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Matrix Multiplications

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1 Vectors

Interpretations of Vectors

- ▷ **Vector**: an ordered list of numbers.
- ▷ Possible notations: $\vec{v} = \mathbf{v}$ are most common.
- ▷ **Dimensionality**: the number of the elements in a vector.
- ▷ **Geometric vector**: an object with a magnitude and direction.
- ▷ **Standard position**: when the vector begins at the origin.
- ▷ Vectors must have same dimensionality for addition and subtraction.
- ▷ **Unit vector**: a vector with a **norm** (length) of 1. Notation: $\hat{u} = \frac{\mathbf{u}}{|\mathbf{u}|}$

Vector Multiplication

- ▷ **Scalar**: scales each element in a vector, does not change direction. Generally represented with greek letters.
- ▷ **Dot product**: a single number that provides information about the relationship between two vectors. Must have **same dimensionality**.
- ▷ Notation for dot product: $\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T \mathbf{b} = \langle \mathbf{ab} \rangle = \sum a_i b_i$
- ▷ *Algebraic* dot product properties:
 - *Associative*: $\mathbf{a}^T (\mathbf{b}^T \mathbf{c}) \neq (\mathbf{a}^T \mathbf{b})^T \mathbf{c}$
 - *Distributive*: $\mathbf{a}^T (\mathbf{b} + \mathbf{c}) = \mathbf{a}^T \mathbf{b} + \mathbf{a}^T \mathbf{c}$
 - *Commutative*: $\mathbf{a}^T \mathbf{b} = \mathbf{b}^T \mathbf{a}$
 - Vector magnitude/length: $\|\mathbf{v}\| = \sqrt{\mathbf{v}^T \mathbf{v}}$
- ▷ *Geometric* dot product properties:
 - Magnitudes of vectors scaled by angle between them. i.e.
 $\vec{a} = |\mathbf{a}| |\mathbf{b}| \cos(\theta_{ab})$
 - Geometric and algebraic are really the same. The above equation can be rewritten as the algebraic vector length, i.e. $\mathbf{a}^T \mathbf{b} = \cos(\theta_{ab}) |\mathbf{a}| |\mathbf{b}|$
- ▷ Dot product features based on θ :
 - If $\cos(\theta) > 0$ then $\alpha > 0$
 - If $\cos(\theta) < 0$ then $\alpha < 0$
 - If $\cos(\theta) = 0$ then $\alpha = 0$; termed **Orthogonal**
 - If $\cos(\theta) = 1$ then $\alpha = |\mathbf{a}| |\mathbf{b}|$

- ▷ *Hadamard vector multiplication*: elementwise multiplication of two vectors of equal dimensionality.
- ▷ **Outer product**: $\mathbf{vw}^T = n \times m$; a matrix resulting from the product of vectors with dimensions n and m .
- ▷ **Cross product**: defined only between two 3D vectors; produces another 3D vector that is perpendicular to both original vectors, or *normal*, to the plane containing them.
- ▷ *Complex conjugate*: the inverse sign of imaginary component of a number.
- ▷ **Hermitian transpose**: or conjugate transpose, a transpose of a vector or matrix containing imaginary numbers using the complex conjugate.
- ▷ Notation for Hermitian transpose on a matrix: \mathbf{M}^H or \mathbf{M}^*

Vector Space

- ▷ **Dimension**: in linear algebra dimension represents a *new element* of information about a vector. Geometrically, each dimension represents a *new direction*.
- ▷ **Fields**: a set of numbers on which a set of arithmetic are valid.
- ▷ Field notations:
 - \mathbb{R} : real numbers.
 - \mathbb{C} : complex numbers.
 - \mathbb{K} : real complex numbers.
 - \mathbb{Z} : integers.
 - \mathbb{N} : positive integers.
 - \mathbb{Q} : rational numbers.
 - Dimensionality in fields is written with superscripts, e.g. \mathbb{R}^N
- ▷ **Subspace**: The set of all vectors that can be created by a linear combination of some vectors or a set of vectors, i.e. $\lambda \mathbf{v}$, $\lambda \in \mathbb{R}$
 - More formal definition: a vector subspace (V) must be closed under addition and scalar multiplication, and must contain the zero vector.
 - $\forall \mathbf{v}, \mathbf{w} \in V; \forall \lambda, \alpha \in \mathbb{R}; \lambda \mathbf{v} + \alpha \mathbf{w} \in V$
 - Geometric: all possible scaled versions of vector produces a line for a single vector, and a 2d plane between two linear independent vectors.
- ▷ **Subset**: a set of points that satisfies some conditions; doesn't need to include the origin, doesn't need to be closed, and can have boundaries.

- ▷ **Span:** all possible linear combinations of vectors within a subspace.
- ▷ **Linear Independence:** a property of an entire set of vectors that is true only if no vector in the set can be written as a linear combination of the others.
 - Geometric perspective: a set of M vectors is independent if each vector points in a geometric dimension not reachable using other vectors in the set.
 - Any set of $M > N$ vectors in \mathbb{R}^N is **dependent**.
 - Any set of $M \leq N$ vectors in \mathbb{R}^N *could be independent*.
- ▷ **Basis:** a combination of span and independence. More formally: if every element of a vector space V over a field is a **linearly independent subset** of V that **spans** V .

2 Introduction to Matrices

Terminology and Dimensionality

▷ Matrix notation:

- **M** for reference to entire matrix.
- $m_{M,N}$ represents an individual element in a matrix.
- **Block matrix**: a matrix that includes matrices itself; useful for matrices with higher level structure and can have computational benefits. e.g:

$$M = \begin{bmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{1} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 1 & 1 & 3 & 0 \\ 1 & 1 & 0 & 4 \end{bmatrix}$$

- **Diagonal**: the diagonal elements of a matrix going from top left to bottom right.
 - *Off-diagonal*: elements not in the diagonal elements.
 - Works for nonsquare matrices.
- Rows (M) first, then (N) columns when describing matrices.
- **Dimensionality**: more open to interpretations:
 - \mathbb{R}^{MN} : simply the product, or total number of elements.
 - $\mathbb{R}^{M \times N}$: more explicit, not the product. $M \times N$ could be different from $N \times M$
 - $C(M) \in \mathbb{R}^M$: *column space* that is the collection of column vectors.
 - $C(M) \in \mathbb{R}^N$: *row space* that is the collection of row vectors.
 - Ambiguity opens up flexibility, but makes terminology more context dependent.
- **Tensor**: higher dimensional cubes, won't be discussed much in early linear algebra.

▷ Types of common matrices:

- **Square**: $M \times M$
 - **Symmetric**: a square matrix that is symmetric across the diagonal.
 - **Skew-symmetric**: diagonal must be zero, and numbers must mirror signs across diagonal.

- **Identity:** square matrix with 1s across the diagonal and zeros everywhere else.
- Notation: I_x .
- x indicates size of matrix; no subscript means it is relevant size to matrix it is being applied to.
- **Rectangular:** non-square, $M \times N$
- **Zero:** matrix consisting of only zeros.
- **Diagonal:** all off diagonal elements are zero.
 - Identity matrix is a special case of diagonal.
- **Triangular:**
 - *Upper:* all elements **below** diagonal are zero.
 - *Lower:* all elements **above** diagonal are zero.
- **Concatenated:** two matrices with same number of rows concatenated. Often a line is placed in the product matrix to indicate concatenation point.

Basic Matrix Arithmetic

- ▷ Addition/subtraction is defined only for two matrices with same number of elements.
 - Simply add/subtract corresponding elements.
 - **Commutative:** $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
 - **Associative:** $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$
- ▷ **Shifting:**
 - Adding a scaled version of identity matrix.
 - Formula: $\mathbf{A} + \lambda \mathbf{I} = \mathbf{C}$
 - Geometrically inflates matrix, pushing it towards a sphere.
 - Tends to **regularize** the matrix.
- ▷ **Matrix-Scalar multiplication:**
 - Elementwise multiplication by the scalar.
 - **Commutative:** $\delta \mathbf{MA} = \mathbf{M} \delta \mathbf{A} = \mathbf{MA} \delta$
 - **Distributive:** $\delta(\mathbf{MA}) = \delta \mathbf{M} + \delta \mathbf{A}$
 - Distributive nature makes it a linear operation.

▷ **Matrix transpose:**

- $(\mathbf{A}^{M \times N})^T = \mathbf{A}^{N \times M}$
- First column becomes first row, and vice versa.
- $\mathbf{A}^{TT} = \mathbf{A}$
- *Symmetric*: $\mathbf{A} = \mathbf{A}^T$
- *Skew-symmetric*: $\mathbf{A} = -\mathbf{A}^T$

▷ **Complex matrices:** similar to vectors, a matrix containing at least one non-zero imaginary component.

- Hermitian transpose only changes the sign (the conjugate) of the complex parts, not the real numbers.

▷ **Diagonal and trace:** extracts the diagonal as a vector.

- Defined both for square and rectangular.
- Not the same as *diagonalizing* a matrix.
- **Trace:** the sum of the vector extracted from a matrix and is defined only for square matrices.
- Diagonal formula: $\mathbf{v}_i = \mathbf{A}_{i,i} \quad i = \{1, 2, \dots, \min(m, n)\}$
- Trace formula: $tr(\mathbf{A}) = \sum \mathbf{A}_{i,i}$

▷ **Broadcasting:** expand, or repeat, a vector until traditional arithmetic is valid between a matrix and the vector.

3 Matrix Multiplications

Standard Matrix Multiplication

- ▷ Order of multiplication often matters and is generally **not commutative**.
 - i.e. $\mathbf{AB} \neq \mathbf{BA}$.
- ▷ Rules for validity:
 - Multiplication is **valid** only when the **inner dimensions** match.
 - Resulting matrix **size** corresponds to the **outer dimensions**.
 - i.e. $\mathbf{M} \times \mathbf{N} \cdot \mathbf{N} \times \mathbf{K} = \mathbf{M} \times \mathbf{K}$
- ▷ Methods of multiplication:
 - **Element perspective**: an order collection of dot products; each element is a dot product between **rows of the left** and **columns of the right** matrix.
 - **Layer perspective**: outer products summed between columns of the left matrix and rows of the right matrix.
 - or the sum of *rank 1 matrices*.
 - **Column perspective**: columns of the left matrix scaled then summed by the elements of the right matrix to make a column in the product matrix.
 - or a *linear weighted combination* of the columns in the left, weighted by the columns in the right.
 - **Rows perspective**: rows of the left matrix scales the rows of right matrix then summed to make a row in the product matrix.

Diagonal Matrix Multiplication

- ▷ \mathbf{AD} → Post-multiplying by a diagonal.
- ▷ \mathbf{DA} → Pre-multiplying by a diagonal.
- ▷ Post-multiplication results in the dense matrix's columns being weighted by the diagonal elements, while pre-multiplication scales the rows.

Order of Operations

- ▷ $(\mathbf{LIVE})^T = \mathbf{E}^T \mathbf{V}^T \mathbf{I}^T \mathbf{L}^T$
- ▷ Matrices multiplied then transposed is equal to the same matrices transposed then multiplied in reverse direction.

Matrix-Vector Multiplication

- ▷ The result is always vector when multiplying by a matrix by a vector.
- ▷ The product vector always has the same orientation as the input vector.
- ▷ The length of the vector is determined by the length of the matrix.
- ▷ $\mathbf{Aw} \rightarrow$ a weighted combination of the **columns** of \mathbf{A} .
- ▷ $\mathbf{w}^T \mathbf{A} \rightarrow$ a weighted combination of the **rows** of \mathbf{A} .
- ▷ Symmetric matrices have the same result.

Additive and Multiplicative Matrix Identities

- ▷ Pre or post multiplication of an identity matrix gives you the same matrix, i.e.
 $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$
 - This is the **multiplicative identity**, which is often just called the identity matrix.
- ▷ Adding a zeros matrix is the **additive identity**, i.e. $\mathbf{A} + \mathbf{0} = \mathbf{A}$
- ▷ Methods of creating symmetric matrices from non-symmetric matrices:
 - Additive method: $\mathbf{S} = (\mathbf{A} + \mathbf{A}^T)/2$
 - Works only if \mathbf{A} is a square matrix.
 - Multiplicative method:
 - $\mathbf{A}^T \mathbf{A} = \mathbf{S}$ that is $n \times n$
 - $\mathbf{AA}^T = \mathbf{S}$ that is $m \times m$
 - Always produces a **symmetric square** matrix.