Problema 1

Să se determine o formulă de cuadratură de forma:

$$\int_0^\infty e^{-t} f(t)dt = A_1 f(0) + A_2 f'(0) + A_3 f(t_3) + A_4 f(t_4) + R(f)$$

care să aibă *grad maxim de exactitate*. Dacă neglijăm restul, integrala se poate aproxima prin:

$\int_0^\infty e^{-t} f(t)dt \approx A_1 f(0) + A_2 f'(0) + A_3 f(t_3) + A_4 f(t_4)$. Fiindca dorim ca formula de cuadratură să aibă grad maxim de exactitate, adică pentru toate polinoamele cu grad <= n, integrala:

$\int_0^\infty e^{-t} f(t)dt = A_1 f(0) + A_2 f'(0) + A_3 f(t_3) + A_4 f(t_4)$, pentru toate polinoamele $\mathbb{P} \in \mathbb{P}_n$.

Căutăm o formulă de cuadratură $F(f) = \int_0^\infty w(t) f(t)dt$ care este exactă, anume aplicând formula asupra funcției, obținem chiar valorile integralei. Integrala de aproximat este apropiată de o cuadratură de tip **Gauss-Laguerre**, insă ne încurcă apariția derivatei $f'(0)$. Alegem funcțiile de test $f(t) = 1, t, t^2, t^3, \ldots$, deoarece acestea se pretează cuadraturilor de tip Gauss-Laguerre, putând face o paralelă cu aceste tipuri de cuadraturi.

Din definiția funcției Gamma avem: $\int_0^\infty e^{-t} t^k dt = \Gamma(k+1) = k!$ Astfel putem, rezultă următorul tabel:

$$\begin{bmatrix} f(t) & f(0) & f'(0) & f(t_3) & f(t_4) & \displaystyle\int_0^\infty e^{-t} f(t)\mathrm{dt} \\ 1 & 1 & 0 & 1 & 1 & 1 \\ t & 0 & 1 & t_3 & t_4 & 1 \\ t^2 & 0 & 0 & t_3^2 & t_4^2 & 2 \\ t^3 & 0 & 0 & t_3^3 & t_4^3 & 6 \\ t^4 & 0 & 0 & t_3^4 & t_4^4 & 24 \\ t^5 & 0 & 0 & t_3^4 & t_4^4 & 120 \end{bmatrix}$$

Rezultă sistemul (puncte fixe t_3, t_4):

$f(t) = 1: \ A_1 \cdot 1 + A_2 \cdot 0 + A_3 \cdot 1 + A_4 \cdot 1 = 1 \Rightarrow A_1 + A_3 + A_4 = 1$

$f(t) = t: A_1 \cdot 0 + A_2 \cdot 1 + A_3 \cdot t_3 + A_4 \cdot t_4 = 1 \Rightarrow A_2 + t_3 \cdot A_3 + t_4 \cdot A_4 = 1$

$f(t) = t^2: A_1 \cdot 0 + A_2 \cdot 0 + A_3 \cdot t_3^2 + A_4 \cdot t_4^2 = 2 \Rightarrow t_3^2 \cdot A_3 + t_4^2 \cdot A_4 = 2$

$f(t) = t^3: A_1 \cdot 0 + A_2 \cdot 0 + A_3 \cdot t_3^3 + A_4 \cdot t_4^3 = 6 \Rightarrow t_3^3 \cdot A_3 + t_4^3 \cdot A_4 = 6$

Pentru $t_3 \& t_4$, putem alege radacinile polinomului Laguerre $L_2(t) = \frac{1}{2}(t^2 - 4t + 2) = \frac{1}{2}t^2 - 2t + 1$ cu radacinile $t_1 = 2 - \sqrt{2}, t_2 = 2 + \sqrt{2}$. Alegem astfel $t_3 = 2 - \sqrt{2}, t_4 = 2 + \sqrt{2}$. Rezolvam sistemul:

$$A_1 = 1 - A_3 - A_4 \Leftrightarrow A_1 = 1 - \frac{4\sqrt{2}-3}{20\sqrt{2}-40} - \frac{1}{2\sqrt{2}\,(2+\sqrt{2})} \approx 1.123223...$$

$$A_2 = 1 - t_3 \cdot A_3 - t_4 \cdot A_4 \Leftrightarrow A_2 = 1 - (2-\sqrt{2})\,\frac{4\sqrt{2}-3}{20\sqrt{2}-40} - \frac{1}{2\sqrt{2}} \approx 0.779288...$$

$$A_3 = \frac{2 - t_4^2 \cdot A_4}{t_3^2} = \frac{2-(2+\sqrt{2})^2 \cdot A_4}{(2-\sqrt{2})^2}$$

$$A_3 = \frac{6-(2+\sqrt{2})^3 \cdot A_4}{(2-\sqrt{2})^3}$$

$$\Rightarrow \frac{6-(2+\sqrt{2})^3 \cdot A_4}{(2-\sqrt{2})^3} = \frac{2-(2+\sqrt{2})^2 \cdot A_4}{(2-\sqrt{2})^2} \Leftrightarrow 6-(2+\sqrt{2})^3 \cdot A_4 = 2(2-\sqrt{2}) - (2-\sqrt{2})(2+\sqrt{2})^2 \cdot A_4 \Leftrightarrow$$

$$\Leftrightarrow A_4 = \frac{4-2\sqrt{2}-6}{(2-\sqrt{2})(2+\sqrt{2})^2-(2+\sqrt{2})^3} = \frac{-2-2\sqrt{2}}{(2+\sqrt{2})^2(2-\sqrt{2}-2-\sqrt{2})} = \frac{2+\sqrt{2}}{2\sqrt{2}\,(2+\sqrt{2})^2} = \frac{1}{2\sqrt{2}\,(2+\sqrt{2})} \approx 0.103553...$$

$$\Rightarrow A_3 = \frac{6-(2+\sqrt{2})^3 \cdot \dfrac{1}{2\sqrt{2}\,(2+\sqrt{2})}}{(2-\sqrt{2})^3} = \frac{12\sqrt{2}-(2+\sqrt{2})^2}{2\sqrt{2}\cdot(2-\sqrt{2})^3} = \frac{12\sqrt{2}-4-4\sqrt{2}-2}{2\sqrt{2}\cdot(8-12\sqrt{2}+12-2\sqrt{2})} \Leftrightarrow$$

$$\Leftrightarrow A_3 = \frac{8\sqrt{2}-6}{40\sqrt{2}-80} = \frac{4\sqrt{2}-3}{20\sqrt{2}-40} \approx -0.226776...$$

Rezulta formula de cuadratura:

$$\int_0^\infty e^{-t} f(t)dt \approx 1.123223 \cdot f(0) + 0.779288 \cdot f'(0) - 0.226776 \cdot f(2-\sqrt{2}) + 0.103553 \cdot f(2+\sqrt{2})$$

$R(f) = \frac{f^{(4)}(\xi)}{4!} * C$, unde $C = \int_0^\infty e^{-t}(t-0)^0(t-0)^0(t-t_3)^2(t-t_4)^2dt$

Rezulta sistemul (toate necunoscutele)

$f(t) = 1 :\ A_1 \cdot 1 + A_2 \cdot 0 + A_3 \cdot 1 + A_4 \cdot 1 = 1 \Rightarrow A_1 + A_3 + A_4 = 1$

$f(t) = t : A_1 \cdot 0 + A_2 \cdot 1 + A_3 \cdot t_3 + A_4 \cdot t_4 = 1 \Rightarrow A_2 + t_3 \cdot A_3 + t_4 \cdot A_4 = 1$

$f(t) = t^2 : A_1 \cdot 0 + A_2 \cdot 0 + A_3 \cdot t_3^2 + A_4 \cdot t_4^2 = 2 \Rightarrow t_3^2 \cdot A_3 + t_4^2 \cdot A_4 = 2$

$f(t) = t^3 : A_1 \cdot 0 + A_2 \cdot 0 + A_3 \cdot t_3^3 + A_4 \cdot t_4^3 = 6 \Rightarrow t_3^3 \cdot A_3 + t_4^3 \cdot A_4 = 6$

$f(t) = t^4 : A_1 \cdot 0 + A_2 \cdot 0 + A_3 \cdot t_3^4 + A_4 \cdot t_4^4 = 24 \Rightarrow t_3^4 \cdot A_3 + t_4^4 \cdot A_4 = 24$

$f(t) = t^5 : A_1 \cdot 0 + A_2 \cdot 0 + A_3 \cdot t_3^5 + A_4 \cdot t_4^5 = 120 \Rightarrow t_3^5 \cdot A_3 + t_4^5 \cdot A_4 = 120$

Rezolvam sistemul cu MATLAB Symbolic.

```
syms A1 A2 A3 A4 t t3 t4

eq1 = A1 + A3 + A4 == 1;
eq2 = A2 + t3 .* A3 + t4 .* A4 == 1;
eq3 = t3 .^ 2 .* A3 + t4 .^ 2 .* A4 == 2;
eq4 = t3 .^ 3 .* A3 + t4 .^ 3 .* A4 == 6;
```

```
eq5 = t3 .^ 4 .* A3 + t4 .^ 4 .* A4 == 24;
eq6 = t3 .^ 5 .* A3 + t4 .^ 5 .* A4 == 120;

solution = solve([eq1, eq2, eq3, eq4, eq5, eq6], [A1, A2, A3, A4, t3, t4]);
fields = fieldnames(solution);
for i = 1:length(fields)
    fprintf('%s = ', fields{i});
    disp(vpa(solution.(fields{i}), 6));
    fprintf('\n');
end
```

```
A1 =
```

$$\begin{pmatrix} 0.611111 \\ 0.611111 \end{pmatrix}$$

```
A2 =
```

$$\begin{pmatrix} 0.166667 \\ 0.166667 \end{pmatrix}$$

```
A3 =
```

$$\begin{pmatrix} 0.0138889 \\ 0.375 \end{pmatrix}$$

```
A4 =
```

$$\begin{pmatrix} 0.375 \\ 0.0138889 \end{pmatrix}$$

```
t3 =
```

$$\begin{pmatrix} 6.0 \\ 2.0 \end{pmatrix}$$

```
t4 =
```

$$\begin{pmatrix} 2.0 \\ 6.0 \end{pmatrix}$$

Solutia finala

$$A_1 = \begin{pmatrix} 0.611111 \\ 0.611111 \end{pmatrix} A_2 = \begin{pmatrix} 0.166667 \\ 0.166667 \end{pmatrix} A_3 = \begin{pmatrix} 0.0138889 \\ 0.375 \end{pmatrix} A_4 = \begin{pmatrix} 0.375 \\ 0.0138889 \end{pmatrix} t_3 = \begin{pmatrix} 6.0 \\ 2.0 \end{pmatrix} t_4 = \begin{pmatrix} 2.0 \\ 6.0 \end{pmatrix}$$

Eroarea:

$$R(f) = \frac{f^{(6)}(\xi)}{6!} \cdot \int_0^\infty e^{-t}\omega(t)dt\,,\ \int_{-1}^1 \omega(t)dt = \int_0^\infty e^{-t}t^2(t-t_3)(t-t_4)dt = 2t_3t_4 - 6t_4 - 6t_3 + 24 = 18.704$$

```
syms t t3 t4

omega = exp(-t) .* t .^ 2 .* (t - t3) .* (t - t4);
int_omega = int(omega, t, 0, Inf);
fprintf("I_omega = ");
```

```
I_omega =
```

```
disp(int_omega);
```

$2\,t_3\,t_4 - 6\,t_4 - 6\,t_3 + 24$

```
fprintf("\n");

omega_known = exp(-t) .* t .^ 2 .* (t - 0.888999) .* (t + 0.00899889);
int_omega_known = int(omega_known, t, 0, Inf);
fprintf("I_omega = ");
```

```
I_omega =
```

```
disp(vpa(int_omega_known, 6));
```

18.704

```
fprintf("\n");
```

$$R_1(f) = R_2(f) = R(f) = \frac{f^{(6)}(\xi)}{6!} \cdot 18.704, \xi \in (-1, 1)$$

Rezulta 2 formule de cuadratura:

$$\int_{-1}^{1} f(t)\mathrm{d}t = 0.611111 \cdot f(-1) + 0.166667 \cdot f'(-1) + 0.0138889 \cdot f(6.0) + 0.375 \cdot f(2.0) + \frac{f^{(6)}(\xi)}{6!} \cdot 18.704, \xi \in (0, \infty)$$

$$\int_{-1}^{1} f(t)\mathrm{d}t = 0.611111 \cdot f(-1) + 0.166667 \cdot f'(-1) + 0.375 \cdot f(2.0) + 0.0138889 \cdot f(6.0) + \frac{f^{(6)}(\xi)}{6!} \cdot 18.704, \xi \in (0, \infty)$$

```
f = @(t) sin(t);
df = @(t) cos(t);
I_exact = integral(@(t) exp(-t) .* f(t), 0, Inf);

A1 = 0.611111;
A2 = 0.166667;
A3 = 0.0138889;
A4 = 0.375;
t3 = 6;
t4 = 2;

I_approx = A1 * f(0) + A2 * df(0) + A3 * f(t3) + A4 * f(t4);
R = I_exact - I_approx;

fprintf("### I_EXACT: %.16e\n", I_exact);
```

```
### I_EXACT: 5.0000000000048506e-01
```

```
fprintf("### I_APPROX: %.16e\n", I_approx);
```

```
### I_APPROX: 5.0377276114669556e-01
```

```
fprintf("### REST: %.16e\n", R);
```

```
### REST: -3.7727611462105015e-03
```

Problema 2

Fie ecuatia $f(x) = 0, f : [a, b] \to \mathbb{R}, f \in C^3[a, b]$ si α o radacina simpla a ei.

a) Sa se arate ca:

$$x_{k+1} = x_k - \frac{f(x_k)}{\sqrt{f'(x_k)^2 - f(x_k) * f''(x_k)}}$$

genereaza un sir care converge cubic.

Fiindca α este o radacina simpla, asta inseamna ca:

- $f(\alpha) = 0;$
- $f'(\alpha) \neq 0;$

Fiindca vrem sa aratam ca metoda iterativa converge cubic, putem arata ca eroarea $R_k = x_k - \alpha$ satisface conditia:

$$|R_{k+1}| = C \cdot R_k^3 + O(R_k^4)\text{, unde C este o constanta}$$

Stiind ca $f \in C^3[a, b]$, ne putem folosi de dezvoltarea Taylor pentru a arata ca $R_{k+1} = O(R_k)$.

$$f(x_k) = f(\alpha + R_k) = f(\alpha) + R_k \cdot f'(\alpha) + \frac{R_k^2}{2} \cdot f''(\alpha) + \frac{R_k^3}{6} \cdot f^{(3)}(\xi_k) = R_k \cdot f'(\alpha) + \frac{R_k^2}{2} \cdot f''(\alpha) + \frac{R_k^3}{6} \cdot f^{(3)}(\xi_k);\ \text{stiind ca}:\ f(\alpha) = 0$$

$$f'(x_k) = f'(\alpha + R_k) = f'(\alpha) + R_k \cdot f''(\alpha) + \frac{R_k^2}{2} \cdot f^{(3)}(\eta_k)$$

$$f''(x_k) = f''(\alpha + R_k) = f''(\alpha) + R_k \cdot f^{(3)}(\gamma_k),$$

$$\text{unde } \xi_k, \eta_k, \gamma_k \in (\alpha, x_k)\ \text{ si } R_k = x_k - \alpha$$

Rezulta ca:

$$x_{k+1} = x_k - \frac{f(x_k)}{\sqrt{f'(x_k)^2 - f(x_k) * f''(x_k)}} = \frac{R_k * f'(\alpha) + \frac{R_k^2}{2} * f''(\alpha) + \frac{R_k^3}{6} * f^{(3)}(\xi_k)}{\sqrt{\left[f'(\alpha) + R_k * f''(\alpha) + \frac{R_k^2}{2} * f^{(3)}(\eta_k)\right] - \left(R_k * f'(\alpha) + \frac{R_k^2}{2} * f''(\alpha) + \frac{R_k^3}{6} * f^{(3)}(\xi_k)\right) *}}$$

$$= \frac{R_k * f'(\alpha) + \frac{R_k^2}{2} * f''(\alpha) + O\left(R_k^3\right)}{f'(\alpha)^2 + R_k * f'(\alpha) * f''(\alpha) + R_k^2 * \left[f''(\alpha)^2 + f'(\alpha) * f^{(3)}(\eta_k) - \frac{1}{2} * f''(\alpha)^2 - f'(\alpha) * f^{(3)}(\gamma_k)\right] + O\left(R_k^3\right)}$$

Se observa ca termenul dominant este: $f'(\alpha)^2 - R_k * f'(\alpha) * f''(\alpha) + R_k^2$

Stiind ca $\sqrt{1-x} \approx 1 - \frac{x}{2}$ =>

$$x_{k+1} \approx x_k - \frac{R_k * f'(\alpha) + O\left(R_k^2\right)}{f'(\alpha)\left(1 - \frac{R_k^2 * f''(\alpha)}{2 * f'(\alpha)} + O\left(R_k^3\right)\right)} = x_k - R_k\left(1 + \frac{R_k * f''(\alpha)}{2 * f'(\alpha)} + O\left(R_k^2\right)\right) =$$

$$= x_k - R_k - \frac{R_k^2 * f''(\alpha)}{2 * f'(\alpha)} + O\left(R_k^3\right)$$

$$\Rightarrow R_{k+1} = x_{k+1} - \alpha = x_k - R_k - \frac{R_k^2 * f''(\alpha)}{2 * f'(\alpha)} + O\left(R_k^3\right) - \alpha = x_k - x_k + \alpha - \frac{R_k^2 * f''(\alpha)}{2 * f'(\alpha)} + O\left(R_k^3\right) - \alpha \Rightarrow$$

$$\Rightarrow R_{k+1} = C * O\left(R_k^3\right), C = -\frac{R_k^2 * f''(\alpha)}{2 * f'(\alpha)}$$

```
f = @(x) x .* exp(x) - 1;
df = @(x) exp(x) + x .* exp(x);
d2f = @(x) 2 * exp(x) + x .* exp(x);

[root, n_iter] = iter_method_cubic(f, df, d2f, 0.5);
fprintf('Root: %.16e | No. iterations: %d\n', root, n_iter);
```

```
Root: 5.6714329040978384e-01 | No. iterations: 3
```

```
function [root, n_iter] = iter_method_cubic(f, df, d2f, x0, tol, max_iter)
    %% ITER_METHOD_CUBIC = implementeaza metoda iterativa descrisa mai sus:
    %  x_{k+1} = x_k - f(x_k) / sqrt(f'(x_k)^2 - f(x_k) * f''(x_k))
    %
    % Inputs:
    %
    % f         - functia de aproximat;
    % df        - prima derivata a functiei;
    % d2f       - a doua derivata a functiei;
    % x0        - nodul de pornire;
    % tol       - eroarea de aproximare admisa;
    % max_iter  - numarul maxim de iteratii;
    %
    % Outputs:
    %
    % root      - radacina aproximata;
    % n_iter    - numarul de iteratii;
    % Eroare: impartire la 0/radical imaginar sau daca nu converge in numarul maxim de iterati

    if nargin < 4
        x0 = 0;
    end
    if nargin < 5
        tol = 1e-6;
    end
    if nargin < 6
        max_iter = 100;
    end

    root = x0;
    for n_iter = 1:max_iter
        fxk = f(root);
        dfxk = df(root);
        d2fxk = d2f(root);

        b = sqrt(dfxk .^ 2 - fxk .* d2fxk);
        if abs(b) < eps || ~isreal(b)
            error('Zero division or non-real radical')
```

```
        end

        xknext = root - fxk/b;
        if abs(xknext - root) < tol
            root = xknext;
            return;
        end

        root = xknext;
    end

    warning('No convergence in %d iterations!', max_iter);
end
```