DRINFELD'S LEMMA FOR ALGEBRAIC STACKS

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Abstract. Drinfeld's lemma is a powerful tool for splitting ℓ-adic local systems defined over a product of connected schemes over a finite field. In this paper, we show that Drinfeld's lemma also holds true for algebraic stacks.

1. Introduction

The main motivation of Drinfeld's lemma is to split ℓ-adic local systems defined over a product of schemes. More precisely, let X_1, X_2 be two connected $\mathbb{F}_q$-schemes, then one would like to get, out of an ℓ-adic local system on $X \coloneqq X_1 \times_{\mathbb{F}_q} X_2$, an ℓ-adic local system coming from local systems on the individual factors X_1, X_2. The problem is easy if one considers complex analytic local systems on a product of complex varieties. Indeed, one can split the local system via the Künneth formula for topological fundamental groups:

$$\pi_1^{\mathrm{top}}(X_1^{\mathrm{an}} \times X_2^{\mathrm{an}}) \xrightarrow{\cong} \pi_1^{\mathrm{top}}(X_1^{\mathrm{an}}) \times \pi_1^{\mathrm{top}}(X_2^{\mathrm{an}})$$

One can also do this for ℓ-adic local systems. Indeed, for any connected schemes X_1, X_2 defined over an algebraically closed field k *of characteristic 0*, one has an isomorphism:

$$\pi_1^{\text{ét}}(X_1 \times_k X_2) \xrightarrow{\cong} \pi_1^{\text{ét}}(X_1) \times \pi_1^{\text{ét}}(X_2)$$

Even when k is of characteristic $p > 0$ (but still algebraically closed), the above Künneth formula still holds true provided that either X_1 or X_2 is proper over k. However, the Künneth formula fails when k is not algebraically closed: take $X_1 = X_2 = \mathrm{Spec}(\mathbb{F}_p)$, then the Künneth formula would mean that the diagonal map $\hat{\mathbb{Z}} \to \hat{\mathbb{Z}} \times \hat{\mathbb{Z}}$ is an isomorphism.

The issue (for finite fields) can be resolved if partial Frobenii actions are brought into play. More precisely, let ϕ_1 (resp. ϕ_2) denote the *partial Frobenius* map $X \to X$, which is the q-absolute Frobenius on X_1 (resp. X_2) and the identity on the other. Consider the category $\mathrm{FÉt}(X/\Phi)$ of triples $(Y, \varphi_1, \varphi_2)$, where Y is a finite étale cover of $X = X_1 \times_{\mathbb{F}_q} X_2$ and φ_i is an isomorphism $Y \xrightarrow{\cong} \phi_i^* Y$ satisfying that $\phi_1^*(\varphi_2) \circ \varphi_1 = \phi_2^*(\varphi_1) \circ \varphi_2$ is the identity (by identitying $(\phi_1 \circ \phi_2)^* Y$ with Y via the absolute Frobenius of Y).

Theorem 1.1 (Drinfeld's lemma for schemes)**.** *Suppose X_1, X_2 are connected quasi-compact and quasi-separated (qcqs) $\mathbb{F}_q$-schemes. Then*

- $\mathrm{FÉt}(X/\Phi)$ *is a Galois category whose Galois group is denoted by* $\pi_1^{\text{ét}}(X/\Phi)$*;*
- *the natural map* $\pi_1^{\text{ét}}(X/\Phi) \to \pi_1^{\text{ét}}(X_1) \times \pi_1^{\text{ét}}(X_2)$ *is an isomorphism.*

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Similarly, one has Drinfeld's lemma for n factors $X_1, \ldots, X_n$. Please refer to [Dri80, Theorem 2.1], [Laf97, IV.2, Theorem 4], [Lau07, Theorem 8.1.4], [Ked17, Theorem 4.2.12], [Laf18, Lemma 0.18], [SW20, Theorem 16.2.4], and [Mül22, Theorem 1.4] for details. Using Drinfeld's Lemma one can split local systems on a product of schemes equipped with commuting partial Frobenius actions.

The notion of fundamental group of algebraic stacks has been introduced and studied by B. Noohi in [Noo00]. The main purpose of this note is to generalize Drinfeld's lemma to algebraic stacks removing the qcqs assumption.

Theorem I (cf. §6)**.** *Let $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n$ be connected algebraic stacks over $\mathbb{F}_q$, and set $\mathcal{X} := \mathcal{X}_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} \mathcal{X}_n$. Then*

(1) The category $\mathrm{F\acute{E}t}(\mathcal{X}/\Phi)$ consisting of finite étale covers of $\mathcal{X}$ equipped with commuting partial Frobenii actions is a Galois category, whose Galois group is denoted by $\pi_1^{\text{ét}}(\mathcal{X}/\Phi)$;

(2) the natural map $\pi_1^{\text{ét}}(\mathcal{X}/\Phi) \longrightarrow \pi_1^{\text{ét}}(\mathcal{X}_1) \times \cdots \times \pi_1^{\text{ét}}(\mathcal{X}_n)$ is an isomorphism.

The key technique that we are using here is the Drinfeld-Lau descent for fibered categories developed in [PTZ20].

2. The G-Connectedness

One of the difficulties in understanding Drinfeld's lemma is that the partial q-Frobenius maps are, in general, not invertible. Thus "the quotient space X/Φ" is only a suggestive symbol but not an honest space – even when X is a scheme! However, we can first forget this if we only look at the actions on the ambient topologial space $|\mathcal{X}|$.

Definition 2.1. Let G be a group, and let X be a topological space equipped with a G-action $\rho_G \colon G \times X \to X$. The pair (X, ρ_G) is called a *G-space*. The space X is called *G-connected* if the quotient space X/G is a connnected topological space. A subspace $S \subseteq X$ is called a *G-component* of X if it is the inverse image of a nonempty clopen subspace of X/G. A ordinary topological space X can be viewed as a G-space via the trivial G-action. In this case, the pair (X, ρ_G) will simply denoted by X.

Let $f \colon (X, \rho_G) \to (S, \rho'_G)$ be a map, *i.e.* a map of topological spaces $f \colon X \to S$ which is compatible with the G-actions. For each G-stable subset $S_1 \subseteq S$ the inverse image $f^{-1}(S_1) \subseteq X$ is a G-stable subspace.

Lemma 2.1. *Let $f \colon (X, \rho_G) \to S$ be map whose fibers are G-connected. Suppose that $X \to S$ is submersive. Then (X, ρ_G) is G-connected if and only if S is connected.*

Proof. The map f induces a continuous map $\bar{f} \colon X/G \to S$ whose fibers are connected (hence nonempty). If (X, ρ_G) is G-connected, then X/G is connected, so S is connnected. Conversely, if (X, ρ_G) is not G-connected, then $X/G = U_1 \sqcup U_2$, where U_1, U_2 are two non-empty open subspaces of X/G. Since the fibers of $\bar{f}$ are connected, any fiber is contained either in U_1 or in U_2. Thus exists subspaces V_1, V_2 of S such that $S = V_1 \sqcup V_2$ and $\bar{f}^{-1}(V_i) = U_i$ $(i = 1, 2)$. That f is submersive implies that $\bar{f}$ is submersive. Hence V_1, V_2 are non-empty opens of S. Then S is not connected. $\square$

3. Frobenius and Universal Homeomorphisms

Recall that for any fibered category $\mathcal{X}$ defined over $\mathbb{F}_p$, the absolute Frobenius map $F_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{X}$ sends a section $x \in \mathcal{X}(T)$ on an $\mathbb{F}_p$-scheme T to the composition $T \xrightarrow{F_T} T \xrightarrow{x} \mathcal{X} \in \mathcal{X}(T)$, where F_T is the absolute Frobenius of the scheme T.

Universal homeomorphisms of schemes are radical maps (cf. [Aut, 01S2]) which play a role as the "opposite" of étale maps. The on the nose extension of this notion to stacks could be misleading in certain situations. For example, consider a finite étale group scheme G over a field k. Any base change of the map $f\colon \mathcal{B}_k G \to \operatorname{Spec}(k)$ is a homeomorphism on the underlying topological spaces, however, the map is étale. This suggests a new definition of the notion of "universal homeomorphism" for algebraic stacks – we should consider not only f but also the diagonal of f which encodes information about the (relative) inertia.

Definition 3.1. A map of algebraic stacks $f\colon \mathcal{X} \to \mathcal{Y}$ is called a *universal homeomorphism on the nose* if for any map of algebraic stacks $\mathcal{T} \to \mathcal{Y}$, the base change $f_{\mathcal{T}}\colon \mathcal{X} \times_{\mathcal{Y}} \mathcal{T} \to \mathcal{T}$ is a homeomorphism on the underlying topological spaces. The map f is called a *universal homeomorphism* if both f and the diagonal Δ_f are universal homeomorphisms on the nose.

Lemma 3.1. *If $\mathcal{X}$ is an algebraic stack over $\mathbb{F}_p$, then $F_{\mathcal{X}}$ is a universal homeomorphism.*

Proof. Suppose $f\colon \mathcal{Y} \to \mathcal{X}$ is a map of algebraic stacks over $\mathbb{F}_p$. Then we have the following cartesian diagram

$$\begin{tikzcd} \mathcal{Y}' \arrow[r, "F'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{Y} \arrow[d, "f"] \\ \mathcal{X} \arrow[r, "F_{\mathcal{X}}"] & \mathcal{X} \end{tikzcd}$$

We want to show that F' induces a bijection $|F'|\colon |\mathcal{Y}'| \to |\mathcal{Y}|$. By the universality of a fibered square, there is a factorization of $F_{\mathcal{Y}}\colon \mathcal{Y} \xrightarrow{F_{\mathcal{Y}/\mathcal{X}}} \mathcal{Y}' \xrightarrow{F'} \mathcal{Y}$. If $F_{\mathcal{Y}}$ induces a homeomorphism of $|\mathcal{Y}|$, then we just have to show that $|F_{\mathcal{Y}/\mathcal{X}}|$ is surjective. Take a point $y' \in |\mathcal{Y}'|$ which is represented by a triple (x, y, α), where $x \in \mathcal{X}(k)$, $y \in \mathcal{Y}(k)$ are geometric points, and α is an isomorphism from $\operatorname{Spec}(k) \xrightarrow{F_k} \operatorname{Spec}(k) \xrightarrow{x} \mathcal{X}$ to $\operatorname{Spec}(k) \xrightarrow{y} \mathcal{Y} \xrightarrow{f} \mathcal{X}$. Consider the k-point $y_1\colon \operatorname{Spec}(k) \xrightarrow{F_k^{-1}} \operatorname{Spec}(k) \xrightarrow{y} \mathcal{Y}$ of $\mathcal{Y}$. Clearly, $F_{\mathcal{Y}}(y_1) = y$ and $x \xrightarrow{\cong} f(y_1)$ via $F_k^{-1}(\alpha)$. We get $(x, y, \alpha) \cong (f(y_1), y, \mathrm{id}_{f(y)}) = F_{\mathcal{Y}/\mathcal{X}}(y_1)$. Thus $|F_{\mathcal{Y}/\mathcal{X}}|$ is surjective.

Now let's show that $|F_{\mathcal{X}}|$ is bijective for any algebraic stack $\mathcal{X}$. If $x \in \mathcal{X}(k)$ is a geometric point, then the k-point $\operatorname{Spec}(k) \xrightarrow{F_k^{-1}} \operatorname{Spec}(k) \xrightarrow{x} \mathcal{X}$ is mapped to x via $F_{\mathcal{X}}$. Hence $|F_{\mathcal{X}}|$ is surjective. Conversely, let $x_1 \in \mathcal{X}(k_1)$ and $x_2 \in \mathcal{X}(k_2)$ are two geometric points such that $F_{\mathcal{X}}(x_1)$ and $F_{\mathcal{X}}(x_2)$ represent the same point in $\mathcal{X}$, *i.e.* $F_{\mathcal{X}}(x_1) \sim F_{\mathcal{X}}(x_2)$. Then, by definition, there is an algebraically closed field k containing both k_1 and k_2 such that $\operatorname{Spec}(k) \to \operatorname{Spec}(k_1) \xrightarrow{F_{k_1}} \operatorname{Spec}(k_1) \xrightarrow{x_1} \mathcal{X}$ is isomorphic to $\operatorname{Spec}(k) \to \operatorname{Spec}(k_2) \xrightarrow{F_{k_2}} \operatorname{Spec}(k_2) \xrightarrow{x_2} \mathcal{X}$. But since $(\operatorname{Spec}(k) \to \operatorname{Spec}(k_i) \xrightarrow{F_{k_i}} \operatorname{Spec}(k_i)) = (\operatorname{Spec}(k) \xrightarrow{F_k} \operatorname{Spec}(k) \to \operatorname{Spec}(k_i))$ for

$i = 1, 2$, we have $\mathrm{Spec}(k) \to \mathrm{Spec}(k_1) \xrightarrow{x_1} \mathcal{X}$ is isomorphic to $\mathrm{Spec}(k) \to \mathrm{Spec}(k_2) \xrightarrow{x_2} \mathcal{X}$, so $x_1 \sim x_2$.

We have seen that $|F'|$, $|F_{\mathcal{Y}/\mathcal{X}}|$, $|F_{\mathcal{Y}}|$ are bijective and continuous. From the very definition of the topology on $|\mathcal{Y}|$ it is clear that $|F_{\mathcal{Y}}|$ is a homeomorphism. The continuity of $|F_{\mathcal{Y}/\mathcal{X}}|$ then implies that $|F'|$ is submersive, hence a homeomorphism.

Let's consider the diagonal $\Delta\colon \mathcal{X} \to \mathcal{X} \times_{F_{\mathcal{X}}} \mathcal{X}$. The map $|\Delta|$ is surjective because any geometric point $(x, y, \alpha) \in (\mathcal{X} \times_{F_{\mathcal{X}}} \mathcal{X})(k)$, where $\alpha\colon (\mathrm{Spec}(k) \xrightarrow{F_k} \mathrm{Spec}(k) \xrightarrow{x} \mathcal{X}) \cong (\mathrm{Spec}(k) \xrightarrow{F_k} \mathrm{Spec}(k) \xrightarrow{y} \mathcal{X})$, induces an isomorphism $F_k^{-1}(\alpha)\colon x \cong y$, so $(x, x, \mathrm{id}_{F_{\mathcal{X}}(x)}) \cong (x, y, \alpha)$. Conversely, if $x \in \mathcal{X}(k_1)$ and $y \in \mathcal{X}(k_2)$ such that $(x, x, \mathrm{id}_{F_{\mathcal{X}}(x)}) \sim (y, y, \mathrm{id}_{F_{\mathcal{X}}(y)})$ in the diagonal, then there is a geometric point $(a, b, \alpha) \in (\mathcal{X} \times_{F_{\mathcal{X}}} \mathcal{X})(k)$, where k contains both k_1 and k_2, such that (a, b, α) is isomorphic to the pullback of both $(x, x, \mathrm{id}_{F_{\mathcal{X}}(x)})$ and $(y, y, \mathrm{id}_{F_{\mathcal{X}}(y)})$ to k. This implies immediately that $x \sim y$. $\square$

Here are some geometric properties of universal homeomorphisms for algebraic stacks. Note that Lemma 3.2 (2) is not true if g is only a universal homeomorphisms on the nose.

Lemma 3.2. *Let $\mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{Z}$ be a sequence of maps of algebraic stacks. Set $h := g \circ f$.*

(1) *If f, g are universal homeomorphisms, then so is f.*
(2) *If g is a universal homeomorphism and h is a universal homeomorphism on the nose, then f is a universal homeomorphism on the nose.*

Proof. (1) is obvious. Let's look at (2). The map f factorizes as $\mathcal{X} \xrightarrow{\Gamma_f} \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \xrightarrow{h \times \mathrm{id}_{\mathcal{Y}}} \mathcal{Y}$, where Γ_f is the graph of f. If h is a universal homeomorphism on the nose and the diagonal g is a universal homeomorphism, then Γ_f and $h \times \mathrm{id}_{\mathcal{Y}}$ are universal homeomorphisms on the nose, and so is the composition f. $\square$

Note that the absolute Frobenius is only $\mathbb{F}_p$-linear. In order to make it $\mathbb{F}_q$-linear we take the q-absolute Frobenius map $\phi_{\mathcal{X}} := F_{\mathcal{X}}^n$, where $n := \log_p q$. Just as in the scheme case, the pullback of the étale maps along the q-absolute Frobenius map of algebraic stacks induces an equivalence of étale sites (actually the identity!).

Proposition 3.3. *Let $\mathcal{X}$ be a Deligne-Mumford stack (resp. an algebraic stack) over $\mathbb{F}_q$, and let $f\colon \mathcal{Y} \to \mathcal{X}$ be an étale map (resp. a representable étale map). Then we have the following cartesian diagram.*

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "\phi_{\mathcal{Y}}"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{Y} \arrow[d, "f"] \\ \mathcal{X} \arrow[r, "\phi_{\mathcal{X}}"] & \mathcal{X} \end{tikzcd}$$

Proof. One just has to show that the natural map $\lambda\colon \mathcal{Y} \to \mathcal{X} \times_{\phi_{\mathcal{X}}} \mathcal{Y}$ is an equivalence. For this, one can take an atlas of $X \twoheadrightarrow \mathcal{X}$, where X is a scheme and show that the pullback of λ to X is an equivalence. Thus we are reduced to the case when $\mathcal{X} = X$ is a scheme. If f is representable by schemes, then we are done. If f is only representable by algebraic spaces, then $\mathcal{Y} = Y$ is an algebraic space. Take an étale atlas $Z \twoheadrightarrow Y$. The following diagram

$$\begin{tikzcd} Z \arrow[r, "\lambda_Z"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & X \times_{\phi_X} Z \arrow[d, two heads] \\ Y \arrow[r, "\lambda"] & X \times_{\phi_X} Y \end{tikzcd}$$

is cartesian, because its extension via the projections $X \times_{\phi_X} Z \to Z$ and $X \times_{\phi_X} Y \to Y$ is cartesian (as $Z \twoheadrightarrow Y$ is representable by schemes). Since λ_Z is an isomorphism by the scheme case, λ is an isomorphism as well. If $\mathcal{Y}$ is only a Deligne-Mumford stack, then we take an étale atlas $Z \twoheadrightarrow \mathcal{Y}$ and repeat the above argument to reduce the problem to the case when f is representable by algebraic spaces which has just been proven. □

4. Partial Frobenius Maps

Let's come back to our original setting. Let $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n$ be connected algebraic stacks over $\mathbb{F}_q$, and set $\mathcal{X} := \mathcal{X}_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} \mathcal{X}_n$. Then there are *partial Frobenius maps* $\phi_i \colon \mathcal{X} \to \mathcal{X}$, which is the q-absolute Frobenius on X_i and the identity on the others. These ϕ_is are commuting endomorphisms of $\mathcal{X}$ (resp. automorphisms of $|\mathcal{X}|$) whose composition $\phi_1 \circ \cdots \circ \phi_n$ is the q-Frobenius of $\mathcal{X}$ (resp. the identity of $|\mathcal{X}|$). Thus each ϕ_i is completely determined by all the others, so we can drop any of the partial Frobenius maps, e.g. ϕ_n. They provide $\mathcal{X}$ (resp. $|\mathcal{X}|$) with a strict action (resp. an action) $\rho_{\mathcal{X}}$ by the monoid $M := \phi_1^{\mathbb{N}} \times \phi_2^{\mathbb{N}} \times \cdots \phi_{n-1}^{\mathbb{N}}$ (resp. the group $G := \phi_1^{\mathbb{Z}} \times \phi_2^{\mathbb{Z}} \times \cdots \phi_{n-1}^{\mathbb{Z}}$).

Lemma 4.1. *Let $G_i := \bigoplus_{j \in \{1,\ldots,n\} \setminus \{i\}} \phi_j^{\mathbb{Z}}$. Then $|\mathcal{X}|$ is G-connected if and only if it is G_i-connected.*

Proof. This is just because that a subset of $|\mathcal{X}|$ is G-stable (resp. G_i-stable) iff it is stable under all the partial Frobenius actions. □

Corollary 4.2. *The space $|\mathcal{X}_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} \mathcal{X}_n|$ equipped with the $\mathbb{Z}^{n-1}$-action by $n-1$ of the partial Frobenius maps is $\mathbb{Z}^{n-1}$-connected.*

Proof. Consider the projection map $|\mathrm{pr}_n| \colon |\mathcal{X}_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} \mathcal{X}_n| \to |\mathcal{X}_n|$, where the former is equipped with the $\mathbb{Z}^{n-1}$-action via the ϕ_is, and the latter is equipped with the trivial $\mathbb{Z}^{n-1}$-action. Let $x \in \mathcal{X}_n(k)$ be a geometric point representing a point in $|\mathcal{X}_n|$. Then the topological space of the stack fiber $\mathcal{X}_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} \mathcal{X}_{n-1} \times_{\mathbb{F}_q} k$ maps surjectively to the topological fiber of x along the map $|\mathrm{pr}_n|$. This surjection moreover preserves the $\mathbb{Z}^{n-1}$-actions. In light of Lemma 2.1, it is enough to show that the space $|\mathcal{X}_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} \mathcal{X}_{n-1} \times_{\mathbb{F}_q} k|$ is $\mathbb{Z}^{n-1}$-connected.

Suppose $\mathcal{U} \subseteq \mathcal{X}_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} \mathcal{X}_{n-1} \times_{\mathbb{F}_q} k$ is a clopen substack which is stable under the $\mathbb{Z}^{n-1}$-action. Then $\mathcal{U}$ is also stable under the $\mathbb{Z}^{n-1}$-action defined by $\phi_1, \ldots, \phi_{n-2}, \phi_k$. By [PTZ20, Theorem 4.1], $\mathcal{U}$ is the preimage of a clopen substack $\mathcal{U}_0 \subseteq \mathcal{X}_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} \mathcal{X}_{n-1}$, and $\mathcal{U}_0$ is obviously stable under the $\mathbb{Z}^{n-2}$-action $\phi_1, \dots, \phi_{n-2}$. Then $\mathcal{U}_0$ is either empty or the whole stack by the connectedness of $\mathcal{X}_{n-1}$ in the case $n = 2$, or by the induction hypothesis in the case $n > 2$. Thus $\mathcal{U}$ is also either empty or the whole stack. This finishes the proof. $\square$

5. Galois Categories

Let $\mathcal{X}$ be an algebraic stack over $\mathbb{F}_q$. When $\mathcal{X}$ is connected, it is known (cf. [Noo00, Theorem 4.2] or [Noo04, §4]) that $\mathrm{F\acute{E}t}(\mathcal{X})$ is a Galois category. Let's now adapt this to our setting. Suppose M is a commutative monoid acting strictly (cf. **Rom05**) on $\mathcal{X}$ via universal homeomorphisms. Let $\rho_{\mathcal{X}}\colon M \to \mathrm{End}_{\mathbb{F}_q}(\mathcal{X})$ be the action map. If G denotes the Grothendieck group associated with M, then $|\mathcal{X}|$ is a G-space. Let $\mathrm{F\acute{E}t}(\mathcal{X}/G)$ denote the category whose objects are 2-commutative diagrams

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "\rho_{\mathcal{Y}}(m)"] \arrow[d, "f"'] \arrow[dr, phantom, "\swarrow \sigma_m"] & \mathcal{Y} \arrow[d, "f"] \\ \mathcal{X} \arrow[r, "\rho_{\mathcal{X}}(m)"] & \mathcal{X} \end{tikzcd}$$

where $\mathcal{Y} \xrightarrow{f} \mathcal{X} \in \mathrm{F\acute{E}t}(\mathcal{X})$ and $\rho_{\mathcal{Y}}$ is a strict action of M on $\mathcal{Y}$ via $\mathbb{F}_q$-endomorphisms of $\mathcal{Y}$ which are universal homeomorphisms, such that $\rho_{\mathcal{Y}}$ is compatible with $\rho_{\mathcal{X}}$, in the sense of **Rom05**, *i.e.* $\sigma_{m_1m_2}$ is equal to the composition

$$(1) \quad \begin{tikzcd} f \circ \rho_{\mathcal{Y}}(m_2) \circ \rho_{\mathcal{Y}}(m_1) \arrow[r, "\sigma_{m_2}"] \arrow[d, equal] & \rho_{\mathcal{X}}(m_2) \circ f \circ \rho_{\mathcal{Y}}(m_1) \arrow[r, "\sigma_{m_1}"] & \rho_{\mathcal{X}}(m_2) \circ \rho_{\mathcal{X}}(m_1) \circ f \arrow[d, equal] \\ f \circ \rho_{\mathcal{Y}}(m_2 m_1) & & \rho_{\mathcal{X}}(m_2 m_1) \circ f \end{tikzcd}$$

and $\sigma_0 = \mathrm{id}_f$. One easier way to characterize $\mathrm{F\acute{E}t}(\mathcal{X}/G)$ is the following.

Lemma 5.1. *The following categories are equivalent*

(a) *the category* $\mathrm{F\acute{E}t}(\mathcal{X}/G)$*;*
(b) *the category of 2-cartesian diagrams*

$$(2) \quad \begin{tikzcd} \mathcal{Y} \arrow[r, "\rho_{\mathcal{Y}}(m)"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner\ \swarrow \sigma_m"] & \mathcal{Y} \arrow[d, "f"] \\ \mathcal{X} \arrow[r, "\rho_{\mathcal{X}}(m)"] & \mathcal{X} \end{tikzcd}$$

where $\mathcal{Y} \xrightarrow{f} \mathcal{X} \in \mathrm{F\acute{E}t}(\mathcal{X})$ *and* $\rho_{\mathcal{Y}}$ *is a strict action of* M *on* $\mathcal{Y}$ *via* $\mathbb{F}_q$*-endomorphisms and* σ *is the compatibility condition,* i.e. $\sigma_{m_1m_2}$ *equals* (1) *and* $\sigma_0 = f$.
(c) *the category of pairs* $(\mathcal{A}, \{\alpha_{\mathcal{A},m}\}_{m \in M})$*, where* $\mathcal{A}$ *is a finite étale* $\mathcal{O}_{\mathcal{X}}$*-algebra and* $\alpha_{\mathcal{A},m}\colon m^*\mathcal{A} \to \mathcal{A}$ $(m^*\mathcal{A} := \rho_{\mathcal{X}}(m)^*\mathcal{A})$ *is an* $\mathcal{O}_{\mathcal{X}}$*-algebra isomorphism satisfying* $\alpha_{\mathcal{A},e} = \mathrm{id}_{\mathcal{A}}$ *and* $\alpha_{\mathcal{A},m_1m_2} = \alpha_{\mathcal{A},m_1} \circ m_1^*\alpha_{\mathcal{A},m_2}$.

Proof. Let's first show (a)$\Leftrightarrow$(b). For any $m \in M$ and $\mathcal{Y} \in \mathrm{F\acute{E}t}(\mathcal{X})$ set $m^*\mathcal{Y} := \mathcal{Y} \times_{\rho_{\mathcal{X}}(m)} \mathcal{X}$. Consider the 2-commutative diagram:

$$\begin{tikzcd}
\mathcal{Y} \arrow[dr, dashed, "\alpha_{\mathcal{Y},m}"] \arrow[drr, dotted, bend left, "\rho_{\mathcal{Y}}(m)"] \arrow[ddr, bend right] & & \\
& m^*\mathcal{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & \mathcal{Y} \arrow[d] \\
& \mathcal{X} \arrow[r, "\rho_{\mathcal{X}}(m)"] & \mathcal{X}
\end{tikzcd} \tag{3}$$

Suppose the square defined by $\rho_{\mathcal{Y}}(m)$ and $\rho_{\mathcal{X}}(m)$ is cartesian, then $\alpha_{\mathcal{Y},m}$ is an equivalence, so $\rho_{\mathcal{Y}}(m)$, as a pullback of $\rho_{\mathcal{X}}(m)$, is a universal homeomorphism, *i.e.* the square belongs to $\mathrm{F\acute{E}t}(\mathcal{X}/G)$. Conversely, if $\rho_{\mathcal{Y}}(m)$ is a universal homeomorphism, then $\alpha_{\mathcal{Y},m}$ is an isomorphism. Indeed, if $\rho_{\mathcal{Y}}(m)$ is a universal homeomorphism, then the fact that $m^*\mathcal{Y} \to \mathcal{Y}$ is a universal homeomorphism implies that $\alpha_{\mathcal{Y},m}$ is a universal homeomorphism on the nose (cf. Lemma 3.2). Since both $\mathcal{Y}$ and $m^*\mathcal{Y}$ are in $\mathrm{F\acute{E}t}(\mathcal{X})$, $\alpha_{\mathcal{Y},m}$ is an isomorphism at all geometric fibers of $\mathcal{X}$, so it is a degree 1 finite étale map, *i.e.* an equivalence, hence the square defined by $\rho_{\mathcal{Y}}(m)$ and $\rho_{\mathcal{X}}(m)$ is cartesian.

The equivalence (b)$\Leftrightarrow$(c) follows from the universality of 2-cartesian diagrams and the equivalence $\mathrm{F\acute{E}t}(\mathcal{X}) \simeq \{\text{finite étale } \mathcal{O}_{\mathcal{X}}\text{-algebras}\}$: $\mathcal{A}$ corresponds to $\mathcal{Y}$; $\alpha_{\mathcal{A},m}$ corresponds to the unique map $\alpha_{\mathcal{Y},m}$; the conditions on α correspond to the conditions on σ. ☐

Lemma 5.2. *Suppose that $\mathcal{X}$ is G-connected, then* $\mathrm{F\acute{E}t}(\mathcal{X}/G)$ *is a Galois category.*

Proof. Set $\mathcal{C} := \mathrm{F\acute{E}t}(\mathcal{X}/G)$. If $\bar{x}$ is a geometric point of $\mathcal{X}$, then take $F\colon \mathrm{F\acute{E}t}(\mathcal{X}/G) \to (\mathrm{Set})$ to be the composition of the forgetful functor with the fiber functor $\mathrm{F\acute{E}t}(\mathcal{X}) \to (\mathrm{Set})$. Let's verify the four axioms in [Aut, 0BMY].

$\mathcal{C}$ has finite limits and finite colimits. Suppose $\{(\mathcal{Y}_j, \rho_{\mathcal{Y}_i}, \sigma_i)\}_{j\in J}$ is a finite projective system in $\mathcal{C}$. For any $m \in M$, the pullback along $\mathcal{X} \xrightarrow{\rho_{\mathcal{X}}(m)} \mathcal{X}$ (denoted by m^*) is obviously a left exact endomorphism of $\mathrm{F\acute{E}t}(\mathcal{X})$. Thus for each $m \in M$, we have natural maps $\rho_{\varprojlim \mathcal{Y}_j}\colon \varprojlim_{j\in J} \mathcal{Y}_j \longrightarrow \varprojlim_{j\in J} \mathcal{Y}_j$ and $\sigma := \varprojlim_{j\in J} \sigma_j$ which make the following diagram

$$\begin{tikzcd}
\varprojlim_{j\in J} \mathcal{Y}_j \arrow[r, "\rho_{\varprojlim \mathcal{Y}_i}"] \arrow[d] \arrow[dr, phantom, "\overset{\sigma}{\Swarrow}"] & \varprojlim_{j\in J} \mathcal{Y}_j \arrow[d] \\
\mathcal{X} \arrow[r, "\rho_{\mathcal{X}}(m)"] & \mathcal{X}
\end{tikzcd}$$

cartesian (as the diagram for each $\rho_{\mathcal{Y}_i}$ is so by Lemma 5.1). Hence the diagram is an object in $\mathrm{F\acute{E}t}(\mathcal{X}/G)$, and it is clearly the limit. As for the colimit, we first conclude, by the proof of [Aut, 0BN9], that the pullback functor m^* is right exact. Indeed, thanks to [Aut, 0GMN] one just has to check that m^* commutes with coproducts and coequalizers in $\mathrm{F\acute{E}t}(\mathcal{X})$. It's easy for coproducts. For coequalizers one considers two maps $a, b\colon \mathcal{Z} \to \mathcal{Y}$

in $\mathrm{F\acute{E}t}(\mathcal{X})$. Suppose $\mathcal{Y} = \mathrm{Spec}_{\mathcal{O}_\mathcal{X}}(\mathcal{B})$ and $\mathcal{Z} = \mathrm{Spec}_{\mathcal{O}_\mathcal{X}}(\mathcal{C})$, where $\mathcal{B}, \mathcal{C}$ are two finite étale $\mathcal{O}_\mathcal{X}$-algebras. Then a, b correspond to $a^\#, b^\# : \mathcal{B} \to \mathcal{C}$. The coequalizer of a, b is $\mathrm{Spec}_{\mathcal{O}_\mathcal{X}}(\mathcal{A})$, where $\mathcal{A} = \mathrm{Ker}(a^\# - b^\#)$. In [Aut, 0BN9] it was shown that $\mathcal{B}, \mathcal{C}$ are, étale locally, sums of $\mathcal{O}_\mathcal{X}$ and $a^\# - b^\#$ is then a map rearranging the summands. Thus $\mathcal{A}$ is a finite étale $\mathcal{O}_\mathcal{X}$-algebra and $\mathcal{B}/\mathcal{A}, \mathcal{C}/\mathrm{im}(a^\# - b^\#)$ are locally free. This implies that m^* preserves the coequalizers. Now the proof of the existence of limits works mutatis mutandis for colimits.

Every object of $\mathcal{C}$ is a finite (possibly empty) coproduct of connected objects. It is easy to check that for any object $f\colon \mathcal{X}' \to \mathcal{X}$ in $\mathrm{F\acute{E}t}(\mathcal{X}/G)$, the induced map $\bar{f}\colon |\mathcal{X}'|/G \to |\mathcal{X}|/G$ enjoys the following property.

Let $u\colon |\mathcal{X}| \to |\mathcal{X}|/G$ (resp. $v\colon |\mathcal{X}'| \to |\mathcal{X}'|/G$) be the quotient map. Then for any subset $S \subseteq |\mathcal{X}'|/G$, $f(v^{-1}(S)) = u^{-1}(\bar{f}(S))$. Indeed, since $u \circ f(v^{-1}(S)) = \bar{f} \circ v(v^{-1}(S)) = \bar{f}(S)$, we have $f(v^{-1}(S)) \subseteq u^{-1}(\bar{f}(S))$. Conversely, suppose $x' \in |\mathcal{X}'|, x \in |\mathcal{X}|$, $v(x') \in S$, and $\bar{f}(v(x')) = u(x)$, then $u(f(x')) = u(x)$, so $\exists g \in G$ such that $x = gf(x') = f(gx')$. That $v(gx') = v(x') \in S$ implies that $gx' \in v^{-1}(S)$, so $x \in f(v^{-1}(S))$.

Then f clopen $\implies$ $\bar{f}$ clopen. Moreover, the fibers of $\bar{f}$ are finite, as for each $\bar{s} \in |\mathcal{X}|/G$ and any lift $x \in |\mathcal{X}|$ of $\bar{s}$, $\bar{f}^{-1}(\bar{s})$ is the v-image of the finite set $f^{-1}(x)$ (take, in the above, $S = \{s\}$, where $\bar{f}(s) = \bar{s}$). Now applying [Aut, 07VB], we see that $|\mathcal{X}'|/G$ is a coproduct of finitely many connected components, *i.e.* $\mathcal{X}'$ is a coproduct of finitely many G-components.

We'll show that a G-connected object $\mathcal{Y} \in \mathcal{C}$ is a connected object in $\mathcal{C}$ (cf. [Aut, 0BMY]). Suppose $a\colon \mathcal{Y}_1 \to \mathcal{Y}$ is a nonempty monomorphism in $\mathcal{C}$. Then the diagonal map $\mathcal{Y}_1 \to \mathcal{Y}_1 \times_\mathcal{Y} \mathcal{Y}_1$ is an isomorphism. This implies that a is also a monomorphism in $\mathrm{F\acute{E}t}(\mathcal{X})$, *i.e.* a exhibits $\mathcal{Y}_1$ as a clopen subset of $\mathcal{Y}$. As $\mathcal{Y}$ is G-connected, $\mathcal{Y}_1 = \mathcal{Y}$, *i.e.* a is an isomorphism.

$F(\mathcal{X}')$ is finite for all $\mathcal{X}' \in \mathcal{C}$. This is clear.

F is conservative and exact. The functor F is, by its very definition, the composition

$$\mathrm{F\acute{E}t}(\mathcal{X}/G) \xrightarrow{\text{Forget}} \mathrm{F\acute{E}t}(\mathcal{X}) \xrightarrow{F_{\bar{x}}} (\mathrm{Set})$$

where Forget is the obvious forgetful functor, which is conservative and preserves finite limits and finite colimits (given how finite limits and finite colimits in $\mathrm{F\acute{E}t}(\mathcal{X}/G)$ are constructed), and $F_{\bar{x}}$ is the classical fiber functor which is conservative and exact. Thus F is conservative and exact. $\square$

6. Drinfeld's lemma

Let $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n$ be algebraic stacks over $\mathbb{F}_q$, and set $\mathcal{X} := \mathcal{X}_1 \times_{\mathbb{F}_q} \mathcal{X}_2 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} \mathcal{X}_n$. Let ϕ_i be the q-Frobenius of $\mathcal{X}_i$ for $1 \leqslant i \leqslant n$. Then $\mathcal{X}$ is equipped with a strict $M := \phi_1^\mathbb{N} \times \phi_2^\mathbb{N} \times \cdots \phi_n^\mathbb{N}$-action. Let M_0 denote the submonoid $\phi_1^\mathbb{N} \times \cdots \times \phi_{n-1}^\mathbb{N}$, and let G (resp. G_0) denote the free abelian group generated by M (resp. M_0). The category $\mathrm{F\acute{E}t}(\mathcal{X}/G_0)$ can be understood as 2-cartesian diagrams

$$\begin{tikzcd}
\mathcal{Y} \arrow[r, "\varphi_{n-1}"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{Y} \arrow[r, "\varphi_{n-2}"] \arrow[d, "f"'] \arrow[dl, phantom, "\Swarrow \sigma_{n-2}"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{Y} \arrow[r, "\cdots"] \arrow[d, "f"'] \arrow[dl, phantom, "\Swarrow \sigma_{n-1}"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{Y} \arrow[r, "\varphi_1"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{Y} \arrow[d, "f"] \arrow[dl, phantom, "\Swarrow \sigma_1"] \\
\mathcal{X} \arrow[r, "\phi_{n-1}"] & \mathcal{X} \arrow[r, "\phi_{n-2}"] & \mathcal{X} \arrow[r, "\cdots"] & \mathcal{X} \arrow[r, "\phi_1"] & \mathcal{X}
\end{tikzcd}$$

where the φ_is (resp. σ_is) are mutually commute. Morally, $\mathrm{F\acute{E}t}(\mathcal{X}/G_0)$ is just a collection of commuting universally homeomorphic $\mathbb{F}_q$-endomorphisms $\{\varphi_1, \varphi_2, \ldots, \varphi_{n-1}\}$ of $\mathcal{Y}$. Note that here we have dropped φ_n in the collection, but we could drop any ϕ_i for $1 \leqslant i \leqslant n$ which would yield the same category.

Lemma 6.1. *Let Y be a scheme. For any geometric point $\bar{x}\colon \mathrm{Spec}(k) \to Y$ the base change functor induces an equivalence:*

$$\varinjlim_{(U,\bar{u})} \mathrm{F\acute{E}t}(\mathcal{X} \times_{\mathbb{F}_q} U/G) \longrightarrow \mathrm{F\acute{E}t}(\mathcal{X} \times_{\mathbb{F}_q} \bar{x}/G)$$

where $(U, \bar{u})$ runs over all the affine étale neighborhood of $\bar{x}$.

Proof. Let $u\colon \mathcal{X}' \to \mathcal{X}$ be an M-equivariant map of $\mathbb{F}_q$-algebraic stacks, and $\mathcal{X}'' := \mathcal{X}' \times_{\mathcal{X}} \mathcal{X}'$, $\mathcal{X}''' := \mathcal{X}'' \times_{\mathcal{X}} \mathcal{X}'$, then there are natural 2-commutative diagrams

$$\begin{tikzcd}[column sep=small]
\varinjlim\limits_{(U,\bar{u})} \mathrm{F\acute{E}t}(\mathcal{X} \times_{\mathbb{F}_q} U/G) \arrow[r] \arrow[d, "\lambda"] & \varinjlim\limits_{(U,\bar{u})} \mathrm{F\acute{E}t}(\mathcal{X}' \times_{\mathbb{F}_q} U/G) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\lambda'"] & \varinjlim\limits_{(U,\bar{u})} \mathrm{F\acute{E}t}(\mathcal{X}'' \times_{\mathbb{F}_q} U/G) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, "\lambda''"] & \varinjlim\limits_{(U,\bar{u})} \mathrm{F\acute{E}t}(\mathcal{X}'' \times_{\mathbb{F}_q} U/G) \arrow[d, "\lambda'''"] \\
\mathrm{F\acute{E}t}(\mathcal{X} \times_{\mathbb{F}_q} \bar{x}/G) \arrow[r] & \mathrm{F\acute{E}t}(\mathcal{X}' \times_{\mathbb{F}_q} \bar{x}/G) \arrow[r, shift left] \arrow[r, shift right] & \mathrm{F\acute{E}t}(\mathcal{X}'' \times_{\mathbb{F}_q} \bar{x}/G) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathrm{F\acute{E}t}(\mathcal{X}'' \times_{\mathbb{F}_q} \bar{x}/G)
\end{tikzcd} \tag{4}$$

Suppose that the horizontal sequences are exact, then λ' is fully faithful (resp. an equivalence) and λ'' is faithful (resp. fully faithful and λ''' is faithful) imply that λ is fully faithful (resp. an equivalence). Note also that by [Aut, 07SK]we have a natural equivalence:

$$\varinjlim_{(U,\bar{u})} \mathrm{F\acute{E}t}(\mathcal{X} \times_{\mathbb{F}_q} U/G) \xrightarrow{\ \simeq\ } \mathrm{F\acute{E}t}(\mathcal{X} \times_{\mathbb{F}_q} \varprojlim_{(U,\bar{u})} U/G) \tag{5}$$

where $\varprojlim_{(U,\bar{u})} U$ is the strict henselization of Y at $\bar{x}$, so the rows of (4) are exact when u is an fpqc-covering (the actions descend together).

Using these simple observations, we'll proceed in several dévissage steps:

Step 1, reduction to $\mathcal{X}_i = X_i$ a connected $\mathbb{F}_q$-scheme of finite type. Replacing each $\mathcal{X}_i$ by an atlas X_i, we may assume that it is an algebraic space. Replacing each algebraic space X_i by an atlas, we may assume that it is a scheme. Replacing each X_i by a disjoint union of affine opens, we may assume that each X_i is affine. Note that although X'' in (4) may not be a disjoint union of affines anymore, we can cover each $X_i' \times_{X_i} X_i'$ by affines and show that λ is fully faithful for any scheme X. Using this and applying affine coverings again, we get the desired equivalence for any scheme X. We then write each affine scheme X_i as a filtered limit of affine schemes of finite type over $\mathbb{F}_q$. Thus we may assume that X_i is of finite type over $\mathbb{F}_q$. Replacing X_i by a connected component, we may assume that X_i is connected.

Step 2, reduction to Y excellent, reduced, and strictly henselian. Since the problem is Zariski-local around $\bar{x}$, we may assume that Y is affine. Writing Y as a filtered limit of affine finite type $\mathbb{F}_q$-schemes and using [Aut, 07SK], we may assume that Y is of finite type over $\mathbb{F}_q$. Replacing Y with its reduced closed structure, we may assume that Y is reduced. Now replacing Y by its strict henselization at $\bar{x}$ (cf. (5)), we may assume that

$Y = \mathrm{Spec}(A)$, where A is strictly henselian and reduced. By [Aut, 06LJ], A is Noetherian; by [Aut, 07QR], A is a G-ring; by [Gro67, 18.8.17, 18.7.5.1], A is universally catenary; and by [Gre76, 5.3] is J-2. In conclusion, A is excellent.

Step 3, the functor (5) is faithful and essentially surjective. Consider the following 2-commutative diagram, where $X := \mathcal{X} = \mathrm{Spec}(A)$.

$$\begin{tikzcd}
 & \mathrm{F\acute{E}t}(X \times_{\mathbb{F}_q} \mathrm{Spec}(A)/G) \arrow[d, "\lambda"] \\
\mathrm{F\acute{E}t}(X/G_0) \arrow[ur, "u"] \arrow[r, "v", "\simeq"'] & \mathrm{F\acute{E}t}(X \times_{\mathbb{F}_q} \mathrm{Spec}(\kappa)/G)
\end{tikzcd}$$

We have to show that λ is an equivalence. Let ϕ_κ denote the absolute Frobenius of κ. Then $\mathrm{F\acute{E}t}(X \times_{\mathbb{F}_q} \mathrm{Spec}(\kappa)/G)$ can also be described via the action of $G_0 \times \phi_\kappa^{\mathbb{Z}}$. By [PTZ20, Theorem II, (C)] we have $\mathrm{F\acute{E}t}(X) \simeq \mathrm{F\acute{E}t}(X \times_{\mathbb{F}_q} \mathrm{Spec}(\kappa)/\phi_\kappa^{\mathbb{Z}})$. This immediately implies that $\mathrm{F\acute{E}t}(X/G_0) \simeq \mathrm{F\acute{E}t}(X \times_{\mathbb{F}_q} \mathrm{Spec}(\kappa)/G)$. Therefore, v is an equivalence. By Lemma 4.2 and Lemma 5.2 all the categories in the above diagram are Galois categories; since all the functors in the diagram commute with the forgetful fiber functors (which are faithful), they are faithful. This implies that u is fully faithful and λ is essentially surjective. Thus to conclude, it is enough to show that λ is fully faithful or u is essentially surjective.

Step 4, reduction to X_1, Y normal and Y has algebraically closed field of fractions. Since v-covers are morphisms of effective descent for étale maps (cf. [HS23, Theorem 1.5]), and "being quasi-compact" as well as "satisfying the valuative criteria for properness" are obviously properties of maps local on the target for the v-topology (cf. [Aut, 02KO]), v-covers are also morphisms of effective descent for finite étale maps. As each X_i is excellent, its normalization map is finite. Using diagram (4) for the product of normalization maps and the fact that λ'' is faithful (as X'' is a finite disjoint union of connected components), we can assume that each X_i is normal.

For Y, we first note that there are finitely many irreducible components of Y, each of which is a spectrum of a strictly henselian excellent domain (cf. [Aut, 0C2Z]). If Y has m irreducible components $Y_1, \cdots, Y_m$, then we set $Y' := Y_1 \sqcup (Y_2 \cup \cdots \cup Y_m)$, which is a v-cover of Y. Consider the following diagram (diagram (4) "with the other factor")

$$\begin{tikzcd}
\mathrm{F\acute{E}t}(X \times_{\mathbb{F}_q} Y/G) \arrow[r] \arrow[d, "\lambda"] & \mathrm{F\acute{E}t}(X \times_{\mathbb{F}_q} Y'/G) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\lambda'"] & \mathrm{F\acute{E}t}(X \times_{\mathbb{F}_q} Y''/G) \arrow[d, "\lambda''"] \\
\mathrm{F\acute{E}t}(X \times_{\mathbb{F}_q} \bar{x}/G) \arrow[r] & \mathrm{F\acute{E}t}(X \times_{\mathbb{F}_q} \bar{x}'/G) \arrow[r, shift left] \arrow[r, shift right] & \mathrm{F\acute{E}t}(X \times_{\mathbb{F}_q} \bar{x}''/G)
\end{tikzcd} \tag{6}$$

where $\bar{x}' := \bar{x} \sqcup \bar{x}$ and $\bar{x}'' := \bar{x}' \times_{\bar{x}} \bar{x}'$. Thanks to v-descent, the rows in (6) are exact. Using induction on m, we are reduced to the case when A is a domain.

Now let K be the field of fractions of A, and let B be the normalization of A in some finite extension L/K. Since

- B is finite over A (as A is excellent) and,
- B as well as $B \otimes_A B$ are local and strictly henselian (cf. [Aut, 04GH]),

applying diagram (6) to the normalization map and $\bar{x}'' = \bar{x}' = \bar{x}$, we are reduced to the case when Y is normal.

Let $\bar{Y} = \operatorname{Spec}(\bar{A})$ be the normalization of Y in $\bar{K}$. Then $\bar{A}$ is normal and strictly henselian (cf. [Aut, 04GI]). If $T_1, T_2 \in \mathrm{F\acute{E}t}(X \times_{\mathbb{F}_q} Y/G)$ and $t\colon T_1 \otimes_A \kappa \to T_2 \otimes_A \kappa$ is a map in $\mathrm{F\acute{E}t}(X \times_{\mathbb{F}_q} \bar{x}/G)$. Suppose t lifts to some $\bar{a} \in \mathrm{F\acute{E}t}(X \times_{\mathbb{F}_q} \bar{Y}/G)$. Since $\bar{a}$ is defined over some finite middle extension $A \subseteq B \subseteq \bar{A}$, we can apply diagram (6) again to conclude that t lifts to a map $a \in \mathrm{F\acute{E}t}(X \times_{\mathbb{F}_q} Y/G)$. Thus we are reduced to the case when the fraction field K of A is algebraically closed and A is normal (but not necessarily Noetherian).

Step 5, final conclusion. Let K be the field of fractions of A. Then the functor

$$\mathrm{F\acute{E}t}(X/G_0) \longrightarrow \mathrm{F\acute{E}t}(X \times_{\mathbb{F}_q} \operatorname{Spec}(K)/G)$$

is an equivalence and

$$\mathrm{F\acute{E}t}(X \times_{\mathbb{F}_q} \operatorname{Spec}(A)/G) \longrightarrow \mathrm{F\acute{E}t}(X \times_{\mathbb{F}_q} \operatorname{Spec}(K)/G)$$

is fully faithful. Thus u is an equivalence, as desired. ∎

Lemma 6.2 (Stein factorization). *Suppose that $\mathcal{X}$ is a G_0-connected algebraic stack and $\mathcal{X}_{n+1}$ is an arbitrary algebraic stack. For any $(\mathcal{Y}, \rho_{\mathcal{Y}}, \sigma) \in \mathrm{F\acute{E}t}(\mathcal{X}/G)$, there exists $\mathcal{T} \in \mathrm{F\acute{E}t}(\mathcal{X}_{n+1})$ and a 2-commutative diagram*

$$\begin{tikzcd}
\mathcal{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{T} \arrow[d] \\
\mathcal{X} \times_{\mathbb{F}_q} \mathcal{X}_{n+1} \arrow[r] & \mathcal{X}_{n+1}
\end{tikzcd} \tag{7}$$

where $\rho_{\mathcal{Y}}$ acts over $\mathcal{T}$ and each geometric fiber $\mathcal{Y}_{\bar{t}}$ of $\mathcal{Y} \to \mathcal{T}$ is G-connected. If diagram (7) exists, then it has the following universal property: For any $\mathcal{T}' \in \mathrm{F\acute{E}t}(\mathcal{X}_{n+1})$ and any commutative diagram (7) with $\mathcal{T}$ replaced by $\mathcal{T}'$, where $\rho_{\mathcal{Y}}$ acts over $\mathcal{T}'$, there is a unique arrow $\lambda\colon \mathcal{T} \to \mathcal{T}'$ making all the natural diagrams 2-commutative. Since the natural commutativity forces λ into an object in $\mathrm{F\acute{E}t}(\mathcal{X}_{n+1})$, which is equivalent to a set, λ is automatically unique up to a unique isomorphism.

Proof. **Step 1, the case when $\mathcal{X}_{n+1}$ is a point.** Let's first consider the case when $\mathcal{X}_{n+1} = \operatorname{Spec}(k)$, where k is an algebraically closed field containing $\mathbb{F}_q$. In this case, we take $\mathcal{T}$ to be $\sqcup_{i \in I} \operatorname{Spec}(k)$, where I is the set of G-components of $|\mathcal{Y}|$. To prove the universality, we first observe that both $\mathcal{T}$ and $\mathcal{T}'$ are finite disjoint unions of $\operatorname{Spec}(k)$. Since each element of I corresponds to a point of $\mathcal{T}$ and it is mapped to a point of $\mathcal{T}'$ by $\mathcal{Y} \to \mathcal{T}'$, this defines a unique map $\mathcal{T} \to \mathcal{T}'$ in $\mathrm{F\acute{E}t}(\mathcal{X}_{n+1})$ making all the natural diagrams commutative.

Step 2, the universal property for $\mathcal{X}_{n+1}$ general. We first observe that $\mathcal{Y} \to \mathcal{T}$ is faithfully flat, so the desired arrow $\mathcal{T} \to \mathcal{T}'$ is unique if it exists (note that $\mathrm{F\acute{E}t}(\mathcal{X}_{n+1})$ is a 1-category, so this makes sense). This allows us to define $\mathcal{T} \to \mathcal{T}'$ fppf locally on

$\mathcal{X}_{n+1}$. Thus we may assume that $\mathcal{T}$ is a finite disjoint union of $\mathcal{X}_{n+1}$. Working with each pieces of $\mathcal{T}$ we may assume that $\mathcal{T} = \mathcal{X}_{n+1}$. Notice that the map $f\colon \mathcal{Y} \to \mathcal{T}'$ is universally open. Indeed, f can be factorized as $\mathcal{Y} \xrightarrow{f_1} \mathcal{X} \times_{\mathbb{F}_q} \mathcal{T}' \xrightarrow{f_2} \mathcal{T}'$, while f_1 is finite étale and f_2 is universally open by [Aut, 0383]. Let $\mathcal{U} := f(\mathcal{Y})$ be the image. We have to show that there exists a unique $\mathcal{X}_{n+1}$-map $\mathcal{T} = \mathcal{X}_{n+1} \to \mathcal{U}$. Now it follows from Step 1 that on each geometric fiber over $\mathcal{X}_{n+1}$ the factorization exists and is bijective, therefore each fiber of the projection map $u\colon \mathcal{U} \to \mathcal{X}_{n+1}$, which is representable and étale, consists of exactly one point. Thus u is an isomorphism and the desired arrow is u^{-1}.

Step 3, reduction to $\mathcal{X}_{n+1}$ affine. By Step 2, to prove the existence of the Stein factorization, it is enough to prove it fppf-locally. Thus we may assume that $\mathcal{X}_{n+1} = X_{n+1} = \operatorname{Spec}(A)$ is affine.

Step 4, reduction to $\mathcal{Y}$ is a pullback from FÉt$(\mathcal{X}/G_0)$. By Lemma 6.1 and [PTZ20, Theorem II, (C)], for any geometric point $\bar{x}$ of X_{n+1} one can find a connected affine étale neighborhood $U \to X_{n+1}$ such that the pullback of $(\mathcal{Y}, \rho_{\mathcal{Y}}, \sigma)$ to $\mathrm{FÉt}(\mathcal{X} \times_{\mathbb{F}_q} U/G)$ comes from an object $\mathcal{Z} \in \mathrm{FÉt}(\mathcal{X}/G_0)$. By Step 2, it is enough to construct the Stein factorization over U. So we assume that $\mathcal{Y}$ comes from $\mathcal{Z} \in \mathrm{FÉt}(\mathcal{X}/G_0)$.

Step 5, final conclusion. Let $\mathcal{Z}_1, \mathcal{Z}_2, \dots, \mathcal{Z}_m$ be the G_0-components of $\mathcal{Z}$ (remember that $\mathrm{FÉt}(\mathcal{X}/G_0)$ is Galois), then for any geometric point $\bar{x} \in X_{n+1}(k)$, the pullback $(\mathcal{Z}_i, \rho_{\mathcal{Z}_i}, \sigma_i)_k$ of $(\mathcal{Z}_i, \rho_{\mathcal{Z}_i}, \sigma_i)$ to $\mathrm{FÉt}(\mathcal{X} \times_{\mathbb{F}_q} k/G)$ has underlying object $\mathcal{Z}_i \times_{\mathbb{F}_q} k$ and action $\rho_{\mathcal{Z}_i} \times \phi_k$. It's G-connected, as any clopen G-equivariant subset of $\mathcal{Z}_i \times_{F_q} k$ is the inverse image of a G_0-equivariant clopen subset of $\mathcal{Z}_i$ (cf. [PTZ20, Theorem 4.1]). Thus we can take $T := \sqcup_{i=1}^{m} X_{n+1}$ and the map $\mathcal{Y} \to T$ to be the sum of the projections $\mathcal{Z}_i \times_{\mathbb{F}_q} X_{n+1} \to X_{n+1}$. $\square$

Theorem 6.3. *Suppose that $\mathcal{X}_i$ $(1 \leqslant i \leqslant n)$ and $\mathcal{X}_{n+1}$ are connected algebraic stacks over $\mathbb{F}_q$. Then for any geometric point $\bar{x}$ of $\mathcal{X} \times_{\mathbb{F}_q} \mathcal{X}_{n+1}$, the top sequence in*

$$\begin{tikzcd}
1 \arrow[r] & \pi_1^{\text{ét}}(\mathcal{X} \times_{\mathbb{F}_q} k/G, \bar{x}) \arrow[r] \arrow[d, "\cong"'] & \pi_1^{\text{ét}}(\mathcal{X} \times_{\mathbb{F}_q} \mathcal{X}_{n+1}/G, \bar{x}) \arrow[r, "\beta"] \arrow[dl, "\alpha"] & \pi_1^{\text{ét}}(\mathcal{X}_{n+1}, \bar{x}) \arrow[r] & 1 \\
& \pi_1^{\text{ét}}(\mathcal{X}/G_0, \bar{x}) & & &
\end{tikzcd} \tag{8}$$

is exact, where k is the field of definition of $\bar{x}$.

Proof. The left exactness of the top sequence is provided by the retraction α, whose existence is a consequence of [PTZ20, Theorem II, (C)]. Thus we are left to show the middle exactness. In view of [Aut, 0BTQ] we only have to show that the following condition holds:

Let $(\mathcal{Y}, \rho_{\mathcal{Y}}) \in \mathtt{FÉt}(\mathcal{X} \times_{\mathbb{F}_q} \mathcal{X}_{n+1}/G)$ be a connected object. Suppose the pullback $(\mathcal{Y}_{\bar{x}}, \rho_{\mathcal{Y}_{\bar{x}}})$ of $(\mathcal{Y}, \rho_{\mathcal{Y}})$ to $\mathtt{FÉt}(\mathcal{X} \times_{\mathbb{F}_q} k/G)$ has a G-component which is mapped isomorphically to $\mathcal{X} \times_{\mathbb{F}_q} k$, then the Stein factorization of $(\mathcal{Y}, \rho_{\mathcal{Y}})$ induces a

`Cartesian diagram.`

$$\begin{tikzcd} \mathcal{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{T} \arrow[d] \\ \mathcal{X} \times_{\mathbb{F}_q} \mathcal{X}_{n+1} \arrow[r] & \mathcal{X}_{n+1} \end{tikzcd}$$

One has to show that $\lambda\colon \mathcal{Y} \to \mathcal{X} \times_{\mathbb{F}_q} \mathcal{T}$ is an isomorphism. For each geometric point $\bar{s}$ of $\mathcal{T}$ the pullback $\mathcal{X} \times_{\mathbb{F}_q} \kappa(\bar{s})$ is G-connected by 4.2. It follows that $\lambda_s\colon \mathcal{Y}_s \to \mathcal{X} \times_{\mathbb{F}_q} \kappa(\bar{s})$ as well as λ are surjective ($\mathcal{Y}_s$ is a G-component, hence is not empty). That $\mathcal{Y}$ is G-connected implies that $\mathcal{X} \times_{\mathbb{F}_q} \mathcal{T}$ is G-connected, thus λ is finite étale of constant degree. Therefore to show that λ is an isomorphism, it is enough to show that the degree of λ equals 1.

The G-component of $(\mathcal{Y}_{\bar{x}}, \rho_{\mathcal{Y}_{\bar{x}}})$ which goes isomorphically to $\mathcal{X} \times_{\mathbb{F}_q} k$ is mapped to a geometric point $\bar{s} \in \mathcal{T}(k)$ lying over $\bar{x} \in \mathcal{X}_{n+1}(k)$. It is moreover the inverse image of $\bar{s}$ by the very construction of the Stein factorization. Thus λ is an isomorphism at $\bar{s}$, and the degree of λ is 1, as desired. □

Proof of Theorem I. By Theorem 8 and diagram (8) we have

$$\pi_1^{\text{ét}}(\mathcal{X}/\phi_1^{\mathbb{Z}} \times \cdots \times \phi_{n-1}^{\mathbb{Z}}) \xrightarrow{\cong} \pi_1^{\text{ét}}(\mathcal{X}_1 \times \cdots \times \mathcal{X}_{n-1}/\phi_1^{\mathbb{Z}} \times \cdots \times \phi_{n-2}^{\mathbb{Z}}) \times \pi_1^{\text{ét}}(\mathcal{X}_n)$$

By induction on n, we get the result. □

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