Error Analysis

Physics 122

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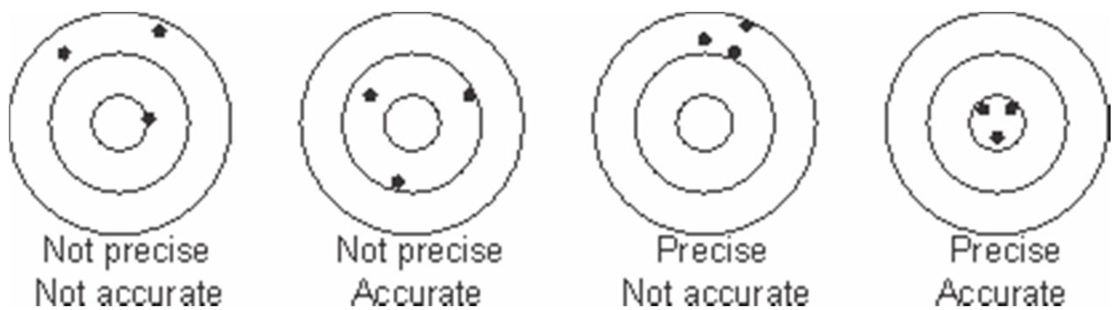
Measuring Errors

- Precision vs. Accuracy
  - Precision : reproducibility, repeatability
  - Accuracy: how close the result is to the correct value



Systematic vs. Random Errors

- Systematic Errors – biases in the measurement, leading to result different from the correct one
  - Calibration error
  - Environmental effect on measurement tool
  - Imperfect method of observation
- Random Errors – Statistical Errors
  - Fluctuations (noise) in measurement apparatus
  - Random operator errors
  - Noise in Nature– Johnson noise, shot noise
  - Usually can be improved by repeated measurements

Quoting Uncertainties in Measurements

- (measured value of x) = $x_{best} \pm \delta x$
- Significant digits

  Ex. 1(measured g) = 9.82 ± 0.02385 m/sec^2

  is absurd estimate of the undercertainty

  (measured g) = 9.82 ± 0.02 m/sec^2

  Ex. 2. (measured speed) = 6051.78 ± 30 m/sec

  => (measured speed) = 6050 ± 30 m/sec

- Fractional uncertainty = $\dfrac{\delta x}{|x_{best}|}$

Sample vs. Parent (Limiting) Distribution

Sample distribution

- Histogram of measurements
- Becomes the Parent (limiting) distribution in the limit of infinite number of measurements

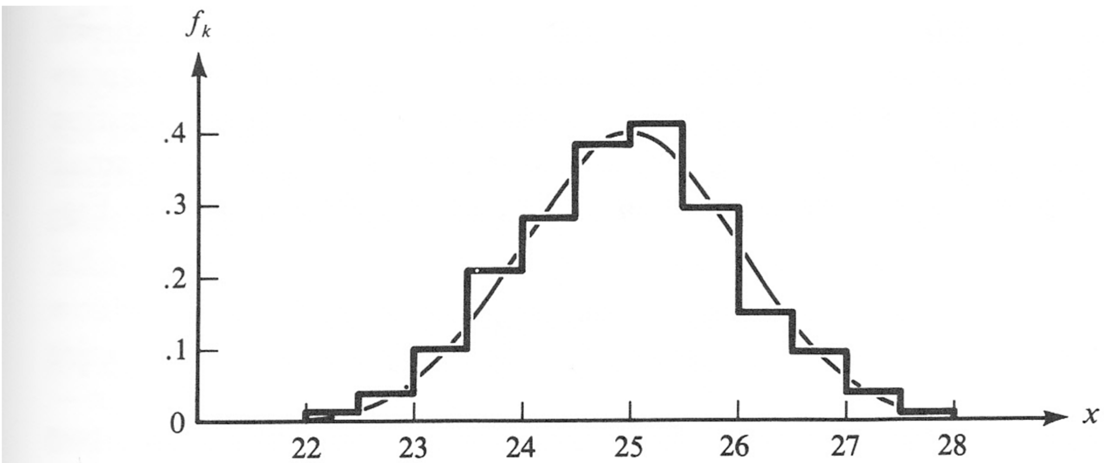


Figure 5.4. Histogram for 1,000 measurements of the same quantity as in Figure 5.3. The broken curve is the limiting distribution.

Properties of Distributions

- Mean of N measurements, $\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$
- Mean μ of parent distribution

$$\mu = \lim_{n \to \infty} \left(\frac{1}{N}\sum_{i=1}^{N} x_i\right)$$

- Median $\mu_{1/2}$, half of measurements are above this value, and half are below

  $P(x_i < \ \mu_{1/2})$ = $P(x_i > \ \mu_{1/2})$ = 0.5
- Mode – most probable value, the one which will occur most often with repeated measurements

Comparison of Mean, Median, Mode

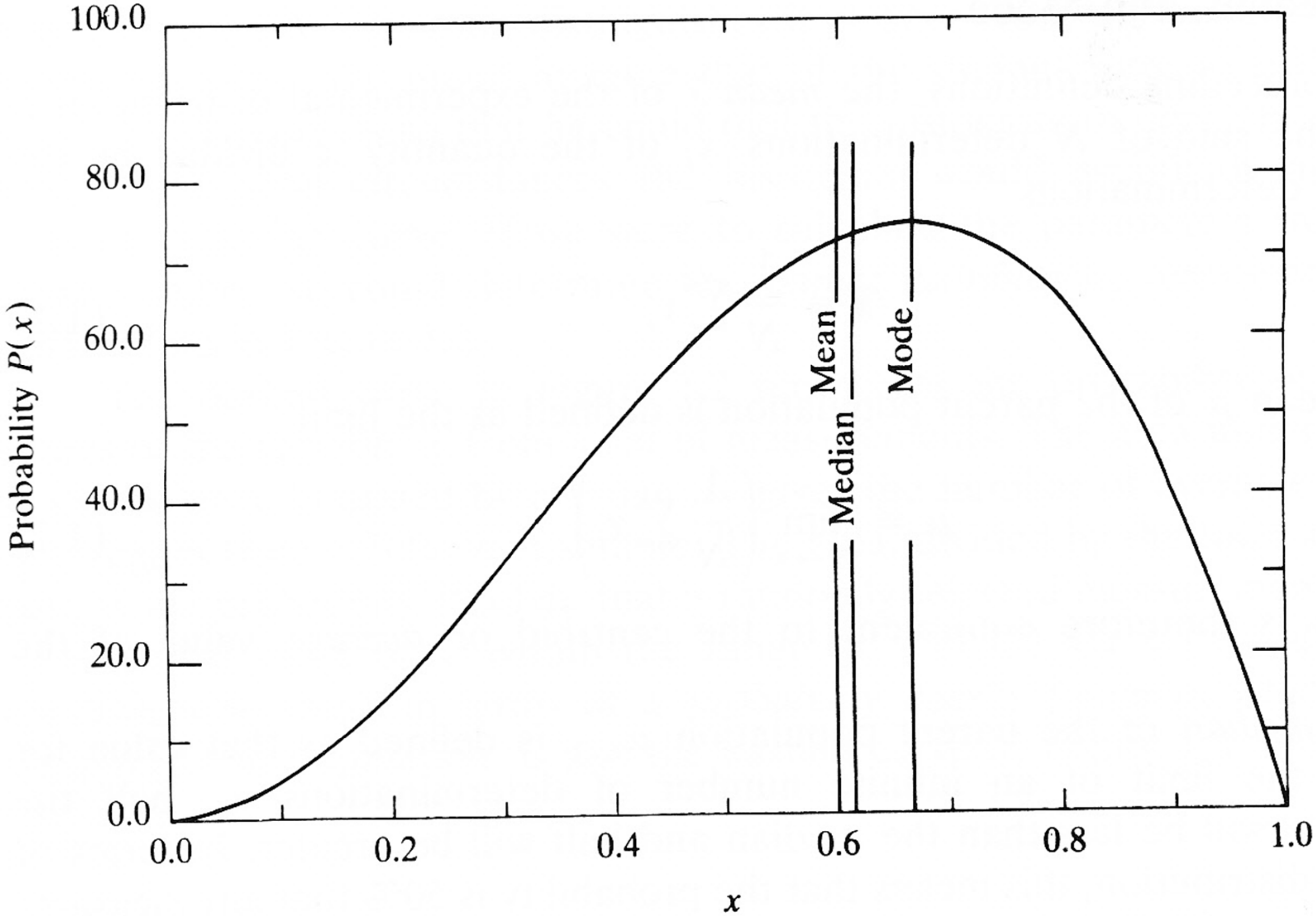


FIGURE 1.3
Asymmetric distribution illustrating the positions of the mean, median, and most probable value of the variable.

Standard Deviation σ

$$\sigma^2 = \lim_{N\to\infty}\left[\frac{1}{N}\sum(x_i - \mu)^2\right]$$

$$= \lim_{N\to\infty}\left[\frac{1}{N}\sum(x_i{}^2 - 2x_i\mu + \mu^2)\quad\right]$$

$$= \lim_{N\to\infty}\left(\frac{1}{N}\sum x_i{}^2\right) - 2\mu^2 + \mu^2$$

$$= \lim_{N\to\infty}\left(\frac{1}{N}\sum x_i{}^2\right) - \mu^2$$

For sample population, $s^2 = \left[\frac{1}{N-1}\sum(x_i - \bar{x})^2\right]$, use factor (N-1) in the denominator since $\bar{x}$ is determined from the data and not independently

Useful Probability Distributions in Physics

- Binomial
- Poisson
- Gaussian
- Lorentzian

Binomial Distribution

$$P_B(x; n, p) = \binom{n}{x} p^x q^{n-x} = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$q = 1 - p.$$

$$(p+q)^n = \sum_{x=0}^{n} \left[\binom{n}{x} p^x q^{n-x} \right]$$

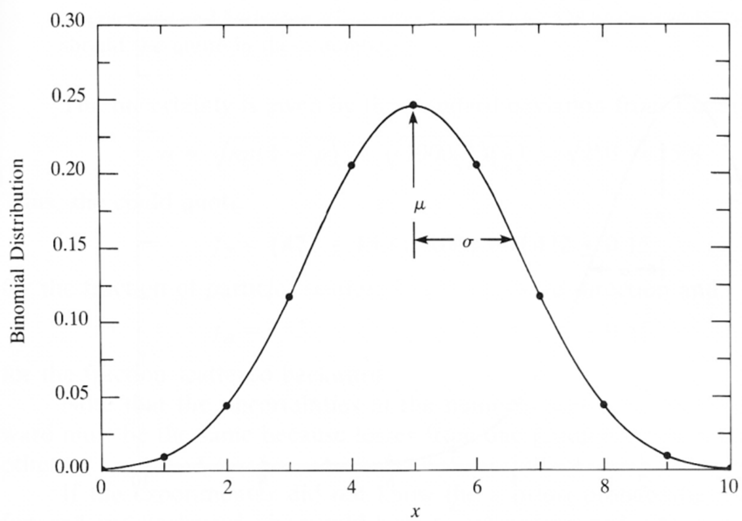


FIGURE 2.1
Binomial distribution for $\mu = 5.0$ and $p = \frac{1}{2}$ shown as a continuous curve although the function only defined at the discrete points indicated by the round dots.

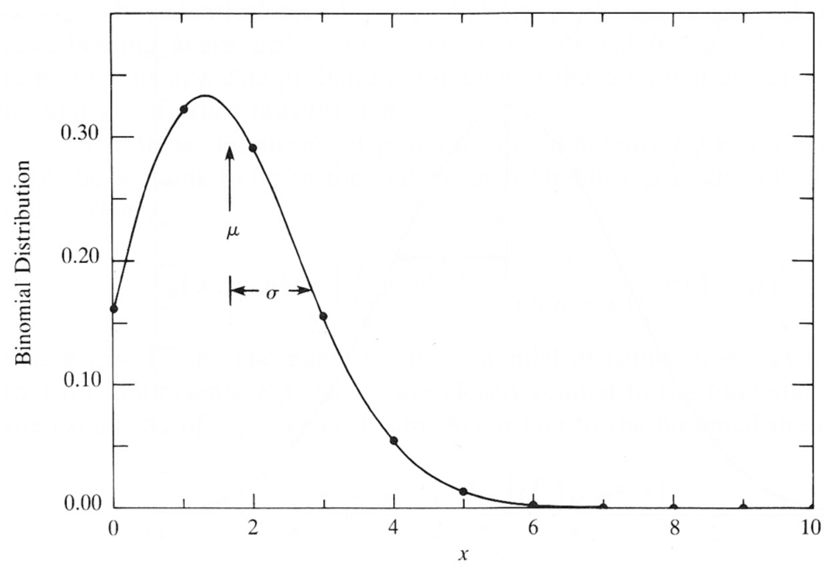


FIGURE 2.2
Binomial distribution for $\mu = 10/6$ and $p = 1/6$ shown as a continuous curve.

Poisson Distribution

Binomial -> Poisson when p->0, $$\lim_{p \to 0} P_B(x; n, p) = P_P(x; \mu) \equiv \frac{\mu^x}{x!} e^{-\mu}$$

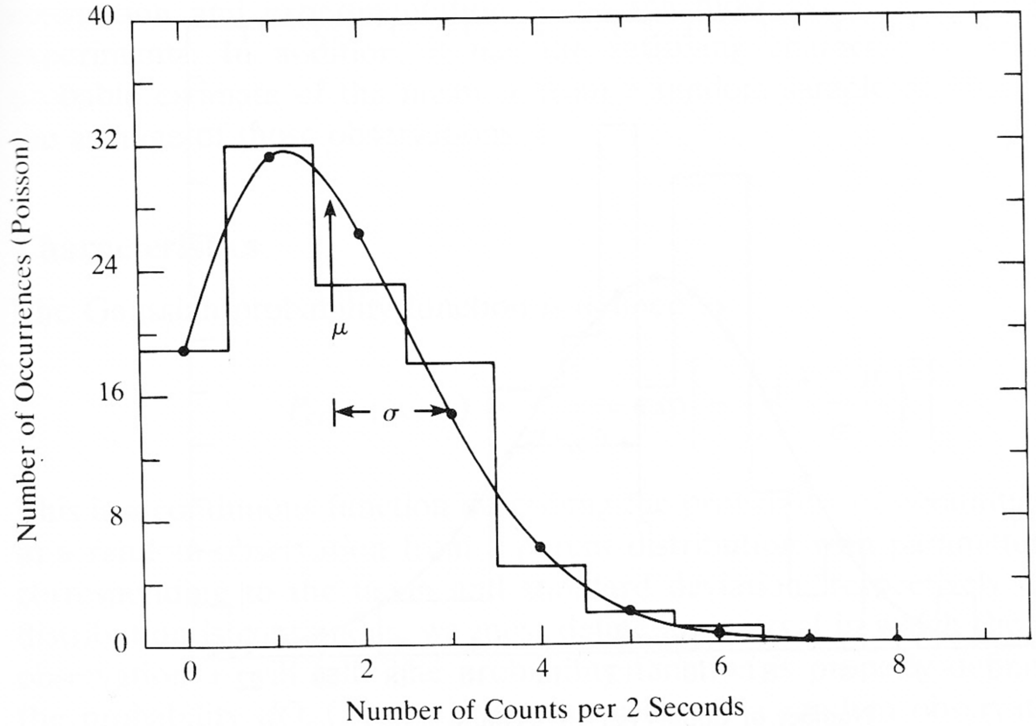


FIGURE 2.3
Histogram of counts in a cosmic ray detector. The Poisson distribution is an estimate of the parent distribution based on the measured mean $\bar{x} = 1.69$. It is shown as a continuous curve although the function is only defined at the discrete points indicated by the round dots. Only the circled calculation points are defined.

Gaussian Distribution

The Normal Distribution (*Chapter 5*)

For any limiting distribution $f(x)$ for measurement of a continuous variable x:

$$f(x)\,dx = \text{probability that any one measurement will give an answer between } x \text{ and } x + dx; \qquad \text{(p. 106)}$$

$$\int_a^b f(x)\,dx = \text{probability that any one measurement will give an answer between } x = a \text{ and } x = b; \qquad \text{(p. 106)}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \text{ is the normalization condition.} \qquad \text{(p. 107)}$$

The normal distribution is

$$f_{X,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-X)^2/2\sigma^2}, \qquad \text{(p. 112)}$$

where

$$\begin{aligned} X &= \text{center of distribution} \\ &= \text{true value of } x \\ &= \text{mean after many measurements,} \\ \sigma &= \text{width of distribution} \\ &= \text{standard deviation after many measurements.} \end{aligned}$$

The probability of a measurement within t standard deviations of X is

$$P(\text{within } t\sigma) = \frac{1}{\sqrt{2\pi}} \int_{-t}^{t} e^{-z^2/2}\, dz = \text{normal error integral}; \qquad \text{(p. 116)}$$

in particular

$$P(\text{within } 1\sigma) = 68\%.$$

Gaussian (Normal) Distribution

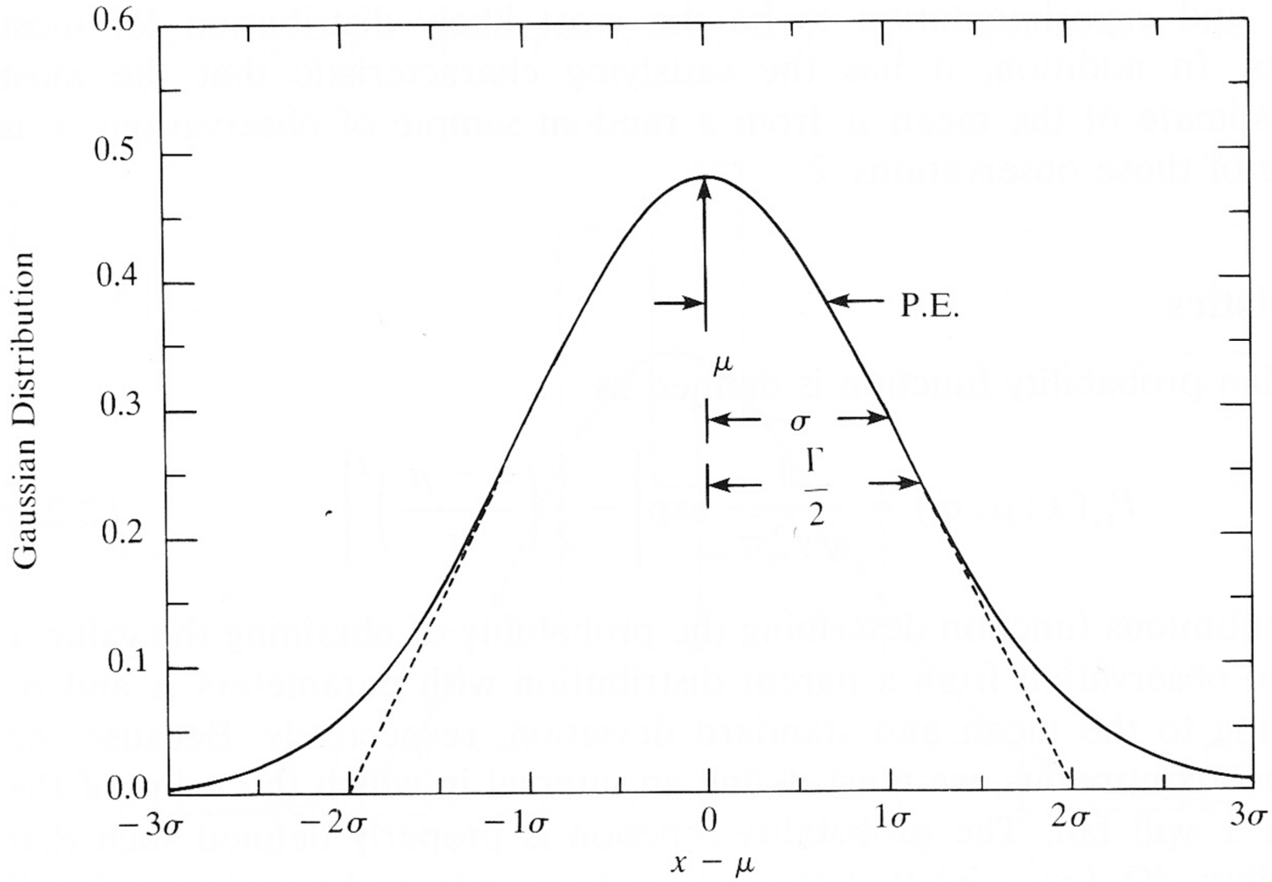


FIGURE 2.5
Gaussian probability distribution illustrating the relation of μ, σ, Γ, and P.E. to the curve. The curve has unit area.

Normal Error Integral or Error Function erf(t)

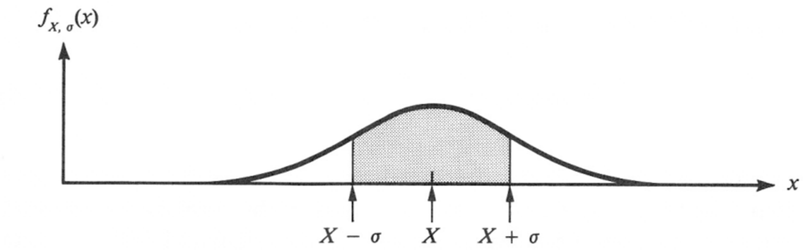
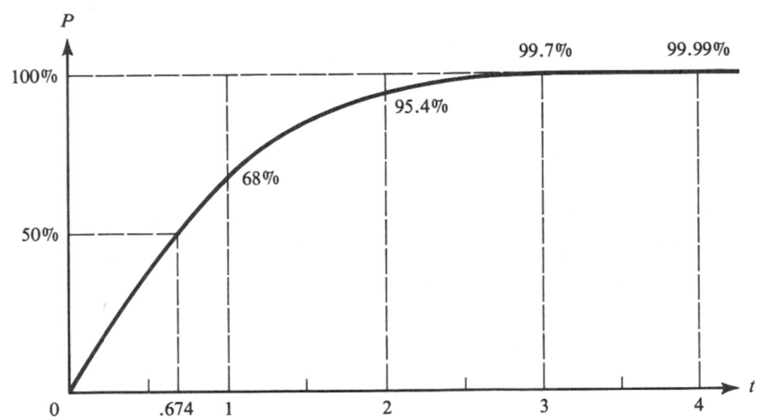


Figure 5.11. The shaded area between $X \pm \sigma$ is the probability of a measurement within one standard deviation of X.



t	0	.25	.5	.75	1.0	1.25	1.5	1.75	2.0	2.5	3.0	3.5	4.0
P(%)	0	20	38	55	68	79	87	92	95.4	98.8	99.7	99.95	99.99

Figure 5.13. The probability $P(\text{within } t\sigma)$ that a measurement of x will fall within t standard deviations of the true value $x = X$. Two common names for this function are the *normal error integral* and the *error function*, erf(t).

Gaussian vs. Lorentzian

Lorentzian: $$P_L(x;\mu,\Gamma) = \frac{1}{\pi}\frac{\Gamma/2}{(x-\mu)^2 + (\Gamma/2)^2}$$

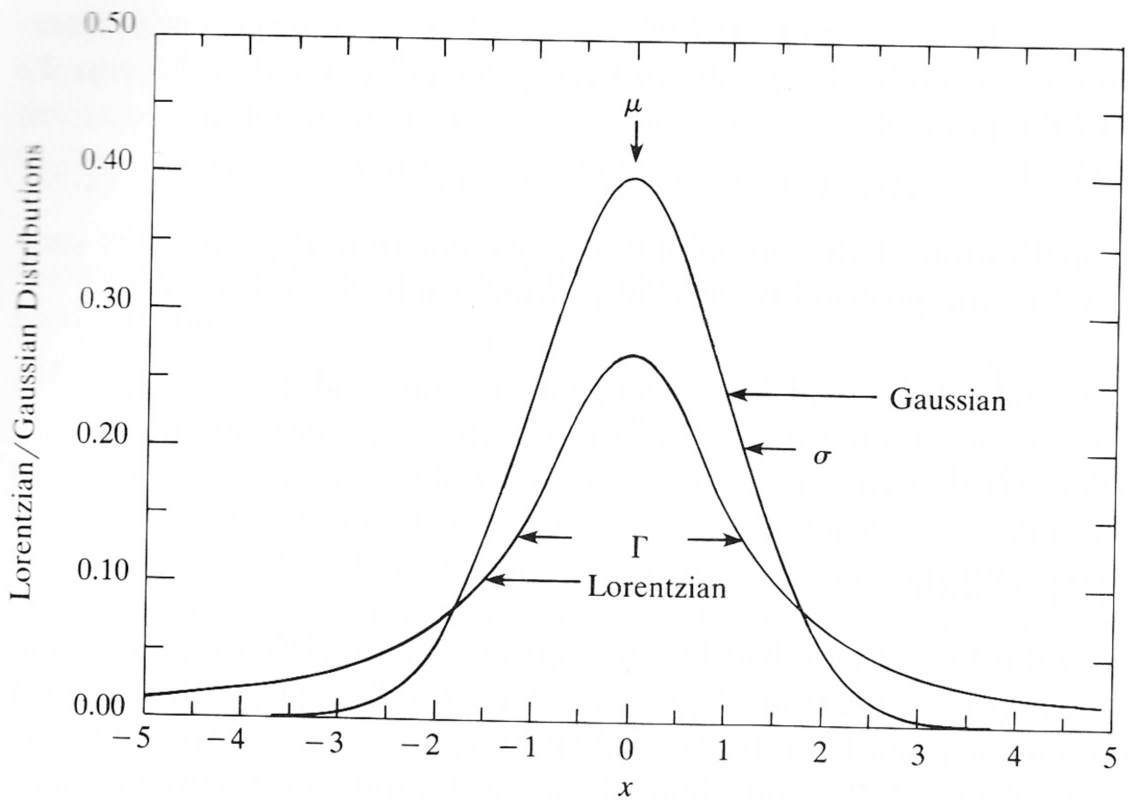


FIGURE 2.6
Comparison of normalized Lorentzian and Gaussian distributions, with $\Gamma = 2.354\sigma$.

Error Propagation

Use partial derivatives

$$x = f(u, v, \ldots)$$

$$x_i - \bar{x} \simeq (u_i - \bar{u})\frac{\partial x}{\partial u} + (v - \bar{v})\frac{\partial x}{\partial v} + \cdots$$

$$\sigma_x^2 = \lim_{N\to\infty}\left[\frac{1}{N}\sum (x_i - \bar{x})^2\right.$$

$$= \lim_{N\to\infty}\frac{1}{N}\left[(u_i - \bar{u})^2\left(\frac{\partial x}{\partial u}\right)^2 + (v_i - \bar{v})^2\left(\frac{\partial x}{\partial v}\right)^2\right.$$

$$\left. + \; 2(u_i - \bar{u})(v_i - \bar{v})\left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial x}{\partial v}\right) + \cdots\right]$$

$$\sigma_u^2 = \lim_{N\to\infty}\left[\frac{1}{N}\sum(u_i - \bar{u})^2\right]$$

$$\sigma_{uv}^2 = \lim_{N\to\infty}\frac{1}{N}\sum\left[(u_i - \bar{u})(v_i - \bar{v})\right]$$

$$\sigma_x^2 = \sigma_u^2\left(\frac{\partial x}{\partial u}\right)^2 + \sigma_v^2\left(\frac{\partial x}{\partial v}\right)^2 + \ldots + 2\sigma_{uv}^2\left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial x}{\partial v}\right) + \ldots$$

0 if fluct in u + v are uncorrelated

Summary of Error Formulas

Covariance: $\sigma_{uv}^2 = \langle (u - \bar{u})(v - \bar{v}) \rangle$.
Propagation of errors: Assume $x = f(u, v)$:

$$\sigma_x^2 = \sigma_u^2 \left(\frac{\partial x}{\partial u} \right)^2 + \sigma_v^2 \left(\frac{\partial x}{\partial v} \right)^2 + 2\sigma_{uv}^2 \left(\frac{\partial x}{\partial u} \right)\left(\frac{\partial x}{\partial v} \right)$$

For u and v uncorrelated, $\sigma_{uv}^2 = 0$.
Specific formulas:

$$x = au \pm bv \qquad \sigma_x^2 = a^2\sigma_u^2 + b^2\sigma_v^2 \pm 2ab\sigma_{uv}^2$$

$$x = \pm auv \qquad \frac{\sigma_x^2}{x^2} = \frac{\sigma_u^2}{u^2} + \frac{\sigma_v^2}{v^2} + 2\frac{\sigma_{uv}^2}{uv}$$

$$x = \pm \frac{au}{v} \qquad \frac{\sigma_x^2}{x^2} = \frac{\sigma_u^2}{u^2} + \frac{\sigma_v^2}{v^2} - 2\frac{\sigma_{uv}^2}{uv}$$

$$x = au^{\pm b} \qquad \frac{\sigma_x}{x} = \pm b\frac{\sigma_u}{u}$$

$$x = ae^{\pm bu} \qquad \frac{\sigma_x}{x} = \pm b\sigma_u$$

$$x = a^{\pm bu} \qquad \frac{\sigma_x}{x} = \pm (b \ln a)\sigma_u$$

$$x = a \ln(\pm bu) \qquad \sigma_x = a\frac{\sigma_u}{u}$$

Fitting Curves

- Method of least squares to fit straight line

Minimize $$\chi^2 = \sum_{i=1}^{N} (y_i - A - Bx_i)^2$$

$$\frac{\partial \chi^2}{\partial A} = -2\sum(y_i - A - Bx_i) = 0$$

$$\frac{\partial \chi^2}{\partial B} = -2\sum x_i(y_i - A - Bx_i) = 0$$

Judging Quality of Fitted Curves

- Compute χ^2 ,

$$\chi^2 = \sum_{i=1}^{n} \frac{(e_i - o_i)^2}{e_i^2}.$$

  where o_i = observed value, e_i = expected value

- Reduced $\chi^2_{reduced} = \frac{\chi^2}{\nu}$, where ν= number of degrees of freedom = N-n-1,

  where N=number of observations

  n=number of fitting parameters
  reduce by 1 for assuming the mean value is another fitted parameter

- A good fit has $\chi^2_{reduced}$ =1, i.e., match between observations and estimates is in accord with error variance

Example from Melissinos, Chap. 10

TABLE 10.4 Observed and Expected Frequencies of the Results of 100 Measurements of a Radioactive Sample

Class	0–75	75–79	79–83	83–87	87–91	91–95	95–∞	Counts/min
o_i	15	11	15	15	18	12	14	Observed freq
e_i	13	12	15	16	16	13	15	Expected freq
$(e_i - o_i)^2/e_i^2$	0.307	0.083	0	0.062	0.25	0.077	0.067	χ^2

we obtain from the data the estimators for the parameters of a Gaussian (1) $\mu^* = \overline{N}$, (2) $\sigma^* = \sqrt{\overline{N}}$, and (3) the overall normalization, namely, $\sum o_i = \sum e_i$; thus the degrees of freedom of χ^2 are four, corresponding to seven classes less three estimators. From the Gaussian distribution we

χ^2 = 0.846, and plot shows that 93% of the cases would have a larger χ^2 than obtained here, i.e., this fit is too good!

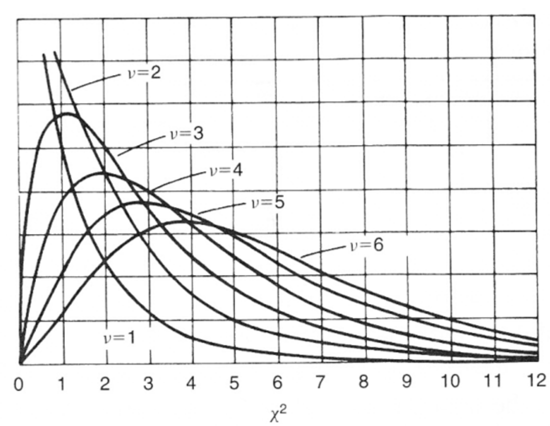


FIGURE 10.7 The frequency function for the distribution of χ^2, for different degrees of freedom. All curves are normalized to the same unit area. Note that for large ν the χ^2 distribution approaches a Gaussian.