Page	Points	Score
2 ↓	6	
3 ↓	8	
4 ↓	4	
5 ↓	4	
6 ↓	4	
7 ↓	4	
Total:	30	

1. Determine whether each statement below is true or false, then explain how you know. Note: that if the statement is false, then it might be easiest to provide a counterexample as justification.

(a) **[/ 2]** If a linear system has n variables and m equations, then the augmented matrix has n columns.

- ✓ True: creating an augmented matrix can easily be done by dropping the variables temporarily, then concatenating the vector of constants onto the matrix of coefficients, i.e.,

$$\left[\begin{array}{cccc|c} a_{1,1} & a_{1,2} & \cdots & a_{1,n} & b_1 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} & b_m \end{array}\right]$$

  where the number of variables are equal to columns (n) of the matrix of coefficients and the numbers of equations are equal to the rows (m).

- Though, if you count the vector of constants as part of the augmented matrix then technically it's false, since the totals columns is $n + 1$.

(b) **[/ 2]** An inconsistent system can be made consistent by performing a sequence of elementary row operations.

- ✘ False: an inconsistent system is always inconsistent. For example, nothing can be down to the following matrix that would make it consistent:

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & -4 & 1 \\ 0 & 5 & 6 & 6 & 1 \\ 0 & 0 & 0 & 0 & 6 \end{array}\right]$$

  One might try to say that multiplying the final row by zero would make it consistent, but such multiplication would not be reversible, making it an invalid elementary row operation.

(c) **[/ 2]** A consistent system whose augmented matrix has 3 rows and 5 columns will have infinitely many solutions.

- ✓ True: a consistent system will have infinitely many solutions if the rank of the matrix is less than the number of columns.

- The max rank of matrix can be defined as a non-negative integer, including zero ($\mathbb{N}_0$), that is equal to the smaller of the two dimensions, either the rows or columns, i.e.,

$$\max(r) = r \in \mathbb{N}_0 \mid 0 \leq r \leq \min(m, n)$$

  Thus, a 3×5 matrix is rank deficient ($r = 3$), meaning $r < n$, which means that there are infinitely many solutions.

2. Create an augmented matrix for the scenarios below or explain why it is impossible to do so. The associated system:

(a) **[/ 2]** has infinite solutions, but the augmented matrix has no row of zeros.

$$A = \left[\begin{array}{cccc|c} \mathbf{6} & 0 & 0 & 0 & 4 \\ 0 & \mathbf{9} & 0 & 0 & 2 \\ 0 & 0 & 0 & \mathbf{1} & 0 \end{array}\right]$$

$$\text{rank}(A) = 3, \quad r < n \rightarrow \text{infinite solutions}$$

(b) **[/ 2]** has exactly one solution, but the augmented matrix has two rows of zeros.

$$B = \left[\begin{array}{cc|c} \mathbf{6} & 0 & 4 \\ 0 & \mathbf{9} & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\text{rank}(B) = 2, \quad r = n \rightarrow \text{one solution}$$

(c) **[/ 2]** is consistent, but has more equations than unknowns.

$$A = \left[\begin{array}{ccc|c} 6 & 0 & 0 & 4 \\ 0 & 9 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$m > n$$

(d) **[/ 2]** is inconsistent, but the augmented matrix has a row of zeros.

$$A = \left[\begin{array}{cccc|c} 6 & 0 & 0 & 0 & 3 \\ 0 & 9 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

3. Consider the linear system whose augmented matrix is

$$\begin{bmatrix} 1 & h & 4 \\ 3 & 6 & 8 \end{bmatrix}$$

(a) Determine a value for h so that the system is consistent.

$$h = 1 \rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 3 & 6 & 8 \end{bmatrix} \quad R_2 - 3R_1 \rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 0 & 3 & -4 \end{bmatrix} \quad \frac{1}{3}R_2 \rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -\frac{4}{3} \end{bmatrix}$$

(b) Determine a value for h so that the system is inconsistent.

$$h = 2 \rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 3 & 6 & 8 \end{bmatrix} \quad 3R_1 - R_2 \rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & -4 \end{bmatrix}$$

4. **[/ 2]** Show that $x = 1$, $y = 2$, and $z = 3$ is not a solution to the following system.

$$\begin{cases} x + y + 2z = 9 \\ 2x + 4y - 2z = 1 \\ 3x + 6y - 5z = 0 \end{cases}$$

$$1 + 2 + 2(3) = 9 \checkmark$$

$$2(1) + 4(2) - 2(3) = 4 \times$$

$$1 \neq 4, \text{ not a solution}$$

5. **[/ 2]** Find a solution to the following system of linear equations:

$$\begin{array}{rcrcrcrcrcrl} -450x_1 & + & -22x_2 & + & 1x_3 & + & 1x_4 & + & 0x_5 & + & 333x_6 & = 0 \\ 3x_1 & + & 2x_2 & + & 1x_3 & + & 0x_4 & + & 900x_5 & + & 0x_6 & = 0 \\ -\pi x_1 & + & 0x_2 & + & 88x_3 & + & 45x_4 & + & 1x_5 & + & 0x_6 & = 0 \\ 7x_1 & + & 12x_2 & + & 300x_3 & + & 0x_4 & + & 9x_5 & + & 0x_6 & = 0 \\ 1x_1 & + & 3x_2 & + & 9x_3 & + & 27x_4 & + & 81x_5 & + & 243x_6 & = 0 \end{array}$$

- Ignoring the trivial solution, here is a nearly trivial solution:

$$\left[\begin{array}{cccccc|c} 1 & 1 & 1 & 1 & 1 & \frac{470}{333} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 + x_2 + x_3 + x_4 + x_5 + \frac{470}{333}x_6 = 0$$

- Computer go brrrr...

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & -3.47 & 0 \\ 0 & 1 & 0 & 0 & 0 & 55.88 & 0 \\ 0 & 0 & 1 & 0 & 0 & -2.15 & 0 \\ 0 & 0 & 0 & 1 & 0 & 3.97 & 0 \\ 0 & 0 & 0 & 0 & 1 & -0.11 & 0 \end{array}\right]$$

$$3.47x_1 - 55.88x_2 + 2.15x_3 + -3.97x_4 + 0.11x_5 + \lambda x_6 = 0$$

Wait, is that right? ↑

6. **[/ 2]** Suppose the matrix below is the augmented matrix of a system of linear equations. Write the general solution in parametric vector form (as a linear combination of vectors some scaled by parameters).

$$\left[\begin{array}{ccccc|c} 1 & 0 & -4 & 0 & 2 & -1 \\ 0 & 1 & 8 & 0 & -7 & 9 \\ 0 & 0 & 0 & 1 & 0 & 3 \end{array}\right]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ 0 \\ 3 \\ 0 \end{bmatrix} + r \begin{bmatrix} 4 \\ -8 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 7 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

7. **[/ 2]** Let $\boldsymbol{A} = \begin{bmatrix} 0 & -4 \\ 3 & 9 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} -9 & 1 \\ -9 & 8 \end{bmatrix}$, and $\boldsymbol{C} = \begin{bmatrix} 7 & 5 \\ 3 & -3 \end{bmatrix}$.

Determine: $-8\boldsymbol{C} + 2(5\boldsymbol{A} - 3\boldsymbol{B}) + 4\boldsymbol{C} - 10\boldsymbol{A} + 4(\boldsymbol{C} + 2\boldsymbol{B})$

- Matrix addition is:
  - ✓ Commutative: $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$
  - ✓ Associative: $\boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} + \boldsymbol{B} + \boldsymbol{C})$
  - ✓ Distributive: $\boldsymbol{A}(\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{AB} + \boldsymbol{AC}$
- Thus:

$$\begin{aligned} -8\boldsymbol{C} + 2(5\boldsymbol{A} - 3\boldsymbol{B}) + 4\boldsymbol{C} - 10\boldsymbol{A} + 4(\boldsymbol{C} + 2\boldsymbol{B}) &= \\ -8\boldsymbol{C} + 10\boldsymbol{A} - 6\boldsymbol{B} + 4\boldsymbol{C} - 10\boldsymbol{A} + 4\boldsymbol{C} + 8\boldsymbol{B} &= \\ 10\boldsymbol{A} - 6\boldsymbol{B} - 10\boldsymbol{A} + 8\boldsymbol{B} &= \\ -6\boldsymbol{B} + 8\boldsymbol{B} &= \\ 2\boldsymbol{B} &= \begin{bmatrix} -18 & 2 \\ -18 & 16 \end{bmatrix} \end{aligned}$$

8. **[/ 2]** Fact: the vector equation below is consistent

$$2\begin{bmatrix}1\\4\end{bmatrix}+3\begin{bmatrix}2\\-12\end{bmatrix}-5\begin{bmatrix}7\\0\end{bmatrix}+6\begin{bmatrix}-3\\5\end{bmatrix}=\begin{bmatrix}-45\\2\end{bmatrix}$$

Use that fact to find a solution to the matrix equation $\boldsymbol{Ax}=\boldsymbol{b}$ where

$$\boldsymbol{A}=\begin{bmatrix}1&2&7&-3\\4&-12&0&5\end{bmatrix}\qquad\text{and}\qquad\boldsymbol{b}=\begin{bmatrix}-45\\2\end{bmatrix}$$

$$\left[\begin{array}{cccc|c}1&2&7&-3&-45\\4&-12&0&5&2\end{array}\right]\quad R_2-4R_1\rightarrow\left[\begin{array}{cccc|c}1&2&7&-3&-45\\0&-20&-28&-17&-178\end{array}\right]$$

A solution ↑

9. **[/ 2]** Define a transformation

$$T:\mathbb{R}^3\rightarrow\mathbb{R}^4$$
$$\boldsymbol{x}\rightarrow\boldsymbol{Ax}$$

where $\boldsymbol{A}=\begin{bmatrix}-5&-4&1\\3&2&-1\\-4&0&8\\7&0&9\end{bmatrix}$. It is a fact that $T\left(\begin{bmatrix}1\\-1\\4\end{bmatrix}\right)=\begin{bmatrix}3\\-3\\28\\43\end{bmatrix}$

Use this fact to produce a solution to the system of linear equations below

$$\begin{aligned}-5x_1&+&-4x_2&+&1x_3&=3\\3x_1&+&2x_2&+&1x_3&=-3\\-4x_1&+&0x_2&+&38x_3&=28\\7x_1&+&0x_2&+&9x_3&=43\end{aligned}$$

$$\text{If }\boldsymbol{x}=\begin{bmatrix}1\\-1\\4\end{bmatrix},\quad\text{then }\boldsymbol{Ax}=\begin{bmatrix}3\\-3\\28\\43\end{bmatrix}$$

$$\text{Thus: if }\boldsymbol{B}=\begin{bmatrix}-5&-4&1\\3&2&-1\\-4&0&38\\7&0&9\end{bmatrix},\quad\text{then }\boldsymbol{Bx}=\begin{bmatrix}3\\-3\\148\\43\end{bmatrix}\text{ is the solution.}$$

Double-checking by computing RREF of both augment matrices ↓

$$\text{rref}\left(\left[\begin{array}{ccc|c} -5 & -4 & 1 & 3 \\ 3 & 2 & -1 & -3 \\ -4 & 0 & 8 & 28 \\ 7 & 0 & 9 & 43 \end{array}\right]\right) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$r = n$, thus the matrix is full rank and only one unique solution exists

$$\text{rref}\left(\left[\begin{array}{ccc|c} -5 & -4 & 1 & 3 \\ 3 & 2 & -1 & -3 \\ -4 & 0 & 28 & 148 \\ 7 & 0 & 9 & 43 \end{array}\right]\right) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$r = n$, thus the matrix is full rank and only one unique solution exists

Does the rank matter here actually? I have not explored linear transformations much, so I'm exploring things here. Hmm, let's create a rank deficient matrix and apply the transformation.

$$\begin{bmatrix} -5 & -4 & 1 \\ 3 & 2 & -1 \\ 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ -3 \end{bmatrix}$$

This results in infinite solutions:

$$\text{rref}\left(\begin{bmatrix} -5 & -4 & 1 \\ 3 & 2 & -1 \\ 3 & 2 & -1 \end{bmatrix}\right) = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

So $\boldsymbol{x}$ maps $\mathbb{R}^2 \to \mathbb{R}^2$ in this case, hmm...maybe I'm getting at nothing here.