Page	Points	Score
2↓	3	
3↓	2	
4↓	2	
5↓	2	
6↓	2	
7↓	3	
8↓	4	
9↓	6	
10↓	2	
11↓	2	
12↓	2	
Total:	30	

1. Use the fact to help respond to the prompts 1(a), 1(b), and 1(c) below.

Fact I: The matrix equation below is consistent:

$$\begin{bmatrix} 7 & -3 \\ 2 & 1 \\ 9 & -6 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ 12 \\ -4 \end{bmatrix}$$

(a) **[/ 1]** Write the matrix equation as an equivalent vector equation.

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

$$\downarrow$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\downarrow$$

Given equation

$$\boxed{-2\boldsymbol{a}_1 - 5\boldsymbol{a}_2 = \boldsymbol{b}}$$

$$\text{where } \boldsymbol{a}_1, \boldsymbol{a}_2 = \begin{bmatrix} 7 \\ 2 \\ 9 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ -6 \\ 2 \end{bmatrix} \text{ and } \boldsymbol{b} = \begin{bmatrix} 1 \\ -9 \\ 12 \\ -4 \end{bmatrix}$$

(b) **[/ 2]** Is $\begin{bmatrix} 1 \\ -9 \\ 12 \\ -4 \end{bmatrix}$ in span$\left\{\begin{bmatrix} 7 \\ 2 \\ 9 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ -6 \\ 2 \end{bmatrix}\right\}$? Justify your response.

- Yes, $\boldsymbol{b}$ is in the span of $\boldsymbol{A}$.
- Span can be defined as set of all finite linear combinations of vectors of $\boldsymbol{A}$ over field K, i.e.,

$$\operatorname{span}(\boldsymbol{A}) = \left\{ \sum_{i=1}^{k} \lambda_i \boldsymbol{v}_i \,\middle|\, k \in \mathbb{N}, \boldsymbol{v}_i \in \boldsymbol{A}, \lambda_i \in K \right\}$$

- Essentially, this is equivalent to asking if there exists such vector $\boldsymbol{x}$ such that $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$. Part (a) showed that there is such vector ($\boldsymbol{x} = \begin{bmatrix} -2 & 5 \end{bmatrix}$), thus $\boldsymbol{b}$ is in the span of $\boldsymbol{A}$.

- This can be confirmed by row reducing, just in case you don't trust random facts:

$$\text{rref}\left(\begin{bmatrix} 7 & -3 & 1 \\ 2 & 1 & -9 \\ 9 & -6 & 12 \\ -3 & 2 & -4 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} -2 \\ -5 \end{bmatrix}$$

(c) **[/ 2]** Recall that a set of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ is linearly independent if the only solution to the equation

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k = 0$$

is the trivial solution $c_1 = c_2 = \cdots = c_k = 0$.

Is the set $S = \left\{\begin{bmatrix} 1 \\ -9 \\ 12 \\ -4 \end{bmatrix}, \begin{bmatrix} 7 \\ 2 \\ 9 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ -6 \\ 2 \end{bmatrix}\right\}$ linearly independent?

- No, the set S is linearly dependent; the column space must span all of $\mathbb{R}^m$ in order to be linearly dependent.
- A good way to test this is using the relationship between the column space and cokernel. The dot with the column space S and a vector from the cokernel must be orthogonal, i.e.,

$$\boldsymbol{x}^T\{\lambda_1 a_1 + \cdots + \lambda_n a_n\} = 0$$

- This implies that the only vector to make this equation true is the zero vector, if the set is linearly independent (trivial solution). Extrapolating using the rank-nullity theorem leads to the conclusion that the kernel must be empty and the set must but full rank in order to be linearly independent.
- First, the number of rows are more than number of columns, so without any calculation, one can tell nullity is out least one. However, using the rref form above clearly shows that dimension of the column space is 2, implying the cokernel's dimension is 2 as well; a basis for the cokernel can be described as:

$$\left(\begin{bmatrix} -21/13 \\ 15/13 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 7/13 \\ -5/13 \\ 0 \\ 1 \end{bmatrix}\right)$$

thus, $(\lambda \in \mathbb{R})$:

$$x_1{}^T\{\lambda a_1 + \lambda a_2 + \lambda \boldsymbol{a}_3\} = 0$$
$$x_2{}^T\{\lambda a_1 + \lambda a_2 + \lambda \boldsymbol{a}_3\} = 0$$

I.e., not trivial solutions $\rightarrow S$ is linearly dependent.

2. Let $\boldsymbol{C} = \begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}$ and $\boldsymbol{D} = \begin{bmatrix} 2 & -3 \\ -2.5 & 4 \end{bmatrix}$ for 2(a) and 2(b).

   (a) **[/ 2]** Show that $\boldsymbol{D} = \boldsymbol{C}^{-1}$ by showing that $\boldsymbol{DC} = \boldsymbol{I}$. You must show some details though you can use a calculator to check your work.

(b) **[/ 2]** Use the fact that $D = C^{-1}$ to solve the system of linear equations (you can check your answer with another method, but full credit will only be awarded to the use of C^{-1}).

$$\begin{cases} 8x_1 + 6x_2 = 2 \\ 5x_1 + 4x_2 = -1. \end{cases}$$

3. Below is a matrix A and the reduced row echelon form oA. Use this A for 3(a) and 3(b).

$$A = \begin{bmatrix} 3 & -1 & 7 & 3 \\ -2 & 2 & -2 & 7 \\ -5 & 9 & 3 & 3 \\ 1 & 1 & 5 & 10 \\ 5 & -3 & 9 & -4 \end{bmatrix} \rightarrow \underbrace{\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{rref of } A}$$

(a) **[/ 2]** Determine a basis for Col A.

$$A = \begin{bmatrix} 3 & -1 & 7 & 3 \\ -2 & 2 & -2 & 7 \\ -5 & 9 & 3 & 3 \\ 1 & 1 & 5 & 10 \\ 5 & -3 & 9 & -4 \end{bmatrix} \to \underbrace{\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{rref of } A}$$

(b) **[/ 2]** Determine a basis for $N(\boldsymbol{A})$.

(c) **[/ 1]** Give the dimension of Col $\boldsymbol{A}$ and the dimension of Nul $\boldsymbol{A}$.

4. **[/ 2]** Is $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$? If so, find the eigenvalue with which it associates.

5. **[/ 2]** Find the eigenvalues of $\begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$.

6. [/ 2] Show that $\lambda = -2$ is an eigenvalue of $\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$ by finding an eigenvector associated with it.

7. [/ 2] Let $\boldsymbol{u} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$, and $\boldsymbol{w} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ for 7(a), 7(b), and 7(c).

(a) [/ 2] Determine the distance between $\boldsymbol{u}$ and $\boldsymbol{v}$.

(b) [/ 2] Which of the vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ are orthogonal to one another?

(c) **[/ 2]** Consider the vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ as points in $\mathbb{R}^3$. Now think about the plane that contains those three points. Give a parametrization of all the points on this plane or give a scalar equation that defines the plane. (You only need to do one, you will not get extra points for doing both, and I'll only grade the first one you do so don't submit both expecting you have a backup plan.) Again for reference:

$$\boldsymbol{u} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}, \quad \text{and} \quad \boldsymbol{w} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

8. **[/ 2]** Consider three geometric objects in $\mathbb{R}^3$, two planes and a line. One plane is given by the equation $2x - z = 0$ and the other is given by the equation $3x + 2z = 0$. The line is parametrized by

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 5 \\ 0 \end{bmatrix}$$

for any real number t. Do the three geometric objects share a common point of intersection? How do you know?

9. **[/ 2]** Suppose $f(x)$ is a degree two polynomial function in one variable. That is, let

$$f(x) = ax^2 + bx + c$$

Determine coefficients a, b, and c so that $f(-1) = -2$, $f(1) = 4$, and $f(3) = 2$.

(This is called an *interpolating* polynomial since it might be used to estimate values between the data points given.)

Hint: The requirement that $f(-1) = -2$ yields the equation

$$-2 = a(-1)^2 + b(-1) + c$$

Use the other requirements to build a system of three equations in three unknowns. Then solve.