Calculus

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Limits and Continuity

Limits

🌐 Limit 🔗 | Thomas's Calculus$^{14\text{th}}$ (2.2–2.4) 📑

- **Limit** $\lim\limits_{x \to c}$: the value of a function (or sequence) approaches as the input (or index) approaches some value (informal definition).

  ○ Limits are used to define continuity↓, derivatives↓, and integrals↓.

Limits of a Functions and Sequences

🌐 Limit of a function 🔗 | Limit of a sequence 🔗 | Essence of Calculus, E7 ▶️

  ○ **Limit of a function**: a fundament concept in calculus and analysis concerning the behavior of a function near a particular input c, i.e.,

$$\lim_{x \to c} f(x) = L$$

  · Reads as "f of x tends to L as x tends to c"

  ○ ε, δ **Limit of function**: a formalized definition, wherein $f(x)$ is defined on an open interval $\mathcal{I}$, except possibly at c itself, leading to above definition, if and only if:

  · For every real measure of closeness $\varepsilon > 0$, there exists a real corresponding $\delta > 0$, such that for all existing further approaches there exist a smaller ε, i.e.,

$$f : \mathbb{R} \to \mathbb{R}, \ c, L \in \mathbb{R} \Rightarrow \lim_{x \to c} f(x) = L$$
$$\Updownarrow$$
$$\forall \varepsilon > 0 \, (\exists \delta > 0 : \forall x \in \mathcal{I} \, (0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon))$$

  ○ Functions do not have a limit when the function:

  · has a unit step, i.e., it "jumps" at a point;

  · is not bounded, i.e., it tends towards infinity;

  · or it oscillates, i.e., does not stay close to any single number.

  ○ **Limit of a sequence**: the value that the terms of a sequence (x_n) "tends to" (and not to any other) as n approaches infinity (or some point), i.e.,

$$\lim_{n \to \infty} x_n = x$$

  ○ ε **Limit of sequence**: for every measure of closeness ε, the sequence's x_n term eventually converge to the limit, i.e.,

$$\forall \varepsilon > 0 \, (\exists N \in \mathbb{N} \, (\forall n \in \mathbb{N} \, (n \geq N \Rightarrow |x_n - x| < \varepsilon)))$$

  · **Convergent**: when a limit of a sequence exists.

  · **Divergent**: a sequence that does not converge.

Properties of Limits

🌐 List of limits 🔗 | Squeeze theorem 🔗

- **Operations on a single known limit**: if $\lim\limits_{x \to c} f(x) = L$ then:
  - $\lim\limits_{x \to c} [f(x) \pm \alpha] = L \pm \alpha$
  - $\lim\limits_{x \to c} \alpha f(x) = \alpha L$
  - $\lim\limits_{x \to c} f(x)^{-1} = L^{-1}, L \neq 0$
  - $\lim\limits_{x \to c} f(x)^n = L^n, n \in \mathbb{N}$
  - $\lim\limits_{x \to c} f(x)^{n^{-1}} = L^{n^{-1}}, n \in \mathbb{N}$, if $n \in \mathbb{N}_e \Rightarrow L > 0$

- **Operations on two known limits**: if $\lim\limits_{x \to c} f(x) = L_1$ and $\lim\limits_{x \to c} g(x) = L_2$
  - $\lim\limits_{x \to c} [f(x) \pm g(x)] = L_1 \pm L_2$
  - $\lim\limits_{x \to c} [f(x)g(x)] = L_1 L_2$
  - $\lim\limits_{x \to c} f(x)g(x)^{-1} = L_1 L_2^{-1}$

- **Squeeze theorem**: used to confirm the limit of a difficult to compute function via comparison with two other functions whose limits are easily known or computed.
  - Let $\mathcal{I}$ be an interval having the point c as a limit point.
  - Let g, f, and h, be functions defined on $\mathcal{I}$, except possibly at c itself.
  - Suppose that $\forall x \in \mathcal{I} \wedge x \neq c \Rightarrow g(x) \leq f(x) \leq h(x)$
  - And suppose that $\lim\limits_{x \to c} g(x) = \lim\limits_{x \to c} h(x) = L$
  - Then, $\lim\limits_{x \to c} f(x) = L$
  - Essentially, the hard to compute limit of the "middle function" is found by finding two other easy functions that "squeeze" the middle function at that point.

One-Sided Limit

🌐 One-Sided Limit 🔗

- **One-sided limit**: one of two limits of $f(x)$ as x approaches a specified point from either the left or from the right.

  - From the left: $\lim\limits_{x \to c^-} = L$ - From the right: $\lim\limits_{x \to c^+} = L$

- If the left and right limits exist and are equal, then

$$\lim\limits_{x \to c} f(x) = L \Leftrightarrow \lim\limits_{x \to c^-} f(x) = L \wedge \lim\limits_{x \to c^+} f(x) = L$$

- Limits can still exist, even if the function is defined at a different point, as long as both one-sided limits approach the same value near the given input.

Continuity

🌐 Continuous function 🔗 | Discontinuities 🔗 | Thomas's Calculus[14th] (2.5) 🗐

- Continuity of functions is one of the core concepts of topology, however, there are definitions in terms of limits that prove useful; the following is only a primer.

Continuous Functions

- ○ **Continuous function**: a function that does not have any abrupt changes in value.

  - · I.e., a function is continuous if and only if arbitrarily small changes in its output can be assured by restricting to sufficiently small changes in its input.

- ○ **Discontinuous**: when a function is not continuous at a point in its domain, leading to a discontinuity; there are three classifications:

  - · **Removable**: when both one-sided limits ↑ exist, are finite, and are equal, but the actual value of $f(x)$ is not equal to the limit and equal to some other value.

    - · The discontinuity can be removed to regain continuity.

    - · Sometimes the term *removable discontinuity* is mistaken for *removable singularity*, or a "whole" in the function (the point is not defined elsewhere).

  - · **Jump**: when a single limit does not exist because the one-sided limits exist and are finite, but not equal.

    - · Points can be defined at the discontinuity, but the function can not be made continuous.

  - · **Essential**: when at least one of two one-sided limits doesn't exist; can be the result of oscillating or unbounded functions.

Intermediate Value Theorem

🌐 Intermediate value theorem 🔗

- ○ **Intermediate value theorem**: if f is a continuous function whose domain contains the interval $[a, b]$, then it takes on any given value between $f(a)$ and $f(b)$ at some point within the intervals.

- ○ Relevant deductions, i.e., important corollaries:

  - · **Bolzano's theorem**: if a continuous function has values of opposite sign inside an interval, then it has a root in that interval.

  - · The image of a continuous function over an interval is itself an interval.

- ○ Thus, the image set $f(\mathcal{I})$ (which has no gaps) is also an interval, and it contains:

$$[\min(f(a), f(b)), \ \max(f(a), f(b))]$$

Limits Involving Infinity

🌐 Limits involving infinity ⸘ | Thomas's Calculus[14th] (2.6) 🗒

- Let $S \subseteq \mathbb{R}$, $x \in S$ and $f : S \mapsto \mathbb{R}$, then limits of these functions can approach arbitrarily large ($\pm$) values, providing a connection to asymptotes, and thus, analysis.

Limits at Infinity and Infinite Limits

- **Limits at infinity**: limits defined as f(x) $\pm$ infinity are defined much like normal limits:

$$\lim_{x \to -\infty} f(x) = L \qquad \lim_{x \to \infty} f(x) = L$$

  - Formally, for all measures of closeness ε there exists a point c such that $|f(x) - L| < \varepsilon$ whenever $x < c \vee x > c$ (respectively), i.e.,

$$\forall \varepsilon > 0 \, (\exists c \, (\forall x \, \{<, >\} \, c : |f(x) - L| < \varepsilon))$$

- Basic rules for rational functions $f(x) = p(x)q(x)^{-1}$, where p and q are polynomials, the degree of each is denoted as $\{p, q\}^\circ$, and the leading coefficients are denoted as P, Q, then:

  - $p^\circ > q^\circ \Rightarrow L$ is $\{+, -\}$ depending on the sign of the leading coefficients.
  - $p^\circ = q^\circ \Rightarrow L = PQ^{-1}$
  - $p^\circ < q^\circ \Rightarrow L = 0$

- **Infinite limits**: the usual limit does not exist for a limit that grows out of bounds, however, limits with infinite values can be introduced:

$$\lim_{x \to a} f(x) = \infty, \quad \text{i.e.,} \quad \forall n > 0 \, (\exists \delta > 0 : f(x) > n \Leftrightarrow 0 < |x - a| < \delta)$$

Asymptotes of functions

🌐 Asymptotes ⸘

- **Asymptote**: a tangent line of a curve at a point at infinity; the distance between the curve and the line approaches zero as a coordinate tends to infinity.

- There are three kinds of asymptotes: *horizontal, vertical* and *oblique*; nature of the asymptote is dependent on a function's relation to infinity.

  - **Horizontal asymptotes**: a result of limits at infinity, i.e., when $x \to \pm\infty$
  - **Vertical asymptotes**: a result of infinite limits, i.e., when $x \to \pm a = \pm\infty$
  - **Oblique asymptotes**: when a linear asymptote is not parallel to either axis. $f(x)$ is asymptotic to the straight line $y = mx + n (m \neq 0)$ if:

$$\lim_{x \to \pm\infty} [f(x) - (mx + n)] = 0$$

Derivatives

Derivative Fundamentals

🌐 Derivative 🔗 | Thomas's Calculus$^{14\text{th}}$ (3.2) 📓

- **Derivative**: the measure of sensitivity to change of the function value with respect to some change in its in argument.

  - Often described as the instantaneous rate of change of a single variable function, since it is the slope a tangent line at a particular point, when it exists.

    - **Tangent line**: the line through a pair of points on a curve (secant line), except the points are infinitely close, thus, it's the rate of change at that "instant".

- Formally, a derivative of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

  - Provided the limit exists.

  - Let $z = x + h$, then $h = z - x \wedge h \to 0 \Leftrightarrow z \to x$; this leads to an equivalent definition of the derivative (sometimes more convenient):

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

- **Differentiation**: the process of finding a derivative; if f' exists at a particular point, then f is said to be differentiable at that point.

  - If f' exists at every point, then f is said to be differentiable.

- **Notation**: there are many ways to denote the derivative, I prefer Leibniz's, but other notations can be useful in various contexts; some common notations (for $y = f(x)$):

$$f'(x) = y' = \dot{y} = \frac{dy}{dx} = \frac{d}{dx}f(x) = D(f)(x) = D_x f(x)$$

Simple Differentiation Rules

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Derivatives of Trigonometric Functions

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Differentiation Rules

Chain Rule

- ○

Product Rule

- ○

Quotient Rule

- ○

Differentials and Related Concepts

Differentials

- ○

Linearization

- ○

Implicit Differentiation

- ○

Related Rates

- ○

Differentials and Related Concepts

Differentials

Applications of Derivatives

Stationary Point

Maxima and Minima

-

Extreme Value Theorem

-

Interior Extremum Theorem

-

Mean Value Theorem

Rolle's Theorem

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Corollaries of the Mean Value Theorem

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Monotonic Functions

Mean Value Theorem

Rolle's Theorem

Derivative Tests

First-Derivative Test

- ○

Second-Derivative Test

- ○

Concavity

- ○

Higher-Order Derivative Test

- ○

Differential Methods

Newton's Method

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Taylor's Theorem

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General Leibniz Rule

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Differential Methods

Newton's Method

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Integrals

Integral Fundamentals

Terminology and Notation

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Primer: Formal Definitions

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Definite Integrals

Riemann Integral

- ○

Integrability

- ○

Properties of Definite Integrals

- ○

Definite Integrals

Riemann Integral

The Fundamental Theorem of Calculus

Fundamental Theorem, Part 1

- ○   ⸺❈ 18 ❈⸺

Fundamental Theorem, Part 2

- ○

The Integral of a Rate

- ○

Total Area

- ○

The Fundamental Theorem of Calculus

Fundamental Theorem, Part 1

Fundamental Theorem, Part 2

Integration By Substitution

Indefinite Integrals

- ○

Definite Integrals

- ○

Symmetric Functions

- ○

Area Between Curves

- ○

Integration By Substitution

Applications of Definite Integrals

Solid of Revolution

Disc Integration

-

Shell Integration

-

Arc Length

Dealing with Discontinuities

-

Differential Arc Length

-

Surface of Revolution

Revolution about the y-Axis

- ○

Transcendental Functions

Inverse Functions

One-to-One Functions

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Derivative Rule for Inverses

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Logarithmic Functions

Natural Logarithm

- ○

Properties of Logarithms

- ○

Trigonometric Integrals

- ○

Logarithmic Differentiation

- ○

Logarithmic Functions

Natural Logarithm

- ○

Exponential Functions

Euler's Number

○

Natural Exponential Function

○

Laws of Exponents

○

General Exponential Function

○

Exponential Change

- **Separable Differential Equations**

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  Examples of Exponential Change

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Indeterminate Forms

Indeterminate Form 0/0

- ○

L'Hôpital's Rule

- ○

Infinite Indeterminate Forms

- ○

Indeterminate Powers

- ○

Indeterminate Forms

Inverse Trigonometric Functions

Principal Trigonometric Values

○

Inverse Trigonometric Tables

○

Hyperbolic Functions

Hyperbolic Function Tables

- ○

Techniques of Integration

Integration by Parts

Definite Integrals by Parts

- ○

Trigonometric Integral Methods

Trigonometric Products and Powers

-

Trigonometric Square Roots

-

Trigonometric Substitutions

-

Trigonometric Integral Methods

Trigonometric Products and Powers

Trigonometric Square Roots

Trigonometric Substitutions

Partial Fraction Decomposition

Partial Fraction Principles

○

General Statement

○

Numerical Integration

Trapezoidal Rule

- ○

Simpson's Rule

- ○

Improper Integrals

Indirect Evaluation

- ○

Infinite Sequences and Series

Parametric Equations and Polar Coordinates

Vectors and Vector-Valued Functions

Partial Derivatives

Multiple Integrals

Vector Calculus

Second-Order Differential Equations