

Calculus



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Second-Order Differential Equations

Limits and Continuity



Limits

🌐 Limit 🌐 | Thomas's Calculus^{14th} (2.2–2.4) 📖

- **Limit** $\lim_{x \rightarrow c}$: the value of a function (or sequence) approaches as the input (or index) approaches some value (informal definition).
 - Limits are used to define **continuity** ↓, **derivatives** ↓, and **integrals** ↓.

Limits of a Functions and Sequences

🌐 Limit of a function 🌐 | Limit of a sequence 🌐 | Essence of Calculus, E7 📺

- **Limit of a function**: a fundamental concept in calculus and analysis concerning the behavior of a function near a particular input c , i.e.,

$$\lim_{x \rightarrow c} f(x) = L$$

- Reads as “ f of x tends to L as x tends to c ”
- ϵ, δ **Limit of function**: a formalized definition, wherein $f(x)$ is defined on an open interval \mathcal{I} , except possibly at c itself, leading to above definition, if and only if:
 - For every real measure of **closeness** $\epsilon > 0$, there exists a real **corresponding** $\delta > 0$, such that for all existing further approaches there exist a smaller ϵ , i.e.,

$$f : \mathbb{R} \rightarrow \mathbb{R}, c, L \in \mathbb{R} \Rightarrow \lim_{x \rightarrow c} f(x) = L$$



$$\forall \epsilon > 0 (\exists \delta > 0 : \forall x \in \mathcal{I} (0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon))$$

- Functions **do not have** a limit when the function:
 - has a **unit step**, i.e., it “jumps” at a point;
 - is **not bounded**, i.e., it tends towards infinity;
 - or it **oscillates**, i.e., does not stay close to any single number.
- **Limit of a sequence**: the value that the terms of a sequence (x_n) “tends to” (and not to any other) as n approaches infinity (or some point), i.e.,

$$\lim_{n \rightarrow \infty} x_n = x$$

- ϵ **Limit of sequence**: for every measure of closeness ϵ , the sequence's x_n term eventually converge to the limit, i.e.,

$$\forall \epsilon > 0 (\exists N \in \mathbb{N} (\forall n \in \mathbb{N} (n \geq N \Rightarrow |x_n - x| < \epsilon)))$$

- **Convergent**: when a limit of a sequence **exists**.
- **Divergent**: a sequence that **does not** converge.

Properties of Limits

📌 List of limits 📌 | Squeeze theorem 📌

- **Operations on a single known limit:** if $\lim_{x \rightarrow c} f(x) = L$ then:
 - $\lim_{x \rightarrow c} [f(x) \pm \alpha] = L \pm \alpha$
 - $\lim_{x \rightarrow c} \alpha f(x) = \alpha L$
 - $\lim_{x \rightarrow c} f(x)^{-1} = L^{-1}, L \neq 0$
 - $\lim_{x \rightarrow c} f(x)^n = L^n, n \in \mathbb{N}$
 - $\lim_{x \rightarrow c} f(x)^{n^{-1}} = L^{n^{-1}}, n \in \mathbb{N}, \text{ if } n \in \mathbb{N}_e \Rightarrow L > 0$
- **Operations on two known limits:** if $\lim_{x \rightarrow c} f(x) = L_1$ and $\lim_{x \rightarrow c} g(x) = L_2$
 - $\lim_{x \rightarrow c} [f(x) \pm g(x)] = L_1 \pm L_2$
 - $\lim_{x \rightarrow c} [f(x)g(x)] = L_1L_2$
 - $\lim_{x \rightarrow c} f(x)g(x)^{-1} = L_1L_2^{-1}$
- **Squeeze theorem:** used to confirm the limit of a difficult to compute function via comparison with two other functions whose limits are easily known or computed.
 - Let \mathcal{I} be an interval having the point c as a limit point.
 - Let g, f , and h , be functions defined on \mathcal{I} , except possibly at c itself.
 - Suppose that $\forall x \in \mathcal{I} \wedge x \neq c \Rightarrow g(x) \leq f(x) \leq h(x)$
 - And suppose that $\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L$
 - Then, $\lim_{x \rightarrow c} f(x) = L$
 - Essentially, the hard to compute limit of the “middle function” is found by finding two other easy functions that “squeeze” the middle function at that point.

One-Sided Limit

📌 One-Sided Limit 📌

- **One-sided limit:** one of two limits of $f(x)$ as x approaches a specified point from either the left or from the right.
 - From the left: $\lim_{x \rightarrow c^-} = L$
 - From the right: $\lim_{x \rightarrow c^+} = L$
- If the left and right limits exist and are equal, then

$$\lim_{x \rightarrow c} f(x) = L \Leftrightarrow \lim_{x \rightarrow c^-} f(x) = L \wedge \lim_{x \rightarrow c^+} f(x) = L$$

- Limits can still exist, even if the function is defined at a different point, as long as both one-sided limits approach the same value near the given input.

Continuity

📌 Continuous function 📌 | Discontinuities 📌 | Thomas's Calculus^{14th} (2.5) 📖

- Continuity of functions is one of the core concepts of topology, however, there are definitions in terms of limits that prove useful; the following is only a primer.

Continuous Functions

- **Continuous function:** a function that does not have any abrupt changes in value.
 - I.e., a function is continuous if and only if arbitrarily small changes in its output can be assured by restricting to sufficiently small changes in its input.
- **Discontinuous:** when a function is not continuous at a point in its domain, leading to a discontinuity; there are three classifications:
 - **Removable:** when both **one-sided limits** \uparrow exist, are finite, and are equal, but the actual value of $f(x)$ is not equal to the limit and equal to some other value.
 - The discontinuity can be removed to regain continuity.
 - Sometimes the term *removable discontinuity* is mistaken for *removable singularity*, or a "whole" in the function (the point is not defined elsewhere).
 - **Jump:** when a single limit does not exist because the one-sided limits exist and are finite, but not equal.
 - Points can be defined at the discontinuity, but the function can not be made continuous.
 - **Essential:** when at least one of two one-sided limits doesn't exist; can be the result of oscillating or unbounded functions.

Intermediate Value Theorem

📌 Intermediate value theorem 📌

- **Intermediate value theorem:** if f is a continuous function whose domain contains the interval $[a, b]$, then it **takes on any given value between $f(a)$ and $f(b)$** at some point within the intervals.
- Relevant deductions, i.e., important corollaries:
 - **Bolzano's theorem:** if a continuous function has values of opposite sign inside an interval, then it **has a root** in that interval.
 - The image of a continuous function over an interval is itself an interval.
- Thus, the image set $f(\mathcal{I})$ (which has no gaps) is also an interval, and it contains:

$$[\min(f(a), f(b)), \max(f(a), f(b))]$$

Limits Involving Infinity

📌 Limits involving infinity 📌 | Thomas's Calculus^{14th} (2.6) 📖

- Let $S \subseteq \mathbb{R}$, $x \in S$ and $f : S \mapsto \mathbb{R}$, then limits of these functions can approach arbitrarily large (\pm) values, providing a connection to asymptotes, and thus, analysis.

Limits at Infinity and Infinite Limits

- Limits at infinity:** limits defined as $f(x) \pm \text{infinity}$ are defined much like normal limits:

$$\lim_{x \rightarrow -\infty} f(x) = L \quad \lim_{x \rightarrow \infty} f(x) = L$$

- Formally, for all measures of closeness ε there exists a point c such that $|f(x) - L| < \varepsilon$ whenever $x < c \vee x > c$ (respectively), i.e.,

$$\forall \varepsilon > 0 (\exists c (\forall x \{<, >\} c : |f(x) - L| < \varepsilon))$$

- Basic rules for rational functions $f(x) = p(x)q(x)^{-1}$, where p and q are polynomials, the degree of each is denoted as $\{p, q\}^\circ$, and the leading coefficients are denoted as P, Q , then:
 - $p^\circ > q^\circ \Rightarrow L$ is $\{+, -\}$ depending on the sign of the leading coefficients.
 - $p^\circ = q^\circ \Rightarrow L = PQ^{-1}$
 - $p^\circ < q^\circ \Rightarrow L = 0$
- Infinite limits:** the usual limit does not exist for a limit that grows out of bounds, however, limits with infinite values can be introduced:

$$\lim_{x \rightarrow a} f(x) = \infty, \quad \text{i.e.,} \quad \forall n > 0 (\exists \delta > 0 : f(x) > n \Leftrightarrow 0 < |x - a| < \delta)$$

Asymptotes of functions

📌 Asymptotes 📌

- Asymptote:** a tangent line of a curve at a point at infinity; the distance between the curve and the line approaches zero as a coordinate tends to infinity.
- There are three kinds of asymptotes: *horizontal*, *vertical* and *oblique*; nature of the asymptote is dependent on a function's relation to infinity.
 - Horizontal asymptotes:** a result of limits at infinity, i.e., when $x \rightarrow \pm\infty$
 - Vertical asymptotes:** a result of infinite limits, i.e., when $x \rightarrow \pm a = \pm\infty$
 - Oblique asymptotes:** when a linear asymptote is not parallel to either axis. $f(x)$ is asymptotic to the straight line $y = mx + n (m \neq 0)$ if:

$$\lim_{x \rightarrow \pm\infty} [f(x) - (mx + n)] = 0$$

Derivatives



Derivative Fundamentals

📖 Derivative 📖 | Thomas's Calculus^{14th} (3.2) 📖

- **Derivative:** the measure of **sensitivity to change** of the function **value** with respect to some change in its **argument**.
 - Often described as the **instantaneous rate of change** of a single variable function, since it is the slope a tangent line at a particular point, when it exists.
 - **Tangent line:** the line through a pair of points on a curve (secant line), except the points are **infinitely close**, thus, it's the rate of change at that "instant".

Definition, Notation

- Formally, a **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is (provided the limit exists)

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

- Let $z = x + h$, then $h = z - x \wedge h \rightarrow 0 \Leftrightarrow z \rightarrow x$; this leads to an equivalent definition of the derivative (sometimes more convenient):

$$f'(x) = \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x}$$

- **Notation:** there are many ways to denote the derivative, I prefer Leibniz's, but other notations can be useful in various contexts; some common notations (for $y = f(x)$):

$$f'(x) = y' = \dot{y} = \frac{dy}{dx} = \frac{d}{dx}f(x) = D(f)(x) = D_x f(x)$$

- **Differentiation:** the process of finding a derivative; if f' exists at a particular point, then f is said to be differentiable at that point.
 - If f' exists at every point on an interval, then f is differentiable on that interval.
 - f' is differentiable on a closed interval $[a, b]$ if both **one-sided limits** \uparrow of the function ($h \rightarrow \{0^+ : a, 0^- : b\}$) exist at the end points and is differentiable on the interior.
 - Not all continuous functions have a derivative, but **functions with a derivative are continuous**; functions with any of following **do not have derivatives**:
 - **corners** (one-sided derivatives differ at a point),
 - **cusps** (slope approaches alternating $\pm\infty$ on both sides of a point),
 - **discontinuities**, or **vertical tangent lines**.

Differentiation Rules

- **Linear, Product, Chain, Inverse**

-

- **Powers, Polynomials, Quotients, Reciprocals**

-

- **Exponential, Logarithmic**

-

- **Trigonometric, Hyperbolic**

-

Differentials and Related Concepts

Differentials

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Linearization

-

Implicit Differentiation

-

Related Rates

-

Applications of Derivatives



Stationary Point

Maxima and Minima

-

Extreme Value Theorem

-

Interior Extremum Theorem

-

Mean Value Theorem

Rolle's Theorem

-

Corollaries of the Mean Value Theorem

-

Monotonic Functions

Derivative Tests

First-Derivative Test

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Second-Derivative Test

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Concavity

-

Higher-Order Derivative Test

-

Differential Methods

Newton's Method

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Taylor's Theorem

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General Leibniz Rule

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Integrals



Integral Fundamentals

Terminology and Notation

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Definite Integrals

Riemann Integral

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Integrability

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The Fundamental Theorem of Calculus

Fundamental Theorem, Part 1

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Fundamental Theorem, Part 2

-

The Integral of a Rate

-

Total Area

-

Integration By Substitution

Indefinite Integrals

-

Definite Integrals

-

Symmetric Functions

-

Area Between Curves

-

Applications of Definite Integrals



Solid of Revolution

Disc Integration

-

Shell Integration

-

Arc Length

Dealing with Discontinuities

-

Differential Arc Length

-

Surface of Revolution

Revolution about the y-Axis

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Transcendental Functions



Inverse Functions

One-to-One Functions

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Derivative Rule for Inverses

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Logarithmic Functions

Natural Logarithm

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Properties of Logarithms

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Exponential Functions

Euler's Number

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Natural Exponential Function

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Laws of Exponents

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General Exponential Function

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Exponential Change

- **Separable Differential Equations**

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Examples of Exponential Change

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Indeterminate Forms

Indeterminate Form $0/0$

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L'Hôpital's Rule

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Infinite Indeterminate Forms

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Indeterminate Powers

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Inverse Trigonometric Functions

Principal Trigonometric Values

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Inverse Trigonometric Tables

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Hyperbolic Functions

Hyperbolic Function Tables

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Techniques of Integration



Integration by Parts

Definite Integrals by Parts

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Trigonometric Integral Methods

Trigonometric Products and Powers

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Trigonometric Square Roots

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Trigonometric Substitutions

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Partial Fraction Decomposition

Partial Fraction Principles

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Numerical Integration

Trapezoidal Rule

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Simpson's Rule

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Improper Integrals

Indirect Evaluation

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Infinite Sequences and Series



First-Order Differential Equations



Parametric Equations and Polar Coordinates



Vectors and Vector-Valued Functions



Partial Derivatives



Multiple Integrals



Vector Calculus



Second-Order Differential Equations

