

Interrogating theoretical models of neural computation with deep learning

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¹ 1 Abstract

² The cornerstone of theoretical neuroscience is the circuit model: a system of equations that captures
³ a hypothesized neural mechanism of scientific importance. Such models are valuable when they give
⁴ rise to an experimentally observed phenomenon – whether behavioral or in terms of neural activity –
⁵ and thus can offer insight into neural computation. The operation of these circuits, like all models,
⁶ critically depends on the choices of model parameters. Historically, the gold standard has been
⁷ to analytically derive the relationship between model parameters and computational properties.
⁸ However, this enterprise quickly becomes infeasible as biologically realistic constraints are included
⁹ into the model, often resulting in *ad hoc* approaches to understanding the relationship between
¹⁰ model and computation. We bring recent machine learning techniques – the use of deep generative
¹¹ models for probabilistic inference – to bear on this problem, learning distributions of parameters
¹² that produce the specified properties of computation. Importantly, the techniques we introduce offer
¹³ a principled means to understand the implications of model parameter choices on computational
¹⁴ properties of interest. We motivate this methodology with a worked example analyzing sensitivity in
¹⁵ the stomatogastric ganglion. We then use it to generate insights into neuron-type input-responsivity
¹⁶ in a model of primary visual cortex, a new understanding of rapid task switching in superior
¹⁷ colliculus models, and attribution of bias in recurrent neural networks solving a toy mathematical
¹⁸ problem. More generally, this work suggests a departure from realism vs tractability considerations
¹⁹ towards the use of modern machine learning for sophisticated interrogation of biologically relevant
²⁰ models.

²¹ 2 Introduction

²² The fundamental practice of theoretical neuroscience is to use a mathematical model to understand
²³ neural computation, whether that computation enables perception, action, or some intermediate
²⁴ processing [1]. In this field, a neural computation is systematized with a set of equations – the
²⁵ model – and these equations are motivated by biophysics, neurophysiology, and other conceptual
²⁶ considerations. The function of this system is governed by the choice of model parameters, which

27 when configured appropriately, give rise to a measurable signature of a computation. The work of
28 analyzing a model then becomes the inverse problem: given a computation of interest, how can we
29 reason about these suitable parameter configurations – their likely values, their uniquenesses and
30 degeneracies, their attractor states and phase transitions, and more?

31 Consider the idealized practice: a theorist considers a model carefully and analytically derives how
32 model parameters govern the computation. Seminal examples of this gold standard include our
33 field’s understanding of memory capacity in associative neural networks [2], chaos and autocorrela-
34 tion timescales in random neural networks [3], and the paradoxical effect in excitatory/inhibitory
35 networks [4]. Unfortunately, as circuit models include more biological realism, theory via analytic
36 derivation becomes intractable. This fact creates an unfavorable tradeoff for the theorist. On the
37 one hand, one may tractably analyze systems of equations with unrealistic assumptions (for ex-
38 ample symmetry or gaussianity), producing accurate inferences about parameters of a too-simple
39 model. On the other hand, one may choose a more biologically relevant model at the cost of *ad hoc*
40 approaches to analysis (simply examining simulated activity), producing questionable or partial
41 inferences about parameters of an appropriately complex, scientifically relevant model.

42 Of course, this same tradeoff has been confronted in many scientific fields and engineering problems
43 characterized by the need to do inference in complex models. In response, the machine learning
44 community has made remarkable progress in recent years, via the use of deep neural networks as a
45 powerful inference engine: a flexible function family that can map observed phenomena (in this case
46 the measurable signal of some computation) back to probability distributions quantifying the likely
47 parameter configurations. One celebrated example of this approach from the machine learning
48 community, from which we draw key inspiration for this work, is the variational autoencoder [5, 6],
49 which uses a deep neural network to induce an (approximate) posterior distribution on hidden
50 variables in a latent variable model, given data. Indeed, these tools have been used to great success
51 in neuroscience as well, in particular for interrogating parameters (sometimes treated as hidden
52 states) in models of both cortical population activity [7, 8, 9, 10] and animal behavior [11, 12, 13].
53 These works have used deep neural networks to expand the expressivity and accuracy of statistical
54 models of neural data [14].

55 However, these inference tools have not significantly influenced the study of theoretical neuroscience
56 models, for at least three reasons. First, at a practical level, the nonlinearities and dynamics of
57 many theoretical models are such that conventional inference tools typically produce a narrow set
58 of insights into these models. Indeed, only in the last few years has the deep learning toolkit

59 expanded to a point of relevance to this class of problem. Second, the object of interest from a
60 theoretical model is not typically data itself, but rather a qualitative phenomenon – inspection of
61 model behavior, or better, a measurable signature of some computation – an *emergent property* of
62 the model. Third, because theoreticians work carefully to construct a model that has biological
63 relevance, such a model as a result often does not fit cleanly into the framing of a statistical model.
64 Technically, because many such models stipulate a noisy system of differential equations that can
65 only be sampled or realized through forward simulation, they lack the explicit likelihood and priors
66 central to the probabilistic modeling toolkit.

67 To address these three challenges, we developed an inference methodology – ‘emergent property
68 inference’ – which learns a distribution over parameter configurations in a theoretical model. Crit-
69 ically, this distribution is such that draws from the distribution (parameter configurations) corre-
70 spond to systems of equations that give rise to a specified emergent property. First, we stipulate a
71 bijective deep neural network that induces a flexible family of probability distributions over model
72 parameterizations with a probability density we can calculate [15, 16, 17]. Second, we quantify
73 the notion of emergent properties as a set of moment constraints on datasets generated by the
74 model. Thus an emergent property is not a single data realization, but a phenomenon or a fea-
75 ture of the model, which is the central object of interest to the theorist (unlike say the statistical
76 neuroscientist). Conditioning on an emergent property requires an extension of deep probabilistic
77 inference methods, which we have produced [18]. Third, because we can not assume the theoretical
78 model has explicit likelihood on data or the emergent property of interest, we use stochastic gradi-
79 ent techniques in the spirit of likelihood free variational inference [19]. Taken together, emergent
80 property inference (EPI) provides a methodology for inferring and then reasoning about parameter
81 configurations that give rise to particular emergent phenomena in theoretical models.

82 Equipped with this methodology, we investigated three models of current importance in theoretical
83 neuroscience. These models were chosen to demonstrate generality through ranges of biological
84 realism (conductance-based biophysics to recurrent neural networks), neural system function (pat-
85 tern generation to abstract cognitive function), and network scale (four to infinite neurons). First,
86 to motivate the contribution of emergent property inference, we investigated network syncing in
87 a classic model of the stomatogastric ganglion [20]. Second, we generated then evaluated a set
88 of verifiable hypotheses of input-responsibility in a four neuron-type dynamical model of primary
89 visual cortex. Third, we demonstrated how the systematic application of EPI to levels of task
90 performance can generate experimentally testable hypotheses regarding connectivity in superior

91 colliculus. Fourth, we leveraged the flexibility of EPI to uncover the sources of bias in a low-rank
 92 recurrent neural network executing a toy computation. The novel scientific insights offered by EPI
 93 contextualize and clarify the previous studies exploring these models [20, 21, 22, 23] and more
 94 generally offer a quantitative grounding for theoretical models going forward, pointing a way to
 95 how rigorous statistical inference can enhance theoretical neuroscience at large.

96 We note that, during our preparation and early presentation of this work [24, 25], another work
 97 has arisen with broadly similar goals: bringing statistical inference to mechanistic models of neural
 98 circuits [26]. We are excited by this broad problem being recognized by the community, and we
 99 emphasize that these works offer complementary neuroscientific contributions and use different
 100 technical methodologies. Scientifically, our work has focused primarily on systems-level theoretical
 101 models, while their focus has been on lower-level cellular models. Secondly, there are several key
 102 technical differences in the approaches (see Section A.1.4) perhaps most notably is our focus on
 103 the emergent property – the measurable signal of the computation in question, vs their focus
 104 on observed datasets; both certainly are worthy pursuits. The existence of these complementary
 105 methodologies emphasizes the increased importance and timeliness of both works.

106 3 Results

107 3.1 Motivating emergent property inference of theoretical models

108 Consideration of the typical workflow of theoretical modeling clarifies the need for emergent prop-
 109 erty inference. First, the theorist designs or chooses an existing model that, it is hypothesized,
 110 captures the computation of interest. To ground this process in a well-known example, consider
 111 the stomatogastric ganglion (STG) of crustaceans, a small neural circuit which generates multiple
 112 rhythmic muscle activation patterns for digestion [27]. A model of the STG [20] is shown schemat-
 113 ically in Figure 1A, and note that the behavior of this model will be critically dependent on its
 114 parameterization – the choices of conductance parameters $z = [g_{el}, g_{synA}]$. Specifically, the two
 115 fast neurons (f_1 and f_2) mutually inhibit one another, and oscillate at a faster frequency than
 116 the mutually inhibiting slow neurons (s_1 and s_2), and the hub neuron (hub) couples with the fast
 117 or slow population or both. Second, once the model is selected, the theorist defines the emergent
 118 property, the measurable signal of scientific interest. To continue our running STG example, one
 119 such emergent property is the phenomenon of *network syncing* – in certain parameter regimes,
 120 the frequency of the hub neuron matches that of the fast and slow populations at an intermediate

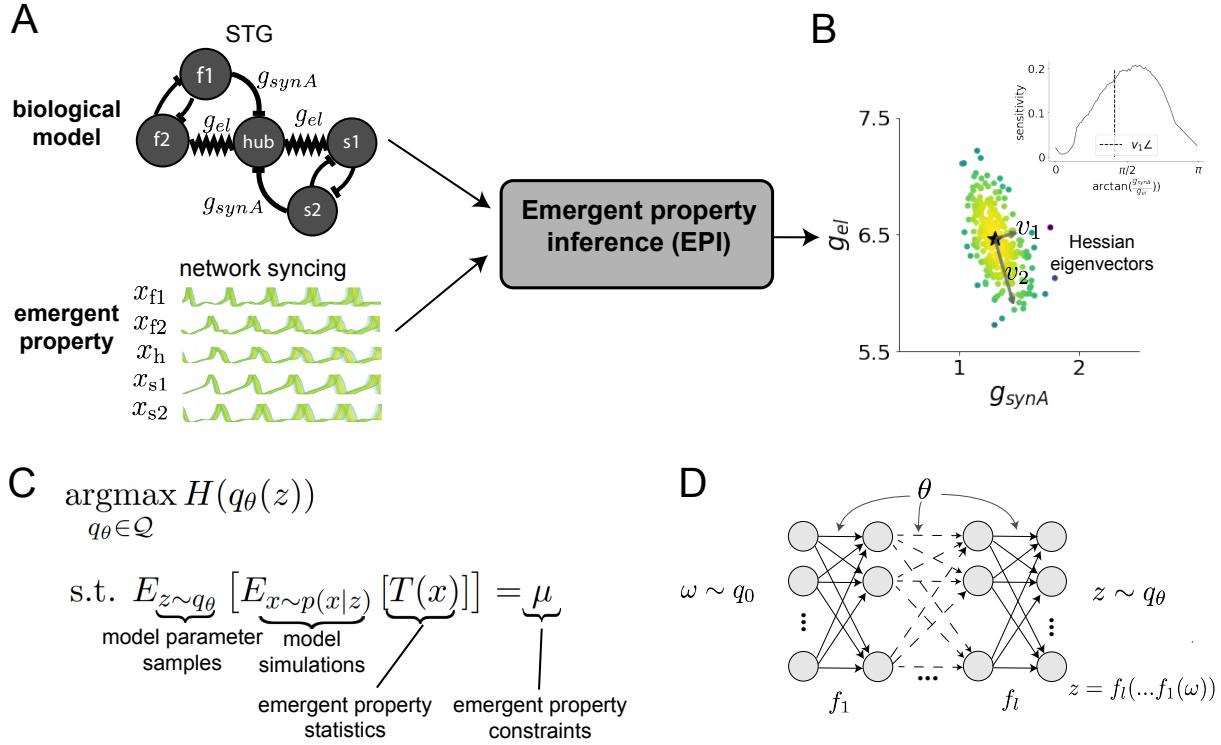


Figure 1: Emergent property inference (EPI) in the stomatogastric ganglion. A. For a choice of model (STG) and emergent property (network syncing), emergent property inference (EPI) learns a posterior distribution of the model parameters $z = [g_{el}, g_{synA}]^\top$ conditioned on network syncing. B. An EPI distribution of STG model parameters producing network syncing. The eigenvectors of the Hessian at the mode of the inferred distribution are indicated as v_1 and v_2 . (Inset) Sensitivity of the system with respect to network syncing along all dimensions of parameter space away from the mode. (see Section A.2.1). C. EPI learns a distribution $q_\theta(z)$ of model parameters that produce an emergent property: the emergent property statistics $T(x)$ are fixed in expectation over parameter distribution samples $z \sim q_\theta(z)$ to particular values μ . EPI distributions maximize randomness via entropy, although other measures are sensible. D. Deep probability distributions map a latent random variable $\omega \sim q_0$, where q_0 is chosen to be simple distribution such as an isotropic gaussian, through a highly expressive function family $f_\theta(\omega) = f_l(\dots f_1(\omega))$ parameterized by the neural network weights and biases $\theta \in \Theta$. This mapping induces an implicit probability model $q(g_\theta(\omega)) \in \mathcal{Q}$

frequency. This emergent property is shown in Figure 1A at a frequency of 0.55Hz. Third, qualitative parameter analysis ensues: since precise mathematical analysis is intractable in this model, a brute force sweep of parameters is done. Subsequently, a qualitative description is formulated to describe of the different parameter configurations that lead to the emergent property. In this last step lies the opportunity for a precise quantification of the emergent property as a statistical feature of the model. Equipped with this methodology, we can infer a probability distribution over parameter configurations that produce this quantified emergent property.

Before presenting technical details (in the following section), let us understand emergent property inference schematically: the black box in Figure 1A takes, as input, the model and the specified emergent property, and produces as output the parameter distribution shown in Figure 1B. This distribution – represented for clarity as samples from the distribution – is then a scientifically meaningful and mathematically tractable object. It conveys parameter regions critical to the emergent property, directions in parameter space that will be invariant (or not) to that property. In the STG model, this distribution can be specifically queried to determine the prototypical parameter configuration for network syncing (the mode; Figure 1B star), and then how quickly network syncing will decay based on changes away that mode (Figure 1B, inset). While it is impossible to determine whether we have converged to the maximum entropy distribution, the emergent property statistics have converged to the emergent property values. For further validation, we apply EPI to condition a two-dimensional linear dynamical system model on a band of oscillations around 1Hz, from which we can analytically derive the contours of the inferred distribution (see Section A.1.1). Taken together, bringing careful inference to theoretical models offers deeper insight into the behavior of these models, and the opportunity to make rigorous this last step in the practice of theoretical neuroscience.

3.2 A deep generative modeling approach to emergent property inference

Emergent property inference (EPI) systematizes the three-step procedure of the previous section. First, we consider the model as a coupled set of differential (and potentially stochastic) equations [20]. In the running STG example the dynamical state $x = [x_{f1}, x_{f2}, x_{hub}, x_{s1}, x_{s2}]$ is the membrane potential for each neuron, which evolves according to the biophysical conductance-based equation:

$$C_m \frac{\partial x}{\partial t} = -h(x; z) = -[h_{leak}(x; z) + h_{Ca}(x; z) + h_K(x; z) + h_{hyp}(x; z) + h_{elec}(x; z) + h_{syn}(x; z)] \quad (1)$$

where $C_m = 1\text{nF}$, and h_{leak} , h_{Ca} , h_K , h_{hyp} , h_{elec} , h_{syn} are the leak, calcium, potassium, hyperpolarization, electrical, and synaptic currents, all of which have their own complicated dependence on x and $z = [g_{\text{el}}, g_{\text{synA}}]$ (see Section A.2.1).

Second, we define the emergent property, which as above is network syncing: the phase locking of the population and its oscillation at an intermediate frequency of our choosing (Figure 1A bottom). Quantifying this phenomenon is straightforward: we define network syncing to be that the spiking frequency of each neuron is close to an intermediate frequency of 0.55Hz. Thus, our measurable signature of computation – the firing frequencies of each neuron $\omega_{f1}(x)$, $\omega_{f2}(x)$, etc.– are statistics of the membrane potential activity x which we insist be near a particular value 0.55Hz. Mathematically, we achieve this via constraints on the mean and variance of $\omega_i(x)$ for each neuron $i \in \{f1, f2, \text{hub}, s1, s2\}$, and thus:

$$E[T(x)] \triangleq E \begin{bmatrix} \omega_{f1}(x) \\ \vdots \\ (\omega_{f1}(x) - 0.55)^2 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0.55 \\ \vdots \\ 0.025^2 \\ \vdots \end{bmatrix} \triangleq \mu, \quad (2)$$

which completes the quantification of the emergent property.

Third, we perform emergent property inference: we find a distribution over parameter configurations z , and insist that samples from this distribution produce the emergent property; in other words, they obey the constraints introduced in Equation 2. This distribution will be chosen from a family of probability distributions $\mathcal{Q} = \{q_\theta(z) : \theta \in \Theta\}$, defined by a deep generative model of the normalizing flow class [15, 16, 17] – neural networks which transform a simple distribution into a suitably complicated distribution (as is needed here). This deep model is represented in Figure 1E (and see Methods for more detail). Then, mathematically, we must solve the following optimization program:

$$\begin{aligned} & \underset{q_\theta \in \mathcal{Q}}{\operatorname{argmax}} H(q_\theta(z)) \\ & \text{s.t. } E_{z \sim q_\theta} [E_{x \sim p(x|z)} [T(x)]] = \mu, \end{aligned} \quad (3)$$

where $T(x), \mu$ are defined as in Equation 3. The purpose of each element in this program is detailed in Figure 1D. Finally, we recognize that many distributions in \mathcal{Q} will respect the emergent property constraints, so we require a normative principle to select amongst them. This principle is captured in Equation 3 by the primal objective H . Here we chose Shannon entropy to model parameter distributions with minimal assumptions beyond some chosen structure [28, 29, 18, 30], but we

¹⁷⁴ emphasize that the EPI method is unaffected by this choice (the results of course will depend on
¹⁷⁵ this choice). Stating such a problem is easy enough; finding a tractable and suitably flexible family
¹⁷⁶ of probability distributions (\mathcal{Q}) is hard.

¹⁷⁷ EPI optimizes the weights and biases θ of the deep neural network (which induces the probability
¹⁷⁸ distribution) by iteratively solving Equation 3. The optimization is complete when the sampled
¹⁷⁹ models with parameters $z \sim q_\theta$ produce activity consistent with the specified emergent property.
¹⁸⁰ Such convergence is evaluated with a hypothesis test that the mean of each emergent property
¹⁸¹ statistic is no different than its emergent property value (see Section A.1.2). Armed with this
¹⁸² method, we now prove out the value of this technology by investigating a range of models and
¹⁸³ using EPI to produce novel scientific insights.

¹⁸⁴ **3.3 Comprehensive input-responsivity in a nonlinear sensory system**

¹⁸⁵ In studies of primary visual cortex (V1), theoretical models with two populations (excitatory (E)
¹⁸⁶ and inhibitory (I) neurons) have reproduced a host of experimentally documented phenomena. In
¹⁸⁷ particular regimes of excitation and inhibition, these E/I models exhibit the paradoxical effect
¹⁸⁸ [4], selective amplification [31], surround suppression [32], and sensory integrative properties [33].
¹⁸⁹ Extending this using experimental evidence of three genetically-defined classes of inhibitory neurons
¹⁹⁰ [34, 35], recent work [21] has investigated a four-population model – excitatory (E), parvalbumin
¹⁹¹ (P), somatostatin (S), and vasointestinal peptide (V) neurons – as shown in Fig. 2A. The dynamical
¹⁹² state of this model is the firing rate of each neuron-type population $x = [x_E, x_P, x_S, x_V]^\top$, which
¹⁹³ evolves according to rectified and exponentiated dynamics:

$$\tau \frac{dx}{dt} = -x + [Wx + h]_+^n \quad (4)$$

¹⁹⁴ with effective connectivity weights W and input h . In our analysis, we set the time constant
¹⁹⁵ $\tau = 20\text{ms}$ and dynamics coefficient $n = 2$. Also, as is fairly standard, we obtain an informative
¹⁹⁶ estimate of the effective connectivities between these neuron-types W in mice by multiplying their
¹⁹⁷ probability of connection with their average synaptic strength [?] (citation on way) (see Section
¹⁹⁸ A.2.2). Given these fixed choices of W , n , and τ , we studied the system’s response to input.

$$h = b + dh \quad (5)$$

¹⁹⁹ where the input h is comprised of a baseline input $b = [b_E, b_P, b_S, b_V]$ and a differential input
²⁰⁰ $dh = [dh_E, dh_P, dh_S, dh_V]$ to each neuron-type population. Throughout subsequent analyses, the
²⁰¹ baseline input is $b = [1, 1, 1, 1]$.

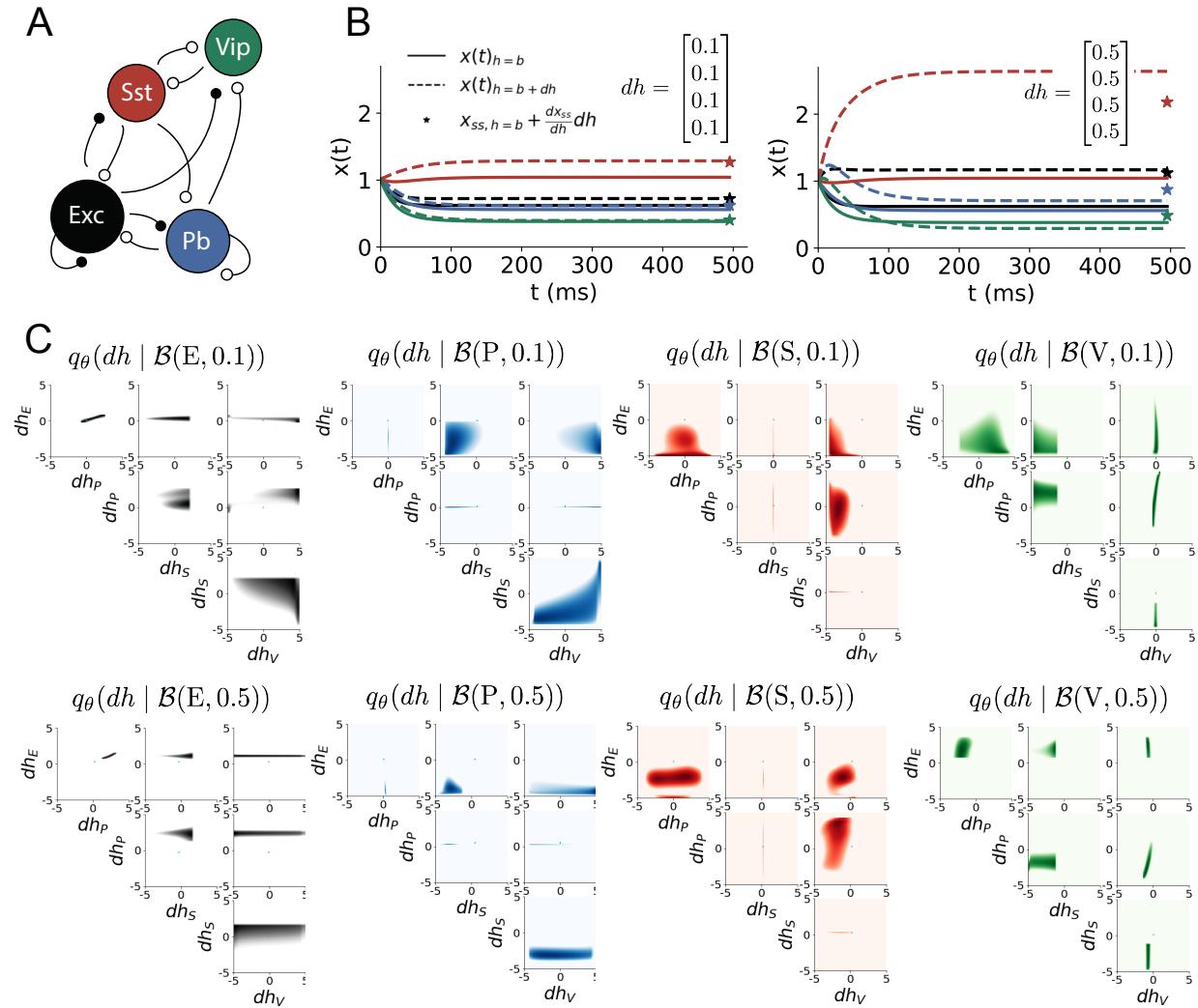


Figure 2: Exploring neuron-type responsivity in V1. A. Four-population model of primary visual cortex with excitatory (black), parvalbumin (blue), somatostatin (red), and vip (green) neurons. Some neuron-types largely do not form synaptic projections to others (excitatory and inhibitory projections filled and unfilled, respectively). B. Linear response predictions become inaccurate with greater input strength. V1 model simulations for input (solid) $h = b$ and (dashed) $h = b + dh$ with $b = [1, 1, 1, 1]^\top$ and (left) $dh = [0.1, 0.1, 0.1, 0.1]^\top$ (right) $dh = [0.5, 0.5, 0.5, 0.5]^\top$. Stars indicate the linear response prediction. C. EPI distributions on differential input dh conditioned on differential response $\mathcal{B}(\alpha, y)$ (see text). The linear prediction from two standard deviations away from y (from negative to positive) is overlaid in cyan (very small, near origin).

Having established our model, we now turn to defining the emergent property. We begin with the linearized response of the system $\frac{dx_{ss}}{dh}$ at fixed points x_{ss} . While this linearization accurately predicts differential responses $dx_{ss} = [dx_{E,ss} \ dx_{P,ss} \ dx_{S,ss} \ dx_{V,ss}]$ for small differential inputs to each population $dh = [0.1 \ 0.1 \ 0.1 \ 0.1]$ (Fig. 2B, left), it can be misleading in such a nonlinear model for a greater differential input strength $dh = [0.1 \ 0.1 \ 0.1 \ 0.1]$ (Fig. 3B, right). In fact, the linearly predicted response for the V-population to $dh = [0.5 \ 0.5 \ 0.5 \ 0.5]$ was actually in the opposite direction of the true response (Fig. 2B, right, green). This shows that currently available approaches to deriving the steady state response of this system are limited.

To get a more comprehensive picture of the input-responsivity of each neuron-type, we used EPI to learn a distribution of differential inputs dh that cause the rate of each neuron-type population $\alpha \in \{E, P, S, V\}$ to increase by a value $y \in 0.1, 0.5$ denoted by the emergent property

$$\mathcal{B}(\alpha, y) \leftrightarrow E \begin{bmatrix} dx_{\alpha,ss} \\ (dx_{\alpha,ss} - y)^2 \end{bmatrix} = \begin{bmatrix} y \\ 0.01^2 \end{bmatrix} \quad (6)$$

Note that we restrict the variance of the emergent property statistic $dx_{\alpha,ss}$ by setting its second moment to a small value. In Fig. 2C, each column visualizes the inferred distribution of dh corresponding to a specific neuron-type increase, while each row corresponds to amounts of increase 0.1 and 0.5. For visualization of this four-dimensional distribution, we show the two-dimensional marginal densities. The inferred distributions suggest a slate of testable hypotheses. 1. As expected, each neuron-type's rate is sensitive to its direct input. 2. The E- and P-populations are largely unaffected by dh_V . 3. Similarly, The S-population is largely unaffected by dh_P . 4. Since EPI showed that negative dh_E results in small $dx_{V,ss}$, but positive dh_E elicited a larger $dx_{V,ss}$ we predict that there is a nonmonotonic response of $dx_{V,ss}$ with dh_E .

We evaluate these hypotheses by taking steps in individual neuron-type input Δh_α away from the modes of the inferred distributions

$$dh^* = z^* = \underset{z}{\operatorname{argmax}} \log q_\theta(z | \mathcal{B}(\alpha, 0.1)) \quad (7)$$

Now, $dx_{\alpha,ss}$ is the steady state response to the system with input $h = b + dh^* + \Delta h_\alpha u_\alpha$ where u_α is a unit vector in the dimension of α . Our hypotheses suggested by EPI are confirmed. 1. the neuron-type responses are sensitive to their direct inputs (Fig. 3A black, 3B blue, 3C red, 3D green), 2. the E- and P-populations are not affected by dh_V (Fig. 3A green, 3B green), 3. the S-population is not affected by dh_P (Fig. 3C blue), and 4. the V-population has a nonmonotonic response to dh_E (Fig. 3D black). All of this validated insight gained beyond what the analytic linear prediction told us (Fig. 2C, cyan).

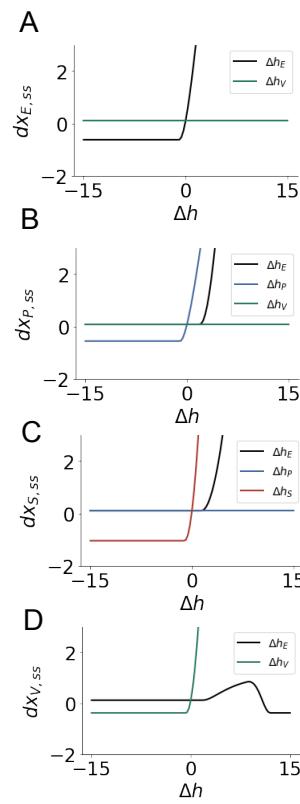


Figure 3: Confirming EPI generated hypotheses in V1. A. Differential responses by the E-population to changes in individual input $\Delta h_\alpha u_\alpha$ away from the mode of the EPI distribution dh^* . B-D Same plots for the P-, S-, and V-populations for the inputs for which hypotheses were formulated.

231 To this point, we have shown the utility of EPI on relatively low-level emergent properties like
 232 network syncing and differential neuron-type population responses. In the remainder of the study,
 233 we focus on using EPI to understand models of more abstract cognitive function.

234 3.4 Identifying neural mechanisms of behavioral learning.

235 Identifying measurable biological changes that result in improved behavior is important for neuro-
 236 science, since they may indicate how the learning brain adapts. In a rapid task switching exper-
 237 iment [36], where rats were to respond right (R) or left (L) to the side of a light stimulus in the
 238 pro (P) task, and oppositely in the anti (A) task predicated by an auditory cue (Fig. 3A), neural
 239 recordings exhibited two population of neurons in each hemisphere of superior colliculus (SC) that
 240 simultaneously represented both task condition and motor response: the Pro/contralateral and
 241 Anti/ipsilateral neurons [22]. Duan et al. proposed a model of SC that, like the V1 model analyzed
 242 in the previous section, is a four-population dynamical system. Here, the neuron-type populations
 243 are functionally-defined as the Pro- and Anti-populations in each hemisphere (left (L) and right
 244 (R)). The Pro- or Anti-populations receive an input determined by the cue, and then the left and
 245 right populations receive an input based on the side of the light stimulus. Activities were bounded

from 0-1, so that a high output (1) of the Pro population in a given hemisphere corresponds to the contralateral response. An additional stipulation is that when one Pro population responds with a high-output, the opposite Pro population must respond with a low output (0). Finally, this circuit operates in the presence of gaussian noise resulting in trial-to-trial variability (see Section A.2.3). The connectivity matrix is parameterized by the geometry of the population arrangement (Fig. 3B).

Here, we used EPI to learn connectivity distributions consistent with various levels of accuracy in the rapid task switching behavioral paradigm. EPI was used to learn distributions of the SC weight matrix parameters $z = W$ conditioned on of various levels of rapid task switching accuracy $\mathcal{B}(p)$ for $p \in \{50\%, 60\%, 70\%, 80\%, 90\%\}$ (see Section A.2.3). There is a decomposition for of the connectivity matrix $W = QAQ^{-1}$, in which the eigenvectors q_i are the same for all W (Fig. 3C). These consistent eigenvectors have intuitive roles in processing for this task, and are accordingly named the *all* - all neurons co-fluctuate, *side* - one side dominates the other, *task* - the Pro or Anti populations dominate the other, and *diag* - Pro- and Anti-populations of opposite hemispheres dominate the opposite pair. The corresponding eigenvalues (e.g. a_{task} , which change according to W) indicate the degree to which activity along that mode is increased or decreased by W .

For greater task accuracies, the task mode eigenvalue increases, indicating the criticality of supporting the task representation in the connectivity of W , (Fig. 4D, purple). Stepping from random chance (50%) networks to marginally task-performing (60%) networks, there is a marked decrease of the side mode eigenvalues (Fig. 3D, orange). Such side mode suppression remains in the models achieving greater accuracy, revealing its importance towards task performance. There were no interesting trends with learning in the all or diag mode. Significantly, we can conclude from our methodology optimized to find all connectivities consistent with a level of accuracy, that side mode suppression in W allows rapid task switching, and that greater task-mode representations in W increase accuracy. These hypotheses are proved out in the model (Fig. 3E). Thus, our EPI-enabled analyses produce novel, experimentally testable predictions that effective connectvity between these populations changes throughout learning in a way that increases its task mode and decreses its side mode eigenvalues.

3.5 Characterizing the sources of bias in RNN computation

So far, each biologically realistic model we have studied was designed from fundamental biophysical principles, genetically- or functionally-defined neuron types. At a more abstract level of modeling,

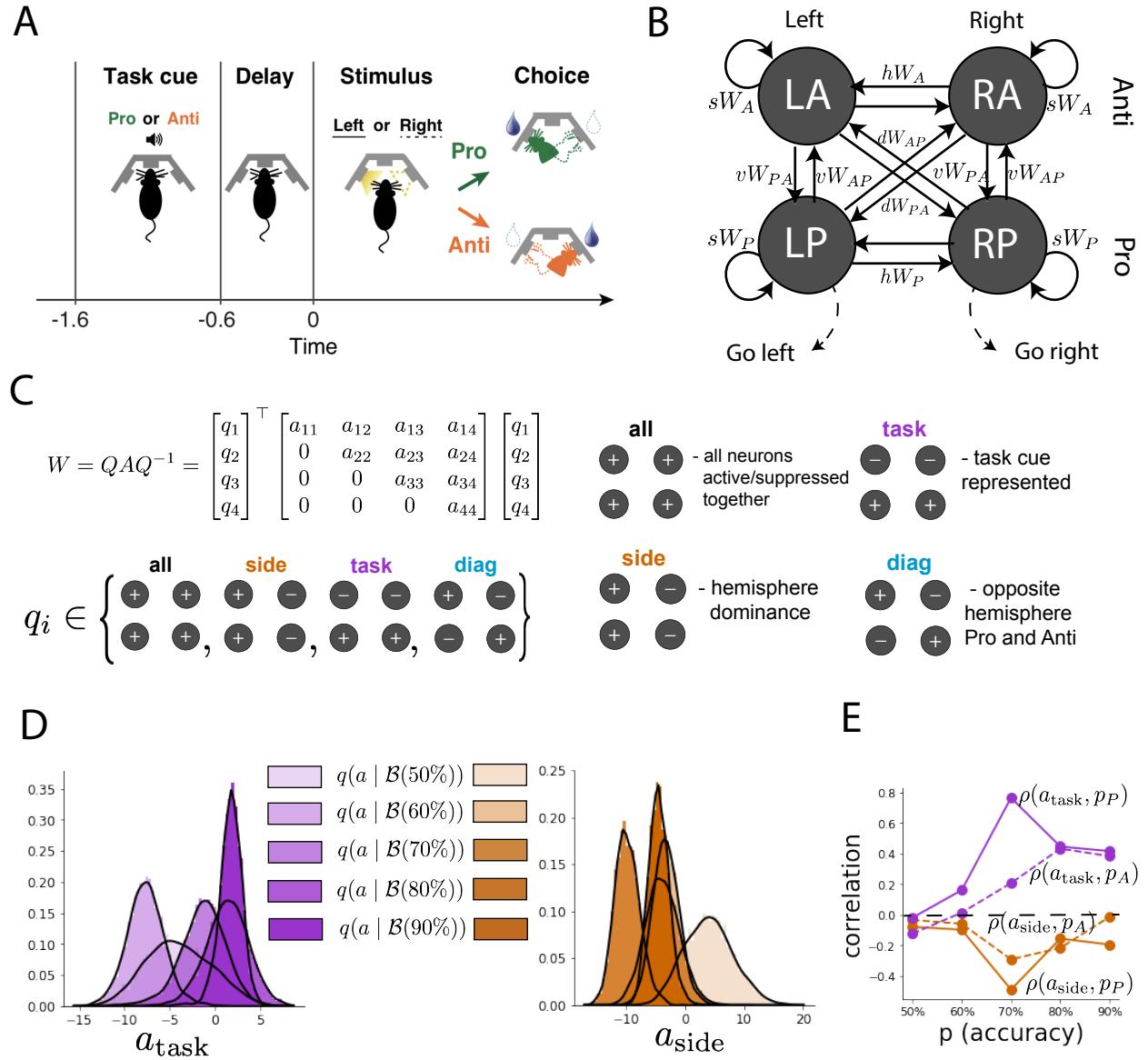


Figure 4: EPI reveals changes in SC [22] connectivity that result in greater task accuracy. A. Rapid task switching behavioral paradigm. In the Pro (Anti) condition indicated by an auditory cue, rats respond by poking into a side port to the same (opposite) side as the light stimulus that is provided after a delay to receive a reward. B. Model of superior colliculus (SC). Neurons: LP - left pro, RP - right pro, LA - left anti, RA - right anti. Parameters: sW - self, hW - horizontal, vW - vertical, dW - diagonal weights. C. The Schur decomposition of the weight matrix $W = QAQ^{-1}$ is a unique decomposition with orthogonal Q and upper triangular A . The invariant Schur eigenmodes (symmetry of W) are labeled by their hypothesized role in computation: q_{all} , q_{task} , q_{side} , and q_{diag} . The values of A are what change for different realizations of W . D. The marginal EPI distributions of the Schur eigenvalues at each level of task accuracy. E. The correlation of Schur eigenvalue with task performance in each learned EPI distribution.

277 recurrent neural networks (RNNs) are high-dimensional models of computation, which have become
 278 increasingly popular in neuroscience research [37]. Typically, RNNs are trained to do a task from a
 279 systems neuroscience experiment, and then the unit activations of the trained RNN are compared
 280 to recorded neural activity. A monumental challenge for this line of work is to link findings at
 281 this level of abstraction with interpretable biophysical mechanisms in the brain. Here we leverage
 282 recent theoretical work to run EPI on interpretable parameterizations of RNN connectivity solving
 283 a toy problem.

284 Importantly, recent work establishes such a link between RNN connectivity weights and the re-
 285 sulting dynamical responses of the network using dynamic mean field theory (DMFT) for neural
 286 networks [3]. Specifically, DMFT describes the properties of activity in infinite-size neural networks
 287 given a distribution on the connectivity weights. This theory has been extended from random neural
 288 networks to low rank RNNs, which have low-dimensional parameterizations of RNN connectivity
 289 via the pairwise correlations of the low-rank vectors (i.e. the low-rank “geometry”) [23]. For
 290 example, the connectivity of a rank-1 RNN J is the sum of a random component with strength
 291 determined by g and a structured component determined by the outer product of vectors m and n :

$$J = g\chi + \frac{1}{N}mn^\top \quad (8)$$

292 where the activity x evolves as

$$\frac{\partial x}{\partial t} = -x(t) + J\phi(x(t)) + I(t) \quad (9)$$

293 $I(t)$ is some input, ϕ is the tanh nonlinearity, and $\chi_{ij} \sim \mathcal{N}(0, \frac{1}{N})$. The entries of m and n are
 294 drawn from gaussian distributions $m_i \sim \mathcal{N}(M_m, 1)$ and $n_i \sim \mathcal{N}(M_n, 1)$, whose parameters M_m and
 295 M_n determine their degree of correlation.

296 Mastrogiuseppe et al. are able to design low-rank connectivities via the pairwise correlations of such
 297 low-rank vectors that solve tasks from behavioral neuroscience. An important detail is that a non-
 298 linear system of equations solver must be used to obtain the task-relevant variables of interest from
 299 the derived consistency equations (see Section A.2.4). However, we can consider the DMFT equa-
 300 tion solver as a black box that takes in a low-rank parameterization z (e.g. $z = [g \ M_m \ M_n]$) and
 301 outputs task-relevant response variables (e.g. average network activity μ , the temporal variability
 302 in the network Δ_T , or network activity along a given dimension κ). Furthermore, we recognize that
 303 the solution produced by the solver is differentiable with respect to the input parameters. Thus, we
 304 are able to combine this DMFT with EPI to learn distributions on such connectivity parameters

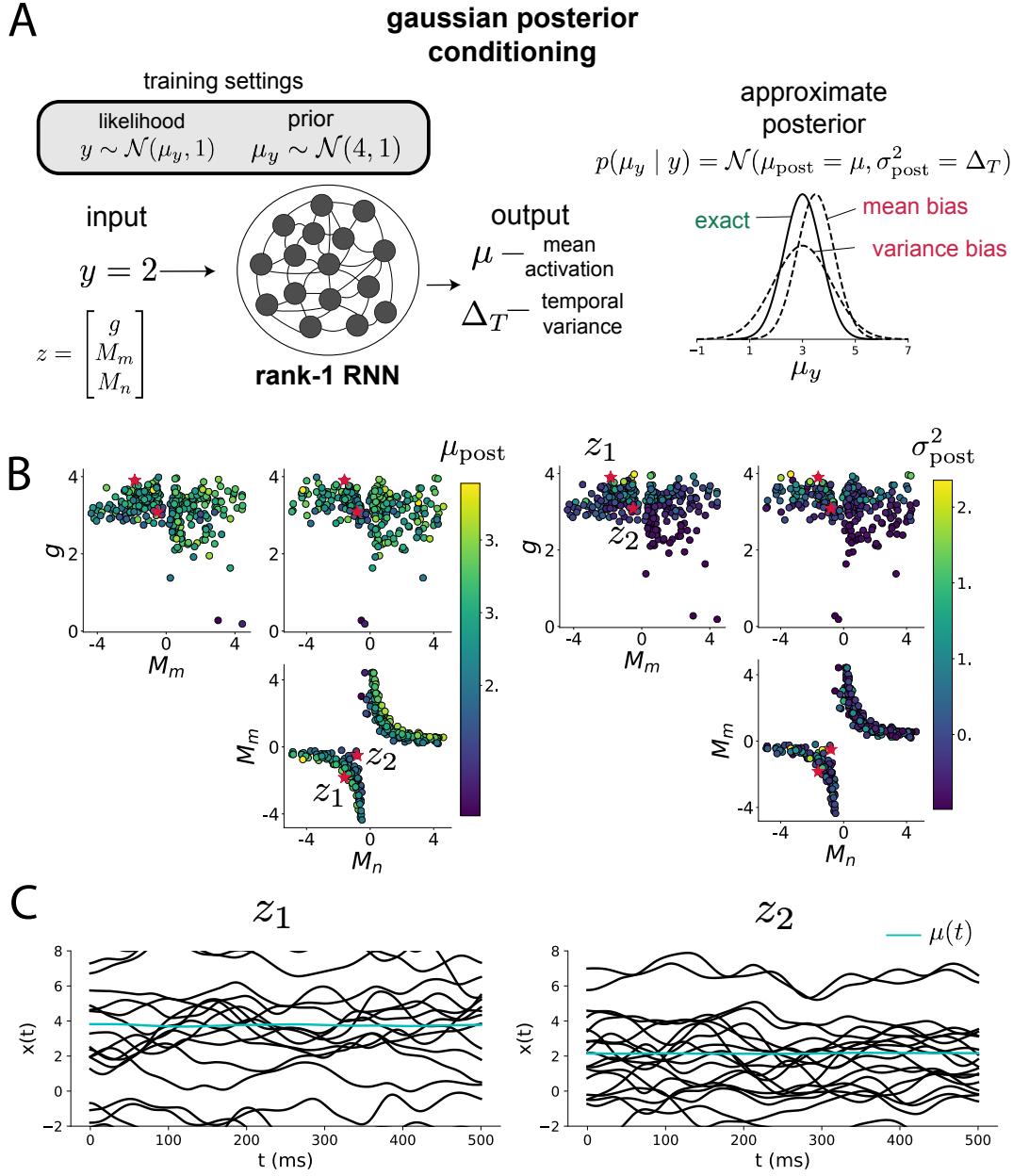


Figure 5: Sources of bias in RNN computation. A. (left) A rank-1 RNN running approximate Bayesian inference on μ_y assuming a gaussian likelihood variance of 1 and a prior of $\mathcal{N}(4, 1)$. (center) The rank-1 RNN represents the computed gaussian posterior mean μ_{post} and variance σ_{post}^2 in its mean activity μ and its temporal variance Δ_T . (right) Bias in this computation can come from over- or under-estimating the posterior mean or variance. B. Distribution of rank-1 RNNs executing approximate Bayesian inference. Samples are colored by (left) posterior mean $\mu_{\text{post}} = \mu$ and (right) posterior variance $\sigma_{\text{post}}^2 = \Delta_T$. C. Finite size realizations agree with the DMFT theory.

305 of RNNs that execute neuroscientific tasks via an emergent property defined on the task-relevant
 306 responses produced by DMFT.

307 For our toy problem, we consider the emergent property of gaussian posterior conditioning. Specif-
 308 ically, we ask the RNN to calculate the parameters of a gaussian posterior distribution on the mean
 309 of a gaussian likelihood μ_y given a single observation of y and a gaussian prior $p(\mu_y) = \mathcal{N}(4, 1)$
 310 (Fig. 5A). Assuming the variance of the gaussian likelihood is 1, the true posterior for an input of
 311 $y = 2$ is $p(\mu_y | y) = \mathcal{N}(3, 0.5)$. We used EPI to learn distributions of RNNs producing the correct
 312 posterior mean and variance in their mean activity $\mu = \mu_{\text{post}}$ and temporal variance $\Delta_T = \sigma_{\text{post}}^2$
 313 given an input of $y = 2$. (see Section A.2.4) (Fig. 5B).

314 When specifying the emergent property of gaussian posterior conditioning, we allowed a substantial
 315 amount of variability in the second moment constraints of the network mean μ and temporal
 316 variance Δ_T . This resulted in a distribution of rank-1 RNN parameterizations having a wide
 317 variety biases in the resulting μ_{post} and σ_{post}^2 (under- or over-estimations of the posterior means
 318 and variances). We can examine the nature of the biases in this toy computation by visualizing the
 319 produced posterior means (Fig. 5B, left) and variances (Fig. 5B, right) in the inferred distribution.
 320 The infererred distribution has roughly symmetric in the M_m - M_n plane, suggesting there is a
 321 degeneracy in the product of M_m and M_n (Fig. 5B). The product of M_m and M_n almost completely
 322 determines the posterior mean (Fig. 5B, left), and the random strength g is the most influential
 323 variable on the temporal variance (Fig. 5B, right). Neither of these observations were obvious from
 324 the consistency equations afforded by DMFT (see Section A.2.4).

325 When working with DMFT, it's important to check that finite-size realizations of these infinite-
 326 size networks match the theoretical predictions. We check 2,000-neuron realizations of drawn
 327 parameters z_1 and z_2 from the inferred distribution. z_1 has relatively high g and high $M_m M_n$,
 328 whereas z_2 has relatively low g and low $M_m M_n$. Confirming our intuition, z_1 overestimates the
 329 posteior mean, since mean activity $\mu(t) > 3$ (Fig. 5C, left cyan). In turn, z_2 underestimates the
 330 posterior mean, since $\mu(t) < 3$ (Fig. 5C, right cyan). Finally, z_1 results in evidently greater temporal
 331 variance than z_2 . This novel procedure of doing inference in interpretable parameterizations of
 332 RNNs conditioned on task execution is straightforwardly generalizable to other tasks like noisy
 333 integration and context-dependent decision making (Fig. S1).

³³⁴ 4 Discussion

³³⁵ 4.1 EPI is a general tool for theoretical neuroscience.

³³⁶ Models of biological systems often have complex nonlinear differential equations, making traditional
³³⁷ statistical inference intractable. In contrast, EPI is capable of learning distributions of parameters
³³⁸ in such models producing measurable signatures of computation. We have demonstrated its utility
³³⁹ on biological models (STG), intermediate-level models of interacting genetically- and functionally-
³⁴⁰ defined neuron-types (V1, SC), and the most abstract of models (RNNs). We are able to condi-
³⁴¹ tion both deterministic and stochastic models on low-level emergent properties like firing rates of
³⁴² membrane potentials, as well as high-level cognitive function like approximate Bayesian inference.
³⁴³ Technically, EPI is tractable when the emergent property statistics are continuously differentiable
³⁴⁴ with respect to the model parameters, which is very often the case; this emphasizes the general
³⁴⁵ utility of EPI.

³⁴⁶ In this study, we have focused on applying EPI to low dimensional parameter spaces of models
³⁴⁷ with low dimensional dynamical state. These choices were made to present the reader with a series
³⁴⁸ of interpretable conclusions, which is more challenging in high dimensional spaces. In fact, EPI
³⁴⁹ should scale reasonably to high dimensional parameter spaces, as the underlying technology has
³⁵⁰ produced state-of-the-art performance on high-dimensional tasks such as texture generation [18].
³⁵¹ Of course, increasing the dimensionality of the dynamical state of the model makes optimization
³⁵² more expensive, and there is a practical limit there as with any machine learning approach. For
³⁵³ systems with high dimensional state, we recommend using theoretical approaches (e.g. [23]) to
³⁵⁴ reason about reduced parameterizations of such high-dimensional systems.

³⁵⁵ There are additional technical considerations when assessing the suitability of EPI for a particu-
³⁵⁶ lar modeling question. First and foremost, as in any optimization problem, the defined emergent
³⁵⁷ property should always be appropriately conditioned (constraints should not have wildly different
³⁵⁸ units). Furthermore, if the program is underconstrained (not enough constraints), the distribution
³⁵⁹ grows (in entropy) unstably unless mapped to a finite support. If overconstrained, there is no pa-
³⁶⁰ rameter set producing the emergent property, and EPI optimization will fail (appropriately). Next,
³⁶¹ one should consider the computational cost of the gradient calculations. In the best circumstance,
³⁶² there is a simple, closed form expression (e.g. Section A.1.1) for the emergent property statistic
³⁶³ given the model parameters. On the other end of the spectrum, many forward simulation iterations
³⁶⁴ may be required before a high quality measurement of the emergent property statistic is available

365 (e.g. Section A.2.1). In such cases, optimization will be expensive.

366 **4.2 Novel hypotheses from EPI**

367 Machine learning has played an effective, multifaceted role in neuroscientific progress. Primarily,
368 it has revealed structure in large-scale neural datasets [39, 40, 41, 42, 43, 44] (see review, [14]).
369 Secondarily, trained algorithms of varying degrees of biological relevance are beginning to be viewed
370 as fully-observable computational systems comparable to the brain [45, 46]. Theorists can use deep
371 learning for probabilistic inference to understand their models and their behavior.

372 For example, consider the fact that we do not yet understand just a four-dimensional, deterministic
373 model of V1 [21]. This should not be surprising, since analytic approaches to studying nonlinear
374 dynamical systems become increasingly complicated when stepping from two-dimensional to three-
375 or four-dimensional systems in the absence of restrictive simplifying assumptions [47]. We promote
376 the recognition of analytic difficulty, and alternatively the use of EPI to gain the desired model
377 insights. In Section 3.3, we showed that EPI was far more informative about neuron-type input
378 responsivity than the predictions afforded through analysis. By flexibly conditioning this V1 model
379 on different emergent properties, we performed an exploratory analysis of a *model* rather than a
380 dataset, which generated and proved out a set of testable predictions.

381 Exploratory analyses can be directed. For example, when interested in model changes during learn-
382 ing, one can use EPI to condition on various levels of an emergent property statistic indicative of
383 performance like task accuracy in a behavioral paradigm (see Section 3.4). This analysis iden-
384 tified experimentally testable predictions (proved out *in-silico*) of changes in connectivity in SC
385 throughout learning of a rapid task switching behavior. Precisely, we predict an initial reduction
386 in side mode eigenvalue, and a steady increase in task mode eigenvalue in the effective connectivity
387 matrices of learning rats.

388 In our final analysis, we present a novel procedure for doing statistical inference on interpretable
389 parameterizations of RNNs executing tasks from behavioral neuroscience. This methodology relies
390 on recently extended theory of responses in random neural networks with minimal structure [23].
391 With this methodology, we can finally open the probabilistic model selection toolkit reasoning
392 about the connectivity of RNNs solving tasks.

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527 **A Methods**

528 **A.1 Emergent property inference (EPI)**

529 Emergent property inference (EPI) learns distributions of theoretical model parameters that pro-
 530 duce emergent properties of interest. EPI combines ideas from likelihood-free variational inference

[19] and maximum entropy flow networks [18]. A maximum entropy flow network is used as a deep probability distribution for the parameters, while these samples often parameterize a differentiable model simulator, which may lack a tractable likelihood function.

Consider model parameterization z and data x generated from some theoretical model simulator represented as $p(x | z)$, which may be deterministic or stochastic. Theoretical models usually have known sampling procedures for simulating activity given a circuit parameterization, yet often lack an explicit likelihood function due to the nonlinearities and dynamics. With EPI, a distribution on parameters z is learned, that yields an emergent property of interest \mathcal{B} ,

$$\mathcal{B} \leftrightarrow E_{z \sim q_\theta} [E_{x \sim p(x|z)} [T(x)]] = \mu \quad (10)$$

by making an approximation $q_\theta(z)$ to $p(z | \mathcal{B})$ (see Section A.1.5). So, over the DSN distribution $q_\theta(z)$ of model $p(x | z)$ for behavior \mathcal{B} , the emergent properties $T(x)$ are constrained in expectation to μ .

In deep probability distributions, a simple random variable $w \sim p_0$ is mapped deterministically via a function f_θ parameterized by a neural network to the support of the distribution of interest where $z = f_\theta(w) = f_l(\dots f_1(w))$. Given a theoretical model $p(x | z)$ and some behavior of interest \mathcal{B} , the deep probability distributions are trained by optimizing the neural network parameters θ to find a good approximation q_θ^* within the deep variational family Q to $p(z | \mathcal{B})$.

In most settings (especially those relevant to theoretical neuroscience) the likelihood of the behavior with respect to the model parameters $p(T(x) | z)$ is unknown or intractable, requiring an alternative to stochastic gradient variational bayes [5] or black box variational inference[48]. These types of methods called likelihood-free variational inference (LFVI, [19]) skate around the intractable likelihood function in situations where there is a differentiable simulator. Akin to LFVI, DSNs are optimized with the following objective for a given theoretical model, emergent property statistics $T(x)$, and emergent property constraints μ :

$$\begin{aligned} q_\theta^*(z) &= \underset{q_\theta \in Q}{\operatorname{argmax}} H(q_\theta(z)) \\ \text{s.t. } E_{z \sim q_\theta} [E_{x \sim p(x|z)} [T(x)]] &= \mu \end{aligned} \quad (11)$$

Optimizing this objective is a technological accomplishment in its own right, the details of which we elaborate in Section A.1.2. Before going through those details, we ground this optimization in a toy example.

557 **A.1.1 Example: 2D LDS**

558 To gain intuition for EPI, consider two-dimensional linear dynamical systems, $\tau \dot{x} = Ax$ with

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$$

559 that produce a band of oscillations. To do EPI with the dynamics matrix elements as the free
 560 parameters $z = [a_1, a_2, a_3, a_4]$, and fixing $\tau = 1$, such that the posterior yields a band of oscillations,
 561 the emergent property statistics $T(x)$ are chosen to contain the first- and second-moments of the
 562 oscillatory frequency Ω and the growth/decay factor d of the oscillating system. To learn the
 563 distribution of real entries of A that yield a distribution of d with mean zero with variance 0.25^2 ,
 564 and oscillation frequency Ω with mean 1 Hz with variance $(0.1\text{Hz})^2$, then we would select the real
 565 part of the complex conjugate eigenvalues $\text{real}(\lambda_1) = d$ (via an arbitrary choice of eigenvalue of the
 566 dynamics matrix λ_1) and the positive imaginary component of one of the eigenvalues $\text{imag}(\lambda_1) =$
 567 $2\pi\Omega$ as the emergent property statistics. Those emergent property statistics are then constrained
 568 to

$$\mu = E \begin{bmatrix} \text{real}(\lambda_1) \\ \text{imag}(\lambda_1) \\ (\text{real}(\lambda_1) - 0)^2 \\ (\text{imag}(\lambda_1) - 2\pi\Omega)^2 \end{bmatrix} = \begin{bmatrix} 0.0 \\ 2\pi\Omega \\ 0.25^2 \\ (2\pi 0.1)^2 \end{bmatrix} \quad (12)$$

569 where $\Omega = 1\text{Hz}$. Unlike the models we study in the paper which calculate $E_{x \sim p(x|z)} [T(x)]$ via
 570 forward simulation, we have a closed form for the eigenvalues of the dynamics matrix. λ can be
 571 calculated using the quadratic formula:

$$\lambda = \frac{\left(\frac{a_1+a_4}{\tau}\right) \pm \sqrt{\left(\frac{a_1+a_4}{\tau}\right)^2 + 4\left(\frac{a_2a_3-a_1a_4}{\tau}\right)}}{2} \quad (13)$$

572 where λ_1 is the eigenvalue of $\frac{1}{\tau}A$ with greatest real part. Even though $E_{x \sim p(x|z)} [T(x)]$ is calculable
 573 directly via a closed form function and does not require simulation, we cannot derive the distribution
 574 q_θ^* directly. This is due to the formally hard problem of the backward mapping: finding the natural
 575 parameters η from the mean parameters μ of an exponential family distribution [49]. Instead, we
 576 can use EPI to learn the linear system parameters producing such a band of oscillations (Fig. S2B).

577 Even this relatively simple system has nontrivial (though intuitively sensible) structure in the
 578 parameter distribution. To validate our method (further than that of the underlying technology
 579 on a ground truth solution [18]) we can analytically derive the contours of the probability density

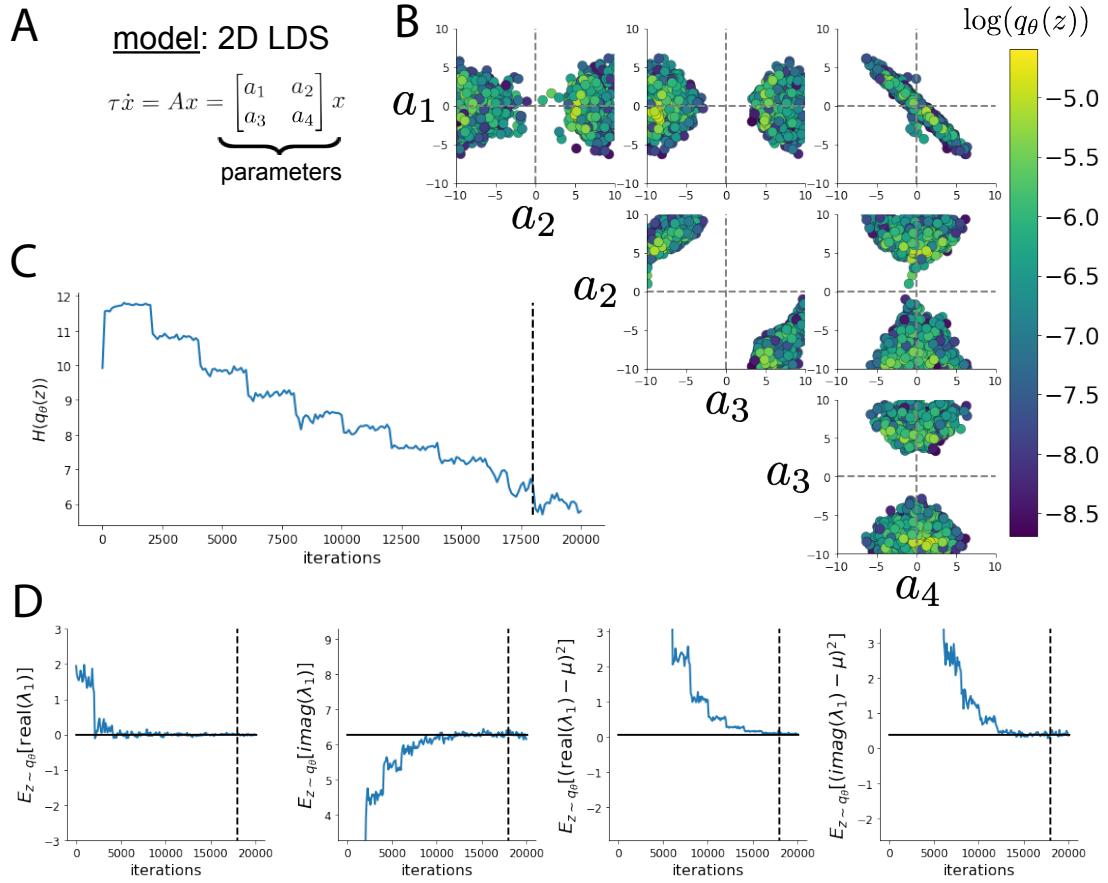


Fig. S2: A. Two-dimensional linear dynamical system model, where real entries of the dynamics matrix A are the parameters. B. The DSN distribution for a 2D LDS with $\tau = 1$ that produces an average of 1Hz oscillations with some small amount of variance. C. Entropy throughout the optimization. At the beginning of each augmented lagrangian epoch (5,000 iterations), the entropy dips due to the shifted optimization manifold where emergent property constraint satisfaction is increasingly weighted. D. Emergent property moments throughout optimization. At the beginning of each augmented lagrangian epoch, the emergent property moments move closer to their constraints.

580 from the emergent property statistics and values (Fig. S3). In the $a_1 - a_4$ plane, is a black line
 581 at $\text{real}(\lambda_1) = \frac{a_1 + a_4}{2} = 0$, a dotted black line at the standard deviation $\text{real}(\lambda_1) = \frac{a_1 + a_4}{2} \pm 1$, and a
 582 grey line at twice the standard deviation $\text{real}(\lambda_1) = \frac{a_1 + a_4}{2} \pm 2$ (Fig. S3A). Here the lines denote the
 583 set of solutions at fixed behaviors, which overlay the posterior obtained through EPI. The learned
 584 DSN distribution precisely reflects the desired statistical constraints and model degeneracy in the
 585 sum of a_1 and a_4 . Intuitively, the parameters equivalent with respect to emergent property statistic
 586 $\text{real}(\lambda_1)$ have similar log densities.

587 To explain the structure in the bimodality of the DSN posterior, we can look at the imaginary
 588 component of λ_1 . When $\text{real}(\lambda_1) = \frac{a_1 + a_4}{2} = 0$, we have

$$\text{imag}(\lambda_1) = \begin{cases} \sqrt{\frac{a_1 a_4 - a_2 a_3}{\tau}}, & \text{if } a_1 a_4 < a_2 a_3 \\ 0 & \text{otherwise} \end{cases} \quad (14)$$

589 When $\tau = 1$ and $a_1 a_4 > a_2 a_3$ (center of distribution above), we have the following equation for the
 590 the other two dimensions:

$$\text{imag}(\lambda_1)^2 = a_1 a_4 - a_2 a_3 \quad (15)$$

591 Since we constrained $E_{q_\theta}[\text{imag}(\lambda)] = 2\pi$ (with $\omega = 1$), we can plot contours of the equation
 592 $\text{imag}(\lambda_1)^2 = a_1 a_4 - a_2 a_3 = (2\pi)^2$ for various $a_1 a_4$ (Fig. S3A). If $\sigma_{1,4} = E_{q_\theta}(|a_1 a_4 - E_{q_\theta}[a_1 a_4]|)$,
 593 then we plot the contours as $a_1 a_4 = 0$ (black), $a_1 a_4 = -\sigma_{1,4}$ (black dotted), and $a_1 a_4 = -2\sigma_{1,4}$
 594 (grey dotted) (Fig. S3B). This validates the curved structure of the inferred distribution learned
 595 through EPI. We take steps in negative standard deviation of $a_1 a_4$ (dotted and gray lines), since
 596 there are few positive values $a_1 a_4$ in the posterior. More subtle model-behavior combinations will
 597 have even more complexity, further motivating the use of EPI for understanding these systems.
 598 Indeed we sample a distribution of systems oscillating near 1Hz (Fig. S4).

599 A.1.2 Augmented Lagrangian optimization

600 To optimize $q_\theta(z)$ in equation 1, the constrained optimization is performed using the augmented
 601 Lagrangian method. The following objective is minimized:

$$L(\theta; \alpha, c) = -H(q_\theta) + \alpha^\top \delta(\theta) + \frac{c}{2} \|\delta(\theta)\|^2 \quad (16)$$

602 where $\delta(\theta) = E_{z \sim q_\theta} [E_{x \sim p(x|z)} [T(x) - \mu]]$, $\alpha \in \mathcal{R}^m$ are the Lagrange multipliers and c is the penalty
 603 coefficient. For a fixed (α, c) , θ is optimized with stochastic gradient descent. A low value of c is

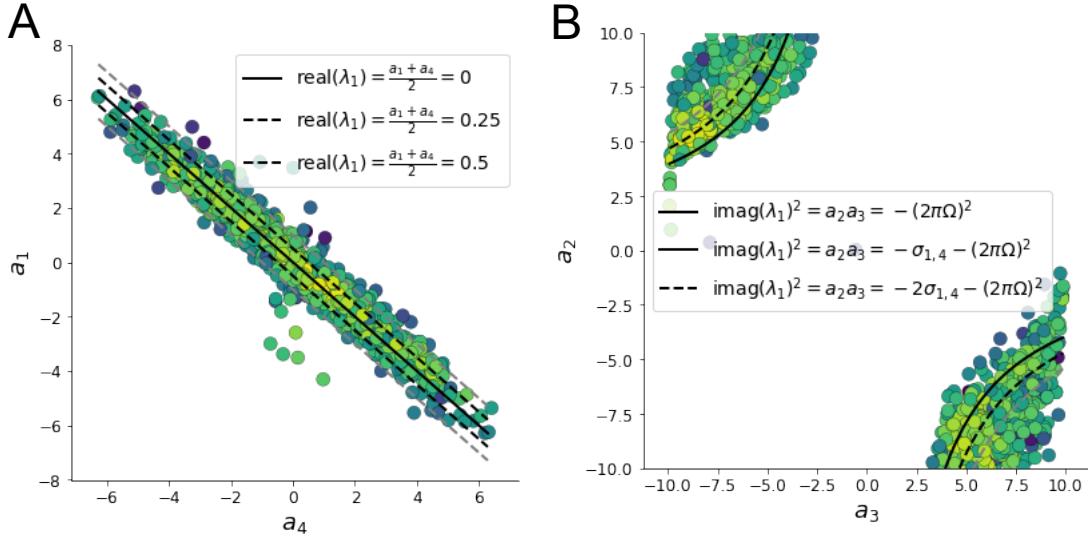


Fig. S3: A. Probability contours in the $a_1 - a_4$ plane can be derived from the relationship to emergent property statistic of growth/decay factor. B. Probability contours in the $a_2 - a_3$ plane can be derived from relationship to the emergent property statistic of oscillation frequency.

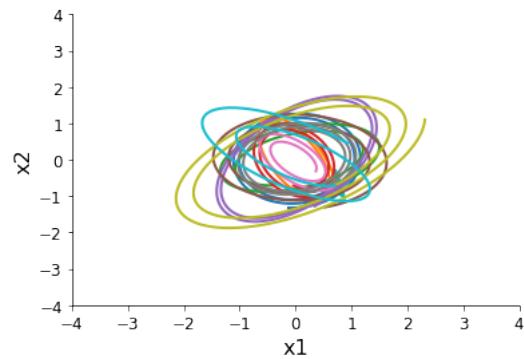


Fig. S4: Sampled dynamical system trajectories from the EPI distribution. Each trajectory is initialized at $x(0) = \left[\frac{\sqrt{2}}{2} \quad -\frac{\sqrt{2}}{2} \right]$.

604 used initially, and increased during each augmented Lagrangian epoch – a period of optimization
 605 with fixed α and c for a given number of stochastic optimziation iterations. Similarly, α is tuned
 606 each epoch based on the constraint violations. For the linear 2-dimensional system (Fig. S2C)
 607 optimization hyperparameters are initialized to $c_1 = 10^{-4}$ and $\alpha_1 = 0$. The penalty coefficient
 608 is updated based on a hypothesis test regarding the reduction in constraint violation. The p-
 609 value of $E[|\delta(\theta_{k+1})|] > \gamma E[|\delta(\theta_k)|]$ is computed, and c_{k+1} is updated to βc_k with probability
 610 $1 - p$. Throughout the project, $\beta = 4.0$ and $\gamma = 0.25$ is used. The other update rule is $\alpha_{k+1} =$
 611 $\alpha_k + c_k \frac{1}{n} \sum_{i=1}^n (T(x^{(i)}) - \mu)$. In this example, each augmented Lagrangian epoch ran for 2,000
 612 iterations. We consider the optimization to have converged when a null hypothesis test of constraint
 613 violations being zero is accepted for all constraints at a significance threshold 0.05. This is the dotted
 614 line on the plots below depicting the optimization cutoff of EPI optimization for the 2-dimensional
 615 linear system. If the optimization is left to continue running, entropy usually decreases, and
 616 structural pathologies in the distribution may be introduced.

617 The intention is that c and α start at values encouraging entropic growth early in optimization.
 618 Then, as they increase in magnitude with each training epoch, the constraint satisfaction terms are
 619 increasingly weighted, resulting in a decrease in entropy. Rather than using a naive initialization,
 620 before EPI, we optimize the deep probability distribution parameters to generate samples of an
 621 isotropic gaussian of a selected variance, such as 1.0 for the 2D LDS example. This provides a
 622 convenient starting point, whose level of entropy is controlled by the user.

623 A.1.3 Normalizing flows

624 Since we are optimizing parameters θ of our deep probability distribution with respect to the
 625 entropy, we will need to take gradients with respect to the log-density of samples from the deep
 626 probability distribution.

$$H(q_\theta(z)) = \int -q_\theta(z) \log(q_\theta(z)) dz = E_{z \sim q_\theta} [-\log(q_\theta(z))] = E_{\omega \sim q_0} [-\log(q_\theta(f_\theta(\omega)))] \quad (17)$$

$$\nabla_\theta H(q_\theta(z)) = E_{\omega \sim q_0} [-\nabla_\theta \log(q_\theta(f_\theta(\omega)))] \quad (18)$$

628 Deep probability models typically consist of several layers of fully connected neural networks.
 629 When each neural network layer is restricted to be a bijective function, the sample density can be
 630 calculated using the change of variables formula at each layer of the network. For $z' = f(z)$,

$$q(z') = q(f^{-1}(z')) \left| \det \frac{\partial f^{-1}(z')}{\partial z'} \right| = q(z) \left| \det \frac{\partial f(z)}{\partial z} \right|^{-1} \quad (19)$$

631 However, this computation has cubic complexity in dimensionality for fully connected layers. By
 632 restricting our layers to normalizing flows [15] – bijective functions with fast log determinant
 633 jacobian computations, we can tractably optimize deep generative models with objectives that are
 634 a function of sample density, like entropy. Most of our analyses use real NVP [50], which have
 635 proven effective in our architecture searches, and have the advantageous features of fast sampling
 636 and fast density evaluation.

637 **A.1.4 Related work**

638 (To come)

639

640 **A.1.5 Emergent property inference as variational inference in an exponential family**

641 (To come)

642

643 **A.2 Theoretical models**

644 In this study, we used emergent property inference to examine several models relevant to theoretical
 645 neuroscience. Here, we provide the details of each model and the related analyses.

646 **A.2.1 Stomatogastric ganglion**

647 Each neuron's membrane potential $x_m(t)$ is the solution of the following differential equation.

$$C_m \frac{\partial x_m}{\partial t} = -[h_{leak}(x; z) + h_{Ca}(x; z) + h_K(x; z) + h_{hyp}(x; z) + h_{elec}(x; z) + h_{syn}(x; z)] \quad (20)$$

648 The membrane potential of each neuron is affected by the leak, calcium, potassium, hyperpolar-
 649 ization, electrical and synaptic currents, respectively. The capacitance of the cell membrane was
 650 set to $C_m = 1nF$. Each current is a function of the the neuron's membrane potential x_m and
 651 the parameters of the circuit such as g_{el} and g_{syn} , whose affect on the circuit is considered in the

652 motivational example of EPI in Fig. 1. Specifically, the currents are the difference in the neuron's
 653 membrane potential and that current type's reversal potential multiplied by a conductance:

$$654 \quad h_{leak}(x; z) = g_{leak}(x_m - V_{leak}) \quad (21)$$

$$655 \quad h_{elec}(x; z) = g_{el}(x_m^{post} - x_m^{pre}) \quad (22)$$

$$656 \quad h_{syn}(x; z) = g_{syn}S_\infty^{pre}(x_m^{post} - V_{syn}) \quad (23)$$

$$657 \quad h_{Ca}(x; z) = g_{Ca}M_\infty(x_m - V_{Ca}) \quad (24)$$

$$658 \quad h_K(x; z) = g_KN(x_m - V_K) \quad (25)$$

$$659 \quad h_{hyp}(x; z) = g_hH(x_m - V_{hyp}) \quad (26)$$

659 The reversal potentials were set to $V_{leak} = -40mV$, $V_{Ca} = 100mV$, $V_K = -80mV$, $V_{hyp} = -20mV$,
 660 and $V_{syn} = -75mV$. The other conductance parameters were fixed to $g_{leak} = 1 \times 10^{-4}\mu S$. g_{Ca} ,
 661 g_K , and g_{hyp} had different values based on fast, intermediate (hub) or slow neuron. Fast: $g_{Ca} =$
 662 1.9×10^{-2} , $g_K = 3.9 \times 10^{-2}$, and $g_{hyp} = 2.5 \times 10^{-2}$. Intermediate: $g_{Ca} = 1.7 \times 10^{-2}$, $g_K = 1.9 \times 10^{-2}$,
 663 and $g_{hyp} = 8.0 \times 10^{-3}$. Intermediate: $g_{Ca} = 8.5 \times 10^{-3}$, $g_K = 1.5 \times 10^{-2}$, and $g_{hyp} = 1.0 \times 10^{-2}$.

664 Furthermore, the Calcium, Potassium, and hyperpolarization channels have time-dependent gating
 665 dynamics dependent on steady-state gating variables M_∞ , N_∞ and H_∞ , respectively.

$$666 \quad M_\infty = 0.5 \left(1 + \tanh \left(\frac{x_m - v_1}{v_2} \right) \right) \quad (27)$$

$$667 \quad \frac{\partial N}{\partial t} = \lambda_N(N_\infty - N) \quad (28)$$

$$668 \quad N_\infty = 0.5 \left(1 + \tanh \left(\frac{x_m - v_3}{v_4} \right) \right) \quad (29)$$

$$669 \quad \lambda_N = \phi_N \cosh \left(\frac{x_m - v_3}{2v_4} \right) \quad (30)$$

$$670 \quad \frac{\partial H}{\partial t} = \frac{(H_\infty - H)}{\tau_h} \quad (31)$$

$$671 \quad H_\infty = \frac{1}{1 + \exp \left(\frac{x_m + v_5}{v_6} \right)} \quad (32)$$

$$672 \quad \tau_h = 272 - \left(\frac{-1499}{1 + \exp \left(\frac{-x_m + v_7}{v_8} \right)} \right) \quad (33)$$

672 where we set $v_1 = 0mV$, $v_2 = 20mV$, $v_3 = 0mV$, $v_4 = 15mV$, $v_5 = 78.3mV$, $v_6 = 10.5mV$,
 673 $v_7 = -42.2mV$, $v_8 = 87.3mV$, $v_9 = 5mV$, and $v_{th} = -25mV$. These are the same parameter
 674 values used in [20].

675 Finally, there is a synaptic gating variable as well:

$$S_\infty = \frac{1}{1 + \exp\left(\frac{v_{th} - x_m}{v_0}\right)} \quad (34)$$

676 When the dynamic gating variables are considered, this is actually a 15-dimensional nonlinear
677 dynamical system.

678 In order to measure the frequency of the hub neuron during EPI, the STG model was simulated
679 for $T = 500$ time steps of $dt = 25ms$. In EPI, since gradients are taken through the simulation
680 process, the number of time steps are kept as modest if possible. The chosen dt and T were the
681 most computationally convenient choices yielding accurate frequency measurement.

682 Our original approach to measuring frequency was to take the max of the fast Fourier transform
683 (FFT) of the simulated time series. There are a few key considerations here. One is resolution
684 in frequency space. Each FFT entry will correspond to a signal frequency of $\frac{F_s k}{N}$, where N is
685 the number of samples used for the FFT, $F_s = \frac{1}{dt}$, and $k \in [0, 1, \dots, N - 1]$. Our resolution is
686 improved by increasing N and decreasing dt . Increasing $N = T - b$, where b is some fixed number of
687 buffer burn-in initialization samples, necessitates an increase in simulation time steps T , which
688 directly increases computational cost. Increasing F_s (decreasing dt) increases system approximation
689 accuracy, but requires more time steps before a full cycle is observed. At the level of $dt = 0.025$,
690 thousands of temporal samples were required for resolution of .01Hz. These challenges in frequency
691 resolution with the discrete Fourier transform motivated the use of an alternative basis of complex
692 exponentials. Instead, we used a basis of complex exponentials with frequencies from 0.0-1.0 Hz at
693 0.01Hz resolution, $\Phi = [0.0, 0.01, \dots, 1.0]^\top$

694 Another consideration was that the frequency spectra of the hub neuron has several peaks. This
695 was due to high-frequency sub-threshold activity. The maximum frequency was often not the firing
696 frequency. Accordingly, subthreshold activity was set to zero, and the whole signal was low-pass
697 filtered with a moving average window of length 20. The signal was subsequently mean centered.
698 After this pre-processing, the maximum frequency in the filter bank accurately reflected the firing
699 frequency.

700 Finally, to differentiate through the maximum frequency identification step, we used a sum-of-
701 powers normalization strategy: Let $\mathcal{X}_i \in \mathcal{C}^{|\Phi|}$ be the complex exponential filter bank dot products
702 with the signal $x_i \in \mathcal{R}^N$, where $i \in \{f1, f2, \text{hub}, s1, s2\}$. The “frequency identification” vector is

$$u_i = \frac{|\mathcal{X}_i|^\alpha}{\sum_{k=1}^N |\mathcal{X}_i(k)|^\alpha} \quad (35)$$

703 The frequency is then calculated as $\Omega_i = u_i^\top \Phi$ with $\alpha = 100$.

704 Network syncing, like all other emergent properties in this work, are defined by the emergent
 705 property statistics and values. The emergent property statistics are the first- and second-moments
 706 of the firing frequencies. The first moments are set to 0.55Hz, while the second moments are set to
 707 0.025Hz².

$$E \begin{bmatrix} \Omega_{f1} \\ \Omega_{f2} \\ \Omega_{hub} \\ \Omega_{s1} \\ \Omega_{s2} \\ (\Omega_{f1} - 0.55)^2 \\ (\Omega_{f2} - 0.55)^2 \\ (\Omega_{hub} - 0.55)^2 \\ (\Omega_{s1} - 0.55)^2 \\ (\Omega_{s2} - 0.55)^2 \end{bmatrix} = \begin{bmatrix} 0.55 \\ 0.55 \\ 0.55 \\ 0.55 \\ 0.55 \\ 0.025^2 \\ 0.025^2 \\ 0.025^2 \\ 0.025^2 \\ 0.025^2 \end{bmatrix} \quad (36)$$

708 For EPI in Fig 2C, we used a real NVP architecture with two coupling layers. Each coupling layer
 709 had two hidden layers of 10 units each, and we mapped onto a support of $z \in \left[\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 10 \\ 8 \end{bmatrix} \right]$. We
 710 have shown the EPI optimization that converged with maximum entropy across 2 random seeds
 711 and augmented Lagrangian coefficient initializations of $c_0=0$, 2, and 5.

712 **A.2.2 Primary visual cortex**

713 The dynamics of each neural populations average rate $x = \begin{bmatrix} x_E \\ x_P \\ x_S \\ x_V \end{bmatrix}$ are given by:

$$\tau \frac{dx}{dt} = -x + [Wx + h]_+^n \quad (37)$$

714 Some neuron-types largely lack synaptic projections to other neuron-types [52], and it is popular

715 to only consider a subset of the effective connectivities [21].

$$W = \begin{bmatrix} W_{EE} & W_{EP} & W_{ES} & 0 \\ W_{PE} & W_{PP} & W_{PS} & 0 \\ W_{SE} & 0 & 0 & W_{SV} \\ W_{VE} & W_{VP} & W_{VS} & 0 \end{bmatrix} \quad (38)$$

716 Estimates of the probability of connection and strength of connection from the Allen institute
 717 result in an estimate of the effective connectivity [?]:

$$W = \begin{bmatrix} 0.0576 & 0.19728 & 0.13144 & 0 \\ 0.58855 & 0.30668 & 0.4285 & 0 \\ 0.15652 & 0 & 0 & 0.2 \\ 0.13755 & 0.0902 & 0.4004 & 0 \end{bmatrix} \quad (39)$$

718 We look at how this four-dimensional nonlinear dynamical model of V1 responds to different inputs,
 719 and compare the predictions of the linear response to the approximate posteriors obtained through
 720 EPI. The input to the system is the sum of a baseline input $b = [1 \ 1 \ 1 \ 1]^\top$ and a differential
 721 input dh :

$$h = b + dh \quad (40)$$

722 All simulations of this system had $T = 100$ time points, a time step $dt = 5\text{ms}$, and time constant
 723 $\tau = 20\text{ms}$. And the system was initialized to a random draw $x(0)_i \sim \mathcal{N}(1, 0.01)$.

724 We can describe the dynamics of this system more generally by

$$\dot{x}_i = -x_i + f(u_i) \quad (41)$$

725 where the input to each neuron is

$$u_i = \sum_j W_{ij}x_j + h_i \quad (42)$$

726 Let $F_{ij} = \gamma_i \delta(i, j)$, where $\gamma_i = f'(u_i)$. Then, the linear response is

$$\frac{\partial x_{ss}}{\partial h} = F(W \frac{\partial x_{ss}}{\partial h} + I) \quad (43)$$

727 which is calculable by

$$\frac{\partial x_{ss}}{\partial h} = (F^{-1} - W)^{-1} \quad (44)$$

728 The emergent property we considered was the first and second moments of the change dr
 729 between the baseline input $h = b$ and $h = b + dh$. We use the following notation to indicate that

730 the emergent property statistics were set to the following values:

$$\mathcal{B}(\alpha, y) \leftrightarrow E \begin{bmatrix} dx_{\alpha,ss} \\ (dx_{\alpha,ss} - y)^2 \end{bmatrix} = \begin{bmatrix} y \\ 0.01^2 \end{bmatrix} \quad (45)$$

731 In the final analysis for this model, we sweep the input one neuron at a time away from the mode
 732 of each inferred distributions $dh^* = z^* = \text{argmax}_z \log q_\theta(z | \mathcal{B}(\alpha, 0.1))$. The differential responses
 733 $dx_{\alpha,ss}$ are examined at perturbed inputs $h = b + dh^* + \Delta h_\alpha u_\alpha$ where u_α is a unit vector in the
 734 dimension of α and $\Delta h_\alpha \in [-15, 15]$.

735 For each $\mathcal{B}(\alpha, y)$ with $\alpha \in \{E, P, S, V\}$ and $y \in \{0.1, 0.5\}$, we ran EPI with five different random
 736 initial seeds using an architecture of four coupling layers, each with two hidden layers of 10 units.
 737 We set $c_0 = 10^5$. The support of the learned distribution was restricted to $z_i \in [-5, 5]$.

738 **A.2.3 Superior colliculus**

739 There are four total units: two in each hemisphere corresponding to the Pro/contralateral and
 740 Anti/ipsilateral populations. Each unit has an activity (x_i) and internal variable (u_i) related by

$$x_i(t) = \left(\frac{1}{2} \tanh \left(\frac{v_i(t) - \epsilon}{\zeta} \right) + \frac{1}{2} \right) \quad (46)$$

741 $\epsilon = 0.05$ and $\zeta = 0.5$ control the position and shape of the nonlinearity, respectively.

742 We can order the elements of x_i and v_i into vectors x and v with elements

$$x = \begin{bmatrix} x_{LP} \\ x_{LA} \\ x_{RP} \\ x_{RA} \end{bmatrix} \quad v = \begin{bmatrix} v_{LP} \\ v_{LA} \\ v_{RP} \\ v_{RA} \end{bmatrix} \quad (47)$$

743 The internal variables follow dynamics:

$$\tau \frac{\partial v}{\partial t} = -v + Wx + h + \sigma \partial B \quad (48)$$

744 with time constant $\tau = 0.09s$ and gaussian noise $\sigma \partial B$ controlled by the magnitude of $\sigma = 1.0$. The
 745 weight matrix has 8 parameters sW_P , sW_A , vW_{PA} , vW_{AP} , hW_P , hW_A , dW_{PA} , and dW_{AP} (Fig.
 746 4B).

$$W = \begin{bmatrix} sW_P & vW_{PA} & hW_P & dW_{PA} \\ vW_{AP} & sW_A & dW_{AP} & hW_A \\ hW_P & dW_{PA} & sW_P & vW_{PA} \\ dW_{AP} & hW_A & vW_{AP} & sW_A \end{bmatrix} \quad (49)$$

⁷⁴⁷ The system receives five inputs throughout each trial, which has a total length of 1.8s.

$$h = h_{\text{rule}} + h_{\text{choice-period}} + h_{\text{light}} \quad (50)$$

⁷⁴⁸ There are rule-based inputs depending on the condition,

$$h_{P,\text{rule}}(t) = \begin{cases} I_{P,\text{rule}} \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^\top, & \text{if } t \leq 1.2s \\ 0, & \text{otherwise} \end{cases} \quad (51)$$

⁷⁴⁹

$$h_{A,\text{rule}}(t) = \begin{cases} I_{A,\text{rule}} \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^\top, & \text{if } t \leq 1.2s \\ 0, & \text{otherwise} \end{cases} \quad (52)$$

⁷⁵⁰ a choice-period input,

$$h_{\text{choice}}(t) = \begin{cases} I_{\text{choice}} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^\top, & \text{if } t > 1.2s \\ 0, & \text{otherwise} \end{cases} \quad (53)$$

⁷⁵¹ and an input to the right or left-side depending on where the light stimulus is delivered.

$$h_{\text{light}}(t) = \begin{cases} I_{\text{light}} \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^\top, & \text{if } t > 1.2s \text{ and Left} \\ I_{\text{light}} \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}^\top, & \text{if } t > 1.2s \text{ and Right} \\ 0, & t \leq 1.2s \end{cases} \quad (54)$$

⁷⁵² The input parameterization was fixed to $I_{P,\text{rule}} = 10$, $I_{A,\text{rule}} = 10$, $I_{\text{choice}} = 2$, and $I_{\text{light}} = 1$

⁷⁵³ TODO: this is probably a good place to explain the intuition behind the naming of the Schur
⁷⁵⁴ eigenmodes.

⁷⁵⁵ To produce a Bernoulli rate of p_{LP} in the Left, Pro condition (we can generalize this to either cue,
⁷⁵⁶ or stimulus condition), let \hat{p}_i be the empirical average steady state (ss) response (final x_{LP} at end
⁷⁵⁷ of task) over M=500 gaussian noise draws for a given SC model parameterization z_i :

$$\hat{p}_i = E_{\sigma \partial B} [x_{LP,ss} \mid s = L, c = P, z_i] = \frac{1}{M} \sum_{j=1}^M x_{LP,ss}(s = L, c = P, z_i, \sigma \partial B_j) \quad (55)$$

⁷⁵⁸ For the first constraint, the average over posterior samples (from $q_\theta(z)$) to be p_{LP} :

$$E_{z_i \sim q_\phi} [E_{\sigma \partial B} [x_{LP,ss} \mid s = L, c = P, z_i]] = E_{z_i \sim q_\phi} [\hat{p}_i] = p_{LP} \quad (56)$$

⁷⁵⁹ We can then ask that the variance of the steady state responses across gaussian draws, is the
⁷⁶⁰ Bernoulli variance for the empirical rate \hat{p}_i .

$$E_{z \sim q\phi} [\sigma_{err}^2] = 0 \quad (57)$$

⁷⁶¹

$$\sigma_{err}^2 = Var_{\sigma \partial B} [x_{LP,ss} | s = L, c = P, z_i] - \hat{p}_i(1 - \hat{p}_i) \quad (58)$$

⁷⁶² We have an additional constraint that the Pro neuron on the opposite hemisphere should have the
⁷⁶³ opposite value. We can enforce this with a final constraint:

$$E_{z \sim q\phi} [d_P] = 1 \quad (59)$$

⁷⁶⁴

$$E_{\sigma \partial W} [(x_{LP,ss} - x_{RP,ss})^2 | s = L, c = P, z_i] \quad (60)$$

⁷⁶⁵ We refer to networks obeying these constraints as Bernoulli, winner-take-all networks. Since the
⁷⁶⁶ maximum variance of a random variable bounded from 0 to 1 is the Bernoulli variance ($\hat{p}(1 - \hat{p})$),
⁷⁶⁷ and the maximum squared difference between two variables bounded from 0 to 1 is 1, we do not
⁷⁶⁸ need to control the second moment of these test statistics. In reality, these variables are dynamical
⁷⁶⁹ system states and can only exponentially decay (or saturate) to 0 (or 1), so the Bernoulli variance
⁷⁷⁰ error and squared difference constraints can only be undershot. This is important to be mindful
⁷⁷¹ of when evaluating the convergence criteria. Instead of using our usual hypothesis testing criteria
⁷⁷² for convergence to the emergent property, we set a slack variable threshold for these technically
⁷⁷³ infeasible constraints to 0.05.

⁷⁷⁴ Training DSNs to learn distributions of dynamical system parameterizations that produce Bernoulli
⁷⁷⁵ responses at a given rate (with small variance around that rate) was harder to do than expected.
⁷⁷⁶ There is a pathology in this optimization setup, where the learned distribution of weights is bimodal
⁷⁷⁷ attributing a fraction p of the samples to an expansive mode (which always sends x_{LP} to 1), and a
⁷⁷⁸ fraction $1 - p$ to a decaying mode (which always sends x_{LP} to 0). This pathology was avoided using
⁷⁷⁹ an inequality constraint prohibiting parameter samples that resulted in low variance of responses
⁷⁸⁰ across noise.

⁷⁸¹ In total, the emergent property of rapid task switching accuracy at level p was defined as

$$\mathcal{B}(p) \leftrightarrow \begin{bmatrix} \hat{p}_P \\ \hat{p}_A \\ (\hat{p}_P - p)^2 \\ (\hat{p}_A - p)^2 \\ \sigma_{P,err}^2 \\ \sigma_{A,err}^2 \\ d_P \\ d_A \end{bmatrix} = \begin{bmatrix} p \\ p \\ 0.15^2 \\ 0.15^2 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad (61)$$

⁷⁸² For each accuracy level p , we ran EPI for 10 different random seeds and selected the maximum
⁷⁸³ entropy solution using an architecture of 10 planar flows with $c_0 = 2$. The support of z was \mathcal{R}^8 .

⁷⁸⁴ **A.2.4 Rank-1 RNN**

⁷⁸⁵ The network dynamics of neuron i 's rate x evolve according to:

$$\dot{x}_i(t) = -x_i(t) + \sum_{j=1}^N J_{ij} \phi(x_j(t)) + I_i \quad (62)$$

⁷⁸⁶ where the connectivity is comprised of a random and structured component:

$$J_{ij} = g\chi_{ij} + P_{ij} \quad (63)$$

⁷⁸⁷ The random all-to-all component has elements drawn from $\chi_{ij} \sim \mathcal{N}(0, \frac{1}{N})$, and the structured
⁷⁸⁸ component is a sum of r unit rank terms:

$$P_{ij} = \sum_{k=1}^r \frac{m_i^{(k)} n_j^{(k)}}{N} \quad (64)$$

⁷⁸⁹ We use this theory to compute $T(x)$ when running EPI.

⁷⁹⁰ Rank-1 vectors m and n have elements drawn

$$m_i \sim \mathcal{N}(M_m, \Sigma_m)$$

⁷⁹¹

$$n_i \sim \mathcal{N}(M_n, \Sigma_n)$$

⁷⁹² The current has the following statistics:

$$I = M_I + \frac{\Sigma_{mI}}{\Sigma_m} x_1 + \frac{\Sigma_{nI}}{\Sigma_n} x_2 + \Sigma_\perp h$$

793 where x_1 , x_2 , and h are standard normal random variables.

794 The $\ddot{\Delta}$ equation is broken into the equation for Δ_0 and Δ_∞ by the autocorrelation dynamics
795 assertions.

$$\ddot{\Delta}(\tau) = -\frac{\partial V}{\partial \Delta}$$

$$796 \quad \ddot{\Delta} = \Delta - \{g^2 \langle [\phi_i(t)\phi_i(t+\tau)] \rangle + \Sigma_m^2 \kappa^2 + 2\Sigma_{mI}\kappa + \Sigma_I^2\}$$

797 We can write out the potential function by integrating the negated RHS.

$$V(\Delta, \Delta_0) = \int \mathcal{D}\Delta \frac{\partial V(\Delta, \Delta_0)}{\partial \Delta}$$

$$798 \quad V(\Delta, \Delta_0) = -\frac{\Delta^2}{2} + g^2 \langle [\Phi_i(t)\Phi_i(t+\tau)] \rangle + (\Sigma_m^2 \kappa^2 + 2\Sigma_{mI}\kappa + \Sigma_I^2)\Delta + C$$

799 We assume that as time goes to infinity, the potential relaxes to a steady state.

$$800 \quad \frac{\partial V(\Delta_\infty, \Delta_0)}{\partial \Delta} = 0$$

$$801 \quad \frac{\partial V(\Delta_\infty, \Delta_0)}{\partial \Delta} = -\Delta + \{g^2 \langle [\phi_i(t)\phi_i(t+\infty)] \rangle + \Sigma_m^2 \kappa^2 + 2\Sigma_{mI}\kappa + \Sigma_I^2\} = 0$$

$$802 \quad \Delta_\infty = g^2 \langle [\phi_i(t)\phi_i(t+\infty)] \rangle + \Sigma_m^2 \kappa^2 + 2\Sigma_{mI}\kappa + \Sigma_I^2$$

$$803 \quad \Delta_\infty = g^2 \int \mathcal{D}z \left[\int \mathcal{D}x \phi(\mu + \sqrt{\Delta_0 - \Delta_\infty}x + \sqrt{\Delta_\infty}z) \right]^2 + \Sigma_m^2 \kappa^2 + 2\Sigma_{mI}\kappa + \Sigma_I^2$$

803 Also, we assume that the energy of the system is perserved throughout the entirety of its evolution.

$$V(\Delta_0, \Delta_0) = V(\Delta_\infty, \Delta_0)$$

$$804 \quad -\frac{\Delta_0^2}{2} + g^2 \langle [\Phi_i(t)\Phi_i(t)] \rangle + (\Sigma_m^2 \kappa^2 + 2\Sigma_{mI}\kappa + \Sigma_I^2)\Delta_0 + C = -\frac{\Delta_\infty^2}{2} + g^2 \langle [\Phi_i(t)\Phi_i(t)] \rangle + (\Sigma_m^2 \kappa^2 + 2\Sigma_{mI}\kappa + \Sigma_I^2)\Delta_\infty + C$$

$$805 \quad \frac{\Delta_0^2 - \Delta_\infty^2}{2} = g^2 (\langle [\Phi_i(t)\Phi_i(t)] \rangle - \langle [\Phi_i(t)\Phi_i(t)] \rangle) + (\Sigma_m^2 \kappa^2 + 2\Sigma_{mI}\kappa + \Sigma_I^2)(\Delta_0 - \Delta_\infty)$$

$$806 \quad \frac{\Delta_0^2 - \Delta_\infty^2}{2} = g^2 \left(\int \mathcal{D}z \Phi^2(\mu + \sqrt{\Delta_0}z) - \int \mathcal{D}z \int \mathcal{D}x \Phi(\mu + \sqrt{\Delta_0 - \Delta_\infty}x + \sqrt{\Delta_\infty}z) \right) \\ + (\Sigma_m^2 \kappa^2 + 2\Sigma_{mI}\kappa + \Sigma_I^2)(\Delta_0 - \Delta_\infty)$$

807 **Consistency equations:**

808

$$\begin{aligned}
 \mu &= F(\mu, \kappa, \Delta_0, \Delta_\infty) = M_m \kappa + M_I \\
 \kappa &= G(\mu, \kappa, \Delta_0, \Delta_\infty) = M_n \langle [\phi_i] \rangle + \Sigma_{nI} \langle [\phi'_i] \rangle \\
 \frac{\Delta_0^2 - \Delta_\infty^2}{2} &= H(\mu, \kappa, \Delta_0, \Delta_\infty) = g^2 \left(\int \mathcal{D}z \Phi^2(\mu + \sqrt{\Delta_0} z) - \int \mathcal{D}z \int \mathcal{D}x \Phi(\mu + \sqrt{\Delta_0 - \Delta_\infty} x + \sqrt{\Delta_\infty} z) \right) \\
 &\quad + (\Sigma_m^2 \kappa^2 + 2\Sigma_{mI} \kappa + \Sigma_I^2)(\Delta_0 - \Delta_\infty) \\
 \Delta_\infty &= L(\mu, \kappa, \Delta_0, \Delta_\infty) = g^2 \int \mathcal{D}z \left[\int \mathcal{D}x \phi(\mu + \sqrt{\Delta_0 - \Delta_\infty} x + \sqrt{\Delta_\infty} z) \right]^2 + \Sigma_m^2 \kappa^2 + 2\Sigma_{mI} \kappa + \Sigma_I^2
 \end{aligned} \tag{65}$$

809 We can solve these equations by simulating the following Langevin dynamical system.

$$\begin{aligned}
 x(t) &= \frac{\Delta_0(t)^2 - \Delta_\infty(t)^2}{2} \\
 \Delta_0(t) &= \sqrt{2x(t) + \Delta_\infty(t)^2} \\
 \dot{\mu}(t) &= -\mu(t) + F(\mu(t), \kappa(t), \Delta_0(t), \Delta_\infty(t)) \\
 \dot{\kappa}(t) &= -\kappa + G(\mu(t), \kappa(t), \Delta_0(t), \Delta_\infty(t)) \\
 \dot{x}(t) &= -x(t) + H(\mu(t), \kappa(t), \Delta_0(t), \Delta_\infty(t)) \\
 \dot{\Delta}_\infty(t) &= -\Delta_\infty(t) + L(\mu(t), \kappa(t), \Delta_0(t), \Delta_\infty(t))
 \end{aligned} \tag{66}$$

810 Then, the temporal variance is simply calculated via

$$\Delta_T = \Delta_0 - \Delta_\infty \tag{67}$$

811 TODO Need to explain the warm starting for the aficionados.

812 TODO explain the density network architectures used.

813 **A.3 Supplementary Figures**

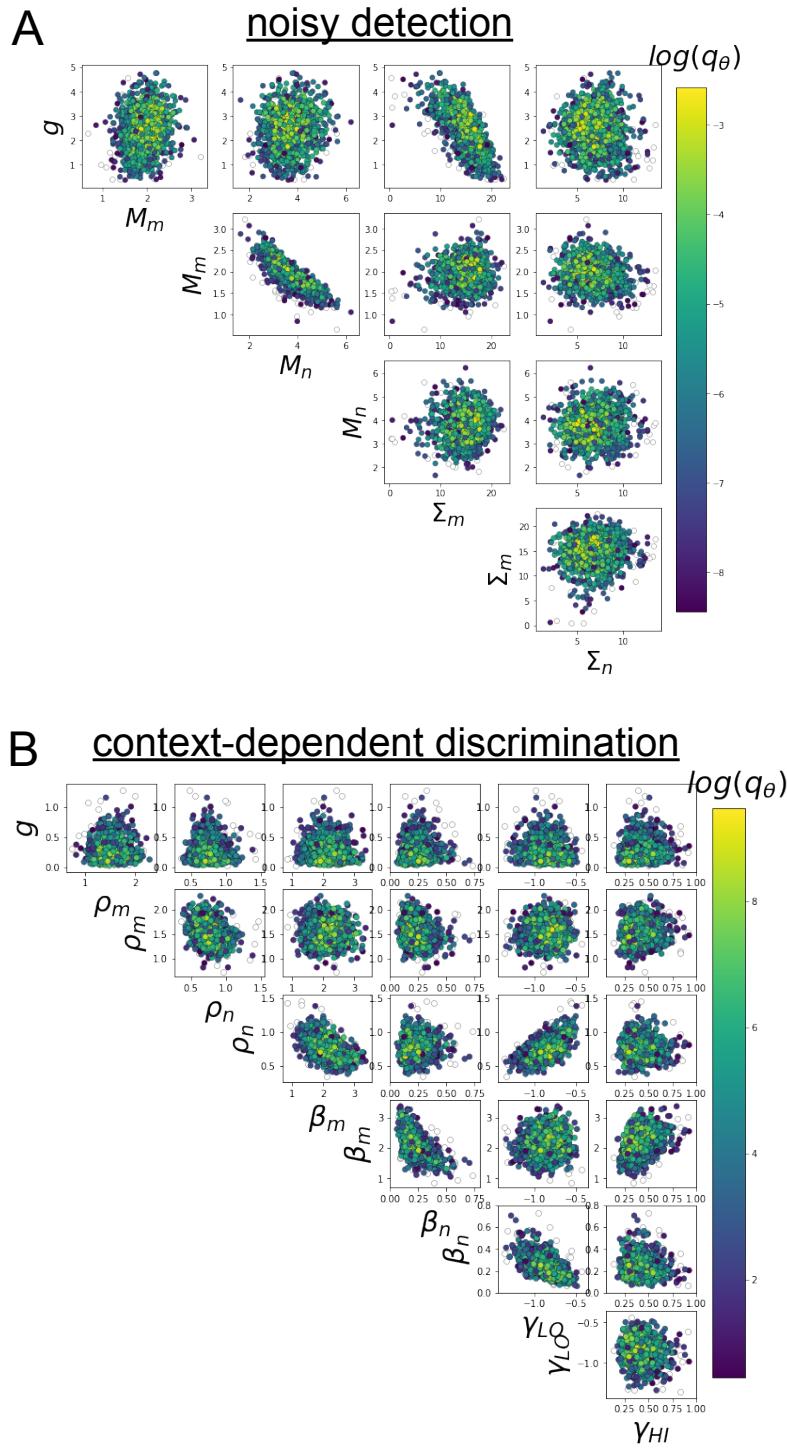


Fig. S1: A. EPI for rank-1 networks doing discrimination. B. EPI for rank-2 networks doing context-dependent discrimination.