DRINFELD'S LEMMA FOR ALGEBRAIC STACKS

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ABSTRACT. Drinfeld's lemma is a powerful tool for splitting ℓ-adic local systems defined over a product of connected schemes over a finite field. In this paper, we show that Drinfeld's lemma also holds true for algebraic stacks.

1. Introduction

The main motivation of Drinfeld's lemma is to split ℓ-adic local systems defined over a product of schemes. More precisely, let X_1, X_2 be two connected $\mathbb{F}_q$-schemes, then one would like to get, out of an ℓ-adic local system on $X := X_1 \times_{\mathbb{F}_q} X_2$, an ℓ-adic local system coming from local systems on the individual factors X_1, X_2. The problem is easy if one considers complex analytic local systems on a product of complex varieties. Indeed, one can split the local system via the Künneth formula for topological fundamental groups:

$$\pi_1^{\mathrm{top}}(X_1^{\mathrm{an}} \times X_2^{\mathrm{an}}) \xrightarrow{\cong} \pi_1^{\mathrm{top}}(X_1^{\mathrm{an}}) \times \pi_1^{\mathrm{top}}(X_2^{\mathrm{an}})$$

One can also do this for ℓ-adic local systems. Indeed, for any connected schemes X_1, X_2 defined over an algebraically closed field k *of characteristic 0*, one has an isomorphism:

$$\pi_1^{\text{ét}}(X_1 \times_k X_2) \xrightarrow{\cong} \pi_1^{\text{ét}}(X_1) \times \pi_1^{\text{ét}}(X_2)$$

Even when k is of characteristic $p > 0$ (but still algebraically closed), the above Künneth formula still holds true provided that either X_1 or X_2 is proper over k. However, the Künneth formula fails when k is not algebraically closed: take $X_1 = X_2 = \mathrm{Spec}(\mathbb{F}_p)$, then the Künneth formula would mean that the diagonal map $\hat{\mathbb{Z}} \to \hat{\mathbb{Z}} \times \hat{\mathbb{Z}}$ is an isomorphism.

The issue (for finite fields) can be resolved if partial Frobenii actions are brought into play. More precisely, let ϕ_1 (resp. ϕ_2) denote the *partial Frobenius* map $X \to X$, which is the q-absolute Frobenius on X_1 (resp. X_2) and the identity on the other. Consider the category $\mathrm{F\acute{E}t}(X/\Phi)$ of triples $(Y, \varphi_1, \varphi_2)$, where Y is a finite étale cover of $X = X_1 \times_{\mathbb{F}_q} X_2$ and φ_i is an isomorphism $Y \xrightarrow{\cong} \phi_i^* Y$ satisfying that $\phi_1^*(\varphi_2) \circ \varphi_1 = \phi_2^*(\varphi_1) \circ \varphi_2$ is the identity (by identitying $(\phi_1 \circ \phi_2)^* Y$ with Y via the absolute Frobenius of Y).

Theorem 1.1 (Drinfeld's lemma for schemes)**.** *Suppose X_1, X_2 are connected quasi-compact and quasi-separated (qcqs) $\mathbb{F}_q$-schemes. Then*

- $\mathrm{F\acute{E}t}(X/\Phi)$ *is a Galois category whose Galois group is denoted by* $\pi_1^{\text{ét}}(X/\Phi)$*;*
- *the natural map* $\pi_1^{\text{ét}}(X/\Phi) \to \pi_1^{\text{ét}}(X_1) \times \pi_1^{\text{ét}}(X_2)$ *is an isomorphism.*

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Similarly, one has Drinfeld's lemma for n factors $X_1, \ldots, X_n$. Please refer to [Dri80, Theorem 2.1], [Laf97, IV.2, Theorem 4], [Lau07, Theorem 8.1.4], [Ked17, Theorem 4.2.12], [Laf18, Lemma 0.18], [SW20, Theorem 16.2.4], and [Mül22, Theorem 1.4] for details. Using Drinfeld's Lemma one can split local systems on a product of schemes equipped with commuting partial Frobenius actions.

The notion of fundamental group of algebraic stacks has been introduced and studied by B. Noohi in [Noo00]. The main purpose of this note is to generalize Drinfeld's lemma to algebraic stacks removing the qcqs assumption.

Theorem I (cf. §5 and §6)**.** *Let $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n$ be connected algebraic stacks over $\mathbb{F}_q$, and set $\mathcal{X} := \mathcal{X}_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} \mathcal{X}_n$. Then*

1. *The category $\mathrm{F\acute{E}t}(\mathcal{X}/\Phi)$ consisting of finite étale covers of $\mathcal{X}$ equipped with commuting partial Frobenii actions is a Galois category, whose Galois group is denoted by $\pi_1^{\text{ét}}(\mathcal{X}/\Phi)$;*
2. *the natural map $\pi_1^{\text{ét}}(\mathcal{X}/\Phi) \longrightarrow \pi_1^{\text{ét}}(\mathcal{X}_1) \times \cdots \times \pi_1^{\text{ét}}(\mathcal{X}_n)$ is an isomorphism.*

The key technique that we are using here is the Drinfeld-Lau descent for fibered categories developed in [PTZ20, Theorem II (C)], which asserts that for any fibered category $\mathcal{X}$ and any algebraically closed field k containing $\mathbb{F}_q$, the natural pullback functor $\mathrm{F\acute{E}t}(\mathcal{X}) \to \mathrm{F\acute{E}t}(\mathcal{X} \times_{\mathbb{F}_q} k/\Phi)$ induces an equivalence. This combined with standard arguments for the scheme case, coupled with some dévissage arguments, yield the proof of Theorem I (2). On top of that, one needs to make sense of the category $\mathrm{F\acute{E}t}(\mathcal{X}/\Phi)$ for $\mathcal{X}$ an algebraic stack. We put this in a more general setting: Let M be a commutative monoid acting on an algebraic stack $\mathcal{X}$, let G denote its Grothendieck group, and let $\mathrm{F\acute{E}t}(\mathcal{X}/G)$ be the category consisting of finite étale covers $\mathcal{Y} \in \mathrm{F\acute{E}t}(\mathcal{X})$ equipped with an M-action compatible with that on $\mathcal{X}$. If the quotient topological space $|\mathcal{X}|/G$ is connected, then $\mathrm{F\acute{E}t}(\mathcal{X}/G)$ is a Galois category (cf. Theorem 5.3). For this, it is invertible to deal with monoid actions on stacks, and we resort to M. Romagny's [Rom05], where group actions on stacks are systematically studied (the definitions work in exactly the same way for monoid). However, as our results show (cf. Lemma 5.1 and Remark 5.2), in our situation the action is much simpler than M. Romagny's setting, as higher compatibilities are automatic.

2. The G-Connectedness

One of the difficulties in understanding Drinfeld's lemma is that the partial q-Frobenius maps are, in general, not invertible. Thus "the quotient space X/Φ" is only a suggestive symbol but not an honest space – even when X is a scheme! However, we can first forget this if we only look at the actions on the ambient topologial space $|\mathcal{X}|$.

Definition 2.1. Let G be a group, and let X be a topological space equipped with a G-action $\rho_G\colon G \times X \to X$. The pair (X, ρ_G) is called a *G-space.* The space X is called *G-connected* if the quotient space X/G is a connnected topological space. A subspace $S \subseteq X$ is called a *G-component* of X if it is the inverse image of a nonempty clopen subspace of X/G. A ordinary topological space X can be viewed as a G-space via the trivial G-action. In this case, the pair (X, ρ_G) will simply denoted by X.

Let $f\colon (X, \rho_G) \to (S, \rho'_G)$ be a map, *i.e.* a map of topological spaces $f\colon X \to S$ which is compatible with the G-actions. For each G-stable subset $S_1 \subseteq S$ the inverse image $f^{-1}(S_1) \subseteq X$ is a G-stable subspace.

Lemma 2.1. *Let $f\colon (X, \rho_G) \to S$ be map whose fibers are G-connected. Suppose that $X \to S$ is submersive. Then (X, ρ_G) is G-connected if and only if S is connected.*

Proof. The map f induces a continuous map $\bar{f}\colon X/G \to S$ whose fibers are connected (hence nonempty). If (X, ρ_G) is G-connected, then X/G is connected, so S is connnected. Conversely, if (X, ρ_G) is not G-connected, then $X/G = U_1 \sqcup U_2$, where U_1, U_2 are two non-empty open subspaces of X/G. Since the fibers of $\bar{f}$ are connected, any fiber is contained either in U_1 or in U_2. Thus exists subspaces V_1, V_2 of S such that $S = V_1 \sqcup V_2$ and $\bar{f}^{-1}(V_i) = U_i$ $(i = 1, 2)$. That f is submersive implies that $\bar{f}$ is submersive. Hence V_1, V_2 are non-empty opens of S. Then S is not connected. □

3. Frobenius and Universal Homeomorphisms

Recall that for any fibered category $\mathcal{X}$ defined over $\mathbb{F}_p$, the absolute Frobenius map $F_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{X}$ sends a section $x \in \mathcal{X}(T)$ on an $\mathbb{F}_p$-scheme T to the composition $T \xrightarrow{F_T} T \xrightarrow{x} \mathcal{X} \in \mathcal{X}(T)$, where F_T is the absolute Frobenius of the scheme T.

Universal homeomorphisms of schemes are radical maps (cf. [Aut, 01S2]) which play a role as the "opposite" of étale maps. The on the nose extension of this notion to stacks could be misleading in certain situations. For example, consider a finite étale group scheme G over a field k. Any base change of the map $f\colon \mathcal{B}_k G \to \mathrm{Spec}(k)$ is a homeomorphism on the underlying topological spaces, however, the map is étale. This suggests a new definition of the notion of "universal homeomorphism" for algebraic stacks – we should consider not only f but also the diagonal of f which encodes information about the (relative) inertia.

Definition 3.1. A map of algebraic stacks $f\colon \mathcal{X} \to \mathcal{Y}$ is called a *universal homeomorphism on the nose* if for any map of algebraic stacks $\mathcal{T} \to \mathcal{Y}$, the base change $f_{\mathcal{T}}\colon \mathcal{X} \times_{\mathcal{Y}} \mathcal{T} \to \mathcal{T}$ is a homeomorphism on the underlying topological spaces. The map f is called a *universal homeomorphism* if both f and the diagonal Δ_f are universal homeomorphisms on the nose.

Lemma 3.1. *If $\mathcal{X}$ is an algebraic stack over $\mathbb{F}_p$, then $F_{\mathcal{X}}$ is a universal homeomorphism.*

Proof. Suppose $f\colon \mathcal{Y} \to \mathcal{X}$ is a map of algebraic stacks over $\mathbb{F}_p$. Then we have the following cartesian diagram

$$\begin{tikzcd} \mathcal{Y}' \arrow[r, "F'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{Y} \arrow[d, "f"] \\ \mathcal{X} \arrow[r, "F_{\mathcal{X}}"] & \mathcal{X} \end{tikzcd}$$

We want to show that F' induces a bijection $|F'|\colon |\mathcal{Y}'| \to |\mathcal{Y}|$. By the universality of a fibered square, there is a factorization of $F_{\mathcal{Y}}\colon \mathcal{Y} \xrightarrow{F_{\mathcal{Y}/\mathcal{X}}} \mathcal{Y}' \xrightarrow{F'} \mathcal{Y}$. If $F_{\mathcal{Y}}$ induces a homeomorphism of $|\mathcal{Y}|$, then we just have to show that $|F_{\mathcal{Y}/\mathcal{X}}|$ is surjective. Take a point $y' \in |\mathcal{Y}'|$

which is represented by a triple (x, y, α), where $x \in \mathcal{X}(k)$, $y \in \mathcal{Y}(k)$ are geometric points, and α is an isomorphism from $\operatorname{Spec}(k) \xrightarrow{F_k} \operatorname{Spec}(k) \xrightarrow{x} \mathcal{X}$ to $\operatorname{Spec}(k) \xrightarrow{y} \mathcal{Y} \xrightarrow{f} \mathcal{X}$. Consider the k-point $y_1\colon \operatorname{Spec}(k) \xrightarrow{F_k^{-1}} \operatorname{Spec}(k) \xrightarrow{y} \mathcal{Y}$ of $\mathcal{Y}$. Clearly, $F_{\mathcal{Y}}(y_1) = y$ and $x \xrightarrow{\cong} f(y_1)$ via $F_k^{-1}(\alpha)$. We get $(x, y, \alpha) \cong (f(y_1), y, \mathrm{id}_{f(y)}) = F_{\mathcal{Y}/\mathcal{X}}(y_1)$. Thus $|F_{\mathcal{Y}/\mathcal{X}}|$ is surjective.

Now let's show that $|F_{\mathcal{X}}|$ is bijective for any algebraic stack $\mathcal{X}$. If $x \in \mathcal{X}(k)$ is a geometric point, then the k-point $\operatorname{Spec}(k) \xrightarrow{F_k^{-1}} \operatorname{Spec}(k) \xrightarrow{x} \mathcal{X}$ is mapped to x via $F_{\mathcal{X}}$. Hence $|F_{\mathcal{X}}|$ is surjective. Conversely, let $x_1 \in \mathcal{X}(k_1)$ and $x_2 \in \mathcal{X}(k_2)$ are two geometric points such that $F_{\mathcal{X}}(x_1)$ and $F_{\mathcal{X}}(x_2)$ represent the same point in $\mathcal{X}$, *i.e.* $F_{\mathcal{X}}(x_1) \sim F_{\mathcal{X}}(x_2)$. Then, by definition, there is an algebraically closed field k containing both k_1 and k_2 such that $\operatorname{Spec}(k) \to \operatorname{Spec}(k_1) \xrightarrow{F_{k_1}} \operatorname{Spec}(k_1) \xrightarrow{x_1} \mathcal{X}$ is isomorphic to $\operatorname{Spec}(k) \to \operatorname{Spec}(k_2) \xrightarrow{F_{k_2}} \operatorname{Spec}(k_2) \xrightarrow{x_2} \mathcal{X}$. But since $(\operatorname{Spec}(k) \to \operatorname{Spec}(k_i) \xrightarrow{F_{k_i}} \operatorname{Spec}(k_i)) = (\operatorname{Spec}(k) \xrightarrow{F_k} \operatorname{Spec}(k) \to \operatorname{Spec}(k_i))$ for $i = 1, 2$, we have $\operatorname{Spec}(k) \to \operatorname{Spec}(k_1) \xrightarrow{x_1} \mathcal{X}$ is isomorphic to $\operatorname{Spec}(k) \to \operatorname{Spec}(k_2) \xrightarrow{x_2} \mathcal{X}$, so $x_1 \sim x_2$.

We have seen that $|F'|$, $|F_{\mathcal{Y}/\mathcal{X}}|$, $|F_{\mathcal{Y}}|$ are bijective and continuous. From the very definition of the topology on $|\mathcal{Y}|$ it is clear that $|F_{\mathcal{Y}}|$ is a homeomorphism. The continuity of $|F_{\mathcal{Y}/\mathcal{X}}|$ then implies that $|F'|$ is submersive, hence a homeomorphism.

Let's consider the diagonal $\Delta\colon \mathcal{X} \to \mathcal{X} \times_{F_{\mathcal{X}}} \mathcal{X}$. The map $|\Delta|$ is surjective because any geometric point $(x, y, \alpha) \in (\mathcal{X} \times_{F_{\mathcal{X}}} \mathcal{X})(k)$, where $\alpha\colon (\operatorname{Spec}(k) \xrightarrow{F_k} \operatorname{Spec}(k) \xrightarrow{x} \mathcal{X}) \cong (\operatorname{Spec}(k) \xrightarrow{F_k} \operatorname{Spec}(k) \xrightarrow{y} \mathcal{X})$, induces an isomorphism $F_k^{-1}(\alpha)\colon x \cong y$, so $(x, x, \mathrm{id}_{F_{\mathcal{X}}(x)}) \cong (x, y, \alpha)$. Conversely, if $x \in \mathcal{X}(k_1)$ and $y \in \mathcal{X}(k_2)$ such that $(x, x, \mathrm{id}_{F_{\mathcal{X}}(x)}) \sim (y, y, \mathrm{id}_{F_{\mathcal{X}}(y)})$ in the diagonal, then there is a geometric point $(a, b, \alpha) \in (\mathcal{X} \times_{F_{\mathcal{X}}} \mathcal{X})(k)$, where k contains both k_1 and k_2, such that (a, b, α) is isomorphic to the pullback of both $(x, x, \mathrm{id}_{F_{\mathcal{X}}(x)})$ and $(y, y, \mathrm{id}_{F_{\mathcal{X}}(y)})$ to k. This implies immediately that $x \sim y$. □

Here are some geometric properties of universal homeomorphisms for algebraic stacks. Note that Lemma 3.2 (2) is not true if g is only a universal homeomorphisms on the nose.

Lemma 3.2. *Let $\mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{Z}$ be a sequence of maps of algebraic stacks. Set $h := g \circ f$.*

(1) *If f, g are universal homeomorphisms, then so is f.*
(2) *If g is a universal homeomorphism and h is a universal homeomorphism on the nose, then f is a universal homeomorphism on the nose.*

Proof. (1) is obvious. Let's look at (2). The map f factorizes as $\mathcal{X} \xrightarrow{\Gamma_f} \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \xrightarrow{h \times \mathrm{id}_{\mathcal{Y}}} \mathcal{Y}$, where Γ_f is the graph of f. If h is a universal homeomorphism on the nose and the diagonal g is a universal homeomorphism, then Γ_f and $h \times \mathrm{id}_{\mathcal{Y}}$ are universal homeomorphisms on the nose, and so is the composition f. □

Note that the absolute Frobenius is only $\mathbb{F}_p$-linear. In order to make it $\mathbb{F}_q$-linear we take the q-absolute Frobenius map $\phi_{\mathcal{X}} := F_{\mathcal{X}}^n$, where $n := \log_p q$. Just as in the scheme case, the pullback of the étale maps along the q-absolute Frobenius map of algebraic stacks induces an equivalence of étale sites (actually the identity!).

Proposition 3.3. *Let $\mathcal{X}$ be a Deligne-Mumford stack (resp. an algebraic stack) over $\mathbb{F}_q$, and let $f\colon \mathcal{Y} \to \mathcal{X}$ be an étale map (resp. a representable étale map). Then we have the following cartesian diagram.*

$$\begin{tikzcd}
\mathcal{Y} \arrow[r, "\phi_{\mathcal{Y}}"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{Y} \arrow[d, "f"] \\
\mathcal{X} \arrow[r, "\phi_{\mathcal{X}}"] & \mathcal{X}
\end{tikzcd}$$

Proof. One just has to show that the natural map $\lambda\colon \mathcal{Y} \to \mathcal{X} \times_{\phi_{\mathcal{X}}} \mathcal{Y}$ is an equivalence. For this, one can take an atlas of $X \twoheadrightarrow \mathcal{X}$, where X is a scheme and show that the pullback of λ to X is an equivalence. Thus we are reduced to the case when $\mathcal{X} = X$ is a scheme. If f is representable by schemes, then we are done. If f is only representable by algebraic spaces, then $\mathcal{Y} = Y$ is an algebraic space. Take an étale atlas $Z \twoheadrightarrow Y$. The following diagram

$$\begin{tikzcd}
Z \arrow[r, "\lambda_Z"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & X \times_{\phi_X} Z \arrow[d, two heads] \\
Y \arrow[r, "\lambda"] & X \times_{\phi_X} Y
\end{tikzcd}$$

is cartesian, because its extension via the projections $X \times_{\phi_X} Z \to Z$ and $X \times_{\phi_X} Y \to Y$ is cartesian (as $Z \twoheadrightarrow Y$ is representable by schemes). Since λ_Z is an isomorphism by the scheme case, λ is an isomorphism as well. If $\mathcal{Y}$ is only a Deligne-Mumford stack, then we take an étale atlas $Z \twoheadrightarrow \mathcal{Y}$ and repeat the above argument to reduce the problem to the case when f is representable by algebraic spaces which has just been proven. □

4. Partial Frobenius Maps

Let's come back to our original setting. Let $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n$ be connected algebraic stacks over $\mathbb{F}_q$, and set $\mathcal{X} := \mathcal{X}_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} \mathcal{X}_n$. Then there are *partial Frobenius maps* $\phi_i\colon \mathcal{X} \to \mathcal{X}$, which is the q-absolute Frobenius on X_i and the identity on the others. These ϕ_is are commuting endomorphisms of $\mathcal{X}$ (resp. automorphisms of $|\mathcal{X}|$) whose composition $\phi_1 \circ \cdots \circ \phi_n$ is the q-Frobenius of $\mathcal{X}$ (resp. the identity of $|\mathcal{X}|$). Thus each ϕ_i is completely determined by all the others, so we can drop any of the partial Frobenius maps, e.g. ϕ_n. They provide $\mathcal{X}$ (resp. $|\mathcal{X}|$) with a strict action (resp. an action) $\rho_{\mathcal{X}}$ by the monoid $M := \phi_1^{\mathbb{N}} \times \phi_2^{\mathbb{N}} \times \cdots \phi_{n-1}^{\mathbb{N}}$ (resp. the group $G := \phi_1^{\mathbb{Z}} \times \phi_2^{\mathbb{Z}} \times \cdots \phi_{n-1}^{\mathbb{Z}}$).

Lemma 4.1. *Let $G_i := \bigoplus_{j \in \{1,\ldots,n\} \setminus \{i\}} \phi_j^{\mathbb{Z}}$. Then $|\mathcal{X}|$ is G-connected if and only if it is G_i-connected.*

Proof. This is just because that a subset of $|\mathcal{X}|$ is G-stable (resp. G_i-stable) iff it is stable under all the partial Frobenius actions. □

Corollary 4.2. *The space $|\mathcal{X}_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} \mathcal{X}_n|$ equipped with the $\mathbb{Z}^{n-1}$-action by $n-1$ of the partial Frobenius maps is $\mathbb{Z}^{n-1}$-connected.*

Proof. Consider the projection map $|\mathrm{pr}_n|\colon |\mathcal{X}_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} \mathcal{X}_n| \to |\mathcal{X}_n|$, where the former is equipped with the $\mathbb{Z}^{n-1}$-action via the ϕ_is, and the latter is equipped with the trivial $\mathbb{Z}^{n-1}$-action. Let $x \in \mathcal{X}_n(k)$ be a geometric point representing a point in $|\mathcal{X}_n|$. Then the topological space of the stack fiber $\mathcal{X}_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} \mathcal{X}_{n-1} \times_{\mathbb{F}_q} k$ maps surjectively to the topological fiber of x along the map $|\mathrm{pr}_n|$. This surjection moreover preserves the $\mathbb{Z}^{n-1}$-actions. In light of Lemma 2.1, it is enough to show that the space $|\mathcal{X}_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} \mathcal{X}_{n-1} \times_{\mathbb{F}_q} k|$ is $\mathbb{Z}^{n-1}$-connected.

Suppose $\mathcal{U} \subseteq \mathcal{X}_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} \mathcal{X}_{n-1} \times_{\mathbb{F}_q} k$ is a clopen substack which is stable under the $\mathbb{Z}^{n-1}$-action. Then $\mathcal{U}$ is also stable under the $\mathbb{Z}^{n-1}$-action defined by $\phi_1, \ldots, \phi_{n-2}, \phi_k$. By [PTZ20, Theorem 4.1], $\mathcal{U}$ is the preimage of a clopen substack $\mathcal{U}_0 \subseteq \mathcal{X}_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} \mathcal{X}_{n-1}$, and $\mathcal{U}_0$ is obviously stable under the $\mathbb{Z}^{n-2}$-action $\phi_1, \ldots, \phi_{n-2}$. Then $\mathcal{U}_0$ is either empty or the whole stack by the connectedness of $\mathcal{X}_{n-1}$ in the case $n = 2$, or by the induction hypothesis in the case $n > 2$. Thus $\mathcal{U}$ is also either empty or the whole stack. This finishes the proof. $\square$

5. Galois Categories

Let $\mathcal{X}$ be an algebraic stack over $\mathbb{F}_q$. When $\mathcal{X}$ is connected, it is known (cf. [Noo00, Theorem 4.2] or [Noo04, §4]) that $\mathrm{F\acute{E}t}(\mathcal{X})$ is a Galois category. Let's now adapt this to our setting. Suppose M is a commutative monoid acting (cf. [Rom05, Def. 1.3 (i)]) on $\mathcal{X}$ via universal homeomorphisms. Let $\rho_{\mathcal{X}}\colon M \to \mathrm{End}_{\mathbb{F}_q}(\mathcal{X})$ be the action map. If G denotes the Grothendieck group associated with M, then $|\mathcal{X}|$ is a G-space. Let $\mathrm{F\acute{E}t}(\mathcal{X}/G)$ denote the category whose objects are 2-commutative diagrams

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "\rho_{\mathcal{Y}}(m)"] \arrow[d, "f"'] \arrow[dr, phantom, "\swarrow \sigma_m"] & \mathcal{Y} \arrow[d, "f"] \\ \mathcal{X} \arrow[r, "\rho_{\mathcal{X}}(m)"] & \mathcal{X} \end{tikzcd}$$

where $\mathcal{Y} \xrightarrow{f} \mathcal{X} \in \mathrm{F\acute{E}t}(\mathcal{X})$ and $\rho_{\mathcal{Y}}$ is an action of M on $\mathcal{Y}$ via $\mathbb{F}_q$-endomorphisms of $\mathcal{Y}$ which are universal homeomorphisms, such that $\rho_{\mathcal{Y}}$ is compatible with $\rho_{\mathcal{X}}$, in the sense of [Rom05, Def. 1.3 (ii)], *i.e.* $\sigma_{m_1 m_2}$ is equal to the composition

$$\begin{tikzcd} f \circ \rho_{\mathcal{Y}}(m_2) \circ \rho_{\mathcal{Y}}(m_1) \arrow[r, "\sigma_{m_2}"] & \rho_{\mathcal{X}}(m_2) \circ f \circ \rho_{\mathcal{Y}}(m_1) \arrow[r, "\sigma_{m_1}"] & \rho_{\mathcal{X}}(m_2) \circ \rho_{\mathcal{X}}(m_1) \circ f \arrow[d, equal] \\ f \circ \rho_{\mathcal{Y}}(m_1 m_2) \arrow[u, "\cong"] & & \rho_{\mathcal{X}}(m_1 m_2) \circ f \end{tikzcd} \tag{1}$$

and σ_1 equals the composition $f \circ \rho_{\mathcal{Y}}(1) \cong f \circ \mathrm{id}_{\mathcal{Y}} = f$. The 1-morphisms $(\mathcal{Y}', \rho'_{\mathcal{Y}}, \sigma') \to (\mathcal{Y}, \rho_{\mathcal{Y}}, \sigma) \in \mathrm{F\acute{E}t}(\mathcal{X}/G)$ are 1-morphisms $g\colon \mathcal{Y}' \to \mathcal{Y} \in \mathrm{F\acute{E}t}(\mathcal{X})$ together with 2-morphisms $\delta_m\colon \rho_{\mathcal{Y}'}(m) \circ g \cong g \circ \rho_{\mathcal{Y}}(m)$ making g an M-equivariant map (cf. [Rom05, Def. 1.3 (ii)]) and are compatible with σ, σ'. The 2-morphisms between two morphisms

$$\begin{tikzcd} (\mathcal{Y}', \rho'_{\mathcal{Y}}, \sigma') \arrow[r, bend left, "(g,\delta)"] \arrow[r, bend right, "(g',\delta')"'] & (\mathcal{Y}, \rho_{\mathcal{Y}}, \sigma) \end{tikzcd}$$

are 2-morphisms in $g \Rightarrow g' \in \mathrm{FÉt}(\mathcal{X})$ which are compatible with δ_m, δ'_m for all m. Note that since $\mathrm{FÉt}(\mathcal{X})$ is anti 2-equivalent to the category of finite étale $\mathcal{O}_{\mathcal{X}}$-algebras, $\mathrm{FÉt}(\mathcal{X})$ is essentially a 1-category, *i.e.* 2-morphisms $g \Rightarrow g'$ are isomorphisms and $\mathrm{Hom}(g, g')$ is either singleton or empty. This implies that $\mathrm{FÉt}(\mathcal{X}/G)$ is essentially a 1-category as well – compatibility conditions don't matter.

Lemma 5.1. *For any $m \in M$ and $\mathcal{Y} \in \mathrm{FÉt}(\mathcal{X})$ set $m^*\mathcal{Y} := \mathcal{Y} \times_{\rho_{\mathcal{X}}(m)} \mathcal{X}$. The following categories are equivalent*

(a) *the category $\mathrm{FÉt}(\mathcal{X}/G)$;*

(b) *the category of 2-cartesian diagrams*

(2)
$$\begin{tikzcd} \mathcal{Y} \arrow[r, "\rho_{\mathcal{Y}}(m)"] \arrow[d, "f"'] \arrow[dr, phantom, very near start, "\lrcorner"] \arrow[dr, phantom, "\overset{\sigma_m}{\Leftarrow}"] & \mathcal{Y} \arrow[d, "f"] \\ \mathcal{X} \arrow[r, "\rho_{\mathcal{X}}(m)"] & \mathcal{X} \end{tikzcd}$$

where $\mathcal{Y} \xrightarrow{f} \mathcal{X} \in \mathrm{FÉt}(\mathcal{X})$ and $\rho_{\mathcal{Y}}$ is an action of M on $\mathcal{Y}$ via $\mathbb{F}_q$-endomorphisms and σ_m is the 2-isomorphism as in the definition of $\mathrm{FÉt}(\mathcal{X}/G)$;

(c) *the category of tuples $(\mathcal{Y}, \alpha_{\mathcal{Y}}, \iota, \tau)$, where $\mathcal{Y} \in \mathrm{FÉt}(\mathcal{X})$, $\alpha_{\mathcal{Y},m} \colon \mathcal{Y} \xrightarrow{\simeq} m^*\mathcal{Y} \in \mathrm{FÉt}(\mathcal{X})$ is an equivalence for each $m \in M$, $\iota_{m_1,m_2} \colon m_1^*\alpha_{\mathcal{Y},m_2} \circ \alpha_{\mathcal{Y},m_1} \cong \alpha_{\mathcal{Y},m_1m_2}$ and $\tau \colon \alpha_{\mathcal{Y},1} \cong \mathrm{id}_{\mathcal{Y}}$ are 2-isomorphisms in $\mathrm{FÉt}(\mathcal{X})$. Moreover, ι, τ satisfy higher compatibilities, namely, $\iota_{m_1m_2,m_3} \circ m_3^*\iota_{m_1,m_2} = \iota_{m_1m_2,m_3}$, $\iota_{1,m} = \alpha_{\mathcal{Y},m}(\tau)$ and $\iota_{m,1} = m^*\tau(\alpha_{\mathcal{Y},m})$. Morphisms are pairs (g, θ), where $g \colon \mathcal{Y}' \to \mathcal{Y} \in \mathrm{FÉt}(\mathcal{X})$, $\theta \colon \alpha_{\mathcal{Y},m} \circ g \cong m^*g \circ \alpha_{\mathcal{Y}',m}$ viewed as a 2-morphism in $\mathrm{FÉt}(\mathcal{X})$, and $g(\tau'^{-1}) \circ \theta \circ \tau(g) = \mathrm{id}_g$;*

(d) *the category of pairs $(\mathcal{Y}, \alpha_{\mathcal{Y}})$, where $\mathcal{Y} \in \mathrm{FÉt}(\mathcal{X})$, $\alpha_{\mathcal{Y},m} \colon \mathcal{Y} \xrightarrow{\simeq} m^*\mathcal{Y} \in \mathrm{FÉt}(\mathcal{X})$ is an equivalence for each $m \in M$. Moreover, we have $m_1^*\alpha_{\mathcal{Y},m_2} \circ \alpha_{\mathcal{Y},m_1} \cong \alpha_{\mathcal{Y},m_1m_2}$ and $\alpha_{\mathcal{Y},1} \cong \mathrm{id}_{\mathcal{Y}}$.*

(e) *the category of pairs $(\mathcal{A}, \alpha_{\mathcal{A}})$, where $\mathcal{A}$ is a finite étale $\mathcal{O}_{\mathcal{X}}$-algebra and $\alpha_{\mathcal{A},m} \colon m^*\mathcal{A} \to \mathcal{A}$ ($m^*\mathcal{A} := \rho_{\mathcal{X}}(m)^*\mathcal{A}$) is an $\mathcal{O}_{\mathcal{X}}$-algebra isomorphism satisfying $\alpha_{\mathcal{A},e} = \mathrm{id}_{\mathcal{A}}$ and $\alpha_{\mathcal{A},m_1m_2} = \alpha_{\mathcal{A},m_1} \circ m_1^*\alpha_{\mathcal{A},m_2}$.*

Proof. First of all, the categories in (a), (b) and (c) are essentially 1-categories as remarked before. Thus we can ignore the natural 2-morphisms in them. Let's first show (a)$\Leftrightarrow$(b). Consider the 2-commutative diagram:

(3)
$$\begin{tikzcd} \mathcal{Y} \arrow[dr, dashed, "\alpha_{\mathcal{Y},m}"] \arrow[drr, dotted, bend left, "\rho_{\mathcal{Y}}(m)"] \arrow[ddr, bend right] & & \\ & m^*\mathcal{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{Y} \arrow[d] \arrow[dl, Rightarrow, shorten=8pt] \\ & \mathcal{X} \arrow[r, "\rho_{\mathcal{X}}(m)"] & \mathcal{X} \end{tikzcd}$$

Suppose the square defined by $\rho_{\mathcal{Y}}(m)$ and $\rho_{\mathcal{X}}(m)$ is cartesian, then $\alpha_{\mathcal{Y},m}$ is an equivalence, so $\rho_{\mathcal{Y}}(m)$, as a pullback of $\rho_{\mathcal{X}}(m)$, is a universal homeomorphism, *i.e.* the square belongs to $\mathrm{F\acute{E}t}(\mathcal{X}/G)$. Conversely, if $\rho_{\mathcal{Y}}(m)$ is a universal homeomorphism, then $\alpha_{\mathcal{Y},m}$ is an isomorphism. Indeed, if $\rho_{\mathcal{Y}}(m)$ is a universal homeomorphism, then the fact that $m^*\mathcal{Y} \to \mathcal{Y}$ is a universal homeomorphism implies that $\alpha_{\mathcal{Y},m}$ is a universal homeomorphism on the nose (cf. Lemma 3.2). Since both $\mathcal{Y}$ and $m^*\mathcal{Y}$ are in $\mathrm{F\acute{E}t}(\mathcal{X})$, $\alpha_{\mathcal{Y},m}$ is an isomorphism at all geometric fibers of $\mathcal{X}$, so it is a degree 1 finite étale map, *i.e.* an equivalence, hence the square defined by $\rho_{\mathcal{Y}}(m)$ and $\rho_{\mathcal{X}}(m)$ is cartesian.

(b)⇔(c) If we ignore the action, then the data of a diagram (2) is equivalent to the data of a pair $(\mathcal{Y}, \alpha_{\mathcal{Y}})$, where $\alpha_{\mathcal{Y},m} \colon \mathcal{Y} \to m^*\mathcal{Y}$ is a 1-morphism for each $m \in M$. The data of the 2-morphisms ι, τ correspond to the data of the 2-morphisms defining the action $\rho_{\mathcal{Y}}$ (cf. [Rom05, Def. 1.3 (i)]) and the compatibility condition defined in diagram (1) and the line after.The higher compatibilities of ι, τ correspond to the higher associativity and other higher compatibility constrains on the action $\rho_{\mathcal{Y}}$.

(c)⇔(d) For this we just have to note that the higher compatibilities are automatic because ι, τ are 2-morphisms in $\mathrm{F\acute{E}t}(\mathcal{X})$ which is essentially a 1-category.

(d)⇔(e) This follows immediately from the anti-equivalence $\mathrm{F\acute{E}t}(\mathcal{X}) \simeq \{$finite étale $\mathcal{O}_{\mathcal{X}}$-algebras$\}$. □

Remark 5.2. The equivalence (c)⇔(d) indicates that the higher compatibility constrains (see the displayed equations in [Rom05, Def. 1.3 (i)]) on the action $\rho_{\mathcal{Y}}$ in (a) or (b) are also redundant. However, it's *not* because that the Frobenius map is representable (actually, it's not, e.g. the Frobenius map of $\mathcal{B}_k\mu_{p,k}$ factors as $\mathcal{B}_k\mu_{p,k} \to \mathrm{Spec}(k) \to \mathcal{B}_k\mu_{p,k}$ where the first map is the projection), but because of the universality of the 2-fibered product and that $\mathrm{F\acute{E}t}(\mathcal{X})$ is essentially a 1-category. Indeed, any automorphism $\rho_{\mathcal{Y},m} \xrightarrow{\cong} \rho_{\mathcal{Y},m}$ which is compatible with σ_m is equal to $\mathrm{id}_{\rho_{\mathcal{Y},m}}$, so $\rho_{\mathcal{Y},m_3} \circ \rho_{\mathcal{Y},m_2} \circ \rho_{\mathcal{Y},m_1} \xrightarrow{\cong} \rho_{\mathcal{Y},m_1m_2m_3}$ is unique.

Theorem 5.3. *Suppose that $\mathcal{X}$ is G-connected, then* $\mathrm{F\acute{E}t}(\mathcal{X}/G)$ *is a Galois category.*

Proof. Set $\mathcal{C} := \mathrm{F\acute{E}t}(\mathcal{X}/G)$. If $\bar{x}$ is a geometric point of $\mathcal{X}$, then take $F \colon \mathrm{F\acute{E}t}(\mathcal{X}/G) \to (\mathrm{Set})$ to be the composition of the forgetful functor with the fiber functor $\mathrm{F\acute{E}t}(\mathcal{X}) \to (\mathrm{Set})$. Let's verify the four axioms in [Aut, 0BMY].

$\mathcal{C}$ has finite limits and finite colimits. Suppose $\{(\mathcal{Y}_j, \rho_{\mathcal{Y}_i}, \sigma_i)\}_{j \in J}$ is a finite projective system in $\mathcal{C}$. For any $m \in M$, the pullback along $\mathcal{X} \xrightarrow{\rho_{\mathcal{X}}(m)} \mathcal{X}$ (denoted by m^*) is obviously a left exact endomorphism of $\mathrm{F\acute{E}t}(\mathcal{X})$. Thus for each $m \in M$, we have natural maps $\rho_{\varprojlim \mathcal{Y}_j} \colon \varprojlim_{j \in J} \mathcal{Y}_j \longrightarrow \varprojlim_{j \in J} \mathcal{Y}_j$ and $\sigma := \varprojlim_{j \in J} \sigma_j$ which make the following diagram

$$\begin{tikzcd}
\varprojlim_{j\in J} \mathcal{Y}_j \arrow[r, "\rho_{\varprojlim \mathcal{Y}_i}"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \varprojlim_{j\in J} \mathcal{Y}_j \arrow[d] \arrow[dl, Rightarrow, "\sigma"'] \\
\mathcal{X} \arrow[r, "\rho_{\mathcal{X}}(m)"] & \mathcal{X}
\end{tikzcd}$$

cartesian (as the diagram for each $\rho_{\mathcal{Y}_i}$ is so by Lemma 5.1). Hence the diagram is an object in $\mathrm{F\acute{E}t}(\mathcal{X}/G)$, and it is clearly the limit. As for the colimit, we first conclude, by the proof of [Aut, 0BN9], that the pullback functor m^* is right exact. Indeed, thanks to [Aut, 0GMN] one just has to check that m^* commutes with coproducts and coequalizers in $\mathrm{F\acute{E}t}(\mathcal{X})$. It's easy for coproducts. For coequalizers one considers two maps $a, b\colon \mathcal{Z} \to \mathcal{Y}$ in $\mathrm{F\acute{E}t}(\mathcal{X})$. Suppose $\mathcal{Y} = \mathrm{Spec}_{\mathcal{O}_\mathcal{X}}(\mathcal{B})$ and $\mathcal{Z} = \mathrm{Spec}_{\mathcal{O}_\mathcal{X}}(\mathcal{C})$, where $\mathcal{B}, \mathcal{C}$ are two finite étale $\mathcal{O}_\mathcal{X}$-algebras. Then a, b correspond to $a^\#, b^\# \colon \mathcal{B} \to \mathcal{C}$. The coequalizer of a, b is $\mathrm{Spec}_{\mathcal{O}_\mathcal{X}}(\mathcal{A})$, where $\mathcal{A} = \mathrm{Ker}(a^\# - b^\#)$. In [Aut, 0BN9] it was shown that $\mathcal{B}, \mathcal{C}$ are, étale locally, sums of $\mathcal{O}_\mathcal{X}$ and $a^\# - b^\#$ is then a map rearranging the summands. Thus $\mathcal{A}$ is a finite étale $\mathcal{O}_\mathcal{X}$-algebra and $\mathcal{B}/\mathcal{A}, \mathcal{C}/\mathrm{im}(a^\# - b^\#)$ are locally free. This implies that m^* preserves the coequalizers. Now the proof of the existence of limits works mutatis mutandis for colimits.

Every object of $\mathcal{C}$ is a finite (possibly empty) coproduct of connected objects. It is easy to check that for any object $f\colon \mathcal{X}' \to \mathcal{X}$ in $\mathrm{F\acute{E}t}(\mathcal{X}/G)$, the induced map $\bar{f}\colon |\mathcal{X}'|/G \to |\mathcal{X}|/G$ enjoys the following property.

Let $u\colon |\mathcal{X}| \to |\mathcal{X}|/G$ (resp. $v\colon |\mathcal{X}'| \to |\mathcal{X}'|/G$) be the quotient map. Then for any subset $S \subseteq |\mathcal{X}'|/G$, $f(v^{-1}(S)) = u^{-1}(\bar{f}(S))$. Indeed, since $u \circ f(v^{-1}(S)) = \bar{f} \circ v(v^{-1}(S)) = \bar{f}(S)$, we have $f(v^{-1}(S)) \subseteq u^{-1}(\bar{f}(S))$. Conversely, suppose $x' \in |\mathcal{X}'|, x \in |\mathcal{X}|$, $v(x') \in S$, and $\bar{f}(v(x')) = u(x)$, then $u(f(x')) = u(x)$, so $\exists g \in G$ such that $x = gf(x') = f(gx')$. That $v(gx') = v(x') \in S$ implies that $gx' \in v^{-1}(S)$, so $x \in f(v^{-1}(S))$.

Then f clopen $\implies$ $\bar{f}$ clopen. Moreover, the fibers of $\bar{f}$ are finite, as for each $\bar{s} \in |\mathcal{X}|/G$ and any lift $x \in |\mathcal{X}|$ of $\bar{s}$, $\bar{f}^{-1}(\bar{s})$ is the v-image of the finite set $f^{-1}(x)$ (take, in the above, $S = \{s\}$, where $\bar{f}(s) = \bar{s}$). Now applying [Aut, 07VB], we see that $|\mathcal{X}'|/G$ is a coproduct of finitely many connected components, *i.e.* $\mathcal{X}'$ is a coproduct of finitely many G-components.

We'll show that a G-connected object $\mathcal{Y} \in \mathcal{C}$ is a connected object in $\mathcal{C}$ (cf. [Aut, 0BMY]). Suppose $a\colon \mathcal{Y}_1 \to \mathcal{Y}$ is a nonempty monomorphism in $\mathcal{C}$. Then the diagonal map $\mathcal{Y}_1 \to \mathcal{Y}_1 \times_\mathcal{Y} \mathcal{Y}_1$ is an isomorphism. This implies that a is also a monomorphism in $\mathrm{F\acute{E}t}(\mathcal{X})$, *i.e.* a exhibits $\mathcal{Y}_1$ as a clopen subset of $\mathcal{Y}$. As $\mathcal{Y}$ is G-connected, $\mathcal{Y}_1 = \mathcal{Y}$, *i.e.* a is an isomorphism.

$F(\mathcal{X}')$ is finite for all $\mathcal{X}' \in \mathcal{C}$. This is clear.

F is conservative and exact. The functor F is, by its very definition, the composition

$$\mathrm{F\acute{E}t}(\mathcal{X}/G) \xrightarrow{\text{Forget}} \mathrm{F\acute{E}t}(\mathcal{X}) \xrightarrow{F_{\bar{x}}} (\mathrm{Set})$$

where Forget is the obvious forgetful functor, which is conservative and preserves finite limits and finite colimits (given how finite limits and finite colimits in $\mathrm{F\acute{E}t}(\mathcal{X}/G)$ are constructed), and $F_{\bar{x}}$ is the classical fiber functor which is conservative and exact. Thus F is conservative and exact. $\square$

6. Drinfeld's lemma

Let $\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_n$ be algebraic stacks over $\mathbb{F}_q$, and set $\mathcal{X} := \mathcal{X}_1 \times_{\mathbb{F}_q} \mathcal{X}_2 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} \mathcal{X}_n$. Let ϕ_i be the q-Frobenius of $\mathcal{X}_i$ for $1 \leqslant i \leqslant n$. Then $\mathcal{X}$ is equipped with a strict $M := \phi_1^\mathbb{N} \times \phi_2^\mathbb{N} \times \cdots \phi_n^\mathbb{N}$-action. Let M_0 denote the submonoid $\phi_1^\mathbb{N} \times \cdots \times \phi_{n-1}^\mathbb{N}$, and let G (resp. G_0) denote the free abelian group generated by M (resp. M_0). The category $\mathrm{F\acute{E}t}(\mathcal{X}/G_0)$

can be understood as 2-cartesian diagrams

$$\begin{tikzcd}
\mathcal{Y} \arrow[r, "\varphi_{n-1}"] \arrow[d, "f"'] \arrow[dr, phantom, "\swarrow \sigma_{n-2}"] & \mathcal{Y} \arrow[r, "\varphi_{n-2}"] \arrow[d, "f"'] \arrow[dr, phantom, "\swarrow \sigma_{n-1}"] & \mathcal{Y} \arrow[r, "\cdots"] \arrow[d, "f"'] \arrow[dr, phantom, "\cdots"] & \mathcal{Y} \arrow[r, "\varphi_1"] \arrow[d, "f"'] \arrow[dr, phantom, "\swarrow \sigma_1"] & \mathcal{Y} \arrow[d, "f"] \\
\mathcal{X} \arrow[r, "\phi_{n-1}"] & \mathcal{X} \arrow[r, "\phi_{n-2}"] & \mathcal{X} \arrow[r, "\cdots"] & \mathcal{X} \arrow[r, "\phi_1"] & \mathcal{X}
\end{tikzcd}$$

where the φ_is (resp. σ_is) are mutually commute. Morally, $\mathrm{F\acute{E}t}(\mathcal{X}/G_0)$ is just a collection of commuting universally homeomorphic $\mathbb{F}_q$-endomorphisms $\{\varphi_1, \varphi_2, \ldots, \varphi_{n-1}\}$ of $\mathcal{Y}$. Note that here we have dropped φ_n in the collection, but we could drop any ϕ_i for $1 \leqslant i \leqslant n$ which would yield the same category.

Lemma 6.1. *Let Y be a scheme. For any geometric point $\bar{x}\colon \mathrm{Spec}(k) \to Y$ the base change functor induces an equivalence:*

$$\varinjlim_{(U,\bar{u})} \mathrm{F\acute{E}t}(\mathcal{X} \times_{\mathbb{F}_q} U/G) \longrightarrow \mathrm{F\acute{E}t}(\mathcal{X} \times_{\mathbb{F}_q} \bar{x}/G)$$

where $(U, \bar{u})$ runs over all the affine étale neighborhood of $\bar{x}$.

Proof. Let $u\colon \mathcal{X}' \to \mathcal{X}$ be an M-equivariant map of $\mathbb{F}_q$-algebraic stacks, and $\mathcal{X}'' := \mathcal{X}' \times_{\mathcal{X}} \mathcal{X}'$, $\mathcal{X}''' := \mathcal{X}'' \times_{\mathcal{X}} \mathcal{X}'$, then there are natural 2-commutative diagrams

$$\begin{tikzcd}
\varinjlim\limits_{(U,\bar{u})} \mathrm{F\acute{E}t}(\mathcal{X} \times_{\mathbb{F}_q} U/G) \arrow[r] \arrow[d, "\lambda"] & \varinjlim\limits_{(U,\bar{u})} \mathrm{F\acute{E}t}(\mathcal{X}' \times_{\mathbb{F}_q} U/G) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\lambda'"] & \varinjlim\limits_{(U,\bar{u})} \mathrm{F\acute{E}t}(\mathcal{X}'' \times_{\mathbb{F}_q} U/G) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, "\lambda''"] & \varinjlim\limits_{(U,\bar{u})} \mathrm{F\acute{E}t}(\mathcal{X}''' \times_{\mathbb{F}_q} U/G) \arrow[d, "\lambda'''"] \\
\mathrm{F\acute{E}t}(\mathcal{X} \times_{\mathbb{F}_q} \bar{x}/G) \arrow[r] & \mathrm{F\acute{E}t}(\mathcal{X}' \times_{\mathbb{F}_q} \bar{x}/G) \arrow[r, shift left] \arrow[r, shift right] & \mathrm{F\acute{E}t}(\mathcal{X}'' \times_{\mathbb{F}_q} \bar{x}/G) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathrm{F\acute{E}t}(\mathcal{X}''' \times_{\mathbb{F}_q} \bar{x}/G)
\end{tikzcd} \tag{4}$$

Suppose that the horizontal sequences are exact, then λ' is fully faithful (resp. an equivalence) and λ'' is faithful (resp. fully faithful and λ''' is faithful) imply that λ is fully faithful (resp. an equivalence). Note also that by [Aut, 07SK]we have a natural equivalence:

$$\varinjlim_{(U,\bar{u})} \mathrm{F\acute{E}t}(\mathcal{X} \times_{\mathbb{F}_q} U/G) \xrightarrow{\simeq} \mathrm{F\acute{E}t}(\mathcal{X} \times_{\mathbb{F}_q} \varprojlim_{(U,\bar{u})} U/G) \tag{5}$$

where $\varprojlim\limits_{(U,\bar{u})} U$ is the strict henselization of Y at $\bar{x}$, so the rows of (4) are exact when u is an fpqc-covering (the actions descend together).

Using these simple observations, we'll proceed in several dévissage steps:

Step 1, reduction to $\mathcal{X}_i = X_i$ a connected $\mathbb{F}_q$-scheme of finite type. Replacing each $\mathcal{X}_i$ by an atlas X_i, we may assume that it is an algebraic space. Replacing each algebraic space X_i by an atlas, we may assume that it is a scheme. Replacing each X_i by a disjoint union of affine opens, we may assume that each X_i is affine. Note that although X'' in (4) may not be a disjoint union of affines anymore, we can cover each $X_i' \times_{X_i} X_i'$ by affines and show that λ is fully faithful for any scheme X. Using this and applying affine coverings again, we get the desired equivalence for any scheme X. We then write each affine scheme X_i as a filtered limit of affine schemes of finite type over $\mathbb{F}_q$. Thus we may assume that X_i is of finite type over $\mathbb{F}_q$. Replacing X_i by a connected component, we may assume that X_i is connected.

Step 2, reduction to Y excellent, reduced, and strictly henselian. Since the problem is Zariski-local around $\bar{x}$, we may assume that Y is affine. Writing Y as a filtered limit of affine finite type $\mathbb{F}_q$-schemes and using [Aut, 07SK], we may assume that Y is of finite type over $\mathbb{F}_q$. Replacing Y with its reduced closed structure, we may assume that Y is reduced. Now replacing Y by its strict henselization at $\bar{x}$ (cf. (5)), we may assume that $Y = \mathrm{Spec}(A)$, where A is strictly henselian and reduced. By [Aut, 06LJ], A is Noetherian; by [Aut, 07QR], A is a G-ring; by [Gro67, 18.8.17, 18.7.5.1], A is universally catenary; and by [Gre76, 5.3] is J-2. In conclusion, A is excellent.

Step 3, the functor (5) is faithful and essentially surjective. Consider the following 2-commutative diagram, where $X := \mathcal{X}$ is affine and of finite type.

$$\begin{tikzcd}
 & \mathrm{F\acute{E}t}(X \times_{\mathbb{F}_q} \mathrm{Spec}(A)/G) \arrow[d, "\lambda"] \\
\mathrm{F\acute{E}t}(X/G_0) \arrow[ur, "u"] \arrow[r, "v", "\simeq"'] & \mathrm{F\acute{E}t}(X \times_{\mathbb{F}_q} \mathrm{Spec}(\kappa)/G)
\end{tikzcd}$$

We have to show that λ is an equivalence. Let ϕ_κ denote the absolute Frobenius of κ. Then $\mathrm{F\acute{E}t}(X \times_{\mathbb{F}_q} \mathrm{Spec}(\kappa)/G)$ can also be described via the action of $G_0 \times \phi_\kappa^{\mathbb{Z}}$. By [Ked17, Lemma 4.2.6] or [SW20, Lemma 16.2.6] we have $\mathrm{F\acute{E}t}(X) \simeq \mathrm{F\acute{E}t}(X \times_{\mathbb{F}_q} \mathrm{Spec}(\kappa)/\phi_\kappa^{\mathbb{Z}})$. This immediately implies that $\mathrm{F\acute{E}t}(X/G_0) \simeq \mathrm{F\acute{E}t}(X \times_{\mathbb{F}_q} \mathrm{Spec}(\kappa)/G)$. Therefore, v is an equivalence. By Lemma 4.2 and Lemma 5.3 all the categories in the above diagram are Galois categories; since all the functors in the diagram commute with the forgetful fiber functors (which are faithful), they are faithful. This implies that u is fully faithful and λ is essentially surjective. Thus to conclude, it is enough to show that λ is fully faithful or u is essentially surjective.

Step 4, reduction to X_1, Y normal and Y has algebraically closed field of fractions. Since v-covers are morphisms of effective descent for étale maps (cf. [HS23, Theorem 1.5]), and "being quasi-compact" as well as "satisfying the valuative criteria for properness" are obviously properties of maps local on the target for the v-topology (cf. [Aut, 02KO]), v-covers are also morphisms of effective descent for finite étale maps. As each X_i is excellent, its normalization map is finite. Using diagram (4) for the product of normalization maps and the fact that λ'' is faithful (as X'' is a finite disjoint union of connected components), we can assume that each X_i is normal.

For Y, we first note that there are finitely many irreducible components of Y, each of which is a spectrum of a strictly henselian excellent domain (cf. [Aut, 0C2Z]). If Y has m irreducible components $Y_1, \cdots, Y_m$, then we set $Y' := Y_1 \sqcup (Y_2 \cup \cdots \cup Y_m)$, which is a v-cover of Y. Consider the following diagram (diagram (4) "with the other factor")

$$
\begin{tikzcd}
\mathrm{F\acute{E}t}(X \times_{\mathbb{F}_q} Y/G) \arrow[r] \arrow[d, "\lambda"] & \mathrm{F\acute{E}t}(X \times_{\mathbb{F}_q} Y'/G) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\lambda'"] & \mathrm{F\acute{E}t}(X \times_{\mathbb{F}_q} Y''/G) \arrow[d, "\lambda''"] \\
\mathrm{F\acute{E}t}(X \times_{\mathbb{F}_q} \bar{x}/G) \arrow[r] & \mathrm{F\acute{E}t}(X \times_{\mathbb{F}_q} \bar{x}'/G) \arrow[r, shift left] \arrow[r, shift right] & \mathrm{F\acute{E}t}(X \times_{\mathbb{F}_q} \bar{x}''/G)
\end{tikzcd} \tag{6}
$$

where $\bar{x}' := \bar{x} \sqcup \bar{x}$ and $\bar{x}'' := \bar{x}' \times_{\bar{x}} \bar{x}'$. Thanks to v-descent, the rows in (6) are exact. Using induction on m, we are reduced to the case when A is a domain.

Now let K be the field of fractions of A, and let B be the normalization of A in some finite extension L/K. Since

- B is finite over A (as A is excellent) and,
- B as well as $B \otimes_A B$ are local and strictly henselian (cf. [Aut, 04GH]),

applying diagram (6) to the normalization map and $\bar{x}'' = \bar{x}' = \bar{x}$, we are reduced to the case when Y is normal.

Let $\bar{Y} = \mathrm{Spec}(\bar{A})$ be the normalization of Y in $\bar{K}$. Then $\bar{A}$ is normal and strictly henselian (cf. [Aut, 04GI]). If $T_1, T_2 \in \mathrm{F\acute{E}t}(X \times_{\mathbb{F}_q} Y/G)$ and $t\colon T_1 \otimes_A \kappa \to T_2 \otimes_A \kappa$ is a map in $\mathrm{F\acute{E}t}(X \times_{\mathbb{F}_q} \bar{x}/G)$. Suppose t lifts to some $\bar{a} \in \mathrm{F\acute{E}t}(X \times_{\mathbb{F}_q} \bar{Y}/G)$. Since $\bar{a}$ is defined over some finite middle extension $A \subseteq B \subseteq \bar{A}$, we can apply diagram (6) again to conclude that t lifts to a map $a \in \mathrm{F\acute{E}t}(X \times_{\mathbb{F}_q} Y/G)$. Thus we are reduced to the case when the fraction field K of A is algebraically closed and A is normal (but not necessarily Noetherian).

Step 5, final conclusion. Let K be the field of fractions of A. Then the functor

$$\mathrm{F\acute{E}t}(X/G_0) \longrightarrow \mathrm{F\acute{E}t}(X \times_{\mathbb{F}_q} \mathrm{Spec}(K)/G)$$

is an equivalence and

$$\mathrm{F\acute{E}t}(X \times_{\mathbb{F}_q} \mathrm{Spec}(A)/G) \longrightarrow \mathrm{F\acute{E}t}(X \times_{\mathbb{F}_q} \mathrm{Spec}(K)/G)$$

is fully faithful. Thus u is an equivalence, as desired. ∎

Lemma 6.2 (Stein factorization)**.** *Suppose that $\mathcal{X}$ is a G_0-connected algebraic stack and $\mathcal{X}_{n+1}$ is an arbitrary algebraic stack. For any $(\mathcal{Y}, \rho_{\mathcal{Y}}, \sigma) \in \mathrm{F\acute{E}t}(\mathcal{X} \times_{\mathbb{F}_q} \mathcal{X}_{n+1}/G)$, there exists $\mathcal{T} \in \mathrm{F\acute{E}t}(\mathcal{X}_{n+1})$ and a 2-commutative diagram*

$$
\begin{tikzcd}
\mathcal{Y} \arrow[r] \arrow[d] & \mathcal{T} \arrow[d] \\
\mathcal{X} \times_{\mathbb{F}_q} \mathcal{X}_{n+1} \arrow[r] & \mathcal{X}_{n+1}
\end{tikzcd} \tag{7}
$$

where $\rho_{\mathcal{Y}}$ acts over $\mathcal{T}$ and each geometric fiber $\mathcal{Y}_{\bar{t}}$ of $\mathcal{Y} \to \mathcal{T}$ is G-connected. If diagram (7) exists, then it has the following universal property: For any $\mathcal{T}' \in \mathrm{F\acute{E}t}(\mathcal{X}_{n+1})$ and any commutative diagram (7) with $\mathcal{T}$ replaced by $\mathcal{T}'$, where $\rho_{\mathcal{Y}}$ acts over $\mathcal{T}'$, there is a unique arrow $\lambda\colon \mathcal{T} \to \mathcal{T}'$ making all the natural diagrams 2-commutative. Since the natural commutativity forces λ into an object in $\mathrm{F\acute{E}t}(\mathcal{X}_{n+1})$, which is equivalent to a set, λ is automatically unique up to a unique isomorphism.

Proof. **Step 1, the case when $\mathcal{X}_{n+1}$ is a point.** Let's first consider the case when $\mathcal{X}_{n+1} = \operatorname{Spec}(k)$, where k is an algebraically closed field containing $\mathbb{F}_q$. In this case, we take $\mathcal{T}$ to be $\sqcup_{i\in I}\operatorname{Spec}(k)$, where I is the set of G-components of $|\mathcal{Y}|$. To prove the universality, we first observe that both $\mathcal{T}$ and $\mathcal{T}'$ are finite disjoint unions of $\operatorname{Spec}(k)$. Since each element of I corresponds to a point of $\mathcal{T}$ and it is mapped to a point of $\mathcal{T}'$ by $\mathcal{Y} \to \mathcal{T}'$, this defines a unique map $\mathcal{T} \to \mathcal{T}'$ in $\mathrm{FÉt}(\mathcal{X}_{n+1})$ making all the natural diagrams commutative.

Step 2, the universal property for $\mathcal{X}_{n+1}$ general. We first observe that $\mathcal{Y} \to \mathcal{T}$ is faithfully flat, so the desired arrow $\mathcal{T} \to \mathcal{T}'$ is unique if it exists (note that $\mathrm{FÉt}(\mathcal{X}_{n+1})$ is a 1-category, so this makes sense). This allows us to define $\mathcal{T} \to \mathcal{T}'$ fppf locally on $\mathcal{X}_{n+1}$. Thus we may assume that $\mathcal{T}$ is a finite disjoint union of $\mathcal{X}_{n+1}$. Working with each pieces of $\mathcal{T}$ we may assume that $\mathcal{T} = \mathcal{X}_{n+1}$. Notice that the map $f\colon \mathcal{Y} \to \mathcal{T}'$ is universally open. Indeed, f can be factorized as $\mathcal{Y} \xrightarrow{f_1} \mathcal{X} \times_{\mathbb{F}_q} \mathcal{T}' \xrightarrow{f_2} \mathcal{T}'$, while f_1 is finite étale and f_2 is universally open by [Aut, 0383]. Let $\mathcal{U} \coloneqq f(\mathcal{Y})$ be the image. We have to show that there exists a unique $\mathcal{X}_{n+1}$-map $\mathcal{T} = \mathcal{X}_{n+1} \to \mathcal{U}$. Now it follows from Step 1 that on each geometric fiber over $\mathcal{X}_{n+1}$ the factorization exists and is bijective, therefore each fiber of the projection map $u\colon \mathcal{U} \to \mathcal{X}_{n+1}$, which is representable and étale, consists of exactly one point. Thus u is an isomorphism and the desired arrow is u^{-1}.

Step 3, reduction to $\mathcal{X}_{n+1}$ affine. By Step 2, to prove the existence of the Stein factorization, it is enough to prove it fppf-locally. Thus we may assume that $\mathcal{X}_{n+1} = X_{n+1} = \operatorname{Spec}(A)$ is affine.

Step 4, reduction to $\mathcal{Y}$ is a pullback from $\mathbf{FÉt}(\mathcal{X}/G_0)$. By Lemma 6.1 and [PTZ20, Theorem II, (C)], for any geometric point $\bar{x}$ of X_{n+1} one can find a connected affine étale neighborhood $U \to X_{n+1}$ such that the pullback of $(\mathcal{Y}, \rho_{\mathcal{Y}}, \sigma)$ to $\mathrm{FÉt}(\mathcal{X} \times_{\mathbb{F}_q} U/G)$ comes from an object $\mathcal{Z} \in \mathrm{FÉt}(\mathcal{X}/G_0)$. By Step 2, it is enough to construct the Stein factorization over U. So we assume that $\mathcal{Y}$ comes from $\mathcal{Z} \in \mathrm{FÉt}(\mathcal{X}/G_0)$.

Step 5, final conclusion. Let $\mathcal{Z}_1, \mathcal{Z}_2, \dots, \mathcal{Z}_m$ be the G_0-components of $\mathcal{Z}$ (remember that $\mathrm{FÉt}(\mathcal{X}/G_0)$ is Galois), then for any geometric point $\bar{x} \in X_{n+1}(k)$, the pullback $(\mathcal{Z}_i, \rho_{\mathcal{Z}_i}, \sigma_i)_k$ of $(\mathcal{Z}_i, \rho_{\mathcal{Z}_i}, \sigma_i)$ to $\mathrm{FÉt}(\mathcal{X} \times_{\mathbb{F}_q} k/G)$ has underlying object $\mathcal{Z}_i \times_{\mathbb{F}_q} k$ and action $\rho_{\mathcal{Z}_i} \times \phi_k$. It's G-connected, as any clopen G-equivariant subset of $\mathcal{Z}_i \times_{F_q} k$ is the inverse image of a G_0-equivariant clopen subset of $\mathcal{Z}_i$ (cf. [PTZ20, Theorem 4.1]). Thus we can take $T \coloneqq \sqcup_{i=1}^m X_{n+1}$ and the map $\mathcal{Y} \to T$ to be the sum of the projections $\mathcal{Z}_i \times_{\mathbb{F}_q} X_{n+1} \to X_{n+1}$. $\square$

Theorem 6.3. *Suppose that $\mathcal{X}_i$ $(1 \leqslant i \leqslant n)$ and $\mathcal{X}_{n+1}$ are connected algebraic stacks over $\mathbb{F}_q$. Then for any geometric point $\bar{x}$ of $\mathcal{X} \times_{\mathbb{F}_q} \mathcal{X}_{n+1}$, the top sequence in*

$$\begin{tikzcd} 1 \arrow[r] & \pi_1^{\text{ét}}(\mathcal{X} \times_{\mathbb{F}_q} k/G, \bar{x}) \arrow[r] \arrow[d, "\cong"'] & \pi_1^{\text{ét}}(\mathcal{X} \times_{\mathbb{F}_q} \mathcal{X}_{n+1}/G, \bar{x}) \arrow[r, "\beta"] \arrow[dl, "\alpha"] & \pi_1^{\text{ét}}(\mathcal{X}_{n+1}, \bar{x}) \arrow[r] & 1 \\ & \pi_1^{\text{ét}}(\mathcal{X}/G_0, \bar{x}) & & & \end{tikzcd} \tag{8}$$

is exact, where k is the field of definition of $\bar{x}$.

Proof. The left exactness of the top sequence is provided by the retraction α, whose existence is a consequence of [PTZ20, Theorem II, (C)]. Thus we are left to show the middle exactness. In view of [Aut, 0BTQ] we only have to show that the following condition holds:

Let $(\mathcal{Y}, \rho_{\mathcal{Y}}) \in \mathtt{F\acute{E}t}(\mathcal{X} \times_{\mathbb{F}_q} \mathcal{X}_{n+1}/G)$ be a connected object. Suppose the pullback $(\mathcal{Y}_{\bar{x}}, \rho_{\mathcal{Y}_{\bar{x}}})$ of $(\mathcal{Y}, \rho_{\mathcal{Y}})$ to $\mathtt{F\acute{E}t}(\mathcal{X} \times_{\mathbb{F}_q} k/G)$ has a G-component which is mapped isomorphically to $\mathcal{X} \times_{\mathbb{F}_q} k$, then the Stein factorization of $(\mathcal{Y}, \rho_{\mathcal{Y}})$ induces a Cartesian diagram.

$$\begin{tikzcd}
\mathcal{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{T} \arrow[d] \\
\mathcal{X} \times_{\mathbb{F}_q} \mathcal{X}_{n+1} \arrow[r] & \mathcal{X}_{n+1}
\end{tikzcd}$$

One has to show that $\lambda \colon \mathcal{Y} \to \mathcal{X} \times_{\mathbb{F}_q} \mathcal{T}$ is an isomorphism. For each geometric point $\bar{s}$ of $\mathcal{T}$ the pullback $\mathcal{X} \times_{\mathbb{F}_q} \kappa(\bar{s})$ is G-connected by 4.2. It follows that $\lambda_s \colon \mathcal{Y}_s \to \mathcal{X} \times_{\mathbb{F}_q} \kappa(\bar{s})$ as well as λ are surjective ($\mathcal{Y}_s$ is a G-component, hence is not empty). That $\mathcal{Y}$ is G-connected implies that $\mathcal{X} \times_{\mathbb{F}_q} \mathcal{T}$ is G-connected, thus λ is finite étale of constant degree. Therefore to show that λ is an isomorphism, it is enough to show that the degree of λ equals 1.

The G-component of $(\mathcal{Y}_{\bar{x}}, \rho_{\mathcal{Y}_{\bar{x}}})$ which goes isomorphically to $\mathcal{X} \times_{\mathbb{F}_q} k$ is mapped to a geometric point $\bar{s} \in \mathcal{T}(k)$ lying over $\bar{x} \in \mathcal{X}_{n+1}(k)$. It is moreover the inverse image of $\bar{s}$ by the very construction of the Stein factorization. Thus λ is an isomorphism at $\bar{s}$, and the degree of λ is 1, as desired. ∎

Proof of Theorem I. (1) is a special case of Theorem 5.3, so let's consider (2). By Theorem 8 and diagram (8) we have

$$\pi_1^{\text{ét}}(\mathcal{X}/\phi_1^{\mathbb{Z}} \times \cdots \times \phi_{n-1}^{\mathbb{Z}}) \xrightarrow{\cong} \pi_1^{\text{ét}}(\mathcal{X}_1 \times \cdots \times \mathcal{X}_{n-1}/\phi_1^{\mathbb{Z}} \times \cdots \times \phi_{n-2}^{\mathbb{Z}}) \times \pi_1^{\text{ét}}(\mathcal{X}_n)$$

By induction on n, we get the result. ∎

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