Calculus

Limits and Continuity

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Applications of Integrals

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Limits and Continuity

Limits

🌐 Limit ⌗ | Thomas' Calculus (2.2–2.4) ⌗

- **Limit** $\lim\limits_{x \to c}$: the value of a function (or sequence) approaches as the input (or index) approaches some value (informal definition).

  - Limits are used to define continuity↓, derivatives↓, and integrals↓.

Limits of a Functions and Sequences

🌐 Limit of a function ⌗ | Limit of a sequence ⌗ | Essence of Calculus, E7 ▶

- **Limit of a function**: a fundament concept in calculus and analysis concerning the behavior L of a function near a particular input c, i.e.,

$$\lim_{x \to c} f(x) = L$$

  - Reads as "f of x tends to L as x tends to c"

- ε, δ **Limit of function**: a formalized definition, wherein $f(x)$ is defined on an open interval I, except possibly at c itself, leading to above definition, if and only if:

  - For every real measure of closeness $\varepsilon > 0$, there exists a real corresponding $\delta > 0$, such that for all existing further approaches there exist a smaller ε, i.e.,

$$f : \mathbb{R} \to \mathbb{R}, \ c, L \in \mathbb{R} \Rightarrow \lim_{x \to c} f(x) = L$$
$$\Updownarrow$$
$$\forall \varepsilon > 0 \left(\exists \delta > 0 : \forall x \in I \left(0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon \right) \right)$$

- Functions do not have a limit when the function:

  - has a unit step, i.e., it "jumps" at a point;

  - is not bounded, i.e., it tends towards infinity;

  - or oscillates, i.e., does not stay close to any single number.

- **Limit of a sequence**: the value that the terms of a sequence (x_n) "tends to" (and not to any other) as n approaches infinity (or some point), i.e.,

$$\lim_{n \to \infty} x_n = x$$

- ε **Limit of sequence**: for every measure of closeness ε, the sequence's x_n term eventually converge to the limit, i.e.,

$$\forall \varepsilon > 0 \left(\exists N \in \mathbb{N} \left(\forall n \in \mathbb{N} \left(n \geq N \Rightarrow |x_n - x| < \varepsilon \right) \right) \right)$$

  - **Convergent**: when a limit of a sequence exists.

  - **Divergent**: a sequence that does not converge.

Properties of Limits
🌐 List of limits ✶ | Squeeze theorem ✶

o **Operations on a single known limit**: if $\lim\limits_{x \to c} f(x) = L$ then:

- $\lim\limits_{x \to c} [f(x) \pm \alpha] = L \pm \alpha$

- $\lim\limits_{x \to c} \alpha f(x) = \alpha L$

- $\lim\limits_{x \to c} f(x)^{-1} = L^{-1}, L \neq 0$

- $\lim\limits_{x \to c} f(x)^n = L^n, n \in \mathbb{N}$

- $\lim\limits_{x \to c} f(x)^{n^{-1}} = L^{n^{-1}}, n \in \mathbb{N}$, if $n \in \mathbb{N}_e \Rightarrow L > 0$

o **Operations on two known limits**: if $\lim\limits_{x \to c} f(x) = L_1$ and $\lim\limits_{x \to c} g(x) = L_2$

- $\lim\limits_{x \to c} [f(x) \pm g(x)] = L_1 \pm L_2$

- $\lim\limits_{x \to c} [f(x)g(x)] = L_1 L_2$

- $\lim\limits_{x \to c} f(x)g(x)^{-1} = L_1 L_2^{-1}$

o **Squeeze theorem**: used to confirm the limit of a difficult to compute function via comparison with two other functions whose limits are easily known or computed.

- Let I be an interval having the point c as a limit point.

- Let g, f, and h, be functions defined on I, except possibly at c itself.

- Suppose that $\forall x \in I \wedge x \neq c \Rightarrow g(x) \leq f(x) \leq h(x)$

- And suppose that $\lim\limits_{x \to c} g(x) = \lim\limits_{x \to c} h(x) = L$

- Then, $\lim\limits_{x \to c} f(x) = L$

- Essentially, the hard to compute limit of the "middle function" is found by finding two other easy functions that "squeeze" the middle function at that point.

One-Sided Limit
🌐 One-Sided Limit ✶

o **One-sided limit**: one of two limits of $f(x)$ as x approaches a specified point from either the left or from the right.

- From the left: $\lim\limits_{x \to c^-} = L$ - From the right: $\lim\limits_{x \to c^+} = L$

o If the left and right limits exist and are equal, then

$$\lim\limits_{x \to c} f(x) = L \Leftrightarrow \lim\limits_{x \to c^-} f(x) = L \wedge \lim\limits_{x \to c^+} f(x) = L$$

o Limits can still exist, even if the function is defined at a different point, as long as both one-sided limits approach the same value near the given input.

Continuity

🌐 Continuous function ✇ | Classification of discontinuities ✇ | Thomas' Calculus (2.5) ✇

- Continuity of functions is one of the core concepts of topology, however, there are definitions in terms of limits that prove useful; the following is only a primer.

Continuous Functions

- **Continuous function**: a function that does not have any abrupt changes in value.

  - I.e., a function is continuous if and only if arbitrarily small changes in its output can be assured by restricting to sufficiently small changes in its input.

- **Discontinuous**: when a function is not continuous at a point in its domain, leading to a discontinuity; there are three classifications:

  - **Removable**: when both one-sided limits ↑ exist, are finite, and are equal, but the actual value of $f(x)$ is not equal to the limit and equal to some other value.

    - The discontinuity can be removed to regain continuity.

    - Sometimes the term *removable discontinuity* is mistaken for *removable singularity*, or a "whole" in the function (the point is not defined elsewhere).

  - **Jump**: when a single limit does not exist because the one-sided limits exist and are finite, but not equal.

    - Points can be defined at the discontinuity, but the function can not be made continuous.

  - **Essential**: when at least one of two one-sided limits doesn't exist; can be the result of oscillating or unbounded functions.

Intermediate Value Theorem

🌐 Intermediate Value Theorem ✇

- **Intermediate value theorem**: if f is a continuous function whose domain contains the interval $[a, b]$, then it takes on any given value between $f(a)$ and $f(b)$ at some point within the intervals.

- Relevant deductions, i.e., important corollaries:

  - **Bolzano's theorem**: if a continuous function has values of opposite sign inside an interval, then it has a root in that interval.

  - The image of a continuous function over an interval is itself an interval.

- Thus, the image set $f(I)$ (which has no gaps) is also an interval, and it contains:

$$[\min(f(a), f(b)), \ \max(f(a), f(b))]$$

Limits Involving Infinity

🌐 Limits involving infinity 🔗 | Thomas' Calculus (2.6) 🔗

- Let $S \subseteq \mathbb{R}, x \in S$ and $f : S \mapsto \mathbb{R}$, then limits of these functions can approach arbitrarily large $(\pm)$ values, providing a connection to asymptotes, and thus, analysis.

Limits at Infinity and Infinite Limits

○ **Limits at infinity**: limits defined as f(x) $\pm$ infinity are defined much like normal limits:

$$\lim_{x \to -\infty} f(x) = L \qquad \lim_{x \to \infty} f(x) = L$$

- Formally, for all measures of closeness ε there exists a point c such that $|f(x) - L| < \varepsilon$ whenever $x < c$ or $x > c$ (respectively), i.e.,

$$\forall \varepsilon > 0 \, (\exists c \, (\forall x \{<, >\} c : |f(x) - L| < \varepsilon))$$

○ Basic rules for rational functions $f(x) = p(x)q(x)^{-1}$, where p and q are polynomials, the degree of each is denoted as $\{p, q\}^\circ$, and the leading coefficients are denoted as P, Q, then:

- $p^\circ > q^\circ \Rightarrow L$ is $\{+, -\}$ depending on the sign of the leading coefficients.
- $p^\circ = q^\circ \Rightarrow L \, PQ^{-1}$
- $p^\circ < q^\circ \Rightarrow L = 0$

○ **Infinite limits**: the usual limit does not exist for a limit that grows out of bounds, however, limits with infinite values can be introduced:

$$\lim_{x \to c} f(x) = \infty, \text{ i.e., } \forall n > 0 \, (\exists \delta > 0 : f(x) > n \Leftrightarrow 0 < |x - a| < \delta)$$

Asymptotes of functions

🌐 Asymptotes 🔗

○ **Asymptote**: a tangent line of a curve at a point at infinity; the distance between the curve and the line approaches zero as a coordinate tends to infinity.

○ There are three kinds of asymptotes: *horizontal, vertical* and *oblique*; nature of the asymptote is dependent on a function's relation to infinity.

- **Horizontal asymptotes**: a result of limits at infinity, i.e., when $x \to \pm\infty$
- **Vertical asymptotes**: a result of infinite limits, i.e., when $x \to \pm a = \pm\infty$
- **Oblique asymptotes**: when a linear asymptote is not parallel to either axis. $f(x)$ is asymptotic to the straight line $y = mx + n(m \neq 0)$ if:

$$\lim_{x \to \pm\infty} [f(x) - (mx + n)] = 0$$

Derivatives

Derivative Fundamentals

Derivative as a Function

- ○

Simple Differentiation Rules

- ○

Instantaneous Rates of Change

- ○

Derivatives of Trigonometric Functions

- ○

Differentiation Rules

Chain Rule

- ○

Product Rule

- ○

Quotient Rule

- ○

Differentials and Related Concepts

Differentials

○

Linearization

○

Implicit Differentiation

○

Related Rates

○

Differentials

○

Linearization

○

Implicit Differentiation

○

Related Rates

○

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