Homework 4

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Question 1

Let

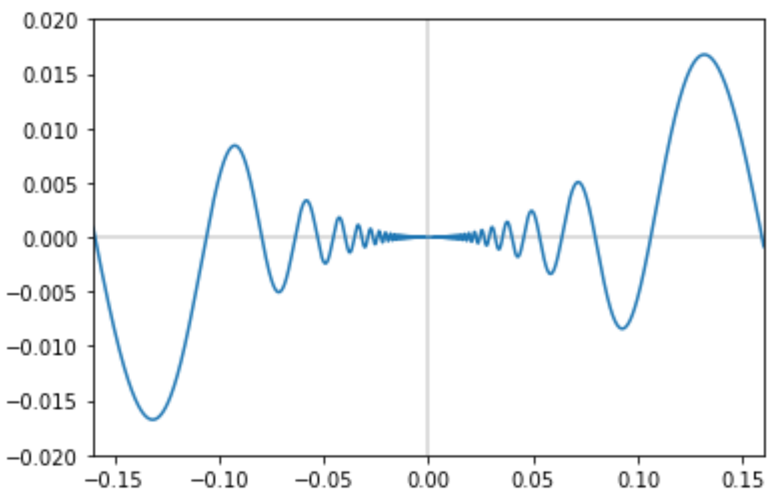
$$f(x) = x^2 sin(1/x), x \neq 0; f(0) = 0 \tag{1}$$

a) Make a rough sketch of the graph:

In [1]:
```
import math
import numpy as np
import matplotlib.pyplot as plt
```

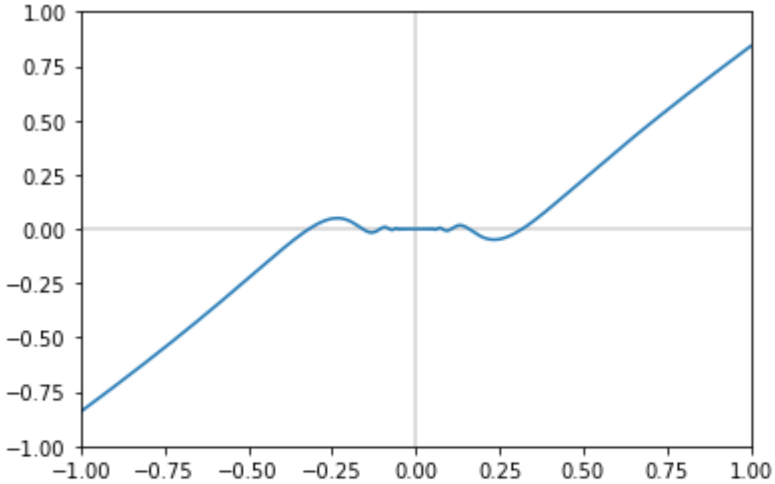
In [13]:
```
x = np.linspace(-0.5, 0.5, 100000)
y = np.array([(i**2)*(math.sin(1/i)) for i in x])
```

In [18]:
```
fig, ax = plt.subplots()
plt.axvline(x=0, color='lightgray')
plt.axhline(y=0, color='lightgray')
plt.xlim(-.16,.16)
plt.ylim(-.02,.02)
_ = ax.plot(x,y)
```



In [26]:
```
x = np.linspace(-2, 2, 100000)
y = np.array([(i**2)*(math.sin(1/i)) for i in x])

fig, ax = plt.subplots()
plt.axvline(x=0, color='lightgray')
plt.axhline(y=0, color='lightgray')
plt.xlim(-1,1)
plt.ylim(-1,1)
_ = ax.plot(x,y)
```



b) show that f is differentiable at 0 and sketch the derivative of $f(x)$:

1)

$$\lim_{h \to 0} \frac{f(x+h)+f(x)}{h} = \lim_{h \to 0} \frac{h^2 sin(1/h)}{h} = \lim_{h \to 0} hsin(1/h) \tag{2}$$

2)

Because sin(1/x) is bounded by 1:

$$-1 <= sin(1/x) <= 1 \tag{3}$$

the following is also true:

$$-\lim_{h \to 0} |h| \leq \lim_{h \to 0} hsin(1/x) \leq \lim_{h \to 0} |h| \tag{4}$$

3) therefore by the squeeze thereom we have:

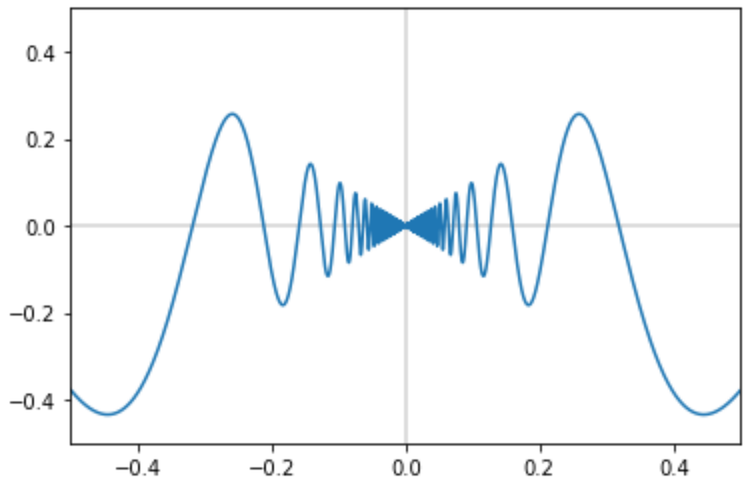
$$\lim_{h \to 0} hsin(1/x) = 0 \tag{5}$$

and therefore: f is differentable at 0

Graph of the Derivative:

$$2xsin(1/x) - cos(1/x) \tag{6}$$

In [34]:
```
x = np.linspace(-0.5, 0.5, 100000)
y = np.array([(2*i)*(math.sin(1/i)*(math.cos(1/i))) for i in x])
fig, ax = plt.subplots()
plt.axvline(x=0, color='lightgray')
plt.axhline(y=0, color='lightgray')
plt.xlim(-0.5,0.5)
plt.ylim(-0.5,0.5)
_ = ax.plot(x,y)
```



Question 2

1) Assume f(x) is constant because the input of 100 distinct points outputs the same value, 0.
2) The derivative of a constant funtcion is 0, and based on the previous assumption $f'(x) = 0$
3) If f(x) is constant and its derivative is 0, then for at least 99 distinct points it will output 0.

It can also be shown by taking two points, a, b out of the set of 100 distinct points, and showing the difference between their outputs, $f(a) - f(b) = 0$. If this holds for any two points in the set then the function within this set is constant $f(x) = c$. As mentioned above, taking the derivative of this function will result in the output zero.

Question 3

Find the nth taylor polynomial and corresponding remainder term for: $f(x) = e^x$:

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + R_n(x) \tag{7}$$

$$where\ R_n(x) = \frac{e^c}{n+1!}x^{n+1} \tag{8}$$

and c = some value between 0 and x

Question 4

a) show that the values converge to e^x

the graph below shows that each sucessive taylor polynomial better approximates e^x

Also can be shown remainder estimation theorem:

$$\exists M s.t. |f^{n+1}(t)| \leq M \tag{9}$$

for all t between x and a

using the remainder function from the previous question $0 < |c| < |x|$ which means

$1 < e^{|c|} < e^{|x|}$

$$\lim_{n\to\infty} |R_n(x)| = \lim_{n\to\infty} \frac{e^{|c|}|x|^{n+1}}{n+1!} \leq \lim_{n\to\infty} \frac{e^{|x|}|x|^{n+1}}{n+1!} = e^{|x|} \lim_{n\to\infty} \frac{|x|^{n+1}}{n+1!} = 0 \qquad (10)$$

Thus,

$\lim_{n\to\infty} R_n(x) = 0$ for all x, and taylor series converges to e^x on $(-\infty, \infty)$

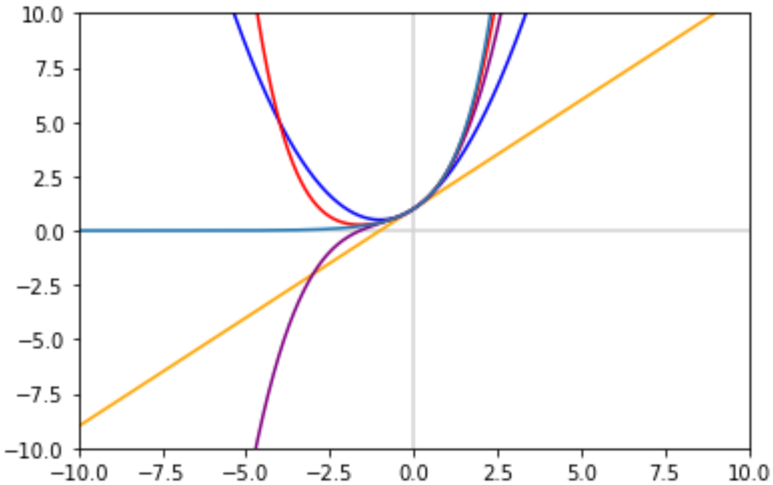
In [58]:

```
first = lambda x: 1 + x
second = lambda x: 1 + x  + (1/2*(x**2))
third = lambda x: 1 + x  + (1/2*(x**2)) + (1/6*(x**3))
fourth = lambda x: 1 + x  + (1/2*(x**2)) + (1/6*(x**3)) + (1/24*(x**4))
```

In [60]:

```
# showing 1,2, and 3 degree taylor polynominal approx of e^x

x = np.linspace(-10, 10, 100000)
y = np.array([math.e**i for i in x])
fig, ax = plt.subplots()
plt.axvline(x=0, color='lightgray')
plt.axhline(y=0, color='lightgray')
plt.xlim(-10,10)
plt.ylim(-10,10)
plt.plot(x, first(x), c='orange')
plt.plot(x, second(x), c='blue')
plt.plot(x, third(x), c='purple')
plt.plot(x, fourth(x), c='red')
_ = ax.plot(x,y)
```



Question 4b

You can calculate the error using:

$$where\ R_n(x) = \frac{e^x}{k+1!} x^{k+1} \qquad (11)$$

For e^1 and $k = 6$ our error is less than 10^{-2}

In [169…
```
def calculate_remainder(r,k):
    return (r**k/math.factorial(k))*math.exp(r)


if calculate_remainder(1,6) < 10**-2:
    print(calculate_remainder(1,6))
```

```
0.0037753914284153404
```

In [174…
```
# approximation using f(x) - R: e^x - taylor series shown

x = 1
x_to_1 = 0
error = 10**-2

for k in range(8):
    x_to_1 += x**k/math.factorial(k)
    delta = abs(x_to_1 - math.exp(1))
    if delta < error:
        print(f'n={k+1} and delta={delta}')
```

```
n=5 and delta=0.009948495125712054
n=6 and delta=0.0016151617923787498
n=7 and delta=0.00022627290348964380
n=8 and delta=2.7860205076724043e-05
```

Question 5

a) Hessian:

$$f(x,y) = 2x^2 + 2a * xy + 3y^2 - 2x + 12y \tag{12}$$

$$\frac{\partial f}{\partial x} = 4x + 2ay - 2 \tag{13}$$

$$\frac{\partial f}{\partial y} = 2ax + 6y + 12 \tag{14}$$

The hessian matrix is:

$$\begin{bmatrix} 4 & 2a \\ 2a & 6 \end{bmatrix} \tag{15}$$

b) the eigenvalues of the hessian are positive, and therefore the hessian is positive-definite, and this means that that the function is concave up.

$$\begin{bmatrix} 4-x & 2a \\ 2a & 6-x \end{bmatrix} \tag{16}$$

$$(4-x)(6-x) - 4a^2 = 0 \tag{17}$$

$$x^2 - 10x + 24 - 4a^2 = 0 \tag{18}$$

solutions:

$$x = \sqrt{4a} + 1 \pm 5 \tag{19}$$

$a = 6$
c)

$$f(x_0, y_0) + fx(x_0, y_0)(x - x_0) + fy(x_0, y_0)(y - y_0) + \frac{fxx(x_0, y_0)}{2}(x - x_0)^2 + fxy($$

In [ ]: