Calculus

Limits and Continuity

Limits

Limit | Thomas' Calculus (2.2–2.4)

- **Limit** $\lim\limits_{x \to c}$: the value of a function (or sequence) approaches as the input (or index) approaches some value (informal definition).
  - Limits are used to define continuity, derivatives, and integrals.

Limits of a Functions and Sequences

Limit of a function | Limit of a sequence | Essence of Calculus, E7

- **Limit of a function**: a fundament concept in calculus and analysis concerning the behavior L of a function near a particular input c, i.e.,

$$\lim_{x \to c} f(x) = L$$

  - Reads as "f of x tends to L as x tends to c"
- ε, δ **Limit of function**: a formalized definition, wherein $f(x)$ is defined on an open interval I, except possibly at c itself, leading to above definition, if and only if:
  - For every real measure of closeness $\varepsilon > 0$, there exists a real corresponding $\delta > 0$, such that for all existing further approaches there exist a smaller ε, i.e.,

$$f : \mathbb{R} \to \mathbb{R},\ c, L \in \mathbb{R} \Rightarrow \lim_{x \to c} f(x) = L$$
$$\Updownarrow$$
$$\forall \varepsilon > 0\, (\exists \delta > 0 : \forall x \in I\, (0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon))$$

- Functions do not have a limit when the function:
  - has a unit step, i.e., it "jumps" at a point;
  - is not bounded, i.e., it tends towards infinity;
  - or does not stay close to any single number, i.e., it oscillates too much.
- **Limit of a sequence**: the value that the terms of a sequence (x_n) "tends to" (and not to any other) as n approaches infinity (or some point), i.e.,

$$\lim_{n \to \infty} x_n = x$$

- ε **Limit of sequence**: for every measure of closeness ε, the sequence's x_n term eventually converge to the limit, i.e.,

$$\forall \varepsilon > 0\, (\exists N \in \mathbb{N}\, (\forall n \in \mathbb{N}\, (n \geq N \Rightarrow |x_n - x| < \varepsilon)))$$

  - **Convergent**: when a limit of a sequence exists.
  - **Divergent**: a sequence that does not converge.

Properties of Limits

List of limits | Squeeze theorem

- **Operations on a single known limit**: if $\lim_{x \to c} f(x) = L$ then:
  - $\lim_{x \to c} [f(x) \pm \alpha] = L \pm \alpha$
  - $\lim_{x \to c} \alpha f(x) = \alpha L$
  - $\lim_{x \to c} f(x)^{-1} = L^{-1}, L \neq 0$
  - $\lim_{x \to c} f(x)^{n} = L^{n}, n \in \mathbb{N}$
  - $\lim_{x \to c} f(x)^{n^{-1}} = L^{n^{-1}}, n \in \mathbb{N},$ if $n \in \mathbb{N}_e \Rightarrow L > 0$
- **Operations on two known limits**: if $\lim_{x \to c} f(x) = L_1$ and $\lim_{x \to c} g(x) = L_2$
  - $\lim_{x \to c} [f(x) \pm g(x)] = L_1 \pm L_2$
  - $\lim_{x \to c} [f(x)g(x)] = L_1 L_2$
  - $\lim_{x \to c} f(x)g(x)^{-1} = L_1 L_2^{-1}$
- **Squeeze theorem**: used to confirm the limit of a function via comparison with two other functions whose limits are easily known or computed.
  - Let I be an interval having the point c as a limit point.
  - Let g, f, and h, be functions defined on I, except possibly at c itself.
  - Suppose that $\forall x \in I \wedge x \neq c \Rightarrow g(x) \leq f(x) \leq h(x)$
  - And suppose that $\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$
  - Then, $\lim_{x \to c} f(x) = L$
  - Essentially, the hard to compute limit of the "middle function" is found by finding two other easy functions that "squeeze" the middle function at that point.

One-Sided Limit

One-Sided Limit

- **One-sided limit**: one of two limits of $f(x)$ as x approaches a specified point from either the left or from the right.
  - From the left: $\lim_{x \to c^-} = L$
  - From the right: $\lim_{x \to c^+} = L$
- If the left and right limits exist and are equal, then

$$\lim_{x \to c} f(x) = L \Leftrightarrow \lim_{x \to c^-} f(x) = L \wedge \lim_{x \to c^+} f(x) = L$$

- Limits can still exist, even if the function is defined at a different point, as long as both one-sided limits approach the same value near the given input.

Continuity

Continuous function | Classification of discontinuities | Thomas' Calculus (2.5)

- Continuity of functions is one of the core concepts of topology, however, there are definitions in terms of limits that prove useful; the following is only a primer.

Continuous Functions

- **Continuous function**: a function that does not have any abrupt changes in value.
  - I.e., a function is continuous if and only if arbitrarily small changes in its output can be assured by restricting to sufficiently small changes in its input.
- **Discontinuous**: when a function is not continuous at a point in its domain, leading to a discontinuity; there are three classifications:
  - **Removable**: when both one-sided limits↑ exist, are finite, and are equal, but the actual value of $f(x)$ is not equal to the limit and equal to some other value.
    - The discontinuity can be removed to regain continuity.
    - Sometimes the term *removable discontinuity* is mistaken for *removable singularity*, or a "whole" in the function (the point is not defined elsewhere).
  - **Jump**: when a single limit does not exist because the one-sided limits exist and are finite, but not equal.
    - Points can be defined at the discontinuity, but the function can not be made continuous.
  - **Essential**: when at least one of two one-sided limits doesn't exist; can be the result of oscillating or unbounded functions.

Intermediate Value Theorem

Intermediate Value Theorem

- **Intermediate value theorem**: if f is a continuous function whose domain contains the interval $[a, b]$, then it takes on any given value between $f(a)$ and $f(b)$ at some point within the intervals.
- Relevant deductions, i.e., important corollaries:
  - **Bolzano's theorem**: if a continuous function has values of opposite sign inside an interval, then it has a root in that interval.
  - The image of a continuous function over an interval is itself an interval.
- Thus, the image set $f(I)$ (which has no gaps) is also an interval, and it contains:

$$[\min(f(a), f(b)),\ \max(f(a), f(b))]$$

Limits Involving Infinity

Limits involving infinity | Thomas' Calculus (2.6)

- Let $S \subseteq \mathbb{R}$, $x \in S$ and $f : S \mapsto \mathbb{R}$, then limits of these functions can approach arbitrarily large $(\pm)$ values, providing a connection to asymptotes, and thus, analysis.

Limits at Infinity and Infinite Limits

- **Limits at infinity**: limits defined as f(x) $\pm$ infinity are defined much like normal limits:

$$\lim_{x \to -\infty} f(x) = L \qquad \lim_{x \to \infty} f(x) = L$$

  - Formally, for all measures of closeness ε there exists a point c such that $|f(x) - L| < \varepsilon$ whenever $x<c$ or $x>c$ (respectively), i.e.,

$$\forall \varepsilon > 0\,(\exists c\,(\forall x\{<,>\}c : |f(x) - L| < \varepsilon))$$

- Basic rules for rational functions $f(x) = p(x)q(x)^{-1}$, where p and q are polynomials, the degree of each is denoted as $\{p, q\}^{\circ}$, and the leading coefficients are denoted as P, Q, then:
  - $p^{\circ} > q^{\circ} \Rightarrow L$ is $\{+, -\}$ depending on the sign of the leading coefficients.
  - $p^{\circ} = q^{\circ} \Rightarrow \mathrm{L}\; PQ^{-1}$
  - $p^{\circ} < q^{\circ} \Rightarrow L = 0$
- **Infinite limits**: the usual limit does not exist for a limit that grows out of bounds, however, limits with infinite values can be introduced:

$$\lim_{x \to c} f(x) = \infty,\ \text{i.e.,} \forall n > 0\,(\exists \delta > 0 : f(x) > n \Leftrightarrow 0 < |x - a| < \delta)$$

Asymptotes of functions

Asymptotes

- **Asymptote**: a tangent line of a curve at a point at infinity; the distance between the curve and the line approaches zero as a coordinate tends to infinity.
- There are three kinds of asymptotes: *horizontal*, *vertical* and *oblique*; nature of the asymptote is dependent on a function's relation to infinity.
  - **Horizontal asymptotes**: a result of limits at infinity, i.e., when $x \to \pm\infty$
  - **Vertical asymptotes**: a result of infinite limits, i.e., when $x \to \pm a = \pm\infty$
  - **Oblique asymptotes**: when a linear asymptote is not parallel to either axis. $f(x)$ is asymptotic to the straight line $y = mx + n(m \neq 0)$ if:

$$\lim_{x \to \pm\infty} [f(x) - (mx + n)] = 0$$

Derivatives

Integrals

Partial Derivatives

Multiple Integrals

Second-Order Differential Equations