

1. Find the determinants in (a), (b), and (c) where $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$

$$(a) \begin{vmatrix} a & b & c \\ 5d & 5e & 5f \\ g & h & i \end{vmatrix} = 7 \cdot 5 = \boxed{35}$$

- The determinant of a matrix \mathbf{A} where a row \mathbf{m}_i (or column \mathbf{n}_i) of \mathbf{A} is multiplied by some scalar α is equal to the determinant of \mathbf{A} multiplied by α , i.e.,

$$\alpha \mathbf{m}_i \vee \alpha \mathbf{n}_i = \alpha \det(\mathbf{A})$$

- Using “Rule of Sarrus” method for 3×3 matrices as a demonstration:

$$\begin{vmatrix} a & b & c \\ 5d & 5e & 5f \\ g & h & i \end{vmatrix} = 5aei + 5bfg + 5cdh - 5ceg - 5bdi - 5afh$$

$$= 5(aei + bfg + cdh - ceg - bdi - afh)$$

$$(b) \begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix} = -1 \cdot 7 = \boxed{-7}$$

- Any distinct permutation of the rows (or columns) of \mathbf{A} multiplies the determinant by -1 , i.e.,

$$\mathbf{m}_i \updownarrow \mathbf{m}_j \vee \mathbf{n}_i \leftrightarrow \mathbf{n}_j = -1 \det(\mathbf{A})$$

- E.g., the determinant of the original matrix:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = ae i + bf g + cd h - ce g - bd i - af h$$

and the determinant of the given matrix with the row swapped:

$$\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix} = bd i + ec g + fa h - gb f - ae i - dh c$$

I moved around some products to make the symmetry more clear, but that doesn't change anything. Now, multiplication by -1 clearly shows change in sign:

$$ae i + bf g + cd h - ce g - bd i - af h$$

$$-ae i - bf g - cd h + ce g + bd i + af h$$

$$(c) \begin{vmatrix} a & b & c \\ d+3g & e+3h & f+3i \\ g & h & i \end{vmatrix} = \boxed{7}$$

- Adding a scalar multiple of one row (or column) to another row (or column) does not change the value of the determinant.

$$\begin{vmatrix} a & b & c \\ d+3g & e+3h & f+3i \\ g & h & i \end{vmatrix} \quad \begin{vmatrix} a & b & c \\ d+3g & e+3h & f+3i \\ g & h & i \end{vmatrix}$$

↓

$$a(e+3h)i + b(f+3i)g + c(d+3g)h - c(e+3h)g - b(d+3g)i - a(f+3i)h$$

multilinearity ↓ alternating

$$aei + bfg + cdh - ceg - bdi - afh$$

2. Construct an example of a 2×2 matrix with only one distinct eigenvalue.

- The eigenvalues of any upper or lower triangular matrix (and any square diagonal matrix) are simply the elements along the diagonal. Thus, all the following examples of the 2×2 matrices have only one distinct eigenvalue:

$$\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \lambda = 6, 6 \quad \begin{bmatrix} 6 & 9 \\ 0 & 6 \end{bmatrix} \lambda = 6, 6 \quad \begin{bmatrix} 6 & 0 \\ 9 & 6 \end{bmatrix} \lambda = 6, 6$$

3. Show that $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$. What is its corresponding eigenvalue?

$$A\mathbf{v} = \lambda\mathbf{v} \rightarrow \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \quad \lambda = 1$$

- Characteristic polynomial of A and corresponding eigenvalues:

$$p(\lambda) = \lambda^3 + 5\lambda^2 - 8\lambda + 4, \quad \lambda = 1, 2, 2$$

- Double-checking with $\lambda = 1$; should yield zero vector given corresponding eigenvalue since $(A - \lambda I)\mathbf{v} = \mathbf{0}$ (I can't bold 0 for some reason, $\mathbf{0}$ = zero vector)

$$\left(\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$