1. Find the determinants in (a), (b), and (c) where $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$

(a) $\begin{vmatrix} a & b & c \\ 5d & 5e & 5f \\ g & h & i \end{vmatrix} = 7 \cdot 5 = \boxed{35}$

- The determinant of a matrix $\boldsymbol{A}$ where a row $\boldsymbol{m}_i$ (or column $\boldsymbol{n}_i$) of $\boldsymbol{A}$ is multiplied by some scalar α is equal to the determinant of $\boldsymbol{A}$ multiplied by α, i.e.,

$$\alpha \boldsymbol{m}_i \vee \alpha \boldsymbol{n}_i = \alpha \det(A)$$

- Using "Rule of Sarrus" method for 3×3 matrices as a demonstration:

$$\begin{vmatrix} a & b & c \\ 5d & 5e & 5f \\ g & h & i \end{vmatrix} - \begin{vmatrix} a & b & c \\ 5d & 5e & 5f \\ g & h & i \end{vmatrix} = 5aei + 5bfg + 5cdh - 5ceg - 5bdi - 5afh$$

$$= 5(aei + bfg + cdh - ceg - bdi - afh)$$

(b) $\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix} = -1 \cdot 7 = \boxed{-7}$

- Any distinct permutation of the rows (or columns) of $\boldsymbol{A}$ multiplies the determinant by -1, i.e.,

$$\boldsymbol{m}_i \updownarrow \boldsymbol{m}_j \vee \boldsymbol{n}_i \leftrightarrow \boldsymbol{n}_j = -1 \det(A)$$

- E.g., the determinant of the original matrix:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} - \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - bdi - afh$$

and the determinant of the given matrix with the row swapped:

$$\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix} - \begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix} = bdi + ecg + fah - gbf - aei - dhc$$

I moved around some products to make the symmetry more clear, but that doesn't change anything. Now, multiplication by -1 clearly shows change in sign:

$$aei + bfg + cdh - ceg - bdi - afh$$
$$-aei - bfg - cdh + ceg + bdi + afh$$

(c) $\begin{vmatrix} a & b & c \\ d+3g & e+3h & f+3i \\ g & h & i \end{vmatrix} = \boxed{7}$

- Adding a scalar multiple of one row (or column) to another row (or column) does not change the value of the determinant.

$$\begin{vmatrix} a & b & c \\ d+3g & e+3h & f+3i \\ g & h & i \end{vmatrix} - \begin{vmatrix} a & b & c \\ d+3g & e+3h & f+3i \\ g & h & i \end{vmatrix}$$

$$\downarrow$$

$$a(e+3h)i + b(f+3i)g + c(d+3g)h - c(e+3h)g - b(d+3g)i - a(f+3i)h$$

$$\downarrow$$

$$aei + bfg + cdh - ceg - bdi - afh$$

2. Construct an example of a 2×2 matrix with only one distinct eigenvalue.

- The eigenvalues of any upper of lower triangular matrix (and any square diagonal matrix) are simply the elements along the diagonal. Thus, all the following examples of the 2×2 matrices have only one distinct eigenvalue:

$$\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \lambda = 6, 6 \quad \begin{bmatrix} 6 & 9 \\ 0 & 6 \end{bmatrix} \lambda = 6, 6 \quad \begin{bmatrix} 6 & 0 \\ 9 & 6 \end{bmatrix} \lambda = 6, 6$$

3. Show that $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of $\boldsymbol{A} = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$. What is its corresponding eigenvalue?

$$\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v} = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \quad \lambda = 1$$

- Characteristic polynomial of $\boldsymbol{A}$ and corresponding eigenvalues:

$$p(\lambda) = \lambda^3 + 5\lambda^2 - 8\lambda + 4, \quad \lambda = 1, 2, 2$$

- Double-checking with $\lambda = 1$

$$\left(\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$