Calculus

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Limits and Continuity

Limits

🌐 Limit 🔗 | Thomas' Calculus (2.2–2.4) 🔗

- **Limit** $\lim\limits_{x \to c}$: the value of a function (or sequence) approaches as the input (or index) approaches some value (informal definition).

  - Limits are used to define continuity↓, derivatives↓, and integrals↓.

  ### Limits of a Functions and Sequences

  🌐 Limit of a function 🔗 | Limit of a sequence 🔗 | Essence of Calculus, E7 ▶

  - **Limit of a function**: a fundament concept in calculus and analysis concerning the behavior L of a function near a particular input c, i.e.,

  $$\lim_{x \to c} f(x) = L$$

    - Reads as "f of x tends to L as x tends to c"

  - ε, δ **Limit of function**: a formalized definition, wherein $f(x)$ is defined on an open interval I, except possibly at c itself, leading to above definition, if and only if:

    - For every real measure of closeness $\varepsilon > 0$, there exists a real corresponding $\delta > 0$, such that for all existing further approaches there exist a smaller ε, i.e.,

    $$f : \mathbb{R} \to \mathbb{R}, \ c, L \in \mathbb{R} \Rightarrow \lim_{x \to c} f(x) = L$$
    $$\Updownarrow$$
    $$\forall \varepsilon > 0 \, (\exists \delta > 0 : \forall x \in I \, (0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon))$$

  - Functions do not have a limit when the function:

    - has a unit step, i.e., it "jumps" at a point;

    - is not bounded, i.e., it tends towards infinity;

    - or does not stay close to any single number, i.e., it oscillates too much.

  - **Limit of a sequence**: the value that the terms of a sequence (x_n) "tends to" (and not to any other) as n approaches infinity (or some point), i.e.,

  $$\lim_{n \to \infty} x_n = x$$

  - ε **Limit of sequence**: for every measure of closeness ε, the sequence's term eventually converge to the limit, i.e.,

  $$\forall \varepsilon > 0 \, (\exists N \in \mathbb{N} \, (\forall n \in \mathbb{N} \, (n \geq N \Rightarrow |x_n - x| < \varepsilon)))$$

    - **Convergent**: when a limit of a sequence exists.

    - **Divergent**: a sequence that does not converge.

Properties of Limits
⊕ List of limits ⁶ᵒ | Squeeze theorem ⁶ᵒ

- **Operations on a single known limit**: if $\lim\limits_{x \to c} f(x) = L$ then:
  - $\lim\limits_{x \to c} [f(x) \pm \alpha] = L \pm \alpha$
  - $\lim\limits_{x \to c} \alpha f(x) = \alpha L$
  - $\lim\limits_{x \to c} f(x)^{-1} = L^{-1}, L \neq 0$
  - $\lim\limits_{x \to c} f(x)^n = L^n, n \in \mathbb{N}$
  - $\lim\limits_{x \to c} f(x)^{n^{-1}} = L^{n^{-1}}, n \in \mathbb{N}$, if $n \in \mathbb{N}_e \Rightarrow L > 0$

- **Operations on two known limits**: if $\lim\limits_{x \to c} f(x) = L_1$ and $\lim\limits_{x \to c} g(x) = L_2$
  - $\lim\limits_{x \to c} [f(x) \pm g(x)] = L_1 \pm L_2$
  - $\lim\limits_{x \to c} [f(x)g(x)] = L_1 L_2$
  - $\lim\limits_{x \to c} f(x)g(x)^{-1} = L_1 L_2^{-1}$

- **Squeeze theorem**: used to confirm the limit of a function via comparison with two other functions whose limits are easily known or computed.
  - Let I be an interval having the point a as a limit point.
  - Let g, f, and h, be functions defined on I, except possibly at a itself.
  - Suppose that $\forall x \in I \wedge x \neq a \Rightarrow g(x) \leq f(x) \leq h(x)$
  - And suppose that $\lim\limits_{x \to a} g(x) = \lim\limits_{x \to a} h(x) = L$
  - Then, $\lim\limits_{x \to a} f(x) = L$
  - Essentially, the hard to compute limit of the "middle function" is found by finding two other easy functions that "squeeze" the middle function at that point.

One-Sided Limit
⊕ One-Sided Limit ⁶ᵒ

- **One-sided limit**: one of two limits of $f(x)$ as x approaches a specified point from either the left or from the right.

  - From the left: $\lim\limits_{x \to c^-} = L$ • From the right: $\lim\limits_{x \to c^+} = L$

- If the left and right limits exist and are equal, then

$$\lim\limits_{x \to c} f(x) = L \Leftrightarrow \lim\limits_{x \to c^-} f(x) = L \wedge \lim\limits_{x \to c^+} f(x) = L$$

- Limits can still exist, even if the function is defined at a different point, as long as both one-sided limits approach the same value near the given input.

Continuity

🌐 Continuous function 🔗 | Classification of discontinuities 🔗 | Thomas' Calculus (2.5) 🔗

- Continuity of functions is one of the core concepts of topology, however, there are definitions in terms of limits that prove useful; the following is only a primer.

Continuous Functions

- ○ **Continuous function**: a function that does not have any abrupt changes in value.
  - · I.e., a function is continuous if and only if arbitrarily small changes in its output can be assured by restricting to sufficiently small changes in its input.
- ○ **Discontinuous**: when a function is not continuous at a point in its domain, leading to a discontinuity; there are three classifications:
  - · **Removable**: when both one-sided limits↑ exist, are finite, and are equal, but the actual value of $f(x)$ is not equal to the limit and equal to some other value.
    - · The discontinuity can be removed to regain continuity.
    - · Sometimes the term *removable discontinuity* is mistaken for *removable singularity*, or a "whole" in the function (the point is not defined elsewhere).
  - · **Jump**: when a single limit does not exist because the one-sided limits exist and are finite, but not equal.
    - · Points can be defined at the discontinuity, but the function can not be made continuous.
  - · **Essential**: when at least one of two one-sided limits doesn't exist; can be the result of oscillating or unbounded functions.

Intermediate Value Theorem

🌐 Intermediate Value Theorem 🔗

- ○ **Intermediate value theorem**: if f is a continuous function whose domain contains the interval $[a, b]$, then it takes on any given value between $f(a)$ and $f(b)$ at some point within the intervals.
- ○ Relevant deductions, i.e., important corollaries:
  - · **Bolzano's theorem**: if a continuous function has values of opposite sign inside an interval, then it has a root in that interval.
  - · The image of a continuous function over an interval is itself an interval.
- ○ Thus, the image set $f(I)$ (which has no gaps) is also an interval, and it contains:

$$[\min(f(a), f(b)), \ \max(f(a), f(b))]$$

Limits Involving Infinity

🌐 Limits involving infinity 🔗 | Thomas' Calculus (2.6) 🔗

- Let $S \subseteq, x \in S$ and $f : S \mapsto \mathbb{R}$, then limits of these functions can approach arbitrarily large $(\pm)$ values, providing a connection to asymptotes, and thus, analysis.

Limits at Infinity

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Infinite Limits

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Asymptotes of functions

🌐 Asymptote 🔗

  ○ **Asymptote**:

Derivatives

Integrals

Infinite Sequences and Series

Vectors and Vector-Valued Functions

Multiple Integrals

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