Group Members:
Muhammad Ahmed (3304158)
Cüneyt Erem (3277992)
Ali Mohammadi (3289515)
Rozhin Bayati (3314202)

4.1	4.2	4.3	Σ
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4.1

a) $\|f_c\|_1 = 1, c = ?$

why can you take the | | away?

$$\left(\int_R |f_c(t)|^p dt\right)^{\frac{1}{p}} = 1 \text{ where } p = 1$$

$$\Rightarrow \int_R |f_c(t)| dt = 1 \Rightarrow \int_{-c}^{0} \left(\frac{t}{c}+1\right) dt \int_{0}^{c} \left(\frac{-t}{c}+1\right) dt = 1$$

$$\Rightarrow \frac{t^2}{2c} + t \Big|_{-c}^{0} + \frac{-t^2}{2c} + t \Big|_0^c = 1$$

$$\Rightarrow 0 - \left(\frac{c^2}{2c} - c\right) + \frac{-c^2}{2c} + c - 0 = 1$$

$$\Rightarrow c = 1$$

2,5/3

b) $\|f_c\|_2 = 1, c = ?$

$$\left(\int_R |f_c(t)|^2 dt\right)^{\frac{1}{2}} = 1 \Rightarrow \left(\int_R |f_c^2(t)| dt\right)^{\frac{1}{2}} = 1$$

$$\Rightarrow \left(\int_{-c}^{0} \left|\frac{t}{c}+1\right|^2 dt + \int_0^c \left|-\frac{t}{c}+1\right|^2 dt\right)^{1/2} = 1$$

$$\Rightarrow \left(\int_{-c}^{0} \left(\frac{t^2}{c^2} + \frac{2t}{c} + 1\right) dt + \int_0^c \left(\frac{t^2}{c^2} - \frac{2t}{c} + 1\right) dt\right)^{1/2} = 1$$

$$\Rightarrow \left(\frac{t^3}{3c^2} + \frac{t^2}{c} + t \Big|_{-c}^{0} + \frac{t^3}{3c^2} - \frac{t^2}{c} + t \Big|_0^c\right)^{1/2} = 1$$

$$\Rightarrow \left(0 - \left(\frac{-c^3}{3c^2} + c - c\right) + \frac{c^3}{3c^2} - c + c - 0\right)^{1/2} = 1$$

$$\left(\frac{2c}{3}\right)^{1/2} = 1 \Rightarrow c = \frac{3}{2}$$

3/3

c) $h \in L^1(R)$ but $h \notin L^2(R)$?

$$h(t) = \begin{cases} \frac{1}{\sqrt{1-t^2}}, & t \in [0,1) \\ 0 & \text{otherwise} \end{cases}$$

$$\left(\int_R |h(t)|^1 dt\right)^1 = \int_0^1 \frac{1}{\sqrt{1-t^2}} dt = \arcsin(t) \Big|_0^1 = \frac{\pi}{2} \Rightarrow h(t) \in L^1(R)$$

erstaunlich

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$$\left(\int_R |h(t)|^2 dt\right)^{\frac{1}{2}} = \left(\int_0^1 \left(\frac{1}{\sqrt{1-t^2}}\right)^2 dt\right)^{\frac{1}{2}} = \left(\int_0^1 \frac{1}{1-t^2} dt\right)^{\frac{1}{2}} = \left(\frac{1}{2} \log\left(\frac{t+1}{1-t}\right)\Big|_0^1\right)^{\frac{1}{2}} = \infty \Rightarrow h(t) \notin L^2(R)$$

4.2)

$i \in$?

Positive or zero

$a.27: \|x\|_p = \left(\sum_i |x_i|^p\right)^{\frac{1}{p}} = a \Rightarrow \sum_i |x_i|^p = a^p$

$\Rightarrow a^p = \begin{cases} 0 \Rightarrow |x_i| = 0 \Rightarrow \sum_i |x_i|^q = 0 \Rightarrow \|x\|_q = \|x\|_p \\ > 0 \Rightarrow a^{-p} \sum_i |x_i|^p = 1 \Rightarrow \sum_i |x_i/a|^p = 1 \end{cases}$

$\Rightarrow \left(\sum_i |x_i/a|^p\right)^{1/p} = 1 = \|x_i/a\|_p$ $*$

$\left(\sum_i |x_i|^q\right)^{1/q} = b \Rightarrow \|x_i/a\|_q = \left(\sum |x_i/a|^q\right)^{1/q}$ =

$= \left(a^{-q} \sum |x_i|^q\right)^{1/q} = \frac{b}{a}$?

$q \geq p \Rightarrow q/p \geq 1$

$\sum_i |x_i/a|^p = 1 \Rightarrow \forall_i : |x_i/a|^p \leq 1 \Rightarrow \left(|x_i/a|^p\right)^{q/p} \leq |x_i/a|^p$

$\Rightarrow \sum_i |x_i/a|^q \leq \sum_i |x_i/a|^p \overset{*}{\Rightarrow} \sum_i |x_i/a|^q \leq 1$

$\Rightarrow \left(\sum_i |x_i/a|^q\right)^{1/q} \leq 1 \Rightarrow \frac{b}{a} \leq 1 \Rightarrow b \leq a \Rightarrow \|x\|_q \leq \|x\|_p$

using L1 we have $\|x\|_q \leq \|x\|_p \Rightarrow (\|x\|_q)^q$

$\leq (\|x\|_p)^q = (\|x\|_p^p)^{q/p}$ ①

in the other hand we have $x \in l_p^p \Rightarrow \sum_i |x_i|^p < \infty$

$\Rightarrow \sum_i |x_i|^p = B < \infty \Rightarrow \left(\left(\sum |x_i|^p\right)^{1/p}\right)^p = B < \infty$

$\Rightarrow \|x\|_p^p = B < \infty$ ②

①, ② $\Rightarrow (\|x\|_q)^q \leq B^{q/p} < \infty \Rightarrow \sum_i |x_i|^q \leq B^{q/p} < \infty$

$\Rightarrow x \in l^q \Rightarrow \forall x \in l^p \Rightarrow x \in l^q \Rightarrow l^p \subseteq l^q$

not easy to follow

2,5/3

b) L1: $\sum_{n=1}^{\infty} 1/n$ diverges. Proof: for any $n \in \mathbb{N}$

$\Rightarrow n \leq 2^{\lceil \log_2^n \rceil}$ $(1 \leq 1, 2 \leq 2, 3 \leq 4, 4 \leq 4, \dots)$ and for

$\forall k \in \mathbb{N}$ ther are 2^{k-1} numbers such as m that

$2^{k-1} < m \leq 2^k$ $(k=1: \{2\}, k=2: \{3,4\}, \dots)$

$\Rightarrow n \leq 2^{\lceil \lg_2^n \rceil} \Rightarrow 1/n \geq 2^{-\lceil \lg^n_2 \rceil} \Rightarrow \sum_{n=1}^{\infty} 1/n \geq \sum_{n=1}^{\infty} 2^{-\lceil \lg n \rceil}$

$= 1 + \sum_{m=1}^{\infty} \sum_{n=2^{m-1}+1}^{2^m} 1/n = 1 + \sum_{m=1}^{k=\infty} 1/2 = 1 + \lim_{k \to \infty} \sum_{m=1}^{k} 1/2$

$= 1 + \lim_{k \to \infty} k/2 = \infty \Rightarrow \sum_{n=1}^{\infty} 1/n \geq (1 + \lim_{k \to \infty} k/2) \longrightarrow \infty$

L2: $\sum_{n=1}^{\infty} n^{-p}$ converges for $\forall p : p > 1$

Proof: $\sum_{n=1}^{\infty} n^{-p} = 1 + \sum_{m=1}^{\infty} \sum_{n=2m}^{2m+1} n^{-p} = 1 + \underbrace{\sum_{m=1}^{\infty} (2m)^{-p}}_{S} + (2m+1)^{-p}$.

$\leq 1 + \sum_{m=1}^{\infty} 2 (2m)^{-p} = 1 + 2 \sum_{m=1}^{\infty} (2m)^{-p} = 1 + 2S$

$\Rightarrow S = \sum_{m=1}^{\infty} (2m)^{-p} = 2^{-p} \sum_{m=1}^{\infty} m^{-p} \leq 2^{-p}(1+2S)$

$\Rightarrow S \leq 2^{-p}(1+2S) \Rightarrow S(2^p - 2) \leq 1 \Rightarrow S \leq \frac{1}{2^p - 2}$

$\Rightarrow \sum_{n=1}^{\infty} n^{-p} \leq 1 + \frac{2}{2^p - 2}$

now by L1 we have $\forall p: x_n = \frac{1}{n^{1/p}} \Rightarrow \sum_{1}^{\infty} x_n^p$ diverges

$\Rightarrow x_n \notin \ell^p$ and by L2 we have $\sum_{1}^{\infty} 1/n^{q/p} = \sum_{1}^{\infty} (1/n^{1/p})^q$ converges if $q/p > 1 \Rightarrow q > p \Rightarrow x_n \in \ell^q$ for $q > p$

✓ 3/3

our motive for this solution

c) from $\underbrace{\text{last part (b)}}$ we know that for

$x_n = \begin{cases} n^{-1/p} & n>0 \\ 0 & n\leq 0 \end{cases}$ — diverges $\Rightarrow x_n \notin \ell^p$, $x_n \in \ell^q, q>p$?

$x_n \notin \ell^p \quad \ell^k \subseteq \ell^p$? for $k<p \Rightarrow x_n \notin \ell^k, 1\leq k\leq p$? k ?

So for $p\to\infty \Rightarrow x_n \in \ell^\infty$ and $x_n \notin \ell^p, p\in[1,\infty)$

$\Rightarrow x_n = \begin{cases} \lim_{p\to\infty} n^{-1/p} & n>0 \\ 0 & n\leq 0 \end{cases} = \begin{cases} 1 & n>0 \\ 0 & n\leq 0 \end{cases}$ is $\in \ell^\infty$

f 0/2