

# Calculus



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## **Transcendental Functions**

## **Techniques of Integration**

## **Infinite Sequences and Series**

## **First-Order Differential Equations**

## **Parametric Equations and Polar Coordinates**

## **Vectors and Vector-Valued Functions**

## **Partial Derivatives**

## **Multiple Integrals**

## **Vector Calculus**

## **Second-Order Differential Equations**

# Limits and Continuity



## Limits

🌐 Limit 🌐 | Thomas' Calculus (2.2–2.4) 🌐

- **Limit**  $\lim_{x \rightarrow c}$ : the value of a function (or sequence) approaches as the input (or index) approaches some value (informal definition).
  - Limits are used to define **continuity** ↓, **derivatives** ↓, and **integrals** ↓.

## Limits of a Functions and Sequences

🌐 Limit of a function 🌐 | Limit of a sequence 🌐 | Essence of Calculus, E7 📺

- **Limit of a function**: a fundamental concept in calculus and analysis concerning the behavior  $L$  of a function near a particular input  $c$ , i.e.,

$$\lim_{x \rightarrow c} f(x) = L$$

- Reads as “ $f$  of  $x$  tends to  $L$  as  $x$  tends to  $c$ ”
- $\epsilon, \delta$  **Limit of function**: a formalized definition, wherein  $f(x)$  is defined on an open interval  $I$ , except possibly at  $c$  itself, leading to above definition, if and only if:
  - For every real measure of closeness  $\epsilon > 0$ , there exists a real corresponding  $\delta > 0$ , such that for all existing further approaches there exist a smaller  $\epsilon$ , i.e.,

$$f : \mathbb{R} \rightarrow \mathbb{R}, c, L \in \mathbb{R} \Rightarrow \lim_{x \rightarrow c} f(x) = L$$



$$\forall \epsilon > 0 (\exists \delta > 0 : \forall x \in I (0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon))$$

- Functions **do not have** a limit when the function:
  - has a **unit step**, i.e., it “jumps” at a point;
  - is **not bounded**, i.e., it tends towards infinity;
  - or it **oscillates**, i.e., does not stay close to any single number.
- **Limit of a sequence**: the value that the terms of a sequence  $(x_n)$  “tends to” (and not to any other) as  $n$  approaches infinity (or some point), i.e.,

$$\lim_{n \rightarrow \infty} x_n = x$$

- $\epsilon$  **Limit of sequence**: for every measure of closeness  $\epsilon$ , the sequence's  $x_n$  term eventually converge to the limit, i.e.,

$$\forall \epsilon > 0 (\exists N \in \mathbb{N} (\forall n \in \mathbb{N} (n \geq N \Rightarrow |x_n - x| < \epsilon)))$$

- **Convergent**: when a limit of a sequence **exists**.
- **Divergent**: a sequence that **does not** converge.

## Properties of Limits

📌 List of limits 📌 | Squeeze theorem 📌

- **Operations on a single known limit:** if  $\lim_{x \rightarrow c} f(x) = L$  then:
  - $\lim_{x \rightarrow c} [f(x) \pm \alpha] = L \pm \alpha$
  - $\lim_{x \rightarrow c} \alpha f(x) = \alpha L$
  - $\lim_{x \rightarrow c} f(x)^{-1} = L^{-1}, L \neq 0$
  - $\lim_{x \rightarrow c} f(x)^n = L^n, n \in \mathbb{N}$
  - $\lim_{x \rightarrow c} f(x)^{n-1} = L^{n-1}, n \in \mathbb{N}, \text{ if } n \in \mathbb{N}_e \Rightarrow L > 0$
- **Operations on two known limits:** if  $\lim_{x \rightarrow c} f(x) = L_1$  and  $\lim_{x \rightarrow c} g(x) = L_2$ 
  - $\lim_{x \rightarrow c} [f(x) \pm g(x)] = L_1 \pm L_2$
  - $\lim_{x \rightarrow c} [f(x)g(x)] = L_1 L_2$
  - $\lim_{x \rightarrow c} f(x)g(x)^{-1} = L_1 L_2^{-1}$
- **Squeeze theorem:** used to confirm the limit of a difficult to compute function via comparison with two other functions whose limits are easily known or computed.
  - Let  $I$  be an interval having the point  $c$  as a limit point.
  - Let  $g, f$ , and  $h$ , be functions defined on  $I$ , except possibly at  $c$  itself.
  - Suppose that  $\forall x \in I \wedge x \neq c \Rightarrow g(x) \leq f(x) \leq h(x)$
  - And suppose that  $\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L$
  - Then,  $\lim_{x \rightarrow c} f(x) = L$
  - Essentially, the hard to compute limit of the “middle function” is found by finding two other easy functions that “squeeze” the middle function at that point.

## One-Sided Limit

📌 One-Sided Limit 📌

- **One-sided limit:** one of two limits of  $f(x)$  as  $x$  approaches a specified point from either the left or from the right.
  - From the left:  $\lim_{x \rightarrow c^-} = L$
  - From the right:  $\lim_{x \rightarrow c^+} = L$
- If the left and right limits exist and are equal, then

$$\lim_{x \rightarrow c} f(x) = L \Leftrightarrow \lim_{x \rightarrow c^-} f(x) = L \wedge \lim_{x \rightarrow c^+} f(x) = L$$

- Limits can still exist, even if the function is defined at a different point, as long as both one-sided limits approach the same value near the given input.

## Continuity

🌱 Continuous function 🌱 | Classification of discontinuities 🌱 | Thomas' Calculus (2.5) 🌱

- Continuity of functions is one of the core concepts of topology, however, there are definitions in terms of limits that prove useful; the following is only a primer.

### Continuous Functions

- **Continuous function:** a function that does not have any abrupt changes in value.
  - I.e., a function is continuous if and only if arbitrarily small changes in its output can be assured by restricting to sufficiently small changes in its input.
- **Discontinuous:** when a function is not continuous at a point in its domain, leading to a discontinuity; there are three classifications:
  - **Removable:** when both **one-sided limits**  $\uparrow$  exist, are finite, and are equal, but the actual value of  $f(x)$  is not equal to the limit and equal to some other value.
    - The discontinuity can be removed to regain continuity.
    - Sometimes the term *removable discontinuity* is mistaken for *removable singularity*, or a “whole” in the function (the point is not defined elsewhere).
  - **Jump:** when a single limit does not exist because the one-sided limits exist and are finite, but not equal.
    - Points can be defined at the discontinuity, but the function can not be made continuous.
  - **Essential:** when at least one of two one-sided limits doesn't exist; can be the result of oscillating or unbounded functions.

### Intermediate Value Theorem

🌱 Intermediate Value Theorem 🌱

- **Intermediate value theorem:** if  $f$  is a continuous function whose domain contains the interval  $[a, b]$ , then it **takes on any given value between  $f(a)$  and  $f(b)$**  at some point within the intervals.
- Relevant deductions, i.e., important corollaries:
  - **Bolzano's theorem:** if a continuous function has values of opposite sign inside an interval, then it **has a root** in that interval.
  - The image of a continuous function over an interval is itself an interval.
- Thus, the image set  $f(I)$  (which has no gaps) is also an interval, and it contains:

$$[\min(f(a), f(b)), \max(f(a), f(b))]$$

## Limits Involving Infinity

📌 Limits involving infinity 📌 | Thomas' Calculus (2.6) 📌

- Let  $S \subseteq \mathbb{R}$ ,  $x \in S$  and  $f : S \mapsto \mathbb{R}$ , then limits of these functions can approach arbitrarily large ( $\pm$ ) values, providing a connection to asymptotes, and thus, analysis.

### Limits at Infinity and Infinite Limits

- Limits at infinity:** limits defined as  $f(x) \pm \text{infinity}$  are defined much like normal limits:

$$\lim_{x \rightarrow -\infty} f(x) = L \quad \lim_{x \rightarrow \infty} f(x) = L$$

- Formally, for all measures of closeness  $\epsilon$  there exists a point  $c$  such that  $|f(x) - L| < \epsilon$  whenever  $x < c \vee x > c$  (respectively), i.e.,

$$\forall \epsilon > 0 (\exists c (\forall x \{<, >\} c : |f(x) - L| < \epsilon))$$

- Basic rules for rational functions  $f(x) = p(x)q(x)^{-1}$ , where  $p$  and  $q$  are polynomials, the degree of each is denoted as  $\{p, q\}^\circ$ , and the leading coefficients are denoted as  $P, Q$ , then:
  - $p^\circ > q^\circ \Rightarrow L$  is  $\{+, -\}$  depending on the sign of the leading coefficients.
  - $p^\circ = q^\circ \Rightarrow L = PQ^{-1}$
  - $p^\circ < q^\circ \Rightarrow L = 0$
- Infinite limits:** the usual limit does not exist for a limit that grows out of bounds, however, limits with infinite values can be introduced:

$$\lim_{x \rightarrow c} f(x) = \infty, \quad \text{i.e.,} \quad \forall n > 0 (\exists \delta > 0 : f(x) > n \Leftrightarrow 0 < |x - a| < \delta)$$

### Asymptotes of functions

📌 Asymptotes 📌

- Asymptote:** a tangent line of a curve at a point at infinity; the distance between the curve and the line approaches zero as a coordinate tends to infinity.
- There are three kinds of asymptotes: *horizontal*, *vertical* and *oblique*; nature of the asymptote is dependent on a function's relation to infinity.
  - Horizontal asymptotes:** a result of limits at infinity, i.e., when  $x \rightarrow \pm\infty$
  - Vertical asymptotes:** a result of infinite limits, i.e., when  $x \rightarrow \pm a = \pm\infty$
  - Oblique asymptotes:** when a linear asymptote is not parallel to either axis.  $f(x)$  is asymptotic to the straight line  $y = mx + n (m \neq 0)$  if:

$$\lim_{x \rightarrow \pm\infty} [f(x) - (mx + n)] = 0$$

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# Derivatives



## Derivative Fundamentals

### Derivative as a Function

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### Simple Differentiation Rules

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### Instantaneous Rates of Change

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### Derivatives of Trigonometric Functions

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## Differentiation Rules

### Chain Rule

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### Product Rule

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### Quotient Rule

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## Differentials and Related Concepts

### Differentials

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### Linearization

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### Implicit Differentiation

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### Related Rates

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# Applications of Derivatives



## Stationary Point

### Maxima and Minima

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### Extreme Value Theorem

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### Interior Extremum Theorem

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## Mean Value Theorem

### Rolle's Theorem

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### Corollaries of the Mean Value Theorem

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### Monotonic Functions

## Derivative Tests

### First-Derivative Test

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### Second-Derivative Test

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### Concavity

- 

### Higher-Order Derivative Test

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## Differential Methods

### Newton's Method

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### Taylor's Theorem

- 

### L'Hôpital's Rule

- 

### General Leibniz Rule

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### Logarithmic Derivative

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# Integrals



## Integral Fundamentals

### Terminology and Notation

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### Primer: Formal Definitions

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## Definite Integrals

### Riemann Integral

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### Integrability

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### Properties of Definite Integrals

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# The Fundamental Theorem of Calculus

## Fundamental Theorem, Part 1

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## Fundamental Theorem, Part 2

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## The Integral of a Rate

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## Total Area

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## Integration By Substitution

### Indefinite Integrals

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### Definite Integrals

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### Symmetric Functions

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### Area Between Curves

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# Applications of Definite Integrals



## Solid of Revolution

### Disc Integration

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### Shell Integration

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## Arc Length

### Dealing with Discontinuities

- 

### Differential Arc Length

-

## Surface of Revolution

### Revolution about the y-Axis

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# Transcendental Functions



# Techniques of Integration



# Infinite Sequences and Series





# First-Order Differential Equations



# Parametric Equations and Polar Coordinates



# Vectors and Vector-Valued Functions



# Partial Derivatives



# Multiple Integrals



# Vector Calculus



# Second-Order Differential Equations

