a daily basis, if not hourly, every minute, or even every second. Had you been asked in the lesson on geometric sequences to graph the function you wrote in Problem 1 Part a, the graph would have looked like the graph shown in Figure 1.

Figure 1 Figure 2

The dots are not connected because t can only take on whole number values. Notice that the ordered pair (4.3, 281839) is not a point on this graph. The graph in Figure 2 was created by assuming that t can take on non-integer values, such as 4.3, 1.67, or 7.3065. It is reasonable to assume that this graph is a more accurate depiction of how Tinyville's population is growing. Both of the graphs show solutions to an exponential function, but it is the latter graph — the one in Figure 2 — that we will be studying at this point.

To help us transition from the discrete aspect of the geometric sequence graphed in Figure 1 to the continuous aspect of the function graphed in Figure 2, we will introduce some notation. The equation for the first figure could be $P_n = 232760(1.0455)^n$, where $n \geq 0$. Unfortunately, this kind of subscripting doesn't allow for $P_{4.3}$: there is no such thing as the 4.3rd term of the sequence. We can get around this difficulty by replacing the P_n with $P(n)$ and writing the equation as $P(n) = 232760(1.0455)^n$. Just as with Twist(x,y) or Oblongness(a,b), you don't multiply P by n. We read this equation as "P of n equals..." $P(2)$ is the population two years after 2006 (i.e., $232760(1.0455)^2 \approx 254423$), and $P(4.3)$ is the population 4.3 years after 2006 (i.e., $232760(1.0455)^{4.3} \approx 281839$). This notation is called $function notation$ or $Euler notation$.

2 Calculate $P(6.3)$ and explain the meaning of this number in the context of the problem.

3 If $f(x) = 5x^2$ and $g(t) = 3 \cdot 2^t$, then calculate each of the following:

a. $f(2)$

b. $g(2)$

c. $f(-1)$

An $exponential function$ is any function that can be written in the form $f(x) = a \cdot b^x$, where $a > 0, b > 0$, and $b \neq 1$. We will take a look at these restrictions on a and b shortly, but first let's explore a couple more problems.

4 Look back at the work you did in Problem 1. When will the population of Tinyville reach 500,000? Use your calculator to help you answer this questions with 3-decimal place accuracy — i.e., correctly rounded to three digits after the decimal point.

5 By what percentage does Tinyville's population grow from year to year? This number is often called the $annual percentage growth rate$. Where does this number show up in your expression describing Tinyville's population t years after 2006?

6

Tinyville's rival town is Punyville. Punyville's population at the end of 2004 had reached 375,000 but was now starting to decline geometrically. Its population at the end of each of the past three years is shown in the table below.

$Year$	2004	2005	2006
$Population$	375000	367500	360150

What will Punyville's population be at the end of 2007? At the end of 2008? t years after 2006? Write your answer to the last question using function notation.

What is the annual percentage decay rate for the population of Punyville? How does this number show up in your expression for Punyville's population t years after 2006?

On a coordinate axis system sketch a graph of Tinyville's population for $t = 0$ years to $t = 15$ years. On the same coordinate axis system sketch a graph of Punyville's population over the same time period.

When will the population of Punyville reach 300,000? Use 3-decimal place accuracy.

Based on your graph, when will Punyville's population first drop below Tinyville's? Now answer the same question but use 3-decimal place accuracy.

7

Not too far from Tinyville and Punyville is the town of Smallville. Smallville's population grew rapidly and reached 610,000 by the end of 2006. Unfortunately, this rapid growth has now precipitated a rapid decline in the population due to overcrowding. The rate of decline is 13% per year.

a. On the axes you drew for Problem 6 Part c, sketch a graph of Smallville's population over the same time period as you drew for Tinyville and Punyville.

b. How does the graph of Smallville's population over time relate to Punyville's? What is similar about them? How can you account for this similarity?

c. Suppose Smallville's population decay rate had been 10% rather than 13%. Without using your calculator, sketch Smallville's population over time on the same set of axes as in Part a. Now check with your calculator.

8 Jill's car is 10 years old and given that it has approximately 170,000 miles on it, she's considering selling it and purchasing a new car. The initial value of the car was $19,500.

a. Based on Jill's research she determines that the car's value has been depreciating at a rate of 25% per year. To the nearest dollar, how much is the car worth now?

b. A used car dealer offers to purchase the car from Jill for $550. Assuming that the dealer believes that this is the actual present value of the car, what depreciation rate did the dealer use to arrive at this price? Solve this problem two ways: using a graph and using algebra. Give both answers accurate to two decimal places.

9 What can you say about the shape of the graph of $f(x) = a \cdot b^x$ if and $0 < b < 1$? As b increases in value from 0 toward 1 how does the graph change?

10 What can you say about the shape of the graph of $f(x) = a \cdot b^x$ if $a > 0$ and $b > 1$? As b increases in value how does the graph change?

11 Suppose $b = 1$ in the equation $f(x) = a \cdot b^x$, where $a > 0$. What does the graph of this function look like? In what way(s) is this graph fundamentally different from the graphs you sketched in Problems 6 or 7? Answer the same question for when $b = 0$.

12 Now suppose $b = -2$ and $a = 1$, so the equation is $f(x) = (-2)^x$.

a. Without using the graphing or table features of your calculator complete the following table. Make sure to enclose -2 in parentheses when doing your calculations.

x	-2	-1	0	1	2	3	4
$f(x)$							

Plot the points you found in Part a and connect them with a smooth curve so as to indicate what the graph of $f(x) = (-2)^x$ might look like.

Describe how the graph in Part b differs from all the other graphs you've sketched in this lesson.

Due to at least one of the differences that you addressed in Problem 12 Part c, mathematicians have decided to limit the values of the base of an exponential function to positive real numbers that are not one.

13

On a coordinate axis system sketch a rough graph of $y = 2^x$. Don't worry about the horizontal scale, but please make sure that the y-intercept is approximately correct. Now on the same axis system, and without the aid of a calculator, sketch graphs of $f(x) = 3 \cdot 2^x$, , $h(x) = 5^x$, , and $m(x) = 0.5^x$.

14

Each function given below models some sort of exponential growth or decay situation. For each function identify the percentage growth or decay rate.

a.

b.

c.

d.

e.

f.

15

Bacteria are growing at a rate of 43% per hour. If there are 500 bacteria now how many bacteria will there be this time tomorrow?

16

How long do you think it will take for the bacteria population in Problem 15 to exceed 10 million? After making an educated guesstimate, come up with a more precise answer as well. Give your answer using three- decimal place accuracy.

17 Suppose you start to slow down your car in such a way that its speed decreases 10% each second. If your initial speed is 65 mph, estimate how long it will take for your speed to drop down below 35 mph. Then calculate the answer more precisely. Give your answer using two-decimal place accuracy.

18 If you want to drop the speed of your car exponentially from 65 mph to 35 mph over a ten-second period, then by what percentage should the speed decrease each second? Solve this problem by using algebra and use two-decimal place accuracy.

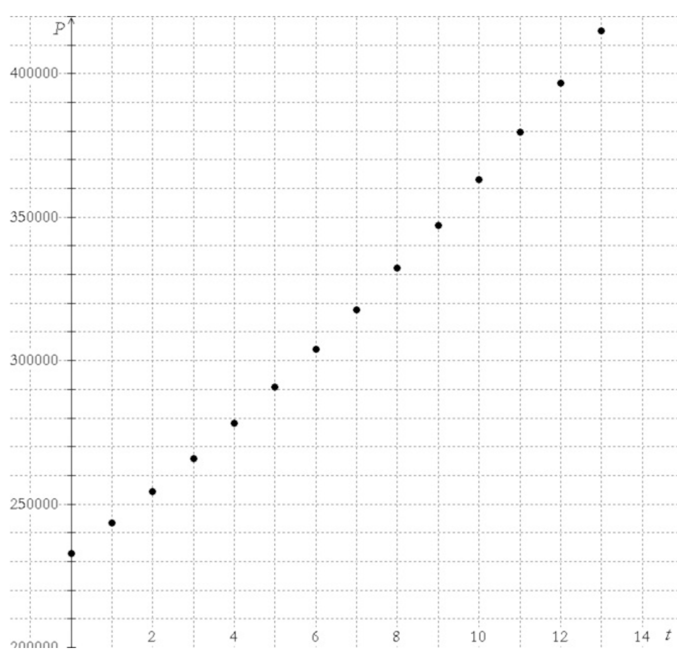
19 Let $f(x) = 4x - x^2$ and $g(x) = 3^x$. Calculate each of the following:

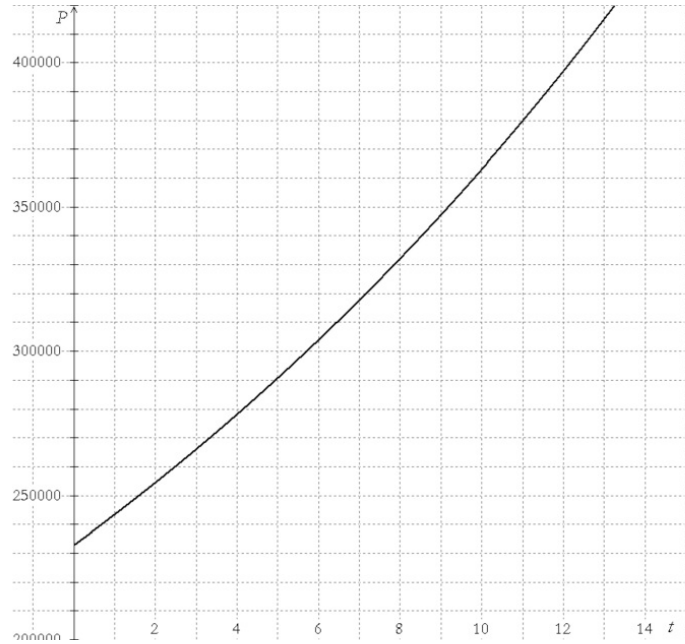
a. $f(-5)$

b. $g(2)$

20 Determine all values of b such that $100 = 24(b)^{10}$} by using graphs. Use two-decimal place accuracy.

21 Determine all values of b such that $1.65 = 5(b)^8$ by using algebra. Use three-decimal place accuracy.




P
400000
350000
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250000
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2
4
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12
14
t

Problems

22 If $f(x) = 3x + 6$ and $h(x) = 2^x$, calculate each of the following. Remember that if parentheses are nested inside of each other, you work on the inside parentheses first.

a. $f(h(0))$

b. $h(f(0))$

c. $h(h(x))$

d. $f\left(\frac{1}{3}x - 2\right)$ y

e. .

23 Determine the values of a and b in $f(x) = a \cdot b^x$ if the following facts are known about f.

a. $(0, 3)$ and $(2, 5)$ are on the graph of the function.

b. $(1, 2)$ and $(3, 10)$ are on the graph of the function.

c. $(2, 3)$ and $(8, 1)$ are on the graph of the function.

d. $(-3, 8)$ and $(1, 1)$ are on the graph of the function.

e. $f(0) = 5$ and $f(3) = 20$.

24 Isaac Newton determined that the positive difference between an object's temperature and its surrounding medium decreases exponentially. You place a cup of hot coffee (initial temperature 180° F) on a table in a room (temperature 70° F). The initial difference in temperature is 110°. After 30 minutes the temperature difference is 60°.

a. Write an equation that models the temperature difference D between the cup of coffee and the room at time t. Use function notation with D and t.

b. How long will it take for the temperature difference to drop to 10° F? Use 3-decimal place accuracy.

c. How long will it take for the coffee to reach a temperature of 75°? Use 3-decimal place accuracy.

25 (Copyright Phillips Exeter Academy) A helium filled balloon is slowly deflating. During any 24-hour period it loses 10% of the helium it had at the beginning of the 24-hour period. At 10 am on Sunday the balloon initially held 10,000 cc of helium.

a. How much helium did the balloon contain 3 days later?

b. Approximately how much time will it take for the balloon to lose half of its initial helium? This time is called the $half-life$. Use 2-decimal place accuracy.

c. Approximately how long will it take for the amount of helium in the balloon to drop to 2,500 cc? Use 2-decimal place accuracy.

d. Approximately how long will it take for the amount of helium in the balloon to drop to 12.5% of its initial amount? Use 2-decimal place accuracy.

In Problems 26-31, you will need to pay close attention to what information you're being given and what question you're trying to solve. With exponential growth and decay problems, the fact that something is decaying by 3% monthly doesn't mean that the base of the exponential function is 3, .03, or even -.03. Often, the natural way of expressing a problem doesn't translate directly into symbolic expressions, so read each problem carefully, pondering the meaning of each term and number that is presented, before venturing into answering the question posed. In doing this, you will be using the habit of $re-examinetheproblem$.

26 An atom of carbon-14 is unstable, meaning that at any moment there is a chance that it will spontaneously change (actually decay) into nitrogen. The half-life of carbon-14 is approximately 5728 years. What is the annual percentage decay rate of carbon-14? Use algebra to solve this problem.

27 The half-life of californium is approximately 898 years. How long does it take for a given mass of californium to decrease by 99%?

28 N grams of a substance has decreased to $.2N$ grams after 60 days. What is the percentage daily decrease of the mass of this substance? Use algebra to solve this problem.

29 Rachel has money in an account that pays her 4% annual interest, compounded yearly.

a. How many years will it take for the money in her account to double?

b. How many years ago was it that the amount of money she had in her account was half of what it is now?

30 You open your locker and discover mold on your favorite snack bread that you inadvertently left in your locker several months ago. The function $M(t) = 25672486(1.18)^t$ describes the number of mold spores found growing on the piece of bread t days after the mold is discovered.

a. How many mold spores were on the bread 5 days ago?

b. What is the daily growth rate of the spore population?

c. What is the hourly growth rate of the spore population? Use $change or simplify the problem$ by thinking about which number in the equation can be changed (made easier) without changing what's really happening with the mold spores.

d. Write a function modeling the number of spores on the bread t hours after the mold was discovered.

31 (Copyright Phillips Exeter Academy) Jennings Bryan Osborne of Little Rock, Arkansas is known for building extravagant displays of Christmas lights. In 1992, Osborne used 1,600,000 light bulbs, which were guaranteed to be of high quality. The probability that any one bulb would burn out in a given 24-hour period was only 0.01 percent. Assuming that Osborne leaves his display on continuously, how many bulbs can be expected to still be working after

a. 1 day?

b. 2 days?

c. 30 days?

32 Explain why a monthly growth rate of 1% is not equivalent to a yearly growth rate of 12%.

33 On the axis system shown below are four graphs (A-D) of exponential functions. Propose four possible equations for the graphs and explain how you arrived at your choices.

34 Define “” to be shorthand for calculating , where there are n number of f 's. For example, . Let . Calculate and .

35 How should we interpret expressions like $4^{3.5}$? Let's take a closer look.

a. Is 4^6 halfway between 4^5 and 4^7? Why or why not?

b. Is $4^{3.5}$ halfway between 4^3 and 4^4?

c. Rewrite $4^{3.5}$ as $4^{fraction}$.

d. Confirm that your answer to part c can be computed without ever using a calculator. Is $4^{3.5}$ closer to 4^3 or 4^4?

e. Use your answer to part a to help answer why your answer to part d makes intuitive sense.

36 There is another kind of mathematical notation that is similar to function notation. This notation, called $mappingnotation$, looks like the following:

$f : x \to x^2 - x.$

This notation is read as "f maps x to $x^2 - x$." Given this definition of f then, $f : 6 \to 30$.

a. Determine what number goes to the right of the arrow. i. $f : -5 \to$ ____ ii.

b. Who came up with this notation? What was the original use for it? Is it still used and in what contexts?

37 A bank offers 6% annual interest on a savings account, but compounded quarterly. What this means is that the bank will calculate the interest on the account each quarter (4 times per year) but will do this by using a 1.5% interest rate for each of these calculations (1/4th of the annual interest rate). Suppose you invest \$1000 into this account.

a. How much money will you have in your account at the end of the year?

b. If the bank changed to giving you interest on a monthly basis, rather than a quarterly basis, how much money would you have in your account at the end of the first year?

c. Complete the table below.

d. Given how our monetary system works, at what point, if any, do you not gain anymore money as the number of times compounding increases? Would you give the same answer if instead of investing \$1000 you invested \\$1 million? Explain.

38 Don't use a calculator for this problem.

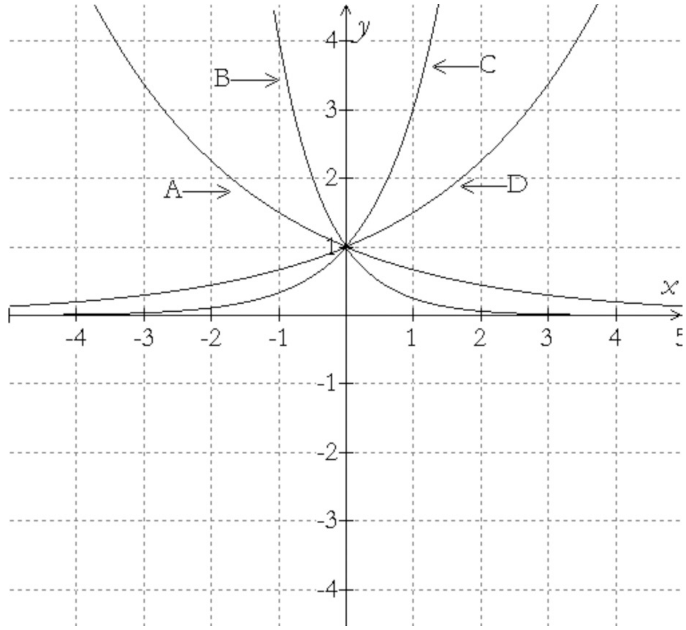
a. Name an angle that has a sine of 1. Then name an angle that has a cosine of -1. How many answers are there to these two questions?

b. Simplify $\frac{3(x-4)+6}{3}$.

c. Simplify $a^2b\sqrt{a^5b^8c^2}$.

d. Find the vertex of the parabola with equation .

e. Simplify .



Exploring in Depth

Compounding	Quarterly	Monthly	Daily	Hourly	Every minute
Total $ after one year	1061.36				

LESSON 3: EXPONENTIAL FUNCTIONS

Introduction: How fast?

The table shown below contains data on a baseball that has been tossed into the air. For each of the times indicated, the corresponding height (above the ground) of the ball is given.

t sec	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
h feet	5	8.84	12.36	15.56	18.44	21	23.24	25.16	26.76

t sec	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7
h feet	28.04	29	29.64	29.96	29.96	29.64	29	28.04	26.76

t sec	1.8	1.9	2.0	2.1	2.2	2.3	2.4	2.5	2.6
h feet	25.16	23.24	21	18.44	15.56	12.36	8.84	5	.84

1 Recall that the average speed of an object over a particular time interval is determined by the distance that it covers during that time period divided by the length of that time period. What was the average speed of the ball during the first second?

2 What was the average speed of the ball during the time interval from $t = 0.6$ to $t = 0.7$?

3 What was the average speed of the ball during the time interval from $t = 0.7$ to $t = 0.8$?

4 Approximately how fast was the ball going at $t = 0.7$ seconds?

5 Approximately how fast was the ball going at $t = 0.3$ seconds?

The average speed of the ball during the time interval $t = 0.6$ to $t = 0.7$ should have been equivalent to calculating the change in the ball's height divided by the amount of time during this change. In other words,

$$\text{averagespeed} = \frac{\text{changeinheight}}{\text{timetaken}} = \frac{(\text{finalheight})-(\text{initialheight})}{(\text{finaltime})-(\text{initialtime})}|.| \;.$$

While you may not have used this exact formula to do Problems 1-3, you probably used the reasoning encapsulated by the formula. But how did you arrive at an answer to Problem 4? Well, your answers to Problems 2 and 3 give you estimates of the ball's speed 1/10th of a second before $t = 0.7$ seconds and 1/10th of a second after $t = 0.7$ seconds. Since the ball probably didn't make any strange movements right at $t = 0.7$(suddenly stop, suddenly jump up to 100 feet and instantaneously fall back to 25.16 feet, et cetera), then averaging your answers to Problems 2 and 3 would be a reasonable way of estimating the speed of the ball at $t = 0.7$ seconds. This is just one way of trying to estimate the speed of an object at a particular time. We will look at a few other ways and also see how average speed (more generally known as average rate of change) relates to some other interesting concepts.

Development

Let's take another look at average speed and speed at a moment in time, this time by investigating an equation.

6 Ben has a wicked fastball — wicked in that it speeds up as it moves toward the plate. A fairly reasonable model for this pitch is $d(t) = 130(2)^t - 130$, where $d(t)$ stands for the distance (in feet) that the ball has traveled since it left Ben's hand after t time (in seconds).

a. What was the average speed of the ball during the time interval from $t = 0.1$ to $t = 0.2$? If you're not sure what to do here then think about how you did Problem 1 and how you can get this problem to be more like Problem 1.

b. What was the average speed of the ball during the time interval from $t = 0.2$ to $t = 0.3$?

c. The calculation you performed in Part b can be seen as the slope of a specific line. Explain why this is true.

d. Below is shown a graph of $d(t) = 130(2)^t - 130$ over the time interval $0 \leq t \leq 0.5$. On the axes below, carefully, and as accurately as possible, draw the line referred to in Part c. Extend the line enough so that someone looking at your work can see that the line is different from the curve.

e. Now, repeat what you did in Part d but draw the line that corresponds to the calculation you did in Part a. Make sure someone looking at your work can tell the difference between the lines.

f. Approximately what was the speed of the ball at $t = 0.2$ seconds?

g. There is a line that just touches the graph of $d(t) = 130(2)^t - 130$ at $t = 0.2$. This line does not pass through the graph. On the axes above, draw this line. Again, make sure to draw this line so that it can be seen to be different from the other two that you drew.

h. Estimate as accurately as possible the slope of the line you drew in Part g. How does this slope compare to your answer in Part f?

i. Estimate the speed of the ball at $t = 0.3$ seconds and then again at $t = 0.4$ seconds? Is the ball speeding up?

j. Estimate the slope of the line that just touches the graph at $t = 0.45$. What does this number tell you about the ball?

The line that you drew in Problem 6 Part g is called a tangent line. By definition we say that the slope of a curve at a specific point is the slope of the tangent line at that point. So, the slope of the curve $d(t) = 130(2)^t - 130$ at $t = 0.2$ is approximately 103.508. Also, since the variable graphed on the vertical axis is distance and the variable graphed on the horizontal axis is time, our slope calculation

$\frac{d_2 - d_1}{t_2 - t_1} = \frac{\text{changeindistance}}{\text{changeintime}}$

gives us the speed of the ball, 103.508 feet/second (approximately 70.6 mph).

If we're looking at a physical situation that involves distance versus time, we have seen how speed and slope of the curve are related. Let's look at what more the slope of a curve can tell us.

7 Graph the function $y = 2^x$ on paper over the interval $-3 \leq x \leq 3$.

a. Estimate the slope of the curve at $x = -2,\ -1,\ 0,\ 1, \text{and} 2$.

b. How will the slope of the curve change as the value of x increases beyond the value of $x = 2$? How will it change as the value of x decreases from $x = -2$?

c. Use your calculations in Part a to explain why $y = 2^x$ is not a linear function.

8 Graph the function $y = 0.7^x$ on paper over the interval $-3 \leq x \leq 3$.

a. Calculate $\frac{0.7^{2.001} - 0.7^2}{2.001 - 2}$ and explain the meaning of this number in the context of this problem.

b. Is it reasonable to think of your calculation in Part a as an estimate of the slope of the curve at $x = 2$? How can you get an even better estimate?

c. Estimate the slope of the curve at $x = -2,\ -1,\ 0,\ 1, \text{and} 2$.

d. Write an expression that would represent an estimate of the slope of the curve at x = b. This expression should have b along with some constants.

e. Looking back at your solution to Part d, for what value(s) of x is the slope of the curve less than -1?

f. Are there any values for x for which the slope of the curve is 0? You might visualize what the tangent line would look like in order to solve this problem.

Your work in Problems 7 and 8 should have given you some idea of how you can use the slope of a curve to determine whether the curve is linear. Also, how the slope of the curve changes can give you some sense of what is happening to the curve.

We will add only one thing more to our understanding of the slope of the curve.

9 A rabbit population is growing according to the model $R(t) = 200(1.02)^t$, where t is measured in days.

a. Estimate the slope of the curve at t = 3. What does this number tell you about the rabbit population? Be specific.

b. At what rate (rabbits/days) is the population increasing at 6 days?

c. What is the daily percentage growth rate of the rabbit population?

d. Why are your numerical answers to Parts a, b, and c all different from each other?

As you may have surmised from the previous problem, "growth rate" is a somewhat ambiguous term: you need to know whether the rate is a percentage growth rate (your answer to Part c and the number that is built into the model) or an absolute growth rate (your answers to Parts a and b). Moreover, the slope of the curve tells you how two quantities change with respect to each other, not necessarily how one quantity changes with respect to itself, which is what percentage change has to do with.

Practice

10 A ball is dropped from the top of a building. Its height h (in meters) at time t (in seconds) is given by the function formula $h(t) = -4.9t^2 + 80$.

a. What was the average speed of the ball during the time interval from t = 1 second to t = 1.3 seconds.

b. Estimate the speed of the ball at 1.5 seconds.

11 Shown below is a graph of .

12 a. Draw, as accurately as possible, the tangent line to the curve at x = 3.

b. Estimate the slope of the line that you drew in Part a.

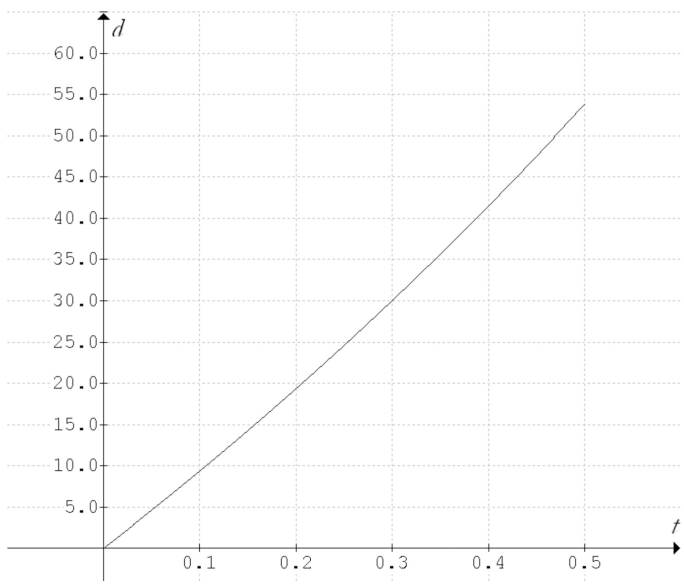
c. Estimate the slope of the curve at x = 3.

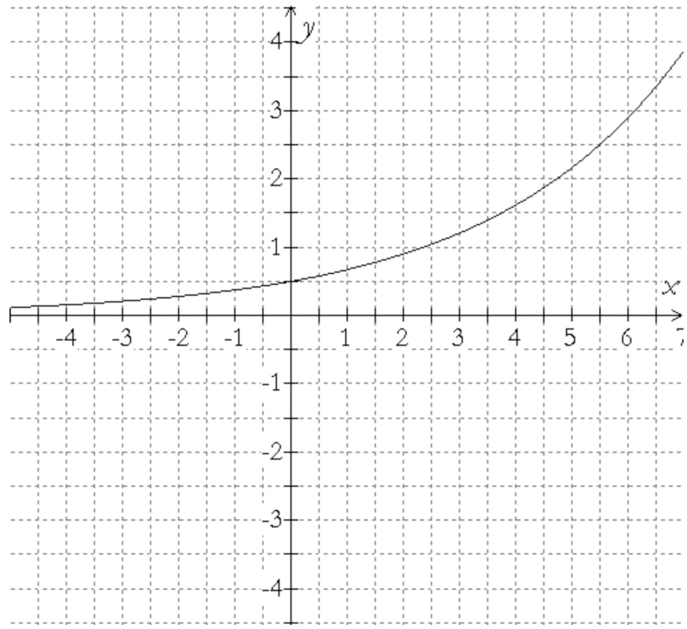
13 You accidentally inhale a toxic gas. The concentration C(t) (in micrograms per milliliter) of poison in your blood at time t (in hours) is given by $C(t) = 3.72(0.9359^t)$.

a. Determine the slope of the curve described by C at t = 2 hours.

b. What does the slope that you calculated in Part a tell you about the concentration of poison in your blood? What are the units for this slope?

c. What is the percentage growth or decay rate of the concentration of poison in your blood?





Year	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Rabbits	1599	2362	2803	2229	1206	594	338	239	210	223	275	382	580	925	1486
Wolves	43	54	80	118	140	135	118	99	82	68	57	48	43	40	42

Year	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
Rabbits	2241	2790	2387	1352	662	365	249	212	218	263	360	540	856	1379	2115
Wolves	51	75	112	138	137	121	102	85	70	58	50	43	40	41	49

Throughout the problems in this section and the next, you'll need to pay close attention to the information presented, particularly what is implied by the information. For example, when you're told that the ball has hit the ground, this should mean to you that its height is zero, and you can bring this information into your calculations. Thinking carefully about the implications of the terms and information in a problem is part of our habit re-examine the problem and will come in handy in solving many problems.

14 A baseball is shot from a cannon so that its height h (in feet) above the ground at time t (in seconds) is given by $h(t) = -16t^2 + 128t + 4$.

a. Determine the speed of the ball when it hits the ground. Use 2-decimal place accuracy.

b. Determine the speed of the ball when it reaches its highest point.

15 The table below shows the populations over time of two species that occupy the same area: rabbits and wolves. Counts are taken at the beginning of each year.

16 a. Calculate the average increase in the wolf population over the first 4 years.

b. Calculate the average decrease in the rabbit population during the 3 years that start with the beginning of the 19th year.

c. Approximately what was the rate of change (wolves per year) of the wolf population at the beginning of year 20?

d. Indicate the longest time period over which both species simultaneously have a positive growth rate.

e. Indicate the longest period over which one species has a positive growth rate while the other has a negative growth rate.

f. When is the first time that the rabbits have zero growth rate? What is happening with the other species and why might this be happening?

17 The following table is for a function $y = b^x$.

x	-2	-1	0	1	2
approximate slope at x	.184	.304	.5	.826	1.363

Sketch a reasonable graph for the function.

Estimate the value of b. Show your work or explain how you arrived at your solution.

18 The graph of a function $f(x) = a \cdot b^x$ is shown below. As accurately as possible estimate the slope of the curve at x = 3 and at x = 6.

19 a.

20 Let $g(x) = 2 \cdot (0.5^x)$. Calculate $\frac{g(2+h)-g(2)}{h}$, where $h = 0.001$. What does this calculation tell you about the function g? Can you come up with an even better estimate in this situation?

21 A bacterium population has been decreasing exponentially. Twenty hours ago the population was 1 million. Ten hours later the population had dropped to half a million. Suppose the bacterium population continues to drop off at the same rate.

a. Write an equation that models the bacterium population.

b. What is the percentage hourly decay rate of the population? Use 2-decimal place accuracy.

c. Exactly ten hours ago, what was the population decay rate (in bacteria per hour)? Use 3-decimal place accuracy.

d. Five hours ago would the population decay rate (in bacteria per hour) have been smaller (in magnitude) than your answer to Part c, larger than it, or effectively the same? Answer this question without doing any calculations. Explain how you arrived at your answer.

e. In how many hours will the population drop to one bacterium? Use 3-decimal place accuracy.

22 On a mathematics quiz the following problem was presented:

Calculate $\frac{2^{2.2001}-2^{2.2}}{.0001}$ using 3-decimal place accuracy and explain the meaning of this number.

In response to this question, Jessica wrote, "3.185; It's the approximate slope of the curve $y = 2^x$ at $y = 2.2$. David wrote, "3.185; It's the approximate slope of the curve $y = 2^x$ at $x = 2.2001$." Meredith wrote, "3.185; It's the approximate slope of the curve $y = 2^x$ at $x = 2.20005$." How would you grade their responses?

23 Tony wishes to estimate the slope of the tangent line to the curve $y = 12(1.3)^x$ at $x = -1.25$. He wants to write a fraction that will complete this task and have 0.0001 in the denominator. Write a fraction that will work for him. Now, write another fraction that will work. Write one more.

24 Peter says that $\frac{0}{0}$ is equal to 17! Here's his argument: another way of writing $5 \times 6 = 30$ is to say that $\frac{30}{6} = 5$. Therefore we can say that another way of writing $17 \times 0 = 0$ is that $\frac{0}{0} = 17$. Do you agree with Peter, or not? Explain!

25 Robert and Allan each throw a baseball at the same time. The height (in feet) of Robert's ball above the ground after t seconds is given by $R(t) = -16t^2 + 50t + 5$. The formula for Allan's ball is $A(t) = -16t^2 + 60t + 4$.

a. What is the average (vertical) speed of Robert's ball during the first second?

b. Estimate the speed of Allan's ball at 1 second.

c. Is there a time when both balls have the same speed?

26 Let $f(x) = \sqrt{25 - x^2}$.

a. Explain why the graph of f is a semicircle. What is the radius of the semicircle?

b. First find the slope of f at the point $(4, 3)$. Then find an equation for the line tangent to the graph going through this point.

c. Find an equation for a line perpendicular to the tangent line, also passing through the point (4,3). Before you do, would you predict that this line crosses the x-axis to left of (0,0), to the right of (0,0), or that it goes through (0,0)?

27 Let $f(x) = \sqrt{x}$ and $g(x) = 4 - 4(0.5^x)$.

a. Kiran was calculating the slope of f at x and thought he could express his answer in two equivalent ways: $\frac{\sqrt{x+.001}-\sqrt{x}}{.001}$ and $\frac{\sqrt{x}+\sqrt{.001}-\sqrt{x}}{.001}$. Give two explanations for why these expressions are not equivalent: one involving algebra and the other involving slope.

b. Kiran then calculated the slope of g at x and wrote $\frac{4-4(0.5^{x+.001})-(4-4(0.5^x))}{.001}$. Show how this expression can be rewritten as $\frac{4\cdot 0.5^x(1-0.5^{.001})}{.001}$.

c. As x increases, what happens to the slope of f? of g?

d. Kiran argues that, given what's happening in Part c, both functions should be doing essentially the same thing as x goes toward infinity. What do you think?

28

Speed and velocity are not synonyms. "What?!" you say. Velocity is speed with direction. For example, if you go forward with a velocity of 5 yd/sec, stop, and then go backwards with a speed of 3 yd/sec, your velocity while going backwards was -3 yd/sec, where the negative merely indicates that you are going in the direction opposite to the positive direction.

Let $h(t) = -16t^2 + 60t + 4$ describe the height, in feet, of a ball above the ground at t seconds. Estimate the velocity of the ball at t = 1 second and at t = 3 seconds.

29

Acceleration is defined to be the change in velocity over the change in time. For the ball in Problem 25, estimate its acceleration at t = 1 second.

30

Let $f(x) = 2^x$.

a. Complete the table below using 3-decimal place accuracy.

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7
Slope of f at x								

On a coordinate axis system plot the ordered pairs from the completed table in Part a. Draw a smooth curve through these points.

Estimate the slope of the curve you drew in Part b at x = 0.5. What does this number tell you about the original curve f at x = 0.5?

31

Don't use a calculator for this problem.

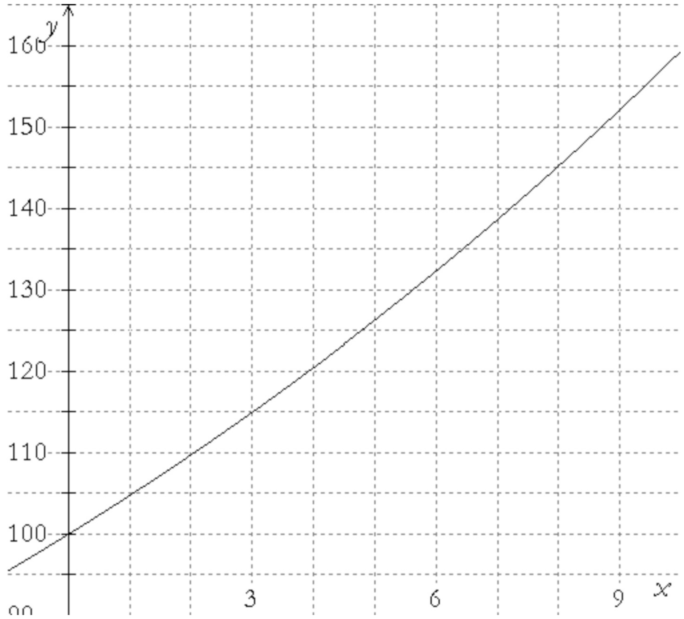
a. Simplify $\frac{\sin 135^\circ + \cos 135^\circ}{\tan 135^\circ}$.

b. *b*. Subtract $\frac{x+3}{6} - \frac{x-5}{8}$.

c. *c*. Simplify $\frac{\left(x^2y\right)^{-2}x^3z}{x^4\left(yz^2\right)^3}$.

d. *d*. Solve for x if $|x-8| \leq 16$.

e. *e*. Solve for x: $x = \frac{1}{x+1} + \frac{3}{2x+2}$



Exploring in Depth

Park School Mathematics

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