

Statistics



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Data



Data Fundamentals

- **Data:** units of qualitative or quantitative information about persons or objects collected via observation.
 - Note: data is different from information—information resolves uncertainty, while data has the potential to be transformed into information post-analysis.
 - Data as a general concept refers to the fact that some existing information or knowledge can be represented in a form suitable for processing.

Data Types

- Data types have two different general meanings:
 - **Data type (computer science):** involves the format of data storage and has implications on operations and storage space.
 - **Data type (statistics):** involves the category of data and has implications on the methods used for analysis.
- There are many data types, with more specific definitions than the following definitions, but for now these are frequently used and adequate for topics covered.

Relevant Statistical Data Types

Category	Type	Description	Example
Numerical	Interval	Degree of difference	Temperature °C
	Ratio	Interval + meaningful zero	Height
	Discrete	Count (integers)	Population
Categorical	Ordinal	Sortable, discrete	Educational level
	Nominal	Non-sortable, discrete	Movie genre

Population vs. Sample Data

- **Population data** μ : data from **all** members of a group.
- **Sample data** $\hat{\mu}$: data from a **subset** of members of a group (hopefully random).
- Statistical procedures generally are designed for sample or population data; wrong conclusions can be drawn if the distinction is not clear.
 - Note: most data are sample data in practice, as generalization of populations using sample data is usually the goal of statistics.
- **Anecdotes:** a case study of a rare occurrence, or a sample size of only one; insights may be possible, but poor confidence in ability to generalize should be noted.

Data Visualization

- **Data visualization:** a mapping between the original data and graphic elements in order to determine how attributes of interest vary according to the data.
 - The design of the mapping can have a significant effect on information extracted from data, in both beneficial and detrimental ways.
- Data visualization is a core tool of statistics and generally considered to be a branch **descriptive statistics**; more techniques will be covered in that chapter.

Visualization Techniques

- Visualizing data can be an art in and of itself, leading to a wide variety of available techniques, i.e., diagram types, in order to better represent the data.
- The following is a rather shallow list of commonly used techniques; in-depth exploration of data visualization will be pursued in other courses.
- **Bar chart:** a representation of **categorical data** with magnitudes proportional to the values they represent.
 - Displays comparisons among **discrete categories** vs. a measured value.
 - Subcategories can be displayed in clusters within each category, with colors/patterns used to differentiate them.
 - Ordering of the categories (chart shape) do not typically matter, excluding aesthetic reasons.
- **Histogram:** a representation of the **distribution** of numerical data via the use of **binning**.
 - **Binning:** a form **quantization of continuous data**, wherein small intervals (bins) of the data are replaced with a value representative of that interval.
 - The bins are usually specified as consecutive, non-overlapping intervals of a variable; they must be adjacent and are often of equal size.
 - Histograms of **counts** are usually better for **qualitative** inspection of raw data, but can be difficult to compare across data sets.
 - Histograms of **proportion** are usually better for **quantitative** analysis, as they are typically easier to compare across data sets, but can take extra effort to create.
- **Scatter plot:** a representation of the **relationship between variables**, often two or three (2D/3D graphs).
 - Points can be coded via color, shape, and/or size to display additional variables.
 - Often used to investigate **correlations** between variables.

- **Network graph:** a representation of data as nodes in a network via analysis of **specialization** of the nodes.
 - Used to discover bridges (information brokers) in a network, relative node influence, and outliers via analysis of how the nodes cluster.
 - Node and tie (connection between nodes) size and color can be used to encode additional information about variables in the data.
- **Pie chart:** a representation of one categorical variable via the division of slices in order to illustrate **numerical proportion**.
- **Box plot:** a representation of numerical data via analysis of their quartiles.
 - **Quartiles:** a quantile (division point) of data points into four parts, or quarters.
 - Q_1 : the middle number between the smallest minimum and the median of the data set; 25% of the data lies below this point.
 - Q_2 : the median of the data set; 50% of the data lies below this point.
 - Q_3 : the middle value between the medium and the maximum of the data set; 75% of the data lies below this point.
 - Often termed box and whisker plot, as the box represents the 50% of the data, and the two whiskers represent the upper and lower 25% of data.
 - **Interquartile range IQR:** the box, i.e., the difference between upper and lower quartiles; $IQR = Q_3 - Q_1$.
 - Outliers may be plotted as individual points.
 - Useful when examining the **variability of samples** without making any assumptions about underlying statistical distributions.

Descriptive Statistics



Descriptive Statistics Fundamentals

Descriptive vs. Inferential Statistics

- **Descriptive statistics:** the processes of using and analyzing summary statistics that quantitatively describes or summarizes features of a collection of information.
 - Methods/measures of descriptive statistics:
 - Distribution shape↓
 - Mean, median, mode↓
 - Variance↓
 - Kurtosis, skew↓
 - No relation to population.
 - No generalization to other data sets.
 - Concerned only with properties of observed data.
- **Inferential statistics:** the process data analysis to deduce properties of an underlying probability distribution.
 - Methods/measures of inferential statistics:
 - P-value↓
 - Hypothesis testing↓
 - T/F/ χ^2 value↓
 - Confidence intervals↓
 - And essentially all of applied statistics.
 - Assumes that the observed data set is sampled from a larger population.
 - Entire purpose is to generalize/relate features to other data sets.

Accuracy, Precision, Resolution

- **Accuracy:** the relationship between the measurement and the actual truth.
 - Inversely related to bias; colloquially interchangeable with accuracy.
- **Precision:** the certainty of each measurement.
 - Inversely related to variance↓
- **Resolution:** the number of data points per unit measurement (e.g., time, space, individual, etc).

Primer: Probability Distributions

- The shapes of data distributions are [functions of probability theory](#)[↓]; a more in-depth explanation will be covered later, but for now coverage of common distribution types might be useful.
- Overall, there is one major distinction of distribution type based on [data types](#)[↑] used, either discrete or continuous.
- **Discrete distribution:**
 - Deals with events that occur in countable sample spaces; contains finite number of outcomes.
 - Summation of values can be done to estimate probability of an interval.
 - Expressed with graphs, piece-wise functions, or tables.
 - Expected values might not be achievable.
 - Common examples:
 - [Bernoulli](#) : a model for the set of possible outcomes of any single binary experiment.
 - [Binomial](#) : a sequence of n independent Bernoulli experiments; a basis for the binomial test.
 - [Uniform](#) : a known, finite number of values are equally likely to be observed.
 - [Poisson](#) : a sequence of independent events over a specified interval with a known constant mean rate.
- **Continuous distribution:**
 - Deals with events that occur in a continuous sample space; contains infinitely many consecutive values.
 - Summation of values in order to determine probability of interval not possible; integrals used instead.
 - Expressed with continuous functions or graphs.
 - Common examples:
 - [Normal \(Gaussian\)](#) : used to represent real-valued random variables who are not known.
 - [Lognormal](#) : distribution of a random variable whose logarithm is normally distributed.
 - [Chi-Squared](#) : the sum of squares of k independent standard normal random variables.
 - [Student's T](#) : estimations of the mean using small sample sizes with unknown standard deviations.
- [Wikipedia's list of probability distributions](#)

Descriptive Techniques

Measures of Central Tendency

- **Mean** \bar{x} : the sum of all measurements x_i divided by the number n of observations in the data set x , i.e.,

$$\bar{x} = n^{-1} \sum_{i=1}^n x_i$$

- Suitable for roughly normally distributed data of continuous data types.

- **Median** $\text{med}(x)$: the middle value of the data, i.e.,

$$x_i, \quad i = \frac{n+1}{2}$$

- Suitable for unimodal distributions of continuous data types.
- Odd number of observations with no distinct middle value are usually defined as the mean of the two middle values.

- **Mode**: most common value.

- Suitable for any discrete distribution, usually used for nominal data types.

Measures of Dispersion

- **Dispersion**: the measure of how distributed, or deviated, data are around a central value.

- **Variance** σ^2, s^2 : the primary measure of dispersion, or more explicitly, the expectation of the squared deviation of a random variable from its mean, i.e.,

$$\sigma^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$$

- Suitable for any distribution; better for normally distributed data.
- Mean centering, i.e., $(x_i - \bar{x})$, is done to capture the dispersion around the average, but not the magnitude of the values themselves.
- The sum of a mean-centered data set would be zero, thus it is squared.

- **Mean absolute difference (MAD)**: when the absolute value of mean-centered data is taken instead of the square value.

- MAD is more robust to outliers, but further from Euclidean distance and less commonly used.

- Division by $n - 1$ is used for sample variance, as often sample sizes can be small and are considered empirical quantities; n^{-1} is used for population variance (a theoretical quantity).

- **Standard Deviation** σ : simply the square root of variance, $\sqrt{\sigma^2}$

Statistical Moments

- **Moments**: a quantitative measure related to shape of a functions graph; relates to physics and statistics.

- Regarding probability distributions, the general formula can be defined as:

$$m_k = n^{-1} \sum_{i=1}^n (x_i - \bar{x})^k$$

- Increments of k define particular moments, i.e.,
 - First moment $k = 1$: expected value, or **mean**↑.
 - Second moment $k = 2$: central moment, or **variance**↑.
 - Third moment $k = 3$: dispersion asymmetry, or skewness.
 - Fourth moment $k = 4$: tail "thickness," or kurtosis.
 - Further moments are possible, but useful applications are less common.
- **Skewness**: a measure of asymmetry of a probability distribution of a real-valued random variable about its mean.
 - Can be positive, zero, negative, or undefined.
 - **Negative skew**: an indication that the tail is on the **left**.
 - Zero skew: an indication that tails **balance** out; can be true for both asymmetric and symmetric distributions depending on kurtosis.
 - **Positive skew**: an indication that the tail is on the **right**.
- **Kurtosis**: a measure of the thickness/curvature of the tail of a probability distribution is; an indication of deviation/outliers.
 - Univariate normal distributions have a kurtosis of 3, leading to a common basis.
 - **Platykurtic** < 3 : a term for **low** kurtosis, indicating that a **lesser degree** of deviations or **outliers** is observed.
 - **Leptokurtic** > 3 : a term for **high** kurtosis, indicating that a **greater degree** of deviations or **outliers** is observed.
 - **Excess kurtosis**: kurtosis minus 3, often colloquially termed as kurtosis; an indication a greater degree outliers compared to a normal distribution.

Visualizations Revisited

- **Q-Q (quantile-quantile) plot:** a graphical method for comparing two probability distributions by plotting their quantiles against each other.
 - **Quantile:** cut points dividing the range of probability distributions into continuous intervals with equal probabilities, e.g.,
 - Percentiles: 0–100
 - Quartiles: 0–4
 - Quantiles: 0– x
 - The points of similar distributions will lie approximately on the line $y = x$;
 - However, other linear relations are possible, meaning points may not necessarily lie on the line $y = x$.
 - Provides a mean for comparing location, scale, and skewness of similarities of differences in two distributions.
- **Histogram bin number k :** there is no “best” number of bins, different bin sizes can reveal different features of the data, but there are several methods of determining k ;
 - Determination via suggested bin width h :

$$k = \left\lceil \frac{\max(x) - \min(x)}{h} \right\rceil$$

- Sturges’ formula: derived from binomial distribution; assumes approximately normal distribution:

$$k = \lceil \log_2(n) \rceil$$

- Freedman-Diaconis’ rule: method of determining h using interquartile range (IQR); often method of choice:

$$h = 2 \frac{\text{IQR}(x)}{\sqrt[3]{n}}$$

- Arbitrary ≈ 42 : often intuitive guesses are sufficient and yield useable results:
- **Violin plot:** similar to a box plot, but rotated with addition of a kernel density plot on each side.
 - **Kernel density plot:** essentially a smoothing estimation based on finite data samples.
 - Statistical and IQR moments can be conveniently shown, sometimes with asymmetric comparisons of similar data sets (rather than a mirrored version).

Introduction to Normalization

- **Normalization of ratings (feature scaling)**: adjusting values measured on different scales to notionally common scale, often prior to averaging.
 - Often in more complicated cases, the adjustments are meant to bring the entire probability distribution of adjusted values into alignment.
- **Normalized values (normalization)**: creation of shifted and scaled versions of samples with the intention of minimizing the effect of gross **anomalies/outliers**↓.
- There are many types of normalization techniques in statistics, each with their own respective applications based on data types and distribution shapes; for now, only standard score and min-max scaling will be covered, with others introduced at more appropriate times.

Z-Score Standardization

- **Z-score (standard score)**: the number of **standard deviations** σ ↑ by which the value of a raw score x_i is above or below the **mean** \bar{x} ↑, i.e.,

$$z_i = \frac{x_i - \bar{x}}{\sigma_x}$$

- Application of z-normalization is best done on data that is roughly **Gaussian**↑.
- The z-score is dimensionless, as units cancel out, leading to main application wherein data of different scales can be meaningfully compared.

Min-Max Scaling

- **Rescaling (min-max normalization)** x' : the simplest method of rescaling the range of features, either from $[0, 1]$ or $[-1, 1]$; the general formula for $[0, 1]$ is given as:

$$x' = \frac{x - \min(x)}{\max(x) - \min(x)}$$

- Rescaling to any arbitrary range $[a, b]$:

$$x' = a + \frac{(x - \min(x))(b - a)}{\max(x) - \min(x)}$$

Outliers

- **Outlier:** a data point that differs significantly from other observations, potentially due to a variety of reasons either, due to the cause of experimental error in observations, random noise, unexplained/surprising phenomena, or simply by natural variability.
- Outliers can cause serious errors in statistical analysis, as many methods square terms, leading to potentially huge errors.
 - Often extremely detrimental impacts on small sample sizes are observed, as significance of the outliers decrease with increasing sample size.
- **Leverage:** a measure of how far away the independent variable values of an observation are from those of other observations.
 - Outliers are worse near the “edges” of the data, compared to the “middle,” as outliers further away increase the leverage.
 - Lower leverage has less influence on statistical analysis, and in particular, it is a large factor in **regression analysis** ↓.
- There are two main strategies for dealing with outliers, either:
 - Identify and **remove outliers** prior to analysis; assuming outliers are **noise or invalid**.
 - **Keep outliers** in and use robust methods that attenuate the negative impact of outliers; assume outliers are **unusual but valid**.
 - Robust methods of retention will be examined when more appropriate.
- Despite strategy chosen, outliers ought to be investigated; sometimes outliers might be an important aspect of the data.

Removing Outliers

- There are many methods of removing outliers, here use of the **z-score** ↑ is explained. Again, more in-depth examinations of methods will be examined when appropriate.
- First, data must be converted to a **normalized** metric, e.g., the z-score.
- Next, a **threshold** must be determined that marks data points for suspect, dealing with them either methods of truncation or winsorization.
 - **Truncation (trimming):** complete removal, with possible replacement of NaN placeholder to maintain indexing.
 - **Winsorization (clipping):** replacement outlier with the nearest or a less suspect “alternative” value.
 - A variety of methods of determining such threshold can be used, even such methods lead to potentially arbitrary choices; 3 is often a default starting point.

- Finally, suspect data are **dealt with iteratively** until no other data pass the given threshold.
- Note, the z-score is generally only useful for roughly **Gaussian distributions**[†], however, a modified z-score using the median can be applied for non-normal distributions, i.e.,

$$z_i = \frac{0.6745(x_i - \text{med}(x))}{\text{med}(|x_i - \text{med}(x)|)}$$

- 0.6745 is a normalization factor equal the standard deviation units of **Q_3** [†] of a Gaussian distribution.
- Deletion of data is generally avoided, with only clear indications of measurement error being the reason to do so.
- Multivariate data sets are dealt in similar way, where the only difference is that the mean of the data set is taken by calculating the Euclidean distance between all points in the set, then applying the method(s) described above.

Probability Theory



Probability Fundamentals

- **Probability**: a measure of the likelihood that an event will occur; used to quantify attitudes towards propositions whose truth are not certain.
 - Quantitatively, probability is a number between 0 and 1, which is often expressed as a percentage.
- **Probability theory**: the axiomatic formalization of probability; widely used in many fields of study from math to philosophy.
- **Probability space** (Ω, \mathcal{F}, P) : a formal construct consisting of three elements that provides a model for a random process.
 - **Sample space** Ω : the set of all possible outcomes.
 - **Event space** \mathcal{F} : all sets of outcomes; all subsets of the sample space.
 - **Probability function** $P(E \in \mathcal{F})$: the assignment of a number between 0 and 1 that represents the probability of each event E in event space.
- **Proportion**: the measure of certainty; a fraction of a whole or the relation between two varying quantities.
 - Proportion *could* involve random variables, so depending on how the question is asked, then proportion could be the same as probability, but ultimately they are not interchangeable.
- **Odds**: the ratio of the number of events that produce an outcome to the number of events that do not; essentially probability reframed in potentially more efficient way.

Probability Theory Axioms

- **First axiom**: the probability of an event is a **non-negative number real number**, i.e.,

$$P(E) \in \mathbb{R}, \quad P(E) \geq 0 \quad \forall E \in \mathcal{F}$$

- **Second axiom**: the assumption of unit measure; the probability that **at least one elementary event** in the entire sample space **will occur** is 1, i.e.,

$$P(\Omega) = 1$$

- **Third axiom**: the assumption of σ -additivity, wherein any **countable** sequence of **disjoint sets**[↓] E_1, E_2, \dots satisfies

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

- Thus, only **discrete data**[↑] are valid for probability; continuous data must be converted to discrete forms in order to be valid.

Independent and Mutually Exclusive Events

- **Stochastically independent:** when an event does not affect the probability of another, i.e.,

$$P(A \text{ and } B) = P(A \cap B) = P(A)P(B)$$

- Two random variables are independent if the realization of one does not affect the probability distribution of the other.
- **Pairwise independent (weak notion):** two specific events in a collection that are independent of each other.
- **Mutually independent (strong notion):** when each event is independent of any combination of other events in the collection.
- Often the stronger notion is simply termed independence, as it implies the weaker version, but not the other way around.
- **Mutually exclusive (disjoint):** two events that cannot occur at the same time, i.e.,
 - Probability of both:

$$P(A \text{ and } B) = P(A \cap B) = 0$$

- Probability of either:

↓

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B)$$

- **Collectively exhaustive (jointly):** when at least one event must occur while exhausting all other possibilities at a given time, or that their union must cover all the events within the entire sample space, i.e.,

$$A \cup B = \Omega$$

Primer: Conditional Probability

- **Conditional probability:** the probability of some event A , given | the occurrence of some other event B , i.e.,

$$P(A | B) = \frac{P(A \cap B)}{P(B)}$$

- Note, $P(A | B)$ typically differs from $P(B | A)$, falsely equating the two often results in errors, termed the base rate fallacy.
- **Bayes' theorem:** probability of an event based on prior knowledge of conditions that might be related to the event; inference using conditional probability, i.e.,

$$P(A | B) = \frac{P(B | A)P(A)}{P(B)}$$

- More on Bayesian statistics may or not be explored in greater depth in this course.

Probability Functions

- Again, a **probability function**[†] is the assignment of a number of **probability space**[†] (sometimes denoted X, \mathcal{A}, P , respectively).
- **Probability distributions**: the product of variations of the probability function based on given event space and properties of data types.
 - As mentioned in the **primer**[†] to this topic, probability distributions are generally divided into two classes based on data, i.e., either **discrete** or **continuous**.
- **Probability mass function (PMF)**: a function that gives the probability that a **discrete** random variable is exactly equal to some value.

- The function $p : \mathbb{R} \rightarrow [0, 1]$ is defined formally as:

$$p(x_i) = P(X = x_i) \quad -\infty < x < \infty$$

- The associated probability values must follow the **Kolmogorov axioms**[†], which means all possible values must be positive and sum up to 1, implying all other probabilities must be 0, i.e.,

$$p(x_i) > 0, \quad \sum p(x_i) = 1, \quad p(x) = 0 \quad \forall x \neq x_i$$

- Thinking of probability as mass helps avoid mistakes since physical mass is conserved, as is total probability for all hypothetical outcomes of x .
- Major associated distributions include the **previously mentioned**[†] Bernoulli and Binomial distributions, but geometric distributions deserve a mention as well.
- **Geometric distribution**: a description of the number of trials/failures needed to get to one/first success.
- **Probability density function (PDF)**: a function that describes **relative** probabilities for a set of **exclusive**[†] **continuous** events, i.e.,

$$P(a \leq X \leq b) = \int_a^b f_X(x) dx$$

- **Cumulative density function (CDF)**: the PDF can also be described as the cumulative sum of continuous probabilities up to a particular point, i.e.,

$$F_X(x) = \int_{-\infty}^x f_X(u) du \quad \text{or (in practice)} \quad C(x_a) = \sum_{i=1}^a p(x_i)$$

- Note: every CDF is non-decreasing and right-continuous
- Note: the sum of CDF is > 1 .

Sampling

- **Sampling distribution:** the probability distribution of a given random variable when derived from a random sample size n .
 - Useful to be considered as the statistic for all possible samples from the same population of a given sample size; is dependent on the underlying distribution of the population.
- Sampling a subset is often an easier and faster way to estimate an entire population, providing a potentially major simplification to statistical inference.
- **Expected mean value** μ , \bar{X} : the expected mean of the **population**↑, or in case of random sampling, the expected mean of numerous samples, i.e.,

$$\bar{X} = \sum_{i=1}^k x_i p_i$$

- Where X is a random variable with a finite number outcomes x_i occurring with respective probabilities p_i .
- Thus, the expected value is the weighted sum, with the probabilities as weights.
- **Sampling variability:** different samples from the same population can have different values of the same measurement.
 - A single measurement may be an unreliable estimate of a population parameter.
 - Potential to randomly select outliers are the main source of sampling variability, but cannot be avoided.
 - Thus, natural variation, measurement noise, and failing to understand complexity of phenomena are all sources of variability.
- **Sampling frame:** the source material, data, or device from which a sample is drawn; ideal frame qualities include:
 - The units have logical, numerical identifiers.
 - The units can be found again, or resampled.
 - The frame is organized, systematically.
 - The frame has additional information about the units and for the potential use of more advanced sampling frames.
 - Every element of the population of interest is present.
 - Every element of the population is only present once.
 - No elements outside the population of interest are present.
 - The data is kept up to date, accepting new information.

Sampling Methods

- **Probability sample:** a sample wherein every unit in the population has a chance ($P > 0$) of being selected in the sample, and the probability can be accurately determined.
- There are numerous methods of sampling, not all of which will be covered, but the various ways have the following two things in common:
 - Every element has a known **nonzero probability** of being sampled.
 - Involves **random selection** at some point.
- Factors that contribute to choice between methods:
 - Nature and quality of the data.
 - Availability of auxiliary information about units on the frame.
 - Accuracy requirements, and need to measure accuracy.
 - Degree of expected analysis.
 - Cost and operational concerns.
- **Simple random sampling:** all subsets of a sampling frame have an equal probability of being selected.
 - Minimizes bias, simplifies analysis.
 - Variance between individual results within the sample is a good indicator of variance of overall population, leading to easy estimations of accuracy.
 - Subject to sampling error, and implicit bias can go unnoticed due to data collection methods.

Systematic sampling:

Law of Large Numbers

◦

Central Limit Theorem

◦

Hypothesis Testing



T-Tests



Confidence Intervals



Correlation



Analysis of Variance



Regression



Statistical Power and Sample Sizes



Clustering and Dimension-Reduction



Signal Detection Theory

