Contents

1 Vectors

Interpretations of Vectors

- **Vector**: an ordered list of numbers.
- Possible notations: $\vec{v} = \boldsymbol{v}$ are most common.
- **Dimensionality**: the number of the elements in a vector.
- **Geometric vector**: an object with a magnitude and direction.
- Standard position: when the vector beings at the origin.
- Vectors must have same dimensionality for addition and subtraction.
- **Unit vector**: a vector with a norm (length) of 1. Notation: $\hat{u} = \frac{\boldsymbol{u}}{|\boldsymbol{u}|}$

Vector Multiplication

- **Scalar**: scales each element in a vector, does not change direction. Generally represented with greek letters.
- **Dot product**: a single number that provides information about the relationship between two vectors. Must have same dimensionality.
- Notation for dot product: $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a}^T\boldsymbol{b} = \langle \boldsymbol{ab} \rangle = \sum a_i b_i$
- *Algebraic* dot product properties:
  - *Associative:* $\boldsymbol{a}^T(\boldsymbol{b}^T\boldsymbol{c}) \neq (\boldsymbol{a}^T\boldsymbol{b})^T\boldsymbol{c}$
  - *Distributive:* $\boldsymbol{a}^T(\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{a}^T\boldsymbol{b} + \boldsymbol{a}^T\boldsymbol{c}$
  - *Commutative:* $\boldsymbol{a}^T\boldsymbol{b} = \boldsymbol{b}^T\boldsymbol{a}$
  - Vector magnitude/length: $||\boldsymbol{v}|| = \sqrt{\boldsymbol{v}^T\boldsymbol{v}}$
- *Geometric* dot product properties:
  - Magnitudes of vectors scaled by angle between them. i.e. $\vec{a} = |a||b|\cos(\theta_{ab})$
  - Geometric and algebraic are really the same. The above equation can be rewritten as the algebraic vector length, i.e. $\boldsymbol{a}^T\boldsymbol{b} = \cos(\theta_{ab})|a||b|$
- Dot product features based on θ:
  - If $\cos(\theta) > 0$ then $\alpha > 0$
  - If $\cos(\theta) < 0$ then $\alpha < 0$
  - If $\cos(\theta) = 0$ then $\alpha = 0$; termed **Orthogonal**
  - If $\cos(\theta) = 1$ then $\alpha = |a||b|$

- *Hadamard vector multiplication*: elementwise multiplication of two vectors of equal dimensionality.
- **Outer product**: $\boldsymbol{v}\boldsymbol{w}^T = n \times m$; a matrix resulting from the product of vectors with dimensions n and m.
- **Cross product**: defined only between two 3D vectors; produces another 3D vector that is perpendicular to both original vectors, or normal, to the plane containing them.
- *Complex conjugate*: the inverse sign of imaginary componenet of a number.
- **Hermitian transpose**: or conjugate transpose, is transpose of a vector or matrix containing imaginary numbers using the complex conjugate.
- Notation for Hermitian transpose on a matrix: $\boldsymbol{M}^H$ or $\boldsymbol{M}^*$

Vector Space

- **Dimension**: in linear algebra dimension represnets a new element of infomation about a vector. Geometrically, each dimension represent a new direction.
- **Fields**: a set of numbers on which a set of arithmetic are valid.
- Field notations:
  - $\mathbb{R}$: real numbers.
  - $\mathbb{C}$: complex numbers.
  - $\mathbb{K}$: real complex numbers.
  - $\mathbb{Z}$: integers.
  - $\mathbb{N}$: poisite integers.
  - $\mathbb{Q}$: rational numbers.
  - Dimensionality in fields is written with superscripts, e.g. $\mathbb{R}^N$
- **Subspace**: The set of all vectors that can be created by a linear combination of some vectors or a set of vectors, i.e. $\lambda\boldsymbol{v}, \ \lambda \in \mathbb{R}$
  - More formal definition: a vector subspace (V) must be closed under addition and scalar multiplication, and must contain the zero vector.
  - $\forall \boldsymbol{v}, \boldsymbol{w} \in V; \ \forall \lambda, \alpha \in \mathbb{R}; \ \lambda\boldsymbol{v} + \alpha\boldsymbol{w} \in V$
  - Geometric: all possible scaled versions of vector produces a line for a single vector, and a 2d plane between two linear independent vectors.
- **Subset**: a set of points that satisfies some conditions; doesn't need to include toe origin, doesn't need to be closed, and can have boundaries.

- **Span**: all possible linear combinations of vectors within a subspace.
- **Linear Independence**: a property of an entire set of vectors that is true only if no vector in the set can be written as a linear combination of the others.
  - Geometric perspective: a set of M vectors is independent if each vector points in a geometric dimension not reachable using other vectors in the set.
  - Any set of $M > N$ vectors in $\mathbb{R}^N$ is dependent.
  - Any set of $M \leq N$ vectors in $\mathbb{R}^N$ *could be* independent.
- **Basis**: a combination of span and independence. More formally: if every element of a vector space V over a field is a linearly independent subset of V that spans V.

2 Introcution to Matrices

Terminology and Dimensionality

- Matrix notation:
  - **M** for reference to entire matrix.
  - $m_{M,N}$ represnets an individual element in a matrix.
  - **Block matrix**: a matrix that includes matrices itself; useful for matrices with higher level structure and can have computational benifits. e.g:
    $$M = \begin{bmatrix} D & 0 \\ 1 & D \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 1 & 1 & 3 & 0 \\ 1 & 1 & 0 & 4 \end{bmatrix}$$
  - **Diagonal**: the diagonal elements of a matrix going form top left to bottom right.
    - *Off-diagonal*: elements not in the diagonal elements.
    - Works for nonsquare matrices.
  - Rows (M) first, then (N) columns when describing matrices.
  - **Dimensionality**: more open to interpretations:
    - $\mathbb{R}^{MN}$: simply the product, or total number of elements.
    - $\mathbb{R}^{M\times N}$: more explicit, not the product. $M \times N$ could be different from $N \times M$
    - $C(M) \in \mathbb{R}^{M}$: *column space* that is the collection of column vectors.
    - $C(M) \in \mathbb{R}^{N}$: *row space* that is the collection of row vectors.
    - Ambiguity opens up flexibility, but makes terminology more context dependent.
  - **Tensor**: higher dimensional cubes, won't be discussed much in early linear algebra.