Calculus

Limits and Continuity

Limits

Limit | Thomas's Calculus 14th (2.2–2.4)

- **Limit** $\lim\limits_{x \to c}$: the value of a function (or sequence) approaches as the input (or index) approaches some value (informal definition).
  - Limits are used to define continuity, derivatives, and integrals.

Limits of a Functions and Sequences

Limit of a function | Limit of a sequence | Essence of Calculus, E7

- **Limit of a function**: a fundament concept in calculus and analysis concerning the behavior of a function near a particular input c, i.e.,

$$\lim_{x \to c} f(x) = L$$

  - Reads as "f of x tends to L as x tends to c"

- ε, δ **Limit of function**: a formalized definition, wherein $f(x)$ is defined on an open interval $\mathcal{I}$, except possibly at c itself, leading to above definition, if and only if:
  - For every real measure of closeness $\varepsilon > 0$, there exists a real corresponding $\delta > 0$, such that for all existing further approaches there exist a smaller ε, i.e.,

$$f : \mathbb{R} \to \mathbb{R},\ c, L \in \mathbb{R} \Rightarrow \lim_{x \to c} f(x) = L$$

$$\Updownarrow$$

$$\forall \varepsilon > 0 \, (\exists \delta > 0 : \forall x \in \mathcal{I} \, (0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon))$$

- Functions do not have a limit when the function:
  - has a unit step, i.e., it "jumps" at a point;
  - is not bounded, i.e., it tends towards infinity;
  - or it oscillates, i.e., does not stay close to any single number.

- **Limit of a sequence**: the value that the terms of a sequence (x_n) "tends to" (and not to any other) as n approaches infinity (or some point), i.e.,

$$\lim_{n \to \infty} x_n = x$$

- ε **Limit of sequence**: for every measure of closeness ε, the sequence's x_n term eventually converge to the limit, i.e.,

$$\forall \varepsilon > 0 \, (\exists N \in \mathbb{N} \, (\forall n \in \mathbb{N} \, (n \geq N \Rightarrow |x_n - x| < \varepsilon)))$$

  - **Convergent**: when a limit of a sequence exists.
  - **Divergent**: a sequence that does not converge.

Properties of Limits

List of limits | Squeeze theorem

- **Operations on a single known limit**: if $\lim_{x\to c} f(x) = L$ then:
  - $\lim_{x\to c}[f(x) \pm \alpha] = L \pm \alpha$
  - $\lim_{x\to c} \alpha f(x) = \alpha L$
  - $\lim_{x\to c} f(x)^{-1} = L^{-1}, L \neq 0$
  - $\lim_{x\to c} f(x)^{n} = L^{n}, n \in \mathbb{N}$
  - $\lim_{x\to c} f(x)^{n^{-1}} = L^{n^{-1}}, n \in \mathbb{N},$ if $n \in \mathbb{N}_e \Rightarrow L > 0$
- **Operations on two known limits**: if $\lim_{x\to c} f(x) = L_1$ and $\lim_{x\to c} g(x) = L_2$
  - $\lim_{x\to c}[f(x) \pm g(x)] = L_1 \pm L_2$
  - $\lim_{x\to c}[f(x)g(x)] = L_1 L_2$
  - $\lim_{x\to c} f(x)g(x)^{-1} = L_1 L_2^{-1}$
- **Squeeze theorem**: used to confirm the limit of a difficult to compute function via comparison with two other functions whose limits are easily known or computed.
  - Let $\mathcal{I}$ be an interval having the point c as a limit point.
  - Let g, f, and h, be functions defined on $\mathcal{I}$, except possibly at c itself.
  - Suppose that $\forall x \in \mathcal{I} \wedge x \neq c \Rightarrow g(x) \leq f(x) \leq h(x)$
  - And suppose that $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$
  - Then, $\lim_{x\to c} f(x) = L$
  - Essentially, the hard to compute limit of the "middle function" is found by finding two other easy functions that "squeeze" the middle function at that point.

One-Sided Limit

One-Sided Limit

- **One-sided limit**: one of two limits of $f(x)$ as x approaches a specified point from either the left or from the right.
  - From the left: $\lim_{x\to c^-} = L$
  - From the right: $\lim_{x\to c^+} = L$
- If the left and right limits exist and are equal, then

$$\lim_{x\to c} f(x) = L \Leftrightarrow \lim_{x\to c^-} f(x) = L \wedge \lim_{x\to c^+} f(x) = L$$

- Limits can still exist, even if the function is defined at a different point, as long as both one-sided limits approach the same value near the given input.

Continuity

Continuous function | Discontinuities | Thomas's Calculus 14th (2.5)

- Continuity of functions is one of the core concepts of topology, however, there are definitions in terms of limits that prove useful; the following is only a primer.

Continuous Functions

- **Continuous function**: a function that does not have any abrupt changes in value.
  - I.e., a function is continuous if and only if arbitrarily small changes in its output can be assured by restricting to sufficiently small changes in its input.
- **Discontinuous**: when a function is not continuous at a point in its domain, leading to a discontinuity; there are three classifications:
  - **Removable**: when both one-sided limits↑ exist, are finite, and are equal, but the actual value of $f(x)$ is not equal to the limit and equal to some other value.
    - The discontinuity can be removed to regain continuity.
    - Sometimes the term *removable discontinuity* is mistaken for *removable singularity*, or a "whole" in the function (the point is not defined elsewhere).
  - **Jump**: when a single limit does not exist because the one-sided limits exist and are finite, but not equal.
    - Points can be defined at the discontinuity, but the function can not be made continuous.
  - **Essential**: when at least one of two one-sided limits doesn't exist; can be the result of oscillating or unbounded functions.

Intermediate Value Theorem

Intermediate value theorem

- **Intermediate value theorem**: if f is a continuous function whose domain contains the interval $[a, b]$, then it takes on any given value between $f(a)$ and $f(b)$ at some point within the intervals.
- Relevant deductions, i.e., important corollaries:
  - **Bolzano's theorem**: if a continuous function has values of opposite sign inside an interval, then it has a root in that interval.
  - The image of a continuous function over an interval is itself an interval.
- Thus, the image set $f(\mathcal{I})$ (which has no gaps) is also an interval, and it contains:

$$[\min(f(a), f(b)),\ \max(f(a), f(b))]$$

Limits Involving Infinity

Limits involving infinity | Thomas's Calculus[14th] (2.6)

- Let $S \subseteq \mathbb{R}$, $x \in S$ and $f : S \mapsto \mathbb{R}$, then limits of these functions can approach arbitrarily large $(\pm)$ values, providing a connection to asymptotes, and thus, analysis.

Limits at Infinity and Infinite Limits

- **Limits at infinity**: limits defined as f(x) $\pm$ infinity are defined much like normal limits:

$$\lim_{x \to -\infty} f(x) = L \qquad \lim_{x \to \infty} f(x) = L$$

  - Formally, for all measures of closeness ε there exists a point c such that $|f(x) - L| < \varepsilon$ whenever $x < c \vee x > c$ (respectively), i.e.,

$$\forall \varepsilon > 0\,(\exists c\,(\forall x\,\{<,>\}\,c : |f(x) - L| < \varepsilon))$$

- Basic rules for rational functions $f(x) = p(x)q(x)^{-1}$, where p and q are polynomials, the degree of each is denoted as $\{p, q\}^{\circ}$, and the leading coefficients are denoted as P, Q, then:
  - $p^{\circ} > q^{\circ} \Rightarrow L$ is $\{+, -\}$ depending on the sign of the leading coefficients.
  - $p^{\circ} = q^{\circ} \Rightarrow$ L = PQ^{-1}
  - $p^{\circ} < q^{\circ} \Rightarrow L = 0$
- **Infinite limits**: the usual limit does not exist for a limit that grows out of bounds, however, limits with infinite values can be introduced:

$$\lim_{x \to a} f(x) = \infty, \quad \text{i.e.,} \quad \forall n > 0\,(\exists \delta > 0 : f(x) > n \Leftrightarrow 0 < |x - a| < \delta)$$

Asymptotes of functions

Asymptotes

- **Asymptote**: a tangent line of a curve at a point at infinity; the distance between the curve and the line approaches zero as a coordinate tends to infinity.
- There are three kinds of asymptotes: *horizontal*, *vertical* and *oblique*; nature of the asymptote is dependent on a function's relation to infinity.
  - **Horizontal asymptotes**: a result of limits at infinity, i.e., when $x \to \pm\infty$
  - **Vertical asymptotes**: a result of infinite limits, i.e., when $x \to \pm a = \pm\infty$
  - **Oblique asymptotes**: when a linear asymptote is not parallel to either axis. $f(x)$ is asymptotic to the straight line $y = mx + n (m \neq 0)$ if:

$$\lim_{x \to \pm\infty} [f(x) - (mx + n)] = 0$$

Derivatives

Derivative Fundamentals

Derivative | Thomas's Calculus[14th] (3.2)

- **Derivative**: the measure of sensitivity to change of the function value with respect to some change in its in argument.
  - Often described as the instantaneous rate of change of a single variable function, since it is the slope a tangent line at a particular point, when it exists.
    - **Tangent line**: the line through a pair of points on a curve (secant line), except the points are infinitely close, thus, it's the rate of change at that "instant".

Definition, Notation

- Formally, a derivative of the function $f(x)$ with respect to the variable x is the function f' whose value at x is (provided the limit exists)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

  - Let $z = x + h$, then $h = z - x \wedge h \to 0 \Leftrightarrow z \to x$; this leads to an equivalent definition of the derivative (sometimes more convenient):

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

- **Notation**: there are many ways to denote the derivative; notations can be useful in various contexts, some common notations (for $y = f(x)$):

$$f'(x) = y' = \dot{y} = \frac{dy}{dx} = \frac{d}{dx} f(x) = D(f)(x) = D_x f(x)$$

- **Differentiation**: the process of finding a derivative; if f' exists at a particular point, then f is said to be differentiable at that point.
  - If f' exists at every point on an interval, then f is differentiable on that interval.
  - f' is differentiable on a closed interval $[a, b]$ if both one-sided limits↑ of the function ($h \to \{0^+ : a,\ 0^- : b\}$) exist at the end points and is differentiable on the interior.
  - Not all continuous functions have a derivative, but functions with a derivative are continuous; functions with any of following do not have derivatives:
    - corners (one-sided derivatives differ at a point),
    - cusps (slope approaches alternating $\pm\infty$ on both sides of a point),
    - discontinuities, or vertical tangent lines.

Differentiation Rules

Differentiation rules | Thomas's Calculus[14th] (3.3, 3.5, 3.7)

- Derivatives can be found by computing its limit, but there are several methods that use of combinations of simpler functions to make computation easier.

Linear, Product, Chain, Inverse

- **Linear**: differentiation of linear functions consists of the constant and sum (& subtraction) rules, given the following

$$\forall(f \wedge g) \wedge \forall(a \wedge b \in \mathbb{R}) \Rightarrow \frac{d(af+bg)}{dx} = a\frac{df}{dx} + b\frac{dg}{dx}$$

Constant	Constant factor	Sum (Difference)
$\frac{d}{dx}(c) = 0$	$(af)' = af'$	$(f +/- g)' = f' +/- g'$

- **Product rule**: used for the product of two functions; can be generalized↓

$$\frac{d(fg)}{dx} = \frac{df}{dx}g + f\frac{dg}{dx}$$

- **Chain rule**: used for the composition of two functions $f(g(x))$; if z depends on y, which is dependent on x, then z depends on x as well, i.e.,

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

The following is used to indicate which points the derivatives have to evaluated at:

$$\left.\frac{dz}{dx}\right|_x = \left.\frac{dz}{dy}\right|_{y(x)} \cdot \left.\frac{dy}{dx}\right|_x$$

  - "Outside-Inside Rule": take the derivative of the "outside" function, leave "inside" alone, and multiply it be the derivative of the "inside."
  - This method must be recursively "chained" when there are further compositions in the inside function, hence the name.

- **Inverse function rule**: can be applied if the function f has an inverse function g, i.e., a function that "undoes" the effect of f.

$$\{g(f(x)) = x \wedge f(g(y)) = y\} \Rightarrow \frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1}$$

  - Application of the chain rule on $f^{-1}(y) = x$ in terms of x clearly shows the result if the derivatives exist and are reciprocal,

$$\frac{dx}{dy} \cdot \frac{dy}{dx} = \frac{dx}{dx} = 1$$

Powers, Polynomials, Quotients, Reciprocals

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Exponential, Logarithmic

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Trigonometric, Hyperbolic

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Differentials and Related Concepts

Differentials

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Linearization

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Implicit Differentiation

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Related Rates

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Applications of Derivatives

Stationary Point

Maxima and Minima

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Extreme Value Theorem

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Interior Extremum Theorem

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Mean Value Theorem

Rolle's Theorem

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Corollaries of the Mean Value Theorem

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Monotonic Functions

Derivative Tests

First-Derivative Test

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Second-Derivative Test

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Concavity

-

Higher-Order Derivative Test

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Differential Methods

Newton's Method

-

Taylor's Theorem

-

General Leibniz Rule

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Integrals

Integral Fundamentals

Terminology and Notation

-

Primer: Formal Definitions

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Definite Integrals

Riemann Integral

-

Integrability

-

Properties of Definite Integrals

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The Fundamental Theorem of Calculus

Fundamental Theorem, Part 1

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Fundamental Theorem, Part 2

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The Integral of a Rate

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Total Area

-

Integration By Substitution

Indefinite Integrals

- ◦

Definite Integrals

- ◦

Symmetric Functions

- ◦

Area Between Curves

- ◦

Applications of Definite Integrals

Solid of Revolution

Disc Integration

- ◦

Shell Integration

- ◦

Arc Length

Dealing with Discontinuities

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Differential Arc Length

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Surface of Revolution

Revolution about the y-Axis

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Transcendental Functions

Inverse Functions

One-to-One Functions

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Derivative Rule for Inverses

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Logarithmic Functions

Natural Logarithm

-

Properties of Logarithms

-

Trigonometric Integrals

-

Logarithmic Differentiation

-

Exponential Functions

Euler's Number

-

Natural Exponential Function

-

Laws of Exponents

-

General Exponential Function

-

Exponential Change

- **Separable Differential Equations**
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  Examples of Exponential Change
    -

Indeterminate Forms

Indeterminate Form 0/0

- ○

L'Hôpital's Rule

- ○

Infinite Indeterminate Forms

- ○

Indeterminate Powers

- ○

Inverse Trigonometric Functions

Principal Trigonometric Values

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Inverse Trigonometric Tables

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Hyperbolic Functions

Hyperbolic Function Tables

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Integration by Parts

Definite Integrals by Parts

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Trigonometric Integral Methods

Trigonometric Products and Powers

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Trigonometric Square Roots

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Trigonometric Substitutions

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Partial Fraction Decomposition

Partial Fraction Principles

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General Statement

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Numerical Integration

Trapezoidal Rule

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Simpson's Rule

-

Improper Integrals

Indirect Evaluation

- ◦

Multiple Integrals

Second-Order Differential Equations