Linear Algebra

Vectors

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Interpretations of Vectors

- **Algebraic vectors** $(\boldsymbol{v}, \ \vec{v})$: an ordered list of numbers.
  - E.g., $\boldsymbol{v} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$
  - Vectors can be written as rows (seen above) or columns (seen below), but differ only at the level of notation and convention.
  - The order of elements in a vector matters:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

- **Dimensionality**: the number of elements in a vector.
- **Geometric vectors**: a line in geometric space that indicates the magnitude and direction from its start point (tail) to its end point (head).
  - Geometric vectors can start at any point in space, but often represented as starting from the origin—such vectors are in standard position.
  - Coordinates are not the same as vectors, but they do indicate where the head of a vector will land if it is in standard position.

Vector Addition and Subtraction

- Algebraically, dimensionality of vectors must be equal. When they are, then addition or subtraction vectors is done on the corresponding elements of each vector, e.g.:

$$\begin{bmatrix} 1 \\ 0 \\ 4 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ -6 \\ 11 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -2 \\ 16 \end{bmatrix}$$

- Geometrically, addition can be thought of translating the tail of one vector to the head of the other—resulting in a new vector.
- Geometric interpretations of subtraction can be thought of in two ways:

1. Multiplying one vector by -1, then applying vector addition method above.
2. Placing both vectors in standard position, with the resulting vector between the two heads being the answer.

Vector-Scalar Multiplication

- **Scalar (α, β, λ)**: an element of a field (typically real numbers) used in scalar-multiplication of vectors.
- Algebraically, scalar-multiplication is the multiplication of each element of a vector by a particular scalar.

$$\lambda \boldsymbol{v} \to 7 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \\ 7 \end{bmatrix}$$

- Geometrically, scalar-multiplication is the extension ($\lambda > 1$) or compression ($\lambda \in (0, 1)$) of a vector.
  - When $\lambda < 0$, then it can be thought of inverting its direction with respect to the origin.

The Dot Product

- **Dot product (scalar product)**: an algebraic operation that takes two equal-length sequences of numbers (usually coordinate vectors), and returns a single number.
  - The result of a dot product is a scalar, so often it is represented as a such (a lower case Greek letter).
  - It can also be represented as multiplication between two vectors ($\boldsymbol{a} \cdot \boldsymbol{b}$).
  - Most commonly it is represented as $\boldsymbol{a}^T \boldsymbol{b}$ – transpose will be explained in more detail when dealing with matrix products.
  - Algebraically: $\sum_{i=1}^{n} \boldsymbol{a}_i \boldsymbol{b}_i$ – where Σ denotes summation and n is the dimension of the vector space, e.g.:

$$\begin{bmatrix} 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} = (1 \cdot 4) + (3 \cdot -2) + (-5 \cdot -1) = 3$$

Properties of the Dot Product

- ✓ Distributive: if **a**, **b**, and **c** and real vectors, then $\boldsymbol{a}^T(\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a}^T \boldsymbol{b} + \boldsymbol{a}^T \boldsymbol{c}$
- ✘ Associative: $\boldsymbol{a}^T(\boldsymbol{b}^T \boldsymbol{c}) \neq (\boldsymbol{a}^T \boldsymbol{b})\boldsymbol{c}$ – in general the associative property does not hold, as the dot product would most likely produce different scalars.
  - Additionally, $\boldsymbol{a}$ could have a different dimensionality than $\boldsymbol{b}$ and $\boldsymbol{c}$. I.e., even if $\boldsymbol{b}$ and $\boldsymbol{c}$ had the same dimensionality ($\boldsymbol{a}^T(\boldsymbol{b}^T \boldsymbol{c})$ would be valid scalar-vector multiplication) then $\boldsymbol{a}^T \boldsymbol{b}$ would be invalid.