Linear Algebra

Vectors

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Interpretations of Vectors

- **Algebraic vectors** ($\boldsymbol{v}$, $\vec{v}$): an ordered list of numbers.
  - E.g., $\boldsymbol{v} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$
  - Vectors can be written as rows (seen above) or columns (seen below), but differ only at the level of notation and convention.
  - The order of elements in a vector matters:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

- **Dimensionality**: the number of elements in a vector.
- **Euclidean (geometric, spatial) vectors**: a line in geometric space that indicates the magnitude and direction from a starting point (tail) to an end point (head).
  - Geometric vectors can start at any point in space, but often represented as starting from the origin—such vectors are in standard position.
  - Coordinates are not the same as vectors, but they do indicate where the head of a vector will land if it is in standard position.

Vector Addition and Subtraction

- Algebraically, dimensionality of vectors must be equal. When they are, then addition or subtraction vectors is done on the corresponding elements of each vector, e.g.,

$$\begin{bmatrix} 1 \\ 0 \\ 4 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ -6 \\ 11 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -2 \\ 16 \end{bmatrix}$$

- Geometrically, addition can be thought of translating the tail of one vector to the head of the other—resulting in a new vector.
- Geometric interpretations of subtraction can be thought of in two ways:

1. Multiplying one vector by -1, then applying vector addition method above.
2. Placing both vectors in standard position, with the resulting vector between the two heads being the answer.

Vector-Scalar Multiplication

- **Scalar**: typically denoted with lower case Greek letters (e.g., α, λ) indicating an element of a field (typically real numbers) used in scalar-multiplication of vectors.
- Algebraically, scalar-multiplication is the multiplication of each element of a vector by a particular scalar, e.g.,

$$\lambda \boldsymbol{v} \rightarrow 7 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \\ 7 \end{bmatrix}$$

- Geometrically, scalar-multiplication can be thought of as the extension ($\lambda > 1$) or compression ($\lambda \in (0, 1)$) of a vector.
    - When $\lambda < 0$, then it can be thought of inverting its direction with respect to the origin.

The Dot Product

- **Dot (scalar, inner) product**: an algebraic operation that takes two equal-length sequences of numbers (usually coordinate vectors), and returns a single number.
    - The result of a dot product is a scalar↑, so often it is represented as such.
        - It can also be represented as multiplication between two vectors ($\boldsymbol{a} \cdot \boldsymbol{b}$).
        - However, it is commonly represented as $\boldsymbol{a}^T \boldsymbol{b}$ — transpose↓ will be explained in more detail when dealing with matrix products↓.
    - Algebraically: $\sum_{i=1}^{n} \boldsymbol{a}_i \boldsymbol{b}_i$ — where Σ denotes summation and n is the dimension of the vector space, e.g.,

$$\begin{bmatrix} 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} = (1 \cdot 4) + (3 \cdot -2) + (-5 \cdot -1) = 3$$

Properties of the Dot Product

- Note: the following properties hold as long as $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are real vectors.
- ✓ Distributive: $\boldsymbol{a}^T(\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a}^T\boldsymbol{b} + \boldsymbol{a}^T\boldsymbol{c}$ — vector multiplication distributes over vector addition.
- ✘ Associative: $\boldsymbol{a}^T(\boldsymbol{b}^T\boldsymbol{c}) \neq (\boldsymbol{a}^T\boldsymbol{b})\boldsymbol{c}$ — in general the associative property does not hold, as the dot product would most likely produce different scalars.
    - Additionally, $\boldsymbol{a}$ could have a different dimensionality than $\boldsymbol{b}$ and $\boldsymbol{c}$. I.e., even if $\boldsymbol{b}$ and $\boldsymbol{c}$ had the same dimensionality ($\boldsymbol{a}^T(\boldsymbol{b}^T\boldsymbol{c})$ would be valid vector-scalar multiplication) then $\boldsymbol{a}^T\boldsymbol{b}$ would be invalid.
- ✓ Commutative: $\boldsymbol{a}^T\boldsymbol{b} = \boldsymbol{b}^T\boldsymbol{a}$ — the order of the vectors does not matter.

Vector Length

- **Vector norm (magnitude, length)**: denoted with double vertical bars $||\boldsymbol{v}||$, indicating length of a vector in euclidean space. Not to be confused with absolute value $|x|$ of a scalar's "norm." However, sometimes the notation $|\boldsymbol{v}|$ is used.
- Calculating $||\boldsymbol{v}||$ is done using the Euclidean norm:

$$||\boldsymbol{v}|| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

  - This is a consequence of the Pythagorean theorem, since the basis vectors↓ $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ are orthogonal↓ unit vectors↓.
- Thus, the norm can easily be found by taking the square root of the dot product of the vector with itself:

$$||\boldsymbol{v}|| = \sqrt{\boldsymbol{v}^T v}$$

Geometric Interpretation of the Dot Product

- The dot product of two Euclidean vectors↑ $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined by:

$$\lambda = \boldsymbol{a}^T \boldsymbol{b} = ||\boldsymbol{a}|| \, ||\boldsymbol{b}|| \cos(\theta)$$

  where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.
- **Features based on θ**:
  - When $\cos\theta > 0$ ($\theta < 90° -$ acute) then $\lambda > 0\ (+)$
  - When $\cos\theta < 0$ ($\theta > 90° -$ obtuse) then $\lambda < 0\ (-)$
  - When $\cos\theta = 0$ ($\theta = 90° -$ perpendicular) then $\lambda = 0$
    - This represents a special case where the vectors are said to be orthogonal.
    - **Orthogonality**: the generalization of the notion of perpendicularity to the linear algebra of bilinear forms↓.
  - When $\cos\theta = 1$ then the vectors are codirectional:

$$\boldsymbol{a}^T \boldsymbol{b} = ||\boldsymbol{a}|| \, ||\boldsymbol{b}||$$

    - Thus, the dot product with a vector $\boldsymbol{v}$ with itself is

$$\boldsymbol{v}^T \boldsymbol{v} = ||\boldsymbol{v}||^2$$

    - Which gives us the norm↑ as defined above, i.e., $||\boldsymbol{v}|| = \sqrt{\boldsymbol{v}^T v}$
    - If $\cos\theta = -1$, then really vectors are still codirectional, but point in opposite directions with respect to the origin.

Other Vector Products

- Matrices↓ are properly defined later. However, vectors are technically single row or column matrices, so many of the following operations also work on vectors, thus use of matrices often appears in this section.

Hadamard Multiplication

- **Hadamard (element-wise) product**: a binary operation (only takes two operands) that matrices of the same dimensions and produces another matrix of the same dimension as the operands, e.g., vector Hadamard multiplication:

$$\begin{bmatrix}1\\0\\4\\5\end{bmatrix} + \begin{bmatrix}2\\3\\-6\\11\end{bmatrix} = \begin{bmatrix}2\\0\\-24\\55\end{bmatrix}$$

Outer Product

- Recall that the dot (scalar, inner) product↑ produces a 1×1 matrix, or rather a scalar, hence "scalar" product.
  - Note: the typical notation used for dot products is $\boldsymbol{v}^T\boldsymbol{w}$ — part of reasoning for the "inner" product.
- **Outer product**: an $N \times M$ matrix that results from the product of two vectors with dimensions n and m.

$$\boldsymbol{v}\boldsymbol{w}^T = N \times M$$

- The subtle change in notation matters in contrast to the dot product, as both represent distinct operations (assuming they are column vectors).
- The outer product allows for the multiplication of vectors with different dimensionality
- Can be thought of in two different ways:
  - The row perspective:

$$\begin{bmatrix}1\\0\\4\\2\end{bmatrix}\begin{bmatrix}a & b & c\end{bmatrix} = \begin{bmatrix}1a & 1b & 1c\\0a & 0b & 0c\\4a & 4b & 4c\\2a & 2b & 2c\end{bmatrix}$$

  - The column perspective:

$$\begin{bmatrix}1\\0\\4\\2\end{bmatrix}\begin{bmatrix}a & b & c\end{bmatrix} = \begin{bmatrix}1a & 1b & 1c\\0a & 0b & 0c\\4a & 4b & 4c\\2a & 2b & 2c\end{bmatrix}$$

Cross Product

- **Cross (vector, directed area) product**: denoted by the symbol ×, indicating a binary operation on two vectors in three-dimensional space $\mathbb{R}^3$.
  - Given two linearly independent vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, then the cross product $\boldsymbol{a} \times \boldsymbol{b}$ produces a new vector that is orthogonal to both $\boldsymbol{a}$ and $\boldsymbol{b}$, or normal to the plane containing them.
  - The direction of the vector is given by the right-hand rule ($\boldsymbol{a}$ = pointer, $\boldsymbol{b}$ = index, thumb = direction).
  - The magnitude of the vector represents the area of the parallelogram that the vectors span.
- The cross product can be defined by the formula:

$$\boldsymbol{a} \times \boldsymbol{b} = \|\boldsymbol{a}\| \, \|\boldsymbol{b}\| \sin(\theta) \, \boldsymbol{v}$$

  - Notice this is similar to the geometric interpretations of the dot product↑ — the contrast between the two leads to an intuitive interpretation:
    - $\cos(\theta)$ in the dot product is used to measure how "parallel" the two vectors are, i.e., they are codirectional when $\theta = 1$, allowing for calculation of the norm↑.
    - $\sin(\theta)$ in the cross product is used to measure how "perpendicular" two vectors are, i.e., they are orthogonal when $\theta = 1$. There are multiple directions of the orthogonal vector, so calculation of the signed area returns vector $\boldsymbol{v}$ that describes both magnitude and direction as described above.
    - The intuition described here will be more clear when the determinant↓ is discussed in more detail.
- An algebraic example of vector e.g.,

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2c & - & 3b \\ 3a & - & 1c \\ 1b & - & 2a \end{bmatrix}$$

Vectors with Complex Numbers

- **Complex number**: a number that can be expressed in the form $a + bi$, where a and b are real numbers, and i (j) is a symbol called the imaginary unit that satisfies the equation $i = \sqrt{-1}$.
- This imaginary unit allows for the imaginary set $(\mathbb{C})$ to combined with the real number set $(\mathbb{R})$ to create a 2-D plane that allows for more efficient calculations.
- He

To be defined

- unit vectors
- basis vectors
- transpose
- matrix products
- determinant