

1. Consider a system of linear equations with augmented matrix \mathbf{A} and coefficient matrix \mathbf{C} . In each case explain why the statement is true or give an example showing that it is false.

(a) If there is more than one solution, \mathbf{A} has a row of zeros.

- If a row with all zeros occurs, then that row added no new information and was only a multiple of another row, i.e., the system is reduced rank.
- Matrix rank corresponds to the maximal number of linearly independent columns.
- Linear independence is when no vector in the matrix can be expressed as a linear combination, i.e.,

$$a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + a_3 \mathbf{v}_3 + a_n \mathbf{v}_n$$

- For example, in a 3-D vector space \mathbb{R}^3 , then any vector in the space is can be made by a linear combination of the following three vectors \mathbf{e}_1 , \mathbf{e}_2 , \mathbf{e}_3 (unit vectors) multiplied by some scalar λ :

$$\lambda \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- If the third column vector was all zeros (making the matrix have a row of all zeros) then it would just be describing a plane in \mathbb{R}^2 , with the last row (or rather, third vector) contributing no information to the system,

$$\lambda \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- That being said, the matrix above could be an augmented matrix with the constant matrix containing values for only the first two vectors, indicating a single solution.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

- And now I've realized I actually answered why the statement in (b) is false and not directly addressing the validity of (a).
- (a) is false, the row of zeros puts no restriction on the **parameter**, but the row doesn't have to be there; the rank just has to be less than the number of columns.

$$\left[\begin{array}{ccc|c} 1 & 0 & \mathbf{2} & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & \mathbf{2} & 4 \\ 0 & 1 & 0 & 2 \end{array} \right] \text{ row zeros doesn't have to exist}$$

- (b) If \mathbf{A} has a row of zeros, there is more than one solution.
- o Not necessarily, see above answer.
- (c) If there is no solution, the reduced row-echelon form of \mathbf{C} has a row of zeros.
- o True, the rref form needs to have a row of zeros in the coefficient matrix and a non-zero in the same row in the constant matrix, i.e., $[0 \ 0 \ 0 \ \cdots \mid a \neq 0]$, in order to have no solution.
- (d) If the row-echelon form of \mathbf{C} has a row of zeros, there is no solution.
- o Not necessarily, a row of zeros is more likely to indicate infinite solutions than no solution or a single solution, albeit, all are still possible.
 - o For example, the following matrix is in ref and contains a row of all zeros.

$$\left[\begin{array}{ccc|c} 1 & 6 & 9 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

2.

$$\text{Verify that } \begin{cases} x_1 = 2s + 12t + 13 \\ x_2 = s \\ x_3 = -s - 3t - 3 \\ x_4 = t \end{cases} \text{ is a solution of } \begin{cases} 2x_1 + 5x_2 + 9x_3 + 3x_4 = -1 \\ x_1 + 2x_2 + 4x_3 = 1. \end{cases}$$

$$\begin{aligned} & \left[\begin{array}{cccc|c} 2 & 5 & 9 & 3 & -1 \\ 1 & 2 & 4 & 0 & 1 \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cccc|c} 1 & 2 & 4 & 0 & 1 \\ 2 & 5 & 9 & 3 & -1 \end{array} \right] \\ & \quad -2R_1 + R_2 \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 4 & 0 & 1 \\ 0 & 1 & 1 & 3 & -3 \end{array} \right] \\ & \quad -2R_2 + R_1 \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 2 & -6 & 7 \\ 0 & 1 & 1 & 3 & -3 \end{array} \right] \end{aligned}$$

Two parameters are left: s, t .

$$x_1 = 2s - 6t + 7$$

$$x_2 = s$$

$$x_3 = -s - 3t - 3$$

$$x_4 = t$$

I am confused.

Let's try something different:

$$\begin{aligned}
 2(2s + 12t + 13) + 5(s) + 9(-s - 3t - 3) + 3(t) &= -1 \\
 4s + 24t + 26 + 5s - 9s - 27t - 27 + 3t &= -1 \\
 24t - 27t + 3t + 26 - 27 &= -1 \\
 26 - 27 &= -1 \\
 -1 &= -1 \\
 2s + 12t + 13 + 2(s) + 4(-s - 3t - 3) &= 1 \\
 2s + 12t + 13 + 2s - 4s - 12t - 12 &= 1 \\
 12t - 12t + 13 - 12 &= 1 \\
 1 &= 1
 \end{aligned}$$

Hmmm...why doesn't rref work.

$$\begin{aligned}
 2(2s - 6t + 7) + 5(s) + 9(-s - 3t - 3) + 3(t) &= -1 \\
 4s - 12t + 14 + 5s - 9s - 27t - 27 + 3t &= -1 \\
 -12t + 14 - 27t - 27 + 3t &= -1 \\
 &\text{yeah, no}
 \end{aligned}$$

Did I do parameters wrong?

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & -6 & 7 \\ 0 & 1 & 1 & 3 & -3 \end{array} \right]$$

$$\begin{aligned}
 x_1 &= 2s + 6t + 7 \\
 x_2 &= -s - 3t - 3 \\
 x_3 &= s \\
 x_4 &= t
 \end{aligned}$$

Idk, let's try it.

$$\begin{aligned}
 2(2s + 6t + 7) + 5(-s - 3t - 3) + 9(s) + 3(t) &= -1 \\
 4s + 12t + 14 - 10s - 15t - 15 + 9s + 3t &= -1 \\
 3s - 1 &= -1 \\
 s &= 0
 \end{aligned}$$

Well, why doesn't that work? I guess rref removes information...or maybe I'm doing something wrong with parameters.

Update after a couple of days later: the rank is 2, with 4 columns, i.e., $r < n$, thus there are infinite solutions. You just gave us one of many possible solutions, hence why $s = 0$ I guess? s , and I bet t if I worked it out, would both be zero, or unrestricted.