MA 570. Homework Assignment 4
Solutions

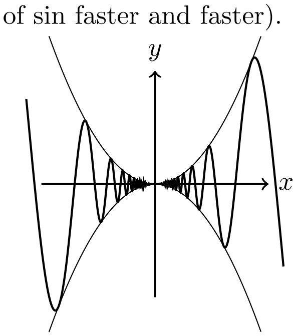
Write and upload your solutions. Make sure to exhibit your reasoning in each problem. Each problem 1–5 is worth 12 points.

Ash's book also has solutions. Feel free to read both. These are a bit more detailed.

(1) [~Ash 5.1.1] Let $f(x) = x^2 \sin(1/x), x \neq 0$; $f(0) = 0$.

(a) Make a rough sketch of graph of f around 0. (Start by drawing parabolas $y = x^2$ and $y = -x^2$. If you are not sure what to do, plot the graph in an automated tool and try to make sense of it.)

▷**Solution.** The main idea is that $f(x)$ goes back and forth between $-x^2$ and x^2, and closer to 0 the faster (since as x approaches 0, $1/x$ goes through periods of sin faster and faster).



(b) Show that f is differentiable at 0. Sketch the derivative of $f(x)$.

▷**Solution.** We have

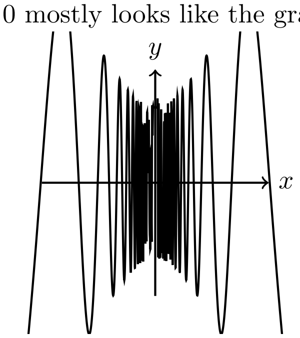
$$f'(0) = \lim_{x\to 0} \frac{x^2 \sin(1/x) - 0}{x} = \lim_{x\to 0} x \sin(1/x) = 0,$$

so the derivative exists and is equal to 0.

To sketch $f'(x)$, we also compute it at nonzero points:

$$f'(x) = -\cos(1/x) + 2x \sin(1/x),$$

so the graph, around 0 mostly looks like the graph of $-\cos(1/x)$.



(2) [~Ash 5.1.2] Suppose a differentiable function $f(x)$ has 100 distinct points where $f(x) = 0$. Show that $f'(x)$ has at least 99 points where $f'(x) = 0$.

▷**Solution.** We note that between any adjacent zeros, my Mean Value Theorem, there is a point where $f'(x) = 0$.

(3) [∼Ash 5.2.4] Find nth Taylor polynomial for the exponential function $f(x) = e^x$ at $x = 0$, and the corresponding remainder term.

▷**Solution.** $f^{(n)}(x) = e^x$. Therefore, the coefficient in front of x^k is $\frac{e^0}{k!} = \frac{1}{k!}$. We therefore get:

$$P_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} = \sum_{k=0}^{n} \frac{x^k}{k!}.$$

The remainder term is

$$R_{n+1}(x) = \frac{f^{(n+1)}(t)}{(n+1)!} x^{n+1} = \frac{e^t}{(n+1)!} x^{n+1},$$

where t is some point between 0 and x that depends on x and n.

(4) [∼Ash 5.2.4] We continue looking at Taylor polynomial and the remainder term for e^x that we found in the previous problem.

(a) Inspect the remainder term for this Taylor polynomial to show that at each point x, the values of Taylor polynomials converge to the value of e^x: $P_n(x) \to e^x$ as $n \to \infty$.

Comment. In other words, here we show that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ is valid for all real x.

▷**Solution.** The remainder term is

$$R_{n+1}(x) = \frac{f^{(n+1)}(t)}{(n+1)!} x^{n+1} = \frac{e^t}{(n+1)!} x^{n+1},$$

where t is some point between 0 and x that depends on x and n. In any case, $e^t \leq e^x$ if $x \leq 0$ and $e^t \leq 1$ if $x \leq 0$. Denote $\max\{1, e^x\} = C$, then we have:

$$|R_{n+1}(x) \leq C \cdot \frac{|x|^{n+1}}{(n+1)!} \to 0 \qquad \text{(as } n \to \infty\text{)}.$$

(To explain that $\frac{|x|^{n+1}}{(n+1)!} \to 0$, notice that when say, $n > 2|x|$, each increment of n by 1 multiplies this value by $\frac{|x|}{n+2} < \frac{1}{2}$.

(b) Using Taylor polynomials and the formula for the remainder term, approximate $e = e^1$ to within 10^{-2}. Make sure to justify that the error of your approximation is $< 10^{-2}$. You can also use automated computation (like WolframAlpha) to check how large the error of your approximation is, exactly.

You can take for granted that $e < 3$.

▷**Solution.** We need to find n s.t. $|R_{n+1}(1)| < 10^{-2}$. Write

$$R_{n+1}(1) \leq e^1 \cdot \frac{1^n}{(n+1)!} < 3 \cdot \frac{1^n}{(n+1)!}.$$

Make the RHS $< 10^{-2}$:

$$\frac{3}{(n+1)!} < 10^{-2},$$

$$(n+1)! > 300,$$

for which $n = 5$ is enough. We have:

$$\begin{aligned} e &\approx 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} = \\ &= 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} = \\ &= \frac{163}{60} = 2.716666\ldots \end{aligned}$$

with error $< \frac{3}{720} < 10^{-2}$. Note that actually

$$|e - \frac{163}{60}| < 0.0017,$$

so this approximation is a bit better than we estimated.

(5) Let $f(x, y) = 2x^2 + 2a \cdot xy + 3y^2 - 2x + 12y$, where a is some constant. Find Hessian of f.

(a) Find Hessian of $f(x, y)$.

▷**Solution.** We have $\frac{\partial^2 f}{\partial x^2} = 4$, and so on. Then Hessian is the matrix:

$$\begin{pmatrix} 4 & 2a \\ 2a & 6 \end{pmatrix}.$$

Notice that here Hessian is constant. This only happens with quadratic functions.

(b) Find for what values of a the function f is guaranteed to be convex (=concave up).

▷**Solution.** In other words, we need to determine when Hessian is positive definite, that is we need to check two conditions:

$$4 > 0,$$

$$\det \begin{pmatrix} 4 & 2a \\ 2a & 6 \end{pmatrix} > 0,$$

that is, $4 \cdot 6 - 2a \cdot 2a > 0$, $24 - 4a^2 > 0$, so $-\sqrt{6} < a < \sqrt{6}$.

(c) Suppose x_0, y_0 is a critical point (i.e., both partial derivatives are 0) of $f(x, y)$, with $f(x_0, y_0) = f_0$. Find second-order approximation $P_2(x, y)$ of f at that point. (You are not asked to actually find x_0, y_0, even though it's possible in this case.)

▷**Solution.** We have, taking into account that first partial derivative are 0:

$$\begin{aligned} P_2(x, y) &= f_0 + \frac{1}{2}(4(x - x_0)^2 + 4a(x - x_0)(y - y_0) + 6(y - y_0)^2) \\ &= f_0 + 2(x - x_0)^2 + 2a(x - x_0)(y - y_0) + 3(y - y_0)^2 \end{aligned}$$