INTRODUCTION TO Bun_G

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1. Stacks and Algebraic Stacks

A sheaf in, say fpqc topology on Aff, is a contra-variant functor

$$\mathcal{F} : (\mathrm{Aff}) \to (\mathrm{Sets})$$

satisfying some gluing conditions. A stack is just a sheaf but taking values in the 2-category of categories instead of in the the 1-category of sets. This means that to define stacks we have to reformulate the sheaf axioms in the 2-category setting.

First we take care of morphisms: For any affine scheme X and any two objects $A, B \in F(X)$ we get a functor

$$\mathcal{F}_{A,B} : (\mathrm{Aff}/X) \to (\mathrm{Sets})$$

which sends any morphism $f^* : Y \to X$ to the set $\mathrm{Hom}_{F(Y)}(f^*A, f^*B)$. The sheaf $\mathcal{F}_{A,B}$ has to be a sheaf on the site (Aff/X). A 2-functor satisfying this property is morally a "presheaf".

For objects: Let X be an object in Aff, and let $\{U_i \to X\}_{i\in I}$ be a covering of X. If $A_i \in \mathcal{F}(U_i)$ are objects and

$$\phi_{ij} : A_i|_{U_i\times_X U_j} \to A_j|_{U_i\times_X U_j}$$

are morphisms in $\mathcal{F}(U_i \times_X U_j)$ which are compatible in a natural way (the cocycle condition), then there exists $A \in \mathcal{F}(X)$ whose restriction to U_i are A_i and whose restrictions to $U_i \times_X U_j$ induces the identifications ϕ_{ij}. A "presheaf" with this property is a stack.

Example 1.1. The 2-functor $\mathcal{F}$ sending any scheme X to the category of quasi-coherent sheaves on X is a stack in the fpqc topology.

A category is called a groupoid if all its morphisms are isomorphisms. A set is a groupoid in which all the arrows are identity morphisms.

Definition 1.2. An algebraic stack, say in fpqc topology, is a 2-functor

$$\mathcal{X} : (\mathrm{Aff}) \to (\mathrm{Groupoids}) \subseteq (\mathrm{Categories})$$

which is an fpqc stack with the following properties:

(1) The diagonal $\mathcal{X} \xrightarrow{\Delta} \mathcal{X}\times\mathcal{X}$ is representable (it would be nicer if they are also quasi-compact and separated).
(2) There is a scheme X and a smooth surjective morphism $X \twoheadrightarrow \mathcal{X}$.

Here the scheme X is called an atlas of $\mathcal{X}$ and the map $X \twoheadrightarrow \mathcal{X}$ is called a presentation.

Example 1.3. (1) A scheme is an algebraic stack.
(2) The stack of quasi-coherent sheaves is not an algebraic stack.

(3) Let G be an affine group scheme over k. Define $\mathcal{B}_k G$ to be the 2-functor sending any k-scheme X to the category of G-torsors over X. This 2-functor is clearly a stack. It is an algebraic stack iff G is linerly algebraic.

Definition 1.4. A morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is called locally of finite type (resp. locally of finite presentation) if for any presentation $Y \twoheadrightarrow \mathcal{Y}$ the fibred product $X \times_{\mathcal{Y}} Y$ has an atlas which is locally of finite type (resp. locally of finite presentation) over Y.

Definition 1.5. An algebraic stack $\mathcal{X}$ is called quasi-compact if there is an atlas X which is quasi-compact.

Definition 1.6. A morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is called quasi-compact if for any map $V \twoheadrightarrow \mathcal{Y}$ with V an affine scheme the fibred product $X \times_{\mathcal{Y}} V$ is quasi-compact.

Definition 1.7. A morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is called of finite type (resp. of finite presentation) if it is locally of finite type (resp. locally of finite presentation) and quasi-compact.

Definition 1.8. Let $\mathcal{X}$ be an algebraic stack. We set $|\mathcal{X}|$ the class

$$\coprod_{K \text{ is a field}} \mathrm{ob}(\mathcal{X}(K))$$

modulo the following equivalent relation: Two elements $x \in \mathrm{ob}(\mathcal{X}(K_1))$ and $y \in \mathrm{ob}(\mathcal{X}(K_2))$ are equivalent iff there is a field K_3 which contains both K_1 and K_2 and the restriction of x, y to K_3 are isomorphic.

Using 2-Yoneda lemma one can rephrase $|\mathcal{X}|$ to be the set of morphisms of the form $\mathrm{Spec}(K) \to \mathcal{X}$ modulo the equivalence relation that $\mathrm{Spec}(K_1) \to \mathcal{X}$ is $\mathrm{Spec}(K_2) \to \mathcal{X}$ iff there are morphisms $\mathrm{Spec}(K_3) \to \mathrm{Spec}(K_1)$ and $\mathrm{Spec}(K_3) \to \mathrm{Spec}(K_2)$ whose compositions with the morphisms we started with are equal.

For any presentation $X \twoheadrightarrow \mathcal{X}$ there is a clear map of sets $|X| \to |\mathcal{X}|$ which is surjective. We give $|\mathcal{X}|$ the quotient topology, i.e. the quotient of $|X|$. The topology is easily checked to be independent of the presentation.

2. The Hom-stack and Bun_G

Let S be a base scheme, and let $\mathcal{X}, \mathcal{Y}$ be two fibered categories over S. We can define a 2-functor

$$\mathcal{H}\mathrm{om}_S(\mathcal{X}, \mathcal{Y}) \colon (\mathrm{Aff}/S) \longrightarrow (\mathrm{Groupoids})$$

by sending any morphism $S' \to S$ to the category of functors $\mathrm{Hom}_{S'}(\mathcal{X} \times_S S', \mathcal{Y} \times_S S')$. One can show easily that $\mathcal{H}\mathrm{om}_S(\mathcal{X}, \mathcal{Y})$ is a stack as soon as $\mathcal{Y}$ is a stack.

Theorem 2.1. *(Hall and Rydh) Let $\mathcal{Y} \to S$ be a morphism of algebraic stacks that is locally of finite presentation, quasi-separated, and has affine stabilizers, with quasi-finite and separated diagonal. Let $\mathcal{X} \to S$ be a morphism of algebraic stacks that is proper, flat, and of finite presentation. Then the S-stack*

$$T \mapsto \mathrm{Hom}_T(\mathcal{X} \times_S T, \mathcal{Y} \times_S T)$$

is algebraic, locally of finite presentation, quasi-separated, with affine diagonal over S.

Definition 2.2. Let G be an affine group scheme over S, and let $\mathcal{X}$ be an algebraic stack over Aff/S. We define

$$\mathrm{Bun}_G(\mathcal{X}) := \mathcal{H}\mathrm{om}_S(\mathcal{X}, \mathcal{B}G)$$

This is a stack over Aff/S. If $G = \mathrm{GL}_r$, then we write $\mathrm{Bun}_r(\mathcal{X})$ for $\mathrm{Bun}_G(\mathcal{X})$.

3. The algebraicity of Bun_r

In this section we are going to show the following theorem:

Theorem 3.1. *Let X be a projective flat scheme over S with S Noetherian. Let G be a closed subgroup scheme of a general linear algebraic group* GL_n *and the fppf-quotient* GL_n/G *is a quasi-projective scheme over S. Then the stack* $\mathrm{Bun}_G(X)$ *is an algebraic stack locally of finite type over S.*

The theorem follows from Theorem 2.1. Here we will give a different but complete proof.

Lemma 3.2. *Let $\mathcal{Y}_1 \to \mathcal{Y}_2$ be a quasi-projective morphism of fibered categories (e.g. algebraic stacks), and let X be a proper flat scheme of finite presentation. Then the natural map*

$$\mathcal{H}\mathrm{om}(X, \mathcal{Y}_1) \to \mathcal{H}\mathrm{om}(X, \mathcal{Y}_2)$$

is representable by schemes which are locally of finite type.

Proof. Let S be a scheme, and let $S \to \mathcal{H}\mathrm{om}(X, \mathcal{Y}_2)$ be a morphism. Then one sees easily that the fibered product

$$S \times_{\mathcal{H}\mathrm{om}(X,\mathcal{Y}_2)} \mathcal{H}\mathrm{om}(X, \mathcal{Y}_1)$$

is equal to the following 2-functor

$$(\mathrm{Aff}/S) \longrightarrow (\mathrm{Groupoids})$$

$$(S' \to S) \mapsto \mathrm{Hom}_{X_S}(X_{S'}, \mathcal{Y}_1 \times_{\mathcal{Y}_2} X_S)$$

Thus the 2-functor over (Aff/S) is actually the space of sections of the projection

$$\mathrm{pr}_2 : \mathcal{Y}_1 \times_{\mathcal{Y}_2} X_S \to X_S$$

which is an open subscheme of the Hilbert scheme $\mathrm{Hilb}_{(\mathcal{Y}_1 \times_{\mathcal{Y}_2} X_S)/S}$ [FGA, pp. 195-13 and pp. 221-19]. □

Corollary 3.3. *If X be a proper flat scheme of finite presentation, then the stack* $\mathrm{Bun}_r(X)$ *has a diagonal which is represented by locally of finite type schemes.*

Proof. The corollary follows immediately from 3.2, and the fact that $\mathrm{Bun}_r(X) \times_S \mathrm{Bun}_r(X) = \mathrm{Bun}_{\mathrm{GL}_r \times_S \mathrm{GL}_r}(X)$ and that $\mathcal{B}\mathrm{GL}_r \to \mathcal{B}\mathrm{GL}_r \times_S \mathcal{B}\mathrm{GL}_r$ is representable by GL_r. □

Proposition 3.4. *Let X be a projective flat scheme over S with S Noetherian. There are open sub-functors $\mathcal{U}_n \hookrightarrow$* $\mathrm{Bun}_r(X)$*, and schemes Y_n locally of finite type with a smooth surjective map $Y_n \twoheadrightarrow \mathcal{U}_n$. Moreover these $\mathcal{U}_n$ cover* $\mathrm{Bun}_r(X)$.

Proof. Let's define $\mathcal{U}_n \subseteq \mathrm{Bun}_r(X)$ to be the subfunctor which sends any morphism $T \to S$ to category of rank r vector bundles E on X_T with the property that $p^*p_*E(n) \to E(n)$ is surjective and $R^s p_*(E(n)) = 0$ for all $s > 0$, where p denotes the projection $p\colon X_T \to T$. In this way we really defined a 2-functor: By [EGA III-1, 2.2.2, pp. 100] $H^q(E(n)|_{X_K}) = 0$ for $q \gg 0$ for all points $\mathrm{Spec}(K) \to T$. Using descending induction on q we see that $E(n)$ satisfies cohomology and base change at all degree $q \geq 0$. To see that $\mathcal{U}_n$ is open we just have to show that our restriction on the vector bundle E is an open condition, i.e. if $t \in T$ is a point for which $p^*p_*E(n) \to E(n)$ is surjective and $R^s p_*(E(n)) = 0$ for all $s > 0$, where $p : X_t \to \mathrm{Spec}(\kappa(t))$ is the projection, then there is an open neighborhood U of t such that for all points in U the condition is satisfied. The condition $R^s p_*(E(n)) = 0$ follows from semi-continuity and $p^*p_*E(n) \twoheadrightarrow E(n)$ then follows from cohomology and base change. Now the fact that $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ covers $\mathrm{Bun}_r(X)$ follows from the following theorem:

Theorem 3.5. [EGA III-1, 2.2.1, pp. 100] *Soient Y un préschéma noethérien, $f : X \to Y$ un morphisme propre, $\mathcal{L}$ un $\mathcal{O}_X$-Module inversible ample pour f. Pour tout $\mathcal{O}_X$-Module $\mathcal{F}$, posons $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ pour tout $n \in \mathbb{Z}$. Alors, pour tout $\mathcal{O}_X$-Module cohérent $\mathcal{F}$:*

(1) Les $R^q f_*(\mathcal{F})$ sont des $\mathcal{O}_Y$-Modules cohérents.
(2) Il existe un entier N tel que pour $n \geq N$, on ait $R^q f_*(\mathcal{F}(n)) = 0$ pour tout $q > 0$.
(3) Il existe un entier N tel que pour $n \geq N$, l'homomorphisme canonique $f^*(f_*(\mathcal{F}(n))) \to \mathcal{F}(n)$ soit surjectif.

Set $\mathcal{U}_{n,d}$ the open substack of $\mathcal{U}_n$ consisting of vector bundles E on X_T whose pushforward $p_*E(n)$ is a vector bundle of rank $d \in \mathbb{N}$. Clearly we have $\bigcup_{d\in\mathbb{N}} \mathcal{U}_{n,d} = \mathcal{U}_n$. Set $Z_{n,d}$ be the 2-functor sending any $T \to S$ to the category of pairs (E, ϕ), where E is in $\mathcal{U}_{n,d}(T)$, $\phi : \mathcal{O}_{X_T}^{\oplus d} \twoheadrightarrow E(n)$. The category is clearly equivalent to a set because of the surjectivity. Thus $Z_{n,d}$ is a 1-functor.

Lemma 3.6. *The 1-functor $Z_{n,d}$ is representable by an open subscheme of the Quot-scheme. Therefore $Z_{n,d}$ is locally of finite type.*

Proof. Let E be an $\mathcal{O}_X$-module of finite presentation. Consider the following two 1-functors:

$$\mathrm{Quot}_{E/X/S}(T) := \{F \in \mathrm{Mod}(\mathcal{O}_{X_T}) \text{ of finite presentation flat over } T \text{ with a surjection } E \twoheadrightarrow F\}$$

$$\mathbb{F}_{E/X/S}(T) := \{F \in \mathrm{Mod}(\mathcal{O}_{X_T}) \text{ of finite presentation flat over } X_T \text{ with a surjection } E \twoheadrightarrow F\}$$

We claim that $\mathbb{F}_{E/X/S}$ is an open substack of $\mathrm{Quot}_{E/X/S}$. Now suppose that $F \in \mathrm{Quot}_{E/X/S}(T)$ and that at a point $t : \mathrm{Spec}(K) \to T$ the pullback of F to X_K is in $\mathbb{F}_{E/X/S}(K)$. One has to show that there exists U containing t such that $F|_U \in \mathbb{F}_{E/X/S}(U)$. Let $A \subseteq X_T$ be the subset of points on which F is not flat. Now by [EGA IV-3, 11.3.10] $U := T \setminus p(A)$ is precisely the open which we are looking for. Finally one checks readily that $Z_{n,d}$ is an open subscheme of $\mathbb{F}_{\mathcal{O}_X(-n)^{\oplus d}/X/S}$. □

Now we look at $Y_{n,d}$ the open subscheme of $Z_{n,d}$ consisting of pairs whose map ϕ induces an isomorphism $\varphi : \mathcal{O}_T^{\oplus d} \to p_*E(n)$. This is open because clearly $\mathrm{Coker}(\varphi) = 0$ is an open condition, so we may assume that φ is surjective. In this case $\mathrm{Ker}(\phi) = 0$ is an open condition. Thus the condition that φ is an isomorphism is an open condition.

Finally we consider the following map $\mathcal{U}_{n,d} \to \mathcal{B}\mathrm{GL}_d$ which sends an object $E \in \mathcal{U}_{n,d}(T)$ to $p_*E(n)$. One checks readily that the following diagram

$$\begin{tikzcd} Y_{n,d} \arrow[r] \arrow[d] & \mathrm{Spec}(k) \arrow[d] \\ \mathcal{U}_{n,d} \arrow[r] & \mathcal{B}\mathrm{GL}_d \end{tikzcd}$$

is Cartesian. In fact it almost follows from the definition of $Y_{n,d}$. The only thing which one has to take care of is that when $\varphi : \mathcal{O}_T^{\oplus d} \to p_*E(n)$ is an isomorphism the corresponding map $\phi : \mathcal{O}_{X_T}^{\oplus d} \twoheadrightarrow E(n)$ is surjective. This is due to the fact that the adjunction $p^*p_*E(n) \to E(n)$ is surjective by the construction of $\mathcal{U}_{n,d}$. Thus we obtain a smooth atlas for each $\mathcal{U}_{n,d}$ □

Proof of Theorem 3.1. It follows from 3.3 and 3.4 that Bun_r is an algebraic stack. Now applying 3.2 to $\mathcal{Y}_1 = \mathcal{B}G$ and $\mathcal{Y}_2 = \mathcal{B}\mathrm{GL}_n$ we get a representable morphism $\mathrm{Bun}_G \to \mathrm{Bun}_n$. Thus the presentation of Bun_n translates to a presentation of Bun_G. □

4. Bun_r is not of finite type

Proposition 4.1. *Let X be the projective space over a field k. There is no surjection from a scheme of finite type to* $\mathrm{Bun}_r(X)$ *for $r \geq 2$.*

Proof. Let $f : Y \to \mathrm{Bun}_r(X)$ be a surjective map with Y of fintie type, and let y_n be the points corresponding to $\mathcal{O}(n) \oplus \mathcal{O}(-n) \oplus \mathcal{O}^{\oplus r-2}$. The map f corresponds to a vector bundle E on X_Y. By the theorem of Serre there exists $n \gg 0$ such that $p^*p_*E(n) \to E(n)$ is surjective. Now left y_{n+1} to a 2-commutative diagram

$$\begin{tikzcd} & Y \arrow[d, "f"] \\ \mathrm{Spec}(K) \arrow[ur, dotted, "h_{n+1}"] \arrow[r, "y_{n+1}"'] & \mathrm{Bun}_2(X) \end{tikzcd}$$

The lift h_{n+1} tells us that E pullbacks to $\mathcal{O}(n+1) \oplus \mathcal{O}(-n-1) \oplus \mathcal{O}^{\oplus r-2}$ and that

$$(\mathcal{O}(n+1) \oplus \mathcal{O}(-n-1) \oplus \mathcal{O}^{\oplus r-2})(n) = \mathcal{O}(2n+1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}^{\oplus r-2}(n)$$

is generated by global sections. But in fact this is false. □

Bibliography

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