

# Calculus



## Limits and Continuity

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# Limits and Continuity



## Limits

🌐 Limit 🌐 | Thomas' Calculus (2.2–2.4) 🌐

- **Limit**  $\lim_{x \rightarrow c}$ : the value of a function (or sequence) approaches as the input (or index) approaches some value (informal definition)
  - Limits are used to define **continuity** ↓, **derivatives** ↓, and **integrals** ↓

## Limits of a Functions and Sequences

🌐 Limit of a function 🌐 | Limit of a sequence 🌐

- **Limit of a function**: a fundamental concept in calculus and analysis concerning the behavior  $L$  of a function near a particular input  $c$ , i.e.,

$$\lim_{x \rightarrow c} f(x) = L$$

- Reads as “ $f$  of  $x$  tends to  $L$  as  $x$  tends to  $c$ ”
- $\epsilon, \delta$  **Limit of function**: a formalized definition, wherein  $f(x)$  is defined on an open interval  $I$ , except possibly at  $c$  itself, leading to above definition, if and only if:
  - For every real measure of **closeness**  $\epsilon > 0$ , there exists a real **corresponding**  $\delta > 0$ , such that for all existing further approaches there exist a smaller  $\epsilon$ , i.e.,

$$f : \mathbb{R} \rightarrow \mathbb{R}, c, L \in \mathbb{R} \Rightarrow \lim_{x \rightarrow c} f(x) = L$$



$$\forall \epsilon > 0 (\exists \delta > 0 : \forall x \in I (0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon))$$

- Functions do not have a limit when the function:
  - has a unit step, i.e., it “jumps” at a point;
  - is not bounded, i.e., it tends towards infinity;
  - or does not stay close to any single number, i.e., it oscillates too much.
- **Limit of a sequence**: the value that the terms of a sequence  $(x_n)$  “tends to” (and not to any other) as  $n$  approaches infinity (or some point), i.e.,

$$\lim_{n \rightarrow \infty} x_n = x$$

- $\epsilon$  **Limit of sequence**: for every measure of closeness  $\epsilon$ , the sequence’s term eventually converge to the limit, i.e.,

$$\forall \epsilon > 0 (\exists N \in \mathbb{N} (\forall n \in \mathbb{N} (n \geq N \Rightarrow |x_n - x| < \epsilon)))$$

- **Convergent**: when a limit of a sequence **exists**.
- **Divergent**: a sequence that **does not** converge.

## Properties of Limits

📌 List of limits 📌 | Squeeze theorem 📌

- **Operations on a single known limit:** if  $\lim_{x \rightarrow c} f(x) = L$  then:
  - $\lim_{x \rightarrow c} [f(x) \pm \alpha] = L \pm \alpha$
  - $\lim_{x \rightarrow c} \alpha f(x) = \alpha L$
  - $\lim_{x \rightarrow c} f(x)^{-1} = L^{-1}, L \neq 0$
  - $\lim_{x \rightarrow c} f(x)^n = L^n, n \in \mathbb{N}$
  - $\lim_{x \rightarrow c} f(x)^{n^{-1}} = L^{n^{-1}}, n \in \mathbb{N}, \text{ if } n \in \mathbb{N}_e \Rightarrow L > 0$
- **Operations on two known limits:** if  $\lim_{x \rightarrow c} f(x) = L_1$  and  $\lim_{x \rightarrow c} g(x) = L_2$ 
  - $\lim_{x \rightarrow c} [f(x) \pm g(x)] = L_1 \pm L_2$
  - $\lim_{x \rightarrow c} [f(x)g(x)] = L_1 L_2$
  - $\lim_{x \rightarrow c} f(x)g(x)^{-1} = L_1 L_2^{-1}$
- **Squeeze theorem:** used to confirm the limit of a function via comparison with two other functions whose limits are easily known or computed.
  - Let  $I$  be an interval having the point  $a$  as a limit point.
  - Let  $g, f$ , and  $h$ , be functions defined on  $I$ , except possibly at  $a$  itself.
  - Suppose that  $\forall x \in I \wedge x \neq a \Rightarrow g(x) \leq f(x) \leq h(x)$
  - And suppose that  $\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} h(x) = L$
  - Then,  $\lim_{x \rightarrow a} f(x) = L$
  - Essentially, the hard to compute limit of the “middle function” is found by finding two other easy functions that “squeeze” the middle function at that point.

## One-Sided Limit

📌 One-Sided Limit 📌

- **One-sided limit:** one of two limits of  $f(x)$  as  $x$  approaches a specified point from either the **left** or from the **right**.
  - From the **left**:  $\lim_{x \rightarrow c^-} = L$
  - From the **right**:  $\lim_{x \rightarrow c^+} = L$
- If the left and right limits exist and are equal, then

$$\lim_{x \rightarrow c} f(x) = L \Leftrightarrow \lim_{x \rightarrow c^-} f(x) = L \wedge \lim_{x \rightarrow c^+} f(x) = L$$

- Limits can still exist, even if the function is defined at a different point, as long as both one-sided limits approach the same value near the given input.

## Continuity

🌐 Continuous function 🌐 | Classification of discontinuities 🌐 | Thomas' Calculus (2.5) 🌐

- Continuity of functions is one of the core concepts of topology, however, there are definitions in terms of limits that prove useful; the following is only a primer.

### Continuous Functions

- **Continuous function:** a function that does not have any abrupt changes in value.
  - I.e., a function is continuous if and only if arbitrarily small changes in its output can be assured by restricting to sufficiently small changes in its input.
- **Discontinuous:** when a function is not continuous at a point in its domain, leading to a discontinuity; there are three classifications:
  - **Removable:** when both **one-sided limits**  $\uparrow$  exist, are finite, and are equal, but the actual value of  $f(x)$  is not equal to the limit and equal to some other value.
    - The discontinuity can be removed to regain continuity.
    - Sometimes the term *removable discontinuity* is mistaken for *removable singularity*, or a "whole" in the function (the point is not defined elsewhere).
  - **Jump:** when a single limit does not exist because the one-sided limits exist and are finite, but not equal.
    - A function may have a point defined at the discontinuity, but can not be made continuous.
  - **Essential:** when at least one of two one-sided limits doesn't exist.
    - Can be the result of oscillating or unbounded functions.

### Intermediate Value Theorem

🌐 Intermediate Value Theorem 🌐

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### Extreme Value Theorem

🌐 Intermediate Value Theorem 🌐

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## Limits Involving Infinity

- Sources:

### Limits at Infinity

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### Infinite Limits

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# Derivatives



# Applications of Derivatives



# Integrals





# Applications of Integrals



# Transcendental Functions



# Techniques of Integration



# Infinite Sequences and Series



# First-Order Differential Equations



# Parametric Equations and Polar Coordinates



# Vectors and Vector-Valued Functions



# Partial Derivatives





# Multiple Integrals



# Vector Calculus



# Second-Order Differential Equations

