ALGEBRAIC AND NORI FUNDAMENTAL GERBES

FABIO TONINI, LEI ZHANG

Abstract. In this paper we extend the generalized algebraic fundamental group constructed in [EH] to general fibered categories using the language of gerbes. As an application we obtain a Tannakian interpretation for the Nori fundamental gerbe defined in [BV] for non smooth non pseudo-proper algebraic stacks.

Introduction

Let k be a field and let X be a smooth and connected scheme over k with a rational point $x \in X(k)$. The algebraic fundamental group of (X, x), denoted by $\pi^{\mathrm{alg}}(X, x)$ is the affine group scheme over k associated with the k-Tannakian category $\mathrm{Dmod}(X/k)$ of $\mathcal{O}_X$-coherent $D_{X/k}$-modules neutralized by the pullback along $x\colon \operatorname{Spec} k \longrightarrow X$. If k is algebraically closed then the profinite quotient of $\pi^{\mathrm{alg}}(X, x)$ is $\pi_1^{\text{ét}}(X, x)$, the Grothendieck's étale fundamental group developed in [SGA1].

On the other hand if X is a connected and reduced scheme over k with a rational point $x \in X(k)$, Nori defined in [Nori] a profinite fundamental group scheme $\pi^{\mathrm{N}}(X, x)$ over k which classifies torsors over X by finite group schemes of k with a trivialization over x. If k is algebraically closed then its proétale quotient is again $\pi_1^{\text{ét}}(X, x)$, so that if X is smooth we have maps

$$\begin{tikzcd}
\pi^{\mathrm{alg},\infty}(X,x) \arrow[r, dashed, two heads, "c"] \arrow[d, dashed, two heads, "d"] & \pi^{\mathrm{N}}(X,x) \arrow[d, two heads, "b"] \\
\pi^{\mathrm{alg}}(X,x) \arrow[r, two heads, "a"] & \pi_1^{\text{ét}}(X,x)
\end{tikzcd}$$

where a is the profinite quotient and b is the pro-étale quotient (and thus an isomorphism if $\operatorname{char} k = 0$). If $\operatorname{char} k > 0$, in [EH] Esnault and Hogadi completed this diagram with dashed arrows from an affine group schemes $\pi^{\mathrm{alg},\infty}(X, x)$ associated with a Tannakian category denoted by $\mathrm{Strat}(X, \infty)$, with c a profinite quotient and d a quotient.

In this paper we would like to generalize the above picture to certain fibered categories over a field k which may not possess a rational point, and this applies in particular to algebraic stacks which are not necessarily smooth. To achieve this we will use the language of gerbes instead of that of affine group schemes, just as how Borne and Vistoli generalized the Nori fundamental group scheme to fibered categories in [BV].

Date: June 5, 2016.

This work was supported by the European Research Council (ERC) Advanced Grant 0419744101 and the Einstein Foundation.

For smooth schemes X there are several equivalent descriptions of the category of $\mathcal{O}_X$ coherent D_X-modules, for instance the category $\mathrm{Crys}(X)$ of crystals on the infinitesimal site of X, or the category $\mathrm{Str}(X)$ of stratified bundles, or, in positive characteristic, the category $\mathrm{Fdiv}(X)$ of F-divided sheaves (see [BO, Prop. 2.11, pp. 2.13] and [Gie, Thm. 1.3, pp. 4]).

Let $\mathcal{X}$ be a quasi-compact, quasi-separated and connected category fibered in groupoids over k (see 2.4 and the section *Notations and Conventions* for the meaning of those adjectives). In order to define an algebraic fundamental gerbe in general, we are going to define k-linear monoidal categories $\mathrm{Crys}(\mathcal{X})$, $\mathrm{Str}(\mathcal{X})$, and, in positive characteristic, $\mathrm{Fdiv}(\mathcal{X})$, and discuss when those are Tannakian categories. More precisely we will define the *big infinitesimal site* $\mathcal{X}_{\mathrm{inf}}$ of $\mathcal{X}$, the *big stratified site* $\mathcal{X}_{\mathrm{str}}$ of $\mathcal{X}$ and the direct limit $\mathcal{X}^{(\infty,k)}$ of relative Frobenius of $\mathcal{X}$. These are fibered categories over k equipped with a morphism from $\mathcal{X}$. The categories $\mathrm{Crys}(\mathcal{X})$, $\mathrm{Str}(\mathcal{X})$ and $\mathrm{Fdiv}(\mathcal{X})$ are then defined as $\mathsf{Vect}(\mathcal{X}_{\mathrm{inf}})$, $\mathsf{Vect}(\mathcal{X}_{\mathrm{str}})$ and $\mathsf{Vect}(\mathcal{X}^{(\infty,k)})$ (see 6.5 and 6.15), where $\mathsf{Vect}(-)$ denotes the category of vector bundles (see the section *Notations and Conventions* for its definition). Since those categories are not equivalent in general when $\mathcal{X}$ is not smooth, we develop an axiomatic language which allow to treat all of them together. The advantage of this language is that all functors involved will be expressed as pullback of vector bundles along certain maps, making proof easier and more conceptual.

Let $\mathcal{X} \longrightarrow \mathcal{X}_{\mathcal{T}}$ be a morphism of fibered categories over k, and let $\mathcal{T}(\mathcal{X}) = \mathsf{Vect}(\mathcal{X}_{\mathcal{T}})$. We will list four axioms A,B,C and D on the given morphism $\mathcal{X} \longrightarrow \mathcal{X}_{\mathcal{T}}$ or, to simplify the exposition, on $\mathcal{T}(\mathcal{X})$ (see 5.2) which imply nice "Tannakian" properties of $\mathcal{T}(\mathcal{X})$. Denote by L_0 the endomorphisms of the unit object of $\mathcal{T}(\mathcal{X})$, that is $L_0 = \mathrm{H}^0(\mathcal{O}_{\mathcal{X}_{\mathcal{T}}})$ and, if $\mathcal{C}$ is a k-Tannakian category, denote by $\Pi_{\mathcal{C}}$ the associated affine gerbe over k. For instance A and B imply that L_0 is a field, that $\mathcal{T}(\mathcal{X})$ is an L_0-Tannakian category and, moreover, that $\Pi_{\mathcal{T}(\mathcal{X})}$ has the following universal property: there is an L_0-map $\mathcal{X}_{\mathcal{T}} \longrightarrow \Pi_{\mathcal{T}(\mathcal{X})}$ which is universal among L_0-morphisms from $\mathcal{X}_{\mathcal{T}}$ to an affine gerbe over L_0 (see 5.8).

The first main application of our axiomatic language is the following:

Theorem I. *[2.6, 6.6, 6.17] Assume that $\mathcal{X}$ is geometrically connected over k and either $\mathrm{H}^0(\mathcal{O}_{\mathcal{X}}) = k$ or there exists a field extension L/k separably generated up to Frobenius (see 6.1) such that $\mathcal{X}(L) \neq \emptyset$.*

(1) If $\mathcal{X}$ admits an fpqc covering $U \to \mathcal{X}$ from a Noetherian scheme U defined over the perfection k^{perf} of k then $\mathrm{Str}(\mathcal{X})$ satisfies axioms A,B and C and it is a k-Tannakian category.

(2) If $\mathcal{X}$ is an algebraic stack locally of finite type over k, then $\mathrm{Crys}(\mathcal{X})$ satisfies axioms A,B and C and it is a k-Tannakian category.

(3) ($\mathrm{char}\, k > 0$) If $\mathcal{X}$ admits an fpqc covering $U \to \mathcal{X}$ from a Noetherian scheme U whose residue fields are separable up to Frobenius over k (see 6.1) then $\mathrm{Fdiv}(\mathcal{X})$ satisfies axioms A,B,C and D and it is a pro-smooth banded (see B.8) k-Tannakian category.

In any of the above situations, taking the gerbe associated with the corresponding Tannakian category, one has a notion of *algebraic fundamental gerbe* for $\mathcal{X}/k$. Notice moreover that all conditions are satisfied in Theorem I if $\mathcal{X}$ is a geometrically connected algebraic stack of finite type over k. In this last situation, in an unpublished result B. Bhatt proved that the three categories $\mathrm{Crys}(\mathcal{X})$, $\mathrm{Str}(\mathcal{X})$ and, in positive characteristic, $\mathrm{Fdiv}(\mathcal{X})$ are all equivalent. This means that the three candidates for algebraic fundamental gerbe coincide for algebraic stacks of finite type over k.

Once we have a notion of an algebraic fundamental gerbe we must compare it with the relative analogous of the Grothendieck's étale fundamental group, namely the Nori étale fundamental gerbe $\Pi^{\mathrm{N,\acute{e}t}}_{\mathcal{X}/k}$ of $\mathcal{X}/k$ (see 4.1) which exists if and only if $\mathcal{X}$ is geometrically connected over k (see 4.3). If $\mathcal{T}(\mathcal{X})$ satisfies axioms A,B and C then $\mathcal{X}$ is geometrically connected over L_0 and $\Pi^{\mathrm{N,\acute{e}t}}_{\mathcal{X}/L_0}$ is the pro-étale quotient of $\Pi_{\mathcal{T}(\mathcal{X})}$. If moreover $\mathcal{T}(\mathcal{X})$ satisfies axiom D, one can use the profinite quotient instead (see 5.8). In the hypothesis of Theorem I we have that $\Pi^{\mathrm{N,\acute{e}t}}_{\mathcal{X}/k}$ is the pro-étale quotient of $\Pi_{\mathrm{Str}(\mathcal{X})}$ and $\Pi_{\mathrm{Crys}(\mathcal{X})}$ in situations (1) and (2) respectively, it is the profinite quotient of $\Pi_{\mathrm{Fdiv}(\mathcal{X})}$ in situation (3).

The fibered category $\mathcal{X}$ admits a Nori fundamental gerbe $\Pi^{\mathrm{N}}_{\mathcal{X}/k}$ over k if and only if it is inflexible over k (see [BV, Definition 5.3 and Theorem 5.7]) and in this case $\Pi^{\mathrm{N,\acute{e}t}}_{\mathcal{X}/k}$ is the pro-étale quotient of $\Pi^{\mathrm{N}}_{\mathcal{X}/k}$. We give a new concrete geometric interpretation of the notion of inflexibility: If $\mathcal{X}$ is reduced (see 2.4) then $\mathcal{X}$ is inflexible if and only if k is integrally closed in $\mathrm{H}^0(\mathcal{O}_{\mathcal{X}})$ (see 4.4).

Assume $\mathcal{X}$ reduced from now on. In characteristic 0 Nori fundamental gerbe and Nori étale fundamental gerbe coincide, so let's assume $\operatorname{char} k = p > 0$. The same procedure used by Esnault and Hogadi in [EH] allows us to consruct a category $\mathcal{T}_\infty(\mathcal{X})$ starting from the functor $\mathcal{T}(\mathcal{X}) \longrightarrow \mathsf{Vect}(\mathcal{X})$ and the pullback of Frobenius on those categories (see 5.9). In particular are defined categories $\mathrm{Crys}_\infty(\mathcal{X})$, $\mathrm{Str}_\infty(\mathcal{X})$ and $\mathrm{Fdiv}_\infty(\mathcal{X})$. If $\mathcal{T}(\mathcal{X})$ satisfies axioms A and B then $\mathcal{T}_\infty(\mathcal{X})$ is an L_∞-Tannakian category, where L_∞ is the purely inseparable closure of L_0 inside $\mathrm{H}^0(\mathcal{O}_{\mathcal{X}})$ and thus, again, a field of definition for $\mathcal{X}$. If $\mathcal{T}(\mathcal{X})$ also satisfies axiom C, then $\mathcal{X}$ is inflexible over L_∞ and we have a diagram

$$\begin{tikzcd} \Pi_{\mathcal{T}_\infty(\mathcal{X})} \arrow[r, "c"] \arrow[d, "d"] & \Pi^{\mathrm{N}}_{\mathcal{X}/L_\infty} \arrow[d, "b"] \\ \Pi_{\mathcal{T}(\mathcal{X})} \arrow[r, "a"] & \Pi^{\mathrm{N,\acute{e}t}}_{\mathcal{X}/L_0} \end{tikzcd}$$

where c is a profinite quotient of L_∞-gerbes (see 5.12). In particular $\mathsf{Rep}(\Pi^{\mathrm{N}}_{\mathcal{X}/L_\infty}) \simeq \mathrm{EFin}(\mathcal{T}_\infty(\mathcal{X}))$, where $\mathrm{EFin}(-)$ denote the full subcategory of essentially finite objects (see [BV, Def 7.7]). Via Theorem I we obtain the following Tannakian interpretation of the Nori fundamental gerbe, which extends the Tannakian interpretation in [BV, Theorem 7.9] to non pseudo-proper fibered categories.

Theorem II. *In the hypothesis of Theorem I assume moreover $\mathcal{X}$ reduced and inflexible. In situation (1) (resp. (2), (3)) of I we have a canonical equivalence of k-Tannakian categories:*

$$\mathsf{Rep}_k(\Pi^{\mathrm{N}}_{\mathcal{X}/k}) \simeq \mathrm{EFin}(\mathrm{Str}_\infty(\mathcal{X}))\ \textit{(resp.}\ \mathrm{EFin}(\mathrm{Crys}_\infty(\mathcal{X})),\ \mathrm{EFin}(\mathrm{Fdiv}_\infty(\mathcal{X})))$$

If $\mathcal{X}$ is inflexible over k and we apply the axiomatic theory to $\mathcal{X} \longrightarrow \mathcal{X}_{\mathcal{T}} = \Pi^{\mathrm{N,ét}}_{\mathcal{X}/k}$ we obtain $\mathsf{Rep}\Pi^{\mathrm{N}}_{\mathcal{X}/k} \simeq \mathcal{T}_{\infty}(\mathcal{X})$. In particular $\mathsf{Rep}(\Pi^{\mathrm{N}}_{\mathcal{X}/k})$ can be reconstructed from the map $\mathsf{Rep}(\Pi^{\mathrm{N,ét}}_{\mathcal{X}/k}) \longrightarrow \mathsf{Vect}(\mathcal{X})$ and the Frobenius pullback of those categories (see 5.14).

Finally we study the infinitesimal part of $\Pi^{\mathrm{N}}_{\mathcal{X}/k}$, that is its pro-local quotient $\Pi^{\mathrm{N,L}}_{\mathcal{X}/k}$ (see B.8), and give a concrete description of its representations in terms of vector bundles on $\mathcal{X}$: applying the axiomatic theory to $\mathcal{X} \longrightarrow \mathcal{X}_{\mathcal{T}} = \operatorname{Spec} k$ we have $\mathsf{Rep}\Pi^{\mathrm{N,L}}_{\mathcal{X}/k} \simeq \mathcal{T}_{\infty}(\mathcal{X})$ (see 7.1).

One of the main ingredient in the proofs of our results regarding the Nori gerbes is the use of a generalized version of Tannaka's duality that can be applied, not only to gerbes, but also to finite stacks. This version of Tannakian duality is discussed §1 in a great generality.

We outline the content of this paper. In the first section we descibe a generalization of classical Tannaka's duality, while in the second and third section we collect some useful results that will be used through all the paper. In section four we introduce different notions of Nori fundamental gerbes and discuss their existence. Section five contains the formalism and general results of the paper, while in section six we determine appropriate conditions under which $\mathrm{Str}(\mathcal{X})$, $\mathrm{Crys}(\mathcal{X})$ and $\mathrm{Fdiv}(\mathcal{X})$ satisfy the axiom of section five. In the last section we study the pro-local Nori fundamental gerbe. In the two appendices we study limit of categories and general results about affine gerbes respectively.

Notations and Conventions

Given a ring R we denote by Aff/R the category of affine R-schemes or, equivalently, the opposite of the category of R-algebras.

If $\mathcal{Z}$ and $\mathcal{Y}$ are categories over a given category $\mathcal{C}$, by a map $\mathcal{Z} \longrightarrow \mathcal{Y}$ we always mean a base preserving functor. Similarly given maps $F, G \colon \mathcal{Z} \longrightarrow \mathcal{Y}$ a natural transformation $\gamma \colon F \longrightarrow G$ will always be a base preserving natural transformation, that is for all $z \in \mathcal{Z}$ over an object $c \in \mathcal{C}$, the map $\gamma_z \colon F(z) \longrightarrow G(z)$ lies over id_c. If $\mathcal{Y}$ is a fibered category we will denote by $\mathrm{Hom}^c_{\mathcal{C}}(\mathcal{Z}, \mathcal{Y})$ the category of base preserving functors $\mathcal{Z} \longrightarrow \mathcal{Y}$ which send all arrows to Cartesian arrows and the maps are the base preserving natural transformations. If $\mathcal{Y}$ is a category fibered in groupoid then $\mathrm{Hom}^c_{\mathcal{C}}(\mathcal{Z}, \mathcal{Y})$ is the category of all base preserving functor and we will simply denote it by $\mathrm{Hom}_{\mathcal{C}}(\mathcal{Z}, \mathcal{Y})$. If $\mathcal{C} = \mathrm{Aff}/R$, where R is a base ring, we will simply write Hom^c_R or Hom^c if the base ring is clear from the context. If $\mathcal{Z}$ is a category over Aff/R the categories

$$\mathsf{Vect}(\mathcal{Z}) \subseteq \mathrm{QCoh}_{\mathrm{fp}}(\mathcal{Z}) \subseteq \mathrm{QCoh}(\mathcal{Z})$$

are defined as $\mathrm{Hom}^c_R(\mathcal{Z}, \mathsf{Vect}) \subseteq \mathrm{Hom}^c_R(\mathcal{Z}, \mathrm{QCoh}_{\mathrm{fp}}) \subseteq \mathrm{Hom}^c_R(\mathcal{Z}, \mathrm{QCoh})$, where $\mathsf{Vect} \subseteq \mathrm{QCoh}_{\mathrm{fp}} \subseteq \mathrm{QCoh}$ are the fiber categories (not in groupoids) over Aff/R of locally free sheaves of finite rank, quasi-coherent sheaves of finite presentation and quasi-coherent sheaves respectively. The categories $\mathsf{Vect}(\mathcal{Z})$, $\mathrm{QCoh}_{\mathrm{fp}}(\mathcal{Z})$ and $\mathrm{QCoh}(\mathcal{Z})$ are R-linear and monoidal categories. Notice that in $\mathrm{QCoh}_{\mathrm{fp}}(\mathcal{Z})$ and $\mathrm{QCoh}(\mathcal{Z})$ all maps have a cokernel (defined pointwise). If $\mathcal{Z} = \operatorname{Spec} B$ is affine we will simply write $\mathsf{Vect}(B)$, $\mathrm{QCoh}_{\mathrm{fp}}(B)$

and $\mathrm{QCoh}(B)$. The writing $\xi \in \mathcal{Z}(A)$ means that ξ is an object of $\mathcal{Z}$ over $\operatorname{Spec} A$, and if $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Z})$, we will denote by $\mathcal{F}_\xi \in \mathrm{QCoh}(A)$ the evaluation of $\mathcal{F}$ in ξ.

If $f\colon \mathcal{Y} \longrightarrow \mathcal{Z}$ is a base preserving map of categories over Aff/R then we have functors

$$f^*\colon \mathsf{Vect}(\mathcal{Z}) \longrightarrow \mathsf{Vect}(\mathcal{Y}),\ f^*\colon\ \mathrm{QCoh}_{\mathrm{fp}}(\mathcal{Z}) \longrightarrow \mathrm{QCoh}_{\mathrm{fp}}(\mathcal{Y}),\ f^*\colon\ \mathrm{QCoh}(\mathcal{Z}) \longrightarrow \mathrm{QCoh}(\mathcal{Y})$$

obtained simply by composing with f and they are R-linear and monoidal.

An fpqc covering $\mathcal{X} \longrightarrow \mathcal{Y}$ between categories fibered in groupoids is a functor representable by fpqc covering of algebraic spaces. A fibered category is called quasi-compact if it is fibered in groupoids and it admits an fpqc covering from an affine scheme. Let $\mathcal{X}$ and $\mathcal{Y}$ be categories fibered in groupoids. A map $f\colon \mathcal{X} \longrightarrow \mathcal{Y}$ is quasi-compact if $\mathcal{X} \times_{\mathcal{Y}} A$ is quasi-compact for all maps $\operatorname{Spec} A \longrightarrow \mathcal{Y}$, it is quasi-separated if its diagonal is quasi-compact. The category $\mathcal{X}$ is called quasi-separated is $\mathcal{X} \longrightarrow \operatorname{Spec} \mathbb{Z}$ is quasi-separated, which implies that all maps $\mathcal{X} \longrightarrow \mathcal{Y}$ are quasi-separated if $\mathcal{Y}$ has affine diagonal. If $f\colon \mathcal{X} \longrightarrow \mathcal{Y}$ is quasi-compact and quasi-separated and $\mathcal{X}$ and $\mathcal{Y}$ admit an fpqc covering from a scheme (resp. affine map between categories fibered in groupoids) then $f^*\colon\ \mathrm{QCoh}(\mathcal{Y}) \longrightarrow \mathrm{QCoh}(\mathcal{X})$ has a right adjoint $f_*\colon\ \mathrm{QCoh}(\mathcal{Y}) \longrightarrow \mathrm{QCoh}(\mathcal{X})$ which is compatible with flat base changes of $\mathcal{Y}$ (resp. any base change of $\mathcal{Y}$) (see [Ton, Prop 1.5 and Prop 1.7]).

Given a category fibered in groupoids $\mathcal{X}$ over $\mathrm{Aff}/\mathbb{F}_p$ we define the absolute Frobenius $F_\mathcal{X}$ of $\mathcal{X}$ as

$$F_\mathcal{X}\colon \mathcal{X} \longrightarrow \mathcal{X},\ \mathcal{X}(A) \ni \xi \longmapsto F_A^*\xi \in \mathcal{X}(A)$$

where $F_A\colon \operatorname{Spec} A \longrightarrow \operatorname{Spec} A$ is the absolute Frobenius of A. The Frobenius is $\mathbb{F}_p$-linear, natural in $\mathcal{X}$ and coincides with the usual Frobenius when $\mathcal{X}$ is a scheme. If $\mathcal{X}$ is defined over a field k of characteristic p we define $\mathcal{X}^{(i,k)} = \mathcal{X} \times_k k$, where $k \longrightarrow k$ is the i-th power of the absolute Frobenius of k, and we regard it as category over k using the second projection. For simplicity when k is clear from the context we will use just $-^{(i)}$ dropping the k. Notice that $(\mathcal{X}^{(i)})^{(j)}$ is canonically equivalent to $\mathcal{X}^{(i+j)}$. The i-th relative Frobenius of $\mathcal{X}$ is the k-linear map $\mathcal{X} \longrightarrow \mathcal{X}^{(i)}$ that, composed with the projection $\mathcal{X}^{(i)} \longrightarrow \mathcal{X}$, is the Frobenius $F^i_\mathcal{X}$. Notice that applying $-^{(j)}$ to the i-th Frobenius of $\mathcal{X}$ one obtains the i-th Frobenius of $\mathcal{X}^{(j)}$ and the composition of 1-th Frobenius

$$\mathcal{X} \longrightarrow \mathcal{X}^{(1)} \longrightarrow \cdots \longrightarrow \mathcal{X}^{(i)}$$

is the i-th Frobenius of $\mathcal{X}$. When $X = \operatorname{Spec} A$ we will also set $A^{(i)} = A \otimes_k k$, where $k \longrightarrow k$ is the i-th power of the absolute Frobenius of k, so that $X^{(i)} = \operatorname{Spec} A^{(i)}$.

All monoidal categories and functors considered will be symmetric unless specified otherwise.

Acknowledgement

We would like to thank B. Bhatt, H. Esnault, M. Olsson, M. Romagny and A. Vistoli for helpful conversations and suggestions received.

1. Tannaka's reconstruction and recognition

Definition 1.1. A *pseudo-abelian* category is an additive category $\mathcal{C}$ endowed with a collection $J_{\mathcal{C}}$ of sequences of the form $c' \longrightarrow c \longrightarrow c''$, where all objects and maps are in $\mathcal{C}$. A linear functor $\Phi\colon \mathcal{C} \longrightarrow \mathcal{D}$ of pseudo-abelian categories is said *exact* if it maps a sequence of $J_{\mathcal{C}}$ to a sequence isomorphic to one of $J_{\mathcal{D}}$.

Let R be a ring. If $\mathcal{X}$ is a category over Aff/R then $\mathsf{Vect}(\mathcal{X})$ will be consider pseudo-abelian with the collection of maps $\mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}''$ which are pointwise short exact sequences. If $\mathcal{C}$ is abelian it is also pseudo-abelian if endowed with its short exact sequences. If $\mathcal{C}$ and $\mathcal{D}$ are R-linear, monoidal and pseudo-abelian categories we denote by $\mathrm{Hom}_{\otimes,R}(\mathcal{C},\mathcal{D})$ the category whose objects are R-linear, exact and monoidal functors and whose arrows are natural monoidal isomorphisms. Notice that if $f\colon \mathcal{Y} \longrightarrow \mathcal{Z}$ is any base preserving map of categories over Aff/R then $f^* \in \mathrm{Hom}_{\otimes,R}(\mathsf{Vect}(\mathcal{Z}),\mathsf{Vect}(\mathcal{Y}))$.

Let $\mathcal{C}$ be a pseudo-abelian monoidal R-linear category. The expression

$$\Pi_{\mathcal{C}}(A/R) = \mathrm{Hom}_{\otimes,R}(\mathcal{C},\mathsf{Vect}(A))$$

defines a stack in groupoids for the fpqc topology over R. There is a functor

$$\mathcal{C} \longrightarrow \mathsf{Vect}(\Pi_{\mathcal{C}}),\ c \longmapsto (\Pi_{\mathcal{C}}(A) \ni \xi \longmapsto \xi(c) \in \mathsf{Vect}(A))$$

which is R-linear, monoidal and exact. This induces a natural functor

$$\mathrm{Hom}_R(\mathcal{Z},\Pi_{\mathcal{C}}) \longrightarrow \mathrm{Hom}_{\otimes,R}(\mathcal{C},\mathsf{Vect}(\mathcal{Z}))$$

for all categories $\mathcal{Z}$ over Aff/R, which is easily seen to be an equivalence.

We say that $\mathcal{C}$ satisfies *Tannakian recognition* if the functor $\Phi\colon \mathcal{C} \longrightarrow \mathsf{Vect}(\Pi_{\mathcal{C}})$ is an equivalence and for all sequences $\chi\colon c' \longrightarrow c \longrightarrow c''$ we have $\Phi(\chi)$ is exact if and only if $\chi \in J_{\mathcal{C}}$ (equivalently Φ has an R-linear, monoidal and exact quasi-inverse).

If $\mathcal{Y}$ is a category over Aff/R there is a base preserving functor $\mathcal{Y} \longrightarrow \Pi_{\mathsf{Vect}(\mathcal{Y})}$, namely

$$\eta \in \mathcal{Y}(A) \longmapsto (\mathsf{Vect}(\mathcal{Y}) \ni \Phi \longmapsto \Phi(\eta) \in \mathsf{Vect}(A))$$

We say that a category fibered in groupoids $\mathcal{Y}$ satisfies *Tannakian reconstruction* if the functor $\mathcal{Y} \longrightarrow \Pi_{\mathsf{Vect}(\mathcal{Y})}$ is an equivalence, or, equivalently, the pullback

$$\mathrm{Hom}^c_R(\mathcal{Z},\mathcal{Y}) \longrightarrow \mathrm{Hom}_{\otimes,R}(\mathsf{Vect}(\mathcal{Y}),\mathsf{Vect}(\mathcal{Z})),\ f \longmapsto f^*$$

is an equivalence for all categories $\mathcal{Z}$ over Aff/R (just apply $\mathrm{Hom}^c_R(\mathcal{Z},-)$ to the map $\mathcal{Y} \longrightarrow \Pi_{\mathsf{Vect}(\mathcal{Y})}$).

Remark 1.2. If $\mathcal{C}$ satisfies Tannakian recognition then $\Pi_{\mathcal{C}}$ satisfies Tannakian reconstruction and if $\mathcal{Y}$ satisfies Tannakian reconstruction then $\mathsf{Vect}(\mathcal{Y})$ satisfies Tannakian recognition. Notice also that those conditions do not depend on the base ring R. Indeed $\mathsf{Vect}(-)$ is insensible to the base ring and if $\mathcal{C}$ is a pseudo-abelian monoidal R-linear category then

$$\Pi_{\mathcal{C}} \longrightarrow \mathrm{Aff}/R \longrightarrow \mathrm{Aff}/\mathbb{Z}$$

coincides with $\Pi_{\mathcal{C}}$ where $\mathcal{C}$ is thought as a $\mathbb{Z}$-linear category.

Definition 1.3. Let $\mathcal{Z}$ be a category fibered in groupoids and $\mathcal{D} \subseteq \mathrm{QCoh}(\mathcal{Z})$ be a full subcategory. We say that $\mathcal{D}$ generates $\mathrm{QCoh}(\mathcal{Z})$ if any object of $\mathrm{QCoh}(\mathcal{Z})$ is a quotient of an arbitrary direct sum of objects of $\mathcal{D}$. We say that $\mathcal{Z}$ has the resolution property if $\mathsf{Vect}(\mathcal{Z})$ generates $\mathrm{QCoh}(\mathcal{Z})$.

Theorem 1.4. [Ton, Cor 5.4] *If $\mathcal{Z}$ is a quasi-compact stack for the fpqc topology over a ring R with quasi-affine diagonal and the resolution property then it satisfies Tannakian reconstruction.*

Example 1.5. let k be a field. Classical Tannaka's duality implies that: if $\mathcal{C}$ is a k-Tannakian category then it satisfies Tannakian recognition and $\Pi_{\mathcal{C}}$ is an affine gerbe (gerbes with affine diagonal) over k. Conversely if Π is an affine gerbe over k then it satisfies Tannakian reconstruction and $\mathsf{Vect}(\Pi)$ is a k-Tannakian category. More precisely Π has the resolution property (see [De3, Cor 3.9, pp. 132]).

Lemma 1.6. *Let $f\colon \mathcal{X} \longrightarrow \mathcal{Y}$ be a map of categories fibered in groupoids over R. If $\mathcal{D} \subseteq \mathrm{QCoh}(\mathcal{X})$ generates $\mathrm{QCoh}(\mathcal{X})$ and f is finite, faithfully flat and finitely presented then $f_*\mathcal{D} = \{f_*\mathcal{E} \mid \mathcal{E} \in \mathcal{D}\}$ generates $\mathrm{QCoh}(\mathcal{Y})$. If $\overline{\mathcal{D}} \subseteq \mathrm{QCoh}(\mathcal{Y})$ generates $\mathrm{QCoh}(\mathcal{Y})$ and f is affine then $f^*\overline{\mathcal{D}} = \{f^*\mathcal{H} \mid \mathcal{H} \in \overline{\mathcal{D}}\}$ generates $\mathrm{QCoh}(\mathcal{X})$.*

Proof. In the second case, if $\mathcal{F} \in \mathrm{QCoh}(\mathcal{X})$ then there is a surjective map $\bigoplus_j \mathcal{H}_j \longrightarrow f_*\mathcal{F}$ with $\mathcal{H}_j \in \overline{\mathcal{D}}$ and therefore a surjective map $\bigoplus_j f^*\mathcal{H}_j \longrightarrow f^*f_*\mathcal{F}$. Since f is affine the map $f^*f_*\mathcal{F} \longrightarrow \mathcal{F}$ is surjective.

Let's consider the first statement. Let $\mathcal{G} \in \mathrm{QCoh}(\mathcal{Y})$ and set $\mathcal{G}_{\mathcal{X}} = \mathcal{G} \otimes_{\mathcal{O}_{\mathcal{Y}}} \underline{\mathrm{Hom}}_{\mathcal{Y}}(f_*\mathcal{O}_{\mathcal{X}}, \mathcal{O}_{\mathcal{Y}})$. The map $\mathcal{O}_{\mathcal{Y}} \longrightarrow f_*\mathcal{O}_{\mathcal{X}}$, which is locally split injective, induces a surjective map

$$\underline{\mathrm{Hom}}_{\mathcal{Y}}(f_*\mathcal{O}_{\mathcal{X}}, \mathcal{O}_{\mathcal{Y}}) \longrightarrow \mathcal{O}_{\mathcal{Y}}$$

and therefore a surjective map $\mathcal{G}_{\mathcal{X}} \longrightarrow \mathcal{G}$. The sheaf $\mathcal{G}_{\mathcal{X}}$ is an $f_*\mathcal{O}_{\mathcal{X}}$-module, so there exists $\mathcal{G}' \in \mathrm{QCoh}(\mathcal{X})$ such that $f_*\mathcal{G}' \simeq \mathcal{G}_{\mathcal{X}}$. Thus, taking a surjection $\bigoplus_j \mathcal{E}_j \longrightarrow \mathcal{G}'$ with $\mathcal{E}'_j \in \mathcal{D}$ and using that f_* is exact we get the result. □

Corollary 1.7. *Let Γ be a finite stack over a field k (see 3.1). Then there exists $\mathcal{E} \in \mathsf{Vect}(\Gamma)$ which generates $\mathrm{QCoh}(\Gamma)$. In particular Γ has the resolution property and it satisfies Tannakian reconstruction.*

Proof. Apply 1.6 to a finite atlas $f\colon U \longrightarrow \Gamma$ with U finite k-scheme and $\mathcal{D} = \{\mathcal{O}_U\}$. □

2. Étale part and geometric connectedness

Through this section we consider given a field k.

Definition 2.1. Given a k-algebra A we set

$$A_{\text{ét},k} = \{a \in A \mid \exists \text{ a separable polynomial } f \in k[x] \text{ s.t. } f(a) = 0\}$$

Alternatively $A_{\text{ét},k}$ is the union of all k-subalgebras of A which are finite and étale over k. When the base field is clear from the context we will simply write $A_{\text{ét}}$.

Remark 2.2. Let A be a k-algebra of characteristic p. The i-th relative Frobenius of A is given by

$$f_i\colon A^{(i)} = A \otimes_k k^{(i)} \longrightarrow A,\ a \otimes \lambda \longmapsto a^p \lambda$$

and a direct computation shows that $x^{p^i} = f_i(x) \otimes 1$ for all $x \in A^{(i)}$. In particular $\mathrm{Ker} f_i = \{x \in A^{(i)} \mid x^{p^i} = 0\}$.

Moreover the map $(A^{(1)})_{\text{ét}} \longrightarrow A_{\text{ét}}$ is an isomorphism. Indeed denote by B the image of $A^{(1)} \longrightarrow A$. Since $A^{(1)} \longrightarrow B$ is surjective with nilpotent kernel the map $(A^{(1)})_{\text{ét}} \longrightarrow B_{\text{ét}}$ is an isomorphism. Since B contains all p-powers of A, we see that $A_{\text{ét}} = B_{\text{ét}}$.

Lemma 2.3. *Let A be a finite k-algebra of characteristic p. There exists $n \in \mathbb{N}$ such that the image of the relative Frobenius $A^{(n)} \longrightarrow A$ is an étale k-algebra. In particular the residue fields of $A^{(n)}$ are separable over k.*

Proof. We can assume that A is local with residue field L. Consider $n \in \mathbb{N}$ such that $p^n \geqslant \dim_k A = \dim_k A^{(n)}$. In particular the p^n-power of the maximal ideal of $A^{(n)}$ is zero. Taking into account 2.2 we see that the image of $A^{(n)} \longrightarrow A$ is the residue field of $A^{(n)}$, which also coincides with the residue field of $L^{(n)}$. If K is the maximal separable extension of k inside L we have that $x^{p^n} \in K$ for all $x \in L$. By 2.2 we see that the image of $L^{(n)} \longrightarrow L$ is contained in K and thus is separable over k. □

Definition 2.4. If $\mathcal{Y}$ and $\mathcal{Z}$ are categories fibered in groupoids we define $\mathcal{Y} \sqcup \mathcal{Z}$ as the category fibered in groupoid whose objects over an affine scheme U are tuples (U', U'', ξ, η) where U', U'' are open subsets of U such that $U = U' \sqcup U''$, $\xi \in \mathcal{Y}(U')$ and $\eta \in \mathcal{Z}(U'')$.

We say that a category fibered in groupoids $\mathcal{X}$ is connected if $\mathrm{H}^0(\mathcal{O}_{\mathcal{X}})$ has no non trivial idempotents. We say it is reduced if any map $U \longrightarrow \mathcal{X}$ from a scheme factors through a reduced scheme fpqc locally in U. We say that a morphism of categories fibered in groupoids $f\colon \mathcal{X} \longrightarrow \mathcal{Y}$ is geometrically connected (resp. reduced) if for all geometric points $\operatorname{Spec} L \longrightarrow \mathcal{Y}$ the fiber $\mathcal{X} \times_{\mathcal{Y}} L$ is connected (resp. reduced).

Remark 2.5. If $\mathcal{X}$ is a stack in groupoids for the Zariski topology and $\mathcal{Y}, \mathcal{Z}$ are open substacks then one can always define a map $\mathcal{Y} \sqcup \mathcal{Z} \longrightarrow \mathcal{X}$. In this situation $\mathcal{X}$ is connected if and only if it cannot be written as a disjoint union of non-empty open substacks.

If $\mathcal{X}$ is a reduced category fibered in groupoids then $\mathrm{H}^0(\mathcal{O}_{\mathcal{X}})$ is a reduced ring. Indeed if $\lambda \in \mathrm{H}^0(\mathcal{O}_{\mathcal{X}})$ then one can define the vanishing substack $\mathcal{Y} \longrightarrow \mathcal{X}$ of λ, so that $\mathcal{Y} \longrightarrow \mathcal{X}$ is a closed immersion which is also nilpotent if λ is so. Let's prove that $\mathcal{Y} = \mathcal{X}$, that is that if $U \longrightarrow \mathcal{X}$ is a map from a scheme then $U \times_{\mathcal{X}} \mathcal{Y} \longrightarrow U$ is an isomorphism. By fpqc descent and the definition of reduceness we reduce the problem to the case when U is reduced, where the result is clear.

If $\mathcal{X}$ is an algebraic stack then the notion of reduceness just defined and the classical one coincides.

Lemma 2.6. *Let $\mathcal{X}$ be a quasi-compact and quasi-separated fibered category over k. Then*

(1) for all field extensions L/k we have

$$\mathrm{H}^0(\mathcal{O}_{\mathcal{X}})_{\text{ét},k} \otimes_k L \simeq \mathrm{H}^0(\mathcal{O}_{\mathcal{X} \times_k L})_{\text{ét},L}$$

(2) the map $\mathcal{X} \longrightarrow \operatorname{Spec} \mathrm{H}^0(\mathcal{O}_{\mathcal{X}})_{\text{ét},k}$ *is geometrically connected;*
(3) the fiber category $\mathcal{X}$ *is geometrically connected over* k *if and only if* $\mathrm{H}^0(\mathcal{O}_{\mathcal{X}})_{\text{ét},k} = k$*.*

Proof. It is clear that 2) $\implies$ 3). Write $A = \mathrm{H}^0(\mathcal{O}_{\mathcal{X}})$ and notice that if $k \subseteq B \subseteq A_{\text{ét}}$ and C is any B-algebra then

$$\mathrm{H}^0(\mathcal{O}_{\mathcal{X}\times_B C}) \simeq A \otimes_B C$$

This follows from the fact that $\mathcal{X} \longrightarrow \operatorname{Spec} B$ is quasi-compact and quasi-separated, so that the notion of push-forward of quasi-coherent sheaves is well defined, and the fact that B is a Von Neumann regular ring, that is all B-modules are flat or, equivalently, all finitely generated ideals are generated by an idempotent: indeed B is a filtered direct limits of its k-etale and finite subalgebras, which are easily seen to be Von Neumann regular rings. This shows that we can assume $\mathcal{X} = \operatorname{Spec} A$ and work only with algebras.

Let's prove 1). We have an inclusion $A_{\text{ét},k} \otimes_k L \subseteq (A \otimes_k L)_{\text{ét},L}$. Given an element $u \in (A \otimes_k L)$ separable over L we must show that $u \in A_{\text{ét},k} \otimes_k L$. Since u can be written with finitely many elements of A and L and the same is true for the separable equation it satisfies, we can assume that A/k is of finite type and L/k is finitely generated. Moreover the result holds for the extension L/k if it holds for all subsequent subextensions in a finite filtration $k = k_0 \subseteq k_1 \subseteq \cdots \subseteq k_l = L$, or if it holds for L'/k, where $L \subseteq L'$, because of the inclusion

$$(A_{\text{ét},k} \otimes_k L) \otimes_L L' \subseteq (A \otimes_k L)_{\text{ét},L} \otimes_L L' \subseteq (A \otimes_k L')_{\text{ét},L'}$$

In conclusion the problem can be split in the following cases: L/k is finite and Galois; $L = k$ and $k \longrightarrow L$ is the Frobenius; k and L are algebraically closed: first assume L algebraically closed, then assume L/k algebraic using the splitting $k \subseteq \overline{k} \subseteq L$, then assume L/k finite and, finally, split in separable and purely inseparable extensions which are subextensions of a Galois extension and of a sequence of Frobenius extension respectively.

Assume first that L/k is finite and Galois with group $G = \operatorname{Gal}(L/k)$. The subalgebra $(A \otimes_k L)_{\text{ét},L}$ of $A \otimes_k L$ is invariant by the action of G and therefore, by Galois descent, we have

$$(A \otimes_k L)_{\text{ét},L} \simeq (A \otimes_k L)^G_{\text{ét},L} \otimes_k L$$

Since $(A \otimes_k L)_{\text{ét},L}$ is etale over L and therefore over k and $(A \otimes_k L)^G_{\text{ét},L} = (A \otimes_k L)_{\text{ét},L} \cap A$ we obtain the result.

Assume now that $L = k$ and $k \longrightarrow L$ is the Frobenius. We have a commutative diagram of k-linear maps

$$\begin{tikzcd}
A_{\text{ét},k} \otimes_k L \arrow[r, "\delta"] \arrow[d, "\gamma_{A_{\text{ét},k}}"] & (A \otimes_k L)_{\text{ét},L} \arrow[r] \arrow[dr, "\mu"] & A \otimes_k L \arrow[d, "\gamma_A"] \\
A_{\text{ét},k} \arrow[rr] & & A
\end{tikzcd}$$

where the γ_* are the relative Frobenius. We must show that δ is an isomorphism. By 2.2 μ and $\gamma_{A_{\text{ét},k}}$ are injective because $(A \otimes_k L)_{\text{ét},L}$ is reduced. By dimension it also follows that $\gamma_{A_{\text{ét},k}}$ is an isomorphism. Since δ is étale and $A_{\text{ét}}$ is the maximum k-étale subalgebra of A it follows that δ is an isomorphism.

Assume now that k and L are algebraically closed. Notice that in this situation A is connected if and only if $A_{\text{ét},k} = k$. Decomposing A into connected components we can assume that A is connected. Since k and L are algebraically closed it follows that also $A \otimes_k L$ is connected and therefore that $(A \otimes_k L)_{\text{ét},L} = L$.

Let's now prove 2). Let $\alpha\colon A_{\text{ét},k} \longrightarrow L$ be a geometric point. We must prove that $(A \otimes_{A_{\text{ét},k}} L)_{\text{ét},L} = L$. Let J be the kernel of α and F be its image, which is easily seen to be a field. Thanks to 1) it is sufficient to prove that $(A \otimes_{A_{\text{ét},k}} F)_{\text{ét},F} = (A/JA)_{\text{ét},F}$ is just F. Let $a \in A$ be such that its quotient lies in $(A/JA)_{\text{ét},F}$. Lifting also a separable equation satisfied by a mod J to $A_{\text{ét},k}$, we can again assume that A is of finite type over k and, moreover, that it is connected. In this case $A_{\text{ét},k}$ is just a field, thus equal to F and the result is obvious. □

3. Some results on finite stacks

Let k be a field. In this section we collect some results about finite stacks that will be used later. For many other properties look at [BV, Section 4].

Definition 3.1. A finite (resp. finite étale) stack Γ over a field k is a stack in the fppf topology on Aff/k which has a finite (resp. finite étale) and faithfully flat morphism $U \longrightarrow \Gamma$ from a finite (resp. finite étale) k-scheme U. Equivalently Γ is the quotient of a flat groupoid of finite (resp. finite étale) k-schemes.

Here is a non-trivial application of the Tannaka's duality discussed in §1 which generalize [BV, Prop 4.3].

Proposition 3.2. *If Γ is a finite and reduced stack over k then $\Gamma \longrightarrow \operatorname{Spec} \mathrm{H}^0(\mathcal{O}_\Gamma)$ is a gerbe.*

Proof. We can assume Γ connected, so that $L = \mathrm{H}^0(\mathcal{O}_\Gamma)$ is a field. Set $\mathcal{C} = \mathsf{Vect}(\Gamma)$. Since Γ is Tannakian reconstructible by 1.7, the functor $\Gamma \longrightarrow \Pi_\mathcal{C}$, which is an L-map, is an equivalence. By [BV, Lemma 7.15] we have $\mathcal{C} = \mathrm{QCoh}_{\mathrm{fp}}(\Gamma)$, which easily implies that $\mathcal{C}$ is an L-Tannakian category and therefore $\Pi_\mathcal{C}$ is a gerbe over L. □

Lemma 3.3. *If L/k is an algebraic extension of fields and Γ is a finite stack over L then there exists a finite subextension F/k, a finite stack Δ over F with an isomorphism $\Gamma \simeq \Delta \times_F L$.*

Proof. The stack Γ is the quotient of a groupoid $s, t\colon R \rightrightarrows U$, where R, U are spectra of finite L-algebras and s, t are faithfully flat. Since everything is of finite presentation, we can descend the groupoid $R \rightrightarrows U$ to a finite sub-extenison F/k, thus also Γ. □

Lemma 3.4. *Let $R \rightrightarrows U$ be a flat groupoid with R and U finite over k. Then $(R \times_{s,t,U} R)_{\text{ét}} = R_{\text{ét}} \times_{s,t,U_{\text{ét}}} R_{\text{ét}}$, the maps defining the groupoid $R \rightrightarrows U$ yields a structure of groupoid on $R_{\text{ét}} \rightrightarrows U_{\text{ét}}$ with a map from $R \rightrightarrows U$. Moreover if the residue fields of R and U are separable over k, the same holds for $(-)_{\text{red}}$ in place of $(-)_{\text{ét}}$ and the resulting groupoids are the same, where $(-)_{\text{red}}$ is the functor which takes, for any scheme X, its reduced closed subscheme structure.*

Proof. Using 2.3 and 2.2, we can Frobenius twist the original groupoid until R and U has separable residue fields, that is their reduced structures are étale. In this case $R_{\mathrm{red}} \longrightarrow R \longrightarrow R_{\text{ét}}$ is an isomorphism and similarly for U. The result follows by expressing a groupoid in terms of commutative and Cartesian diagrams and using the following fact: if V, W, Z are finite k-scheme whose reduced structures are étale and $V, W \longrightarrow Z$ are maps then

$$(V \times_Z W)_{\mathrm{red}} = V_{\mathrm{red}} \times_{Z_{\mathrm{red}}} W_{\mathrm{red}} = V_{\text{ét}} \times_{Z_{\text{ét}}} W_{\text{ét}} = (V \times_Z W)_{\text{ét}}$$

The above equalities follows because a product of étale schemes is étale and thus reduced. □

Definition 3.5. Let Γ be a finite stack over k and let $U \longrightarrow \Gamma$ be a finite atlas where U is affine. We define $\Gamma_{\text{ét},k}$ as the quotient of the groupoid constructed in 3.4 with respect to the groupoid $R = U \times_\Gamma U \rightrightarrows U$. When k is clear from the context we will drop the $-_k$. By 3.6 below this notion does not depend on the choice of the finite atlas.

Lemma 3.6. *Let Γ/k be a finite stack and E/k be a finite and étale stack. Then the functor* $\mathrm{Hom}_k(\Gamma_{\text{ét}}, E) \longrightarrow \mathrm{Hom}_k(\Gamma, E)$ *is an equivalence. Moreover for all $j \in \mathbb{N}$ the map $\Gamma_{\text{ét}} \longrightarrow (\Gamma^{(j)})_{\text{ét}}$ is an equivalence and for $j \gg 0$ the functor $\Gamma^{(j)} \longrightarrow (\Gamma^{(j)})_{\text{ét}}$ has a section. In particular for $j \gg 0$ the relative Frobenius $\Gamma \longrightarrow \Gamma^{(j)}$ factors through $\Gamma_{\text{ét}}$.*

Proof. The second part follows from 2.3 and 2.2. For the first part is enough to show that, if U is a finite k-scheme, then $E(U_{\text{ét}}) \longrightarrow E(U)$ is an equivalence. Since $U \longrightarrow U_{\text{ét}}$ is finite, flat and geometrically connected by 2.6, it follows that the diagonal $U \longrightarrow R = U \times_{U_{\text{ét}}} U$ is a nilpotent closed immersion, so that $E(R) \longrightarrow E(U)$ and the two maps $E(U) \rightrightarrows E(R)$ induced by the projections $R \rightrightarrows U$ are equivalences. Computing $E(U_{\text{ét}})$ on the flat groupoid $R \rightrightarrows U$ we get the result. □

Remark 3.7. If Γ is a finite stack over k and L/k is a field extension then, by 2.6 and the definition of $\Gamma_{\text{ét},k}$, we have $\Gamma_{\text{ét},k} \times_k L \simeq (\Gamma \times_k L)_{\text{ét},L}$.

Remark 3.8. If Γ is a finite stack over F and F/k is a finite and purely inseparable field extension then the natural morphism $\Gamma_{\text{ét},F} \longrightarrow \Gamma_{\text{ét},k} \times_k F \cong (\Gamma \times_k F)_{\text{ét},F}$ is an equivalence. Indeed $\Gamma \to \Gamma \times_k F$ is the base change of the diagonal of $\mathrm{Spec}\,(F)$ by $\Gamma_{\text{ét},F}$, thus it is a nilpotent thickening, so it induces an equivalence on the étale quotients.

Definition 3.9. A finite stack Γ over k is called *local* if $\Gamma_{\text{ét},k} = \mathrm{Spec}\, k$.

Remark 3.10. A closed substack of a finite and local stack is always local. Indeed if Δ is a closed substack of a finite and local stack Γ then, since Γ is connected and thus topologically a point, the map $\Delta \longrightarrow \Gamma$ is a nilpotent closed immersion: using the definition of the étale part from a presentation follows that $\Delta_{\text{ét}} = \Gamma_{\text{ét}}$.

4. Nori fundamental gerbes

Definition 4.1. [BV, Section 5] If $\mathcal{Z}$ is a category over Aff/k the Nori fundamental gerbe (resp. étale Nori fundamental gerbe, local Nori fundamental gerbe) of $\mathcal{Z}/k$ is a profinite (resp. pro-étale, pro-local) gerbe Π over k together with a map $\mathcal{Z} \longrightarrow \Pi$ such that for all finite (resp. finite and étale, finite and local) stacks Γ over k the pullback functor

$$\mathrm{Hom}_k(\Pi, \Gamma) \longrightarrow \mathrm{Hom}_k(\mathcal{Z}, \Gamma)$$

is an equivalence. If this gerbe exists it is unique up to a unique isomorphism and it will be denoted by $\Pi^{\mathrm{N}}_{\mathcal{Z}/k}$ (resp. $\Pi^{\mathrm{N,\acute{e}t}}_{\mathcal{Z}/k}$, $\Pi^{\mathrm{N,L}}_{\mathcal{Z}/k}$) or by dropping the $/k$ if it is clear from the context.

Remark 4.2. If $\mathcal{Z}$ is a category fibered in groupoids over k a Nori gerbe exists over k if and only if $\mathcal{Z}$ is inflexible over k, that is all maps from $\mathcal{Z}$ to a finite stack over k factors through an affine gerbe over k (see [BV, Definition 5.3 and Theorem 5.7]). This is the case if $\mathcal{Z}$ is an affine gerbe over k. Moreover if $\mathcal{Z}$ is inflexible also the étale and local Nori gerbe exist, $\Pi^{\mathrm{N,\acute{e}t}}_{\mathcal{Z}} = (\Pi^{\mathrm{N}}_{\mathcal{Z}})_{\mathrm{\acute{e}t}}$ and $\Pi^{\mathrm{N,L}}_{\mathcal{Z}} = (\Pi^{\mathrm{N}}_{\mathcal{Z}})_{\mathrm{L}}$ (use 3.10 for the local case).

The following result, although not stated elsewhere, is known by experts.

Proposition 4.3. *Let $\mathcal{Z}$ be a quasi-compact and quasi-separated fibered category. Then $\mathcal{Z}$ admits a Nori étale fundamental gerbe if and only if $\mathcal{Z}$ is geometrically connected over k.*

Proof. Assume a Nori étale fundamental gerbe exists. If $k \subseteq A \subseteq \mathrm{H}^0(\mathcal{O}_{\mathcal{Z}})$ with A/k étale, then by definition $\mathcal{Z} \longrightarrow \operatorname{Spec} A$ factors through $\Pi^{\mathrm{N,\acute{e}t}}_{\mathcal{Z}}$. Since $\mathrm{H}^0(\mathcal{O}_{\Pi^{\mathrm{N,\acute{e}t}}_{\mathcal{Z}}}) = k$, the factorization tells us that $A \longrightarrow \mathrm{H}^0(\mathcal{O}_{\mathcal{Z}})$ factors through k. Thus $\mathrm{H}^0(\mathcal{O}_{\mathcal{Z}})_{\mathrm{\acute{e}t}} = k$ and $\mathcal{Z}$ is geometrically connected by 2.6.

Assume now $\mathcal{Z}$ geometrically connected. The proof of the existence of $\Pi^{\mathrm{N,\acute{e}t}}_{\mathcal{Z}}$ follows the same proof given in [BV, Proof of Theorem 5.7]. In our case I is the 2-category of Nori reduced maps $\mathcal{Z} \longrightarrow \Delta$ where Δ is an étale gerbe. The only thing that must be check is that if $\mathcal{Z} \xrightarrow{f} \Gamma$ is a map to a finite and étale stack then there exists a factorization $\mathcal{Z} \xrightarrow{\alpha} \Delta \longrightarrow \Gamma$ where Δ is an étale gerbe and α is Nori reduced. Consider $\Delta' = \operatorname{Spec}(\mathcal{O}_\Gamma/\mathcal{I})$ where $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_\Gamma \longrightarrow f_*\mathcal{O}_{\mathcal{Z}})$. The stack Δ' is finite, étale and $\mathrm{H}^0(\mathcal{O}_{\Delta'})$ is étale over k and contained in $\mathrm{H}^0(\mathcal{O}_{\mathcal{Z}})$, thus equal to k. So Δ'/k is a gerbe thanks to 3.2. The map $\mathcal{Z} \longrightarrow \Delta'$ factors through a Nori reduced map $\mathcal{Z} \xrightarrow{\alpha} \Delta$, where Δ is a finite gerbe, and $\Delta \longrightarrow \Delta'$ is faithful. It follows that Δ is étale because faithfulness means that the map on stabilizers is injective. □

The following result generalize [BV, Proposition 5.5]

Theorem 4.4. *Let $\mathcal{Z}$ be a reduced, quasi-compact and quasi-separated fibered category. Then $\mathcal{Z}$ is inflexible if and only if k is integrally closed inside $\mathrm{H}^0(\mathcal{O}_{\mathcal{Z}})$.*

Proof. The only if part is [BV, Prop 5.4, a)]. For the if part consider a map $f \colon \mathcal{Z} \longrightarrow \Gamma$ where Γ is a finite stack. If $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_\Gamma \longrightarrow f_*\mathcal{O}_{\mathcal{Z}})$ then f factors through $\operatorname{Spec}(\mathcal{O}_\Gamma/\mathcal{I})$, so that we can assume $\mathcal{O}_\Gamma \longrightarrow f_*\mathcal{O}_{\mathcal{Z}}$ injective. So Γ is reduced, finite and $\mathrm{H}^0(\mathcal{O}_\Gamma)$ is a subalgebra of $\mathrm{H}^0(\mathcal{O}_{\mathcal{Z}})$ finite over k, thus equal to k by our assumption. By 3.2 it follows that Γ is a finite gerbe. □

5. Formalism for algebraic and Nori fundamental gerbes

Let k be a field and consider two categories $\mathcal{X}$ and $\mathcal{X}_\mathcal{T}$ over Aff/k together with a base preserving functor $\pi_\mathcal{T} : \mathcal{X} \longrightarrow \mathcal{X}_\mathcal{T}$.

Definition 5.1. Set $\mathcal{T}_k(\mathcal{X}) = \mathsf{Vect}(\mathcal{X}_\mathcal{T})$, which is a pseudo-abelian, rigid, monoidal and k-linear category. Moreover the functor $\pi_\mathcal{T}^*\colon \mathcal{T}_k(\mathcal{X}) \longrightarrow \mathsf{Vect}(\mathcal{X})$ is k-linear, monoidal and exact. More generally if $\mathcal{Y}$ is a fibered category over Aff/k we have a natural functor

$$\mathrm{Hom}_k^c(\mathcal{X}_\mathcal{T}, \mathcal{Y}) \longrightarrow \mathrm{Hom}_k^c(\mathcal{X}, \mathcal{Y})$$

By 1.1 $\Pi_{\mathcal{T}_k(\mathcal{X})}$ comes equipped with a k-map $\mathcal{X}_\mathcal{T} \longrightarrow \Pi_{\mathcal{T}_k(\mathcal{X})}$ inducing $\mathrm{id}\colon \mathcal{T}_k(\mathcal{X}) \longrightarrow \mathsf{Vect}(\mathcal{X}_\mathcal{T})$. We will drop the $-_k$ when k is clear from the context.

We consider categories over k instead of just fibered categories over k in order to apply this theory also to categories $\mathcal{X}$ like small sites of algebraic stacks.

We now introduce a list of axioms that will ensure nice Tannakian properties of $\mathcal{T}(\mathcal{X})$. In what follows by a finite (étale) stack over a ring R we mean a stack which is an fppf quotient of an fppf groupoid of finite (étale), faithfully flat and finitely presented R-schemes.

Axioms 5.2. Set $L = \mathrm{H}^0(\mathcal{O}_{\mathcal{X}_\mathcal{T}}) = \mathrm{End}_{\mathcal{T}(X)}(1_{\mathcal{T}(\mathcal{X})})$ and consider:

- A: $\mathcal{T}(\mathcal{X}) = \mathrm{QCoh}_{\mathrm{fp}}(\mathcal{X}_\mathcal{T})$;
- B: the functor $\mathcal{T}(\mathcal{X}) \longrightarrow \mathsf{Vect}(\mathcal{X})$ is faithful;
- C: for all finite and étale stack Γ over L the following functor is an equivalence

$$\mathrm{Hom}_L(\mathcal{X}_\mathcal{T}, \Gamma) \longrightarrow \mathrm{Hom}_L(\mathcal{X}, \Gamma)$$

- D: all L-maps from $\mathcal{X}_\mathcal{T}$ to a finite gerbe over L factors trough a finite and étale gerbe over L.

Remark 5.3. If $\mathrm{char}\, k = 0$ and $L = \mathrm{H}^0(\mathcal{O}_{\mathcal{X}_\mathcal{T}})$ is a field then axiom D is automatic, because all finite gerbes are also étale.

Lemma 5.4. *Assume axiom A. Then $\mathcal{T}(\mathcal{X})$ is a k-linear, abelian, monoidal and rigid category and the exact sequences are pointwise exact.*

Proof. We already know that $\mathcal{T}(\mathcal{X})$ is k-linear, rigid and monoidal. In the category $\mathrm{QCoh}_{\mathrm{fp}}(\mathcal{X}_\mathcal{T})$ cokernel can be taken pointwise. The result then follows because if $\alpha\colon \mathcal{F} \longrightarrow \mathcal{G}$ is a map of locally free sheaves over $\mathrm{Spec}\,(R)$ whose cokernel is locally free, then $\mathrm{Ker}(\alpha)$ is locally free and the formation of the kernel commutes with arbitrary base change. □

Remark 5.5. If $\mathcal{C}$ is a k-linear and monoidal category and $R = \mathrm{End}_\mathcal{C}(1_\mathcal{C})$ then $\mathcal{C}$ has a natural structure of R-linear category: if $\lambda \in R$ and $\phi\colon x \longrightarrow y$ is a morphism in $\mathcal{C}$ we define

$$\lambda\phi\colon x \simeq x \otimes 1_\mathcal{C} \xrightarrow{\phi\otimes\lambda} y \otimes 1_\mathcal{C} \simeq y$$

Lemma 5.6. *Let $\mathcal{C}$ be a k-linear, rigid, abelian and monoidal category and let $F\colon \mathcal{C} \longrightarrow \mathsf{Vect}(\mathcal{Z})$, where $\mathcal{Z}$ is a non-empty category over Aff/k, be a k-linear, exact and monoidal functor. If $\mathcal{Z}$ is connected and F is faithful then $\mathrm{End}_\mathcal{C}(1_\mathcal{C})$ is a field. If $L = \mathrm{End}_\mathcal{C}(1_\mathcal{C})$ is a field then $\mathcal{C}$, with its natural L-linear structure, is an L-Tannakian category and F is faithful. In particular $\mathcal{C}$ is Tannakian recognizable and $\Pi_\mathcal{C}$ is an affine gerbe over L.*

Proof. Assume $\mathcal{Z}$ connected, F faithful and set $R = \mathrm{End}_{\mathcal{C}}(1_{\mathcal{C}})$. Let's show that it is a field proving that if $\alpha \in R$ is non zero then it is invertible in R. Since $\mathcal{C}$ is abelian consider the exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow 1_{\mathcal{C}} \xrightarrow{\alpha} 1_{\mathcal{C}} \longrightarrow \mathcal{Q} \longrightarrow 0$$

Since F is exact, $F(\alpha)$ is an element of $\mathrm{End}(\mathcal{O}_{\mathcal{Z}}) = \mathrm{H}^0(\mathcal{O}_{\mathcal{Z}})$ whose kernel and cokernel are locally free. Thus for all $\xi \in \mathcal{Z}$, we get a natural decomposition $\mathrm{Spec}\,((\mathcal{O}_{\mathcal{Z}})_\xi) = U_\xi \sqcup V_\xi$ where U_ξ, V_ξ are the opens where $F(\alpha)_\xi$ is invertible and 0 respectively. This determines an idempotent $e \in \mathrm{H}^0(\mathcal{O}_{\mathcal{Z}})$. Since this ring is connected by hypothesis then $e = 0$ or $e = 1$, that is one of the following situations occur: $F(\alpha) = 0$ so that $\alpha = 0$ since F is faithful; $F(\alpha)$ is an isomorphism, so that $F(\mathcal{K}) = F(\mathcal{Q}) = 0$ and, again by faithfulness of F, $\mathcal{K} = \mathcal{Q} = 0$, which implies that α is an isomorphism. Thus $R = L$ is a field.

Assume now $L = \mathrm{End}_{\mathcal{C}}(1_{\mathcal{C}})$ is a field. Since $\mathcal{Z}$ is non empty there exists $\xi \in \mathcal{Z}(A)$ for some k-algebra A and the functor

$$G : \mathcal{C} \xrightarrow{F} \mathsf{Vect}(\mathcal{Z}) \xrightarrow{-_\xi} \mathsf{Vect}(A)$$

is k-linear, exact and monoidal. The L-linear structure on $\mathcal{C}$ induces an L-algebra structure on A such that G is L-linear. From [De3, 1.9, pp. 114] it follows that $\mathcal{C}$ is an L-Tannakian category and G is faithful. In particular also F is faithful. $\square$

As a consequence of 5.4 and 5.6 we obtain:

Proposition 5.7. *Assume axiom A and set $L = \mathrm{H}^0(\mathcal{O}_{\mathcal{X}_{\mathcal{T}}})$. If $\mathcal{X}$ is connected and axiom B holds then L is a field. If L is a field then axiom B holds, $\mathcal{T}(\mathcal{X})$ is an L-Tannakian category, $\Pi_{\mathcal{T}(\mathcal{X})}$ is an affine gerbe over L and $\mathcal{X} \longrightarrow \mathcal{X}_{\mathcal{T}} \longrightarrow \Pi_{\mathcal{T}(\mathcal{X})}$ can be considered as L-maps.*

Theorem 5.8. *Assume axioms A and that $L = \mathrm{H}^0(\mathcal{O}_{\mathcal{X}_{\mathcal{T}}})$ is a field (for instance if B holds and $\mathcal{X}$ is connected), so that $\mathcal{T}(\mathcal{X})$ is an L-Tannakian category by 5.7, where $L = \mathrm{H}^0(\mathcal{O}_{\mathcal{X}_{\mathcal{T}}})$. If $R \longrightarrow L$ is a map of rings and Γ is any stack in groupoids over R satisfying Tannakian reconstruction, then the functor*

$$\mathrm{Hom}_R(\Pi_{\mathcal{T}(\mathcal{X})}, \Gamma) \longrightarrow \mathrm{Hom}_R(\mathcal{X}_{\mathcal{T}}, \Gamma)$$

is an equivalence. In particular $\mathcal{X}_{\mathcal{T}} \longrightarrow \Pi_{\mathcal{T}(\mathcal{X})}$ is universal among L-maps from $\mathcal{X}_{\mathcal{T}}$ to an affine gerbe over L.

If axiom C also holds then $\mathcal{X} \longrightarrow (\Pi_{\mathcal{T}(\mathcal{X})})_{\text{ét}}$ is the étale Nori fundamental gerbe of $\mathcal{X}$ over L, so that $\mathsf{Rep}(\Pi^{\mathrm{N,ét}}_{\mathcal{X}/L}) \simeq \text{Ét}(\mathcal{T}(\mathcal{X}))$ (see B.8).

If both axioms C and D also holds, then $\widehat{\Pi}_{\mathcal{T}(\mathcal{X})} = (\Pi_{\mathcal{T}(\mathcal{X})})_{\text{ét}}$, so that $\mathsf{Rep}\Pi^{\mathrm{N,ét}}_{\mathcal{X}/L} \simeq \mathrm{EFin}(\mathcal{T}(\mathcal{X}))$ (see B.8).

Proof. Since $\mathcal{T}(\mathcal{X})$ and Γ are Tannakian recognizable and reconstructible respectively, we have equivalences

$$\mathrm{Hom}_R(\Pi_{\mathcal{T}(\mathcal{X})}, \Gamma) \simeq \mathrm{Hom}_{\otimes,R}(\mathsf{Vect}(\Gamma), \mathcal{T}(\mathcal{X})) \simeq \mathrm{Hom}_R(\mathcal{X}_{\mathcal{T}}, \Gamma)$$

The above map is easily seen to coincide with the map induced by $\mathcal{X}_{\mathcal{T}} \longrightarrow \Pi_{\mathcal{T}(\mathcal{X})}$. Since affine gerbes satisfies Tannakian reconstruction we get the universality of $\mathcal{X}_{\mathcal{T}} \longrightarrow \Pi_{\mathcal{T}(\mathcal{X})}$.

Assume now C. Since finite stacks are Tannakian reconstructible by 1.7, for all finite and étale stacks Γ the maps $\mathcal{X} \longrightarrow \mathcal{X}_{\mathcal{T}} \longrightarrow \Pi_{\mathcal{T}(\mathcal{X})} \longrightarrow (\Pi_{\mathcal{T}(\mathcal{X})})_{\text{ét}}$ induces equivalences

$$\mathrm{Hom}_L((\Pi_{\mathcal{T}(\mathcal{X})})_{\text{ét}}, \Gamma) \simeq \mathrm{Hom}_L(\Pi_{\mathcal{T}(\mathcal{X})}, \Gamma) \simeq \mathrm{Hom}_L(\mathcal{X}_{\mathcal{T}}, \Gamma) \simeq \mathrm{Hom}_L(\mathcal{X}, \Gamma)$$

as desired, where the first equivalence follows because $(\Pi_{\mathcal{T}(\mathcal{X})})_{\text{ét}}$ is the Nori étale quotient of $\Pi_{\mathcal{T}(\mathcal{X})}$ (see 4.2). Finally axiom D tells exactly that a morphism from $\Pi_{\mathcal{T}(\mathcal{X})}$ to a finite stack factors through an étale stack, which implies the result. $\square$

From now on we assume that k has positive characteristic p. If $\mathcal{Z}$ is any category over Aff/k we define the Frobenius pullback

$$F^*\colon \mathsf{Vect}(\mathcal{Z}) \longrightarrow \mathsf{Vect}(\mathcal{Z})$$

applying the pullback of the absolute Frobenius pointwise. The functor F^* is $\mathbb{F}_p$-linear, exact and monoidal.

Definition 5.9. Given $i \in \mathbb{N}$ we define $\mathcal{T}_i(\mathcal{X})$ as the category of tuples $(\mathcal{F}, \mathcal{G}, \lambda)$ where $\mathcal{F} \in \mathsf{Vect}(\mathcal{X})$, $\mathcal{G} \in \mathcal{T}(\mathcal{X})$ and $\lambda\colon F^{i*}\mathcal{F} \longrightarrow \mathcal{G}_{|\mathcal{X}}$ is an isomorphism. A morphism from $(\mathcal{F}, \mathcal{G}, \lambda)$ to $(\mathcal{F}', \mathcal{G}', \lambda')$ is a pair of morphisms $\phi : \mathcal{F} \to \mathcal{F}'$ and $\varphi : \mathcal{G} \to \mathcal{G}'$ which are compatible with λ and λ' in an obvious way. The category $\mathcal{T}_i(\mathcal{X})$ is $\mathbb{F}_p$-linear, monoidal and rigid. We endow $\mathcal{T}_i(\mathcal{X})$ with a k-structure via

$$k \longrightarrow \mathrm{End}_{\mathcal{T}_i(\mathcal{X})}(\mathcal{O}_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}_{\mathcal{T}}}, \mathrm{id}_{\mathcal{O}_{\mathcal{X}}}),\ \lambda \longmapsto (\lambda, \lambda^{p^i})$$

Finally we regard $\mathcal{T}_i(\mathcal{X})$ as a pseudo-abelian category with the distinguished set of sequences which are exact pointwise. The forgetful functor $\mathcal{T}_i(\mathcal{X}) \longrightarrow \mathsf{Vect}(\mathcal{X})$ is k-linear, monoidal and exact.

There is a k-linear, monoidal and exact functor

$$\mathcal{T}_i(\mathcal{X}) \longrightarrow \mathcal{T}_{i+1}(\mathcal{X}),\ (\mathcal{F}, \mathcal{G}, \lambda) \longmapsto (\mathcal{F}, F^*\mathcal{G}, F^*\lambda)$$

We define $\mathcal{T}_\infty(\mathcal{X})$ as the direct limit of the categories $\mathcal{T}_i(\mathcal{X})$. The category $\mathcal{T}_\infty(\mathcal{X})$ is k-linear, monoidal and rigid.

Remark 5.10. Given a category fibered in groupoids Γ over $\mathbb{F}_p$ we denote by $\mathrm{Hom}(\mathcal{X}, \mathcal{X}_{\mathcal{T}}, i, \Gamma)$ the category of $\mathbb{F}_p$-linear 2-commutative diagrams

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\pi"] \arrow[d, "f"] & \mathcal{X}_{\mathcal{T}} \arrow[d, "g"] \\ \Gamma \arrow[r, "F_\Gamma"] & \Gamma \end{tikzcd}$$

where F_Γ is the absolute Frobenius. Pulling back along f and g one obtains a functor $\Phi_i^\Gamma\colon \mathrm{Hom}(\mathcal{X}, \mathcal{X}_{\mathcal{T}}, i, \Gamma) \longrightarrow \mathrm{Hom}_{\otimes, \mathbb{F}_p}(\mathsf{Vect}(\Gamma), \mathcal{T}_i(\mathcal{X}))$ which is an equivalence if Γ is Tannakian reconstructible. On the other hand using the universal property of Π_* in 1.1 and the definition of $\mathcal{T}_i(\mathcal{X})$ we see that $\Pi_{\mathcal{T}_i(\mathcal{X})}$ comes equipped with a 2-commutative diagram $\chi \in \mathrm{Hom}(\mathcal{X}, \mathcal{X}_{\mathcal{T}}, i, \Pi_{\mathcal{T}_i(\mathcal{X})})$ such that $\mathcal{T}_i(\mathcal{X}) \longrightarrow \mathsf{Vect}(\Pi_{\mathcal{T}_i(\mathcal{X})}) \xrightarrow{J_i} \mathcal{T}_i(\mathcal{X})$ is the identity, where $J_i = \Phi_i^{\Pi_{\mathcal{T}_i(\mathcal{X})}}(\chi)$. Composing with χ we obtain a functor

$$\mathrm{Hom}_{\mathbb{F}_p}(\Pi_{\mathcal{T}_i(\mathcal{X})}, \Gamma) \longrightarrow \mathrm{Hom}(\mathcal{X}, \mathcal{X}_{\mathcal{T}}, i, \Gamma)$$

which is an equivalence if Γ and $\mathcal{T}_i(\mathcal{X})$ are Tannakian reconstructible and recognizable respectively.

Lemma 5.11. *There is a 2-commutative diagram*

$$\begin{tikzcd} \mathsf{Vect}(\Pi_{\mathcal{T}_i(\mathcal{X})}) \arrow[r, "J_i"] \arrow[d, "F^*"] & \mathcal{T}_i(\mathcal{X}) \arrow[d, "F_{\mathcal{T}}"] \\ \mathsf{Vect}(\Pi_{\mathcal{T}_i(\mathcal{X})}) \arrow[r, "J_i"] & \mathcal{T}_i(\mathcal{X}) \end{tikzcd}$$

where $\mathcal{F}_{\mathcal{T}}(\mathcal{F},\mathcal{G},\lambda) = (F^*_{\mathcal{X}}\mathcal{F}, F^*_{\mathcal{X}_{\mathcal{T}}}\mathcal{G}, F^*_{\mathcal{X}}\lambda)$*,* F *is the absolute Frobenius of* $\Pi_{\mathcal{T}_i(\mathcal{X})}$ *and* J_i *is defined in 5.10. Moreover* $\mathcal{F}^i_{\mathcal{T}}$ *factors as* $\mathcal{T}_i(\mathcal{X}) \xrightarrow{\alpha} \mathcal{T}_0(\mathcal{X}) \xrightarrow{\beta} \mathcal{T}_i(\mathcal{X})$*, where* α *is the projection* $(\mathcal{F},\mathcal{G},\lambda) \mapsto \mathcal{G}$ *and* β *is the transition morphism defined in 5.9.*

Proof. The commutativity of the first diagram follows from the naturality of Frobenius pullbacks and the definition of J_i. The second claim follows from the formula

$$(F^{i*}_{\mathcal{X}}\mathcal{F}, F^{i*}_{\mathcal{X}}\mathcal{G}, F^{i*}_{\mathcal{X}}\lambda) \xrightarrow{(\lambda,\mathrm{id})} (\mathcal{G}|_{\mathcal{X}}, F^{i*}_{\mathcal{X}}\mathcal{G}, \mathrm{id})$$

$\square$

Theorem 5.12. *Assume axiom A, that* $L = L_0 = \mathrm{H}^0(\mathcal{O}_{\mathcal{X}_{\mathcal{T}}})$ *is a field and the following property:*

$$\forall \mathcal{F} \in \mathrm{QCoh}_{\mathrm{fp}}(\mathcal{X}),\ \text{if } F^*\mathcal{F} \in \mathsf{Vect}(\mathcal{X}) \text{ then } \mathcal{F} \in \mathsf{Vect}(\mathcal{X})$$

Then for all $j \in \mathbb{N} \cup \{\infty\}$ *the ring* $L_j = \mathrm{End}_{\mathcal{T}_j(\mathcal{X})}(1_{\mathcal{T}_j(\mathcal{X})})$ *is a field,* $\mathcal{T}_j(\mathcal{X})$ *is an* L_j*-Tannakian category, the functors*

$$\begin{tikzcd} \mathcal{T}_j(\mathcal{X}) \arrow[r] \arrow[dr] & \mathcal{T}_{j+1}(\mathcal{X}) \arrow[r] \arrow[d] & \mathcal{T}_\infty(\mathcal{X}) \arrow[dl] \\ & \mathsf{Vect}(\mathcal{X}) & \end{tikzcd}$$

are faithful, monoidal and exact, $\Pi_{\mathcal{T}_j(\mathcal{X})}$ *is an affine gerbe over* L_j *and the functor* $\mathcal{T}_j(\mathcal{X}) \longrightarrow \mathsf{Vect}(\mathcal{X})$ *induces a map* $\mathcal{X} \longrightarrow \Pi_{\mathcal{T}_j(\mathcal{X})}$*, so that* $\mathcal{X}$ *is defined over* L_∞*. Moreover*

$$L_\infty = \{x \in \mathrm{H}^0(\mathcal{O}_{\mathcal{X}}) \mid \exists i \in \mathbb{N} \text{ such that } x^{p^i} \in L_0\}$$

is purely inseparable over L_0*,*

$$\mathrm{EFin}(\mathcal{T}_i(\mathcal{X})) = \{(\mathcal{F},\mathcal{G},\lambda) \in \mathcal{T}_i(\mathcal{X}) \mid \mathcal{G} \in \mathrm{EFin}(\mathcal{T}_0(\mathcal{X}))\},\ \mathrm{EFin}(\mathcal{T}_\infty(\mathcal{X})) \simeq \varinjlim_i \mathrm{EFin}(\mathcal{T}_i(\mathcal{X}))$$

and $\mathcal{X} \longrightarrow (\Pi_{\mathcal{T}_\infty(\mathcal{X})})_{\mathrm{L}}$ *is the pro-local Nori fundamental gerbe of* $\mathcal{X}$ *over* L_∞*.*

If we also assume axiom C then $\mathcal{X} \longrightarrow \widehat{\Pi}_{\mathcal{T}_\infty(\mathcal{X})}$ *is the Nori fundamental gerbe of* $\mathcal{X}$ *over* L_∞*, so that* $\mathsf{Rep}(\Pi^{\mathrm{N}}_{\mathcal{X}/L_\infty}) \simeq \mathrm{EFin}(\mathcal{T}_\infty(\mathcal{X}))$*, where all the notations here are in B.8.*

Proof. Notice that $\mathcal{T}(\mathcal{X}) = \mathcal{T}_0(\mathcal{X})$ is L_0-Tannakian and $\mathcal{T}(\mathcal{X}) \longrightarrow \mathsf{Vect}(\mathcal{X})$ is faithful thanks to 5.7. Let's show that the category $\mathcal{T}_i(\mathcal{X})$ is abelian. If $(\mathcal{F},\mathcal{G},\lambda) \xrightarrow{(\alpha,\beta)} (\mathcal{F}',\mathcal{G}',\lambda')$ is a map in $\mathcal{T}_i(\mathcal{X})$, then there is an induced isomorphism $\delta\colon F^{i*}(\mathrm{Coker}\,\alpha) \longrightarrow (\mathrm{Coker}\,\beta)_{|\mathcal{X}}$, which implies that $\operatorname{Coker}\alpha \in \mathsf{Vect}(\mathcal{X})$ and that $(\operatorname{Coker}\alpha, \operatorname{Coker}\beta, \delta) \in \mathcal{T}_i(\mathcal{X})$ is a cokernel. In this situation also kernels can be taken pointwise so that we obtain a kernel for (α, β). The map $\mathcal{T}_i(\mathcal{X}) \longrightarrow \mathsf{Vect}(\mathcal{X})$ is faithful because if (α, β) is a map as above with $\alpha = 0$, then $\beta_{|\mathcal{X}} = 0$, which implies $\beta = 0$ because $\mathcal{T}(\mathcal{X}) \longrightarrow \mathsf{Vect}(\mathcal{X})$ is faithful. This implies that all the functors in the statements are faithful and that $\mathcal{T}_\infty(\mathcal{X})$ is an abelian category. In particular for $i \in \mathbb{N}$ the functor $\mathcal{T}_i(\mathcal{X}) \longrightarrow \mathsf{Vect}(\mathcal{X})$ induces an isomorphism

$$L_i = \{(x,y) \mid x \in \mathrm{H}^0(\mathcal{O}_\mathcal{X}),\ y \in L_0,\ x^{p^i} = y\} \longrightarrow \{x \in \mathrm{H}^0(\mathcal{O}_\mathcal{X}) \mid x^{p^i} \in L_0\}$$

Notice that $\mathrm{H}^0(\mathcal{O}_\mathcal{X})$ is reduced: if $u \in \mathrm{H}^0(\mathcal{O}_\mathcal{X})$ with $u^n = 0$, then for i large $F^{i*}(\mathcal{O}_\mathcal{X}/u\mathcal{O}_\mathcal{X}) \simeq \mathcal{O}_\mathcal{X}/u^{p^i}\mathcal{O}_\mathcal{X} \simeq \mathcal{O}_\mathcal{X}$, and this implies that $(\mathcal{O}_\mathcal{X}/u\mathcal{O}_\mathcal{X}) \in \mathsf{Vect}(\mathcal{X})$ which is possible only if $u = 0$. In particular it follows that L_i is a field. Moreover L_∞ is the union of the L_i, which implies that it is a field and that the description in the statement holds. By 5.6 we conclude that the categories $\mathcal{T}_i(\mathcal{X})$ and $\mathcal{T}_\infty(\mathcal{X})$ are L_i-Tannakian and L_∞-Tannakian respectively.

Let's consider now the equality about essentially finite objects of $\mathcal{T}_i(\mathcal{X})$ in the statement. The projection $\mathcal{T}_i(\mathcal{X}) \xrightarrow{\alpha} \mathcal{T}_0(\mathcal{X})$ is $\mathbb{Z}$-linear, exact and monoidal. This gives the inclusion $\subseteq$. For the converse let $\chi = (\mathcal{F}, \mathcal{G}, \lambda) \in \mathcal{T}_i(\mathcal{X})$ such that $\mathcal{G} \in \mathrm{EFin}(\mathcal{T}_0(\mathcal{X}))$ and denote by Γ the monodromy gerbe of χ, which is an L_i-gerbe of finite type such that $\mathsf{Rep}\Gamma = \langle \chi \rangle \subseteq \mathcal{T}_i(\mathcal{X})$ (see B.8). We have to show that Γ is finite. Using 5.11 and its notation we have a 2-commutative diagram

$$\begin{array}{ccc} \mathsf{Vect}(\Gamma) & \longrightarrow & \mathcal{T}_i(\mathcal{X}) \\ \downarrow{\scriptstyle F^{i*}} & & \downarrow{\scriptstyle F^i_\mathcal{T}} \\ \mathsf{Vect}(\Gamma) & \longrightarrow & \mathcal{T}_i(\mathcal{X}) \end{array}$$

and, moreover, $F^i_\mathcal{T}(\chi)$ is essentially finite. Since $\mathsf{Rep}\Gamma$ is a sub Tannakian category of $\mathcal{T}_i(\mathcal{X})$ it follows that $F^{i*}\chi$ is essentially finite in $\mathsf{Rep}\Gamma$ and, since $\mathsf{Rep}\Gamma = \langle \chi \rangle$, it follows that the i-th absolute and therefore relative Frobenius of Γ factors through a finite gerbe. Such a factorization continues to hold if we base change to $\overline{L_i}$, so that $\Gamma \times_{L_i} \overline{L_i} \simeq \mathrm{B}\,G$, where G is an affine group of finite type over $\overline{L_i}$ whose relative Frobenius $G \longrightarrow G^{(i)}$ factors through a finite group scheme. Since the relative Frobenius is topologically surjective, we conclude that G is a finite group scheme as desired.

Let's now prove the isomorphism between $\mathrm{EFin}(\mathcal{T}_\infty(\mathcal{X}))$ and the limit in the statement. Let $\chi \in \mathcal{T}_\infty(\mathcal{X})$ and $\chi_i \in \mathcal{T}_i(\mathcal{X})$ mapping to χ for some i. If χ is finite then clearly χ_i will be finite up to replace i. If χ is instead a kernel of a map between finite objects, then those objects and this map will be image of a map u of finite objects in some $\mathcal{T}_j(\mathcal{X})$. The kernel of u is then a essentially finite objects of $\mathcal{T}_j(\mathcal{X})$ mapping to χ.

We now consider the claims about Nori gerbes. Let Φ be a finite stack over L_∞ and consider the map

$$\mathrm{Hom}_{L_\infty}(\Pi_{\mathcal{T}_\infty(\mathcal{X})}, \Phi) \longrightarrow \mathrm{Hom}_{L_\infty}(\mathcal{X}, \Phi)$$

We have to prove that this is an equivalence if Φ is local and an equivalence in general when axiom C holds. We can moreover assume that $L_0 = k$. Using 3.3, we can find a finite extension F/k, a finite stack Γ over F with an isomorphism $\Phi \simeq \Gamma \times_F L_\infty$. The above map then becomes

$$\Psi_{\Gamma,F}\colon \mathrm{Hom}_F(\Pi_{\mathcal{T}_\infty(\mathcal{X})}, \Gamma) \longrightarrow \mathrm{Hom}_F(\mathcal{X}, \Gamma)$$

Notice that F/k is a finite purely inseparable extension and thus Γ/k is finite. Moreover if Φ is local then Γ/k is also local thanks to 3.7 and 3.8.

Thus if we know that $\Psi_{\Gamma,k}$ and $\Psi_{\mathrm{Spec}\,(F),k}$ are equivalences we can conclude that $\Psi_{\Gamma,F}$ is an equivalence. This shows that we can assume $F = k$. Set also $\Psi = \Psi_{\Gamma,k}$.

We are going to use that finite stacks satisfies Tannakian reconstruction by 1.7. Moreover the map $\mathrm{Hom}_k(\mathcal{X}_\mathcal{T}, \Gamma_{\text{ét}}) \longrightarrow \mathrm{Hom}_k(\mathcal{X}, \Gamma_{\text{ét}})$ is an equivalence if Γ is local (that is $\Gamma_{\text{ét}} = \mathrm{Spec}\, k$) or in general if axiom C holds. Thus we can assume it is an equivalence.

Ψ *essentially surjective.* Let $\mathcal{X} \xrightarrow{a} \Gamma$ be a k-map and consider the factorization $\Gamma \longrightarrow \Gamma_{\text{ét}} \longrightarrow \Gamma^{(j)}$ of 3.6. We can extend the map $\mathcal{X} \longrightarrow \Gamma_{\text{ét}}$ to $\mathcal{X}_\mathcal{T}$ obtaining a 2-commutative diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d, "a"] & \mathcal{X}_\mathcal{T} \arrow[d] & \\ \Gamma \arrow[r] & \Gamma_{\text{ét}} \arrow[r] & \Gamma^{(j)} \end{tikzcd}$$

and therefore, by 5.10, a map $e\colon \Pi_{\mathcal{T}_j(\mathcal{X})} \longrightarrow \Gamma$ inducing $a\colon \mathcal{X} \longrightarrow \Gamma$. The map e is automatically k-linear because a is so and $\mathcal{T}_j(\mathcal{X}) \longrightarrow \mathsf{Vect}(\mathcal{X})$ is faithful.

Ψ *fully faithful.* We are going to show that a map $\Pi_{\mathcal{T}_\infty(\mathcal{X})} \longrightarrow \Gamma$ factors through a map $\Pi_{\mathcal{T}_i(\mathcal{X})} \longrightarrow \Gamma$. Before doing that we show how to conclude that Ψ is fully faithful. Let $\alpha, \beta\colon \Pi_{\mathcal{T}_\infty(\mathcal{X})} \longrightarrow \Gamma$ be two maps and $\delta\colon \alpha_{|\mathcal{X}} \longrightarrow \beta_{|\mathcal{X}}$ be an isomorphism of functors $\mathcal{X} \longrightarrow \Gamma$. The uniqueness of an extension is easy, because α, β correspond to maps $\mathsf{Vect}(\Gamma) \longrightarrow \mathcal{T}_\infty(\mathcal{X})$ and $\mathcal{T}_\infty(\mathcal{X}) \longrightarrow \mathsf{Vect}(\mathcal{X})$ is faithful. We can assume that both α, β factors through $\Pi_{\mathcal{T}_i(\mathcal{X})}$, so that, by 5.10, they correspond to 2-commutative diagrams

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d, bend right, "\alpha_{|\mathcal{X}}"'] \arrow[d, bend left, "\beta_{|\mathcal{X}}"] & \mathcal{X}_\mathcal{T} \arrow[d, bend right, "u"'] \arrow[d, bend left, "v"] \\ \Gamma \arrow[r] & \Gamma^{(i)} \end{tikzcd}$$

where u, v are k-linear. By 3.6 there exists $j > i$ and a factorization $\Gamma^{(i)} \longrightarrow \Gamma^{(i)}_{\text{ét}} = \Gamma_{\text{ét}} \longrightarrow \Gamma^{(j)}$. Replacing i by j we can assume that u, v factor through $\Gamma_{\text{ét}} \longrightarrow \Gamma^{(i)}$. Since $\mathrm{Hom}_k(\mathcal{X}_\mathcal{T}, \Gamma_{\text{ét}}) \simeq \mathrm{Hom}_k(\mathcal{X}, \Gamma_{\text{ét}})$ we can lift the isomorphism $\delta\colon \alpha_{|\mathcal{X}} \longrightarrow \beta_{|\mathcal{X}}$ to an isomorphism $u \longrightarrow v$ as required.

It remains to show that a k-linear, monoidal and exact map $F\colon \mathsf{Vect}(\Gamma) \longrightarrow \mathcal{T}_\infty(\mathcal{X})$ factors through some $\mathcal{T}_i(\mathcal{X})$. We will use a slight modification of [BV, Prop 3.8] and its proof. Pick $S = \mathrm{Spec}\, K \longrightarrow \Pi_{\mathcal{T}_\infty}$, where K is a field, an object corresponding to $\xi\colon \mathcal{T}_\infty(\mathcal{X}) \longrightarrow \mathsf{Vect}(K)$ and set $R_j = S \times_{\Pi_{\mathcal{T}_j(\mathcal{X})}} S$. Given a K-scheme T an object of $R_\infty(T)$ is a triple (u, v, γ) where $u, v\colon T \longrightarrow S$ and $\gamma\colon u^* \circ \xi \longrightarrow v^* \circ \xi$ is a monoidal isomorphism. Similarly, using the functor $\chi = \chi_{\mathsf{Vect}(T)}$ of A.2, an object of $(\varprojlim_j R_j)(T)$ is a triple $(u, v, \tilde{\gamma})$ where $u, v\colon T \longrightarrow S$ and $\tilde{\gamma}\colon \chi(u^* \circ \xi) \longrightarrow \chi(v^* \circ \xi^*)$ is an isomorphism given by monoidal natural transformations. From this we can deduce that $R_\infty \simeq \varprojlim_j R_j$. Since a map from a scheme to a gerbe is an fpqc covering, the map $\Pi_{\mathcal{T}_\infty(\mathcal{X})} \longrightarrow \Gamma$ is given by an object $z \in \Gamma(S)$ with an identification of the two projections in $\Gamma(R_\infty) \simeq \varinjlim_i \Gamma(R_j)$ satisfying the cocycle condition. Here we use that Γ is finitely presented. This identification lies in some $\Gamma(R_j)$. Up to replace this j, we can also assume that this identification satisfies the cocycle condition, which yields the desired factorization. $\square$

Remark 5.13. If $\mathcal{X}$ is a reduced category fibered in groupoid over k and $\mathcal{F} \in \mathrm{QCoh}_{\mathrm{fp}}(\mathcal{X})$ then $F^*\mathcal{F} \in \mathsf{Vect}(\mathcal{X})$ implies that $\mathcal{F} \in \mathsf{Vect}(\mathcal{X})$. Indeed let $\phi\colon V \longrightarrow \mathcal{X}$ be a map from a scheme. We must show that $\phi^*\mathcal{F}$ is a vector bundle. Since $\mathcal{X}$ is reduced, by fpqc descent we can assume that ϕ factors through a reduced scheme. This allow to assume that $\mathcal{X}$ is a reduced scheme and also that $\mathcal{X} = \operatorname{Spec} R$, where R is a local ring, so that $F^*\mathcal{F}$ is free of some rank r. Since the Frobenius is an homeomorphism, it follows that for all $p \in \mathcal{X}$ we have $\dim_{k(p)} \mathcal{F} \otimes k(p) = r$. Nakayama's lemma gives a surjective morphism $\phi\colon R^r \longrightarrow \mathcal{F}$. If $v \in \mathrm{Ker}\phi$, since ϕ is an isomorphism on each minimal prime ideal of R, it follows that all entries of v are nilpotent and thus $v = 0$.

Theorem 5.14. *Let $\mathcal{Z}$ be a reduced and inflexible category fibered in groupoids over k and denote by $\pi\colon \mathcal{Z} \longrightarrow \Pi^{\mathrm{N,\acute{e}t}}_{\mathcal{Z}/k}$ the structure morphism. Denote also by $\mathcal{C}_i$ the monoidal and additive category of triples $(\mathcal{E}, V, \lambda)$ where $\mathcal{E} \in \mathsf{Vect}(\mathcal{Z})$, $V \in \mathsf{Rep}(\Pi^{\mathrm{N,\acute{e}t}}_{\mathcal{Z}/k})$ and $\lambda\colon F^{i*}\mathcal{E} \longrightarrow \pi^*V$ is an isomorphism and regard $\mathcal{C}_i$ as a k-linear category via $k \longrightarrow \mathrm{End}(\mathcal{O}_{\mathcal{Z}}, \mathcal{O}_{\Pi^{\mathrm{N,\acute{e}t}}_{\mathcal{Z}/k}}, 1)$, $x \mapsto (x, x^{p^i})$. By pulling back along the Frobenius of $\Pi^{\mathrm{N,\acute{e}t}}_{\mathcal{Z}/k}$ we obtain k-linear monoidal functors $\mathcal{C}_i \longrightarrow \mathcal{C}_{i+1}$. Then the $\mathcal{C}_i$ are k-Tannakian categories and there is an equivalence of k-Tannakian categories $\varinjlim_i \mathcal{C}_i \simeq \mathsf{Rep}\Pi^{\mathrm{N}}_{\mathcal{Z}/k}$, where the structure morphism $\mathsf{Rep}\Pi^{\mathrm{N}}_{\mathcal{Z}/k} \longrightarrow \mathsf{Vect}(\mathcal{Z})$ corresponds to the forgetful functor.*

Proof. Consider $\mathcal{Z} = \mathcal{X} \longrightarrow \mathcal{X}_{\mathcal{T}} = \Pi^{\mathrm{N,\acute{e}t}}_{\mathcal{Z}/k}$. It is easy to see that this map satisfies axioms A, B, C and D and $L = \mathrm{H}^0(\mathcal{O}_{\Pi^{\mathrm{N,\acute{e}t}}_{\mathcal{Z}/k}}) = k$. By [BV, Prop 5.4, a)] it follows that k is integrally closed in $\mathrm{H}^0(\mathcal{O}_{\mathcal{Z}})$ and the result then follows from 5.12: we have $k = L_0 = L_\infty$, $\mathcal{C}_i = \mathcal{T}_i(\mathcal{Z})$ and $\mathrm{EFin}(\mathcal{T}_0(\mathcal{X})) = \mathcal{T}_0(\mathcal{X})$ implies $\mathrm{EFin}(\mathcal{T}_\infty(\mathcal{X})) = \mathcal{T}_\infty(\mathcal{X})$. $\square$

6. Stratification, crystal and Frobenius divided structures

In this section we apply the result of the previous section to find explicit morphisms $\pi_{\mathcal{T}} : \mathcal{X} \longrightarrow \mathcal{X}_{\mathcal{T}}$ for which the general theory works properly. In the next sections when we talk about axioms we will always refer to the list of axioms 5.2.

We start by introducing some geometric notions that will be used in the whole section.

Definition 6.1. A field extension L/k is called separably generated (resp. separable) up to Frobenius if $\mathrm{char}\, k = 0$ or $\mathrm{char}\, k = p > 0$ and there exists $i \in \mathbb{N}$ such that L^{p^i} is contained in a separably generated (resp. separable) field extension of k (see [SP, 030I]).

Definition 6.2. Let X be a scheme. A point $p \in X$ is called adically separated if the local ring $(\mathcal{O}_{X,p}, m_p)$ is m_p-adically separated, that is $\bigcap_n m_p^n = 0$.

We introduced this notion instead of considering just Noetherian rings because, when studying F-divided sheaves, we have to consider Frobenius twists $X^{(i)}$ of a scheme X, which may be not Noetherian even though X is so. Instead adically separatedness is maintained by Frobenius twists:

Lemma 6.3. *If (R,m) is an m-adically separated local ring defined over a field k of positive characteristic then, for all $i \in N$, $R^{(i)}$ is a local ring separated for the topology of its maximal ideal.*

Proof. We can assume $i = 1$. Since the Frobenius of k is purely inseparable, $R^{(1)}$ is again a local ring. Denote by m_1 its maximal ideal, set $L = k$ with the k-structure given by the Frobenius and consider a k-basis $\{\lambda_i\}_i$ of L. Let $x \in \cap_n m_1^n$ and write it as (a finite sum) $x = \sum_i x_i \otimes \lambda_i$, with $x_i \in R$. Given $l \in \mathbb{N}$ the ring $(R/m^l R)^{(1)}$ has nilpotent maximal ideal, which implies that $x \in m^l \otimes_k L$. By the uniqueness of the x_i, we can conclude that for all l and i we have $x_i \in m^l$. By hypothesis this means $x_i = 0$ for all i and therefore $x = 0$. □

Lemma 6.4. *Let (R,m) be an m-adically separated ring and M be a finitely generated R-module. Then M is free if and only if M/m^nM is a free R/m^n-module for all $n \in \mathbb{N}$.*

Proof. We have to prove $\Longleftarrow$. Lifting a basis of M/mM we can define a surjective morphism $\phi\colon R^l \longrightarrow M$, which will be an isomorphism after tensoring by R/m^nR by hypothesis. So if $v \in \mathrm{Ker}\phi$, it becomes 0 on all the quotients $(R/m^nR)^l$ and therefore $v \in (\cap_n m^n)^l = 0$ as desired. □

6.1. **Stratifications and crystals.**

Definition 6.5. Let $\pi\colon \mathcal{X} \longrightarrow \mathrm{Aff}/k$ be a category over Aff/k. We define the big infinitesimal site $\mathcal{X}_{\mathrm{inf}/k}$ of $\mathcal{X}$ as the category of pairs (ξ, j) where $\xi \in \mathcal{X}$ and $j\colon \pi(\xi) \longrightarrow T$, where T is an affine k-scheme, is a nilpotent closed immersion. A morphism $(\xi, \pi(\xi) \xrightarrow{j} T) \longrightarrow (\xi', \pi(\xi') \xrightarrow{j'} T')$ is a pair (α, β), where $\alpha\colon \xi \longrightarrow \xi'$ and $\beta\colon T \longrightarrow T'$ are such that the following diagram is commutative

$$\begin{tikzcd} \pi(\xi) \arrow[r, "j"] \arrow[d, "\pi(\alpha)"'] & T \arrow[d, "\beta"] \\ \pi(\xi') \arrow[r, "j'"] & T' \end{tikzcd}$$

An object $(\xi, \pi(\xi) \xrightarrow{j} T) \in \mathcal{X}_{\mathrm{inf}/k}$ is called *extendable* if there exists a map $\xi \longrightarrow \eta$ in $\mathcal{X}$ such that $\pi(\xi) \longrightarrow \pi(\eta)$ factors through $\pi(\xi) \longrightarrow T$. If $\mathcal{X}$ is a fibered category this simply means that $\xi\colon \pi(\xi) \longrightarrow \mathcal{X}$ extends along $\pi(\xi) \longrightarrow T$. We define the big stratified site $\mathcal{X}_{\mathrm{str}/k}$ of $\mathcal{X}$ as the full sub category of extendable objects of $\mathcal{X}_{\mathrm{inf}/k}$. We will consider $\mathcal{X}_{\mathrm{str}/k}$ and $\mathcal{X}_{\mathrm{inf}/k}$ as categories over k via the association $(\xi, j : \pi(\xi) \longrightarrow T) \longmapsto T$. Notice that there is a canonical map $\mathcal{X} \longrightarrow \mathcal{X}_{\mathrm{str}/k} \subseteq \mathcal{X}_{\mathrm{inf}/k}$ of categories over Aff/k given by $\xi \longmapsto (\xi, \mathrm{id}_{\pi(\xi)})$. If $\mathcal{X}$ is a fibered category over k then also $\mathcal{X}_{\mathrm{str}}$ and $\mathcal{X}_{\mathrm{inf}}$ are fibered categories.

Let $\mathcal{Y}$ be a fibered category over k. Following notations and definitions from 5.1 we define the following objects:

- if $\mathcal{X} \longrightarrow \mathcal{X}_{\mathcal{T}} = \mathcal{X}_{\mathrm{str}/k}$ then $\mathcal{T}_k$ will be replaced by Str_k and an object of $\mathrm{Str}_k(\mathcal{X}, \mathcal{Y}) = \mathrm{Hom}^c_k(\mathcal{X}_{\mathrm{str}/k}, \mathcal{Y})$ will be called a stratified map, while an object of $\mathrm{Str}_k(\mathcal{X}) = \mathsf{Vect}(\mathcal{X}_{\mathrm{str}/k})$ a stratified sheaf on $\mathcal{X}$.
- if $\mathcal{X} \longrightarrow \mathcal{X}_{\mathcal{T}} = \mathcal{X}_{\mathrm{inf}/k}$ then $\mathcal{T}_k$ will be replaced by Crys_k and an object of $\mathrm{Crys}_k(\mathcal{X}, \mathcal{Y}) = \mathrm{Hom}^c_k(\mathcal{X}_{\mathrm{inf}/k}, \mathcal{Y})$ will be called a crystal map, while an object of $\mathrm{Crys}_k(\mathcal{X}) = \mathsf{Vect}(\mathcal{X}_{\mathrm{inf}/k})$ a crystal of sheaves on $\mathcal{X}$.

When k is clear from the context it will be omitted.

If $\mathcal{Z}$ is a scheme and $\mathcal{X}$ is the category of open subsets of $\mathcal{Z}$ then $\mathrm{Crys}(\mathcal{X})$ is, by construction, the usual category of crystals of sheaves. Although we don't prove it here, it is possible to show that the restriction $\mathrm{Crys}(\mathcal{Z}) \longrightarrow \mathrm{Crys}(\mathcal{X})$ is an equivalence. In this paper we prefer to consider the big site $\mathrm{Aff}/\mathcal{Z}$ instead of the small Zariski site to extends the theory to algebraic stacks and fibered categories.

The main result of this section is the following Theorem:

Theorem 6.6. *Let $\mathcal{Z}$ be a category fibered in groupoids over k. Then:*

(1) axiom C holds for $\mathcal{Z} \longrightarrow \mathcal{Z}_{\mathrm{str}}$ and $\mathcal{Z} \longrightarrow \mathcal{Z}_{\mathrm{inf}}$;
(2) axiom A implies axiom B for $\mathcal{Z} \longrightarrow \mathcal{Z}_{\mathrm{str}}$ and $\mathcal{Z} \longrightarrow \mathcal{Z}_{\mathrm{inf}}$;
(3) axioms A and B hold for $\mathcal{Z} \longrightarrow \mathcal{Z}_{\mathrm{str}}$ if $\mathcal{Z}$ admits an fpqc covering $U \longrightarrow \mathcal{Z}$ where U is a scheme over k^{perf} such that all its closed subsets contain an adically separated point (see 6.2); if moreover $\mathcal{Z}$ is connected and there exists a map $\operatorname{Spec} L \longrightarrow \mathcal{Z}$ where L/k is a field extension which is separably generated up to Frobenius (see 6.1) then $\mathrm{H}^0(\mathcal{O}_{\mathcal{Z}_{\mathrm{str}/k}}) = \mathrm{H}^0(\mathcal{O}_{\mathcal{Z}})_{\text{ét},k}$;
(4) axiom A and B holds for $\mathcal{Z} \longrightarrow \mathcal{Z}_{\mathrm{inf}}$ if $\mathcal{Z}$ is an algebraic stack locally of finite type over k; moreover in this case $\mathrm{H}^0(\mathcal{O}_{\mathcal{Z}_{\mathrm{inf}/k}}) = \mathrm{H}^0(\mathcal{O}_{\mathcal{Z}})_{\text{ét},k}$.

Proof of Theorem 6.6,1). This follows by definition, because a map to something étale extends uniquely along a nilpotent closed immersion. □

Remark 6.7. If $\mathcal{Y}$ is a fibered category there are functors $\mathrm{Crys}_k(\mathcal{X}, \mathcal{Y}), \mathrm{Str}_k(\mathcal{X}, \mathcal{Y}) \longrightarrow \mathrm{Hom}^c_k(\mathcal{X}, \mathcal{Y})$ and there is a map $\mathrm{Crys}_k(\mathcal{X}, \mathcal{Y}) \longrightarrow \mathrm{Str}_k(\mathcal{X}, \mathcal{Y})$ over $\mathrm{Hom}^c_k(\mathcal{X}, \mathcal{Y})$. Moreover if $\mathcal{X}$ is defined over a field extension L of k there is a forgetful functor $\mathcal{X}_{\mathrm{inf}/L} \longrightarrow \mathcal{X}_{\mathrm{inf}/k}$ maintaining the stratified sites and inducing maps

$$\mathrm{Crys}_k(\mathcal{X}, \mathcal{Y}) \longrightarrow \mathrm{Crys}_L(\mathcal{X}, \mathcal{Y} \times_k L),\ \mathrm{Str}_k(\mathcal{X}, \mathcal{Y}) \longrightarrow \mathrm{Str}_L(\mathcal{X}, \mathcal{Y} \times_k L)$$

Lemma 6.8. *Let $i \colon \mathcal{X} \longrightarrow \mathcal{X}'$ be a nilpotent closed immersion of categories fibered in groupoids over k and $\mathcal{Y}$ be a fibered category over k. Then the restriction $\mathrm{Crys}(\mathcal{X}', \mathcal{Y}) \longrightarrow \mathrm{Crys}(\mathcal{X}, \mathcal{Y})$ is an equivalence. If i admits a retraction then also $\mathrm{Str}(\mathcal{X}', \mathcal{Y}) \longrightarrow \mathrm{Str}(\mathcal{X}, \mathcal{Y})$ is an equivalence.*

Proof. We will consider only the stratified case since the crystal one is completely analogous. There is a restriction functor $\psi \colon \mathcal{X}_{\mathrm{str}} \longrightarrow \mathcal{X}'_{\mathrm{str}}$ obtained by composing with $i \colon \mathcal{X} \longrightarrow \mathcal{X}'$. Using the pullback along i we also get a morphism $\phi \colon \mathcal{X}'_{\mathrm{str}} \longrightarrow \mathcal{X}_{\mathrm{str}}$ (the extendability condition is preserved). It is easy to define base-preserving natural transformations $\phi \circ \psi \longrightarrow \mathrm{id}$ and $\psi \circ \phi \longrightarrow \mathrm{id}$. Since stratified maps sends all arrows to Cartesian arrows, it follows that $\mathrm{Str}(\mathcal{X},\mathcal{Y}) \longrightarrow \mathrm{Str}(\mathcal{X}',\mathcal{Y})$ obtained by composing with ϕ is a quasi-inverse of the map in the statement. $\square$

Usually stratified sheaves for a scheme are defined using higher diagonals. In the following proposition we show that our definition is equivalent to the classical one. Recall that, given a k-scheme X, the n-th diagonal at level $r \in \mathbb{N}$, denoted $P^n_{X/k}(r)$ is defined as follows: pick an open $U \subseteq X^{\times r}$ containing the diagonal as a closed subscheme with ideal sheaf $\mathcal{I}$ and set $P^n_{X/k}(r) = \mathrm{Spec}\,(\mathcal{O}_U/\mathcal{I}^n)$.

Proposition 6.9. *Let X be a k-scheme and $\mathcal{Y}$ be a Zariski stack over* Aff/k. *The category* $\mathrm{Str}(X,\mathcal{Y})$ *is canonically equivalent to the category* $\hat{\mathrm{Str}}(X,\mathcal{Y})$ *whose objects are tuples* $(\eta,\sigma_n)_{n\in\mathbb{N}}$ *where* $\eta \in \mathcal{Y}(X)$ *and* $(\sigma_n)_{n\in\mathbb{N}}$ *is a compatible system of isomorphisms between the two pullbacks of* η *to* $\mathcal{Y}(P^n_{X/k})$ *satisfying the cocycle condition on* $\mathcal{Y}(P^n_{X/k}(2))$*, while the morphisms are maps in* $\mathcal{Y}(X)$ *compatible with the* σ_n.

Proof. Let $\mathcal{F} \in \mathrm{Str}(X,\mathcal{Y})$ be an object and consider the maps

$$\begin{tikzcd} X \arrow[r, "j_n"] & P^n_{X/k}(2) \arrow[r, bend left, "p_{12}"] \arrow[r, "p_{23}"] \arrow[r, bend right, "p_{13}"'] & P^n_{X/k} \arrow[r, bend left, "p_1"] \arrow[r, bend right, "p_2"'] & X \end{tikzcd}$$

where p_i and p_{ij} are the projections. Since all the maps $X \longrightarrow P^n_{X/k}(r)$ are nilpotent closed immersion with a retraction for all $n, r \in \mathbb{N}$, by 6.8 we see that applying $\mathrm{Str}(-,\mathcal{Y})$ to the above sequence of maps we get a sequence of equivalences. This easily yields compatible maps $\sigma_n\colon p_2^*\mathcal{F} \xrightarrow{\simeq} p_1^*\mathcal{F}$ in $\mathrm{Str}(P^n_{X/k},\mathcal{Y})$ satisfying the cocycle condition in $\mathrm{Str}(P^n_{X/k}(2),\mathcal{Y})$. Applying the natural functor $\mathrm{Str}(-,\mathcal{Y}) \longrightarrow \mathrm{Hom}(-,\mathcal{Y}) \simeq \mathcal{Y}(-)$ we obtain an object of $\hat{\mathrm{Str}}(X,\mathcal{Y})$. The association just defined extends to a functor $\mathrm{Str}(X,\mathcal{Y}) \to \hat{\mathrm{Str}}(X,\mathcal{Y})$.

A quasi-inverse can be defined as follows. For all $\chi = (U \longrightarrow T) \in X_{\mathrm{str}}$, where U is an X-scheme, choose an extension $g_\chi\colon T \longrightarrow X$. Given $(\eta,\sigma_n)_{n\in\mathbb{N}} \in \hat{\mathrm{Str}}(X,\mathcal{Y})$ define $\Phi_X((\eta,\sigma_n)_{n\in\mathbb{N}}) = \mathcal{F} \in \mathrm{Str}(X,\mathcal{Y})$ as follows. For $\chi \in X_{\mathrm{str}}$ set $\mathcal{F}(\chi) = g_\chi^*\eta$. Given a map $\psi\colon \chi \longrightarrow \chi'$ over $T \xrightarrow{\alpha} T'$ we have to specify a Cartesian arrow $\mathcal{F}(\psi)\colon g_\chi^*\eta \longrightarrow g_{\chi'}^*\eta$ over α. By construction $U \longrightarrow T \xrightarrow{(g_{\chi'}\alpha, g_\chi)} X \times X$ factors through the diagonal. Since $U \longrightarrow T$ is nilpotent, we get a factorization $(g_{\chi'}\alpha, g_\chi)\colon T \xrightarrow{\beta} P^n_{X/k} \subseteq X \times X$ for some $n \in \mathbb{N}$. The map $\mathcal{F}(\psi)$ is $g_\chi^*\eta \simeq \beta^*\,\mathrm{pr}_2^*\,\eta \xrightarrow{\beta^*\sigma_n} \beta^*\,\mathrm{pr}_1^*\,\eta \simeq \alpha^* g_{\chi'}^*\eta \longrightarrow g_{\chi'}^*\eta$. The compatibility among the σ_j tell us that $\mathcal{F}(\psi)$ does not depend on the choice of n, while the cocycle condition and a similar argument show that $\mathcal{F}$ is indeed a functor.

One can show that the two functors are quasi-inverse of each other. $\square$

Lemma 6.10. *Let R be a k-algebra, I be a nilpotent ideal and L/k be a field extension with a k-map $L \longrightarrow R/I$. Then there exists an fpqc covering $R \longrightarrow R'$ and an isomorphism $R'/IR' \simeq R/I \otimes_L L^{\mathrm{perf}}$, where L^{perf} is the perfect completion of L.*

Proof. A proof is required only if $p = \mathrm{char}\,k > 0$. We show how to construct the ring R' when L^{perf} is replaced by $L^{1/p}$. A simple induction on $\mathbb{N}$ will then give the desired algebra.

By Zorn's lemma there exists a maximal subset $S \subseteq L - L^p$ such that

$$\text{for all finite } T \subseteq S \text{ the map } L_T = L[X_t]_{t\in T}/(X_t^p - t) \longrightarrow L^{1/p} \text{ is injective}$$

Moreover it is easy to show that $L^{1/p} \simeq \lim_T L_T$. For all $t \in S$ let $\hat{t} \in R$ be a lifting and set

$$R_T = R[X_t]_{t\in T}/(X_t^p - \hat{t}) \text{ for } T \subseteq S \text{ finite and } R' = \lim_T R_T$$

It is now easy to prove that $R'/IR' \simeq R/I \otimes_L L^{1/p}$ as required. $\square$

Proof of Theorem 6.6,2). Let $\mathcal{C} = \mathrm{Str}(\mathcal{Z})$ or $\mathcal{C} = \mathrm{Crys}(\mathcal{Z})$, By 5.4 the category $\mathcal{C}$ is abelian, thus one has to show that if $\mathcal{F} \in \mathcal{C}$ and $\mathcal{F}_{|\mathcal{Z}} = 0$ then $\mathcal{F} = 0$. This follows because if $j\colon U \longrightarrow T$ is a nilpotent closed immersion and $\mathcal{E} \in \mathsf{Vect}(T)$ is such that $j^*\mathcal{E} = 0$ then $\mathcal{E} = 0$. $\square$

Proof of Theorem 6.6,3), first sentence. Let $\mathcal{F} \in \mathrm{Str}(\mathcal{Z}, \mathrm{QCoh}_{\mathrm{fp}})$. Since the nilpotent closed immersions in $\mathcal{Z}_{\mathrm{str}}$ have a retraction, it is enough to show that $\mathcal{F}_{|\mathcal{Z}} \in \mathrm{QCoh}_{\mathrm{fp}}(\mathcal{Z})$ is locally free in oder to conclude that $\mathcal{F} \in \mathrm{Str}(\mathcal{Z})$. Using the existence of an atlas as in the statement, fpqc descent and 6.4 we can reduce the problem to the case $\mathcal{Z} = \operatorname{Spec} R$, where (R, m) is a local ring defined over k^{perf} and with m nilpotent. Using the restriction $\mathrm{Str}_k(\mathcal{Z}) \longrightarrow \mathrm{Str}_{k^{\mathrm{perf}}}(\mathcal{Z})$ we can also assume k-perfect. By 6.10 applied when I the maximal ideal of R and L is its residue field we can assume that L is perfect. Since an extension of perfect fields is formally smooth (see [SP, 031U]), we can assume that the nilpotent closed immersion $\operatorname{Spec} L \longrightarrow \operatorname{Spec} R$ has a retraction $\sigma\colon \operatorname{Spec} R \longrightarrow \operatorname{Spec} L$. Thanks to 6.8, there exists $\mathcal{G} \in \mathrm{Str}(\operatorname{Spec} L, \mathrm{QCoh}_{\mathrm{fp}})$ restricting to $\mathcal{F}$ along the retraction σ. In particular

$$\mathcal{F}(\mathrm{id}_R, \operatorname{Spec} R \xrightarrow{\mathrm{id}} \operatorname{Spec} R) \simeq \mathcal{G}(\sigma, \operatorname{Spec} R \xrightarrow{\mathrm{id}} \operatorname{Spec} R) \simeq \sigma^*\mathcal{G}(\mathrm{id}_L, \operatorname{Spec} L \xrightarrow{\mathrm{id}} \operatorname{Spec} L)$$

is free as required. $\square$

Example 6.11. If we don't assume that the scheme U in 6.6, 2) is defined over k^{perf} then the conclusion is false, even if U is the spectrum of an Artinian ring. Consider $k = \mathbb{F}_p(z)$, $L = k^{\mathrm{perf}}$ and $A = L[x]/(x^2)$. We regard A as a k-algebra via the morphism $\lambda\colon k \longrightarrow A$ mapping z to $z - x$. We are going to construct an object $\mathcal{F} \in \mathrm{Str}_k(A, \mathrm{QCoh}_{\mathrm{fp}})$ which is not a vector bundle. Write $x = z - \lambda(z)$ and let $y_n \in L$ such that $y_n^{p^n} = z$. If J is the ideal of the diagonal in $A \otimes_k A$ we have

$$x \otimes 1 - 1 \otimes x = z \otimes 1 - 1 \otimes z = (y_n \otimes 1 - 1 \otimes y_n)^{p^n} \in J^{p^n}$$

By 6.9 we can conclude that $A \xrightarrow{x} A$ is a map in $\mathrm{Str}_k(A)$, where A is the trivial object. Since in $\mathrm{Str}_k(A, \mathrm{QCoh})$ we can take cokernels pointwise, we can conclude that A/x has a stratification, even though is not locally free.

This also show that $\mathrm{Str}_k(A) \longrightarrow \mathrm{Str}_k(L)$ is not an equivalence even though $\operatorname{Spec} L \longrightarrow \operatorname{Spec} A$ is a nilpotent closed immersion. To see this we prove that $\mathrm{Str}_k(L) \simeq \mathsf{Vect}(L)$. Indeed for $l \in \mathbb{N}$ let $J_l = \mathrm{Ker}(L^{\otimes_k l} \xrightarrow{\mu_l} L)$. If $x \in J_l$, since $L^{\otimes_k l}$ is perfect, there exists $y \in L^{\otimes_k l}$ such that $y^p = x$. Since $\mu_l(x) = \mu_l(y)^p$ we see that $y \in J_l$, so that $J_l^2 = J_l$. Thus all higher diagonals of $\operatorname{Spec} L$ over k are trivial, which implies the result.

Lemma 6.12. *If V is an affine scheme over k and $\chi \in V_{\mathrm{inf}}$ there exists $\chi' = (\mathrm{id}_V, V \longrightarrow T') \in V_{\mathrm{inf}}$ and a map $\chi \longrightarrow \chi'$. If V is of finite type over k we can furthermore assume that T' is of finite type too.*

Proof. Set $V = \operatorname{Spec} A$, and consider a ring B with a nilpotent ideal I and a map $\phi\colon A \longrightarrow B/I$. Set $\pi\colon B \longrightarrow B/I$ the projection and write $A = k[\underline{x}]/J$, where $\underline{x} = (x_s)_{s\in S}$ is a set of variables. For all $s \in S$ choose $b_s \in B$ such that $\pi(b_s) = \phi(x_s)$ and denote $\psi\colon k[\underline{x}] \longrightarrow B$ the map such that $\psi(x_s) = b_s$. Since $\psi(J) \subseteq I$ and I is nilpotent, there exists $n \in \mathbb{N}$ such that $J^n \subseteq \operatorname{Ker}(\psi)$. We can therefore choose $T' = \operatorname{Spec}(k[\underline{x}]/J^n)$. If V is of finite type then S can be chosen finite and therefore also the last claim holds. □

Proof of Theorem 6.6,4), first sentence. Let $\mathcal{F} \in \operatorname{Crys}(\mathcal{Z}, \operatorname{QCoh}_{\mathrm{fp}})$ and $(\xi, V \xrightarrow{j} T) \in \mathcal{Z}_{\mathrm{inf}}$. We must show that $\mathcal{F}(\xi, j) \in \operatorname{QCoh}_{\mathrm{fp}}(T)$ is locally free. Let $U \longrightarrow \mathcal{Z}$ be a smooth atlas. There are Cartesian diagrams

$$\begin{tikzcd} & & T' \arrow[rr] & & T \\ & & & & \\ V' \arrow[r] \arrow[uurr, "j'"] & V_U \arrow[r] \arrow[d] & V \arrow[uurr, "j"'] \arrow[d] & & \\ & U \arrow[r] & \mathcal{Z} & & \end{tikzcd}$$

where the map $T' \longrightarrow T$ is étale and surjective. The above diagram is obtained using that the smooth surjective map $V_U \longrightarrow V$ has sections in the étale topology and that the étale map $V' \longrightarrow V$ always extends along a nilpotent closed immersion by [SGA4, Exposé VIII, Thm 1.1]. By descent we can assume $\mathcal{Z} = U$ and, since the problem is Zariski local, that U is affine. By 6.12 we can further assume that $V = U$, $\xi = \mathrm{id}$ and that T is of finite type over k. Using 6.8, there exists $\mathcal{G} \in \operatorname{Crys}(T)$ restricting to our $\mathcal{F} \in \operatorname{Crys}(U)$. In particular

$$\mathcal{F}(\mathrm{id}_U, U \xrightarrow{j} T) \simeq \mathcal{G}(j, U \xrightarrow{j} T) = \mathcal{G}(\mathrm{id}_T, T \xrightarrow{\mathrm{id}} T) = (\mathcal{G}_{|T_{\mathrm{str}}})(\mathrm{id}_T, T \xrightarrow{\mathrm{id}} T)$$

This sheaf is locally free because $T \times_k k^{\mathrm{perf}} \longrightarrow T$ is an fpqc atlas, $T \times_k k^{\mathrm{perf}}$ is Noetherian and therefore, by 6.6,2), $\mathcal{G}_{|T_{\mathrm{str}}} \in \operatorname{Str}(T)$. □

Example 6.13. Let $k = \mathbb{F}_p(z)$ and $L = k^{\mathrm{perf}}$. We are going to construct a $\mathcal{F} \in \operatorname{Crys}_k(L, \operatorname{QCoh}_{\mathrm{fp}})$ such that $\mathcal{F} \notin \operatorname{Crys}_k(L)$. More precisely we construct a non zero $a\colon \mathcal{O}_{(\operatorname{Spec} L)_{\mathrm{inf}}} \longrightarrow \mathcal{O}_{(\operatorname{Spec} L)_{\mathrm{inf}}}$ which is 0 in $\operatorname{Str}_k(L) = \mathsf{Vect}(L)$ (see 6.11) and show that $\mathcal{F} = \operatorname{Coker}(a)$ (pointwise) satisfies the requirements. In particular if follows that $\operatorname{Crys}_k(L) \longrightarrow \operatorname{Str}_k(L)$ is not faithful. Let $\chi \in (\operatorname{Spec} L)_{\mathrm{inf}}$ given by a map $L \longrightarrow B/I$, where B is a k-algebra and I a nilpotent ideal. Using the Frobenius it is easy to show that there exists a unique $\mathbb{F}_p$-linear map $\phi_\chi\colon L \longrightarrow B$ lifting the given k-map $L \longrightarrow B/I$. The map a we are looking for is given by $a(\chi) = \phi_\chi(z) - z$. We have $a(\mathrm{id}_L, \mathrm{id}_L) = 0$ so that $a = 0$ in $\operatorname{Str}_k(L)$. Consider $B = L[x]/(x^2)$ with the k-structure $\lambda\colon k \longrightarrow B$, $\lambda(z) = z - x$, $I = (x)$. If $\chi \in (\operatorname{Spec} L)_{\mathrm{inf}}$ is the corresponding object, by construction $a(\chi) = x \neq 0$ in B and therefore $\mathcal{F}(\chi) = B/x$ which is not locally free.

Proposition 6.14. *Let L/k be a field extension separably generated up to Frobenius (see 6.1). Then* $\mathrm{End}_{\mathrm{Str}(L/k)}(1) = L_{\text{ét},k}$.

Proof. Let's assume $\mathrm{char}\, k = p > 0$. We want to show that, if $k \subseteq K \subseteq L$ and $L^{p^i} \subseteq K$ for some $i \in \mathbb{N}$ then $\mathrm{End}_{\mathrm{Str}(L/k)}(1) = \mathrm{End}_{\mathrm{Str}(K/k)}(1)$.

The first step is the case $K = k$. Set $A = L \otimes_k \overline{k}$, which is a local $\overline{k}$-algebra with residue field $\overline{k}$. Since $L^{p^i} \subseteq k$ for some $i \in \mathbb{N}$ we see also that A has nilpotent maximal ideal. By 6.8 we see that $\mathrm{End}_{\mathrm{Str}(A/\overline{k})}(1) = \overline{k}$. Since there is a functor $\mathrm{Str}(L/k) \longrightarrow \mathrm{Str}(A/\overline{k})$ we can conclude that $\mathrm{End}_{\mathrm{Str}(L/k)}(1)$ is contained in the intersection of L and $\overline{k}$ inside $A = L \otimes_k \overline{k}$. By fpqc descent this intersection is k.

For a field Q over k denote by $\phi_{i,Q} \colon Q^{(i)} \longrightarrow Q$ the relative Frobenius. The second step is showing that

$$\mathrm{End}_{\mathrm{Str}(Q/k)}(1) = \bigcap_j \mathrm{Im}\, \phi_{j,Q}$$

Since $Q^{p^j} \subseteq \mathrm{Im}\, \phi_{j,Q}$ and there is a functor $\mathrm{Str}(Q/k) \longrightarrow \mathrm{Str}(Q/\mathrm{Im}\, \phi_{j,Q})$ we obtain the inclusion $\subseteq$. For the other inclusion, by 6.9 we have that $\mathrm{End}_{\mathrm{Str}(Q/k)}(1)$ is the intersection of the subrings $Q_n = \mathrm{Ker}(Q \longrightarrow (Q \otimes_k Q/J^n))$, where J is the ideal of the diagonal. Since

$$\phi_{j,Q}(\sum_q z_q \otimes \lambda_q) \otimes 1 - 1 \otimes \phi_{j,Q}(\sum_q z_q \otimes \lambda_q) = \sum_q \lambda_q (z_q \otimes 1 - 1 \otimes z_q)^{p^j} \in J^{p^j}$$

we get $\mathrm{Im}\, \phi_{j,Q} \subseteq Q_j$ as required.

Now let K/k be any extension with $L^{p^i} \subseteq K$ for some i. A direct computation shows that $\mathrm{Im}\, \phi_{i+j,L} \subseteq \mathrm{Im}\, \phi_{j,K}$ for all j, so that $\mathrm{End}_{\mathrm{Str}(L/k)}(1) = \mathrm{End}_{\mathrm{Str}(K/k)}(1)$ as desired.

We come back now to the general statement. If F/k is a separable and finite field extension then $(\mathrm{Spec}\, F)_{\mathrm{Str}/k} = (\mathrm{Spec}\, F)_{\mathrm{Str}/F}$ and therefore $\mathrm{Str}(F/k) = \mathsf{Vect}(F)$. By functoriality this implies the inclusion $L_{\text{ét},k} \subseteq \mathrm{End}_{\mathrm{Str}(L/k)}(1)$. Consider the other inclusion. By discussion above and the hypothesis we can assume that L/k is separably generated. Let $x \in F = \mathrm{End}_{\mathrm{Str}(L/k)}(1)$. We are going to show that $x \in L$ is algebraic over k. Notice that, since L/k is separably generated, this will imply $x \in L_{\text{ét},k}$.

Assume by contradiction that x is transcendental and let $(z_s)_{s \in S}$ a transcendence basis of L/k such that $L/k(z_s)_{s \in S}$ is separable. Let $f(X) = X^n + a_1 X^{n-1} + \cdots + a_n$ be the minimal polynomial of x over $k(z_s)_{s \in S}$. Since $x \in \mathrm{End}_{\mathrm{Str}(L/k)}(1)$ and thanks to 6.9, we have $0 = d(x) = x \otimes 1 - 1 \otimes x \in I/I^2 \simeq \Omega_{L/k}$ where I is the ideal of the diagonal in $L \otimes_k L$. Thus $0 = d(f(x)) = d(a_1)x^{n-1} + \cdots + d(a_n)$. Since $L/k(z_s)_{s \in S}$ is separable, $\{d(z_s)\}_{s \in S}$ is a free basis of $\Omega_{L/k}$. As x is transcendental over k, we have $a_i \notin k$ for some i, that is $\partial a_i / \partial z_s \neq 0$ for some i, s. So the component of $d(a_1)X^{n-1} + \cdots + d(a_n)$ with respect to $d(z_s)$ is a non-zero polynomial over $k(z_s)_s$ which kills x. This contradicts the minimality of $f(X)$. □

Proof of Theorem 6.6,3), second sentence. Since axioms A and B holds and $\mathcal{Z}$ is connected, by 5.7 we know that $\mathrm{H}^0(\mathcal{O}_{\mathcal{Z}_{\mathrm{str}}}) = F \subseteq \mathrm{H}^0(\mathcal{O}_{\mathcal{Z}})$ is a field. By pulling back via $\mathrm{Spec}\, L \longrightarrow \mathcal{Z}$ we get a map $F \longrightarrow \mathrm{H}^0(\mathcal{O}_{(\mathrm{Spec}\, L)_{\mathrm{str}}}) = \mathrm{End}_{\mathrm{Str}(L/k)}(1) = L_{\text{ét},k}$, where we have used 6.14. So

$F \subseteq \mathrm{H}^0(\mathcal{O}_{\mathcal{Z}})_{\text{ét},k}$. The other inclusion follows pulling back along $\mathcal{Z} \longrightarrow \operatorname{Spec} \mathrm{H}^0(\mathcal{O}_{\mathcal{Z}})_{\text{ét},k}$ and using again 6.14. $\square$

Proof of Theorem 6.6,4), second sentence. We can assume $\mathcal{Z}$ connected and set

$$F := \mathrm{H}^0(\mathcal{O}_{\mathcal{Z}_{\inf}}) \subseteq \mathrm{H}^0(\mathcal{O}_{\mathcal{Z}}).$$

Using the map $\mathrm{Crys}(\mathcal{Z}) \longrightarrow \mathrm{Str}(\mathcal{Z})$ we can conclude that $F \subseteq \mathrm{H}^0(\mathcal{O}_{\mathcal{Z}})_{\text{ét}}$. The other inclusion follows pulling back along $\mathcal{Z} \longrightarrow \operatorname{Spec} \mathrm{H}^0(\mathcal{O}_{\mathcal{Z}})_{\text{ét},k}$: If Q/k is a separable and finite field extension then $(\operatorname{Spec} Q)_{\inf/k} = (\operatorname{Spec} Q)_{\inf/Q}$ so that $\mathrm{Crys}(Q/k) = \mathsf{Vect}(Q)$. $\square$

6.2. **F-divided structures.** In this section we fix a base field k with positive characteristic p.

Definition 6.15. Let $\mathcal{Z}$ be a category fibered in groupoids over k. The chain of relative Frobenius of $\mathcal{Z}$

$$\mathcal{Z} \longrightarrow \mathcal{Z}^{(1,k)} \longrightarrow \mathcal{Z}^{(2,k)} \longrightarrow \cdots$$

defines a direct system of fibered categories over k indexed by $\mathbb{N}$ and we will denote by $\mathcal{Z}^{(\infty,k)}$ its limit, which is a category fibered in groupoids over k (see A.4). Let $\mathcal{Y}$ be a fibered category over k. Following notations and definitions from 5.1 we define the following objects: if $\mathcal{X} = \mathcal{Z}$, $\mathcal{X}_{\mathcal{T}} = \mathcal{Z}^{(\infty,k)}$ and the map $\mathcal{X} \longrightarrow \mathcal{X}_{\mathcal{T}}$ is the one induced by the limit, then $\mathcal{T}_k$ will be replaced by Fdiv_k and an object of $\mathrm{Fdiv}_k(\mathcal{Z},\mathcal{Y}) = \mathrm{Hom}_k^c(\mathcal{Z}^{(\infty,k)},\mathcal{Y})$ will be called an F-divided map, while an object of $\mathrm{Fdiv}_k(\mathcal{Z}) = \mathsf{Vect}(\mathcal{Z}^{(\infty,k)})$ an F-divided sheaf. When k is clear from the context it will be omitted.

Using A.3 and A.4 we have a more concrete description, which will be the one used in this paper.

Proposition 6.16. *Let $\mathcal{Y}$ be a fibered category and $\mathcal{Z}$ be a category fibered in groupoids. Then $\mathrm{Fdiv}(\mathcal{Z},\mathcal{Y})$ is equivalent to the category of objects (Q_n,σ_n) where $Q_n\colon \mathcal{Z}^{(n)} \longrightarrow \mathcal{Y}$ is a map and $\sigma_n\colon Q_{n+1}\circ R_n \longrightarrow Q_n$ are isomorphisms, where $R_n\colon \mathcal{Z}^{(n)} \longrightarrow \mathcal{Z}^{(n+1)}$ is the relative Frobenius. Under this equivalence the functor $\mathrm{Fdiv}(\mathcal{Z},\mathcal{Y}) \longrightarrow \mathrm{Hom}(\mathcal{Z},\mathcal{Y})$ is given by $(Q_n,\sigma_n) \longmapsto Q_0$.*

The main result of this section is the following Theorem:

Theorem 6.17. *Let $\mathcal{Z}$ be a category fibered in groupoids over k. Then:*

(1) if $\mathcal{Z}$ is connected then axiom A implies axioms B,C and D for $\mathcal{Z} \longrightarrow \mathcal{Z}^{(\infty)}$ and that $\Pi_{\mathrm{Fdiv}(\mathcal{Z})}$ is a pro-smooth banded gerbe (see B.8).

(2) axioms A and B holds for $\mathcal{Z} \longrightarrow \mathcal{Z}^{(\infty)}$ if $\mathcal{Z}$ admits an fpqc covering $U \longrightarrow \mathcal{Z}$ from a scheme U such that all its closed subsets contains an adically separated point q (see 6.2) with $k(q)/k$ separable up to Frobenius (see 6.1); if moreover $\mathcal{Z}$ is connected and there exists a map $\operatorname{Spec} L \longrightarrow \mathcal{Z}$ where L/k is a field extension which is separably generated up to Frobenius (see 6.1) then $\mathrm{H}^0(\mathcal{O}_{\mathcal{Z}^{(\infty)}}) = \mathrm{H}^0(\mathcal{O}_{\mathcal{Z}})_{\text{ét},k}$;

Lemma 6.18. *Let $f : \mathcal{Z} \longrightarrow \mathcal{Z}'$ be a nilpotent closed immersion. Then the induced functor $\mathcal{Z}^{(\infty)} \longrightarrow \mathcal{Z}'^{(\infty)}$ is an equivalence. In particular for any fiber category $\mathcal{Y}$ the restriction $\mathrm{Fdiv}(\mathcal{Z},\mathcal{Y}) \to \mathrm{Fdiv}(\mathcal{Z}',\mathcal{Y})$ is an equivalence.*

Proof. Let $N \in \mathbb{N}$ such that the N-th power of the ideals defining $\mathcal{Z} \longrightarrow \mathcal{Z}'$ are all 0. For a given $i \in \mathbb{N}$ set $\mathcal{X} = \mathcal{Z}^{(i)}$ and $\mathcal{X}' = \mathcal{Z}'^{(i)}$. In particular this N works also for the nilpotent closed immersion $\mathcal{X} \longrightarrow \mathcal{X}'$. Let $i_0 \in \mathbb{N}$ with $p^{i_0} \geqslant N$. Notice that $F_{\mathcal{X}}^{i_0} : \mathcal{X}' \longrightarrow \mathcal{X}'$ factors through $\mathcal{X} \subseteq \mathcal{X}'$. This yield a k-map $\mathcal{X}' \longrightarrow \mathcal{X}^{(i_0)}$ making the following diagram commutative

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X}^{(i_0)} \arrow[d] \\ \mathcal{X}' \arrow[r] \arrow[ur] & \mathcal{X}'^{(i_0)} \end{tikzcd}$$

Thus we get k-maps $\mathcal{Z}'^{(i)} \longrightarrow \mathcal{Z}^{(i+i_0)}$ with the above property. This yields a k-map $\mathcal{Z}'^{(\infty)} \longrightarrow \mathcal{Z}^{(\infty)}$ which is easily seen to be a quasi-inverse of $\mathcal{Z}^{(\infty)} \longrightarrow \mathcal{Z}'^{(\infty)}$. □

Corollary 6.19. *Let $\mathcal{Z}$ be a category fibered in groupoids. Then there exists a natural k-functor $\psi : \mathcal{Z}_{\mathrm{inf}} \longrightarrow \mathcal{Z}^{(\infty)}$ making the following diagram commutative*

$$\begin{tikzcd} & \mathcal{Z} \arrow[dl] \arrow[dr] & \\ \mathcal{Z}_{\mathrm{inf}} \arrow[rr, "\psi"] & & \mathcal{Z}^{(\infty)} \end{tikzcd}$$

In particular if $\mathcal{Y}$ is a fibered category we obtain a restriction functor $\mathrm{Fdiv}(\mathcal{Z}, \mathcal{Y}) \longrightarrow \mathrm{Crys}(\mathcal{Z}, \mathcal{Y})$.

Proof. Given $(\xi, U \xrightarrow{j} T) \in \mathcal{Z}_{\mathrm{inf}}$ we obtain an arrow

$$T \longrightarrow T^{(\infty)} \xrightarrow{(j^{(\infty)})^{-1}} U^{(\infty)} \xrightarrow{\xi^{(\infty)}} \mathcal{Z}^{(\infty)}$$

In a similar way an arrow in $\mathcal{Z}_{\mathrm{inf}}$ can be mapped to an arrow in $\mathcal{Z}^{(\infty)}$. □

Lemma 6.20. *If $\mathcal{Z}$ is a category fibered in groupoids then, for $\mathcal{Z} \longrightarrow \mathcal{Z}^{(\infty)}$, axiom A implies axiom B.*

Proof. By 5.4 the category $\mathrm{Fdiv}(\mathcal{Z})$ is abelian, thus one has to show that if $\mathcal{F} = (\mathcal{F}_n, \sigma_n) \in \mathrm{Fdiv}(\mathcal{Z})$ and $\mathcal{F}_0 = 0$ then $\mathcal{F} = 0$. If C is an $\mathbb{F}_p$-algebra and $\xi : \mathrm{Spec}\,(C) \longrightarrow \mathcal{Z}^{(n)}$ a map, there exists $\eta : \mathrm{Spec}\,(C) \longrightarrow \mathcal{Z}$, namely the composition $\mathrm{Spec}\,(C) \longrightarrow \mathcal{Z}^{(n)} \longrightarrow \mathcal{Z}$ and a factorization of ξ as $\mathrm{Spec}\,(C) \longrightarrow V = \mathcal{Z}^{(n)} \times_{\mathcal{Z}} \mathrm{Spec}\, C \longrightarrow \mathcal{Z}^{(n)}$. If C has the k-structure induced by $\eta : \mathrm{Spec}\, C \longrightarrow \mathcal{Z}$, then $V = (\mathrm{Spec}\, C)^{(n)}$, so that the pullback of $(\mathcal{F}_n)_{|V}$ along the relative Frobenius of C coincides with $\eta^* \mathcal{F}_0 = 0$ on $\mathrm{Spec}\, C$. Since the relative Frobenius for affine schemes is a homeomorphism, we can conclude that $(\mathcal{F}_n)_{|V} = 0$, so that $\xi^* \mathcal{F}_n = 0$. □

Proof of Theorem 6.17, 2), first sentence. Let $(\mathcal{E}_n, \sigma_n)_{n \in \mathbb{N}} \in \mathrm{Fdiv}(\mathcal{Z}, \mathrm{QCoh}_{\mathrm{fp}})$ and $U \longrightarrow \mathcal{Z}$ be the atlas of the statement. We have to show that all $\mathcal{E}_i$ are locally free. Since all $U^{(i)} \to \mathcal{Z}^{(i)}$ are fpqc coverings we can assume $\mathcal{Z} = U$. Moreover, since the relative Frobenius is a homeomorphism, we can moreover assume $\mathcal{Z} = \mathrm{Spec}\, R$, where R is adically separated with maximal ideal m and residue field L. As for each $i \in \mathbb{N}$, $(\mathcal{E}_n, \sigma_n)_{n \geqslant i}$ is in $\mathrm{Fdiv}(\mathrm{Spec}\,(R)^{(i)}, \mathrm{QCoh}_{\mathrm{fp}})$ and $R^{(i)}$ is again adically separated by 6.3, we see that we can always replace R by $R^{(i)}$ and, using 6.4, that we can assume m nilpotent. Since L/k is separable up to Frobenius, it follows that, for $i \gg 0$, $R^{(i)}$ has separable residue field. Thus we can assume L/k separable. By 6.18 applied on the nilpotent closed immersion $\operatorname{Spec} L \longrightarrow \operatorname{Spec} R$ we obtain $(\operatorname{Spec} L)^{(\infty)} \simeq (\operatorname{Spec} R)^{(\infty)}$. Thus we may assume $R = L$ a field. Since L/k is separable all $L^{(i)}$ are reduced 0-dimensional local rings (see [SP, 05DT]), that is fields. Thus all $\mathcal{E}_n$ are vector spaces and thus locally free. $\square$

Example 6.21. Without the hypothesis on the residue fields in 6.17 the conclusion is false. Indeed if $k = \mathbb{F}_p(z)$ and $L = k^{\mathrm{perf}}$ then $\mathrm{Fdiv}_k(L) \neq \mathrm{Fdiv}_k(L, \mathrm{QCoh}_{\mathrm{fp}})$. Let $\phi_i \colon L^{(i+1)} \longrightarrow L^{(i)}$ the relative Frobenius, that is $\phi_i(a \otimes \lambda) = a^p \otimes \lambda$, and consider $x_i = z^{1/p^i} \otimes 1 - 1 \otimes z \in L^{(i)}$. A direct computation shows that $\phi_i(x_{i+1}) = x_i$ and $x_0 = 0$. The collection $x = (x_i)_{i \in \mathbb{N}}$ defines a morphism $\mathcal{O}_{(\operatorname{Spec} L)^{(\infty)}} \longrightarrow \mathcal{O}_{(\operatorname{Spec} L)^{(\infty)}}$. Its cokernel is not in $\mathrm{Fdiv}_k(L)$ because $x_0 = 0$ but $x_1 \neq 0$.

Proof of Theorem 6.6, 2), second sentence. Since axioms A and B holds and $\mathcal{Z}$ is connected, by 5.7 we know that $\mathrm{H}^0(\mathcal{O}_{\mathcal{Z}^{(\infty)}}) = F \subseteq \mathrm{H}^0(\mathcal{O}_{\mathcal{Z}})$ is a field. The inclusion $\mathrm{H}^0(\mathcal{O}_{\mathcal{Z}})_{\text{ét},k} \subseteq F$ follows pulling back along $\mathcal{Z} \longrightarrow \operatorname{Spec} \mathrm{H}^0(\mathcal{O}_{\mathcal{Z}})_{\text{ét},k}$: if Q/k is a separable and finite extension of k then $\operatorname{Spec} Q = (\operatorname{Spec} Q)^{(\infty,k)}$ so that $\mathrm{Fdiv}(Q/k) = \mathsf{Vect}(Q)$. For the other inclusion, pulling back via $\operatorname{Spec} L \longrightarrow \mathcal{Z}$ we get a map $F \longrightarrow \mathrm{H}^0((\operatorname{Spec} L)^{(\infty)}) = \mathrm{End}_{\mathrm{Fdiv}(L/k)}(1) = L'$. Using the map $\mathrm{Fdiv}(L/k) \longrightarrow \mathrm{Str}(L/k)$ and 6.6, 2) we see that $L' \subseteq L_{\text{ét},k}$ as desired. $\square$

Proof of Theorem 6.17, 1), A $\implies$ C. By 5.7 and 6.20 L is a field. In what follows we will use the following notation. If $\mathcal{W}$ is a category fibered in groupoid over L we will use $\mathcal{W}^{(i)}$ for $\mathcal{W}^{(i,k)}$ for $i \in \mathbb{N} \cup \{\infty\}$ and denote by $\mathcal{W}_{(i,L)}$, for $i \in \mathbb{N}$, the fibered category $\mathcal{W}$ with L-structure $\mathcal{W} \xrightarrow{F^i_{\mathcal{W}}} \mathcal{W} \xrightarrow{\pi} \operatorname{Spec} L$, where $F_{\mathcal{W}}$ is the absolute Frobenius, π is the structure map.

We need to show that the pullback functor $\mathrm{Hom}_L(\mathcal{Z}^{(\infty)}, \Gamma) \to \mathrm{Hom}_L(\mathcal{Z}, \Gamma)$ is an equivalence for a finite and étale stack Γ over L. By A.3 it is enough to prove that $\phi^*_{(i,\mathcal{Z})} : \mathrm{Hom}_L(\mathcal{Z}^{(i)}, \Gamma) \to \mathrm{Hom}_L(\mathcal{Z}, \Gamma)$ for all i, where $\mathcal{Z}^{(i)}$ is equipped with the L-structure via $\mathcal{Z}^{(\infty)}$ and $\phi_{(i,\mathcal{Z})}$ is the relative Frobenius. Denote by $\varphi \colon \mathcal{Z}^{(i)} \longrightarrow \mathcal{Z}$ the projection and consider the following 2-commutative diagram.

$$\begin{tikzcd}
 & (\mathcal{Z})_{(i,L)} \arrow[r, "\phi_{(i,\mathcal{Z})}"] \arrow[d, "F^i_{\mathcal{Z}}"] & (\mathcal{Z}^{(i)})_{(i,L)} \arrow[r, "\varphi"] \arrow[d, "F^i_{\mathcal{Z}^{(i)}}"] & \mathcal{Z} \\
(\mathcal{Z}^{(i)})_{(i,L)} \arrow[r, "\varphi"'] & \mathcal{Z} \arrow[r, "\phi_{(i,\mathcal{Z})}"'] & \mathcal{Z}^{(i)} &
\end{tikzcd}$$

We have $\varphi \circ \phi_{(i,\mathcal{Z})} = F^i_{\mathcal{Z}}$, $\phi_{(i,\mathcal{Z})} \circ \varphi = F^i_{\mathcal{Z}^{(i)}}$, and that the morphisms in the above diagram are L-linear. The result then follows upon applying $\mathrm{Hom}_L(-, \Gamma)$ to the diagram, provided that the following is true: If $\mathcal{W}$ is a category fibered in groupoids over L then $\mathrm{Hom}_L(\mathcal{W}, \Gamma) \xrightarrow{F^i_{\mathcal{W}} \circ -} \mathrm{Hom}_L(\mathcal{W}_{(i,L)}, \Gamma)$ is an equivalence. By construction $\mathrm{Hom}_L(\mathcal{W}_{(i,L)}, \Gamma) \simeq \mathrm{Hom}_L(\mathcal{W}, \Gamma^{(i,L)})$ and a direct check shows that $F^i_{\mathcal{W}} \circ -$ corresponds to the composition along the relative Frobenius $\Gamma \longrightarrow \Gamma^{(i,L)}$, which is an equivalence because Γ is étale over L. $\square$

Proof of Theorem 6.17,1), A $\Longrightarrow$ D and last sentence. By 5.7 and 6.20 we have that $L = \mathrm{H}^0(\mathcal{O}_{\mathcal{Z}^{(\infty,k)}})$ is a field and $\Pi_{\mathrm{Fdiv}(\mathcal{Z})}$ an L-gerbe. We must prove that, if Γ is a quotient L-gerbe of $\Pi_{\mathrm{Fdiv}(\mathcal{Z})}$ of finite type, then Γ is smooth banded. We have an L-map $\phi\colon \mathcal{Z}^{(\infty,k)} \longrightarrow \Gamma$ such that $\phi^*\colon \mathsf{Rep}\Gamma \longrightarrow \mathrm{Fdiv}(\mathcal{Z})$ is fully faithful. Set $\overline{\mathcal{Z}} = \mathcal{Z} \times_k \overline{k}$ and $\overline{\Gamma} = \Gamma \times_k \overline{k} \xrightarrow{\pi} \Gamma$. Since $\overline{\mathcal{Z}}^{(i,\overline{k})} \simeq \mathcal{Z}^{(i,k)} \times_k \overline{k}$, using the definition of limit it is easy to see that $\overline{\mathcal{Z}}^{(\infty,\overline{k})} \simeq \mathcal{Z}^{(\infty,k)} \times_k \overline{k}$. Denote by $\overline{\phi}\colon \overline{\mathcal{Z}}^{(\infty,\overline{k})} \longrightarrow \overline{\Gamma}$ the base change of ϕ. We claim that

$$\overline{\phi}^*\colon \mathsf{Vect}(\overline{\Gamma}) \longrightarrow \mathsf{Vect}(\overline{\mathcal{Z}}^{(\infty,\overline{k})}) = \mathrm{Fdiv}_{\overline{k}}(\overline{\mathcal{Z}})$$

is fully faithful. Let $\overline{V}, \overline{W} \in \mathsf{Vect}(\overline{\Gamma})$. Since ϕ^* is faithful, it is enough to prove that

$$\mathrm{Hom}_\Gamma(\pi_*\overline{V}, \pi_*\overline{W}) \longrightarrow \mathrm{Hom}_{\mathcal{Z}^{(\infty,k)}}(\phi^*\pi_*\overline{V}, \phi^*\pi_*\overline{W})$$

is bijective. The pushforward $\pi_*\overline{V}$ can be written as a direct sum of vector bundles on Γ. Indeed let k'/k be a finite extension for which there exists $V' \in \mathsf{Vect}(\Gamma \times_k k')$ inducing $\overline{V}$ and consider $\overline{\Gamma} \xrightarrow{\alpha} \Gamma \times_k k' \xrightarrow{\beta} \Gamma$. We have that $\pi_*\overline{V} = \beta_*(V' \otimes_{k'} \overline{k})$, which is a direct sum of copies of β_*V', and β_*V' is a vector bundle because it is a coherent sheaf on Γ, which is an L-gerbe. Writing $\pi_*V = \bigoplus_i V_i$ and $\pi_*W = \bigoplus_j W_j$ and using that ϕ^* is fully faithful on vector bundles, the proof of the bijectivity of the above map translates into the following statement: given a collection of maps $\lambda_{i,j}\colon V_i \longrightarrow W_j$ for all i,j such that $\phi^*\lambda_{i,j}$ induces a map $\mu\colon \bigoplus_i \phi^*V_i \longrightarrow \bigoplus_j \phi^*W_j$, then it also induces a map $\bigoplus_i V_i \longrightarrow \bigoplus_j W_j$. If $\xi\colon \operatorname{Spec} B \longrightarrow \mathcal{Z}^{(\infty,k)}$ is any object, since $\xi^*\mu$ is defined and $\operatorname{Spec} B$ is quasi-compact, we can conclude that for all i the set $\{j \mid \xi^*\phi^*\lambda_{i,j} \neq 0\}$ is finite. Since $\operatorname{Spec} B \xrightarrow{\phi\xi} \Gamma$ is faithfully flat, the same holds over Γ and therefore the map $\bigoplus_i V_i \longrightarrow \bigoplus_j W_j$ is well defined.

As $\overline{k}$ is perfect, the absolute Frobenius of $\overline{\mathcal{Z}}^{(\infty,\overline{k})}$ is also an equivalence. By the discussion above, we conclude that $F^*\colon \mathsf{Vect}(\overline{\Gamma}) \longrightarrow \mathsf{Vect}(\overline{\Gamma})$ is fully faithful, where F is the absolute Frobenius of $\overline{\Gamma}$. We show that $u\colon \mathcal{O}_{\overline{\Gamma}} \longrightarrow F_*\mathcal{O}_{\overline{\Gamma}}$ is surjective. For all $V \in \mathsf{Vect}(\overline{\Gamma})$ we have a bijection

$$\mathrm{Hom}_{\overline{\Gamma}}(V, \mathcal{O}_{\overline{\Gamma}}) \longrightarrow \mathrm{Hom}_{\overline{\Gamma}}(F^*V, F^*\mathcal{O}_{\overline{\Gamma}}) \simeq \mathrm{Hom}_{\overline{\Gamma}}(V, F_*\mathcal{O}_{\overline{\Gamma}})$$

which is induced by u. By [De3, Cor 3.9, pp. 132] and 1.6 the sheaf $F_*\mathcal{O}_{\overline{\Gamma}}$ is a quotient of a direct sum of vector bundles. This easily implies that u is surjective.

Recall that if $\mathcal{X}$ is a category fibered in groupoids over a scheme S and we set $\mathcal{X}^{(1,S)}$ for the base change of $\mathcal{X} \longrightarrow S$ along the absolute Frobenius of S, then the absolute Frobenius factors as $\mathcal{X} \longrightarrow \mathcal{X}^{(1,S)} \longrightarrow \mathcal{X}$. Moreover if $T \longrightarrow S$ is a map and we apply $- \times_S T$ to the map $\mathcal{X} \longrightarrow \mathcal{X}^{(1,S)}$ we get $\mathcal{X} \times_S T \longrightarrow (\mathcal{X} \times_S T)^{(1,T)}$. The stack $\overline{\Gamma} = \Gamma \times_k \overline{k}$ is a stack over $S = \operatorname{Spec}(L \otimes_k \overline{k})$. Thus the absolute Frobenius of $\overline{\Gamma}$ factors as $\overline{\Gamma} \xrightarrow{\alpha} \overline{\Gamma}^{(1,S)} \xrightarrow{\beta} \overline{\Gamma}$. Since β is affine and $\mathcal{O}_{\overline{\Gamma}} \longrightarrow \beta_*\mathcal{O}_{\overline{\Gamma}^{(1,S)}} \longrightarrow \beta_*\alpha_*\mathcal{O}_{\overline{\Gamma}} = F_*\mathcal{O}_{\overline{\Gamma}}$ is surjective, we can conclude that $\mathcal{O}_{\overline{\Gamma}^{(1,S)}} \longrightarrow \alpha_*\mathcal{O}_{\overline{\Gamma}}$ is surjective. Since α is the base change of $\Gamma \xrightarrow{\delta} \Gamma^{(1,L)}$ along the flat map $S \longrightarrow \operatorname{Spec} L$, it also follow that $\mathcal{O}_{\Gamma^{(1,L)}} \longrightarrow \delta_*\mathcal{O}_\Gamma$ is surjective. In particular $\delta_*\mathcal{O}_\Gamma$ is of finite type and thus locally free, which implies that $\mathcal{O}_{\Gamma^{(1,L)}} \longrightarrow \delta_*\mathcal{O}_\Gamma$ is an isomorphism. Using B.2, (1) and B.7 it follows that $\Gamma \longrightarrow \Gamma^{(1,L)}$ is a quotient. We claim that this implies that Γ is smooth banded. For this we can assume $L = k$ algebraically closed and $\Gamma = \mathrm{B}\,G$, for an affine group scheme G of finite type over k. The relative Frobenius is a quotient means that $G \longrightarrow G^{(1)}$ is faithfully flat, which implies that G is reduced and thus smooth. $\square$

7. The Local Quotient of the Nori Fundamental Gerbe

Let k be a field of characteristic $p > 0$, $\mathcal{X}$ be a category fibered in groupoid over k and denote by $F\colon \mathcal{X} \longrightarrow \mathcal{X}$ the absolute Frobenius. For $i \in \mathbb{N}$ denote by $\mathcal{D}_i$ the category of triples $(\mathcal{F}, V, \lambda)$ where $\mathcal{F} \in \mathsf{Vect}(\mathcal{X})$, $V \in \mathsf{Vect}(k)$ and $\lambda\colon V \otimes_k \mathcal{O}_\mathcal{X} \longrightarrow F^{i*}\mathcal{F}$ is an isomorphism. The category $\mathcal{D}_i$ is monoidal, rigid and k-linear via $k \longrightarrow \mathrm{End}_{\mathcal{D}_i}(\mathcal{O}_\mathcal{X}, k, \mathrm{id})$, $x \longmapsto (x, x^{p^i})$. Moreover the association

$$\mathcal{D}_i \longrightarrow \mathcal{D}_{i+1},\ (\mathcal{F}, V, \lambda) \longmapsto (\mathcal{F}, F_k^* V, F^*\lambda)$$

where F_k is the absolute Frobenius of k, is k-linear and monoidal. We can therefore define

$$\mathcal{D}_\infty = \varinjlim_{i \in \mathbb{N}} \mathcal{D}_i$$

Theorem 7.1. *Let $\mathcal{X}$ be a reduced category fibered in groupoids over k. Then $\mathcal{X}$ admits a Nori local fundamental gerbe over k if and only if $\mathrm{H}^0(\mathcal{O}_\mathcal{X})$ does not contain non trivial purely inseparable field extensions of k. In this case $\mathcal{D}_\infty$ is a k-Tannakian category and the map $\mathcal{X} \longrightarrow \Pi_{\mathcal{D}_\infty}$, induced by the forgetful functor $\mathcal{D}_\infty \longrightarrow \mathsf{Vect}(\mathcal{X})$, is the pro-local Nori fundamental gerbe of $\mathcal{X}$.*

If $\mathrm{H}^0(\mathcal{O}_\mathcal{X}) = k$ then $\mathsf{Rep}(\Pi^{\mathrm{N,L}}_{\mathcal{X}/k}) \longrightarrow \mathsf{Vect}(\mathcal{X})$ is an equivalence onto the full subcategory of $\mathsf{Vect}(\mathcal{X})$ of sheaves $\mathcal{F}$ such that $F^{i*}_\mathcal{X}\mathcal{F}$ is free for some $i \in \mathbb{N}$.

Proof. The only if part in the first claim is very similar to the proof in 4.3, taking into account that a finite and purely inseparable field extension is a finite and local stack. For the if part it is enough to show the remaining claims in the statement. We apply 5.12 on the map $\mathcal{X} \longrightarrow \mathcal{X}_\mathcal{T} = \operatorname{Spec} k$, which satisfies axiom A and $L = \mathrm{H}^0(\mathcal{O}_{\mathcal{X}_\mathcal{T}}) = k$ is a field. We have $\mathcal{T}_i(\mathcal{X}) = \mathcal{D}_i$ for all $i \in \mathbb{N} \cup \{\infty\}$ and that $\mathcal{X} \longrightarrow (\Pi_{\mathcal{D}_\infty})_\mathrm{L}$ is the local Nori fundamental gerbe of $\mathcal{X}/L_\infty$. Since $L_0 = k$ and $L_0 \subseteq L_i$ are purely inseparable inside $\mathrm{H}^0(\mathcal{O}_\mathcal{X})$ we also have $L_i = k$ for all $i \in \mathbb{N} \cup \{\infty\}$. Thus it remains to show $\Pi_{\mathcal{D}_\infty}$ is pro-local. Thanks to 5.11, for any $V \in \mathcal{D}_\infty = \mathsf{Vect}(\Pi_{\mathcal{D}_\infty})$ there exists an index $i \in \mathbb{N}$ such that $F^{i*}_{\Pi_{\mathcal{D}_\infty}} V$ is free, where $F_{\Pi_{\mathcal{D}_\infty}}$ is the absolute Frobenius of $\Pi_{\mathcal{D}_\infty}$, and by 5.12 plus the fact that $\mathcal{T}(\mathcal{X}) = \mathsf{Vect}(k)$ is made of finite objects, V is essentially finite. Let Γ be the monodromy gerbe of $V \in \mathcal{D}_\infty$ (see B.8). Then the absolute Frobenius F^i_Γ factors as $\Gamma \xrightarrow{\pi} \operatorname{Spec}(k) \to \Gamma$, where π is the structure map of Γ/k. This implies immediately that Γ is local.

In the last claim we have to show that $\mathcal{D}_\infty \longrightarrow \mathsf{Vect}(\mathcal{X})$ is full. Actually one can easily check that $\mathcal{D}_i \longrightarrow \mathsf{Vect}(\mathcal{X})$ is fully faithful for all $i \in \mathbb{N}$. $\square$

Appendix A. Limit of categories and fibered categories

Definition A.1. Let I be a filtered category. A directed system of categories indexed by I is a pseudo-functor $\mathcal{D}_*\colon I \longrightarrow (\mathrm{Cat})$ [Vis, Def. 3.10]. Concretely this is the assignment of data $(\mathcal{D}_i, \mathcal{D}_\alpha, \lambda_{\alpha,\beta}, \lambda_i)$: categories $\mathcal{D}_i$ for all $i \in I$, functors $\mathcal{D}_\alpha\colon \mathcal{D}_i \longrightarrow \mathcal{D}_j$ for all $i \xrightarrow{\alpha} j$ in I and natural isomorphisms $\lambda_{\alpha,\beta}\colon \mathcal{D}_\beta \circ \mathcal{D}_\alpha \longrightarrow \mathcal{D}_{\beta\alpha}$ for all composable arrows $i \xrightarrow{\alpha} j \xrightarrow{\beta} k$ and $\lambda_i\colon \mathcal{D}_{\mathrm{id}_i} \longrightarrow \mathrm{id}_{\mathcal{D}_i}$ for all $i \in I$. This data is subject to compatibility conditions (see [Vis, Def. 3.10]).

We define the limit of $\mathcal{D}_*$, written $\lim_{i\in I} \mathcal{D}_i$ or $\mathcal{D}_\infty$, in the following way. The category $\mathcal{D}_\infty$ has pairs (i,x), where $i \in I$ and $x \in \mathcal{D}_i$, as objects. Given $(i,x),(j,y) \in \mathcal{D}_\infty$ the set $\mathrm{Hom}_{\mathcal{D}_\infty}((i,x)),(j,y))$ is the limit on the category of pairs $(i \xrightarrow{\alpha} k, j \xrightarrow{\beta} k)$ (which is a filtered category) of the sets $\mathrm{Hom}_{\mathcal{D}_k}(\mathcal{D}_\alpha(x), \mathcal{D}_\beta(y))$. Composition is defined in the obvious way. For all $i \in I$ there are functors $F_i\colon \mathcal{D}_i \longrightarrow \mathcal{D}_\infty$, $F_i(x) = (i,x)$ and, for all $i \xrightarrow{\alpha} j$ in I, there are canonical isomorphism $\mu_\alpha\colon F_j \circ \mathcal{D}_\alpha \longrightarrow F_i$.

Given a category $\mathcal{C}$ we define the category $\mathcal{C}^{\mathcal{D}}$ in the following way. The objects are collections $(H_i, \delta_\alpha)_{i, i\xrightarrow{\alpha} j}$ where: $H_i\colon \mathcal{D}_i \longrightarrow \mathcal{C}$ are functors for all $i \in I$, $\delta_\alpha\colon H_j \circ \mathcal{D}_\alpha \longrightarrow H_i$ are natural isomorphisms for all arrows $i \xrightarrow{\alpha} j$ in I. This data is subject to the following compatibilities. For all $i \in I$ we have $\delta_{\mathrm{id}_i} = H_i \circ \lambda_i\colon H_i \circ \mathcal{D}_{\mathrm{id}_i} \longrightarrow H_i$. For all composable arrows $i \xrightarrow{\alpha} j \xrightarrow{\beta} k$ the following diagram commutes

$$\begin{tikzcd} H_k \circ \mathcal{D}_{\beta\alpha} \arrow[r, "\delta_{\beta\alpha}"] & H_i \\ H_k \circ \mathcal{D}_\beta \circ \mathcal{D}_\alpha \arrow[u, "H_k\circ\lambda_{\alpha,\beta}"] \arrow[r, "\delta_\beta\circ\mathcal{D}_\alpha"] & H_j \circ \mathcal{D}_\alpha \arrow[u, "\delta_\alpha"'] \end{tikzcd}$$

The arrows in $\mathcal{C}^{\mathcal{D}}$ are the obvious ones.

Given a functor $G\colon \mathcal{C} \longrightarrow \mathcal{C}'$ one can easily define a functor $G^{\mathcal{D}}\colon \mathcal{C}^{\mathcal{D}} \longrightarrow \mathcal{C}'^{\mathcal{D}}$. Moreover the data (F_i, μ_α) defined above is an object of $\mathcal{D}_\infty^{\mathcal{D}}$. In particular we obtain a functor

$$\chi_{\mathcal{C}}\colon \mathrm{Hom}(\mathcal{D}_\infty, \mathcal{C}) \longrightarrow \mathcal{C}^{\mathcal{D}},\ (\mathcal{D}_\infty \xrightarrow{G} \mathcal{C}) \longmapsto G^{\mathcal{D}}(F_i, \mu_\alpha)$$

Proposition A.2. *The functor $\chi_{\mathcal{C}}$ in A.1 is an isomorphism of categories.*

Proof. Let's define a functor $\iota\colon \mathcal{C}^{\mathcal{D}} \longrightarrow \mathrm{Hom}(\mathcal{D}_\infty, \mathcal{C})$. Given $a = (H_i, \delta_\alpha) \in \mathcal{C}^{\mathcal{D}}$ defines $\iota(a)\colon \mathcal{D}_\infty \longrightarrow \mathcal{C}$ as follows. For $(i,x) \in \mathcal{D}_\infty$ set $\iota(a)(i,x) = H_i(x)$. For $\phi\colon (i,x) \longrightarrow (j,y)$ in $\mathcal{D}_\infty$ choose $i \xrightarrow{f} k, j \xrightarrow{g} k$ such that ϕ is induced by the arrow $v\colon \mathcal{D}_f(x) \longrightarrow \mathcal{D}_g(y)$ in $\mathcal{D}_k$. Set $\iota(a)(\phi)$ as the only dashed arrow making the following diagram commutative

$$\begin{tikzcd} H_k \circ D_f(x) \arrow[r, "H_k(v)"] \arrow[d, "\delta_f"] & H_k \circ D_g(y) \arrow[d, "\delta_g"] \\ H_i(x) \arrow[r, dashed] & H_j(y) \end{tikzcd}$$

A direct check shows that this arrows does not depend on the choices of f, g, v. In particular $\iota(a)$ is easily seen to be a functor $\mathcal{D}_\infty \longrightarrow \mathcal{C}$. The action of ι on arrows is the obvious one:

the required compatibilities follows from the compatibilities of arrows in $\mathcal{C}^{\mathcal{D}}$. In conclusion one get a functor $\iota\colon \mathcal{C}^{\mathcal{D}} \longrightarrow \mathrm{Hom}(\mathcal{D}_\infty, \mathcal{C})$. The equality $\chi_{\mathcal{C}} \circ \iota = \mathrm{id}$ can be checked directly.

For the converse let $G\colon \mathcal{D}_\infty \longrightarrow \mathcal{C}$ be a functor. We have $\chi_C(G) = (G \circ F_i, G \circ \mu_\alpha)$ and set $\widetilde{G} = \iota(\chi_C(G))$. We must show that $G = \widetilde{G}$. For $i \in I$ and $x \in \mathcal{D}_i$ we have $G(i,x) = G(F_i(x)) = \widetilde{G}(x)$. Let now $\phi\colon (i,x) \longrightarrow (j,y)$ be an arrow in $\mathcal{D}_\infty$ and $i \xrightarrow{\alpha} k, j \xrightarrow{\beta} k$ arrows, $v\colon \mathcal{D}_\alpha(x) \longrightarrow \mathcal{D}_\beta(y)$ inducing ϕ. This can be expressed in the following commutative diagram

$$\begin{tikzcd}
(k, \mathcal{D}_\alpha(x)) \arrow[r, "\mu_\alpha(x)"] \arrow[d, "F_k(v)"] & (i,x) \arrow[d, "\phi"] \\
(k, \mathcal{D}_\beta(y)) \arrow[r, "\mu_\beta(y)"] & (j,y)
\end{tikzcd}$$

We have $G(F_k(v)) = \widetilde{G}(F_k(v))$, $G \circ \mu_\alpha = \widetilde{G} \circ \mu_\alpha$ and $G \circ \mu_\beta = \widetilde{G} \circ \mu_\beta$ by construction. It follows that $G(\phi) = \widetilde{G}(\phi)$. $\square$

Remark A.3. When $I = \mathbb{N}$ with the usual order a directed system $\mathcal{D}_*$ of categories indexed by $\mathbb{N}$ is just an infinite sequence of categories and functors:

$$\mathcal{D}_0 \xrightarrow{G_0} \mathcal{D}_1 \xrightarrow{G_1} \mathcal{D}_2 \xrightarrow{G_2} \cdots$$

Moreover if $\mathcal{C}$ is a category then $\mathcal{C}^{\mathcal{D}}$ is equivalent to the category whose objects are tuples (H_n, σ_n) where: $H_n\colon \mathcal{D}_n \longrightarrow \mathcal{C}$ is a functor, $\sigma_n\colon H_{n+1} \circ G_n \longrightarrow H_n$ a natural isomorphism.

Let $\mathcal{D}_*$ be a direct system of categories indexed by I. We have the following fact which are easy to check:

- If for all arrows α in I the functor $\mathcal{D}_\alpha$ is faithful (resp. fully faithful, equivalence) then for all $i \in I$ the functor F_i is faithful (resp. fully faithful, equivalence);
- If for all $i \in I$ the category $\mathcal{D}_i$ is a groupoid then $\mathcal{D}_\infty$ is a groupoid;
- If R is a ring, for all $i \in I$ the category $\mathcal{D}_i$ is R-linear and for all arrows α in I the functor $\mathcal{D}_\alpha$ is R-linear then $\mathcal{D}_\infty$ is naturally an R-linear category and for all $i \in I$ the functor F_i is R-linear;
- If for all $i \in I$ the category $\mathcal{D}_i$ is abelian and for all arrows α the functor $\mathcal{D}_\alpha$ is additive and exact, then $\mathcal{D}_\infty$ is an abelian category and for all $j \in I$ the functor F_j is also additive and exact.
- If for all $i \in I$ the category $\mathcal{D}_i$ is monoidal and for all arrows α, β in I the functor D_α has a monoidal structure and the $\lambda_{\alpha,\beta}$ are monoidal then we can endow $\mathcal{D}_\infty$ and, for all $i \in I$, F_i with a monoidal structure in the following way. Given $i, j \in I$ choose $k_{i,j} \in I$, maps $i \xrightarrow{\alpha_{i,j}} k_{i,j}$, $j \xrightarrow{\beta_{i,j}} k_{i,j}$ and define
$$(i,x) \otimes (j,y) = (k_{i,j}, \mathcal{D}_{\alpha_{i,j}}(x) \otimes_{\mathcal{D}_{k_{i,j}}} \mathcal{D}_{\beta_{i,j}}(y))$$
and $(i_0, 1_{\mathcal{D}_{i_0}})$ as unit for a chosen $i_0 \in I$. All the maps required in order to have a monoidal structure are easy to define.

Proposition A.4. *Let $\mathcal{C}$ be a category with fiber products, I be a filtered category and $\mathcal{X}_*$ be a directed system of fibered categories over $\mathcal{C}$, that is a direct system of categories $\mathcal{X}_*$ given* *by data $(\mathcal{X}_i, \mathcal{X}_\alpha, \lambda_{\alpha,\beta}, \lambda_i)$ such all $\pi_i \colon \mathcal{X}_i \longrightarrow \mathcal{C}$ are fibered categories, all $\mathcal{X}_\alpha \colon \mathcal{X}_i \longrightarrow \mathcal{X}_j$ are maps of fibered categories and all $\lambda_{\alpha,\beta}$, λ_i are base preserving natural transformations. Then the induced functor $\mathcal{X}_\infty \longrightarrow \mathcal{C}$ makes $\mathcal{X}_\infty$ into a fibered category, the functor $F_i \colon \mathcal{X}_i \longrightarrow \mathcal{X}_\infty$ are maps of fibered categories and μ_α are base preserving natural transformations. Moreover if all $\mathcal{X}_i$ are fibered in groupoids (resp. sets) then so is $\mathcal{X}_\infty$.*

If $c \in \mathcal{C}$ then the direct system $\mathcal{X}_*$ induces a direct system of categories $\mathcal{X}(c)_* \colon I \longrightarrow (cat)$ and the $F_i \colon \mathcal{X}_i \longrightarrow \mathcal{X}_\infty$ and the natural transformations μ_α induces an equivalence

$$\mathcal{X}(c)_\infty \simeq \mathcal{X}_\infty(c)$$

If $\mathcal{Y}$ is another fiber category over $\mathcal{C}$ then $\chi_{\mathcal{X}}$ restricts to an isomorphism between $\mathrm{Hom}_{\mathcal{C}}(\mathcal{X}_\infty, \mathcal{Y})$ and the full subcategory of $\mathcal{Y}^{\mathcal{X}}$ of objects (H_i, δ_α) such that H_i are maps of fibered categories and the δ_α are base preserving natural transformations.

Proof. We have that $(\pi_i, \omega_\alpha) \in \mathcal{C}^{\mathcal{X}}$, where we set $\omega_\alpha = \mathrm{id}$ for all α, because the π_i strictly commutes with the $\mathcal{X}_\alpha$. We therefore get a functor $\pi_\infty \colon \mathcal{X}_\infty \longrightarrow \mathcal{C}$ such that $\pi_i = \pi_\infty \circ F_i$ and $\pi_\infty(\mu_\alpha) = \mathrm{id}$. The first equation assures that the F_i strictly commutes over $\mathcal{C}$, the second assures that the μ_α are base preserving natural transformations. Moreover it is easy to see that the F_i map Cartesian arrows to Cartesian arrows, which in particular implies that $\mathcal{X}_\infty$ is a fibered category.

The system $\mathcal{X}_*$ together with the structure morphisms π_i can be seen as a pseudo-functor from I to the 2-category $\mathrm{Fib}(\mathcal{C})$ of fibered categories over $\mathcal{C}$. Given $c \in \mathcal{C}$ the evaluation in c yields a functor $\mathrm{Fib}(\mathcal{C}) \longrightarrow (\mathrm{cat})$ and, composing, we obtain the direct system $\mathcal{X}(c)_*$. It is easy to see that $\mathcal{X}_\infty(c)$ and $\mathcal{X}(c)_\infty$ are the same categories. In particular if all $\mathcal{X}_i$ are fibered in groupoids (resp. sets) then so is $\mathcal{X}_\infty$.

Let $G \colon \mathcal{X}_\infty \longrightarrow \mathcal{Y}$ any functor and $\chi_{\mathcal{X}}(G) = (G \circ F_i, G \circ \mu_\alpha) \in \mathcal{Y}^{\mathcal{X}}$. It is easy to see that G is base preserving if and only if the $G \circ F_i$ and $G(\mu_\alpha)$ are base preserving. In this case, assuming that the $G \circ F_i$ preserve Cartesian arrows, we have to show that G does the same. This follows from the fact that a Cartesian arrow γ in $\mathcal{X}_\infty$ is, up to isomorphism, determined by the target of γ and $\pi_\infty(\gamma)$, which implies that γ is image of a Cartesian arrow in some $\mathcal{X}_i$. □

Appendix B. Affine gerbes and Tannakian categories

Let k be a field. In this appendix we collect useful results about affine gerbes and Tannakian categories. Recall that an affine gerbe Γ over k is a gerbe for the fpqc topology $\Gamma \longrightarrow \mathrm{Aff}/k$ with affine diagonal. If L/k is a field extension and $\xi \in \Gamma(L)$ then Γ is affine if and only if $\underline{\mathrm{Aut}}_\Gamma(\xi)$ is an affine scheme. Moreover any map from a scheme $X \longrightarrow \Gamma$ is an fpqc covering which is affine if X is affine. (See [BV, Prop 3.1] for details.)

A k-Tannakian category is a k-linear, monoidal, rigid and abelian category $\mathcal{C}$ such that $\mathrm{End}_{\mathcal{C}}(1_{\mathcal{C}}) = k$ (where $1_{\mathcal{C}}$ is the unit) and there exists a field extension L/k and a k-linear, exact and monoidal functor $\mathcal{C} \longrightarrow \mathsf{Vect} L$.

Classical Tannaka's duality states that the functors $\mathsf{Vect}(-)$ and Π_* between the 2-categories of affine gerbes over k and k-Tannakian categories are "quasi-inverses" of each other. See Section 1 for the definition of Π_* and of the natural functors $\mathcal{C} \longrightarrow \mathsf{Vect}(\Pi_{\mathcal{C}})$ and $\Gamma \longrightarrow \Pi_{\mathsf{Vect}(\Gamma)}$.

Given an affine gerbe Γ we will often use the notation $\mathsf{Rep}\Gamma$ instead of $\mathsf{Vect}(\Gamma)$.

Definition B.1. A map of affine group schemes $G \longrightarrow G'$ over k is a quotient if it is faithfully flat or equivalently if $\mathrm{H}^0(\mathcal{O}_{G'}) \longrightarrow \mathrm{H}^0(\mathcal{O}_G)$ is injective (see [Wat, Chapter 14]).

A map of affine gerbes $\Gamma \xrightarrow{\phi} \Gamma'$ over k is a quotient (resp. faithful) if there exists a field L and $\xi \in \Gamma(L)$ such that the map of affine group schemes $\underline{\mathrm{Aut}}_\Gamma(\xi) \longrightarrow \underline{\mathrm{Aut}}_{\Gamma'}(\phi(\xi))$ is a quotient (a monomorphism or equivalently a closed immersion by [Wat, Section 15.3]). This notion does not depend on the choice of ξ and L. Moreover ϕ is faithful if and only if it is faithful as a functor.

Proposition B.2. *Let $\phi\colon \Gamma \longrightarrow \Gamma'$ be a map of affine gerbes. Then*

(1) the map $\mathcal{O}_{\Gamma'} \longrightarrow \phi_*\mathcal{O}_\Gamma$ is an isomorphism if and only if $\phi^*\colon \mathsf{Rep}\Gamma' \longrightarrow \mathsf{Rep}\Gamma$ is fully faithful;
(2) the following are equivalent: a) ϕ is a quotient; b) ϕ is a relative gerbe; c) the functor $\phi^*\colon \mathsf{Rep}\Gamma' \longrightarrow \mathsf{Rep}\Gamma$ is fully faithful and its image is stable under quotients;
(3) the functor ϕ is faithful if and only if all $V \in \mathsf{Rep}\Gamma$ is a subquotient of ϕ^*W for some $W \in \mathsf{Rep}\Gamma'$.

Proof. For (1), the map $\rho\colon \mathcal{O}_{\Gamma'} \longrightarrow \phi_*\mathcal{O}_\Gamma$ induces maps

$$\mathrm{Hom}_{\Gamma'}(V, W) \longrightarrow \mathrm{Hom}_{\Gamma'}(V, W \otimes \phi_*\mathcal{O}_\Gamma) \simeq \mathrm{Hom}_\Gamma(\phi^*V, \phi^*W) \text{ for } V, W \in \mathsf{Rep}\Gamma'$$

So if ρ is an isomorphism then ϕ^* is fully faithful. Conversely assume the above map bijective for all V, W and choose $W = \mathcal{O}_{\Gamma'}$. The map ρ is injective since ϕ is faithfully flat. The surjectivity follows using that $\mathsf{Rep}\Gamma$ generates $\mathrm{QCoh}(\Gamma)$ by [De3, Cor 3.9, pp. 132].

For $(2), a) \iff c)$ and (3) see [Saa, 3.3.3 c), pp. 205]. For $(2), a) \iff b)$ we can assume $\Gamma = \mathrm{B}\,G$, $\Gamma' = \mathrm{B}\,G'$ and ϕ induced by $G \longrightarrow G'$. If ϕ is a quotient then $\mathrm{B}\,G \times_{\mathrm{B}\,G'} \operatorname{Spec} k \simeq \mathrm{B}\,K$, where K is the kernel of $G \longrightarrow G'$, and thus ϕ is a relative gerbe. For the converse, one can replace Γ by the image of $G \longrightarrow G'$ and assume $G \subseteq G'$ a closed subgroup. In this case $\mathrm{B}\,G \times_{\mathrm{B}\,G'} \operatorname{Spec} k \simeq G'/G$ and $G'/G \longrightarrow \operatorname{Spec} k$ is a gerbe if and only if it is an isomorphism, that is $G' = G$. ☐

Definition B.3. Given a Tannakian category $\mathcal{C}$ a full Tannakian subcategory of $\mathcal{C}$ is a sub-abelian, sub-monoidal and rigid full subcategory $\mathcal{D} \subseteq \mathcal{C}$ which is stable under quotients (in other words is the image of a functor $\mathsf{Rep}\Gamma' \longrightarrow \mathcal{C}$ induced by a quotient map $\Pi_{\mathcal{C}} \longrightarrow \Gamma'$.

Given a subset T of objects of $\mathcal{C}$ we denote by $\langle T \rangle$ the full subcategory of $\mathcal{C}$ whose objects are subquotients of objects of the form $P(X)$ or $P(X^\vee)$ for $X \in T$ and $P \in \mathbb{N}[t]$. It is easy to see that $\langle T \rangle$ is the smallest full Tannakian subcategory of $\mathcal{C}$ containing T. For this reason we call $\langle T \rangle$ the sub Tannakian category spanned by T.

Definition B.4. If $\phi\colon \Gamma \longrightarrow \Gamma'$ is a map of affine gerbe there exists a unique (up to a unique isomorphism) factorization of ϕ as $\Gamma \xrightarrow{\alpha} \Delta \xrightarrow{\beta} \Gamma'$, where α is a quotient and β is faithful. We call Δ the image of ϕ.

Definition B.5. A finite gerbe over k is an affine gerbe over k which is a finite stack. An affine gerbe Γ over k is finite and étale (resp. local) if it is finite and étale (resp. local) in the sense of 3.1 (resp. 3.9).

Proposition B.6. *Let Γ be an affine gerbe over k, L/k be a field extension and $\xi \in \Gamma(L)$.*

(1) They are equivalent: a) Γ is an algebraic stack; b) $\underline{\mathrm{Aut}}_\Gamma(\xi)/L$ is of finite type; c) there exists $V \in \mathsf{Rep}\Gamma$ such that $\langle V \rangle = \mathsf{Rep}\Gamma$.

(2) The gerbe Γ is finite if and only if there exists $V \in \mathsf{Rep}\Gamma$ generating $\mathrm{QCoh}(\Gamma)$ (see 1.3);

(3) The gerbe Γ is finite (resp. finite and étale, finite and local) if and only if $\underline{\mathrm{Aut}}_\Gamma(\xi)/L$ is finite (resp. finite and étale, finite and local).

Proof. Implications 1), b) $\iff$ c) $\implies$ a) follows from [Saa, Chapter III, 3.3.1.1] and fpqc descent. For a) $\implies$ b), we choose an fppf atlas $X \to \Gamma$ with X a k-scheme. Since $X \times_\Gamma X$ is an fppf X-algebraic space, the map $X \times_\Gamma X \to X \times_k X$ is also fppf. This implies that the diagonal of Γ is fppf, whence the result.

Item 2) is proved in [Saa, Chapter III, 3.3.3 a)], while 3) follows from 1) and 3.7. ☐

Remark B.7. Let ϕ be a map of gerbes factorizing as $\Gamma \xrightarrow{\alpha} \Delta \xrightarrow{\beta} \Gamma'$, where α is a quotient and β is faithful. If β is affine then ϕ is a quotient if and only if $\phi^*\colon \mathsf{Rep}\Gamma' \longrightarrow \mathsf{Rep}\Gamma$ is fully faithful. Indeed in this last case also $\beta^*\colon \mathsf{Rep}\Gamma' \longrightarrow \mathsf{Rep}\Delta$ would be fully faithful, that is $\mathcal{O}_{\Gamma'} \simeq \beta_*\mathcal{O}_\Delta$ thanks to B.2: if β is affine than it is an isomorphism.

The map β is affine in the following cases: Δ is finite, for instance if Γ or Γ' is finite; Γ is of finite type and ϕ is a relative Frobenius. Moreover, if L/k is a field extension, $\xi \in \Gamma(L)$, $v\colon G = \underline{\mathrm{Aut}}_\Gamma(\xi) \longrightarrow \underline{\mathrm{Aut}}_{\Gamma'}(\phi(\xi)) = G'$ and H its image, then β is affine if and only if G'/H is affine, which is true in the following cases: H is normal in G', for instance if Γ' is abelian; G' is of finite type and the closed immersion $H \longrightarrow G'$ is nilpotent.

This can be proved when $L = k$ is algebraically closed, so that $\Gamma = \mathrm{B}\,G$, $\Gamma' = \mathrm{B}\,G'$ and ϕ is induced by $v\colon G \longrightarrow G'$. The map β is $\mathrm{B}\,H \longrightarrow \mathrm{B}\,G'$ and we have a 2-Cartesian diagram

$$\begin{array}{ccc} G'/H & \longrightarrow & \operatorname{Spec} k \\ \downarrow & & \downarrow \\ \mathrm{B}\,H & \xrightarrow{\beta} & \mathrm{B}\,G' \end{array}$$

So β is affine if and only if G'/H is affine. This is the case if H is finite (see [SP, 03BM]) or if H is normal (see [Wat, Section 16.3]). If $H(k) = G'(k)$, as for the relative Frobenius, we have that G'/H is an algebraic space of finite type and with only one rational section $p \in G'/H$. The complement of p is an algebraic space of finite type without rational points and thus empty. Since quasi-separated algebraic spaces are generically schemes, we can conclude that G'/H is a scheme of finite type over k with just one point, thus a finite k-scheme.

Definition B.8. Given an affine gerbe Γ over k and $E \in \mathsf{Vect}(\Gamma)$, the *monodromy gerbe* of E, denoted by Γ_E, is the gerbe corresponding to $\langle E \rangle$, or, equivalently, the image of the map $\Gamma \longrightarrow \mathrm{B}\,\mathrm{GL}_n$ induced by E (where $n = \mathrm{rk}\, E$). By B.6 Γ_E is of finite type over k.

Let $\mathcal{C}$ be a Tannakian category, we denote $\mathrm{EFin}(\mathcal{C})$ (resp. $\mathrm{\acute{E}t}(\mathcal{C})$, $\mathrm{Loc}(\mathcal{C})$) the full sub category of $\mathcal{C}$ consisting of objects with finite (resp. finite and étale, finite and local) monodromy gerbe. Given $E, F \in \mathcal{C}$ the monodromy gerbe of $E \oplus F$ is the image of $\Pi_{\mathcal{C}} \longrightarrow (\Pi_{\mathcal{C}})_E \times_k (\Pi_{\mathcal{C}})_F$. In particular $\mathrm{EFin}(\mathcal{C})$ (resp. $\mathrm{\acute{E}t}(\mathcal{C})$, $\mathrm{Loc}(\mathcal{C})$) is a full Tannakian subcategory of $\mathcal{C}$. Let Γ be an affine gerbe. We say that Γ is profinite (resp. pro-étale, pro-local) if it is a filtered projective limit (in the sense of [BV, Section 3]) of finite (resp. finite and étale, finite and local) gerbes. We denote by $\widehat{\Gamma}$ (resp. $\Gamma_{\mathrm{\acute{e}t}}$, Γ_{L}) the quotient gerbe $\Pi_{\mathrm{EFin}(\mathsf{Rep}\Gamma)}$ (resp. $\Pi_{\mathrm{\acute{E}t}(\mathsf{Rep}\Gamma)}$, $\Pi_{\mathrm{Loc}(\mathcal{C})}$) and call it the profinite (resp. pro-étale, pro-local) quotient of Γ. Notice that Γ is profinite (resp. pro-étale, pro-local) if and only if $\Gamma = \widehat{\Gamma}$ (resp. $\Gamma = \Gamma_{\mathrm{\acute{e}t}}$, $\Gamma = \Gamma_{\mathrm{L}}$).

In general, if $\mathcal{C}$ is an additive and monoidal category we denote by $\mathrm{EFin}(\mathcal{C})$ the full subcategory of $\mathcal{C}$ consisting of essentially finite objects (see [BV, Def 7.7]). When $\mathcal{C}$ is k-Tannakian the two notions just introduced agree thanks to [BV, Thm 7.9] and, if Γ is an affine gerbe over k, $\widehat{\Gamma} = \Pi^{\mathrm{N}}_{\Gamma/k}$, $\Gamma_{\mathrm{\acute{e}t}} = \Pi^{\mathrm{N,\acute{e}t}}_{\Gamma/k}$ and $\Gamma_{\mathrm{L}} = \Pi^{\mathrm{N,L}}_{\Gamma/k}$.

We say that Γ is smooth (pro-smooth) banded if there exists L/k field extension and $\xi \in \Gamma(L)$ such that $\underline{\mathrm{Aut}}_{\Gamma}(\xi)$ is a smooth group scheme over L (a projective limit of smooth group schemes over L).

Remark B.9. An affine gerbe Γ is pro-smooth banded if and only if any finite type quotient of Γ is smooth banded. The implication "$\Leftarrow$" follows from the fact that affine gerbes are projective limit of gerbes of finite type. For the other, we can reduce to the neutral case, so that one has to prove that if $v\colon G_\infty = \varprojlim_j G_j \longrightarrow G$ is a quotient, G is of finite type and the G_j are smooth then G is smooth. But v factors through a quotient map $G_j \longrightarrow G$. Since $G_j \longrightarrow G$ is faithfully flat and G_j is smooth it follows that G is smooth.

References

[BO] P. Berthelot and A. Ogus, *Notes on crystalline cohomology*, Princeton University Press, 1978.

[BV] N. Borne, A. Vistoli, *The Nori fundamental gerbe of a fibered category*, J. Algebraic Geometry, S 1056-3911, 00638-X, 2014.

[De1] P. Deligne, *Le Groupe Fondamental de la Droit Projective Moins Trois Points*: in *Galois Group over* $\mathbb{Q}$, Springer-Verlag, 1989.

[De2] P. Deligne and J. Milne, *Tannakian categories*: in Lecture Notes in Math., number 900, pages 101-228. Springer Verlag, 1982.

[De3] P. Deligne, *Catégories Tannakiannes*, The Grothendieck Festschrift, Vol. II, Progr. Math., vol. 87, Birkhäuser Boston, Boston, MA, pp. 111-195, 1990.

[EGA3] A. Grothendieck, J. A. Dieudonné, EGA III: *Étude cohomologique des schéma et des faisceaux cohérents*, Seconde partie, Publications mathématiques de l'I.H.É.S., 17 (1963), pp. 5-91.

[EGA4] A. Grothendieck, J. A. Dieudonné, EGA IV: *Étude locale des schémas et des morphismes de schémas*, Quatriéme partie, Publications mathématiques de l'I.H.É.S., 32 (1967), pp. 5-361.

[EH] H. Esnault, A. Hogadi, *On the algebraic fundamental group of smooth varieties in characteristic* $p > 0$, Transactions of the American Mathematical Society, Volume 364, Number 5, pp. 2429-2442, 2012.

[Gie] D. Gieseker, *Flat vector bundles and the fundamental group in nonzero characteristics*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 2 (1975), no. 1, pp 1-31. MR MR0382271 (52 #3156).

[Nori] M. Nori, *The fundamental group schemes*, Proc.Indian Aacd.Sci. 91, pp. 73-122, 1982.

[Saa] N. Saavedra, *Catégories Tannakiennes*, LNM 265, Springer Verlag, 1972.

[SGA1] A. Grothendieck, M. Raynaud, *Revêtements Étales et Groupe Fondamental*, SGA 1, Springer-Verlag, 1971.

[SGA4] M. Artin, A. Grothendieck, J-L. Verdier, *Théorie de Topos et Cohomologie Etale des Schémas*, SGA 4, Springer-Verlag, 1963/64.

[SP] Stack Project Authors, *Stack Project*, `http://stacks.math.columbia.edu/`.

[Ton] F. Tonini, *Sheafification functors and Tannaka's reconstruction*, arXiv:1409.4073, pp. 35, September 2014.

[Vis] A. Vistoli, *Grothendieck Topologies, Fibred Categories and Descent theory*, in *Fundamental Algebraic Geometry*, American Mathematical Society, 2006.

[Wat] W.C. Waterhouse, *Introduction to Affine Group Schemes (Graduate Texts in Mathematics)*, pp. 184, Springer, 1979.

Fabio Tonini, Freie Universität Berlin, FB Mathematik und Informatik, Arnimallee 3, Zimmer 112A, 14195 Berlin, Deutschland

E-mail address: `tonini@zedat.fu-berlin.de`

Lei Zhang, Freie Universität Berlin, FB Mathematik und Informatik, Arnimallee 3, Zimmer 112A, 14195 Berlin, Deutschland

E-mail address: `l.zhang@fu-berlin.de`