Report on 'The arithmetic local Nori fundamental group' by Matthieu Romagny, Fabio Tonini, Lei Zhang

1 Summary

Let $\mathfrak{X}$ be a reduced algebraic stack defined over a perfect field k. The authors study the arithmetic variant of the Nori local fundamental group scheme of $\mathfrak{X}$ that they defined in several previous works. The main focus is on the case $\mathfrak{X} = \operatorname{spec} K$ for a field extension K/k, where the corresponding group scheme $\pi^L(K/k)$ is shown to contain a lot of information about purely inseparable extensions of K. There are two main results: a partial analog of the Galois correspondence, and a generic surjectivity theorem for $\pi^L(\mathfrak{X}/k)$.

2 Evaluation

The article strikes by its originality. The Tannakian description of the Nori local fundamental group scheme of a field is very nice. The authors open a completely new path and I have no doubt that this will stimulate further investigations. The last part on generic surjectivity is more technical but also quite relevant as it enables to relate this article to Otabe's recent work on Abhyankar's conjecture. The level of the exposition is excellent. I definitely recommend this article for publication in the *Transactions of the AMS*.

3 Comments

3.1 General criticism

I have two general remarks/questions.

1. There exists a traditional approach of Galois theory for purely inseparable extensions named 'Jacobson correspondence'. It is natural to ask if there is a hidden relationship between the two a priori very distinct strategies.

2. One of the main points of the article is that a local gerbe of a perfect field is uniquely neutral (Lemma 2.7). The neutrality is proved using Tannaka theory. One may wonder if there is a simpler proof. It seems to me that this is the case: namely the last part of the proof of Lemma 2.7, which does not resort to Tannaka theory, seems to imply more or less immediately the neutrality. If this is exact this could be mentioned.

3.2 Specific criticism

1. p.2, Theorem I: I think $\pi^L(E_1/K)$ should be replaced by $\pi^L(E_1/k)$.
2. p.8, end of the Proof of Proposition 2.5, 'By 2.2 it follows that $\beta_{ij} \in k$': the formulation is a bit strange.
3. p.9, Definition 2.10: how does this definition differ from the notion of 'Nori reduced' torsor ?
4. p.12, Definition 3.3: this seems redundant with, precisely, Definition 2.10.
5. p.13, Lemma 3.6: it seems wise to exclude the trivial case $E = 0$.
6. p.14, line 4 and 6: P stands for $\mathcal{P}$.
7. p.14, line 5: there is a typo 'isomophism' instead of isomorphism.