NORI'S FUNDAMENTAL GROUP OVER A NON-ALGEBRAICALLY CLOSED FIELD

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ABSTRACT. Let X be a connected reduced scheme over a field k, $x \in X(k)$ be a k-rational point. M. V. Nori constructed in his Ph.D thesis a fundamental group scheme $\pi^N(X,x)$ which generalizes A. Grothendieck's étale fundamental group $\pi_1^{\text{ét}}(X,x)$ by including infinitesimal coverings. However, Nori's fundamental group scheme carries little arithmetic information, and it behalves like the étale fundamental group only when k is algebraically closed. For example, if $X = \operatorname{Spec}(k)$, then Nori's fundamental group scheme is always trivial while the étale fundamental group $\pi_1^{\text{ét}}(X,x) = \operatorname{Gal}(\bar{k}/k)$. In this paper, we study a slightly modified version of Nori's fundamental group scheme: We take x to be a geometric point instead of a rational point. It is very suprising to the author that this tiny little modification of Nori's original definition brings a lot of arithmetic information and makes the fundamental group scheme more like $\pi_1^{\text{ét}}(X,x)$. For example, now if we take $X = \operatorname{Spec}(k)$ again, with $\bar{x} \in X(\bar{k})$, then we get a profinite group scheme $\pi^N(k/k,\bar{x})$ over k which takes $\operatorname{Gal}(\bar{k}/k)$ as its (pro-constant) quotient. Thus not only the Galois extensions, but also the purely inseparable extensions of k are encoded into $\pi^N(k/k,\bar{x})$. We call $\pi^N(k/k,\bar{x})$ the *Nori-Galois group* of k. We also studied the fundamental sequence which relates the Nori-Galois group to the geometric fundamental group. It turns out that the expected fundamental exact sequence is always a complex and exact on the right, but fails to be exact in the middle and on the left. Then we give conditions to determine when the exactness holds.

0. INTRODUCTION

Let X be a connected scheme, $x \in X(\bar{k})$ be a geometric point. Let $\operatorname{ECov}(X)$ be the category of finite étale coverings of X. Then we have a fibre functor F from $\operatorname{ECov}(X)$ to the category of finite sets by sending any finite étale covering $f : Y \to X$ to its fibres $f^{-1}(x)$. In [SGA1][Exposé V] A. Grothendieck proved that $\operatorname{ECov}(X)$ together with the fibre functor F forms a Galois category. Then he defined the étale fundamental group $\pi_1^{\text{ét}}(X,x) := \operatorname{Aut}(F)$ to be the group of automorphisms of F. This is a profinite group, i.e. a topological group of the form $\varprojlim_{i\in I} G_i$ where I is a small cofiltered category and G_i is a finite group for each $i \in I$. This profinite group classifies all torsors under finite groups, in particular when G is a finite abelian group we have $H^1_{\text{ét}}(X,G) = \operatorname{Hom}_{\text{cont}}(\pi_1^{\text{ét}}(X,x),G)$. If $X = \operatorname{Spec}(k)$ and $x \in X(\bar{k})$ corresponds to the field extension $k \subseteq \bar{k}$, then I can be chosen as the set of finite Galois sub-extensions of $k \subseteq \bar{k}$, and for each $i = (k \subseteq K) \in I$, $G_i := \operatorname{Gal}(K/k)$, so $\pi_1^{\text{ét}}(X,x) = \varprojlim_{i\in I} G_i$ is just the absolute Galois group $\operatorname{Gal}(\bar{k}/k)$.

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Let X be a proper reduced connected scheme over a field k, $x \in X(k)$ be a rational point. In [Nori, Part I, Chapter I] M. V. Nori constructed a full subcategory $\mathrm{EFin}(X) \subseteq \mathrm{Vec}(X)$ of the category of vector bundles on X. Objects in $\mathrm{EFin}(X)$ are called essentially finite vector bundles. He proved that $\mathrm{EFin}(X)$ with the fibre functor ω from $\mathrm{EFin}(X)$ to the category of finite dimensional vector spaces sending $V \mapsto V|_x$ is a Tannakian category over k. Then he defined $\pi^N(X, x) := \mathrm{Aut}^{\otimes}(\omega)$ to be the group of k-linear tensor automorphisms of ω. This is a profinite k-group scheme which classifies all k-pointed torsors over X under finite k-group schemes. In particular when G is a finite abelian k-group scheme we have $H^1_{\mathrm{fppf}}(X, G) = \mathrm{Hom}_{\mathrm{grp.sch}}(\pi_1^N(X, x), G)$. However, in this construction the properness assumption is vital, it does not apply to non-proper schemes. To remedy this M. V. Nori introduced in [Nori, Part I, Chapter II] another construction. For any reduced connected scheme X over a field k with a rational point $x \in X(k)$, let $N(X/k, x)$ be the category of torsors under finite $k-$group schemes with a fixed k-point lying over x. Then Nori defined $\pi^N(X, x) := \varprojlim_{i \in N(X/k,x)} G_i$, where G_i is the finite group scheme corresponding to the pointed torsor i. It is not hard to prove that this definition coincides with the Tannakian one (See [Nori, Part I, Chapter I, Proposition 3.11] and 2.1 for an explanation).

There is a comparison between Grothendieck's fundamental group and Nori's fundamental group scheme: If X is a connected reduced scheme over an algebraically closed field k with a rational point $x \in X(k)$, then $\pi^N(X, x)(k) \cong \pi_1^{\text{ét}}(X, x)$ as topological groups, where $\pi^N(X, x)(k)$ is equipped with the Zariski topology. This means, *over an algebraically closed field*, Nori's fundamental group scheme is a generalization of Grothendieck's étale fundamental group. However, when the base field is not algebraically closed, Nori's definition is quite different from Grothendieck's:

(1) If $X = \mathrm{Spec}\,(k)$, then Nori's fundamental group scheme is always trivial while the étale fundamental group $\pi_1^{\text{ét}}(X, x) = \mathrm{Gal}(\bar{k}/k)$.
(2) In [Nori, Part I, Chapter II, Proposition 5] Nori proved that the fundamental group scheme satisfies base change by separable field extensions. But this does hold for the étale fundamental group. Take the projective space for example, if we see $\pi_1^{\text{ét}}(\mathbb{P}^n_{\mathbb{Q}})$ as a profinite group scheme over $\mathbb{Q}$, then we have $\pi_1^{\text{ét}}(\mathbb{P}^n_{\mathbb{Q}}) \times_{\mathbb{Q}} \bar{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \times_{\mathbb{Q}} \bar{\mathbb{Q}} \neq \{1\} = \pi_1^{\text{ét}}(\mathbb{P}_{\bar{\mathbb{Q}}})$.
(3) By [Nori, Chapter II, Proposition 4], $\pi^N(X, x_1)$ and $\pi^N(X, x_2)$ differ by an inner twist for different rational points $x_1, x_2 \in X(k)$, and they only become isomorphic after base change to $\bar{k}$.

The first two properties reveals that Nori's fundamental group scheme is in some sense a *geometric fundamental group*, i.e. it is better designed for schemes over an algebraically closed field. In this paper, we are going to study an arithmetic variant of this fundamental group scheme which brings it closer to $\pi_1^{\text{ét}}$. In Nori's second definition of the fundamental group scheme, instead of taking a rational point $x \in X(k)$, we take a geometric point $\bar{x} \in X(\bar{k})$. It is really surprising that this tiny little modification makes the fundamental group scheme contain extremely rich arithmetic information.

To simplify the study we first split the fundamental group scheme into several different parts and study each of them. Let $N(X/k, \bar{x})$ be as before, except that x is now replaced by a geometric point $\bar{x}$, and denote $\pi^N(X/k,\bar{x})$ the group scheme $\varprojlim_{i\in N(X/k,\bar{x})} G_i$. Let $I_{\text{ét}}(X/k,\bar{x})$ (resp. $I_{co}(X/k,\bar{x}), I_{lc}(X/k,\bar{x})$) be the full subcategory of $N(X/k,\bar{x})$ consisting of those pointed torsors whose group schemes are étale (resp. constant, local), and the corresponding fundamental group is denoted by $\pi^E(X/k,\bar{x})$ (resp. $\pi^G(X/k,\bar{x})$, $\pi^L(X/k,\bar{x})$). Then according to 2.6, we have the following canonical surjections:

$$\begin{tikzcd} & \pi^E(X/k,\bar{x}) \arrow[r, two heads] & \pi^G(X/k,\bar{x}) \\ \pi^N(X/k,\bar{x}) \arrow[ur, two heads] \arrow[dr, two heads] & & \\ & \pi^L(X/k,\bar{x}) & \end{tikzcd}$$

$$\pi^N(X/k,\bar{x}) \twoheadrightarrow \pi^E(X/k,\bar{x}) \times_k \pi^L(X/k,\bar{x}) \ .$$

In fact, $\pi^G(X/k,\bar{x})$ is nothing but a "group scheme version" of $\pi_1^{\text{ét}}(X,\bar{x})$ (see 2.4 (ii)). Although $\pi^E(X/k,\bar{x})$ and $\pi_1^{\text{ét}}(X,\bar{x})$ are all fundamental groups classifying étale coverings they are indeed largely different. For example $\pi_1^{\text{ét}}(\mathrm{Spec}\,(\mathbb{R}),\bar{x}) = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$ and the universal covering of $\mathrm{Spec}\,(\mathbb{R})$ under $\pi_1^{\text{ét}}(\mathrm{Spec}\,(\mathbb{R}),\bar{x})$ is $\mathrm{Spec}\,(\mathbb{C})$, while we have

Theorem 0.1. *(See 2.14). Let $\mathbb{R}$ be the field of real numbers, $\bar{x} : \mathrm{Spec}\,(\mathbb{C}) \to \mathrm{Spec}\,(\mathbb{R})$ be the morphism corresponding to the natural inclusion $\mathbb{R} \subset \mathbb{C}$. Then*

$$\pi^N(\mathbb{R}/\mathbb{R},\bar{x}) = \pi^E(\mathbb{R}/\mathbb{R},\bar{x}) = \varprojlim_{n\in\mathbb{N}^+} \mu_{n,\mathbb{R}}$$

is an infinite $\mathbb{R}-$group scheme, and the universal covering corresponding to $\pi^E(\mathbb{R}/\mathbb{R},\bar{x})$ is a non-Noetherian affine scheme with infinitely many connected components.

Although the so defined fundamental group scheme is quite complicated, it does behalve well as an arithmetic fundamental group scheme:

Proposition 0.2. *(See 2.3). Let $X = \mathrm{Spec}\,(k)$ be a field, $\bar{x} \in X(\bar{k})$ be the geometric point $k \subseteq \bar{k}$. Then:*

(i) $\pi^L(k/k,\bar{x}) = \{1\},\ \pi^N(k/k,\bar{x}) = \pi^E(k/k,\bar{x})$, *when k is a perfect field;*
(ii) $\pi^E(k/k,\bar{x}) = \{1\},\ \pi^N(k/k,\bar{x}) = \pi^L(k/k,\bar{x})$, *when k is a separably closed field;*
(iii) $\pi^N(X/k,\bar{x}) = \pi^E(X/k,\bar{x}) = \pi^L(X/k,\bar{x}) = \{1\}$, *when X is $\mathbb{A}^n_{\bar{k}}$ with k a field of characteristic 0 or X is $\mathbb{P}^n_{\bar{k}}$ with k a field of arbitrary characteristic.*

In [Ro] you can find an interesting application of 0.2.

The following is an analogue of [SGA1, Exposé X, Corollarie 1.8., pp. 204]:

Proposition 0.3. *(See 2.23). Let X be a scheme geometrically connected proper separable over a field k, $k \subseteq l \subseteq l'$ be a sequence of field extensions, where l and l' are algebraically closed fields. Let $\bar{x} : \mathrm{Spec}\,(l') \to X$ be a geometric point. Then the following natural map*

$$\pi_l^{l'} : \pi^E(X \times_k l'/k, \bar{x}) \longrightarrow \pi^E(X \times_k l/k, \bar{x})$$

is an isomorphism of k-group schemes.

By contrast $\pi^L(X/k, \bar{x})$ (hence also $\pi^N(X/k, \bar{x})$) doesn't satisfy base change by algebraically closed field extensions. This can be deduced from a famous counterexample by Mehta and Subramanian in [MS] which was used to show that base change by algebraically closed field extensions fails for Nori's original definition (See 2.24 for details).

The following theorem shows that the arithmetic fundamental group scheme we are considering here deserves the name *fundamental group*.

Proposition 0.4. *(See 2.21). Let X be any connected reduced scheme over k, $\bar{x}_1 : \mathrm{Spec}\,(\bar{l}_1) \to X$ and $\bar{x}_2 : \mathrm{Spec}\,(\bar{l}_2) \to X$ be two geometric points of X. Then there are (non-canonical) isomorphisms between the following k-group schemes:*

(i) $$\pi^E(X/k, \bar{x}_1) \cong \pi^E(X/k, \bar{x}_2)$$

(ii) $$\pi^L(X/k, \bar{x}_1) \cong \pi^L(X/k, \bar{x}_2)$$

(iii) $$\pi^N(X/k, \bar{x}_1) \cong \pi^N(X/k, \bar{x}_2).$$

A very powerful tool to understand arithmetic fundamental groups is the so called *fundamental exact sequence* which relates the geometric part (the geometric fundamental group) to the arithmetic part (the Galois group). Unlike the fundamental exact sequence for $\pi_1^{\text{ét}}$ in [SGA1, Exposé IX, Théorème 6.1], ours is exact only in certain cases. Nonetheless it does provide some valuable information about the arithmetic fundamental group, e.g. it follows immediately from the following theorem that our arithmetic fundamental group will *never* satisfy base change by separable field extensions if $k \neq k^{\text{sep}}$.

Theorem 0.5. *(See §3). Let X be a geometrically connected scheme which is separable over a field k and let $\bar{x} \in X(\bar{k})$ be a geometric point, then there is a complex of k-group schemes*

(1) $$1 \to \pi^I(\bar{X}/k, \bar{x}) \to \pi^I(X/k, \bar{x}) \to \pi^I(k/k, \bar{x}) \to 1$$

where $I = N, L$ or E. This sequence is always exact on the right, not always exact on the left (3.12), and is exact in the middle if and only if for any object $(P, G, p) \in I(X/k, \bar{x})$ both of the following conditions are satisfied:

(i) If (P, G, p) is saturated, then the image of the composition of the natural homomorphisms

$$\pi^I(\bar{X}/k, \bar{x}) \to \pi^I(X/k, \bar{x}) \twoheadrightarrow G$$

is a normal subgroup of G;

(ii) Whenever the pull-back of (P, G, p) along $\bar{X} \to X$ is trivial there is an object $(Q, H, q) \in I(k/k, \bar{x})$ whose pull-back along $X \to \mathrm{Spec}\,(k)$ is isomorphic to (P, G, p).

Moreover, condition (i) holds for triples (P, G, p) where G is étale and P is connected (3.7) or G is local and k is perfect (3.12). But as the example (3.9) shows, (i) fails when P is not connected while G is still étale. The condition (ii) holds when k is perfect or X is proper or G is étale (3.6), but fails (3.5) when k is not perfect, X is not proper and G is not étale.

Here, I think the state of the art is the counterexample 3.9. In fact the normality problem as in (i) is always very difficult in the study of the homotopy sequence of fundamental groups. For example, in [Zh2] an essential part of the proof is devoted to the normality problem, and it is also the key issue in [EHV]. Here we give a very delicate example to show that the normality condition breaks in many cases. But of course, if one only restricts to the abelian quotient π^N_{ab} of the whole fundamental group scheme π^N then one gets an exact sequence as long as condition (ii) holds.

The following is another special case in which (i) and (ii) hold.

Corollary 0.6. *(See 3.4). If either $X = \mathbb{A}^n_k$ with k is a field of characteristic 0 or X is a complete rational variety over an arbitrary field k, then the canonical map*

$$\pi^N(X/k, \bar{x}) \to \pi^N(k/k, \bar{x})$$

is an isomorphism. In other words, any finite flat torsor over X descends uniquely to k.

As we will see in 2.5, there are two geometric fundamental group schemes corresponding to this arithmetic fundamental group scheme. Here is the fundamental sequence with respect to the other geometric fundamental group scheme.

Theorem 0.7. *(See §4). Let X be a scheme geometrically connected separable over a field k, $\bar{x} \in X(\bar{k})$ be a geometric point, then there is a natural sequence of $\bar{k}$-group schemes*

$$1 \to \pi^I(\bar{X}/\bar{k}, \bar{x}) \to \pi^I(X/k, \bar{x}) \times_k \bar{k} \to \pi^I(k/k, \bar{x}) \times_k \bar{k} \to 1. \tag{2}$$

It is a complex, always exact on the right, exact on the left when k is perfect and X is q.s. and q.c., but it is in general not exact in the middle for $I = N, E, L$.

In the end we apply our discussions to construct a possibly smaller subset $\mathrm{Section}^N_{\sim}(k, X)$ of the full set of section classes of the fundamental exact sequence of the étale fundamental group (see pp. 5). In fact this subset contains all the "geometric sections", i.e. those sections which come from the rational points of X. Thus if one expect that there is a one-to-one correspondence between the rational points and the section classes, then a priori one should expect a one-to-one correspondence between the rational points and $\mathrm{Section}^N_{\sim}(k, X)$. Therefore, we formulate this possibly weaker version of the section conjecture (5.4). Indeed this formulation has an advantage when one deals with the problem in characteristic $p > 0$ (see 5.5).

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Notations and Conventions

(i) We always use k to denote a field, $\bar{k}$ to denote its algebraic closure.

(ii) Let $f: S' \to S$ be a morphism of schemes, X' be a scheme over S'. We say X' possess an *S-form* if there is a scheme X over S whose pull-back along f is isomorphic to X'.

(iii) When X is a scheme over k, we use $\bar{X}$ to denote $X \times_k \bar{k}$. If $k \subseteq K$ is a field extension we use X_K to denote $X \times_k K$. Sometimes we also use $\bar{X}$ (resp.X_K) to denote something over $\bar{k}$ (resp. K) which does not necessarily possess a k-form X. This depends on the situation we are in.

(iv) Let $X \times_S Y$ be a fibred product of schemes. We use pr_1 to denote the first projection $X \times_S Y \to X$ and pr_2 to denote the second projection.

(v) Let G be a group scheme over k. In this note, a *G-torsor* over a k-scheme X is an X-scheme P equipped with a right action $\rho: P \times_k G \to P$, where ρ is a morphism of X-schemes which induces an isomorphism $pr_1 \times \rho: P \times_k G \to P \times_X P$. Moreover we require that the structure map $P \to X$ of the X-scheme P is faithfully flat and quasi-compact.

(vi) Let X be a scheme. We use X_{red} to denote the reduced closed subscheme structure of X.

(vii) Let $f: X \to S$ be a morphism of schemes. We call f *separable* [SGA1, Exposé X, Définition 1.1] if it is flat and all its geometric fibres are reduced. A k-scheme X is called separable if and only if its structure map is separable.

(viii) Let G be a group scheme over k, $H \subseteq G$ be a subgroup scheme. We say $H \subseteq G$ is a *normal subgroup scheme* if for any k-scheme T, $H(T) \subseteq G(T)$ is a normal subgroup. Note that $H \subseteq G$ is normal if and only if $\bar{H} \subseteq \bar{G}$ is normal.

(ix) Let $f: X \to Y$ be a morphism of schemes. We say f is *surjective* if for each morphism $y: \mathrm{Spec}\,(k) \to Y$ the fibred product of y with f is a non-empty scheme. If $f: X \to Y$ is a morphism of group schemes, then we say f is *surjective* when f is surjective as a morphism of FPQC-sheaves of groups. In the case when X, Y are affine group schemes over k, f is surjective $\Leftrightarrow$ the corresponding map of Hopf-algebras is injective.

(x) Let $S' \to S$ be a Galois covering, i.e. a connected finite étale covering which is a torsor under its own automorphism group $\mathrm{Aut}_S(S')$. Let $\pi': X' \to S'$ be a morphism of schemes. A *twisted action* of $\mathrm{Aut}_S(S')$ on X' is a group homomorphism $f: \mathrm{Aut}_S(S') \to \mathrm{Aut}(X')$, where $\mathrm{Aut}(X')$ is the group of scheme automorphisms of X', such that for any $\sigma \in \mathrm{Aut}_S(S')$ the following diagram

$$\begin{tikzcd} X' \arrow[r, "f(\sigma)"] \arrow[d, "\pi'"'] & X' \arrow[d, "\pi'"] \\ S' \arrow[r, "\sigma"] & S' \end{tikzcd}$$

is commutative. By Grothendieck's general descent theory [BLR, 6.2, Example B, pp. 139], there is an equivalence of categories between the category of affine S'-schemes equipped with a twisted action from $\mathrm{Aut}_S(S')$ and the category of affine S-schemes. We often refer to this as *Galois descent*.

(xi) Here is another version of *Galois descent*. Let $k \subseteq K$ be a finite Galois extension. There is an equivalence of categories between the category of finite abstract groups equipped with a continous action from $\mathrm{Gal}(\bar{k}/k)$ (resp. an action from $\mathrm{Gal}(K/k)$) via group automorphisms and the category of finite étale k-group schemes (resp. k-group schemes whose pull-back to K are finite constant). [AV, 3.25-3.26].

1. The Arithmetic Nori's Fundamental Group

Let X be a reduced connected scheme over a field k, $x : S \to X$ be a morphism of k-schemes with S non-empty.

Definition 1. Consider the triples (P, G, p) where G is a finite group scheme over k, P is a G-torsor over X, $p : S \to P$ is a k-morphism lifting $x : S \to X$. A morphism from (P_1, G_1, p_1) to (P_2, G_2, p_2) is a pair (s, t) where $t : G_1 \to G_2$ is a k-group scheme homomorphism, $s : P_1 \to P_2$ is an X-scheme morphism which intertwines the group action and sends $p_1 \mapsto p_2$. We denote the category consisting of such triples by $N(X/k, x)$.

Definition 2. [SGA4, Exposé I, Définition 2.7] A category I is called cofiltered if it satisfies the following three conditions:

(i) it is non-empty;
(ii) for any objects $i, j \in I$, there exists an object $k \in I$ and two arrows $k \to i$, $k \to j$;
(iii) for any two morphisms

$$\begin{tikzcd} j \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] & i \end{tikzcd}$$

there exists a morphism $c : k \to j$ satisfying $a \circ c = b \circ c$.

Remark 1.1. The category $N(X/k, x)$ has finite fibred products and a final object $(X, \{1\}, x)$, so in particular it is cofiltered. The proof is due to M.V.Nori. Considering the importance of the fact to our construction, we would like to reproduce his proof in our settings.

Proposition 1.2. [Nori, Chapter II, Proposition 1 and Proposition 2] *Fibred products exist in $N(X/k, x)$.*

Proof. We have to show that given any two morphisms

$$(\phi_i, h_i) : (P_i, G_i, p_i) \to (Q, G, q) \in N(X/k, x)$$

where $i = 1, 2$, the triple $(P_1 \times_Q P_2, G_1 \times_G G_2, p_1 \times_q p_2)$ is again an object in $N(X/k, x)$.

The action of G_1 on P_1 (resp. G_2 on P_2) induces a morphism of k-schemes

$$\lambda : (P_1 \times_Q P_2) \times_k (G_1 \times_G G_2) \to (P_1 \times_Q P_2) \times_X (P_1 \times_Q P_2)$$
$$(x_1, x_2) \times (g_1, g_2) \mapsto (x_1, x_2) \times (x_1 g_1, x_2 g_2).$$

By a purely abstract nonsense argument, we see that the induced morphism is an isomorphism. Now the problem is to show that the projection $\phi : P_1 \times_Q P_2 \to X$ is FPQC.

Let Y be the quotient of $P_1 \times_Q P_2$ by $G_1 \times_G G_2$,

$$\varphi : P_1 \times_Q P_2 \to Y$$

be the quotient map. Then there is a unique morphism of schemes $i : Y \to X$ through which the projection ϕ factors. Consider the following commutative diagram:

$$\begin{tikzcd}
(P_1 \times_Q P_2) \times_k (G_1 \times_G G_2) \arrow[r, "\lambda", "\cong"'] \arrow[d, "\varphi \circ pr_1"'] & (P_1 \times_Q P_2) \times_X (P_1 \times_Q P_2) . \arrow[d, "\varphi \times \varphi"] \\
Y \arrow[r, "\Delta"] & Y \times_X Y
\end{tikzcd}$$

As $i : Y \to X$ is finite, Δ is of finite presentation [EGA, 1.4.3.1, pp.231]. Since φ is finite faithfully flat [SGA3, Exposé V, Théorème 4.1, pp.259], $\varphi \circ pr_1$, $\varphi \times \varphi$, λ are all finite and faithfully flat. So Δ is also faithfully flat. But Δ is already a closed immersion, so it has to be an isomorphism. Hence the finite morphism $i : Y \to X$ is a monomorphism [EGA, 17.2.6] in the category of schemes. Thus it has to be a closed immersion [EGA, 18.12.6]. Now look at the following diagram

$$\begin{tikzcd}
P_1 \times_Q P_2 \arrow[r, hook] \arrow[d, two heads, "\varphi"'] & P_1 \times_X P_2 . \arrow[d, two heads, "\psi"] \\
Y \arrow[r, "i"] & X
\end{tikzcd}$$

Since $P_1 \times_Q P_2$ is the fibre of the neutral element of G under the following map

$$P_1 \times_X P_2 \xrightarrow{(\phi_1 \times \phi_2)} Q \times_X Q \xrightarrow{\cong} Q \times_k G \xrightarrow{pr_2} G,$$

$P_1 \times_Q P_2 \subseteq P_1 \times_X P_2$ must be both open and closed as a sub topological space (but not as a subscheme). The map ψ is finite flat and of finite presentation, so the underlying topological space of the scheme Y, as the image of $P_1 \times_Q P_2$ under ψ, is both open and closed in X. Since $P_1 \times_Q P_2$ admits a morphism from a *non-empty* scheme S, it must be non-empty as well. Thus $Y \neq \varnothing$. Combining this with the condition that X is connected and reduced we conclude that $i : Y \to X$ is an isomorphism. Now $\phi = i \circ \varphi$ is finite locally free and surjective, so in particular FPQC. $\square$

Remark 1.3. (i) Proposition 1.2 implies that $N(X/k, x)$ is cofiltered[1]. Indeed, conditions (i), (ii) of Definition 2 are directly checked. For (iii), suppose we have two maps $a, b : j \to i$ as in 2, then we could make the following cartesian diagram

$$\begin{tikzcd}
k \arrow[r, "c"] \arrow[d] & j \arrow[d, "a \times b"] \\
i \arrow[r, "\Delta"] & i \times i
\end{tikzcd}$$

where Δ stands for the diagonal map. The map c in the diagram is precisely what we are looking for.

(ii) It is rather important that we require $S \neq \varnothing$, otherwise the category is not cofiltered. For example, let's take $X = \mathrm{Spec}\,(k)$ to be a field, $Q = (\mathbb{Z}/2\mathbb{Z})_k$ be the trivial torsor under

[1]This was suggested to us by Jilong Tong. We were using a non-standard notion of cofilteredness in the earlier version. We thank him for this suggestion.

the constant group scheme $(\mathbb{Z}/2\mathbb{Z})_k$, $P_1 = P_2 = \mathrm{Spec}\,(k)$ be the trivial torsor under the trivial k-group scheme $\{1\}$, $\phi_i : P_i \to Q$ $(i = 1, 2)$ be two maps sending P_i to the two different points of Q. If the category was cofiltered, then there should be two morphisms of torsors $\psi_i : P \to P_i$ $(i = 1, 2)$ which equalize ϕ_1 and ϕ_2. But $P_1 \times_Q P_2 = \varnothing$, so this can't happen.

(iii) In 2 the reducedness and connectedness assumptions are actually quite important. For example, we could take $X := \alpha_{p,k}$ where k is a field of characteristic p, $Q := X \times_k \alpha_{p,k}$ the trivial $\alpha_{p,k}$-torsor over X, $P_1 = P_2 = X$ is the trivial torsor over X under the trivial group scheme $\{1\}$. Let $\phi_1 : P_1 \to Q$ be the diagonal map $X \to X \times_k \alpha_{p,k} = X \times_k X$, and let $\phi_2 : P_2 \to Q$ be the map $id \times 0 : X \to X \times_k \alpha_{p,k}$. Then $P_1 \times_Q P_2 = \mathrm{Spec}\,(k)$ and the projection $P_1 \times_Q P_2 \to X$ is just the imbedding of the identity point $0 : \mathrm{Spec}\,(k) \hookrightarrow \alpha_{p,k}$ which certainly can't be flat. Now let's equip X with a geometric point $x : \mathrm{Spec}\,(\bar{k}) \to \mathrm{Spec}\,(k) \xrightarrow{0} X = \alpha_{p,k}$, then there are unique liftings $p_1 \in P_1(\bar{k})$, $p_2 \in P_2(\bar{k})$, $q \in Q(\bar{k})$ of x. If we had a triple $(T, H, t) \in N(X/k, x)$ and a commutative diagram

$$\begin{tikzcd} (T,H,t) \arrow[r] \arrow[d] & (P_1,\{1\},p_1) \arrow[d, "(\phi_1,0)"] \\ (P_2,\{1\},p_2) \arrow[r, "(\phi_2,0)"] & (Q,\alpha_{p,k},q) \end{tikzcd}$$

then the structure map $T \to X$ of the X-scheme T would factor through $0 : \mathrm{Spec}\,(k) \to X = \alpha_{p,k}$ because $\mathrm{Spec}\,(k) = P_1 \times_Q P_2$. Then $T \to X$ can not be faithfully flat, but this contradicts to the assumption that T is a torsor over X. Thus $N(X/k, x)$ is not cofiltered. If X is allowed to be non-connected, then take any morphism of k-schemes $S \to Y$ with S non-empty, let $Y_1 = Y_2 = Y_3 = Y_4 = Y_5 = Y_6 = Y$, $X := Y_1 \coprod Y_2$, $P_1 = P_2 = Y_1 \coprod Y_2 = X$ be the trivial $\{1\}$-torsor, $Q = Y_3 \coprod Y_4 \coprod Y_5 \coprod Y_6$ be the trivial $(\mathbb{Z}/2\mathbb{Z})_k$-torsor with structure map $Y_3 \mapsto Y_1, Y_4 \mapsto Y_2, Y_5 \mapsto Y_1, Y_6 \mapsto Y_2$. Now set $x : S \to Y_1 \subseteq X$ $p_1 : S \to Y_1 \subseteq P_1$, $p_2 : S \to Y_1 \subseteq P_2$, $q : S \to Y_3 \subseteq Q$. Let $\phi_1 : P_1 \to Q$ be the map sending $Y_1 \mapsto Y_3$, $Y_2 \mapsto Y_4$, $\phi_2 : P_2 \to Q$ be the map sending $Y_1 \mapsto Y_3$, $Y_2 \mapsto Y_6$. If we had a triple $(T, H, t) \in N(X/k, x)$ and a commutative diagram

$$\begin{tikzcd} (T,H,t) \arrow[r] \arrow[d] & (P_1,\{1\},p_1) \arrow[d, "(\phi_1,0)"] \\ (P_2,\{1\},p_2) \arrow[r, "(\phi_2,0)"] & (Q,\alpha_{p,k},q) \end{tikzcd}$$

Then the structure map $T \to X$ of the X-scheme would factor through $Y_1 \subsetneq X$. Therefore, T is flat but *not* faithfully flat over X, a contradiction! So $N(X/k, x)$ can not be cofiltered.

(iv) In 1.2, if G_1 and G_2 are étale then ϕ is automatically FPQC ([SGA1, Exposé I, Corollaire 4.8., pp.4]) even when X is non-reduced. However, connectedness is still vital.

Definition 3. Let X be a reduced connected scheme over a field k, $x : S \to X$ be a morphism of k-schemes with S non-empty, $I(X/k, x) \subseteq N(X/k, x)$ be a cofiltered full subcategory. The forgetful functor $i := (P_i, G_i, p_i) \longmapsto G_i$ from $I(X/k, x)$ to the category of k-group schemes defines a small cofiltered projective system of finite k-group schemes.

We define the arithmetic Nori fundamental group scheme $\pi^I(X/k,x)$ to be $\pi^I(X/k,x) := \varprojlim_{i\in I(X/k,x)} G_i$.

2. First Properties of $\pi^I(X/k,x)$

2.1. **The Universal Covering.** As in [Nori, Chapter II, Proposition 2] we can define the universal covering for our fundamental group scheme.

Proposition 2.1. *Let X be a connected reduced scheme over a field k, $x : S \to X$ be a morphism of k-schemes with S non-empty, $I(X/k,x) \subseteq N(X/k,x)$ be a cofiltered full subcategory. Then there exists a triple $(\widetilde{X_x}, \pi^I(X/k,x), \tilde{x})$, where $\widetilde{X_x}$ is a $\pi^I(X/k,x)$-torsor over X, $\tilde{x} : S \to \widetilde{X_x}$ is an S-point of $\widetilde{X_x}$ lying above x, which satisfies that for any $(P,G,p) \in I$ there exists a unique morphism*

$$(\phi, h) : (\widetilde{X_x}, \pi^I(X/k,x), \tilde{x}) \to (P,G,p),$$

where $h : \pi^I(X/k,x) \to G$ is homomorphism of k-group schemes and $\phi : \widetilde{X_x} \to P$ is a morphism of X-schemes which sends $\tilde{x}$ to p and intertwines the group actions.

Proof. Consider the following functors

$$F_X : I(X/k,x) \to \mathrm{Aff}(X), \qquad (P,G,p) \mapsto P$$

$$F_k : I(X/k,x) \to \mathrm{Grsch}(k), \qquad (P,G,p) \mapsto G$$

where $\mathrm{Aff}(X)$ denotes the category of affine schemes over X, and $\mathrm{Grsch}(k)$ denotes the category of finite group schemes over k. We have by 3 that

$$\pi^I(X/k,x) = \varprojlim_{i\in I(X/k,x)} F_k(i)$$

Now let

$$\widetilde{X_x} := \varprojlim_{i\in I(X/k,x)} F_X(i).$$

Then $\widetilde{X_x}$ is an affine scheme over X which admits a point $\tilde{x} : S \to \widetilde{X_x}$ lying above x.

Now we get a triple $(\widetilde{X_x}, \pi^I(X/k,x), \tilde{x})$ which has the property that for any $i := (P,G,p) \in I(X/k,x)$ there is a morphism

$$(\phi_i, h_i) : (\widetilde{X_x}, \pi^I(X/k,x), \tilde{x}) \to (P,G,p)$$

defined by the projection to the index $i \in I(X/k,x)$. Let H be the image of h_i, then we get a factorization of h_i

$$\pi^I(X/k,x) \xrightarrow{\ f\ } \!\!\!\!\rightarrow H \xhookrightarrow{\ g\ } G$$

and a commutative diagram

$$\begin{tikzcd}
 & (\widetilde{X}_x, \pi^I(X/k,x), \tilde{x}) \arrow[dl, two heads, "(\psi, f)"'] \arrow[dr, "(\phi_i, h_i)"] & \\
(Q, H, q) \arrow[rr, hook, "(\varphi, g)"] & & (P, G, p)
\end{tikzcd},$$

where $Q := \widetilde{X}_x \times_f^{\pi^I(X/k,x)} H$ is the contracted producted along f, and ψ, φ are canonical maps induced by the contracted product. Let $j := (Q, H, q)$. There is a projection map

$$(\phi_j, h_j) : (\widetilde{X}_x, \pi^I(X/k,x), \tilde{x}) \longrightarrow (Q, H, q)$$

which also makes the above diagram commutative after replacing (ψ, f) by (ϕ_j, h_j). Since φ and g are closed imbeddings, we must have $(\psi, f) = (\phi_j, h_j)$. Hence the affine ring of $\pi^I(X/k,x)$ is in fact a filtered inductive limit of its sub Hopf-algebras which are induced by those $j \in I(X/k,x)$ whose h_j are surjective, and the same thing happens for $\widetilde{X}_x$. This implies that if there is another morphism

$$(\phi, h) : (\widetilde{X}_x, \pi^I(X/k,x), \tilde{x}) \to (P, G, p)$$

then we can find an index $i' := (P', G', p') \in I$ such that ϕ, h factor through the projection morphisms

$$\phi_{i'} : \widetilde{X}_x \twoheadrightarrow P' \qquad \text{and} \qquad h_{i'} : \pi^I(X/k,x) \twoheadrightarrow G',$$

in other words, we have a commutative diagram

$$\begin{tikzcd}
 & (\widetilde{X}_x, \pi^I(X/k,x), \tilde{x}) \arrow[dl, "(\phi_{i'}, h_{i'})"'] \arrow[dr, "(\phi, h)"] & \\
(P', G', p') \arrow[rr, "(\varphi, g)"] & & (P, G, p)
\end{tikzcd}.$$

But by the very definition of a projective limit, we know that $(\varphi, g) \circ (\phi_{i'}, h_{i'}) = (\phi_i, h_i)$. Thus $(\phi_i, h_i) = (\phi, h)$. This completes the proof. □

Corollary 2.2. *Let* $\mathrm{Hom}_{\mathrm{grp.sch}}(\pi^N(X/k,x), -)$ *be the category whose objects are finite k-group schemes equipped with k-group scheme homomorphisms from $\pi^N(X/k,x)$, and whose morphisms are k-group scheme homomorphisms which are comptible with the homomorphisms from $\pi^N(X/k,x)$. Then there is an equivalence of categories*

$$\mathrm{Hom}_{\mathrm{grp.sch}}(\pi^N(X/k,x), -) \xrightarrow{\cong} N(X/k,x).$$

A similar statement holds if one replaces $N(X/k,x)$ by some smaller cofiltered subcategory.

Proof. Given a k-group scheme homomorphism $f : \pi^N(X/k,x) \to G$, we get a contracted product

$$(\widetilde{X}_x \times_f^{\pi^N(X/k,x)} G, G, \tilde{x}) \in N(X/k,x),$$

and given a morphism in $\mathrm{Hom}_{\mathrm{grp.sch}}(\pi^N(X/k,x),-)$, we get a morphism in $N(X/k,x)$ defined by the universal property of the contracted product. This defines a functor

$$\mathrm{Hom}_{\mathrm{grp.sch}}(\pi^N(X/k,x),-)\xrightarrow{\cong}N(X/k,x).$$

The quasi-inverse of this functor is given by 2.1. ☐

Definition 4. [2] Let X be a reduced connected scheme over a field k, $x:S\to X$ be a morphism of k-schemes with S non-empty, $I(X/k,x)\subseteq N(X/k,x)$ be a cofiltered full subcategory. We call a triple $(P,G,p)\in I(X/k,x)$ an I-saturated object if the corresponding projection map $\pi^I(X/k,x)\to G$ is surjective.

Lemma 2.3. *Let X be a reduced connected scheme over a field k, $x:S\to X$ be a morphism of k-schemes with $S\neq\varnothing$, $I(X/k,x)\subseteq N(X/k,x)$ be a cofiltered full subcategory. Then the full subcategory of $I(X/k,x)$ consisting of I-saturated objects is cofinal in $I(X/k,x)$, i.e. for any object $(P,G,p)\in I(X/k,x)$ there is a morphism*

$$(Q,H,q)\to(P,G,p)\in I(X/k,x)$$

where (Q,H,q) is an I-saturated object. So when we study projective limits indexed by $I(X/k,x)$ we can restrict ourselves to this smaller category of I-saturated objects.

Proof. Given a triple $(P,G,p)\in I(X/k,x)$ we get a homomorphism $\pi^I(X/k,x)\to G$. Since $\pi^I(X/k,x)$ and G are affine group schemes, there is a unique decomposition

$$\pi^I(X/k,x)\twoheadrightarrow H\subseteq G.$$

By 2.2 we have a morphism $(Q,H,q)\subseteq(P,G,p)\in I(X/k,x)$, where (Q,H,q) corresponds to the surjection $\pi^I(X/k,x)\twoheadrightarrow H$. This finishes the proof. ☐

2.2. Relations among π^N, π^L, π^E, π^G.

Definition 5. There are various choices of $I(X/k,x)\subseteq N(X/k,x)$. We will list some of them which will be frequently used in the rest of this paper.

(i) $\pi^N(X/k,x):=\pi^I(X/k,x)$ when $I(X/k,x)=N(X/k,x)$;

(ii) $\pi^E(X/k,x):=\pi^I(X/k,x)$ when $I(X/k,x)=I_{\text{ét}}(X/k,x)$ is the subcategory consisting of triples (P,G,p) where G is an *étale group scheme* over k;

(iii) $\pi^G(X/k,x):=\pi^I(X/k,x)$ when $I(X/k,x)=I_{\mathrm{co}}(X/k,x)$ is the subcategory consisting of triples (P,G,p) where G is a *constant group scheme* over k;

(iv) $\pi^L(X/k,x):=\pi^I(X/k,x)$ when $I(X/k,x)=I_{\mathrm{lc}}(X/k,x)$ is the subcategory consisting of triples (P,G,p) where G is a *local* (i.e. *connected*) *group scheme.*

[2]The terminology *saturated* is taken from [EHV]. We also used it in [Zh]. In [Nori] such objects are called *reduced*.

Remark 2.4. (i) As we have seen in 1.3 (iv), $\pi^E(X/k,x)$ can be defined without the assumption that X is reduced.

(ii) When $x : S \to X$ is taken to be a geometric point in $X(\bar{k})$, $\pi^G(X/k,x)$ is a profinite affine group scheme whose group of k-points is just Grothendieck's étale fundamental group $\pi_1^{\text{ét}}(X,x)$. The only difference between $\pi_1^{\text{ét}}(X,x)$ and $\pi^G(X/k,x)$ is that $\pi_1^{\text{ét}}(X,x)$ is a projective limit of finite groups, where the limit is taken in the category of topological groups in which each finite group has the discrete topology while $\pi^G(X/k,x)$ is a projective limit of finite groups, where the limit is taken in the category of affine group schemes in which each finite group is regarded as a constant group scheme over k. In other words, $\pi^G(X/k,x)$ is none other than a linearization of $\pi_1^{\text{ét}}(X,x)$.

Lemma 2.5. *Let $I_1 \subseteq I_2 \subseteq N(X/k,x)$ be two cofiltered full subcategories and (P,G,p) be an object in I_1. If for any imbedding*

$$(Q,H,q) \hookrightarrow (P,G,p) \in I_2$$

(i.e. $H \subseteq G$ is a subgroup), $(P,G,p) \in I_1$ implies $(Q,H,q) \in I_1$, then we have a surjection

$$\pi^{I_2}(X/k,x) \twoheadrightarrow \pi^{I_1}(X/k,x).$$

Proof. Let $(P,G,p) \in I_1$ be an I_1-saturated object. Then we can take the image of the composition

$$\pi^{I_2}(X/k,x) \to \pi^{I_1}(X/k,x) \twoheadrightarrow G$$

and denote it by H. By 2.2 we get an inclusion $(Q,H,q) \hookrightarrow (P,G,p) \in I_2$. So by the assumption this inclusion lives in I_1. This implies that the surjection $\pi^{I_1}(X/k,x) \twoheadrightarrow G$ factors through $H \hookrightarrow G$. Thus $H = G$. This concludes the proof. $\square$

Proposition 2.6. *The following natural k-group scheme homomorphisms*

(i) $\pi^N(X/k,x) \twoheadrightarrow \pi^E(X/k,x) \twoheadrightarrow \pi^G(X/k,x)$
(i) $\pi^N(X/k,x) \twoheadrightarrow \pi^L(X/k,x)$
(i) $\pi^N(X/k,x) \twoheadrightarrow \pi^E(X/k,x) \times_k \pi^L(X/k,x)$

are all surjections.

Proof. In the view of 2.5, only the last statement needs to be explained. For this, we take in 2.5 $I_2 := N(X/k,x)$ and I_1 to be the triples (P,G,p) whose group G is isomorphic to a direct product of an étale k-group scheme and a local k-group scheme, i.e. $G = G^0 \times_k G_{\text{ét}}$. Now suppose $H \subseteq G$ is a subgroup scheme. Then the connected-étale sequence for H splits because $H_{\text{red}} \subseteq G_{\text{red}} = G_{\text{ét}} \Rightarrow H_{\text{red}} = H_{\text{ét}}$. But since $G_{\text{ét}}$ acts trivially on G^0 and the action of $H_{\text{ét}}$ on H^0 is compatible with that of $G_{\text{ét}}$ on G^0, $H_{\text{ét}}$ must act trivially on H^0, or in other words, $H = H^0 \times_k H_{\text{ét}}$. $\square$

Exemple 2.7. Here we want to point out that all the above surjections are, in general, not isomorphisms.

(i). For $\pi^E_G : \pi^E(X/k,x) \twoheadrightarrow \pi^G(X/k,x)$. Let's take $X = \operatorname{Spec}(k) = \operatorname{Spec}(\mathbb{Q})$, $\bar{x} : \operatorname{Spec}(\bar{\mathbb{Q}}) \to \operatorname{Spec}(\mathbb{Q})$ is the natural field extension. Let $\alpha \in \mathbb{Q}$, $n \in \mathbb{N}\backslash\{0\}$, and suppose $x^n - \alpha$ has no root in $\mathbb{Q}$, then $P := \operatorname{Spec}(\mathbb{Q}[x]/(x^n - \alpha))$ is a non-trivial μ_n-torsor over $\mathbb{Q}$. Choosing any point $p \in P(\bar{\mathbb{Q}})$, we get a triple $(P,\mu_n,p) \in N(X/k,\bar{x})$. Let $\varphi : \pi^E(X/k,\bar{x}) \to$

μ_n be the homomorphism corresponding to (P, μ_n, p) as in 2.2. If the map π^E_G was an isomorphism, then there should be a k-group scheme homomorphism $\phi : \pi^G(X/k, \bar{x}) \to \mu_n$ satisfying $\phi \circ \pi^E_G = \varphi$. But since $\pi^G(X/k, \bar{x})$ is a cofiltered projective limit of finite constant group schemes, there must be a factorization

$$\begin{tikzcd} & H \arrow[dd, dotted, "\lambda"] \\ \pi^G(X/k, \bar{x}) \arrow[ur, two heads] \arrow[dr, "\phi"'] & \\ & \mu_n \end{tikzcd},$$

where H is a constant group scheme. But when n is a prime number, μ_n is a $\mathbb{Q}$-scheme of two connected components. Thus the fact that P is a non-trivial torsor would imply that φ is surjective, and so is ϕ. Therefore, the map $\lambda : H \to \mu_n$ should also be sujective, and hence μ_n has to be a constant group scheme. But this is not the case when $n > 2$.
(ii). For $\pi^N_E : \pi^N(X/k, x) \twoheadrightarrow \pi^E(X/k, x)$. If it was an isomorphism then any torsor with local group scheme will be dominated by an étale torsor, then the local torsor has to be trivial. Hence any non-trivial local torsor gives a counterexample. Yet we would like to point out that if $X = \mathrm{Spec}\,(k)$ where k is perfect, $\bar{x} : \mathrm{Spec}\,(\bar{k}) \to \mathrm{Spec}\,(k)$ is the natural field extension, then π^N_E is an isomorphism (see 2.8). But if k is not perfect and $\mathrm{char}(k) = p$, one can choose $\alpha \in \bar{k}$ such that $\alpha \notin k$ but $\alpha^p \in k$. Thus the field extension $k \subseteq k(\alpha)$ is a non-trivial μ_p-torsor.
(iii). For $\pi^N_L : \pi^N(X/k, x) \twoheadrightarrow \pi^L(X/k, x)$. As in (ii) any non-trivial étale torsor provides a counterexample. And also (iii) is implied by (iv).
(iv). For $\pi^N(X/k, x) \twoheadrightarrow \pi^E(X/k, x) \times_k \pi^L(X/k, x)$. There is a perfect counterexample in [EPS][Remark 4.3].

2.3. **The Nori-Galois group of a Field.**

Definition 6. Let k be a field, $\bar{x}$ be the map $\mathrm{Spec}\,(\bar{k}) \to \mathrm{Spec}\,(k)$ coresponding to the natural field extension $k \subseteq \bar{k}$. We call $\pi^N(k/k, \bar{x})$ the Nori-Galois group of k.

Proposition 2.8. *Let k be a perfect field, $\bar{x} : \mathrm{Spec}\,(\bar{k}) \to \mathrm{Spec}\,(k)$ be the natural field extension $k \subseteq \bar{k}$. Then the canonical surjection*

$$\pi^N_E : \pi^N(k/k, \bar{x}) \longrightarrow \pi^E(k/k, \bar{x})$$

is an isomorphism.

Proof. Let $(P, G, p) \in N(k/k, \bar{x})$ be an object. Then there is a canonical isomorphism $P \times_k G \cong P \times_k P$. Let P_{red} be the reduced closed subscheme of P and G_{red} be the reduced closed subscheme of G. As k is perfect,

$$P_{\mathrm{red}} \times_k G_{\mathrm{red}} \subseteq P \times_k G \qquad \text{and} \qquad P_{\mathrm{red}} \times_k P_{\mathrm{red}} \subseteq P \times_k P$$

are the unique reduced closed subschemes of the underlying spaces. This induces a diagram

$$\begin{tikzcd} P_{\mathrm{red}} \times_k G_{\mathrm{red}} \arrow[r] \arrow[d] & P_{\mathrm{red}} \times_k P_{\mathrm{red}} \arrow[d] \\ P \times_k G \arrow[r, "\cong"] & P \times_k P \end{tikzcd}$$

in which the upper horizontal arrow is an isomorphism. But G_{red} is étale, as k is perfect. Therefore, we get a morphsim

$$(P_{\mathrm{red}}, G_{\mathrm{red}}, p) \subseteq (P, G, p) \in N(k/k, \bar{x})$$

where $(P_{\mathrm{red}}, G_{\mathrm{red}}, p) \in I_{\text{ét}}(k/k, x)$. Hence $I_{\text{ét}}(k/k, x)$ is cofinal inside $N(k/k, x)$. Thus π_E^N is an isomorphism. $\square$

Corollary 2.9. *Assumptions and notations being as in 2.8, we have*

$$\pi^L(k/k, \bar{x}) = \{1\}.$$

Proof. Let $(P, G, p) \in I_{\mathrm{lc}}(k/k, \bar{x})$ be an object. Then, as in the proof of 2.8, we see that there is an imbedding

$$(P_{\mathrm{red}}, G_{\mathrm{red}}, p) \subseteq (P, G, p) \in N(k/k, \bar{x}).$$

But since G is connected, $(P_{\mathrm{red}}, G_{\mathrm{red}}, p)$ is just the trivial triple. This finishes the proof. $\square$

Proposition 2.10. *Let k be a separably closed field, and $\bar{x} : \mathrm{Spec}\,(\bar{k}) \to \mathrm{Spec}\,(k)$ be the natural field extension. Then we have*

$$\pi^E(k/k, \bar{x}) = \{1\},$$

and the canonical surjection

$$\pi_L^N : \pi^N(k/k, \bar{x}) \longrightarrow \pi^L(k/k, \bar{x})$$

is an isomorphism.

Proof. Let $(P, G, p) \in N(k/k, \bar{x})$ be an object, $G_{\text{ét}}$ be the maximal étale quotient of G. Then the quotient map $h : G \twoheadrightarrow G_{\text{ét}}$ induces, by 2.2, a triple $(P_{\text{ét}}, G_{\text{ét}}, p) \in I_{\text{ét}}(k/k, \bar{x})$ and a morphism

$$(\phi, h) : (P, G, p) \twoheadrightarrow (P_{\text{ét}}, G_{\text{ét}}, p) \in N(k/k, \bar{x}).$$

Since $P_{\text{ét}}$ is an étale scheme over a separably closed field, every point of $P_{\text{ét}}$ is a k-rational point. This means that $P_{\text{ét}}$ is a trivial $G_{\text{ét}}$-torsor, and hence $\pi^E(k/k, \bar{x}) = \{1\}$. Now we can pull back the map $\phi : P \to P_{\text{ét}}$ along the k-rational point $p \in P_{\text{ét}}(k)$. Then we get a triple $(P^0, G^0, p) \in I_{\mathrm{lc}}(k/k, \bar{x})$ and a morphism

$$(P^0, G^0, p) \hookrightarrow (P, G, p) \in N(k/k, \bar{x}).$$

This means that $I_{\mathrm{lc}}(k/k, \bar{x})$ is cofinal inside $N(k/k, \bar{x})$. By the same argument as in 2.8, we see that π_L^N is an isomorphism. $\square$

Proposition 2.11. *Let k be a field, X be a complete rational variety over $\bar{k}$, $n \in \mathbb{N}^+$, $x : S \to X$ be any morphism with S connected and non-empty. Then we have*

$$\pi^N(X/k, x) = \pi^E(X/k, x) = \pi^L(X/k, x) = \pi^G(X/k, x) = \{1\}.$$

Proof. Let $(P, G, p) \in N(X/k, x)$ be an object. Then by [Nori, Chapter II, lemma, pp. 92] plus Künneth formula [MS, Theorem 2.3], P is a trivial G-torsor, i.e. $P \cong X \times_k G$. Since S is connected, it is mapped to a connected component Q of $X \times_k G$ via $p : S \to P$. As $X \times_k G \cong X \times_{\bar{k}} \bar{G}$, the composition

$$Q_{\text{red}} \subseteq Q \subseteq X \times_k G \to X$$

must be an isomorphism, thus the map p factors through a section of the structure map $P \to X$. This means that there is a unique morphism

$$(X, \{1\}, x) \to (P, G, p) \in N(X/k, x).$$

Therefore $(X, \{1\}, x)$ is a cofinal object in $N(X/k, x)$. By 3, $\pi^N(X/k, x) = \{1\}$. □

Remark 2.12. The connectedness assumption on S in the above proposition is quite important. Otherwise, we could take $x : P \to X$ to be the natural projection, $P := \mathbb{P}^1_{\bar{k}} \coprod \mathbb{P}^1_{\bar{k}}$ to be the trivial torsor under $\mathbb{Z}/2\mathbb{Z}$, and $p : P \to P$ to be the identity. In this way, there is no morphism $(X, \{1\}, x) \to (P, G, p) \in N(X/k, x)$. Thus the homomorphism $\pi^N(X/k, x) \to (\mathbb{Z}/2\mathbb{Z})_k$ corresponding to $(P, \mathbb{Z}/2\mathbb{Z}, p)$ is not tivial, but surjective. Therefore, $\pi^N(X/k, x)$ is not trivial.

Proposition 2.13. *Let k be a field of characteristic 0, $X := \mathbb{A}^n_{\bar{k}}$, $n \in \mathbb{N}^+$, $x : S \to X$ be any morphism with S connected and non-empty. Then we have*

$$\pi^N(X/k, x) = \pi^E(X/k, x) = \pi^L(X/k, x) = \pi^G(X/k, x) = \{1\}.$$

Proof. The point is that in this case any finite torsor over X is étale and any étale torsor over X is trivial. Then we do 2.11 again. □

2.4. **The Étale Piece of the Arithmetic Fundamental Group Scheme.**

Theorem 2.14. *Let $\mathbb{R}$ be the field of real numbers, $\bar{x} : \mathrm{Spec}\,(\mathbb{C}) \to \mathrm{Spec}\,(\mathbb{R})$ be the morphism corresponding to the natural inclusion $\mathbb{R} \subset \mathbb{C}$. Then*

$$\pi^E(\mathbb{R}/\mathbb{R}, \bar{x}) = \varprojlim_{n \in \mathbb{N}^+} \mu_{n,\mathbb{R}}$$

is an infinite $\mathbb{R}-$group scheme, and the universal covering corresponding to $\pi^E(\mathbb{R}/\mathbb{R}, \bar{x})$ is a non-Noetherian affine scheme with infinitely many connected components.

Proof. Let $(P, G, p) \in I_{\text{ét}}(\mathbb{R}/\mathbb{R}, \bar{x})$. Then $P(\mathbb{C})$ is a principal homogeous space under $G(\mathbb{C})$. By Galois descent, there is an action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \langle \sigma \rangle$ on $P(\mathbb{C})$ via set-theoretical automorphisms and an action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \langle \sigma \rangle$ on $G(\mathbb{C})$ via group automorphisms such that these two actions are compatible. Let $\sigma(p) = pa$ for some $a \in G(\mathbb{C})$, $n \in \mathbb{N}^+$ denote the order of a. Then for any $b \in G(\mathbb{C})$, $\sigma(pb) = \sigma(p)\sigma(b) = pa\sigma(b)$. But $\sigma^2 = id$ is trivial, so $pb = \sigma^2(pb) = \sigma(pa\sigma(b)) = \sigma(p)\sigma(a)b = pa\sigma(a)b$. Thus $a\sigma(a) = e$ is trivial, so $\sigma(a) = a^{-1}$. Let $Q_n(\mathbb{C}) \subseteq P(\mathbb{C})$ be the subset $\{pa^i | i \in \mathbb{N}\}$, $H_n(\mathbb{C}) \subseteq G(\mathbb{C})$ be the subgroup $\langle a \rangle$. These substructures are clearly stable under the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-actions, so they descend to $\mathbb{R}$, i.e. we have a subobject

$$(Q_n, H_n, p) \subseteq (P, G, p) \in I_{\text{ét}}(\mathbb{R}/\mathbb{R}, \bar{x}),$$

where the set of $\mathbb{C}$-points of Q_n is $Q_n(\mathbb{C})$ and the group of $\mathbb{C}$-points of H_n is $H_n(\mathbb{C})$. Let

$$(P_n, \mu_{n,\mathbb{R}}, p_n) := (\mathrm{Spec}\,(\mathbb{R}[x]/(x^n+1)), \mathrm{Spec}\,(\mathbb{R}[x]/(x^n-1)), \mathrm{e}^{\frac{(2n-1)\pi i}{n}}) \in I_{\text{ét}}(\mathbb{R}/\mathbb{R}, \bar{x})$$

where the action of $\mu_{n,\mathbb{R}}$ on P_n is defined simply by multiplying a $n-$th root of unity on a root of $x^n + 1 = 0$ in $\mathbb{C}$ and $\mathrm{e}^{\frac{(2n-1)\pi i}{n}}$ is the n-th root $\cos(\frac{(2n-1)\pi i}{n}) + i\sin(\frac{(2n-1)\pi i}{n})$. By sending $a \mapsto \mathrm{e}^{\frac{2\pi i}{n}}$ we get an isomorphism $h : H_n \cong \mu_{n,\mathbb{R}} = \mathrm{Spec}\,(\mathbb{R}[x]/(x^n-1))$. By sending $p \mapsto \mathrm{e}^{\frac{(2n-1)\pi i}{n}}$ we get an isomorphism of $\mathbb{R}$-schemes $\phi : Q_n \cong \mathrm{Spec}\,(\mathbb{R}[x]/(x^n+1))$ which is compatible with h under the actions. This means that the full subcategory of $I_{\text{ét}}(\mathbb{R}/\mathbb{R}, \bar{x})$ consisting of objects of the form $(P_n, \mu_{n,\mathbb{R}}, p_n)$ is cofinal.

On the other hand, the triple $(P_n, \mu_{n,\mathbb{R}}, p_n)$ is $I_{\text{ét}}$-saturated. If we have a subobject

$$(Q, H, p) \subseteq (P_n, \mu_{n,\mathbb{R}}, p_n) \in I_{\text{ét}}(\mathbb{R}/\mathbb{R}, \bar{x})$$

then $p_n = p \in Q(\mathbb{C})$ implies $\mathrm{e}^{\frac{\pi i}{n}} \in Q(\mathbb{C})$ for $Q(\mathbb{C})$ should always contain the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-orbit, i.e. the complex conjugation, of $p = p_n = \mathrm{e}^{\frac{(2n-1)\pi i}{n}}$. Therefore, by the equation

$$p_n \mathrm{e}^{\frac{2\pi i}{n}} = \mathrm{e}^{\frac{(2n-1)\pi i}{n}} \cdot \mathrm{e}^{\frac{2\pi i}{n}} = \mathrm{e}^{\frac{\pi i}{n}}$$

we have $\mathrm{e}^{\frac{2\pi i}{n}} \in H(\mathbb{C})$. Since $H(\mathbb{C})$ contains the generator of the n-th cyclic group $\mu_{n,\mathbb{R}}(\mathbb{C})$, we have $H(\mathbb{C}) = \mu_{n,\mathbb{R}}(\mathbb{C})$. Or equivalently, $H = \mu_{n,\mathbb{R}}$ and $Q = P_n$. Thus $(P_n, \mu_{n,\mathbb{R}}, p_n)$ is an $I_{\text{ét}}$-saturated object.

Now if $m, n \in \mathbb{N}^+$ and $m|n$, then we can define a "raise to $\frac{n}{m}$-power" map

$$(P_n, \mu_{n,\mathbb{R}}, p_n) \to (P_m, \mu_{m,\mathbb{R}}, p_m)$$

by sending $x \mapsto x^{\frac{n}{m}}$ in the affine coordinate ring. This defines a projective system in $I_{\text{ét}}(\mathbb{R}/\mathbb{R}, \bar{x})$. By taking projective limit in the category of affine schemes (resp. group schemes) over $\mathbb{R}$, we get a triple

$$(\varprojlim_{n\in\mathbb{N}^+} P_n, \varprojlim_{n\in\mathbb{N}^+} \mu_{n,\mathbb{R}}, \tilde{p}).$$

Let $(\widetilde{X}_{\bar{x}}, \pi^E(\mathbb{R}/\mathbb{R}, \bar{x}), \tilde{x})$ be the universal triple defined in 2.1. Then by the universality, we get a morphism

$$(\widetilde{X}_{\bar{x}}, \pi^E(\mathbb{R}/\mathbb{R}, \bar{x}), \tilde{x}) \longrightarrow (\varprojlim_{n\in\mathbb{N}^+} P_n, \varprojlim_{n\in\mathbb{N}^+} \mu_{n,\mathbb{R}}, \tilde{p})$$

which is indeed an isomorphism because of the fact that $\{(P_n, \mu_{n,\mathbb{R}}, p_n)|n \in \mathbb{N}^+\}$ is cofinal and saturated in $I_{\text{ét}}(\mathbb{R}/\mathbb{R}, \bar{x})$. This proves that $\pi^E(\mathbb{R}/\mathbb{R}, \bar{x})$ is infinite and also that $\widetilde{X}_{\bar{x}}$ has infinitely many connected components. Since $\widetilde{X}_{\bar{x}}$ is affine, it must be quasi-compact. But then the connected components of $\widetilde{X}_{\bar{x}}$ can not be open, otherwise there should be finitely many of them. Therefore $\widetilde{X}_{\bar{x}}$ is not Noetherian. $\square$

Proposition 2.15. *Let k be field whose Galois group* $\mathrm{Gal}(\bar{k}/k)$ *admits* $\mathbb{Z}/l\mathbb{Z}$ *as a quotient for some prime number $l > 3$. Let $X = \mathrm{Spec}\,(k)$, $\bar{x} : \mathrm{Spec}\,(\bar{k}) \to X$ be a geometric point. Then $\pi^E(k/k, \bar{x})$ is a non-commutative k-group scheme.*

Proof. Let $k \subseteq K \subseteq \bar{k}$ a finite Galois subextension so that $\mathrm{Gal}(K/k) = \langle \sigma \rangle \cong \mathbb{Z}/l\mathbb{Z}$. Let $G_K := (\mathbb{Z}/l\mathbb{Z} \times \mathbb{Z}/l\mathbb{Z}) \rtimes \langle b \rangle$, where $\langle b \rangle \cong \mathbb{Z}/l\mathbb{Z}$ acts on $\mathbb{Z}/l\mathbb{Z} \times \mathbb{Z}/l\mathbb{Z}$ by

$$b \quad \longmapsto \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We define an action of $\mathrm{Gal}(K/k)$ on G_K by letting $\sigma(z) = z$ for all $z \in \mathbb{Z}/l\mathbb{Z} \times \mathbb{Z}/l\mathbb{Z}$ and $\sigma(b) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} b$. This action corresponds, by Galois descent, to a k-group scheme G which is a k-form of the K-group scheme G_K.

The constant K-group scheme G_K can be written as

$$G_K := \coprod_{i \in G_K} Y_i$$

where $Y_i = \mathrm{Spec}\,(K)$. G_K acts on itself by right translations, i.e. for any $j \in G_K$, j acts on Y_i by the identity map $\mathrm{Spec}\,(K) = Y_i \to Y_{ij} = \mathrm{Spec}\,(K)$. Now we define a twisted action of $\mathrm{Gal}(K/k)$ on the K-scheme G_K. We define the action of σ on Y_i to be the morphism τ in the following commutative diagram

$$\begin{tikzcd} Y_i \arrow[r, "\tau"] \arrow[d, equal] & Y_{b\sigma(i)} \arrow[d, equal] \\ \mathrm{Spec}\,(K) \arrow[r, "t_\sigma"] & \mathrm{Spec}\,(K) \end{tikzcd}$$

where t_σ is the map obtained by applying the functor $\mathrm{Spec}\,(-)$ to the field automorphism $\sigma : K \to K$. We have the following compatiblity between the action of $\mathrm{Gal}(K/k)$ on the K-group scheme G_K and that on the K-scheme G_K, i.e. the diagram

$$\begin{tikzcd} Y_i \arrow[r, "j"] \arrow[d, "\tau"'] & Y_{ij} \arrow[d, "\tau"] \\ Y_{b\sigma(i)} \arrow[r, "\sigma(j)"] & Y_{b\sigma(ij)} \end{tikzcd}$$

is commutative for any $j \in G_K$. By Galois descent, the K-scheme G_K descends to a k-scheme P and there is an action of G on P which makes P a G-torsor over k. Picking $p \in Y_e(\bar{k})$ to be the inclusion $K \subseteq \bar{k}$, we get an object $(P, G, p) \in I_{\text{ét}}(k/k, \bar{x})$. This object induces a k-homomorphism

$$\lambda : \pi^E(k/k, \bar{x}) \longrightarrow G.$$

Let $N \subseteq G$ be the image. Then we get a subobject $(Q, N, p) \subseteq (P, G, p)$. As $p \in Q$, $Y_e \subseteq Q_K \Rightarrow Y_b = \sigma(Y_e) \subseteq Q_K \Rightarrow b \in N_K \subseteq G_K$. But $N_K \subseteq G_K$ is stable under the Galois action, so $\sigma(b) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} b \in N_K \Longrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in N_K \Longrightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} = b \begin{pmatrix} 0 \\ 1 \end{pmatrix} b^{l-1} \in N_K$.

Thus $N_K = G_K$ for $\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, b\}$ generates G_K. Therefore λ is surjective. Then $\pi^E(k/k, \bar{x})$ must be non-commutative for G is. $\square$

Remark 2.16. The point of the assumption $l > 3$ is that one needs the action of σ^l on G_K to be trivial, i.e. one needs that $b\sigma(b)\sigma^2(b)\cdots\sigma^{l-1}(b)$ to be trivial in G_K. For this one needs

$$1^2 + 2^2 + 3^2 + \cdots + (l-1)^2 = \frac{1}{6}(l-1)l(2l-1)$$

to be divisible by l. This is OK only when the prime number $l > 3$.

Exemple 2.17. The notion of $\pi_1^{\text{ét}}(X, x)$ is *absolute*, i.e. it has no reference to the base field, so X could even be a scheme of mixed characteristic. The notion of $\pi^N(X, x)$ in [Nori] depends only on the base field where X is defined. However, the fundamental group we are considering here depends also on the field where the group structure is defined.

By a theorem of Serre-Lang, it is known that for $X/\bar{k}$ an abelian variety $\pi_1^{\text{ét}}(X, 0) = \varprojlim_{n\in\mathbb{N}^+} X[n](\bar{k})$, or more generally, Nori proved in [Nori2] that $\pi^N(X, 0) = \varprojlim_{n\in\mathbb{N}^+} X[n]$. Since our fundamental group is a generalization of [Nori], we still have $\pi^N(X/\bar{k}, 0) = \pi^N(X, 0) = \varprojlim_{n\in\mathbb{N}^+} X[n]$. However, if we see X as a scheme over k via $X \to \operatorname{Spec}(\bar{k}) \to \operatorname{Spec}(k)$ then we really get something different. In this example we take an abelian variety X over $\mathbb{C}$ and view it as a scheme over $\mathbb{R}$, then show that $\pi^N(X/\mathbb{R}, 0)$ is non-commutative.

Let A be an abelian variety over $k := \mathbb{R}$, $K := \mathbb{C}$, $\bar{x} \in A_K(K)$. Take any Galois covering $Y \to A_K$ with Galois group $\mathbb{Z}/2\mathbb{Z}$. Let $G_K := \langle a\rangle \rtimes \langle b\rangle$, where $\langle a\rangle \cong \mathbb{Z}/n\mathbb{Z}$ for $n \geqslant 3 \in \mathbb{N}^+$ and $\langle b\rangle \cong \mathbb{Z}/2\mathbb{Z}$ acts on $\langle a\rangle$ by $b(z) = z^{-1}$ for all $z \in \langle a\rangle$. We define an action of $\mathrm{Gal}(K/k)$ on G_K by letting $\sigma(z) = z^{-1}$ for all $z \in \langle a\rangle$ and $\sigma(b) = ab$. Then there is a k-form G of the K-group scheme G_K which corresponds to this action.

Let $H_K \subset G_K$ denote the subgroup $\langle a\rangle$, and let

$$P_K := \coprod_{i\in H_K} Y_i$$

where $Y_i = Y$. Now we define an action of G_K on P_K. Take any $g \in G_K$, we can write it uniquely as $g = b^r j$, where $r \in \{0, 1\}$ and $j \in H_K$, then the action of g on Y_i is defined to be the morphism τ in the following commutative diagram

$$\begin{tikzcd} Y_i \arrow[r, "\tau"] \arrow[d, equal] & Y_{b^{2-r}(i)j} \arrow[d, equal] \\ Y \arrow[r, "b^r"] & Y \end{tikzcd}$$

where b^r is the map defined by the non-trivial A_K-automorphism of Y if $r = 1$, the identity if $r = 0$. In this way, P_K becomes a G_K-torsor over A_K. Now viewing G_K as a constant K-group scheme we get a morphism

$$\rho : P_K \times_{\operatorname{Spec}(K)} G_K \longrightarrow P_K$$

defined by the above action. Composing ρ with the following isomorphism

$$P_K \times_{\mathrm{Spec}\,(K)} (\mathrm{Spec}\,(K) \times_{\mathrm{Spec}\,(k)} G) \xrightarrow{\cong} P_K \times_{\mathrm{Spec}\,(K)} G_K$$

we get an action of G on P_K which makes P_K a G-torsor over A_K. Picking any k-morphism $p : \mathrm{Spec}\,(K) \to Y_e = Y$ (where $e \in H_K$ is the identity element) over $\bar{x}$, we get an object $(P_K, G, p) \in I_{\text{ét}}(A_K/k, \bar{x})$. This object induces a k-homomorphism

$$\lambda : \qquad \pi^E(A_K/k, \bar{x}) \longrightarrow G.$$

Let $N \subseteq G$ be the image. Then we get a subobject $(Q, N, p) \subseteq (P_K, G, p)$. As $p \in Q$, $Y_e \subseteq Q$. Thus $b \in N_K \subseteq G_K$ (because $Y_e \subseteq P_K$ is a torsor under $\langle b \rangle \subseteq G_K$). But $N_K \subseteq G_K$ is stable under the Galois action, so $\sigma(b) = ab \in N_K \Longrightarrow a \in N_K \Longrightarrow N_K = G_K$. Therefore λ is surjective. Then $\pi^N(A_K/k, \bar{x}) = \pi^E(A_K/k, \bar{x})$ must be non-commutative for G is.

2.5. **Comparison of the Geometric Fundamental Groups.** Let X be a separable geometrically connected scheme over a field k, and $\bar{x} : \mathrm{Spec}\,(\bar{k}) \hookrightarrow X$ be a geometric point. Associated to the arithmetic fundamental group scheme $\pi^N(X/k, \bar{x})$, there are two geometric fundamental group schemes $\pi^N(\bar{X}/k, \bar{x})$ and $\pi^N(\bar{X}/\bar{k}, \bar{x})$, the later being the *classical* Nori's fundamental group. We would like to understand the relation between these two.

Proposition 2.18. *Notations and assumptions being as above, if X is moreover quasi-compact and k is perfect, then we have an imbedding*

$$\chi^N_{X/k} : \qquad \pi^N(\bar{X}/\bar{k}, \bar{x}) \hookrightarrow \pi^N(\bar{X}/k, \bar{x}) \times_k \bar{k}$$

of $\bar{k}-$group schemes. A similar statement holds if one replaces N by E, G, L.

Proof. Given $(P, G, p) \in N(\bar{X}/k, \bar{x})$, $(P, G \times_k \bar{k}, p)$ is naturally an object in $N(\bar{X}/\bar{k}, \bar{x})$. In this way we get a functor F which makes the following diagram 2-commutative

$$\begin{tikzcd} N(\bar{X}/k, \bar{x}) \arrow[rr, "F"] \arrow[dr, "\varphi"'] & & N(\bar{X}/\bar{k}, \bar{x}) \arrow[dl, "\psi"] \\ & \mathrm{Grsch}(\bar{k}) & \end{tikzcd}$$

where φ is the functor sending $(P, G, p) \in N(\bar{X}/k, \bar{x})$ to $G \times_k \bar{k}$, and ψ is the forgetful functor sending $(Q, H, q) \in N(\bar{X}/\bar{k}, \bar{x})$ to H. Since base change is compatible with taking projective limit, we have

$$\varprojlim_{i \in N(\bar{X}/k, \bar{x})} \varphi(i) = \pi^N(\bar{X}/k, \bar{x}) \times_k \bar{k}.$$

Therefore, we get the homomorphism

$$\chi^N_{X/k} : \qquad \pi^N(\bar{X}/\bar{k}, \bar{x}) \longrightarrow \pi^N(\bar{X}/k, \bar{x}) \times_k \bar{k}$$

by passing to the limit. The injectivity of $\chi^N_{X/k}$ is proved in 4.1. $\square$

Proposition 2.19. *If X is a connected scheme over any field k with a geometric point $\bar{x} \in X(\bar{k})$, and if X is also a $\bar{k}$-scheme (e.g. $X = Y \times_k \bar{k}$ for some k-scheme Y), then the immbedding*

$$\chi^E_{X/\bar{k}/k} : \ \pi^G(X/k, \bar{x}) \times_k \bar{k} = \pi^G(X/\bar{k}, \bar{x}) = \pi^E(X/\bar{k}, \bar{x}) \hookrightarrow \pi^E(X/k, \bar{x}) \times_k \bar{k}$$

of $\bar{k}-$group schemes is a section of the quotient map

$$\pi^E_G \times_k \bar{k} : \ \pi^E(X/k, \bar{x}) \times_k \bar{k} \twoheadrightarrow \pi^G(X/k, \bar{x}) \times_k \bar{k}.$$

Proof. Let's first redo the construction in 2.18. Given $(P, G, p) \in I_{\text{ét}}(X/k, \bar{x})$, let G' be the abstract group associated to $G \times_k \bar{k}$. Viewing G' as a constant group scheme over k, we get an object $(P, G', p) \in I_{\text{co}}(X/k, \bar{x})$. In this way we get a functor which makes the following diagram 2-commutative

$$\begin{tikzcd}
I_{\text{ét}}(X/k, \bar{x}) \arrow[rr, "F"] \arrow[dr, "\varphi"'] & & I_{\text{co}}(X/k, \bar{x}) \arrow[dl, "\psi"] \\
& \mathrm{Grsch}(\bar{k}) &
\end{tikzcd}$$

where φ is the functor sending $(P, G, p) \in I_{\text{ét}}(X/k, \bar{x})$ to $G \times_k \bar{k}$, and ψ is the functor sending $(P, G, p) \in I_{\text{co}}(X/k, \bar{x})$ to the abstract group G regarded as a group scheme over $\bar{k}$. Since base change is compatible with projective limit, we have

$$\varprojlim_{i \in I_{\text{ét}}(X/k, \bar{x})} \varphi(i) = \pi^E(X/k, \bar{x}) \times_k \bar{k} \qquad \text{and} \qquad \varprojlim_{i \in I_{\text{co}}(X/k, \bar{x})} \psi(i) = \pi^G(X/k, \bar{x}) \times_k \bar{k}$$

This defines the homomorphism $\chi^E_{X/\bar{k}/k}$ which is then easily seen as a section of π^E_G, because the (right) composition of F with the inclusion

$$i : I_{\text{co}}(X/k, \bar{x}) \longrightarrow I_{\text{ét}}(X/k, \bar{x})$$

is isomorphic to the identity functor on $I_{\text{co}}(X/k, \bar{x})$. □

Remark 2.20. We have seen from 2.17 that both $\chi^N_{X/k}$ and $\chi^E_{X/\bar{k}/k}$ are not, in general, isomorphisms.

2.6. **The Geometric Base Point.**

Proposition 2.21. *Let X be any connected reduced scheme over k, $\bar{x}_1 : \mathrm{Spec}\,(\bar{l}_1) \to X$ and $\bar{x}_2 : \mathrm{Spec}\,(\bar{l}_2) \to X$ be two geometric points of X. Then there are (non-canonical) isomorphisms between the following k-group schemes:*

(i) $$\pi^E(X/k, \bar{x}_1) \cong \pi^E(X/k, \bar{x}_2)$$

(ii) $$\pi^L(X/k, \bar{x}_1) \cong \pi^L(X/k, \bar{x}_2)$$

(iii) $$\pi^N(X/k, \bar{x}_1) \cong \pi^N(X/k, \bar{x}_2).$$

Proof. Let's first examine (i). Let $I_{\text{ét}}(X/k)$ be the category of pairs (P, G), where P is a torsor over X under a finite étale k-group scheme G and let $\mathrm{Ecov}(X)$ be the category of finite étale coverings of X. Consider the following functors

$$I_{\text{ét}}(X/k) \xrightarrow{F} \mathrm{Ecov}(X) \underset{F_{\bar{x}_2}}{\overset{F_{\bar{x}_1}}{\rightrightarrows}} ((\mathrm{Sets}))$$

where F is the forgetful functor (forgetting the group) and $F_{\bar{x}_1}, F_{\bar{x}_2}$ are the fibre functors induced by $\bar{x}_1, \bar{x}_2$. The category $I_{\text{ét}}(X/k, \bar{x}_1)$ is just the opposite category of representable presheaves over the presheaf $F_{\bar{x}_1} \circ F$ on $I_{\text{ét}}(X/k)^{\mathrm{o}}$, i.e. its objects are pairs (A, a) where $A \in I_{\text{ét}}(X/k)$ and $a : A \to F_{\bar{x}_1} \circ F$ is a morphism of presheaves on $I_{\text{ét}}(X/k)^{\mathrm{o}}$. But from [SGA1, Exposé V, Corollaire 5.7, pp. 107], there is an isomorphism of functors $F_{\bar{x}_1} \cong F_{\bar{x}_2}$, hence an isomorphism $F_{\bar{x}_1} \circ F \cong F_{\bar{x}_2} \circ F$. Therefore we get an equivalence $I_{\text{ét}}(X/k, \bar{x}_1) \cong I_{\text{ét}}(X/k, \bar{x}_2)$ which is compatible with the forgetful functors to $I_{\text{ét}}(X/k)$. This gives the isomorphism (i).

The isomorphism (ii) is clear. The reason is that surjective purely inseparable morphisms are homeomorphisms on the underlying topological spaces.

Finally we consider the sequence

$$N(X/k) \xrightarrow{q} I_{\text{ét}}(X/k) \xrightarrow{F} \mathrm{Ecov}(X) \underset{F_{\bar{x}_2}}{\overset{F_{\bar{x}_1}}{\rightrightarrows}} ((\mathrm{Sets}))$$

where $N(X/k)$ the category of pairs (P, G) in which G is a finite k-group scheme, P is a torsor over X under G, and q is the the functor sending any pair (P, G) to its étale quotient $(P_{\text{ét}}, G_{\text{ét}})$. Now replacing $I_{\text{ét}}(X/k)$ by $N(X/k)$ and F by $F \circ q$ we can do the same argument as that in the proof of (i) to get the isomorphism (iii). □

Remark 2.22. From the proof of (ii) we see that actually in the definition of π^L the base point is not necessary, as for any two different base points $\bar{x_1}, \bar{x_2}$ of X, the isomorphism $\pi^L(X/k, \bar{x}_1) \xrightarrow{\cong} \pi^L(X/k, \bar{x}_2)$ is *canonical*.

2.7. **Base Change.**

Proposition 2.23. *Let X be a scheme geometrically connected proper separable over a field k, $k \subseteq l \subseteq l'$ be a sequence of field extensions, where l and l' are algebraically closed fields. Let $\bar{x} : \mathrm{Spec}\,(l') \to X$ be a geometric point. Then the following natural map*

$$\pi_l^{l'} : \pi^E(X \times_k l'/k, \bar{x}) \longrightarrow \pi^E(X \times_k l/k, \bar{x})$$

is an isomorphism of k-group schemes.

Proof. Let $Y' \to X \times_k l'$ be a G'-torsor with a fixed point $\mathrm{Spec}\,(l') \to Y'$ lying over $\bar{x}$. By [SGA1, Exposé X, Corollaire 1.7],

$$\pi_1^{\text{ét}}(X \times_k l', \bar{x}) \cong \pi_1^{\text{ét}}(X \times_k l, \bar{x}).$$

Thus by [SGA1, Exposé V,Théorème 4.1], the base change functor $- \times_l l'$ induces an equivalence of categories between the categories of finite étale coverings $\mathrm{ECov}(X \times_k l)$ and

$\mathrm{ECov}(X \times_k l')$. Thus there is a finite étale covering $Y \to X \times_k l$ such that $Y \times_l l' = Y'$. Now from the full faithfulness of $- \times_l l'$ and the fact that $G \times_k l$ and $G \times_k l'$ are constant group schemes, the action

$$(Y \times_l l') \times_{l'} (G \times_k l') = Y \times_l l' \times_k G = Y' \times_k G \to Y' = Y \times_l l'$$

descends to an action $Y \times_k G \to Y$ and makes Y a G-torsor. This means that the pull back functor

$$F^l_{l'} : N(X \times_k l/k, \bar{x}) \to N(X \times_k l'/k, \bar{x})$$

is essentially surjective. But by the fully faithfulness of $- \times_l l'$ the pull back functor $F^l_{l'}$ is also fully faithful. Hence $F^l_{l'}$ is an equivalence, and therefore the canonical morphism $\pi^{l'}_l$ is an isomorphism. □

Remark 2.24. Unfortunately the similar statement for π^L is false. This is due to an example by V. Mehta and S. Subramanian. Let X be an integral projective curve over $k = \bar{k}$ of characteristic $p > 0$ with at least one cuspidal singularity. Let $x \in X(k)$ be a rational point, and $k \subsetneq l$ be an extension of algebraically closed fields. We have the following commutative diagram

$$\begin{tikzcd} \pi^L(X \times_k l/l, x) \arrow[rr] \arrow[dr] & & \pi^L(X \times_k l/k, x) \times_k l \arrow[dl, "\pi^l_k \times \mathrm{id}"] \\ & \pi^L(X/k, x) \times_k l & \end{tikzcd}$$

with canonical morphisms. In [MS, §3], Mehta and Subramanian constructed a homomorphism $\phi : \pi^L(X \times_k l/l, x) \to \mu_{p,l}$ which does not come from a homomorphism $\pi^L(X/k, x) \to \mu_{p,k}$ by base change. If π^l_k was an isomorphism, then ϕ does not come from a homomorphism $\pi^L(X \times_k l/k, x) \to \mu_{p,k}$. But this is a contradiction, since any $\mu_{p,l}$-torsor over $X \times_k l$ comes from a $\mu_{p,k}$-torsor over $X \times_k l$.

2.8. **The Étale Universal Covering.** In this subsection we want to emphasize a big difference between $\pi^E(X/k, x)$ and $\pi^G(X/k, x)$ (or $\pi_1^{\text{ét}}(X, x)$) via comparing their universal coverings. The following statement is well known in the literature.

Statement 2.25. *Let X be a connected Noetherian scheme, $x \in X(\mathrm{Spec}\,(\bar{k}))$ be any geometric point. Then the universal covering $\widetilde{X}_x$ corresponding to $\pi_1^{\text{ét}}(X, x)$ is connected.*

The major reason behind this phenomenon is the following:

Fact. If X is a locally Noetherian connected scheme, and $x \in X(\mathrm{Spec}\,(\bar{k}))$ is any geometric point, then for any triple $(P, G, p) \in I_{\mathrm{co}}(X, x)$ the corresponding map $\pi_1^{\text{ét}}(X, x) \to G$ is surjective if and only if P is connected.

But for universal coverings under π^E, they are usually highly non-connected. We have seen some examples in 2.4 which are caused by complicated structures of the étale group schemes. Here is another example which is caused by the choice of the point on the torsor.

Exemple 2.26. Let $X = \mathrm{Spec}\,(\mathbb{Q})$, $\bar{x} : \mathbb{Q} \subseteq \bar{\mathbb{Q}}$. Consider a prime number $p > 2$. Then $\mu_p \cong \mathrm{Spec}\,(\mathbb{Q}) \coprod \mathrm{Spec}\,(K)$ as a scheme, where K is the p-th cyclotomic field. Let (μ_p, μ_p, q) be the trivial μ_p-torsor equipped with the point $q : \mathrm{Spec}\,(\bar{\mathbb{Q}}) \to \mathrm{Spec}\,(K)$. Obviously μ_p is not connected, but the unique map

$$(\phi, h) : \ (\widetilde{X_x}, \pi^E(X/k, \bar{x}), \tilde{x}) \to (\mu_p, \mu_p, q)$$

can not be trivial on h, for otherwise (ϕ, h) would factor through the trivial triple $(X, \{1\}, \bar{x})$, and then q has to be the trivial point $\mathrm{Spec}\,(\bar{\mathbb{Q}}) \to \mathrm{Spec}\,(\mathbb{Q})$. But if h is non-trivial then it has to be surjective. Therefore (μ_p, μ_p, q) is saturated but not connected. Since h is surjective, ϕ must be faithfully flat. But μ_p is not connected so $\widetilde{X_x}$ can not be either.

Proof of the fact. "$\Rightarrow$" If P was not connected then we can take the connected component $Q \subsetneq P$ containing p. Let $H \subsetneq G$ be the stabilizer of Q, then $(Q, H, p) \subsetneq (P, G, p)$. Therefore we have a factorization $\pi_1^{\text{ét}}(X, x) \to H \subsetneq G$ which contradicts to the assumption that $\pi_1^{\text{ét}}(X, x) \to G$ is surjective. "$\Leftarrow$" Suppose $\pi_1^{\text{ét}}(X, x) \to G$ factorizes as $\pi_1^{\text{ét}}(X, x) \to H \subseteq G$. Then we would have an imbedding $(Q, H, p) \subseteq (P, G, p)$. But $Q \subseteq P$ is finite étale, so it's both open and closed. Therefore $Q = P$ for $Q \neq \varnothing$. Hence $H = G$. □

Proof of the statement. Let $I \subseteq I_{\mathrm{co}}(X, x)$ be the full subcategory consisting of saturated objects. By 2.3, the category I is cofiltered. Then $\widetilde{X}_x = \varprojlim_{i \in I} P_i$, where $i = (P_i, G_i, p_i) \in I$. Because of the above *Fact*, these P_i are connected. The scheme $\widetilde{X}_x$ is connected if and only if $H^0(\widetilde{X}_x, O_{\widetilde{X}_x})$ has no non-trivial idempotents. Since X is quasi-compact and $\varinjlim$ is an exact functor we know that

$$H^0(\widetilde{X}_x, O_{\widetilde{X}_x}) = \varinjlim_{i \in I} H^0(P_i, O_{P_i}).$$

As each P_i is connected, there is no non-trivial idempotent in $H^0(P_i, O_{P_i}) \subseteq H^0(\widetilde{X}_x, O_{\widetilde{X}_x})$, hence there is no non-trivial idempotent in $H^0(\widetilde{X}_x, O_{\widetilde{X}_x})$. □

3. The First Fundamental Sequence

3.1. **The General Case.**

Proposition 3.1. *Let X be a geometrically connected separable scheme over a field k, and $\bar{x} : \mathrm{Spec}\,(\bar{k}) \hookrightarrow X$ be a geometric point. Then the natural k-group scheme homomorphism*

$$\pi^I(X/k, \bar{x}) \to \pi^I(k/k, \bar{x})$$

is surjective for $I = E, G, N, L$.

Proof. Suppose that we have an object $(l, G, t) \in I(k/k)$ and that we have a morphism

$$(\lambda, i) : (Q, H, s) \to (l \times_k X, G, t) \in I(X/k),$$

where the group homomorphism $i : H \to G$ is a closed imbedding. Then we have a section in the category of X-schemes

$$X = Q/H \hookrightarrow (l \times_k X)/H = (l/H) \times_k X.$$

As l/H is finite over k, its connected components are single points. Let $x \in l/H$ be the image of $t \in l$ under the projection $l \to l/H$. Since X is connected, reduced and λ sends $s \mapsto t$, the map

$$X \hookrightarrow (l/H) \times_k X \xrightarrow{pr_1} l/H$$

factors through $x : \mathrm{Spec}\,(\kappa(x)) \hookrightarrow l/H$ where $\kappa(x)$ is the residue field of x. Hence X is a scheme over $\kappa(x)$. But X is geometrically connected and geometrically reduced over k, so the extension $k \subseteq \kappa(x)$ has to be trivial, i.e. $k = \kappa(x)$. In other words, x is a k-rational point of l/H. Now pull back the projection map $l \to l/H$ along $x : \mathrm{Spec}\,(k) \to l/H$, we get a map $(q, H, t) \to (l, G, t) \in I(k/k)$ in which the group homomorphism is the imbedding $i : H \hookrightarrow G$. In particular if the map $\pi^I(k/k, \bar{x}) \to G$ corresponding to (l, G, t) is surjective, then the composition

$$\pi^I(X/k, \bar{x}) \to \pi^I(k/k, \bar{x}) \twoheadrightarrow G$$

has to be surjective too. This means precisely that $\pi^I(X/k, \bar{x}) \to \pi^I(k/k, \bar{x})$ is surjective. □

Proposition 3.2. *Let X be a geometrically connected separable scheme over a field k, and $\bar{x} : \mathrm{Spec}\,(\bar{k}) \hookrightarrow X$ be a geometric point. Then the natural sequence of k-group schemes*

$$1 \to \pi^I(\bar{X}/k, \bar{x}) \to \pi^I(X/k, \bar{x}) \to \pi^I(k/k, \bar{x}) \to 1 \tag{1}$$

is a complex, and it is exact in the middle if and only if the following two conditions are satisfied.

(i) For any I-saturated object $(P, G, p) \in I(X/k, \bar{x})$, the image of the composition of the natural homomorphisms

$$\pi^I(\bar{X}/k, \bar{x}) \to \pi^I(X/k, \bar{x}) \twoheadrightarrow G$$

is a normal subgroup of G.

(ii) Whenever there is an object $(P, G, p) \in I(X/k, \bar{x})$ whose pull-back along $\bar{X} \to X$ is trivial then there is an object $(Q, H, q) \in I(k/k, \bar{x})$ whose pull-back along $X \to \mathrm{Spec}\,(k)$ is isomorphic to (P, G, p).

Proof. For the first statement it is enough to see that the pull-back functor

$$\mathfrak{C}(k/k, \bar{x}, I) \to \mathfrak{C}(\bar{X}/k, \bar{x}, I)$$

sends any object in $\mathfrak{C}(k/k, \bar{x}, I)$ to a trivial object in $\mathfrak{C}(\bar{X}/k, \bar{x}, I)$. But this is indeed the case, for the pull-back functor $I(k/k, \bar{x}) \to I(\bar{X}/k, \bar{x})$ is trivial.

Now for the second statement. "⇒" (i) is clear, because a normal subgroup is still normal in any quotient. (ii) follows directly from 2.2. Indeed, given $(P, G, p) \in I(X/k, \bar{x})$, there is a unique morphism $\phi : \pi^I(X/k, \bar{x}) \to G$ corresponding to (P, G, p). The pull-back of (P, G, p) is trivial means that the composition

$$\pi^I(\bar{X}/k, \bar{x}) \to \pi^I(X/k, \bar{x}) \xrightarrow{\phi} G$$

is trivial. By the exactness there is a unique map $\varphi : \pi^I(k/k, \bar{x}) \to G$ making the diagram

$$\begin{tikzcd} \pi^I(X/k,\bar{x}) \arrow[rr] \arrow[dr, "\phi"'] & & \pi^I(k/k,\bar{x}) \arrow[dl, dotted, "\varphi"] \\ & G & \end{tikzcd}$$

commutative. Therefore, φ defines an object in $I(k/k, \bar{x})$ whose pull-back is isomorphic to (P, G, p).

"$\Leftarrow$" Let $(P, G, p) \in I(X/k, \bar{x})$ be an I-saturated object. By 2.2, it corresponds to a k-homomorphism $\phi : \pi^I(X/k, \bar{x}) \twoheadrightarrow G$. Let H be the image of the composition

$$\pi^I(\bar{X}/k, \bar{x}) \to \pi^I(X/k, \bar{x}) \xrightarrow{\phi} G.$$

By (i), $H \subseteq G$ is a normal subgroup. Thus we get an object $(P/H, G/H, p) \in I(X/k, \bar{x})$. Since the composition

$$\pi^I(\bar{X}/k, \bar{x}) \to \pi^I(X/k, \bar{x}) \to G \to G/H$$

is trivial by definition, the pull-back of $(P/H, G/H, p)$ to $\bar{X}$ is a trivial object. By (ii), $(P/H, G/H, p)$ descends to an object in $I(k/k, \bar{x})$, or equivalently, there is a commutative diagram

$$\begin{tikzcd} \pi^I(\bar{X}/k,\bar{x}) \arrow[r] \arrow[d] & \pi^I(X/k,\bar{x}) \arrow[d, dotted] \\ G \arrow[r] & G/H \end{tikzcd} .$$

Let N be the image of the kernel of $\pi^I(X/k, \bar{x}) \to \pi^I(k/k, \bar{x})$ under the map

$$\phi : \pi^I(X/k, \bar{x}) \twoheadrightarrow G.$$

The above diagram implies that $N \subseteq H$ and the first statement of this proposition implies that $H \subseteq N$. Therefore $H = N$. But since this is valid for all I-saturated objects, we can conclude the middle exactness. □

Remark 3.3. The sequence is in general not exact on the left. See 3.15 for an example when k is perfect $X = \mathbb{A}^1_k$ and $I = L$. The example does not work for π^E. However, one can not use the injectivity for π^G (or $\pi_1^{\text{ét}}$) to conclude the injectivity for π^E either. The injectivity for π^G (or $\pi_1^{\text{ét}}$) was deduced from the theory of *Galois closure* for Galois coverings [Sz, Proposition 5.3.9, pp. 169]. But we can not find an analogue for π^E.

Corollary 3.4. *If either $X = \mathbb{A}^n_k$ with k is a field of characteristic 0 or X is a complete rational variety over an arbitrary field k, then the canonical map*

$$\pi^N(X/k, \bar{x}) \to \pi^N(k/k, \bar{x})$$

is an isomorphism.

Proof. By 2.11 and 2.13, $\pi^N(\bar{X}/k, \bar{x}) = \{1\}$. Then the corollary follows from 3.2 and 3.6. □

Exemple 3.5. In this part we would like to give an example to show that the condition (ii) of 3.2 is not always satisfied.

Let's just take $k = \mathbb{F}_p(s,t)$ (the function field in two variables over $\mathbb{F}_p$), $X = \mathbb{A}^1_k \backslash \{a\}$, and

$$P = \operatorname{Spec}\left(A[T]/(T^p - (s+tx^p))\right)$$

be the μ_p-torsor over X in a natural way, where $A := O_X(X)$ and $a \in \mathbb{A}^1_k$ is the closed point determined by the polynomial $s + tx^p \in k[x]$. Since P is a local torsor the base point plays no role. For this reason we are going to omit the base point in the following discussion. Clearly, the equation

$$T^p - (s + tx^p) = 0$$

has no solution in A, thus P is a non-trivial μ_p-torsor. Furthermore, $P \times_k \bar{k}$ is a tivial torsor over $\bar{X}$, the section being given by the solution of the above equation in $\bar{k}[x]$. But $P \to X$ can not descent to a μ_p-torsor over $\operatorname{Spec}(k)$. In fact, the two μ_p-torsors

$$P_0 = \operatorname{Spec}\left(k[T]/(T^p - s)\right) \qquad \text{and} \qquad P_1 = \operatorname{Spec}\left(k[T]/(T^p - s - t)\right)$$

which are fibres of $P \to X$ at $x = 0$ and $x = 1$ respectively, can not be isomorphic. Suppose there was an isomorphism of torsors

$$f : k[T]/(T^p - s) \longrightarrow k[T]/(T^p - s - t)$$

sending $T \mapsto f(T)$, where $f(T) \in k[T]$ is a polynomial of degree less than p. Let $\mu_p = k[Y]/(Y^p - 1)$, then $\operatorname{Aut}_{k-\mathrm{grp.sch}}(\mu_p) = (\mathbb{Z}/p\mathbb{Z})^*$, where $m \in (\mathbb{Z}/p\mathbb{Z})^*$ stands for $Y \mapsto Y^m$. Thus we should have a commutative diagram

$$\begin{tikzcd} k[T]/(T^p - s) \arrow[r, "f"] \arrow[d, "\rho_0"'] & k[T]/(T^p - s - t) \arrow[d, "\rho_1"] \\ k[T]/(T^p - s) \otimes_k k[Y]/(Y^p - 1) \arrow[r, "f\otimes m"] & k[T]/(T^p - s - t) \otimes_k k[Y]/(Y^p - 1) \end{tikzcd} ,$$

where ρ_0 and ρ_1 are defined by the action of μ_p on P_0 and P_1 respectively. Tracing the image of T in the above diagram, we get $f(T) \otimes Y^m = f(T \otimes Y)$. This implies that $f(T)$ is a polynomial of the form αT^m with $\alpha \in k$. Then we should have

$$f: \quad T^p - s \longmapsto f(T)^p - s = \alpha^p T^{mp} - s = 0 \in k[T]/(T^p - s - t).$$

But $T^p = s + t \in k[T]/(T^p - s - t)$. Hence we should have $\alpha^p (s+t)^m - s = 0 \in k \subset k[T]/(T^p - s - t)$. However, this equation can not happen in k because $\mathbb{F}_p[s,t]$ is a UFD.

However, under some conditions (ii) actually holds.

Proposition 3.6. *If in 3.2, one of the following conditions hold,*

- *the field k is perfect and X is in addition quasi-compact;*
- *the scheme X is proper;*
- *the group G is étale,*

then condition (ii) *holds for $N(X/k, \bar{x})$.*

Proof. The last case will be proved in 3.11. Let's show the first two. Let $(P, G, p) \in N(X/k, \bar{x})$ be an object whose pull-back to $N(\bar{X}/k, \bar{x})$ is trivial, i.e. there is a morphism

$$\lambda : (\bar{X}, \{1\}, \bar{x}) \to (\bar{P}, G, p) \in N(\bar{X}/k, \bar{x}).$$

First assume that k is perfect and X is quasi-compact. Let $(P_{\text{ét}}, G_{\text{ét}}, p) \in I_{\text{ét}}(X/k, \bar{x})$ be the étale quotient of (P, G, p). Then $(\bar{P}_{\text{ét}}, G_{\text{ét}}, p)$ is also trivial. Thus by 3.11 there is a triple $(Q, H, q) \in I_{\text{ét}}(k/k, \bar{x}) \subseteq N(k/k, \bar{x})$ such that the pull-back of (Q, H, q) to X is isomorphic to $(P_{\text{ét}}, G_{\text{ét}}, p)$. Let n be the order of the k-group scheme $G_{\text{ét}}$. Then $\bar{P}_{\text{ét}}$ can be written as n-copies of $\bar{X}$:

$$\bar{P}_{\text{ét}} = \coprod_{i=1,\cdots,n} \bar{X}_i$$

where $\bar{X}_i = \bar{X}$. The quotient $\pi : P \to P_{\text{ét}}$ makes P as G^0-torsor over $P_{\text{ét}}$, and we have a decomposition

$$\bar{P} = \coprod_{i=1,\cdots,n} \bar{P}_i$$

where $\bar{P}_i$ is just $(\pi \times_k \bar{k})^{-1}(\bar{X}_i)$. Suppose $p \in \bar{P}_1(\bar{k})$. Then the map λ makes $\bar{P}_1$ a trivial G^0-torsor over $\bar{X}_1$. Since G^0 is local and $\bar{X}_1$ is reduced, the closed imbedding $(\bar{P}_1)_{\text{red}} \hookrightarrow \bar{P}_1$ composing with the projection $\bar{P}_1 \to \bar{X}_1$ has to be an isomorphism. As $G_{\text{ét}}(\bar{k})$ acts transitively on the components $\bar{X}_i$, for any $1 \leqslant i \leqslant n$ there is an element $g \in G_{\text{ét}}(\bar{k}) = G(\bar{k})$ making the diagram

$$\begin{tikzcd}
\bar{P}_0 \arrow[r, "g"] \arrow[d] & \bar{P}_i \arrow[d] \\
\bar{X}_0 \arrow[r, "g"] & \bar{X}_i
\end{tikzcd}$$

commutative. Hence the closed imbedding $(\bar{P}_i)_{\text{red}} \hookrightarrow \bar{P}_i$ composing with the projection $\bar{P}_i \to \bar{X}_i$ is also an isomorphism for each i. Therefore the composition $\bar{P}_{\text{red}} \hookrightarrow \bar{P} \to \bar{P}_{\text{ét}}$ is an isomorphism. But as k is perfect, $\bar{P}_{\text{red}} = P_{\text{red}} \times_k \bar{k}$. Thus the composition $P_{\text{red}} \hookrightarrow P \to P_{\text{ét}}$ has to be an isomorphism too. This defines a section $s : P_{\text{ét}} \hookrightarrow P$ for the projection $\pi : P \to P_{\text{ét}}$. The universality of the reduced closed subscheme structure $P_{\text{red}} \subseteq P$ tells us that there is a unique arrow $P_{\text{ét}} \times_k G_{\text{ét}} \dashrightarrow P_{\text{ét}}$ making the following diagram

$$\begin{tikzcd}
P_{\text{ét}} \times_k G_{\text{ét}} \arrow[r, dotted] \arrow[d, tail, "s \times i"] \arrow[dd, equal, bend right=60] & P_{\text{ét}} \arrow[d, tail, "s"'] \arrow[dd, equal, bend left=60] \\
P \times_k G \arrow[r, "\rho"] \arrow[d, two heads, "\pi \times o"] & P \arrow[d, two heads, "\pi"'] \\
P_{\text{ét}} \times_k G_{\text{ét}} \arrow[r, "\rho_{\text{ét}}"] & P_{\text{ét}}
\end{tikzcd}$$

commutative, where $i : G_{\text{ét}} \subseteq G$ is the inclusion of the reduced subscheme structure of G, $\rho_{\text{ét}}$ is action of $G_{\text{ét}}$ on $P_{\text{ét}}$ induced by ρ, $o : G \twoheadrightarrow G_{\text{ét}}$ is the étale quotient map. Therefore we obtain a morphism

$$(P_{\text{ét}}, G_{\text{ét}}, p) \to (P, G, p) \in N(X/k, \bar{x}).$$

In view of the isomorphism $(P_{\text{ét}}, G_{\text{ét}}, p) \cong (Q \times_k X, H, q)$, we can equip the k-scheme G with a left action from H via $H \cong G_{\text{ét}} \xrightarrow{i} G$, then the contracted product $Q \times^H G$ provides a k-form for the G-torsor P over X. This finishes the proof the first case.

Now suppose that X is proper. Let $f : X \to \operatorname{Spec}(k)$ be the structure morphism, and $\mathcal{A}$ be the push-forward of $\mathcal{O}_P$ to X along $P \to X$. Then $\mathcal{A}$ is a locally free coherent $\mathcal{O}_X$-algebra equipped with a G-action map

$$\rho : \mathcal{A} \longrightarrow \mathcal{A} \otimes_k k[G]$$

which makes the induced map $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A} \xrightarrow{\mathrm{id}\otimes\rho} \mathcal{A} \otimes_k k[G]$ into an isomorphism. Since $\bar{P}$ is a trivial G-torsor over $\bar{X}$, $\bar{\mathcal{A}} := \mathcal{A} \otimes_k \bar{k}$ is a free $\mathcal{O}_{\bar{X}}$-module. But X is proper separable and geometrically connected over k, so the adjunction map $\bar{f}^* \bar{f}_* \bar{\mathcal{A}} \to \bar{\mathcal{A}}$ is an isomorphism. By FPQC-descent, we have that $f^* f_* \mathcal{A} \to \mathcal{A}$ is an isomorphism. Let $A := f_* \mathcal{A} = \Gamma(X, \mathcal{O}_X)$. Then the action $f_* \rho : A \to A \otimes_k k[G]$ makes $\operatorname{Spec}(A)$ into a G-torsor whose pull-back to X is precisely $P = \operatorname{Spec}(\mathcal{A})$. ☐

3.2. **The Étale Case.**

Proposition 3.7. *Let X be a Noetherian scheme, which is geometrically connected over a field k, and let $\bar{x} : \operatorname{Spec}(\bar{k}) \hookrightarrow X$, $(P, G, p) \in I_{\acute{e}t}(X/k, \bar{x})$ be a saturated object. Let N be the image of the following composition*

$$\pi^E(\bar{X}/k, \bar{x}) \to \pi^E(X/k, \bar{x}) \twoheadrightarrow G.$$

Then we get an imbedding $(\bar{P}', N, p) \subseteq (\bar{P}, G, p) \in I_{\acute{e}t}(\bar{X}/k, \bar{x})$. If one of the following conditions is satisfied, then $N \subseteq G$ is a normal subgroup scheme.

(i) *P, as a scheme, is connected.*
(i) *$\bar{P}'$, as a scheme, is connected.*

Proof. By [SGA1, Éxposé IX, Théorème 4.10] we may assume that k is a perfect field. There is a finite Galois subextenison $k \subseteq K$ of $k \subseteq \bar{k}$ such that G_K is constant over K and the number of connected components of P_K is the same as that of $\bar{P}$. In this case the image of $\pi^E(\bar{X}/k, \bar{x})$ and $\pi^E(X_K/k, \bar{x})$ are the same in G. Thus replacing $\bar{k}$ by K, we may assume that $k \subseteq \bar{k}$ is a finite Galois extension.

Suppose that we have an $I_{\text{ét}}$-saturated object $(P, G, p) \in I_{\text{ét}}(X/k, \bar{x})$. Let $\bar{G} := G \times_k \bar{k}$, $\bar{P} := P \times_k \bar{k}$, $\bar{P}_0$ be the connected component of $\bar{P}$ containing p. Let $\bar{H} \subseteq \bar{G}$ be the subgroup which fixes $\bar{P}_0$, i.e.

$$\bar{H} := \{\ g \in \bar{G} \mid \bar{P}_0\ g = \bar{P}_0\ \}.$$

Then $\bar{N} \subseteq \bar{G}$ is the smallest subgroup which contains the subset

$$\bigcup_{\sigma \in \mathrm{Gal}(\bar{k}/k)} \sigma(\bar{H}) \quad \subseteq \ \bar{G}.$$

Now let T be the subset of $\bar{G}$ whose elements are those t_σ which send $p \mapsto \sigma(p)$ for some $\sigma \in \mathrm{Gal}(\bar{k}/k)$. Let $\bar{M}$ be the smallest subgroup of $\bar{G}$ containing T. Since for any $\sigma \in \mathrm{Gal}(\bar{k}/k)$ and $t_\tau \in T$ sending $p \mapsto \tau(p)$ we have $t_\sigma \circ \sigma(t_\tau) = t_{\sigma\tau}$. So $\sigma(t_\tau) = t_\sigma^{-1} \circ t_{\sigma\tau} \in \bar{M}$, then it follows that $\sigma(\bar{M}) \subseteq \bar{M}$. Let $\bar{M}\bar{N} \subseteq \bar{G}$ be the smallest subgroup of $\bar{G}$ containing $\bar{M}$ and $\bar{N}$. Let

$$\bar{P}'' \;:=\; \bigcup_{g\in\bar{M}\bar{N}} (\bar{P}_0)g \quad\subseteq\; \bar{P},$$

i.e. the $\bar{M}\bar{N}$-obits of $\bar{P}_0$ in $\bar{P}$. Then $\bar{P}''$ is a torsor under $\bar{M}\bar{N}$. Since both $\bar{P}''$ and $\bar{M}\bar{N}$ are stable under the induced Galois action, they both descend to k, i.e. there exists a k-form $MN \subseteq G$ of $\bar{M}\bar{N}$ such that $\bar{P}''$ descends to an MN-torsor $P'' \subseteq P$ over X. Then there is a homomorphism

$$\pi^E(X/k, \bar{x}) \to MN \subseteq G.$$

But $\pi^E(X/k, \bar{x}) \to G$ is already surjective by the assumption, this immediately implies that $MN = G$ or equivalently $\bar{M}\bar{N} = \bar{G}$.

Next we show that $\bar{N}$ is a normal subgroup of $\bar{G}$. From the above discussion it is enough to check $m^{-1}\bar{H}m \subseteq \bar{N}$ for $\forall m \in \bar{M}$. If (i) is satisfied, then $\mathrm{Gal}(\bar{k}/k)$ acts transitively on the connected components of $\bar{P}$, so any element $g \in \bar{G}$ can be written as $h \circ t_\sigma$, where $h \in \bar{H}$, $t_\sigma \in T$. If (ii) is satisfied, then $\bar{H} = \bar{N}$. In either case it is already enough to check $t_\sigma^{-1}\bar{H}t_\sigma \subseteq \bar{N}$ for $\forall t_\sigma \in T$. From the very definition of t_σ we have

$$\sigma(p)t_\sigma^{-1}ht_\sigma = (ph)t_\sigma = \sigma(p)h'$$

where h' is contained in the stabilizer of $\sigma(P_0)$, i.e. $\sigma(H)$. Thus $t_\sigma h t_\sigma^{-1} \in \sigma(H) \subseteq \bar{N}$. □

Remark 3.8. The two conditions in 3.7 are already satisfied by most interesting étale torsors. For example it is known in the literature 2.8 that any triple $(P, G, p) \in I_{\text{co}}(X/k, \bar{x})$ is I_{co}-saturated if and only if it is connected. Thus in view of 3.11 and 3.2 this proposition can be seen as a generalization of the fundamental exact sequence of the étale fundamental group [SGA1, Exposé IX, Théorème 6.1].

Exemple 3.9. In this part, we would like to construct an example showing that for a saturated object $(P, G, p) \in I_{\text{ét}}(X/k, \bar{x})$ which does not satisfy any of the conditions in 3.7 the image of the composition

$$\pi^E(\bar{X}/k, \bar{x}) \to \pi^E(X/k, \bar{x}) \twoheadrightarrow G$$

needs not be a normal subgroup of G.

Let X be any scheme which is geometrically connected over a field k, and let $Y \to X$ be a torsor under the constant group scheme $(\mathbb{Z}/2\mathbb{Z})_k$ with Y geometrically connected over k. Now we are going to construct a finite Galois field extention K/k with Galois group M, a torsor P'_K over X_K under an abstract group G'_K which contains the X_K-scheme Y_K as a connected component. We will also construct a twisted action of M on the X_K-scheme P'_K and an action of M on the abstract group G'_K in such a way that these two actions are compatible.

Let $n \in \mathbb{N}^+$ be an even number which is equal to or larger than 2. Let

$$G'_K = (((\langle a_1\rangle \times \langle a_2\rangle \times \langle a_3\rangle \times \langle a_4\rangle) \rtimes (\langle b_1\rangle \times \langle b_2\rangle)) \rtimes \langle \xi\rangle,$$

where $\langle a_1\rangle = \langle a_2\rangle = \langle a_3\rangle = \langle a_4\rangle \cong \mathbb{Z}/2n\mathbb{Z}$, $\langle b_1\rangle = \langle b_2\rangle = \langle \xi\rangle \cong \mathbb{Z}/2\mathbb{Z}$. The actions are defined by the following relations.

$$b_1a_1 = a_2b_1 \qquad b_1a_2 = a_1b_1 \qquad b_1a_3 = a_3b_1 \qquad b_1a_4 = a_4b_1$$

$$b_2a_1 = a_1b_2 \qquad b_2a_2 = a_2b_2 \qquad b_2a_3 = a_4b_2 \qquad b_2a_4 = a_3b_2$$

$$\xi b_1 = b_1\xi \qquad \xi b_2 = a_3^n a_4^n b_2 \xi \qquad \xi a_i = a_i^{n+1}\xi \qquad i = 1, 2, 3, 4.$$

In addition we can define an action of $\mathbb{Z}/2\mathbb{Z} = \{e, \sigma\}$ on G'_K (via group automorphisms). The action is given by the following equations.

$$\sigma(a_1) = a_3 \qquad \sigma(a_2) = a_4 \qquad \sigma(a_3) = a_1 \qquad \sigma(a_4) = a_2$$

$$\sigma(b_1) = b_2 \qquad \sigma(b_2) = b_1 \qquad \sigma(\xi) = a_1^n a_3^n \xi.$$

Next we construct the G'_K-torsor P'_K. Let $H'_K \subseteq G'_K$ be the subgroup

$$(\langle a_1\rangle \times \langle a_2\rangle \times \langle a_3\rangle \times \langle a_4\rangle) \rtimes (\langle b_1\rangle \times \langle b_2\rangle) \subseteq G'_K$$

and let

$$P' := \coprod_{i \in H'_K} Y_i$$

be the disjoint union of copies of Y (i.e. $Y_i = Y$). We define a right action of G'_K on P' in the following way. If $j \in H'_K$ then the action of j on Y_i is defined by the identity morphism $Y = Y_i \to Y_{ij} = Y$. If $j \notin H'_K$, then ij is uniquely written as a product $ij = \xi k$ with $k \in H'_K$. Then the action of j on Y_i will be the morphism $Y = Y_i \to Y_k = Y$ given by the action of the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$ (remember that Y is a $\mathbb{Z}/2\mathbb{Z}$-torsor over X). Viewing G'_K as a constant group scheme over k, one gets a morphism

$$\rho : P' \times_k G'_K \to P'$$

over X defining the right action. This action actually defines P' as a G'_K-torsor over X. Indeed, one can take a geometric point $\bar{x} \in X$. Then the fibre of $\bar{x}$ under the projection $Y_e = Y \to X$ consists of two points. We pick any point in the fibre and denote it by p. Then the other point is $p\xi$. We can also translate p and $p\xi$ by the group H'_K. In this way we get all the fibres of $\bar{x}$ under the projection $P' \to X$. Each fibre can be written uniquely as pi or $p\xi i$ for some $i \in H'_K$. By the very definition of the action of G'_K on the set of fibres of $\bar{x}$, one sees that the set of all fibres of $\bar{x}$ is a principal homogeneous space under G'_K. Hence the X-morphism

$$P' \times_k G'_K \xrightarrow{id \times \rho} P' \times_X P'$$

induces an isomorphism at the fibre of $\bar{x} \in X$. Since $P' \times_k G'_K$ and $P' \times_X P'$ are all finite étale X-schemes and X is connected, the morphism $id \times \rho$ is an isomorphism by [SGA1, Exposé V, Théorème 4.1]. Therefore P' is a G'_K-torsor over X. Moreover, we would like to introduce two actions on P' by $\mathbb{Z}/2n\mathbb{Z} = \langle u\rangle$ and $\mathbb{Z}/2\mathbb{Z} = \langle v\rangle$ respectively. The action of u on a component Y_i is defined to be the identity morphism $Y = Y_i \to Y_{a_1 a_3 \sigma(i)} = Y$. Notice that we have a commutative diagram

(☠)
$$\begin{tikzcd} P' \arrow[r, "g"] \arrow[d, "u"'] & P' \arrow[d, "u"] \\ P' \arrow[r, "\sigma(g)"] & P' \end{tikzcd}$$

for all $g \in G'_K$, even when $g \notin H'_K$. The action of v on a component Y_i is defined to be the identity morphism $Y = Y_i \to Y_{b_1 i} = Y$. Similarly, we have a commutative diagram

(♕)
$$\begin{tikzcd} P' \arrow[r, "g"] \arrow[d, "v"'] & P' \arrow[d, "v"] \\ P' \arrow[r, "g"] & P' \end{tikzcd}$$

for all $g \in G'_K$.

Next we would like to construct a finite group M generated by two elements $\{x, y\}$ and two group homomorphisms

$$f : M \to \mathrm{Aut}_X(P') \qquad \text{and} \qquad g : M \to \mathrm{Aut}(G'_K)$$

such that $f(x) = u, f(y) = v$ and $g(x) = \sigma, g(y) = id$, where P' is considered as an object in the category of X-schemes and G'_K is considered as an object in the category of abstract groups. There should be some general procedure to obtain such M and f, g, because all the automorphism groups are finite. But unfortunately the author has to rely on some brutal computational methods. For simplicity we treat only the case when $n = 2$. In this case, we first consider the following equations.

(i)
$$xyx^2yx = yxyx^2yxy = yx^3yx^2yx^3y$$

(ii)
$$yx^2yx^2 = x^2yx^2y$$

(iii)
$$x^3yxy = yx^3yx$$
$$yxyx^3 = xyx^3y$$

(iv)
$$x^3yx^3yx^2yxy = yx^3yx^3yx^3y = yxyx^2yx^3yx^3$$
$$xyxyx^2yx^3y = yxyxyxy = yx^3yx^2yxyx$$

(v)
$$x^3yx^2yxyx^2y = xyx^2yx^3yx^2y = yx^2yx^3yx^2yx = yx^2yxyx^2yx^3$$

(vi)
$$yxyxyxyx = x^3yx^3yx^3yx^3y$$

(vii)
$$x^4 = 1 \qquad\qquad y^2 = 1$$

One observes first that the above equations hold in $\mathrm{Aut}(G'_K)$ when one replaces x by σ, y by id, and the same hold in $\mathrm{Aut}_X(P')$ when one replaces x by u, y by v. The former is somewhat clear, the latter needs some computation. To verify the above equations for u, v, we choose a geometric point $\bar{x} \in X$ and a fibre p of $\bar{x}$ under $Y_e = Y \to X$, then check whether the actions from both sides of the equation are agree on p. If so, one could then use ♛, ☠ and the compatibility in $\mathrm{Aut}(G'_K)$ to move p around, through all the fibres of $\bar{x}$ under $P'_K \to X_K$, and finally conclude that the actions from both sides of the equation agree on all fibres. Then the equations for u, v follow from [SGA1, Exposé V, Théorème 4.1]. Here is the result of the calculations.

(i)
$$uvu^2vu(p) = vuvu^2vuv(p) = vu^3vu^2vu^3v(p) = p(a_3a_4)^2$$
(ii)
$$vu^2vu^2(p) = u^2vu^2v(p) = p(a_1a_2)^2$$
(iii)
$$\begin{aligned} u^3vuv = vu^3vu(p) &= pa_3^3a_4b_1b_2 \\ vuvu^3(p) = uvu^3v(p) &= pa_3a_4^3b_1b_2 \end{aligned}$$
(iv)
$$\begin{aligned} u^3vu^3vu^2vuv(p) = vu^3vu^3vu^3v(p) = vuvu^2vu^3vu^3(p) &= pa_1^3a_3^3a_2^2a_4^2 \\ uvuvu^2vu^3v(p) = vuvuvuv(p) = vu^3vu^2vuvu(p) &= pa_1a_3a_2^2a_4^2 \end{aligned}$$
(v)
$$vuvuvuvu(p) = u^3vu^3vu^3vu^3v(p) = p(a_1a_2a_3a_4)^2$$
(vi)
$$\begin{aligned} &u^3vu^2vuvu^2v(p) = uvu^2vu^3vu^2v(p) \\ =&vu^2vu^3vu^2vu(p) = vu^2vuvu^2vu^3(p) \\ =&p(a_1a_2a_3a_4)^2 \end{aligned}$$
(vii)
$$u^4(p) = p \qquad\qquad v^2(p) = p$$

Let M be the free group generated by x, y modulo the relations (i)-(vii). One can see without too much difficulty that M is a finite group generated by x, y. Clearly there are group homomorphisms $f : x \mapsto u, y \mapsto v$ and $g : x \mapsto \sigma, y \mapsto id$.

Now let L be any field of any characteristic. Choose an imbedding $M \subseteq S_m$ for some $m \in \mathbb{N}^+$, we get a faithful action of M on $L(X_1, X_2, \cdots, X_m)$. Let
$$K := L(X_1, X_2, \cdots, X_m) \qquad\qquad k := K^M$$
where $K^M \subseteq K$ denotes the subfield of invariant elements under the action of M. Then K/k is a finite Galois extension with Galois group M.

Let $P'_K := P' \times_k K$. Then P'_K is a G'_K-torsor over X_K. We also have an imbedding $Y_K = Y_e \times_k K \subseteq P' \times_k K = P'_K$. Since the connected covering $Y_K \to X_K$ comes, via base change, from a Galois covering $Y \to X$ over k, it has to be again a Galois covering

[SGA1, Exposé V, §4, f), (ii)]. Thus the inclusion $\mathrm{Aut}_X(Y) \subseteq \mathrm{Aut}_{X_K}(Y_K)$ has to be an isomorphism (because Y/X and Y_K/X_K are of the same degree).

Now for each element $\alpha \in M$ we can define a twisted action on P'_K via

$$P'_K = P' \times_k K \xrightarrow{f(\alpha)\times\alpha} P' \times_k K = P'_K \ .$$

By ♛ and ☠ this twisted action is compatible with the action of M on G'_K.

Viewing G'_K as a constant group scheme over K and applying Galois descent we get a k-group scheme G'_k and a right G'_k-torsor P'_k over X such that the pull-back of (P'_k, G'_k) to K is (P'_K, G'_K). Choosing a geometric point $\bar{x} : \mathrm{Spec}\,(\bar{K}) \to X_K$ and a lifting $p : \mathrm{Spec}\,(\bar{K}) \to Y_e \times_k K \subseteq P'_K$, we get a triple $(P'_k, G'_k, p) \in I_{\text{ét}}(X/k, \bar{x})$. This triple corresponds to a homomorphism $\pi^E(X/k, \bar{x}) \to G'_k$. Let $G \subseteq G'_k$ be the image, $(P, G, p) \subseteq (P'_k, G'_k, p)$ be the triple in $I_{\text{ét}}(X/k, \bar{x})$ corresponding to $\pi^E(X/k, \bar{x}) \twoheadrightarrow G \subseteq G'_k$.

In this case, (P, G, p) is a saturated object by definition, and the pull-back $G_K \subseteq G'_K$ is a subgroup stable under the action of M. Since $P_K \subseteq P'_K$ is a subscheme containing $p \in Y_e \times_k K$ and is stable under the action of M, P_K contains

$$Y_e \times_k K, \qquad Y_{b_1} \times_k K = y(Y_e \times_k K), \qquad Y_{a_1a_3} \times_k K = x(Y_e \times_k K).$$

As $Y_{b_1} = Y_e b_1$ and $Y_{a_1a_3} = Y_e a_1 a_3$, G_K contains ξ, b_1, a_1a_3. Just like in 3.7, we denote by N the image of the homomorphism

$$\pi^N(\bar{X}/k, \bar{x}) \to G$$

corresponding to the triple $(\bar{P}, G, p)$. Then $\bar{N}$ is generated by the $\mathrm{Gal}(\bar{k}/k)$-orbit of $\{e, \xi\} \subseteq G_K = \bar{G}$, or equivalently by the M-orbit of $\{e, \xi\} \subseteq G_K$, i.e.

$$\bar{N} := \{e, \xi, (a_1a_3)^n, (a_1a_3)^n\xi\}.$$

However, $\bar{N} \subseteq \bar{G}$ is not a normal subgroup for $b_1(a_1a_3)^n b_1^{-1} = (a_2a_3)^n \notin \bar{N}$.

Remark 3.10. (i) In the above example, we could take $X := \mathbb{A}^1_k$ and $Y \to X$ to be the Artin-Schreier covering under $\mathbb{F}_2$ if k is of characteristic 2.
(ii) We could also take A to be an abelian variety over a field l of characteristic $\neq 2$, then $D := A[2] \times_l \bar{l}$ is a constant group scheme of order $2^{2\dim(A)}$. Suppose $A[2]$ stays as a constant group scheme over $l \subseteq L \subseteq \bar{l}$ and $D \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$ is a surjective homomorphism. Let $k := L(X_1, X_2, \cdots, X_m)^M$ be as in the example, $X := A \times_l k$. Then we could define the $\mathbb{Z}/2\mathbb{Z}$-torsor $Y \to X$ to be the one obtained by taking the contracted product of the D-torsor $X \xrightarrow{2\cdot} X$ along $D \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$. Clearly Y is geometrically connected over k. In this example k is allowed to be of any characteristic $\neq 2$.

Lemma 3.11. *Let X be a geometrically connected quasi-compact scheme over a field k, G be a finite étale group scheme over k, and P be a G-torsor over X. If $\bar{P}$ is a trivial G-torsor over $\bar{X}$, then there is a G-torsor Q over k whose pull-back along $X \to k$ is P.*

Proof. Let K be an intermediate finite Galois extension of $k \subseteq \bar{k}$ over which G_K becomes a constant group scheme and P_K remains a trivial torsor. Since P_K is a trivial G-torsor over X_K, by choosing an X_K-section for the projection $\pi : P_K \to X_K$ we get isomorphisms (in the category of X_K-schemes)

$$P_K \cong X_K \times_k G = X_K \times_K G_K = \coprod_{i \in G_K} X_K.$$

By Galois descent, giving the X-scheme P is equivalent to giving a twisted action of $\mathrm{Gal}(K/k)$ on P_K, i.e. a homomorphism

$$f : \mathrm{Gal}(K/k) \to \mathrm{Aut}_X(P_K)$$

such that the following diagram

$$(*) \qquad \begin{tikzcd} P_K \cong \coprod_{i \in G_K} X_K \arrow[r, "f(\sigma)"] \arrow[d, "\pi"'] & \coprod_{i \in G_K} X_K \cong P_K \arrow[d, "\pi"] \\ X_K \arrow[r, "id \times \sigma"] \arrow[d] & X_K \arrow[d] \\ K \arrow[r, "\sigma"] & K \end{tikzcd}$$

is commutative for all $\sigma \in \mathrm{Gal}(K/k)$. One observes that, as X_K is connected, such a twisted action on $P_K \cong \coprod_{i \in G_K} X_K$ is none other than a permutation of the connected components in a twisted manner. The observation can be written more formally.

Let $n \in \mathbb{N}^+$ be the order of the k-group scheme G, S_n be the n-th permutation group. Then there is a unique group homomorphism

$$\lambda_X : S_n \times \mathrm{Gal}(K/k) \to \mathrm{Aut}_X(P_K)$$

whose restriction to S_n is the permutation of the connected components of P_K and whose restriction to $\mathrm{Gal}(K/k)$ is

$$\sigma \quad \longmapsto \quad P_K \cong (\coprod_{i \in G_K} X) \times_k K \xrightarrow{id \times \sigma} (\coprod_{i \in G_K} X) \times_k K \cong P_K.$$

The observation means that there is a group homomorphism $\theta : \mathrm{Gal}(K/k) \to S_n$ making the following diagram

$$\begin{tikzcd} \mathrm{Gal}(K/k) \arrow[rr, "f"] \arrow[dr, "\theta \times id"'] & & \mathrm{Aut}_X(P_K) \\ & S_n \times \mathrm{Gal}(K/k) \arrow[ur, "\lambda_X"'] & \end{tikzcd}$$

commutative.

In the above, we could replace P_K by the k-scheme $G_K = \coprod_{i \in G_K} \mathrm{Spec}\,(K)$ to obtain a homomorphism

$$\lambda_k : S_n \times \mathrm{Gal}(K/k) \to \mathrm{Aut}_k(G_K)$$

where G_K is regarded as an object in the category of k-schemes. In this way, we get a homomorphism

$$g : \mathrm{Gal}(K/k) \xrightarrow{\theta \times id} S_n \times \mathrm{Gal}(K/k) \xrightarrow{\lambda_k} \mathrm{Aut}_k(G_K)$$

making the following diagram

$$\begin{tikzcd} G_K = \coprod_{i \in G_K} \mathrm{Spec}\,(K) \arrow[r, "g(\sigma)"] \arrow[d] & \coprod_{i \in G_K} \mathrm{Spec}\,(K) = G_K \arrow[d] \\ K \arrow[r, "\sigma"] & K \end{tikzcd}$$

commutative for each $\sigma \in \mathrm{Gal}(K/k)$. In other words, we get a twisted action of $\mathrm{Gal}(K/k)$ on G_K. This defines a k-form Q for the K-scheme G_K.

The X-scheme $X \times_k Q$ is an X-form of the X_K-scheme $P_K \cong X_K \times_K G_K = X \times_k G_K$. From the very definition of Q we see that the twisted action of $\mathrm{Gal}(K/k)$ on P_K corresponding to the two X-forms $X \times_k Q$ and P are the same. Therefore, by Galois descent $P \cong X \times_k Q$ as X-schemes. On the other hand since G is a k-form of the K-group scheme G_K, there is an action via group automorphisms

$$\phi : \mathrm{Gal}(K/k) \to \mathrm{Aut}_{\mathrm{grp}}(G_K)$$

corresponding to G. As P is a G-torsor we have the following commutative diagram

$$\begin{tikzcd} P_K \arrow[r, "a"] \arrow[d, "f(\sigma)"'] & P_K \arrow[d, "f(\sigma)"] \\ P_K \arrow[r, "\phi(\sigma)(a)"] & P_K \end{tikzcd}$$

for all $\sigma \in \mathrm{Gal}(K/k)$ and $a \in G_K$. Since the identification $P_K \cong X_K \times_K G_K$ is equivariant under the right actions and the action of G_K on P_K is also just a permutation of connected components, we have another commutative diagram

$$\begin{tikzcd} G_K \arrow[r, "a"] \arrow[d, "g(\sigma)"'] & G_K \arrow[d, "g(\sigma)"] \\ G_K \arrow[r, "\phi(\sigma)(a)"] & G_K \end{tikzcd}$$

for all $\sigma \in \mathrm{Gal}(K/k)$ and $a \in G_K$. Therefore, by Galois descent there is an action of G on Q over k which makes the isomorphism $P \cong X \times_k Q$ equivariant under G. Hence Q is a G-torsor over k whose pull-back to X is the G-torsor P. □

3.3. **The Infinitesimal Case.**

Proposition 3.12. *Let X be a geometrically connected separable scheme over a field k. Let $\bar{x} : \mathrm{Spec}\,(\bar{k}) \to X$ be a geometric point. Then the canonical map*

$$\pi_k^{\bar{k}} : \pi^L(\bar{X}/k, \bar{x}) \to \pi^L(X/k, \bar{x})$$

is surjective, but not*, in general, an isomorphism.*

Proof. Suppose we have a saturated object $(P, G, p) \in I_{\mathrm{lc}}(X/k, \bar{x})$. We take the image of the composition

$$\pi^L(\bar{X}/k, \bar{x}) \to \pi^L(X/k, \bar{x}) \to G,$$

and denote it by H. By 2.2 there is $(\bar{Q}, H, q) \in N(\bar{X}/k, \bar{x})$ with a morphism

$$(\bar{Q}, H, q) \hookrightarrow (\bar{P}, G, p).$$

As $H \subseteq G$ is an infinitesimal closed imbedding (closed imbedding with nilpotent ideal sheaf), so is $\bar{Q} \subseteq \bar{P}$. Now take the quotient by H on both sides. We get a section:

$$\bar{s} : \bar{X} \cong \bar{Q}/H \to \bar{P}/H.$$

Since the projection $\bar{P} \to \bar{P}/H$ is faithfully flat, the ideal sheaf of the section $\bar{s}$ is contained in the ideal sheaf of the infinitesimal imbedding $\bar{Q} \subseteq \bar{P}$. Hence imbedding $\bar{s} : \bar{X} = \bar{Q}/H \hookrightarrow \bar{P}/H$ is also infinitesimal. As $\bar{X}$ is reduced, $\bar{X} = (\bar{P}/H)_{\mathrm{red}}$ is the unique reduced closed subscheme of $\bar{P}/H$. Because k is perfect, we have

$$(\bar{P}/H)_{\mathrm{red}} = (P/H)_{\mathrm{red}} \times_k \bar{k} \hookrightarrow (P/H) \times_k \bar{k} = \bar{P}/H.$$

We have known that the composition

$$\bar{X} = (\bar{P}/H)_{\mathrm{red}} \hookrightarrow \bar{P}/H \to \bar{X}$$

is an isomorphism, so $(P/H)_{\mathrm{red}} \to X$ is also an isomorphism. In this way we get a section s for the X-scheme P/H. Now we pull back the H-torsor $P \to P/H$ via s.

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & X \arrow[d, "s"] \\ P \arrow[r] & P/H \end{tikzcd}$$

Then we get a triple $(Q, H, q) \in I_{\mathrm{lc}}(X/k, \bar{x})$ which dominates (P, G, p). In other words, the map $\pi^L(X/k, \bar{x}) \to G$ factors through the imbedding $H \subseteq G$. Hence if (P, G, p) is saturated in $I_{\mathrm{lc}}(X/k, \bar{x})$ then $(\bar{P}, G, p)$ is saturated in $I_{\mathrm{lc}}(\bar{X}/k, \bar{x})$. This means that $\pi_k^{\bar{k}}$ is surjective. For the failure of the injectivity see 3.15. $\square$

Corollary 3.13. *Let X be a geometrically connected separable scheme over a field k. Let $\bar{x} : \mathrm{Spec}\,(\bar{k}) \to X$ be a geometric point. The canonical map*

$$\pi_k^{\bar{k}} : \pi^L(\bar{X}/k, \bar{x}) \to \pi^L(X/k, \bar{x})$$

is an isomorphism if and only if for any G-torsor $Y \to \bar{X}$ with G a finite local k-group scheme, there exists a G-torsor P over X whose pull-back is isomorphic to Y as a G-torsor.

Proof. This is an immediate consequence of 2.2 and 3.12. $\square$

Lemma 3.14. *Let X be a scheme over a perfect field k of characteristic p. If there is a* reduced *X-scheme Y whose pull-back $\bar{Y}$ is a torsor over $\bar{X}$ under an infinitesimal k-group scheme G, and if Y' is an X-scheme, then any $\bar{X}$-isomorphism $\phi : \bar{Y}' \cong \bar{Y}$ descends to X.*

Proof. The claim is true if only if there is a map $\varphi : Y \to Y'$ fitting into the following diagram

$$\begin{tikzcd} \bar{Y} \arrow[r, "\phi"] \arrow[d] & \bar{Y}' \arrow[d] \\ Y \arrow[r, dotted, "\varphi"] & Y' \end{tikzcd}.$$

The problem being local on X, we could assume $X = \mathrm{Spec}\,(A)$, $Y = \mathrm{Spec}\,(B)$, $Y' = \mathrm{Spec}\,(B')$. We have to show that the image of the composition $\iota : B' \to B' \otimes_k \bar{k} \to B \otimes_k \bar{k}$ lands on B.

Since $\bar{Y}$ is a torsor over $\bar{X}$ under an infinitesimal group scheme, for any $x \in B \otimes_k \bar{k}$, $x^{p^n} \in A \otimes_k \bar{k}$ for $n \in \mathbb{N}$ sufficiently large. This implies that for any $x \in B$, $x^{p^n} \in A$ for $n \in \mathbb{N}$ sufficiently large, because $A \otimes_k \bar{k} \cap B = A$ inside $B \otimes_k \bar{k}$. Conversely, if $x \in B \otimes_k \bar{k}$ and $x^{p^n} \in A$ for some $n \in \mathbb{N}$, then $x \in B$. Indeed, as k is perfect, we can assume $x \in B \otimes_k l$ for some finite separable extension l/k of degree m. Let $l = k(\alpha)$ for some primitive element $\alpha \in l$. Then x can be uniquely written as $x = s_0 + s_1 \otimes \alpha + s_2 \otimes \alpha^2 + \cdots + s_{m-1} \otimes \alpha^{m-1}$ with $s_i \in B$. Since α^{p^n} is still a primitive element in l, i.e. $l = k(\alpha) = k(\alpha^{p^n})$, $x^{p^n} \in A$ implies that $s_i^{p^n} = 0$ for all $i > 0$. As B is reduced, $s_i = 0$ for all $i > 0$, hence $x \in B$. Thus $B \subseteq B \otimes_k \bar{k}$ is the subset consisting of elements whose p^n-th power is in A.

By the same argument as above, any element $x \in B'$ has p^n-th power in A. Hence $\iota(B') \subseteq B \otimes_k \bar{k}$ is contained in B. This completes the proof. ☐

Corollary 3.15. *Let k be a perfect but not separably closed field of characteristic p. If X is $\mathbb{A}^1_k$ or an elliptic curve such that $X(F) = \alpha_{p,k}$ (where F is the relative Frobenius), then the surjective map*

$$\pi^L(\bar{X}/k, \bar{x}) \to \pi^L(X/k, \bar{x})$$

is not an isomorphism.

Proof. In any case, we have a non-trivial $\alpha_{p,k}$-torsor $F : X \to X$ defined by the relative Frobenius. Taking any $a \in \bar{k}\backslash k$ we can define a $\bar{k}-$automorphism of $\alpha_{p,\bar{k}}$ by the following map of Hopf-algebras: $\bar{k}[x]/x^p \to \bar{k}[x]/x^p$ sending $x \mapsto ax$. This automorphism of $\alpha_{p,\bar{k}}$ defines a new action of $\alpha_{p,k}$ on $F : \bar{X} \to \bar{X}$ which makes $\bar{X}$ an $\alpha_{p,k}$-torsor over itself. But the new action $\bar{X} \times_k \alpha_{p,k} \to \bar{X}$ certainly does not descend to $X \times_k \alpha_{p,k} \to X$. However, if $\pi^L(\bar{X}/k, \bar{x}) \to \pi^L(X/k, \bar{x})$ was an isomorphism, then by 3.13 and 3.14 the morphism $\bar{X} \times_k \alpha_{p,k} \to \bar{X}$ descends to $X \times_k \alpha_{p,k} \to X$, a contradiction! ☐

4. The Second Fundamental Sequence

Theorem 4.1. *Let X be a geometrically connected separable scheme over a field k, and $\bar{x} : \mathrm{Spec}\,(\bar{k}) \hookrightarrow X$ be a geometric point. Then there is a natural sequence of $\bar{k}$-group schemes*

$$1 \to \pi^I(\bar{X}/\bar{k}, \bar{x}) \to \pi^I(X/k, \bar{x}) \times_k \bar{k} \to \pi^I(k/k, \bar{x}) \times_k \bar{k} \to 1. \tag{2}$$

It is a complex, always exact on the right, exact on the left if k is perfect and if X is quasi-compact and quasi-separated, but it is in general not exact in the middle for $I = N, E, L$.

Proof. The homomorphism $\theta : \pi^I(\bar{X}/\bar{k},\bar{x}) \to \pi^I(X/k,\bar{x}) \times_k \bar{k}$ is obtained via composing $\chi^I_{\bar{k}/k}$ (cf. 2.18) with the canonical morphism

$$\delta : \quad \pi^I(\bar{X}/k,\bar{x}) \times_k \bar{k} \to \pi^I(X/k,\bar{x}) \times_k \bar{k}$$

obtained by base-change from the morphism $\pi^I(\bar{X}/k,\bar{x}) \to \pi^I(X/k,\bar{x})$ in the first fundamental sequence (1). The fact that (2) is a complex and that the right map is surjective follows from 3.2. As the image of θ is contained in the image of δ the failure of exactness of (2) follows from that of (1). Now we show the left injectivity assuming k perfect and X q.c. and q.s..

Since X is q.c. and q.s. and k is perfect, any saturated triple $(P,G,p) \in I(\bar{X}/\bar{k},\bar{x})$ is defined over some $X \times_k l$, where l is a finite separable extension of k. The Weil restriction $\mathrm{Res}_{X\times_k l/X}(P)$ is then a torsor under $\mathrm{Res}_{l/k}(G)$ over X [BLR, 7.6, Theorem 4 and Proposition 5], and there are canonical adjunction maps

$$\phi : \mathrm{Res}_{X\times_k l/X}(P) \times_k \bar{k} \to P \qquad \text{and} \qquad h : \mathrm{Res}_{l/k}(G) \times_k \bar{k} \to G$$

where ϕ has to be surjective for P is a connected scheme. By choosing a $\bar{k}$-point q in the fibre of $p \in P(\bar{k})$ we get a morphism

$$(\phi,h) : \ (\mathrm{Res}_{X\times_k l/X}(P) \times_k \bar{k}, \mathrm{Res}_{l/k}(G) \times_k \bar{k}, q) \to (P,G,p) \in I(\bar{X}/\bar{k},\bar{x}).$$

This means that for any surjective $\bar{k}$-homomorphism $\pi^I(\bar{X}/\bar{k},\bar{x}) \twoheadrightarrow G$ we can find a k-group scheme H (namely $\mathrm{Res}_{l/k}(G)$) and a homomorphism $\pi^I(X/k) \to H$ with a commutative diagram

$$\begin{tikzcd}
\pi^I(\bar{X}/\bar{k},\bar{x}) \arrow[rr, two heads] \arrow[dr, "\phi"] & & G \\
& H \times_k \bar{k} \arrow[ur, two heads] &
\end{tikzcd}$$

where ϕ is the natural composition $\pi^I(\bar{X}/\bar{k},\bar{x}) \xrightarrow{\theta} \pi^I(X/k) \times_k \bar{k} \to H \times_k \bar{k}$. Thus θ induces a surjection of the Hopf-algebras, and is therefore a closed imbedding. $\square$

5. Further Remarks

Since Nori's original definition is very geometric, it is hard to adapt somearithmetic problems arised from the étale fundamental group to Nori's setting. One of such arithmetic problems is the section conjecture:

Let X be a smooth projective geometrically connected curve of genus $\geqslant 2$ over a field k finitely generated over $\mathbb{Q}$. Let $\bar{x}$ be a geometric point of X. Consider the fundamental exact sequence.

$$\text{(FES)} \qquad 1 \to \pi_1^{\text{ét}}(\bar{X},\bar{x}) \to \pi_1^{\text{ét}}(X,\bar{x}) \xrightarrow{\pi} \mathrm{Gal}(\bar{k}/k) \to 1$$

Let $\mathrm{Section}(k,X)$ be the set of sections of π, i.e. continous group homomorphisms from $\mathrm{Gal}(\bar{k}/k) \to \pi_1^{\text{ét}}(X,\bar{x})$ whose composition with π is the identity of $\mathrm{Gal}(\bar{k}/k)$. In $\mathrm{Section}(k,X)$ we define an equivalence relation: Two sections f,g are equivalent if there exists an element $a \in \pi_1^{\text{ét}}(\bar{X},\bar{x})$ such that f and g differ by the inner automorphism of $\pi_1^{\text{ét}}(X,\bar{x})$ defined by a.

We denote $\mathrm{Section}_{\sim}(k,X)$ the set of *sections classes*. If X has a rational point $y \in X(k)$, then we get a section class $y_* \in \mathrm{Section}_{\sim}(k,X)$ by the functoriality of $\pi_1^{\text{ét}}$. It can be shown that the so defined map $X(k) \to \mathrm{Section}_{\sim}(k,X)$ is injective. The section classes of the form y_* are called *geometric sections*.

Conjecture 5.1. *(Grothendieck's section conjecture) All sections in* $\mathrm{Section}_{\sim}(k,X)$ *are geometric sections.*

Since in Nori's original definition the fundamental group scheme of a field is trivial, it is not possible to directly reformulate the section conjecture in this setting. In [EHai] and [EHai2], H. Esnault and P. H. Hai successfully used the language of fundamental groupoid scheme to get some arithmetic information from Nori's geometric fundamental group scheme, and using this they reformulated the section conjecture and proved the Packet conjecture. In [BV], N. Borne and A. Vistoli greatly generalized Nori's definition, and using the language of gerbes they also gave a reformulation of this conjecture. Here we would like to suggest another thought on this conjecture.

Lemma 5.2. *Let X be a connected reduced scheme over a field k, and $\bar{x} \in X(\bar{k})$. Then we have a canonical map of sets*

$$\Delta : \mathrm{Section}^N(k,X) \longrightarrow \mathrm{Section}(k,X)$$

where $\mathrm{Section}^N(k,X)$ *denotes the set of sections of the canonical surjection* $\pi^N(X/k,\bar{x}) \twoheadrightarrow \pi^N(k/k,\bar{x})$.

Proof. Suppose $f \in \mathrm{Section}^N(k,X)$ is a section. Consider the following commutative diagram

$$\begin{tikzcd}
\pi^N(k/k,\bar{x}) \arrow[r, "f"] \arrow[d, two heads, "\pi^N_G"] & \pi^N(X/k,\bar{x}) \arrow[r, two heads] \arrow[d, two heads, "\pi^N_G"] & \pi^N(k/k,\bar{x}) \arrow[d, two heads, "\pi^N_G"] \\
\pi^G(k/k,\bar{x}) \arrow[r, dotted, "\phi"] & \pi^G(X/k,\bar{x}) \arrow[r, two heads] & \pi^G(k/k,\bar{x})
\end{tikzcd}$$

where ϕ is obtained by the universality of $\pi^N(k/k,\bar{x}) \xrightarrow{\pi^N_G} \pi^G(k/k,\bar{x})$: For any homomorphism $\lambda : \pi^N(k/k,\bar{x}) \to M$ where M is a pro-constant group scheme, there is a unique homomorphism $\delta : \pi^G(k/k,\bar{x}) \to M$ such that $\delta \circ \pi^N_G = \lambda$. Clearly $\phi \in \mathrm{Section}(k,X)$. □

Now denote $\mathrm{Section}^N_{\sim}(k,X)$ the image of the following composition.

$$\mathrm{Section}^N(k,X) \xrightarrow{\Delta} \mathrm{Section}(k,X) \to \mathrm{Section}_{\sim}(k,X)$$

Then $\mathrm{Section}^N_{\sim}(k,X) \subseteq \mathrm{Section}_{\sim}(k,X)$ becomes a subset.

Lemma 5.3. *The subset* $\mathrm{Section}^N_{\sim}(k,X)$ *contains all the geometric sections.*

Proof. Let $y \in X(k)$ be a rational point, and $\bar{y} \in X(\bar{k})$ be the composition of y with $\mathrm{Spec}\,(\bar{k}) \to \mathrm{Spec}\,(k)$. Then we have two fibre functors $\bar{F}_{\bar{x}}, \bar{F}_{\bar{y}}$ from the category of finite étale covers $\mathrm{Ecov}(\bar{X})$ to the category of sets and also the following diagram of categories.

$$\begin{tikzcd}
\mathrm{Ecov}(X) \arrow[r] \arrow[rr, bend left, "F_{\bar{x}}"] \arrow[rr, bend right, "F_{\bar{y}}"'] & \mathrm{Ecov}(\bar{X}) \arrow[r, shift left, "\bar{F}_{\bar{x}}"] \arrow[r, shift right, "\bar{F}_{\bar{y}}"'] & ((\mathrm{Sets}))
\end{tikzcd}$$

Now fix an isomorphism $\bar{F}_{\bar{x}} \xrightarrow{\cong} \bar{F}_{\bar{y}}$, it will then induce an isomorphism $\phi : F_{\bar{x}} \xrightarrow{\cong} F_{\bar{y}}$. Going through the proof of 2.21, we get an isomorphism $\pi^N(X/k, \bar{x}) \xrightarrow{\cong_\phi} \pi^N(X/k, \bar{y})$ which fits into the following commutative diagram.

$$\begin{tikzcd}
\pi^N(X/k, \bar{x}) \arrow[r, "\cong_\phi"] \arrow[d, two heads] & \pi^N(X/k, \bar{y}) \arrow[d, two heads] \\
\pi^G(X/k, \bar{x}) \arrow[r, "\cong_\phi"] & \pi^G(X/k, \bar{y})
\end{tikzcd}$$

This implies immediately that the geometric section y_* comes from $\mathrm{Section}^N_\sim(k, X)$. □

Thus $\mathrm{Section}^N_\sim(k, X)$ is a possibly smaller subset of $\mathrm{Section}_\sim(k, X)$, and the section conjecture would immediately imply:

Conjecture 5.4. *If X is a proper smooth and geometrically connected curve of genus $\geqslant 2$ over a field k finitely generated over $\mathbb{Q}$, then all sections in $\mathrm{Section}^N_\sim(k, X)$ are geometric sections.*

Remark 5.5. The above conjecture could also be formulated in the case when k is a global field. If k is of characteristic $p > 0$, there is a little subtlety in the original formulation of the section conjecture, as there might be closed points in X whose residue field are non-trivial purely inseparable extensions of k, and such points also contribute to sections of the fundamental exact sequence (because the Galois group is insensitive to purely inseparable extensions). Thus in positive characteristic one may expect a smaller subset of sections which correspond to the rational points. In this situation, $\mathrm{Section}^N_\sim(k, X)$ might do this job as the Nori-Galois group does sensitive to purely inseparable extensions. There is another formualtion by F. Pop (see Anabelian Phenomena, pp. 32), where he extended the set of rational points to all closed points whose residues fields are purely inseparable.

There is a more general philosophy behind this section conjecture, namely the anabelian conjecture:

Conjecture 5.6. *(Grothendieck's anabelian conjecture) Let k be a field finitely generated over $\mathbb{Q}$. Let X, Y be two (proper) anabelian schemes over k, and $\bar{x}, \bar{y}$ are geometric points of X and Y. Then the natural map*

$$\mathrm{Hom}_{\mathrm{Sch}/k}(X, Y) \to \mathrm{Hom}_{\mathrm{Gal}(k)}(\pi_1^{\text{ét}}(X, \bar{x}), \pi_1^{\text{ét}}(Y, \bar{y}))/\mathrm{Inn}(\pi_1^{\text{ét}}(\bar{Y}, \bar{y}))$$

is bijective, where $\pi_1^{\text{ét}}(X,\bar{x})$ *and* $\pi_1^{\text{ét}}(Y,\bar{y})$ *are viewed as groups over* $\mathrm{Gal}(k)$ *and the quotient is with respect to the action of* $\pi_1^{\text{ét}}(\bar{Y},\bar{y})$ *on the target via inner automorphisms.*

Roughly speaking the conjecture predicts that there is a full subcategory of k-schemes, i.e. the conjectural *anabelian scheme*, which are reconstructible from their étale fundamental groups. If X is taken to be the base field, then this is more or less just the section conjecture. In my opinion, Nori's fundamental group scheme carries more information than the étale fundamental group, thus a scheme should be more reconstructible from its fundamental group scheme. To start with, M. Romagny, G. Zalamansky and me, we formulated the following conjecture which is an analog of the Neukirch-Uchida theorem ([NSW, 12.2.1, pp. 792]) in the purely inseparable settings.

Conjecture 5.7. *Let* $k=\bar{k}$ *be an algebraically closed field. Let* K/k *be a field extension, and* $\mathrm{PI(K)}$ *be the category of finite purely inseparable extensions of* K *whose morphisms are just* K*-algebra homomorphisms. Let* $Gr.Sch_k/\pi^L(K/k)$ *be the category of* k*-group schemes over the fixed* k*-group scheme* $\pi^L(K/k)$*, i.e. the category of* k*-homomorphisms from some* k*-group scheme to* $\pi^L(K/k)$*. Then the canonical contravariant functor*

$$F : \mathrm{PI(K)} \longrightarrow Gr.Sch_k/\pi^L(K/k)$$

$$L/K \mapsto \pi^L(L/k)/\pi^L(K/k)$$

is fully faithful.

Note that since here we only consider torsors under finite infinitesimal group schemes, by 2.22, the fundamental group schemes are *canonically* isomorphic when we choose different base points. Thus we don't need the quotient by the inner automorphisms to erase the effect brought by the choice of the base points.

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