

Calculus



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Limits and Continuity



Limits

🌐 Limit 🌐 | Thomas' Calculus (2.2–2.4) 🌐

- **Limit** $\lim_{x \rightarrow c}$: the value of a function (or sequence) approaches as the input (or index) approaches some value (informal definition).
 - Limits are used to define **continuity** ↓, **derivatives** ↓, and **integrals** ↓.

Limits of a Functions and Sequences

🌐 Limit of a function 🌐 | Limit of a sequence 🌐 | Essence of Calculus, E7 📺

- **Limit of a function**: a fundamental concept in calculus and analysis concerning the behavior L of a function near a particular input c , i.e.,

$$\lim_{x \rightarrow c} f(x) = L$$

- Reads as “ f of x tends to L as x tends to c ”
- ϵ, δ **Limit of function**: a formalized definition, wherein $f(x)$ is defined on an open interval I , except possibly at c itself, leading to above definition, if and only if:
 - For every real measure of **closeness** $\epsilon > 0$, there exists a real **corresponding** $\delta > 0$, such that for all existing further approaches there exist a smaller ϵ , i.e.,

$$f : \mathbb{R} \rightarrow \mathbb{R}, c, L \in \mathbb{R} \Rightarrow \lim_{x \rightarrow c} f(x) = L$$



$$\forall \epsilon > 0 (\exists \delta > 0 : \forall x \in I (0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon))$$

- Functions **do not have** a limit when the function:
 - has a **unit step**, i.e., it “jumps” at a point;
 - is **not bounded**, i.e., it tends towards infinity;
 - or it **oscillates**, i.e., does not stay close to any single number.
- **Limit of a sequence**: the value that the terms of a sequence (x_n) “tends to” (and not to any other) as n approaches infinity (or some point), i.e.,

$$\lim_{n \rightarrow \infty} x_n = x$$

- ϵ **Limit of sequence**: for every measure of closeness ϵ , the sequence's x_n term eventually converge to the limit, i.e.,

$$\forall \epsilon > 0 (\exists N \in \mathbb{N} (\forall n \in \mathbb{N} (n \geq N \Rightarrow |x_n - x| < \epsilon)))$$

- **Convergent**: when a limit of a sequence **exists**.
- **Divergent**: a sequence that **does not** converge.

Properties of Limits

📌 List of limits 📌 | Squeeze theorem 📌

- **Operations on a single known limit:** if $\lim_{x \rightarrow c} f(x) = L$ then:
 - $\lim_{x \rightarrow c} [f(x) \pm \alpha] = L \pm \alpha$
 - $\lim_{x \rightarrow c} \alpha f(x) = \alpha L$
 - $\lim_{x \rightarrow c} f(x)^{-1} = L^{-1}, L \neq 0$
 - $\lim_{x \rightarrow c} f(x)^n = L^n, n \in \mathbb{N}$
 - $\lim_{x \rightarrow c} f(x)^{n^{-1}} = L^{n^{-1}}, n \in \mathbb{N}, \text{ if } n \in \mathbb{N}_e \Rightarrow L > 0$
- **Operations on two known limits:** if $\lim_{x \rightarrow c} f(x) = L_1$ and $\lim_{x \rightarrow c} g(x) = L_2$
 - $\lim_{x \rightarrow c} [f(x) \pm g(x)] = L_1 \pm L_2$
 - $\lim_{x \rightarrow c} [f(x)g(x)] = L_1 L_2$
 - $\lim_{x \rightarrow c} f(x)g(x)^{-1} = L_1 L_2^{-1}$
- **Squeeze theorem:** used to confirm the limit of a difficult to compute function via comparison with two other functions whose limits are easily known or computed.
 - Let I be an interval having the point c as a limit point.
 - Let g, f , and h , be functions defined on I , except possibly at c itself.
 - Suppose that $\forall x \in I \wedge x \neq c \Rightarrow g(x) \leq f(x) \leq h(x)$
 - And suppose that $\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L$
 - Then, $\lim_{x \rightarrow c} f(x) = L$
 - Essentially, the hard to compute limit of the “middle function” is found by finding two other easy functions that “squeeze” the middle function at that point.

One-Sided Limit

📌 One-Sided Limit 📌

- **One-sided limit:** one of two limits of $f(x)$ as x approaches a specified point from either the left or from the right.
 - From the left: $\lim_{x \rightarrow c^-} = L$
 - From the right: $\lim_{x \rightarrow c^+} = L$
- If the left and right limits exist and are equal, then

$$\lim_{x \rightarrow c} f(x) = L \Leftrightarrow \lim_{x \rightarrow c^-} f(x) = L \wedge \lim_{x \rightarrow c^+} f(x) = L$$

- Limits can still exist, even if the function is defined at a different point, as long as both one-sided limits approach the same value near the given input.

Continuity

🌱 Continuous function 🌱 | Classification of discontinuities 🌱 | Thomas' Calculus (2.5) 🌱

- Continuity of functions is one of the core concepts of topology, however, there are definitions in terms of limits that prove useful; the following is only a primer.

Continuous Functions

- **Continuous function:** a function that does not have any abrupt changes in value.
 - I.e., a function is continuous if and only if arbitrarily small changes in its output can be assured by restricting to sufficiently small changes in its input.
- **Discontinuous:** when a function is not continuous at a point in its domain, leading to a discontinuity; there are three classifications:
 - **Removable:** when both **one-sided limits** \uparrow exist, are finite, and are equal, but the actual value of $f(x)$ is not equal to the limit and equal to some other value.
 - The discontinuity can be removed to regain continuity.
 - Sometimes the term *removable discontinuity* is mistaken for *removable singularity*, or a “whole” in the function (the point is not defined elsewhere).
 - **Jump:** when a single limit does not exist because the one-sided limits exist and are finite, but not equal.
 - Points can be defined at the discontinuity, but the function can not be made continuous.
 - **Essential:** when at least one of two one-sided limits doesn't exist; can be the result of oscillating or unbounded functions.

Intermediate Value Theorem

🌱 Intermediate Value Theorem 🌱

- **Intermediate value theorem:** if f is a continuous function whose domain contains the interval $[a, b]$, then it **takes on any given value between $f(a)$ and $f(b)$** at some point within the intervals.
- Relevant deductions, i.e., important corollaries:
 - **Bolzano's theorem:** if a continuous function has values of opposite sign inside an interval, then it **has a root** in that interval.
 - The image of a continuous function over an interval is itself an interval.
- Thus, the image set $f(I)$ (which has no gaps) is also an interval, and it contains:

$$[\min(f(a), f(b)), \max(f(a), f(b))]$$

Limits Involving Infinity

📌 Limits involving infinity 📌 | Thomas' Calculus (2.6) 📌

- Let $S \subseteq \mathbb{R}$, $x \in S$ and $f : S \mapsto \mathbb{R}$, then limits of these functions can approach arbitrarily large (\pm) values, providing a connection to asymptotes, and thus, analysis.

Limits at Infinity and Infinite Limits

- Limits at infinity:** limits defined as $f(x) \pm \text{infinity}$ are defined much like normal limits:

$$\lim_{x \rightarrow -\infty} f(x) = L \quad \lim_{x \rightarrow \infty} f(x) = L$$

- Formally, for all measures of closeness ϵ there exists a point c such that $|f(x) - L| < \epsilon$ whenever $x < c \vee x > c$ (respectively), i.e.,

$$\forall \epsilon > 0 (\exists c (\forall x \{<, >\} c : |f(x) - L| < \epsilon))$$

- Basic rules for rational functions $f(x) = p(x)q(x)^{-1}$, where p and q are polynomials, the degree of each is denoted as $\{p, q\}^\circ$, and the leading coefficients are denoted as P, Q , then:
 - $p^\circ > q^\circ \Rightarrow L$ is $\{+, -\}$ depending on the sign of the leading coefficients.
 - $p^\circ = q^\circ \Rightarrow L = PQ^{-1}$
 - $p^\circ < q^\circ \Rightarrow L = 0$
- Infinite limits:** the usual limit does not exist for a limit that grows out of bounds, however, limits with infinite values can be introduced:

$$\lim_{x \rightarrow c} f(x) = \infty, \quad \text{i.e.,} \quad \forall n > 0 (\exists \delta > 0 : f(x) > n \Leftrightarrow 0 < |x - a| < \delta)$$

Asymptotes of functions

📌 Asymptotes 📌

- Asymptote:** a tangent line of a curve at a point at infinity; the distance between the curve and the line approaches zero as a coordinate tends to infinity.
- There are three kinds of asymptotes: *horizontal*, *vertical* and *oblique*; nature of the asymptote is dependent on a function's relation to infinity.
 - Horizontal asymptotes:** a result of limits at infinity, i.e., when $x \rightarrow \pm\infty$
 - Vertical asymptotes:** a result of infinite limits, i.e., when $x \rightarrow \pm a = \pm\infty$
 - Oblique asymptotes:** when a linear asymptote is not parallel to either axis. $f(x)$ is asymptotic to the straight line $y = mx + n (m \neq 0)$ if:

$$\lim_{x \rightarrow \pm\infty} [f(x) - (mx + n)] = 0$$

Derivatives



Derivative Fundamentals

Derivative as a Function

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Simple Differentiation Rules

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Instantaneous Rates of Change

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Derivatives of Trigonometric Functions

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Differentiation Rules

Chain Rule

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Product Rule

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Quotient Rule

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Differentials and Related Concepts

Differentials

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Linearization

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Implicit Differentiation

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Related Rates

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Applications of Derivatives



Stationary Point

Maxima and Minima

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Extreme Value Theorem

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Interior Extremum Theorem

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Mean Value Theorem

Rolle's Theorem

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Corollaries of the Mean Value Theorem

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Monotonic Functions

Derivative Tests

First-Derivative Test

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Second-Derivative Test

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Concavity

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Higher-Order Derivative Test

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Differential Methods

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Taylor's Theorem

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Integrals



Applications of Integrals



Transcendental Functions



Techniques of Integration



Infinite Sequences and Series



First-Order Differential Equations



Parametric Equations and Polar Coordinates



Vectors and Vector-Valued Functions



Partial Derivatives



Multiple Integrals



Vector Calculus



Second-Order Differential Equations

