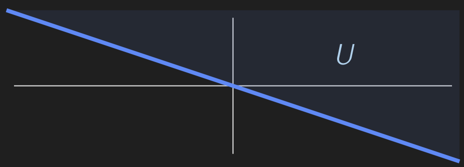
1. Consider the set of all points in the region U shown in $\mathbb{R}^2$ below. Assume the set includes the boundary line. Give a specific reason why the set U is *not* a subspace of $\mathbb{R}^2$.



- If the set of vectors that describe $\mathbb{R}^2$ is defined as V, then U is just a subset of V. All subspaces are subsets, but not all subsets are subspaces.
  - Subsets don't need to include the origin, don't need to be closed under addition and scalar multiplication, or could have boundaries.
- That being said, a vector subspace L must be closed under addition and scalar λ, α multiplication while also containing the zero vector, i.e.,

$$\forall\, \boldsymbol{v}, \boldsymbol{w} \in L; \quad \forall\, \lambda, \alpha \in \mathbb{R}; \quad \lambda \boldsymbol{v} + \alpha \boldsymbol{w} \in L$$

  - This shows how the region U is not a subspace, because if you take two vectors in U, then some linear combinations would yield points that were not in U.
- Interestingly, the boundary line of U is a linear subspace of V.

2. Let W be the set of all points on either the x- or y- axis. That is, W is all the points of the form $\begin{bmatrix} a \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ b \end{bmatrix}$ for any real numbers a and b. Show that W is not a subspace of $\mathbb{R}^2$

- W is not a subspace of $\mathbb{R}^2$ because W is $\mathbb{R}^2$. This is because any vector in $\mathbb{R}^2$ can be made via a linear combination of $\begin{bmatrix} a \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ b \end{bmatrix}$.
- W would have to be limited to only the x- or y- axis, but not both, in order to be a subspace of $\mathbb{R}^2$.
- If W contains both axes, but not any other points on the plane, then one is putting boundaries the space and limiting the ability to take linear combination of vectors in the subspace, making it only subset.

3. Consider A and its reduced row echelon form below.

$$A = \begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix} \to \underbrace{\begin{bmatrix} 1 & -3 & 0 & 3/2 \\ 0 & 0 & 1 & 5/4 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{rref of } A}$$

Let $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, $\boldsymbol{b}_3$, and $\boldsymbol{b}_4$ be the columns of rref(A). Note that

$$-3\boldsymbol{b}_1 = \boldsymbol{b}_2 \quad \text{and} \quad \frac{3}{2}\boldsymbol{b}_1 + \frac{5}{4}\boldsymbol{b}_3 = \boldsymbol{b}_4$$

Now let $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, and $\boldsymbol{a}_4$ be the columns of A.

(a) Show that $-3\boldsymbol{a}_1 = \boldsymbol{a}_2$.

$$-3\boldsymbol{a}_1 = -3\begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ -6 \\ -9 \end{bmatrix} = \boldsymbol{a}_2$$

(b) Find a linear combination of $\boldsymbol{a}_4$ in terms of $\boldsymbol{a}_1$ and $\boldsymbol{a}_3$.

$$\frac{3}{2}\boldsymbol{a}_1 + \frac{5}{4}\boldsymbol{a}_3 = \frac{3}{2}\begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix} + \frac{5}{4}\begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} -4.5 \\ 3 \\ 4.5 \end{bmatrix} + \begin{bmatrix} -2.5 \\ 5 \\ -2.5 \end{bmatrix} = \begin{bmatrix} -7 \\ 8 \\ 2 \end{bmatrix} = \boldsymbol{b}_4$$

(c) Show that the set $A\left\{\boldsymbol{a}_1,\ \boldsymbol{a}_2,\ \boldsymbol{a}_3,\ \boldsymbol{a}_4\right\}$ is linearly dependent.

- Algebraically, a sequence of vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, . . . , $\boldsymbol{v}_k$ from a vector space are linearly dependent if there exist scalars (not all zero) λ_1, $\lambda_2 \dots s$, λ_k such that they form the zero vector $\boldsymbol{o}$, i.e.,

$$\lambda_1\boldsymbol{v}_1, \lambda_2\boldsymbol{v}_2, \dots, \lambda_k\boldsymbol{v}_k = \boldsymbol{o} \qquad \lambda \in \mathbb{R}$$

- We already showed that at least one vector in the set can be expressed as a linear combination of another, meaning A is linearly dependent, but it can still be done explicitly:

$$3\boldsymbol{a}_1 + \boldsymbol{a}_2 + 0\boldsymbol{a}_3 + 0\boldsymbol{a}_4 = \boldsymbol{o}$$

or

$$-\frac{3}{2}\boldsymbol{a}_1 + 0\boldsymbol{a}_2 - \frac{5}{4}\boldsymbol{a}_3 + \boldsymbol{a}_4 = \boldsymbol{o}$$