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# 1 Vectors

## Interpretations of Vectors

- ▷ **Vector**: an ordered list of numbers.
- ▷ Possible notations:  $\vec{v} = \boldsymbol{v}$  are most common.
- ▷ **Dimensionality**: the number of the elements in a vector.
- ▷ **Geometric vector**: an object with a magnitude and direction.
- ▷ **Standard position**: when the vector begins at the origin.
- ▷ Vectors must have same dimensionality for addition and subtraction.
- ▷ **Unit vector**: a vector with a **norm** (length) of 1. Notation:  $\hat{u} = \frac{\boldsymbol{u}}{|\boldsymbol{u}|}$

## Vector Multiplication

- ▷ **Scalar**: scales each element in a vector, does not change direction. Generally represented with greek letters.
- ▷ **Dot product**: a single number that provides information about the relationship between two vectors. Must have **same dimensionality**.
- ▷ Notation for dot product:  $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a}^T \boldsymbol{b} = \langle \boldsymbol{a} \boldsymbol{b} \rangle = \sum a_i b_i$
- ▷ *Algebraic* dot product properties:
  - **Associative**:  $\boldsymbol{a}^T (\boldsymbol{b}^T \boldsymbol{c}) \neq (\boldsymbol{a}^T \boldsymbol{b})^T \boldsymbol{c}$
  - **Distributive**:  $\boldsymbol{a}^T (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a}^T \boldsymbol{b} + \boldsymbol{a}^T \boldsymbol{c}$
  - **Commutative**:  $\boldsymbol{a}^T \boldsymbol{b} = \boldsymbol{b}^T \boldsymbol{a}$
  - Vector magnitude/length:  $\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v}^T \boldsymbol{v}}$
- ▷ *Geometric* dot product properties:
  - Magnitudes of vectors scaled by angle between them. i.e.  
 $\vec{a} = |\boldsymbol{a}| |\boldsymbol{b}| \cos(\theta_{ab})$
  - Geometric and algebraic are really the same. The above equation can be rewritten as the algebraic vector length, i.e.  $\boldsymbol{a}^T \boldsymbol{b} = \cos(\theta_{ab}) |\boldsymbol{a}| |\boldsymbol{b}|$
- ▷ Dot product features based on  $\theta$ :
  - If  $\cos(\theta) > 0$  then  $\alpha > 0$
  - If  $\cos(\theta) < 0$  then  $\alpha < 0$
  - If  $\cos(\theta) = 0$  then  $\alpha = 0$ ; termed **Orthogonal**
  - If  $\cos(\theta) = 1$  then  $\alpha = |\boldsymbol{a}| |\boldsymbol{b}|$

- ▷ *Hadamard vector multiplication*: elementwise multiplication of two vectors of equal dimensionality.
- ▷ **Outer product**:  $\mathbf{vw}^T = n \times m$ ; a matrix resulting from the product of vectors with dimensions  $n$  and  $m$ .
- ▷ **Cross product**: defined only between two 3D vectors; produces another 3D vector that is perpendicular to both original vectors, or *normal*, to the plane containing them.
- ▷ *Complex conjugate*: the inverse sign of imaginary component of a number.
- ▷ **Hermitian transpose**: or conjugate transpose, a transpose of a vector or matrix containing imaginary numbers using the complex conjugate.
- ▷ Notation for Hermitian transpose on a matrix:  $\mathbf{M}^H$  or  $\mathbf{M}^*$

## Vector Space

- ▷ **Dimension**: in linear algebra dimension represents a *new element* of information about a vector. Geometrically, each dimension represents a *new direction*.
- ▷ **Fields**: a set of numbers on which a set of arithmetic are valid.
- ▷ Field notations:
  - $\mathbb{R}$ : real numbers.
  - $\mathbb{C}$ : complex numbers.
  - $\mathbb{K}$ : real complex numbers.
  - $\mathbb{Z}$ : integers.
  - $\mathbb{N}$ : positive integers.
  - $\mathbb{Q}$ : rational numbers.
  - Dimensionality in fields is written with superscripts, e.g.  $\mathbb{R}^N$
- ▷ **Subspace**: The set of all vectors that can be created by a linear combination of some vectors or a set of vectors, i.e.  $\lambda \mathbf{v}$ ,  $\lambda \in \mathbb{R}$ 
  - More formal definition: a vector subspace ( $V$ ) must be closed under addition and scalar multiplication, and must contain the zero vector.
  - $\forall \mathbf{v}, \mathbf{w} \in V; \forall \lambda, \alpha \in \mathbb{R}; \lambda \mathbf{v} + \alpha \mathbf{w} \in V$
  - Geometric: all possible scaled versions of vector produces a line for a single vector, and a 2d plane between two linear independent vectors.
- ▷ **Subset**: a set of points that satisfies some conditions; doesn't need to include the origin, doesn't need to be closed, and can have boundaries.

- ▷ **Span:** all possible linear combinations of vectors within a subspace.
- ▷ **Linear Independence:** a property of an entire set of vectors that is true only if no vector in the set can be written as a linear combination of the others.
  - Geometric perspective: a set of  $M$  vectors is independent if each vector points in a geometric dimension not reachable using other vectors in the set.
  - Any set of  $M > N$  vectors in  $\mathbb{R}^N$  is **dependent**.
  - Any set of  $M \leq N$  vectors in  $\mathbb{R}^N$  *could be independent*.
- ▷ **Basis:** a combination of span and independence. More formally: if every element of a vector space  $V$  over a field is a **linearly independent subset** of  $V$  that **spans**  $V$ .

## 2 Introduction to Matrices

### Terminology and Dimensionality

▷ Matrix notation:

- **M** for reference to entire matrix.
- $m_{M,N}$  represents an individual element in a matrix.
- **Block matrix**: a matrix that includes matrices itself; useful for matrices with higher level structure and can have computational benefits. e.g:

$$M = \begin{bmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{1} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 1 & 1 & 3 & 0 \\ 1 & 1 & 0 & 4 \end{bmatrix}$$

- **Diagonal**: the diagonal elements of a matrix going from top left to bottom right.
  - *Off-diagonal*: elements not in the diagonal elements.
  - Works for nonsquare matrices.
- Rows (M) first, then (N) columns when describing matrices.
- **Dimensionality**: more open to interpretations:
  - $\mathbb{R}^{MN}$ : simply the product, or total number of elements.
  - $\mathbb{R}^{M \times N}$ : more explicit, not the product.  $M \times N$  could be different from  $N \times M$
  - $C(M) \in \mathbb{R}^M$ : *column space* that is the collection of column vectors.
  - $C(M) \in \mathbb{R}^N$ : *row space* that is the collection of row vectors.
  - Ambiguity opens up flexibility, but makes terminology more context dependent.
- **Tensor**: higher dimensional cubes, won't be discussed much in early linear algebra.

▷ Types of common matrices:

- **Square**:  $M \times M$ 
  - **Symmetric**: a square matrix that is symmetric across the diagonal.
  - **Skew-symmetric**: diagonal must be zero, and numbers must mirror signs across diagonal.

- **Identity:** square matrix with 1s across the diagonal and zeros everywhere else.
- Notation:  $I_x$ .
- $x$  indicates size of matrix; no subscript means it is relevant size to matrix it is being applied to.
- **Rectangular:** non-square,  $M \times N$
- **Zero:** matrix consisting of only zeros.
- **Diagonal:** all off diagonal elements are zero.
  - Identity matrix is a special case of diagonal.
- **Triangular:**
  - *Upper:* all elements **below** diagonal are zero.
  - *Lower:* all elements **above** diagonal are zero.
- **Concatenated:** two matrices with same number of rows concatenated. Often a line is placed in the product matrix to indicate concatenation point.

### Basic Matrix Arithmetic

- ▷ Addition/subtraction is defined only for two matrices with same number of elements.
  - Simply add/subtract corresponding elements.
  - **Commutative:**  $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
  - **Associative:**  $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$
- ▷ **Shifting:**
  - Adding a scaled version of identity matrix.
  - Formula:  $\mathbf{A} + \lambda \mathbf{I} = \mathbf{C}$
  - Geometrically inflates matrix, pushing it towards a sphere.
  - Tends to **regularize** the matrix.
- ▷ **Matrix-Scalar multiplication:**
  - Elementwise multiplication by the scalar.
  - **Commutative:**  $\delta \mathbf{MA} = \mathbf{M} \delta \mathbf{A} = \mathbf{MA} \delta$
  - **Distributive:**  $\delta(\mathbf{MA}) = \delta \mathbf{M} + \delta \mathbf{A}$
  - Distributive nature makes it a linear operation.

▷ **Matrix transpose:**

- $(\mathbf{A}^{M \times N})^T = \mathbf{A}^{N \times M}$
- First column becomes first row, and vice versa.
- $\mathbf{A}^{TT} = \mathbf{A}$
- *Symmetric*:  $\mathbf{A} = \mathbf{A}^T$
- *Skew-symmetric*:  $\mathbf{A} = -\mathbf{A}^T$

▷ **Complex matrices:** similar to vectors, a matrix containing at least one non-zero imaginary component.

- Hermitian transpose only changes the sign (the conjugate) of the complex parts, not the real numbers.

▷ **Diagonal and trace:** extracts the diagonal as a vector.

- Defined both for square and rectangular.
- Not the same as *diagonalizing* a matrix.
- **Trace:** the sum of the vector extracted from a matrix and is defined only for square matrices.
- Diagonal formula:  $\mathbf{v}_i = \mathbf{A}_{i,i} \quad i = \{1, 2, \dots, \min(m, n)\}$
- Trace formula:  $tr(\mathbf{A}) = \sum \mathbf{A}_{i,i}$

▷ **Broadcasting:** expand, or repeat, a vector until traditional arithmetic is valid between a matrix and the vector.

## 3 Matrix Multiplications

### Standard Matrix Multiplication

- ▷ Order of multiplication often matters and is generally **not commutative**.
  - i.e.  $\mathbf{AB} \neq \mathbf{BA}$ .
- ▷ Rules for validity:
  - Multiplication is **valid** only when the **inner dimensions** match.
  - Resulting matrix **size** corresponds to the **outer dimensions**.
  - i.e.  $\mathbf{M} \times \mathbf{N} \cdot \mathbf{N} \times \mathbf{K} = \mathbf{M} \times \mathbf{K}$
- ▷ Methods of multiplication:
  - **Element perspective**: an order collection of dot products; each element is a dot product between **rows of the left** and **columns of the right** matrix.
  - **Layer perspective**: outer products summed between columns of the left matrix and rows of the right matrix.
    - or the sum of *rank 1 matrices*.
  - **Column perspective**: columns of the left matrix scaled then summed by the elements of the right matrix to make a column in the product matrix.
    - or a *linear weighted combination* of the columns in the left, weighted by the columns in the right.
  - **Rows perspective**: rows of the left matrix scales the rows of right matrix then summed to make a row in the product matrix.

### Diagonal Matrix Multiplication

- ▷  $\mathbf{AD}$  → Post-multiplying by a diagonal.
- ▷  $\mathbf{DA}$  → Pre-multiplying by a diagonal.
- ▷ Post-multiplication results in the dense matrix's columns being weighted by the diagonal elements, while pre-multiplication scales the rows.

### Order of Operations

- ▷  $(\mathbf{LIVE})^T = \mathbf{E}^T \mathbf{V}^T \mathbf{I}^T \mathbf{L}^T$
- ▷ Matrices multiplied then transposed is equal to the same matrices transposed then multiplied in reverse direction.



**Matrix-Vector Multiplication**

- ▷ The result is always vector when multiplying by a matrix by a vector.
- ▷ The product vector always has the same orientation as the input vector.
- ▷ The length of the vector is determined by the length of the matrix.
- ▷  $\mathbf{Aw} \rightarrow$  a weighted combination of the columns of  $\mathbf{A}$ .
- ▷  $\mathbf{w}^T \mathbf{A} \rightarrow$  a weighted combination of the rows of  $\mathbf{A}$ .
- ▷ Symmetric matrices have the same result.