1. Consider a system of linear equations with augmented matrix $\boldsymbol{A}$ and coefficient matrix $\boldsymbol{C}$. In each case explain why the statement is true or give an example showing that it is false.

(a) If there is more than one solution, $\boldsymbol{A}$ has a row of zeros.

- If a row with all zeros occurs, then that row added no new information and was only a multiple of another row, i.e., the system is reduced rank.
- Matrix rank corresponds to the maximal number of linearly independent columns.
- Linear independence is when no vector in the matrix can be expressed as a linear combination, i.e.,

$$a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + a_3\boldsymbol{v}_3 + a_n\boldsymbol{v}_n$$

- For example, in a 3-D vector space $\mathbb{R}^3$, then any vector in the space is can be made by a linear combination of the following three vectors $\boldsymbol{e}_1,\ \boldsymbol{e}_2,\ \boldsymbol{e}_3$ (unit vectors) multiplied by some scalar λ:

$$\lambda\begin{bmatrix}1\\0\\0\end{bmatrix} + \lambda\begin{bmatrix}0\\1\\0\end{bmatrix} + \lambda\begin{bmatrix}0\\0\\1\end{bmatrix}$$

- If the third column vector was all zeros (making the matrix have a row of all zeros) then it would just be describing a plane in $\mathbb{R}^2$, with the last row (or rather, third vector) contributing no information to the system,

$$\lambda\begin{bmatrix}1\\0\\0\end{bmatrix} + \lambda\begin{bmatrix}0\\1\\0\end{bmatrix} + \lambda\begin{bmatrix}0\\0\\0\end{bmatrix}$$

- That being said, the matrix above could be an augmented matrix with the constant matrix containing values for the first two vectors, indicating a single solution.

$$\left[\begin{array}{ccc|c}1 & 0 & 0 & y\\0 & 1 & 0 & x\\0 & 0 & 0 & 0\end{array}\right]$$

- And now I've realized I actually answered why the statement in (b) is false and not directly addressing the validity of (a).
- For (a) to be true, then there does have to be a row of zeros, given the original system of equations, as well as parameters in at least one of the other systems, as a row of all zeros puts no restriction on z.

$$\left[\begin{array}{ccc|c}1 & 0 & 2 & x\\0 & 1 & 0 & y\\0 & 0 & 0 & 0(z)\end{array}\right]$$

(b) If $\boldsymbol{A}$ has a row of zeros, there is more than one solution.

- Not necessarily, see above answer.

(c) If there is no solution, the reduced row-echelon form of $\boldsymbol{C}$ has a row of zeros.

- True, the rref form needs to have a row of zeros in the coefficient matrix and a non-zero in the same row in the constant matrix, i.e., $\left[\begin{array}{cccc|c} 0 & 0 & 0 & \cdots & a \neq 0 \end{array}\right]$, in order to have no solution.

(d) If the row-echelon form of $\boldsymbol{C}$ has a row of zeros, there is no solution.

- Not necessarily, a row of zeros is more likely to indicate infinite solutions than no solution or a single solution, albeit, all are still possible.
- For example, the following matrix is in ref and contains a row of all zeros.

$$\left[\begin{array}{ccc|c} 1 & 6 & 9 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

2.

Verify that $\begin{cases} x_1 &= 2s + 12t + 13 \\ x_2 &= s \\ x_3 &= -s - 3t - 3 \\ x_4 &= t \end{cases}$ is a solution of $\begin{cases} 2x_1 + 5x_2 + 9x_3 + 3x_4 &= -1 \\ x_1 + 2x_2 + 4x_3 &= 1. \end{cases}$

$$\begin{aligned} \left[\begin{array}{cccc|c} 2 & 5 & 9 & 3 & -1 \\ 1 & 2 & 4 & 0 & 1 \end{array}\right] R_1 \updownarrow R_2 & \left[\begin{array}{cccc|c} 1 & 2 & 4 & 0 & 1 \\ 2 & 5 & 9 & 3 & -1 \end{array}\right] \\ -2R_1 + R_2 & \left[\begin{array}{cccc|c} 1 & 2 & 4 & 0 & 1 \\ 0 & 1 & 1 & 3 & -3 \end{array}\right] \\ -2R_2 + R_1 & \left[\begin{array}{cccc|c} 1 & 0 & 2 & -6 & 7 \\ 0 & 1 & 1 & 3 & -3 \end{array}\right] \end{aligned}$$

Two parameters are left: $s,\ t$.

$$\begin{aligned} x_1 &= 2s - 6t + 7 \\ x_2 &= s \\ x_3 &= -s - 3t - 3 \\ x_4 &= t \end{aligned}$$

I am confused.

Let's try something different:

$$\begin{aligned}
2(2s + 12t + 13) + 5(s) + 9(-s - 3t - 3) + 3(t) &= -1 \\
4s + 24t + 26 + 5s - 9s - 27t - 27 + 3t &= -1 \\
24t - 27t + 3t + 26 - 27 &= -1 \\
26 - 27 &= -1 \\
-1 &= -1 \\
2s + 12t + 13 + 2(s) + 4(-s - 3t - 3) &= 1 \\
2s + 12t + 13 + 2s - 4s - 12t - 12 &= 1 \\
12t - 12t + 13 - 12 &= 1 \\
1 &= 1
\end{aligned}$$

Hmmm...why doesn't rref work.

$$\begin{aligned}
2(2s - 6t + 7) + 5(s) + 9(-s - 3t - 3) + 3(t) &= -1 \\
4s - 12t + 14 + 5s - 9s - 27t - 27 + 3t &= -1 \\
-12t + 14 - 27t - 27 + 3t &= -1 \\
&\text{yeah, no}
\end{aligned}$$

Did I do parameters wrong?

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & -6 & 7 \\ 0 & 1 & 1 & 3 & -3 \end{array}\right]$$

$$\begin{aligned}
x_1 &= 2s + 6t + 7 \\
x_2 &= -s - 3t - 3 \\
x_3 &= s \\
x_4 &= t
\end{aligned}$$

Idk, let's try it.

$$\begin{aligned}
2(2s + 6t + 7) + 5(-s - 3t - 3) + 9(s) + 3(t) &= -1 \\
4s + 12t + 14 - 10s - 15t - 15 + 9s + 3t &= -1 \\
3s - 1 &= -1 \\
s &= 0
\end{aligned}$$

Well, why doesn't that work? I guess rref removes information...or maybe I'm doing something wrong with parameters.