

DRINFELD'S LEMMA FOR ALGEBRAIC STACKS

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ABSTRACT. Drinfeld's lemma is a powerful tool for splitting ℓ -adic local systems defined over a product of connected schemes over a finite field. In this paper, we show that Drinfeld's lemma also holds true for algebraic stacks.

1. INTRODUCTION

The main motivation of Drinfeld's lemma is to split ℓ -adic local systems defined over a product of schemes. More precisely, let X_1, X_2 be two connected \mathbb{F}_q -schemes, then one would like to get, out of an ℓ -adic local system on $X := X_1 \times_{\mathbb{F}_q} X_2$, an ℓ -adic local system coming from local systems on the individual factors X_1, X_2 . The problem is easy if one considers complex analytic local systems on a product of complex varieties. Indeed, one can split the local system via the Künneth formula for topological fundamental groups:

$$\pi_1^{\text{top}}(X_1^{\text{an}} \times X_2^{\text{an}}) \xrightarrow{\cong} \pi_1^{\text{top}}(X_1^{\text{an}}) \times \pi_1^{\text{top}}(X_2^{\text{an}})$$

One can also do this for ℓ -adic local systems. Indeed, for any connected schemes X_1, X_2 defined over an algebraically closed field k of characteristic 0, one has an isomorphism:

$$\pi_1^{\text{ét}}(X_1 \times_k X_2) \xrightarrow{\cong} \pi_1^{\text{ét}}(X_1) \times \pi_1^{\text{ét}}(X_2)$$

Even when k is of characteristic $p > 0$ (but still algebraically closed), the above Künneth formula still holds true provided that either X_1 or X_2 is proper over k . However, the Künneth formula fails when k is not algebraically closed: take $X_1 = X_2 = \text{Spec}(\mathbb{F}_p)$, then the Künneth formula would mean that the diagonal map $\hat{\mathbb{Z}} \rightarrow \hat{\mathbb{Z}} \times \hat{\mathbb{Z}}$ is an isomorphism.

The issue (for finite fields) can be resolved if partial Frobenii actions are brought into play. More precisely, let ϕ_1 (resp. ϕ_2) denote the *partial Frobenius* map $X \rightarrow X$, which is the q -absolute Frobenius on X_1 (resp. X_2) and the identity on the other. Consider the category $\text{F}\acute{\text{E}}\text{t}(X/\Phi)$ of triples $(Y, \varphi_1, \varphi_2)$, where Y is a finite étale cover of $X = X_1 \times_{\mathbb{F}_q} X_2$ and φ_i is an isomorphism $Y \xrightarrow{\cong} \phi_i^* Y$ satisfying that $\phi_1^*(\varphi_2) \circ \varphi_1 = \phi_2^*(\varphi_1) \circ \varphi_2$ is the identity (by identifying $(\phi_1 \circ \phi_2)^* Y$ with Y via the absolute Frobenius of Y).

Theorem 1.1 (Drinfeld's lemma for schemes). *Suppose X_1, X_2 are connected quasi-compact and quasi-separated (qcqs) \mathbb{F}_q -schemes. Then*

- $\text{F}\acute{\text{E}}\text{t}(X/\Phi)$ is a Galois category whose Galois group is denoted by $\pi_1^{\text{ét}}(X/\Phi)$;
- the natural map $\pi_1^{\text{ét}}(X/\Phi) \rightarrow \pi_1^{\text{ét}}(X_1) \times \pi_1^{\text{ét}}(X_2)$ is an isomorphism.

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Similarly, one has Drinfeld's lemma for n factors X_1, \dots, X_n . Please refer to [Dri80, Theorem 2.1], [Laf97, IV.2, Theorem 4], [Lau07, Theorem 8.1.4], [Ked17, Theorem 4.2.12], [Laf18, Lemma 0.18], [SW20, Theorem 16.2.4], and [Mül22, Theorem 1.4] for details. Using Drinfeld's Lemma one can split local systems on a product of schemes equipped with commuting partial Frobenius actions.

The notion of fundamental group of algebraic stacks has been introduced and studied by B. Noohi in [Noo00]. The main purpose of this note is to generalize Drinfeld's lemma to algebraic stacks removing the qcqs assumption.

Theorem I (cf. §6). *Let $\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_n$ be connected algebraic stacks over \mathbb{F}_q , and set $\mathcal{X} := \mathcal{X}_1 \times_{\mathbb{F}_q} \dots \times_{\mathbb{F}_q} \mathcal{X}_n$. Then*

- (1) *The category $\text{F}\acute{\text{E}}\text{t}(\mathcal{X}/\Phi)$ consisting of finite étale covers of \mathcal{X} equipped with commuting partial Frobenius actions is a Galois category, whose Galois group is denoted by $\pi_1^{\acute{\text{E}}\text{t}}(\mathcal{X}/\Phi)$;*
- (2) *the natural map $\pi_1^{\acute{\text{E}}\text{t}}(\mathcal{X}/\Phi) \longrightarrow \pi_1^{\acute{\text{E}}\text{t}}(\mathcal{X}_1) \times \dots \times \pi_1^{\acute{\text{E}}\text{t}}(\mathcal{X}_n)$ is an isomorphism.*

The key technique that we are using here is the Drinfeld-Lau descent for fibered categories developed in [PTZ20].

2. THE G -CONNECTEDNESS

One of the difficulties in understanding Drinfeld's lemma is that the partial q -Frobenius maps are, in general, not invertible. Thus “the quotient space X/Φ ” is only a suggestive symbol but not an honest space – even when X is a scheme! However, we can first forget this if we only look at the actions on the ambient topological space $|\mathcal{X}|$.

Definition 2.1. Let G be a group, and let X be a topological space equipped with a G -action $\rho_G: G \times X \rightarrow X$. The pair (X, ρ_G) is called a G -space. The space X is called G -connected if the quotient space X/G is a connected topological space. A subspace $S \subseteq X$ is called a G -component of X if it is the inverse image of a nonempty clopen subspace of X/G . A ordinary topological space X can be viewed as a G -space via the trivial G -action. In this case, the pair (X, ρ_G) will simply denoted by X .

Let $f: (X, \rho_G) \rightarrow (S, \rho'_G)$ be a map, *i.e.* a map of topological spaces $f: X \rightarrow S$ which is compatible with the G -actions. For each G -stable subset $S_1 \subseteq S$ the inverse image $f^{-1}(S_1) \subseteq X$ is a G -stable subspace.

Lemma 2.1. *Let $f: (X, \rho_G) \rightarrow S$ be map whose fibers are G -connected. Suppose that $X \rightarrow S$ is submersive. Then (X, ρ_G) is G -connected if and only if S is connected.*

Proof. The map f induces a continuous map $\bar{f}: X/G \rightarrow S$ whose fibers are connected (hence nonempty). If (X, ρ_G) is G -connected, then X/G is connected, so S is connected. Conversely, if (X, ρ_G) is not G -connected, then $X/G = U_1 \sqcup U_2$, where U_1, U_2 are two non-empty open subspaces of X/G . Since the fibers of \bar{f} are connected, any fiber is contained either in U_1 or in U_2 . Thus exists subspaces V_1, V_2 of S such that $S = V_1 \sqcup V_2$ and $\bar{f}^{-1}(V_i) = U_i$ ($i = 1, 2$). That f is submersive implies that \bar{f} is submersive. Hence V_1, V_2 are non-empty opens of S . Then S is not connected. \square

3. FROBENIUS AND UNIVERSAL HOMEOMORPHISMS

Recall that for any fibered category \mathcal{X} defined over \mathbb{F}_p , the absolute Frobenius map $F_{\mathcal{X}}: \mathcal{X} \rightarrow \mathcal{X}$ sends a section $x \in \mathcal{X}(T)$ on an \mathbb{F}_p -scheme T to the composition $T \xrightarrow{F_T} T \xrightarrow{x} \mathcal{X} \in \mathcal{X}(T)$, where F_T is the absolute Frobenius of the scheme T .

Universal homeomorphisms of schemes are radical maps (cf. [Aut, 01S2]) which play a role as the “opposite” of étale maps. The on the nose extension of this notion to stacks could be misleading in certain situations. For example, consider a finite étale group scheme G over a field k . Any base change of the map $f: \mathcal{B}_k G \rightarrow \mathrm{Spec}(k)$ is a homeomorphism on the underlying topological spaces, however, the map is étale. This suggests a new definition of the notion of “universal homeomorphism” for algebraic stacks – we should consider not only f but also the diagonal of f which encodes information about the (relative) inertia.

Definition 3.1. A map of algebraic stacks $f: \mathcal{X} \rightarrow \mathcal{Y}$ is called a *universal homeomorphism on the nose* if for any map of algebraic stacks $\mathcal{T} \rightarrow \mathcal{Y}$, the base change $f_{\mathcal{T}}: \mathcal{X} \times_{\mathcal{Y}} \mathcal{T} \rightarrow \mathcal{T}$ is a homeomorphism on the underlying topological spaces. The map f is called a *universal homeomorphism* if both f and the diagonal Δ_f are universal homeomorphisms on the nose.

Lemma 3.1. *If \mathcal{X} is an algebraic stack over \mathbb{F}_p , then $F_{\mathcal{X}}$ is a universal homeomorphism.*

Proof. Suppose $f: \mathcal{Y} \rightarrow \mathcal{X}$ is a map of algebraic stacks over \mathbb{F}_p . Then we have the following cartesian diagram

$$\begin{array}{ccc} \mathcal{Y}' & \xrightarrow{F'} & \mathcal{Y} \\ \downarrow f' & \lrcorner & \downarrow f \\ \mathcal{X} & \xrightarrow{F_{\mathcal{X}}} & \mathcal{X} \end{array}$$

We want to show that F' induces a bijection $|F'|: |\mathcal{Y}'| \rightarrow |\mathcal{Y}|$. By the universality of a fibered square, there is a factorization of $F_{\mathcal{Y}}: \mathcal{Y} \xrightarrow{F_{\mathcal{Y}/\mathcal{X}}} \mathcal{Y}' \xrightarrow{F'} \mathcal{Y}$. If $F_{\mathcal{Y}}$ induces a homeomorphism of $|\mathcal{Y}|$, then we just have to show that $|F_{\mathcal{Y}/\mathcal{X}}|$ is surjective. Take a point $y' \in |\mathcal{Y}'|$ which is represented by a triple (x, y, α) , where $x \in \mathcal{X}(k)$, $y \in \mathcal{Y}(k)$ are geometric points, and α is an isomorphism from $\mathrm{Spec}(k) \xrightarrow{F_k} \mathrm{Spec}(k) \xrightarrow{x} \mathcal{X}$ to $\mathrm{Spec}(k) \xrightarrow{y} \mathcal{Y} \xrightarrow{f} \mathcal{X}$. Consider the k -point $y_1: \mathrm{Spec}(k) \xrightarrow{F_k^{-1}} \mathrm{Spec}(k) \xrightarrow{y} \mathcal{Y}$ of \mathcal{Y} . Clearly, $F_{\mathcal{Y}}(y_1) = y$ and $x \xrightarrow{\cong} f(y_1)$ via $F_k^{-1}(\alpha)$. We get $(x, y, \alpha) \cong (f(y_1), y, \mathrm{id}_{f(y)}) = F_{\mathcal{Y}/\mathcal{X}}(y_1)$. Thus $|F_{\mathcal{Y}/\mathcal{X}}|$ is surjective.

Now let's show that $|F_{\mathcal{X}}|$ is bijective for any algebraic stack \mathcal{X} . If $x \in \mathcal{X}(k)$ is a geometric point, then the k -point $\mathrm{Spec}(k) \xrightarrow{F_k^{-1}} \mathrm{Spec}(k) \xrightarrow{x} \mathcal{X}$ is mapped to x via $F_{\mathcal{X}}$. Hence $|F_{\mathcal{X}}|$ is surjective. Conversely, let $x_1 \in \mathcal{X}(k_1)$ and $x_2 \in \mathcal{X}(k_2)$ are two geometric points such that $F_{\mathcal{X}}(x_1)$ and $F_{\mathcal{X}}(x_2)$ represent the same point in \mathcal{X} , i.e. $F_{\mathcal{X}}(x_1) \sim F_{\mathcal{X}}(x_2)$. Then, by definition, there is an algebraically closed field k containing both k_1 and k_2 such that $\mathrm{Spec}(k) \rightarrow \mathrm{Spec}(k_1) \xrightarrow{F_{k_1}} \mathrm{Spec}(k_1) \xrightarrow{x_1} \mathcal{X}$ is isomorphic to $\mathrm{Spec}(k) \rightarrow \mathrm{Spec}(k_2) \xrightarrow{F_{k_2}} \mathrm{Spec}(k_2) \xrightarrow{x_2} \mathcal{X}$. But since $(\mathrm{Spec}(k) \rightarrow \mathrm{Spec}(k_i) \xrightarrow{F_{k_i}} \mathrm{Spec}(k_i)) = (\mathrm{Spec}(k) \xrightarrow{F_k} \mathrm{Spec}(k) \rightarrow \mathrm{Spec}(k_i))$ for

$i = 1, 2$, we have $\mathrm{Spec}(k) \rightarrow \mathrm{Spec}(k_1) \xrightarrow{x_1} \mathcal{X}$ is isomorphic to $\mathrm{Spec}(k) \rightarrow \mathrm{Spec}(k_2) \xrightarrow{x_2} \mathcal{X}$, so $x_1 \sim x_2$.

We have seen that $|F'|$, $|F_{\mathcal{Y}/\mathcal{X}}|$, $|F_{\mathcal{Y}}|$ are bijective and continuous. From the very definition of the topology on $|\mathcal{Y}|$ it is clear that $|F_{\mathcal{Y}}|$ is a homeomorphism. The continuity of $|F_{\mathcal{Y}/\mathcal{X}}|$ then implies that $|F'|$ is submersive, hence a homeomorphism.

Let's consider the diagonal $\Delta: \mathcal{X} \rightarrow \mathcal{X} \times_{F_{\mathcal{X}}} \mathcal{X}$. The map $|\Delta|$ is surjective because any geometric point $(x, y, \alpha) \in (\mathcal{X} \times_{F_{\mathcal{X}}} \mathcal{X})(k)$, where $\alpha: (\mathrm{Spec}(k) \xrightarrow{F_k} \mathrm{Spec}(k) \xrightarrow{x} \mathcal{X}) \cong (\mathrm{Spec}(k) \xrightarrow{F_k} \mathrm{Spec}(k) \xrightarrow{y} \mathcal{X})$, induces an isomorphism $F_k^{-1}(\alpha): x \cong y$, so $(x, x, \mathrm{id}_{F_{\mathcal{X}}(x)}) \cong (x, y, \alpha)$. Conversely, if $x \in \mathcal{X}(k_1)$ and $y \in \mathcal{X}(k_2)$ such that $(x, x, \mathrm{id}_{F_{\mathcal{X}}(x)}) \sim (y, y, \mathrm{id}_{F_{\mathcal{X}}(y)})$ in the diagonal, then there is a geometric point $(a, b, \alpha) \in (\mathcal{X} \times_{F_{\mathcal{X}}} \mathcal{X})(k)$, where k contains both k_1 and k_2 , such that (a, b, α) is isomorphic to the pullback of both $(x, x, \mathrm{id}_{F_{\mathcal{X}}(x)})$ and $(y, y, \mathrm{id}_{F_{\mathcal{X}}(y)})$ to k . This implies immediately that $x \sim y$. \square

Here are some geometric properties of universal homeomorphisms for algebraic stacks. Note that Lemma 3.2 (2) is not true if g is only a universal homeomorphisms on the nose.

Lemma 3.2. *Let $\mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{Z}$ be a sequence of maps of algebraic stacks. Set $h := g \circ f$.*

- (1) *If f, g are universal homeomorphisms, then so is h .*
- (2) *If g is a universal homeomorphism and h is a universal homeomorphism on the nose, then f is a universal homeomorphism on the nose.*

Proof. (1) is obvious. Let's look at (2). The map f factorizes as $\mathcal{X} \xrightarrow{\Gamma_f} \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \xrightarrow{h \times \mathrm{id}_{\mathcal{Y}}} \mathcal{Y}$, where Γ_f is the graph of f . If h is a universal homeomorphism on the nose and the diagonal g is a universal homeomorphism, then Γ_f and $h \times \mathrm{id}_{\mathcal{Y}}$ are universal homeomorphisms on the nose, and so is the composition f . \square

Note that the absolute Frobenius is only \mathbb{F}_p -linear. In order to make it \mathbb{F}_q -linear we take the q -absolute Frobenius map $\phi_{\mathcal{X}} := F_{\mathcal{X}}^n$, where $n := \log_p q$. Just as in the scheme case, the pullback of the étale maps along the q -absolute Frobenius map of algebraic stacks induces an equivalence of étale sites (actually the identity!).

Proposition 3.3. *Let \mathcal{X} be a Deligne-Mumford stack (resp. an algebraic stack) over \mathbb{F}_q , and let $f: \mathcal{Y} \rightarrow \mathcal{X}$ be an étale map (resp. a representable étale map). Then we have the following cartesian diagram.*

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\phi_{\mathcal{Y}}} & \mathcal{Y} \\ \downarrow f & \lrcorner & \downarrow f \\ \mathcal{X} & \xrightarrow{\phi_{\mathcal{X}}} & \mathcal{X} \end{array}$$

Proof. One just has to show that the natural map $\lambda: \mathcal{Y} \rightarrow \mathcal{X} \times_{\phi_{\mathcal{X}}} \mathcal{Y}$ is an equivalence. For this, one can take an atlas of $\mathcal{X} \rightarrow \mathcal{X}$, where X is a scheme and show that the pullback of λ to X is an equivalence. Thus we are reduced to the case when $\mathcal{X} = X$ is a scheme. If f is

representable by schemes, then we are done. If f is only representable by algebraic spaces, then $\mathcal{Y} = Y$ is an algebraic space. Take an étale atlas $Z \rightarrow Y$. The following diagram

$$\begin{array}{ccc} Z & \xrightarrow{\lambda_Z} & X \times_{\phi_X} Z \\ \downarrow & \lrcorner & \downarrow \\ Y & \xrightarrow{\lambda} & X \times_{\phi_X} Y \end{array}$$

is cartesian, because its extension via the projections $X \times_{\phi_X} Z \rightarrow Z$ and $X \times_{\phi_X} Y \rightarrow Y$ is cartesian (as $Z \rightarrow Y$ is representable by schemes). Since λ_Z is an isomorphism by the scheme case, λ is an isomorphism as well. If \mathcal{Y} is only a Deligne-Mumford stack, then we take an étale atlas $Z \rightarrow \mathcal{Y}$ and repeat the above argument to reduce the problem to the case when f is representable by algebraic spaces which has just been proven. \square

4. PARTIAL FROBENIUS MAPS

Let's come back to our original setting. Let $\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_n$ be connected algebraic stacks over \mathbb{F}_q , and set $\mathcal{X} := \mathcal{X}_1 \times_{\mathbb{F}_q} \dots \times_{\mathbb{F}_q} \mathcal{X}_n$. Then there are *partial Frobenius maps* $\phi_i: \mathcal{X} \rightarrow \mathcal{X}$, which is the q -absolute Frobenius on \mathcal{X}_i and the identity on the others. These ϕ_i s are commuting endomorphisms of \mathcal{X} (resp. automorphisms of $|\mathcal{X}|$) whose composition $\phi_1 \circ \dots \circ \phi_n$ is the q -Frobenius of \mathcal{X} (resp. the identity of $|\mathcal{X}|$). Thus each ϕ_i is completely determined by all the others, so we can drop any of the partial Frobenius maps, e.g. ϕ_n . They provide \mathcal{X} (resp. $|\mathcal{X}|$) with a strict action (resp. an action) $\rho_{\mathcal{X}}$ by the monoid $M := \phi_1^{\mathbb{N}} \times \phi_2^{\mathbb{N}} \times \dots \times \phi_{n-1}^{\mathbb{N}}$ (resp. the group $G := \phi_1^{\mathbb{Z}} \times \phi_2^{\mathbb{Z}} \times \dots \times \phi_{n-1}^{\mathbb{Z}}$).

Lemma 4.1. *Let $G_i := \bigoplus_{j \in \{1, \dots, n\} \setminus \{i\}} \phi_j^{\mathbb{Z}}$. Then $|\mathcal{X}|$ is G -connected if and only if it is G_i -connected.*

Proof. This is just because that a subset of $|\mathcal{X}|$ is G -stable (resp. G_i -stable) iff it is stable under all the partial Frobenius actions. \square

Corollary 4.2. *The space $|\mathcal{X}_1 \times_{\mathbb{F}_q} \dots \times_{\mathbb{F}_q} \mathcal{X}_n|$ equipped with the \mathbb{Z}^{n-1} -action by $n-1$ of the partial Frobenius maps is \mathbb{Z}^{n-1} -connected.*

Proof. Consider the projection map $|\mathrm{pr}_n|: |\mathcal{X}_1 \times_{\mathbb{F}_q} \dots \times_{\mathbb{F}_q} \mathcal{X}_n| \rightarrow |\mathcal{X}_n|$, where the former is equipped with the \mathbb{Z}^{n-1} -action via the ϕ_i s, and the latter is equipped with the trivial \mathbb{Z}^{n-1} -action. Let $x \in \mathcal{X}_n(k)$ be a geometric point representing a point in $|\mathcal{X}_n|$. Then the topological space of the stack fiber $\mathcal{X}_1 \times_{\mathbb{F}_q} \dots \times_{\mathbb{F}_q} \mathcal{X}_{n-1} \times_{\mathbb{F}_q} k$ maps surjectively to the topological fiber of x along the map $|\mathrm{pr}_n|$. This surjection moreover preserves the \mathbb{Z}^{n-1} -actions. In light of Lemma 2.1, it is enough to show that the space $|\mathcal{X}_1 \times_{\mathbb{F}_q} \dots \times_{\mathbb{F}_q} \mathcal{X}_{n-1} \times_{\mathbb{F}_q} k|$ is \mathbb{Z}^{n-1} -connected.

Suppose $\mathcal{U} \subseteq \mathcal{X}_1 \times_{\mathbb{F}_q} \dots \times_{\mathbb{F}_q} \mathcal{X}_{n-1} \times_{\mathbb{F}_q} k$ is a clopen substack which is stable under the \mathbb{Z}^{n-1} -action. Then \mathcal{U} is also stable under the \mathbb{Z}^{n-1} -action defined by $\phi_1, \dots, \phi_{n-2}, \phi_k$. By [PTZ20, Theorem 4.1], \mathcal{U} is the preimage of a clopen substack $\mathcal{U}_0 \subseteq \mathcal{X}_1 \times_{\mathbb{F}_q} \dots \times_{\mathbb{F}_q} \mathcal{X}_{n-1}$,

and \mathcal{U}_0 is obviously stable under the \mathbb{Z}^{n-2} -action $\phi_1, \dots, \phi_{n-2}$. Then \mathcal{U}_0 is either empty or the whole stack by the connectedness of \mathcal{X}_{n-1} in the case $n = 2$, or by the induction hypothesis in the case $n > 2$. Thus \mathcal{U} is also either empty or the whole stack. This finishes the proof. \square

5. GALOIS CATEGORIES

Let \mathcal{X} be an algebraic stack over \mathbb{F}_q . When \mathcal{X} is connected, it is known (cf. [Noo00, Theorem 4.2] or [Noo04, §4]) that $\mathrm{F}\acute{\mathrm{E}}\mathrm{t}(\mathcal{X})$ is a Galois category. Let's now adapt this to our setting. Suppose M is a commutative monoid acting strictly (cf. **Rom05**) on \mathcal{X} via universal homeomorphisms. Let $\rho_{\mathcal{X}}: M \rightarrow \mathrm{End}_{\mathbb{F}_q}(\mathcal{X})$ be the action map. If G denotes the Grothendieck group associated with M , then $|\mathcal{X}|$ is a G -space. Let $\mathrm{F}\acute{\mathrm{E}}\mathrm{t}(\mathcal{X}/G)$ denote the category whose objects are 2-commutative diagrams

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\rho_{\mathcal{Y}}(m)} & \mathcal{Y} \\ f \downarrow & \swarrow \sigma_m & \downarrow f \\ \mathcal{X} & \xrightarrow{\rho_{\mathcal{X}}(m)} & \mathcal{X} \end{array}$$

where $\mathcal{Y} \xrightarrow{f} \mathcal{X} \in \mathrm{F}\acute{\mathrm{E}}\mathrm{t}(\mathcal{X})$ and $\rho_{\mathcal{Y}}$ is a strict action of M on \mathcal{Y} via \mathbb{F}_q -endomorphisms of \mathcal{Y} which are universal homeomorphisms, such that $\rho_{\mathcal{Y}}$ is compatible with $\rho_{\mathcal{X}}$, in the sense of **Rom05**, i.e. $\sigma_{m_1 m_2}$ is equal to the composition

$$(1) \quad \begin{array}{ccc} f \circ \rho_{\mathcal{Y}}(m_2) \circ \rho_{\mathcal{Y}}(m_1) & \xrightarrow{\sigma_{m_2}} & \rho_{\mathcal{X}}(m_2) \circ f \circ \rho_{\mathcal{Y}}(m_1) \xrightarrow{\sigma_{m_1}} \rho_{\mathcal{X}}(m_2) \circ \rho_{\mathcal{X}}(m_1) \circ f \\ \parallel & & \parallel \\ f \circ \rho_{\mathcal{Y}}(m_2 m_1) & & \rho_{\mathcal{X}}(m_2 m_1) \circ f \end{array}$$

and $\sigma_0 = \mathrm{id}_f$. One easier way to characterize $\mathrm{F}\acute{\mathrm{E}}\mathrm{t}(\mathcal{X}/G)$ is the following.

Lemma 5.1. *The following categories are equivalent*

- (a) *the category $\mathrm{F}\acute{\mathrm{E}}\mathrm{t}(\mathcal{X}/G)$;*
- (b) *the category of 2-cartesian diagrams*

$$(2) \quad \begin{array}{ccc} \mathcal{Y} & \xrightarrow{\rho_{\mathcal{Y}}(m)} & \mathcal{Y} \\ f \downarrow \quad \color{red}\lrcorner & \swarrow \sigma_m & \downarrow f \\ \mathcal{X} & \xrightarrow{\rho_{\mathcal{X}}(m)} & \mathcal{X} \end{array}$$

where $\mathcal{Y} \xrightarrow{f} \mathcal{X} \in \mathrm{F}\acute{\mathrm{E}}\mathrm{t}(\mathcal{X})$ and $\rho_{\mathcal{Y}}$ is a strict action of M on \mathcal{Y} via \mathbb{F}_q -endomorphisms and σ is the compatibility condition, i.e. $\sigma_{m_1 m_2}$ equals (1) and $\sigma_0 = f$.

- (c) *the category of pairs $(\mathcal{A}, \{\alpha_{\mathcal{A}, m}\}_{m \in M})$, where \mathcal{A} is a finite étale $\mathcal{O}_{\mathcal{X}}$ -algebra and $\alpha_{\mathcal{A}, m}: m^* \mathcal{A} \rightarrow \mathcal{A}$ ($m^* \mathcal{A} := \rho_{\mathcal{X}}(m)^* \mathcal{A}$) is an $\mathcal{O}_{\mathcal{X}}$ -algebra isomorphism satisfying $\alpha_{\mathcal{A}, e} = \mathrm{id}_{\mathcal{A}}$ and $\alpha_{\mathcal{A}, m_1 m_2} = \alpha_{\mathcal{A}, m_1} \circ m_1^* \alpha_{\mathcal{A}, m_2}$.*

Proof. Let's first show (a) \Leftrightarrow (b). For any $m \in M$ and $\mathcal{Y} \in \mathbf{F\acute{E}t}(\mathcal{X})$ set $m^*\mathcal{Y} := \mathcal{Y} \times_{\rho_{\mathcal{X}}(m)} \mathcal{X}$. Consider the 2-commutative diagram:

$$(3) \quad \begin{array}{ccccc} & & \rho_{\mathcal{Y}}(m) & & \\ & \swarrow \alpha_{\mathcal{Y},m} & & \searrow & \\ \mathcal{Y} & & m^*\mathcal{Y} & \xrightarrow{\quad} & \mathcal{Y} \\ & \searrow & \downarrow & \swarrow \scriptstyle \text{red } \perp & \downarrow \\ & & \mathcal{X} & \xrightarrow{\rho_{\mathcal{X}}(m)} & \mathcal{X} \end{array}$$

Suppose the square defined by $\rho_{\mathcal{Y}}(m)$ and $\rho_{\mathcal{X}}(m)$ is cartesian, then $\alpha_{\mathcal{Y},m}$ is an equivalence, so $\rho_{\mathcal{Y}}(m)$, as a pullback of $\rho_{\mathcal{X}}(m)$, is a universal homeomorphism, *i.e.* the square belongs to $\mathbf{F\acute{E}t}(\mathcal{X}/G)$. Conversely, if $\rho_{\mathcal{Y}}(m)$ is a universal homeomorphism, then $\alpha_{\mathcal{Y},m}$ is an isomorphism. Indeed, if $\rho_{\mathcal{Y}}(m)$ is a universal homeomorphism, then the fact that $m^*\mathcal{Y} \rightarrow \mathcal{Y}$ is a universal homeomorphism implies that $\alpha_{\mathcal{Y},m}$ is a universal homeomorphism on the nose (cf. Lemma 3.2). Since both \mathcal{Y} and $m^*\mathcal{Y}$ are in $\mathbf{F\acute{E}t}(\mathcal{X})$, $\alpha_{\mathcal{Y},m}$ is an isomorphism at all geometric fibers of \mathcal{X} , so it is a degree 1 finite étale map, *i.e.* an equivalence, hence the square defined by $\rho_{\mathcal{Y}}(m)$ and $\rho_{\mathcal{X}}(m)$ is cartesian.

The equivalence (b) \Leftrightarrow (c) follows from the universality of 2-cartesian diagrams and the equivalence $\mathbf{F\acute{E}t}(\mathcal{X}) \simeq \{\text{finite étale } \mathcal{O}_{\mathcal{X}}\text{-algebras}\}$: \mathcal{A} corresponds to \mathcal{Y} ; $\alpha_{\mathcal{A},m}$ corresponds to the unique map $\alpha_{\mathcal{Y},m}$; the conditions on α correspond to the conditions on σ . \square

Lemma 5.2. *Suppose that \mathcal{X} is G -connected, then $\mathbf{F\acute{E}t}(\mathcal{X}/G)$ is a Galois category.*

Proof. Set $\mathcal{C} := \mathbf{F\acute{E}t}(\mathcal{X}/G)$. If \bar{x} is a geometric point of \mathcal{X} , then take $F: \mathbf{F\acute{E}t}(\mathcal{X}/G) \rightarrow (\mathbf{Set})$ to be the composition of the forgetful functor with the fiber functor $\mathbf{F\acute{E}t}(\mathcal{X}) \rightarrow (\mathbf{Set})$. Let's verify the four axioms in [Aut, OBM9].

\mathcal{C} has finite limits and finite colimits. Suppose $\{(\mathcal{Y}_j, \rho_{\mathcal{Y}_j}, \sigma_j)\}_{j \in J}$ is a finite projective system in \mathcal{C} . For any $m \in M$, the pullback along $\mathcal{X} \xrightarrow{\rho_{\mathcal{X}}(m)} \mathcal{X}$ (denoted by m^*) is obviously a left exact endomorphism of $\mathbf{F\acute{E}t}(\mathcal{X})$. Thus for each $m \in M$, we have natural maps $\rho_{\varprojlim \mathcal{Y}_j} : \varprojlim_{j \in J} \mathcal{Y}_j \longrightarrow \varprojlim_{j \in J} \mathcal{Y}_j$ and $\sigma := \varprojlim_{j \in J} \sigma_j$ which make the following diagram

$$\begin{array}{ccc} \varprojlim_{j \in J} \mathcal{Y}_j & \xrightarrow{\rho_{\varprojlim \mathcal{Y}_j}} & \varprojlim_{j \in J} \mathcal{Y}_j \\ \downarrow & \swarrow \scriptstyle \text{red } \perp \quad \sigma & \downarrow \\ \mathcal{X} & \xrightarrow{\rho_{\mathcal{X}}(m)} & \mathcal{X} \end{array}$$

cartesian (as the diagram for each $\rho_{\mathcal{Y}_j}$ is so by Lemma 5.1). Hence the diagram is an object in $\mathbf{F\acute{E}t}(\mathcal{X}/G)$, and it is clearly the limit. As for the colimit, we first conclude, by the proof of [Aut, OBN9], that the pullback functor m^* is right exact. Indeed, thanks to [Aut, OGMN] one just has to check that m^* commutes with coproducts and coequalizers in $\mathbf{F\acute{E}t}(\mathcal{X})$. It's easy for coproducts. For coequalizers one considers two maps $a, b: \mathcal{Z} \rightarrow \mathcal{Y}$

in $\mathbf{F}\acute{\mathbf{E}}\mathbf{t}(\mathcal{X})$. Suppose $\mathcal{Y} = \mathrm{Spec}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{B})$ and $\mathcal{Z} = \mathrm{Spec}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{C})$, where \mathcal{B}, \mathcal{C} are two finite étale $\mathcal{O}_{\mathcal{X}}$ -algebras. Then a, b correspond to $a^{\#}, b^{\#}: \mathcal{B} \rightarrow \mathcal{C}$. The coequalizer of a, b is $\mathrm{Spec}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{A})$, where $\mathcal{A} = \mathrm{Ker}(a^{\#} - b^{\#})$. In [Aut, 0BN9] it was shown that \mathcal{B}, \mathcal{C} are, étale locally, sums of $\mathcal{O}_{\mathcal{X}}$ and $a^{\#} - b^{\#}$ is then a map rearranging the summands. Thus \mathcal{A} is a finite étale $\mathcal{O}_{\mathcal{X}}$ -algebra and $\mathcal{B}/\mathcal{A}, \mathcal{C}/\mathrm{im}(a^{\#} - b^{\#})$ are locally free. This implies that m^* preserves the coequalizers. Now the proof of the existence of limits works mutatis mutandis for colimits.

Every object of \mathcal{C} is a finite (possibly empty) coproduct of connected objects. It is easy to check that for any object $f: \mathcal{X}' \rightarrow \mathcal{X}$ in $\mathbf{F}\acute{\mathbf{E}}\mathbf{t}(\mathcal{X}/G)$, the induced map $\bar{f}: |\mathcal{X}'|/G \rightarrow |\mathcal{X}|/G$ enjoys the following property.

Let $u: |\mathcal{X}| \rightarrow |\mathcal{X}|/G$ (resp. $v: |\mathcal{X}'| \rightarrow |\mathcal{X}'|/G$) be the quotient map. Then for any subset $S \subseteq |\mathcal{X}'|/G$, $f(v^{-1}(S)) = u^{-1}(\bar{f}(S))$. Indeed, since $u \circ f(v^{-1}(S)) = \bar{f} \circ v(v^{-1}(S)) = \bar{f}(S)$, we have $f(v^{-1}(S)) \subseteq u^{-1}(\bar{f}(S))$. Conversely, suppose $x' \in |\mathcal{X}'|, x \in |\mathcal{X}|, v(x') \in S$, and $\bar{f}(v(x')) = u(x)$, then $u(f(x')) = u(x)$, so $\exists g \in G$ such that $x = gf(x') = f(gx')$. That $v(gx') = v(x') \in S$ implies that $gx' \in v^{-1}(S)$, so $x \in f(v^{-1}(S))$.

Then f clopen $\implies \bar{f}$ clopen. Moreover, the fibers of \bar{f} are finite, as for each $\bar{s} \in |\mathcal{X}|/G$ and any lift $x \in |\mathcal{X}|$ of \bar{s} , $\bar{f}^{-1}(\bar{s})$ is the v -image of the finite set $f^{-1}(x)$ (take, in the above, $S = \{s\}$, where $\bar{f}(s) = \bar{s}$). Now applying [Aut, 07VB], we see that $|\mathcal{X}'|/G$ is a coproduct of finitely many connected components, *i.e.* \mathcal{X}' is a coproduct of finitely many G -components.

We'll show that a G -connected object $\mathcal{Y} \in \mathcal{C}$ is a connected object in \mathcal{C} (cf. [Aut, 0BMY]). Suppose $a: \mathcal{Y}_1 \rightarrow \mathcal{Y}$ is a nonempty monomorphism in \mathcal{C} . Then the diagonal map $\mathcal{Y}_1 \rightarrow \mathcal{Y}_1 \times_{\mathcal{Y}} \mathcal{Y}_1$ is an isomorphism. This implies that a is also a monomorphism in $\mathbf{F}\acute{\mathbf{E}}\mathbf{t}(\mathcal{X})$, *i.e.* a exhibits \mathcal{Y}_1 as a clopen subset of \mathcal{Y} . As \mathcal{Y} is G -connected, $\mathcal{Y}_1 = \mathcal{Y}$, *i.e.* a is an isomorphism.

$F(\mathcal{X}')$ is finite for all $\mathcal{X}' \in \mathcal{C}$. This is clear.

F is conservative and exact. The functor F is, by its very definition, the composition

$$\mathbf{F}\acute{\mathbf{E}}\mathbf{t}(\mathcal{X}/G) \xrightarrow{\mathrm{Forget}} \mathbf{F}\acute{\mathbf{E}}\mathbf{t}(\mathcal{X}) \xrightarrow{F_{\bar{x}}} (\mathrm{Set})$$

where Forget is the obvious forgetful functor, which is conservative and preserves finite limits and finite colimits (given how finite limits and finite colimits in $\mathbf{F}\acute{\mathbf{E}}\mathbf{t}(\mathcal{X}/G)$ are constructed), and $F_{\bar{x}}$ is the classical fiber functor which is conservative and exact. Thus F is conservative and exact. \square

6. DRINFELD'S LEMMA

Let $\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_n$ be algebraic stacks over \mathbb{F}_q , and set $\mathcal{X} := \mathcal{X}_1 \times_{\mathbb{F}_q} \mathcal{X}_2 \times_{\mathbb{F}_q} \dots \times_{\mathbb{F}_q} \mathcal{X}_n$. Let ϕ_i be the q -Frobenius of \mathcal{X}_i for $1 \leq i \leq n$. Then \mathcal{X} is equipped with a strict $M := \phi_1^{\mathbb{N}} \times \phi_2^{\mathbb{N}} \times \dots \times \phi_n^{\mathbb{N}}$ -action. Let M_0 denote the submonoid $\phi_1^{\mathbb{N}} \times \dots \times \phi_{n-1}^{\mathbb{N}}$, and let G (resp. G_0) denote the free abelian group generated by M (resp. M_0). The category $\mathbf{F}\acute{\mathbf{E}}\mathbf{t}(\mathcal{X}/G_0)$ can be understood as 2-cartesian diagrams

$$\begin{array}{ccccccc} \mathcal{Y} & \xrightarrow{\varphi_{n-1}} & \mathcal{Y} & \xrightarrow{\varphi_{n-2}} & \mathcal{Y} & \xrightarrow{\dots} & \mathcal{Y} & \xrightarrow{\varphi_1} & \mathcal{Y} \\ f \downarrow & \lrcorner & \swarrow \sigma_{n-2} & \lrcorner & \swarrow \sigma_{n-1} & \lrcorner & \swarrow \dots & \lrcorner & \swarrow \sigma_1 & \downarrow f \\ \mathcal{X} & \xrightarrow{\phi_{n-1}} & \mathcal{X} & \xrightarrow{\phi_{n-2}} & \mathcal{X} & \xrightarrow{\dots} & \mathcal{X} & \xrightarrow{\phi_1} & \mathcal{X} \end{array}$$

where the φ_i s (resp. σ_i s) are mutually commute. Morally, $\mathrm{F}\acute{\mathrm{E}}\mathrm{t}(\mathcal{X}/G_0)$ is just a collection of commuting universally homeomorphic \mathbb{F}_q -endomorphisms $\{\varphi_1, \varphi_2, \dots, \varphi_{n-1}\}$ of \mathcal{Y} . Note that here we have dropped φ_n in the collection, but we could drop any ϕ_i for $1 \leq i \leq n$ which would yield the same category.

Lemma 6.1. *Let Y be a scheme. For any geometric point $\bar{x}: \mathrm{Spec}(k) \rightarrow Y$ the base change functor induces an equivalence:*

$$\varinjlim_{(U, \bar{u})} \mathrm{F}\acute{\mathrm{E}}\mathrm{t}(\mathcal{X} \times_{\mathbb{F}_q} U/G) \longrightarrow \mathrm{F}\acute{\mathrm{E}}\mathrm{t}(\mathcal{X} \times_{\mathbb{F}_q} \bar{x}/G)$$

where (U, \bar{u}) runs over all the affine étale neighborhood of \bar{x} .

Proof. Let $u: \mathcal{X}' \rightarrow \mathcal{X}$ be an M -equivariant map of \mathbb{F}_q -algebraic stacks, and $\mathcal{X}'' := \mathcal{X}' \times_{\mathcal{X}} \mathcal{X}'$, $\mathcal{X}''' := \mathcal{X}'' \times_{\mathcal{X}} \mathcal{X}'$, then there are natural 2-commutative diagrams

$$(4) \quad \begin{array}{ccccccc} \varinjlim_{(U, \bar{u})} \mathrm{F}\acute{\mathrm{E}}\mathrm{t}(\mathcal{X} \times_{\mathbb{F}_q} U/G) & \longrightarrow & \varinjlim_{(U, \bar{u})} \mathrm{F}\acute{\mathrm{E}}\mathrm{t}(\mathcal{X}' \times_{\mathbb{F}_q} U/G) & \longrightarrow & \varinjlim_{(U, \bar{u})} \mathrm{F}\acute{\mathrm{E}}\mathrm{t}(\mathcal{X}'' \times_{\mathbb{F}_q} U/G) & \rightrightarrows & \varinjlim_{(U, \bar{u})} \mathrm{F}\acute{\mathrm{E}}\mathrm{t}(\mathcal{X}''' \times_{\mathbb{F}_q} U/G) \\ \downarrow \lambda & & \downarrow \lambda' & & \downarrow \lambda'' & & \downarrow \lambda''' \\ \mathrm{F}\acute{\mathrm{E}}\mathrm{t}(\mathcal{X} \times_{\mathbb{F}_q} \bar{x}/G) & \longrightarrow & \mathrm{F}\acute{\mathrm{E}}\mathrm{t}(\mathcal{X}' \times_{\mathbb{F}_q} \bar{x}/G) & \longrightarrow & \mathrm{F}\acute{\mathrm{E}}\mathrm{t}(\mathcal{X}'' \times_{\mathbb{F}_q} \bar{x}/G) & \rightrightarrows & \mathrm{F}\acute{\mathrm{E}}\mathrm{t}(\mathcal{X}''' \times_{\mathbb{F}_q} \bar{x}/G) \end{array}$$

Suppose that the horizontal sequences are exact, then λ' is fully faithful (resp. an equivalence) and λ'' is faithful (resp. fully faithful and λ''' is faithful) imply that λ is fully faithful (resp. an equivalence). Note also that by [Aut, 07SK] we have a natural equivalence:

$$(5) \quad \varinjlim_{(U, \bar{u})} \mathrm{F}\acute{\mathrm{E}}\mathrm{t}(\mathcal{X} \times_{\mathbb{F}_q} U/G) \xrightarrow{\simeq} \mathrm{F}\acute{\mathrm{E}}\mathrm{t}(\mathcal{X} \times_{\mathbb{F}_q} \varprojlim_{(U, \bar{u})} U/G)$$

where $\varprojlim_{(U, \bar{u})} U$ is the strict henselization of Y at \bar{x} , so the rows of (4) are exact when u is an fpqc-covering (the actions descend together).

Using these simple observations, we'll proceed in several dévissage steps:

Step 1, reduction to $\mathcal{X}_i = X_i$ a connected \mathbb{F}_q -scheme of finite type. Replacing each \mathcal{X}_i by an atlas X_i , we may assume that it is an algebraic space. Replacing each algebraic space X_i by an atlas, we may assume that it is a scheme. Replacing each X_i by a disjoint union of affine opens, we may assume that each X_i is affine. Note that although X'' in (4) may not be a disjoint union of affines anymore, we can cover each $X'_i \times_{X_i} X'_i$ by affines and show that λ is fully faithful for any scheme X . Using this and applying affine coverings again, we get the desired equivalence for any scheme X . We then write each affine scheme X_i as a filtered limit of affine schemes of finite type over \mathbb{F}_q . Thus we may assume that X_i is of finite type over \mathbb{F}_q . Replacing X_i by a connected component, we may assume that X_i is connected.

Step 2, reduction to Y excellent, reduced, and strictly henselian. Since the problem is Zariski-local around \bar{x} , we may assume that Y is affine. Writing Y as a filtered limit of affine finite type \mathbb{F}_q -schemes and using [Aut, 07SK], we may assume that Y is of finite type over \mathbb{F}_q . Replacing Y with its reduced closed structure, we may assume that Y is reduced. Now replacing Y by its strict henselization at \bar{x} (cf. (5)), we may assume that

$Y = \text{Spec}(A)$, where A is strictly henselian and reduced. By [Aut, 06LJ], A is Noetherian; by [Aut, 07QR], A is a G-ring; by [Gro67, 18.8.17, 18.7.5.1], A is universally catenary; and by [Gre76, 5.3] is J-2. In conclusion, A is excellent.

Step 3, the functor (5) is faithful and essentially surjective. Consider the following 2-commutative diagram, where $X := \mathcal{X} = \text{Spec}(A)$.

$$\begin{array}{ccc}
 & & \text{F}\acute{\text{E}}\text{t}(X \times_{\mathbb{F}_q} \text{Spec}(A)/G) \\
 & \nearrow u & \downarrow \lambda \\
 \text{F}\acute{\text{E}}\text{t}(X/G_0) & & \text{F}\acute{\text{E}}\text{t}(X \times_{\mathbb{F}_q} \text{Spec}(\kappa)/G) \\
 & \searrow v & \\
 & & \text{F}\acute{\text{E}}\text{t}(X \times_{\mathbb{F}_q} \text{Spec}(\kappa)/G)
 \end{array}$$

We have to show that λ is an equivalence. Let ϕ_κ denote the absolute Frobenius of κ . Then $\text{F}\acute{\text{E}}\text{t}(X \times_{\mathbb{F}_q} \text{Spec}(\kappa)/G)$ can also be described via the action of $G_0 \times \phi_\kappa^\mathbb{Z}$. By [PTZ20, Theorem II, (C)] we have $\text{F}\acute{\text{E}}\text{t}(X) \simeq \text{F}\acute{\text{E}}\text{t}(X \times_{\mathbb{F}_q} \text{Spec}(\kappa)/\phi_\kappa^\mathbb{Z})$. This immediately implies that $\text{F}\acute{\text{E}}\text{t}(X/G_0) \simeq \text{F}\acute{\text{E}}\text{t}(X \times_{\mathbb{F}_q} \text{Spec}(\kappa)/G)$. Therefore, v is an equivalence. By Lemma 4.2 and Lemma 5.2 all the categories in the above diagram are Galois categories; since all the functors in the diagram commute with the forgetful fiber functors (which are faithful), they are faithful. This implies that u is fully faithful and λ is essentially surjective. Thus to conclude, it is enough to show that λ is fully faithful or u is essentially surjective.

Step 4, reduction to X_1 , Y normal and Y has algebraically closed field of fractions. Since v-covers are morphisms of effective descent for étale maps (cf. [HS23, Theorem 1.5]), and “being quasi-compact” as well as “satisfying the valuative criteria for properness” are obviously properties of maps local on the target for the v-topology (cf. [Aut, 02KO]), v-covers are also morphisms of effective descent for finite étale maps. As each X_i is excellent, its normalization map is finite. Using diagram (4) for the product of normalization maps and the fact that λ'' is faithful (as X'' is a finite disjoint union of connected components), we can assume that each X_i is normal.

For Y , we first note that there are finitely many irreducible components of Y , each of which is a spectrum of a strictly henselian excellent domain (cf. [Aut, 0C2Z]). If Y has m irreducible components Y_1, \dots, Y_m , then we set $Y' := Y_1 \sqcup (Y_2 \cup \dots \cup Y_m)$, which is a v-cover of Y . Consider the following diagram (diagram (4) “with the other factor”)

$$\begin{array}{ccccc}
 \text{F}\acute{\text{E}}\text{t}(X \times_{\mathbb{F}_q} Y/G) & \longrightarrow & \text{F}\acute{\text{E}}\text{t}(X \times_{\mathbb{F}_q} Y'/G) & \rightrightarrows & \text{F}\acute{\text{E}}\text{t}(X \times_{\mathbb{F}_q} Y''/G) \\
 \downarrow \lambda & & \downarrow \lambda' & & \downarrow \lambda'' \\
 \text{F}\acute{\text{E}}\text{t}(X \times_{\mathbb{F}_q} \bar{x}/G) & \longrightarrow & \text{F}\acute{\text{E}}\text{t}(X \times_{\mathbb{F}_q} \bar{x}'/G) & \rightrightarrows & \text{F}\acute{\text{E}}\text{t}(X \times_{\mathbb{F}_q} \bar{x}''/G)
 \end{array}
 \tag{6}$$

where $\bar{x}' := \bar{x} \sqcup \bar{x}$ and $\bar{x}'' := \bar{x}' \times_{\bar{x}} \bar{x}'$. Thanks to v-descent, the rows in (6) are exact. Using induction on m , we are reduced to the case when A is a domain.

Now let K be the field of fractions of A , and let B be the normalization of A in some finite extension L/K . Since

- B is finite over A (as A is excellent) and,
- B as well as $B \otimes_A B$ are local and strictly henselian (cf. [Aut, 04GH]),

applying diagram (6) to the normalization map and $\bar{x}'' = \bar{x}' = \bar{x}$, we are reduced to the case when Y is normal.

Let $\bar{Y} = \text{Spec}(\bar{A})$ be the normalization of Y in \bar{K} . Then \bar{A} is normal and strictly henselian (cf. [Aut, 04GI]). If $T_1, T_2 \in \text{F}\acute{\text{E}}\text{t}(X \times_{\mathbb{F}_q} Y/G)$ and $t: T_1 \otimes_A \kappa \rightarrow T_2 \otimes_A \kappa$ is a map in $\text{F}\acute{\text{E}}\text{t}(X \times_{\mathbb{F}_q} \bar{x}/G)$. Suppose t lifts to some $\bar{a} \in \text{F}\acute{\text{E}}\text{t}(X \times_{\mathbb{F}_q} \bar{Y}/G)$. Since \bar{a} is defined over some finite middle extension $A \subseteq B \subseteq \bar{A}$, we can apply diagram (6) again to conclude that t lifts to a map $a \in \text{F}\acute{\text{E}}\text{t}(X \times_{\mathbb{F}_q} Y/G)$. Thus we are reduced to the case when the fraction field K of A is algebraically closed and A is normal (but not necessarily Noetherian).

Step 5, final conclusion. Let K be the field of fractions of A . Then the functor

$$\text{F}\acute{\text{E}}\text{t}(X/G_0) \longrightarrow \text{F}\acute{\text{E}}\text{t}(X \times_{\mathbb{F}_q} \text{Spec}(K)/G)$$

is an equivalence and

$$\text{F}\acute{\text{E}}\text{t}(X \times_{\mathbb{F}_q} \text{Spec}(A)/G) \longrightarrow \text{F}\acute{\text{E}}\text{t}(X \times_{\mathbb{F}_q} \text{Spec}(K)/G)$$

is fully faithful. Thus u is an equivalence, as desired. \square

Lemma 6.2 (Stein factorization). *Suppose that \mathcal{X} is a G_0 -connected algebraic stack and \mathcal{X}_{n+1} is an arbitrary algebraic stack. For any $(\mathcal{Y}, \rho_{\mathcal{Y}}, \sigma) \in \text{F}\acute{\text{E}}\text{t}(\mathcal{X}/G)$, there exists $\mathcal{T} \in \text{F}\acute{\text{E}}\text{t}(\mathcal{X}_{n+1})$ and a 2-commutative diagram*

$$(7) \quad \begin{array}{ccc} \mathcal{Y} & \xrightarrow{\quad} & \mathcal{T} \\ \downarrow \lrcorner & & \downarrow \\ \mathcal{X} \times_{\mathbb{F}_q} \mathcal{X}_{n+1} & \xrightarrow{\quad} & \mathcal{X}_{n+1} \end{array}$$

where $\rho_{\mathcal{Y}}$ acts over \mathcal{T} and each geometric fiber $\mathcal{Y}_{\bar{t}}$ of $\mathcal{Y} \rightarrow \mathcal{T}$ is G -connected. If diagram (7) exists, then it has the following universal property: For any $\mathcal{T}' \in \text{F}\acute{\text{E}}\text{t}(\mathcal{X}_{n+1})$ and any commutative diagram (7) with \mathcal{T} replaced by \mathcal{T}' , where $\rho_{\mathcal{Y}}$ acts over \mathcal{T}' , there is a unique arrow $\lambda: \mathcal{T} \rightarrow \mathcal{T}'$ making all the natural diagrams 2-commutative. Since the natural commutativity forces λ into an object in $\text{F}\acute{\text{E}}\text{t}(\mathcal{X}_{n+1})$, which is equivalent to a set, λ is automatically unique up to a unique isomorphism.

Proof. **Step 1, the case when \mathcal{X}_{n+1} is a point.** Let's first consider the case when $\mathcal{X}_{n+1} = \text{Spec}(k)$, where k is an algebraically closed field containing \mathbb{F}_q . In this case, we take \mathcal{T} to be $\sqcup_{i \in I} \text{Spec}(k)$, where I is the set of G -components of $|\mathcal{Y}|$. To prove the universality, we first observe that both \mathcal{T} and \mathcal{T}' are finite disjoint unions of $\text{Spec}(k)$. Since each element of I corresponds to a point of \mathcal{T} and it is mapped to a point of \mathcal{T}' by $\mathcal{Y} \rightarrow \mathcal{T}'$, this defines a unique map $\mathcal{T} \rightarrow \mathcal{T}'$ in $\text{F}\acute{\text{E}}\text{t}(\mathcal{X}_{n+1})$ making all the natural diagrams commutative.

Step 2, the universal property for \mathcal{X}_{n+1} general. We first observe that $\mathcal{Y} \rightarrow \mathcal{T}$ is faithfully flat, so the desired arrow $\mathcal{T} \rightarrow \mathcal{T}'$ is unique if it exists (note that $\text{F}\acute{\text{E}}\text{t}(\mathcal{X}_{n+1})$ is a 1-category, so this makes sense). This allows us to define $\mathcal{T} \rightarrow \mathcal{T}'$ fppf locally on

\mathcal{X}_{n+1} . Thus we may assume that \mathcal{T} is a finite disjoint union of \mathcal{X}_{n+1} . Working with each pieces of \mathcal{T} we may assume that $\mathcal{T} = \mathcal{X}_{n+1}$. Notice that the map $f: \mathcal{Y} \rightarrow \mathcal{T}'$ is universally open. Indeed, f can be factorized as $\mathcal{Y} \xrightarrow{f_1} \mathcal{X} \times_{\mathbb{F}_q} \mathcal{T}' \xrightarrow{f_2} \mathcal{T}'$, while f_1 is finite étale and f_2 is universally open by [Aut, 0383]. Let $\mathcal{U} := f(\mathcal{Y})$ be the image. We have to show that there exists a unique \mathcal{X}_{n+1} -map $\mathcal{T} = \mathcal{X}_{n+1} \rightarrow \mathcal{U}$. Now it follows from Step 1 that on each geometric fiber over \mathcal{X}_{n+1} the factorization exists and is bijective, therefore each fiber of the projection map $u: \mathcal{U} \rightarrow \mathcal{X}_{n+1}$, which is representable and étale, consists of exactly one point. Thus u is an isomorphism and the desired arrow is u^{-1} .

Step 3, reduction to \mathcal{X}_{n+1} affine. By Step 2, to prove the existence of the Stein factorization, it is enough to prove it fppf-locally. Thus we may assume that $\mathcal{X}_{n+1} = X_{n+1} = \text{Spec}(A)$ is affine.

Step 4, reduction to \mathcal{Y} is a pullback from $\mathbf{F}\acute{\text{E}}\mathbf{t}(\mathcal{X}/G_0)$. By Lemma 6.1 and [PTZ20, Theorem II, (C)], for any geometric point \bar{x} of X_{n+1} one can find a connected affine étale neighborhood $U \rightarrow X_{n+1}$ such that the pullback of $(\mathcal{Y}, \rho_{\mathcal{Y}}, \sigma)$ to $\mathbf{F}\acute{\text{E}}\mathbf{t}(\mathcal{X} \times_{\mathbb{F}_q} U/G)$ comes from an object $\mathcal{Z} \in \mathbf{F}\acute{\text{E}}\mathbf{t}(\mathcal{X}/G_0)$. By Step 2, it is enough to construct the Stein factorization over U . So we assume that \mathcal{Y} comes from $\mathcal{Z} \in \mathbf{F}\acute{\text{E}}\mathbf{t}(\mathcal{X}/G_0)$.

Step 5, final conclusion. Let $\mathcal{Z}_1, \mathcal{Z}_2, \dots, \mathcal{Z}_m$ be the G_0 -components of \mathcal{Z} (remember that $\mathbf{F}\acute{\text{E}}\mathbf{t}(\mathcal{X}/G_0)$ is Galois), then for any geometric point $\bar{x} \in X_{n+1}(k)$, the pullback $(\mathcal{Z}_i, \rho_{\mathcal{Z}_i}, \sigma_i)_k$ of $(\mathcal{Z}_i, \rho_{\mathcal{Z}_i}, \sigma_i)$ to $\mathbf{F}\acute{\text{E}}\mathbf{t}(\mathcal{X} \times_{\mathbb{F}_q} k/G)$ has underlying object $\mathcal{Z}_i \times_{\mathbb{F}_q} k$ and action $\rho_{\mathcal{Z}_i} \times \phi_k$. It's G -connected, as any clopen G -equivariant subset of $\mathcal{Z}_i \times_{\mathbb{F}_q} k$ is the inverse image of a G_0 -equivariant clopen subset of \mathcal{Z}_i (cf. [PTZ20, Theorem 4.1]). Thus we can take $T := \sqcup_{i=1}^m X_{n+1}$ and the map $\mathcal{Y} \rightarrow T$ to be the sum of the projections $\mathcal{Z}_i \times_{\mathbb{F}_q} X_{n+1} \rightarrow X_{n+1}$. \square

Theorem 6.3. *Suppose that \mathcal{X}_i ($1 \leq i \leq n$) and \mathcal{X}_{n+1} are connected algebraic stacks over \mathbb{F}_q . Then for any geometric point \bar{x} of $\mathcal{X} \times_{\mathbb{F}_q} \mathcal{X}_{n+1}$, the top sequence in*

$$(8) \quad \begin{array}{ccccccc} 1 & \rightarrow & \pi_1^{\acute{\text{e}}\text{t}}(\mathcal{X} \times_{\mathbb{F}_q} k/G, \bar{x}) & \longrightarrow & \pi_1^{\acute{\text{e}}\text{t}}(\mathcal{X} \times_{\mathbb{F}_q} \mathcal{X}_{n+1}/G, \bar{x}) & \xrightarrow{\beta} & \pi_1^{\acute{\text{e}}\text{t}}(\mathcal{X}_{n+1}, \bar{x}) \longrightarrow 1 \\ & & \downarrow \cong & & \swarrow \alpha & & \\ & & \pi_1^{\acute{\text{e}}\text{t}}(\mathcal{X}/G_0, \bar{x}) & & & & \end{array}$$

is exact, where k is the field of definition of \bar{x} .

Proof. The left exactness of the top sequence is provided by the retraction α , whose existence is a consequence of [PTZ20, Theorem II, (C)]. Thus we are left to show the middle exactness. In view of [Aut, 0BTQ] we only have to show that the following condition holds:

Let $(\mathcal{Y}, \rho_{\mathcal{Y}}) \in \mathbf{F}\acute{\text{E}}\mathbf{t}(\mathcal{X} \times_{\mathbb{F}_q} \mathcal{X}_{n+1}/G)$ be a connected object. Suppose the pullback $(\mathcal{Y}_{\bar{x}}, \rho_{\mathcal{Y}_{\bar{x}}})$ of $(\mathcal{Y}, \rho_{\mathcal{Y}})$ to $\mathbf{F}\acute{\text{E}}\mathbf{t}(\mathcal{X} \times_{\mathbb{F}_q} k/G)$ has a G -component which is mapped isomorphically to $\mathcal{X} \times_{\mathbb{F}_q} k$, then the Stein factorization of $(\mathcal{Y}, \rho_{\mathcal{Y}})$ induces a

Cartesian diagram.

$$\begin{array}{ccc} \mathcal{Y} & \longrightarrow & \mathcal{T} \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{X} \times_{\mathbb{F}_q} \mathcal{X}_{n+1} & \longrightarrow & \mathcal{X}_{n+1} \end{array}$$

One has to show that $\lambda: \mathcal{Y} \rightarrow \mathcal{X} \times_{\mathbb{F}_q} \mathcal{T}$ is an isomorphism. For each geometric point \bar{s} of \mathcal{T} the pullback $\mathcal{X} \times_{\mathbb{F}_q} \kappa(\bar{s})$ is G -connected by 4.2. It follows that $\lambda_s: \mathcal{Y}_s \rightarrow \mathcal{X} \times_{\mathbb{F}_q} \kappa(\bar{s})$ as well as λ are surjective (\mathcal{Y}_s is a G -component, hence is not empty). That \mathcal{Y} is G -connected implies that $\mathcal{X} \times_{\mathbb{F}_q} \mathcal{T}$ is G -connected, thus λ is finite étale of constant degree. Therefore to show that λ is an isomorphism, it is enough to show that the degree of λ equals 1.

The G -component of $(\mathcal{Y}_{\bar{x}}, \rho_{\mathcal{Y}_{\bar{x}}})$ which goes isomorphically to $\mathcal{X} \times_{\mathbb{F}_q} k$ is mapped to a geometric point $\bar{s} \in \mathcal{T}(k)$ lying over $\bar{x} \in \mathcal{X}_{n+1}(k)$. It is moreover the inverse image of \bar{s} by the very construction of the Stein factorization. Thus λ is an isomorphism at \bar{s} , and the degree of λ is 1, as desired. \square

Proof of Theorem I. By Theorem 8 and diagram (8) we have

$$\pi_1^{\text{ét}}(\mathcal{X}/\phi_1^{\mathbb{Z}} \times \cdots \times \phi_{n-1}^{\mathbb{Z}}) \xrightarrow{\cong} \pi_1^{\text{ét}}(\mathcal{X}_1 \times \cdots \times \mathcal{X}_{n-1}/\phi_1^{\mathbb{Z}} \times \cdots \times \phi_{n-2}^{\mathbb{Z}}) \times \pi_1^{\text{ét}}(\mathcal{X}_n)$$

By induction on n , we get the result. \square

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