Report on 'A generalized Abhyankar's conjecture for simple Lie algebras in characteristic $p > 5$'

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1 Recommendation

This is an interesting paper and I heartily recommend its publication in this journal. The main results are meaningful and display beautiful mathematics. It is well-written and covers enough material to be useful for those coming to it. It brings the theory of non-classical Lie algebras in characteristic p to the realm of Nori's theory of the π_1; the authors take their time to explain this intricate, beautiful, and often hermetic theory in brief but detailed fashion; I find this praiseworthy (although in some places the exposition can be improved).

In order to improve the paper, I would like to see the comments below addressed, specially those of Section 3.

2 Comments on writing, syntax, etc.

1. (1,22). 'prove *its* validity for all finite local ...' sounds better. This also occurs elsewhere, e.g. (3,18). Please check allover.

2. (2,11). It is probably better to avoid talking about Abhyankar's conjecture as 'describing the quotients'. A quotient is a group plus a morphism, and we don't know much about the morphisms: suppose $f, f' : \pi^N \mathbb{A}^1 \to G$ are quots, what can be said about the kernels? can I suggest that you follow the presentation of [14] by working with π_A^{loc}?

3. (2,12) remove 'possible' (in a conjecture we're already on the realm of possibility).

4. (2,13) After U, make full stop and start new sentence.

5. (2,31). *gives*$\to$*is*.

6. (2,34) p *and* t *are* ≥ 3.

7. (3,18). 'would be determined by the abelian quotients' is confusing. Either remove it or explain what you mean.

8. (3,34) 'Let k...' is unnecessary. That $k = \overline{k}$ was already fixed at the start of the section.

9. (3,42) Can I suggest that you give reference to a place where the theorem is fully proved instead of [26]? (Say in Mumford's Abelian Varieties or Demazure-Gabriel.)

10. (3,55) 'Toward more precise...' can be improved: *Moving* toward *a* more ...

11. (3,42) I cannot find Theorem 2.3 in [26]. Same problem on line 50. It seems to me that you're using the ArXiv version of [26]. This needs correction.

12. In several places you talk about non-abelian simple Lie algebras, which is redundant. See comment on the next section.

13. It is perhaps useful to introduce some notation for the following situation which occurs all the time. L is a simple Lie algebra over k, then $L_{[p]}$ is a restricted Lie algebra, and then you get a local group scheme $\mathfrak{E}(L_{[p]})$ (notation of Demazure-Gabriel). It is perhaps useful to give $\mathfrak{E}(L_{[p]})$ a name, say $G(L)$. For example, your important corollary 2.9 is less attractive since it is not automatic to find the definitino of 'the finite local k-group scheme of height one associated to L'.

14. (5,10). handle *this* case.

15. (5,14) …the *pull-back*…

16. (5,18) 'have to do further works' sounds odd. Suggestion: Therefore, more work is needed to settle the problem.

17. (5,22) This may be improved. You should probably mention that in order to apply the criterion you just talked about (Thm 3.9) you shall require certain filtrations on the Lie algebra (and here you can mention Cor. 3.11), which fortunately accompany many non-classical Lie algebras.

18. (5,27) The *obstruction for our construction* is vague. What are you talking about? The obstruction to the application of Theorem 3.9? Of 3.11?

19. (5,38) the phrase 'we give a sufficient … *so that*…' is odd. Maybe: we give a sufficient condition on a simple Lie algebra to have its associated group scheme appear as a quotient of $\pi_1^{loc}(A^1)$? (Note here that if you use $G(L)$ suggested above, the sentence is simpler: Give a sufficient condition on a simple algebra L for $G(L)$ to be a quotient of π. Even better, if you use the phrasing of [14], then you just need $G(L) \in \pi_A^{loc}$.)

20. (5,44) criterions→criteria. Also, I suppose that you want the group scheme of $L_{[p]}$ to appear as a quotient.

21. (6,34) 'naturally bijective into' is odd. naturally in bijection with?

22. (6,41) You did not define 'pro-finite local'...

23. (7,17) *image* of *a* basis. Also, the sentence containing this bit can be improved (it is too long).

24. In many places (e.g. (7,18)) you use the notion of "injective" map of group schemes. It is worth employing a more standard terminology as "closed embedding" or "monomorphism". (One can argue that the Frobenius is "injective". In EGA I, injective morphisms is defined by the obvious useless notion.)

25. (7,22) You should be careful here. If $F : G_a \to G_a$ is the Frobenius, what is $Ker F$ and what is $Ker\, Lie \to Lie$?

26. (7,33) Here you'll encounter the problem of 'what is a simple' Lie algebra and what is a simple group scheme. See the corresponding remark in Section 3.

27. (7,38) Everyone knows what a derivation is (or at least those who will have enough education to read your paper), what the adjoint is and that the center is the kernel of the adjoint ... This is distracting.

28. (11,32) The Artin-Hasse exponential, at least in your reference to Serre, usually depends on a Witt vector and what you call the AH-series is an auxiliary thing. Also, 'As well-understood'→'As is well-understood'.

29. (13,45) Suppress 'for some finite dimensional k-vector space'.

30. (16,40) Why write $F^{(1)}$ and not F?

31. (16,53) This is a confuse sentence. Suggestion: For that, it suffices to show that for any factor $W_{n_i} \to W_{\mathbf{n}}^{(1)} \times \mathbf{G}_a^{n(1)}$, resp. any $\mathbf{G}_a \to W_{\mathbf{n}}^{(1)} \times G_a^n$, the image of $\pi^{loc} W_{n_i} \to \pi^{loc}(W_{\mathbf{n}}^{(1)} \times G_a^n(1)) \to GL(V)_{(1)}$, resp. $\pi^{loc} G_a \to \pi^{loc}(W_{\mathbf{n}}^{(1)} \times G_a^n(1)) \to GL(V)_{(1)}$, is ...

32. (17,36) Please make the argument properly with points having value in the correct algebras.

33. (17,6) Please remind the reader why is it that W_l being reduced as a scheme implies that $F : W_l \to W_l^{(1)}$ is "Nori reduced" principal bundle.

34. (19,17) and (19,24). take*s* and Therefore, *to prove the Corollary*, it suffices ...

35. (19,27) Remove *However*

36. (21,11) any→each.

37. (22,26) free basis → a basis.

38. (25,44) It is perhaps worth pointing out that the action of $W(m, \mathbf{n})$ on Ω^* is just the Lie derivative, 'contract derivate plus derivate contract', of differential geometry.

39. (26,17) Remove one of the 'then'.

40. (26,29) subalgebra of $W(m, \mathbf{n})\ldots \to$ subalgebra of $W(m, \mathbf{n})$ spanned by $\{D_{ij}(f) | 1 \leq i < j \leq m,\ f \in A(m, \mathbf{n})\}$.

41. (27,44) The final sentence of the remark is odd. You *adopt* a convention to the *simplicity*? Suggestion: Remove Rmk 4.6 and add in Rmk 4.4 that $S(2, \mathbf{n}) = H(2, \mathbf{n})$, and that the fact that authors usually exclude the case $S(2, \mathbf{n})$ from the analysis is because it is covered by the case $H(2, \mathbf{n})$, therefore making the statements clearer.

42. (28,24) Your reference to [23] is (4.2.10). This part is also too densely packed and you should write things more orderly, for example by copying [23].

43. (28,29) You should give the correct reference to [23]. Better yet, you should go for 'Strade-Farnsteiner' since in [23] all he's doing is quoting from this reference.

44. (29,21 and following) Improve the exposition. 'Then, according ... Then $K(2r+1, \mathbf{n})$ can be ... Namely ...' is poor. Why not: According to [some reference], we have $K(2r+1, \mathbf{n})^{(1)} = \ldots$ Also, I don't see why you introduce all the notations concerning $\langle -, - \rangle$ and $|| - ||$ here; you'll only use it in the proof of Lemma 4.7.

45. (29,54) Here, we *offer*...

3 More important suggestions

1. (3,59) There exist no *abelian* simple Lie algebras according to the standard definition. (See Seligman, Strade-Farnsteiner, etc.) I can only suppose that this is also the definition that [26] uses. Now, one should also suppose that the definition of a *simple restricted Lie algebra* in [26] also excludes the *abelian.* This is used in the proof of [26, Theorem 4]. (The definition of simple in [26] is wrong from all aspects.) On the other hand, there is no good reason to exclude α_p and μ_p from simple group schemes (Oort's book doesn't). *You'll need to clean this up*, specially because this causes more confusion in your section 2.2.

2. (4,5) To understand "the" p-envelope, you refer the reader to the definition of the *relative* p-envelope. This is confusing. You should clearly state what the correspondence assures. (Suggestion: introduce some notation, $p\mathrm{Env}(L, \mathrm{ad})$, or $L_{\mathrm{ad},[p]}$, etc, for the p-envelope of L in $\mathfrak{gl}(L)$). The reference [26] is also confusing since he calls *universal* envelope what Strade-Farnsteiner call the *minimal* envelope (see p.97 of this book). You should profit to have that corrected.

3. (5,47 on). This can be improved. You should tell the reader that you're *not only* recalling the definition (it should then suffice to give references) but you're looking at these algebras with a view towards application of Th. 3.9 or Cor 3.11. Even in the text, this is confusing, since you go on for pages to digress on unmotivated properties of Lie algebras just to find out in section 5 that you want to *apply these* to *employ* Cor. 3.11. For example, in the crucial proof of Tmh 5.5, you say *we will prove that the graded Lie ... in Corollary 3.11. However, as we recalled in §4.2, all the conditions (GI), (G2) and (GIII) are fulfilled...* So, all the work is done in §4.2, but you just don't mention it, and then you collect its fruits somewhere else. Why not mention after eqs. (4.7)–(4.9) that these are enough to prove that the Witt algebras are quotients of $\pi(A^1)$? This should explain the reader what the strategy is. (See also the last item.)

4. You should modify the bijection in (3,58) and make it correspond to Thm 2.2. (in one place you say isomorphism classes, in the other you do not, for example).

5. Lemma 3.7-(2). No need for U^+ and U^- to be abelian. Please also specify always what you mean by "generated". (Suggestion: Introduce some notation, say $\langle A, B, \ldots\rangle_{Lie}$, to mean "Lie algebra generated by ...")

6. Theorem 3.1. Your reference to [22] is wrong. Moreover, in [22], the result which you need is a transcript of results proved somewhere else! You should avoid working with secondary sources; this increases the chance of misinterpretation. Please refer to Proud's paper mentioned in [22]. Moreover, in [22] and in Proud one find the assumption that k is algebraically closed. Either argument, saying that you went to through the proof of these results and found that they hold if $k \neq \overline{k}$, or fix your k to be algebraically closed.

7. (15,17) What is c? You mean conjugation by the K-point of GL_V defined by δ I guess, so it is $\mathrm{GL}_{V,K} \to \mathrm{GL}_{V,K}$?

8. (16,34). The definition of $\langle \omega(W_{\mathbf{n}(1)}), \delta_1 \ldots\rangle$ is not in sight. What does this mean? (It is always a tricky thing to define subgroup generated by something...) You should

probably include some comments on

$$Lie(\langle G_1, \ldots, G_n \rangle) = \langle Lie(G_1), \ldots, Lie(G_n) \rangle_{pLie}$$

where $\langle G_1, \ldots, G_n \rangle$ is the subgroup generated by $\{G_i\}$ (inside some H) and $\langle L_i \rangle_{pLie}$ is the pLie-subalgebra generated by $\{L_i\}$ (inside $LieH$). Also, $\delta(\alpha_p)$ looks odd; $Im(\delta)$?

9. (19,61) It is difficult to see how Example 3.13 gives a counter-example to the question you mentioned. It seems that *your method of proof* breaks down, not that the result is plainly false. Please elaborate.

10. (22,42 onwards) It is difficult to understand what you want to do here and the reader will only find it out by going to the proof of theorem 5.5. Can I suggest that you *tell the reader* that the calculations on $W(m, \mathbf{n})$ here *already show a theorem* saying that the p-envelope of ad : $W(m, \mathbf{n}) \to gl(W(m, \mathbf{n}))$ gives a quotient of $\pi_1 A^1$? The same is true for the other calculations with the other algebras, as I can infer from (4.17), etc.