

Interrogating theoretical models of neural computation with deep inference

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¹ 1 Abstract

² The cornerstone of theoretical neuroscience is the circuit model: a system of equations that captures
³ a hypothesized neural mechanism. Such models are valuable when they give rise to an experimen-
⁴ tally observed phenomenon – whether behavioral or in terms of neural activity – and thus can offer
⁵ insights into neural computation. The operation of these circuits, like all models, critically depends
⁶ on the choices of model parameters. Historically, the gold standard has been to analytically derive
⁷ the relationship between model parameters and computational properties. However, this enterprise
⁸ quickly becomes infeasible as biologically realistic constraints are included into the model increas-
⁹ ing its complexity, often resulting in *ad hoc* approaches to understanding the relationship between
¹⁰ model and computation. We bring recent machine learning techniques – the use of deep generative
¹¹ models for probabilistic inference – to bear on this problem, learning distributions of parameters
¹² that produce the specified properties of computation. Importantly, the techniques we introduce
¹³ offer a principled means to understand the implications of model parameter choices on compu-
¹⁴ tational properties of interest. We motivate this methodology with a worked example analyzing
¹⁵ sensitivity in the stomatogastric ganglion. We then use it to generate insights into neuron-type
¹⁶ input-responsivity in a model of primary visual cortex, a new understanding of rapid task switch-
¹⁷ ing in superior colliculus models, and attribution of bias in recurrent neural networks solving a toy
¹⁸ mathematical problem. More generally, this work offers a quantitative grounding for theoretical
¹⁹ models going forward, pointing a way to how rigorous statistical inference can enhance theoretical
²⁰ neuroscience at large.

²¹ 2 Introduction

²² The fundamental practice of theoretical neuroscience is to use a mathematical model to understand
²³ neural computation, whether that computation enables perception, action, or some intermediate
²⁴ processing [1]. In this field, a neural computation is systematized with a set of equations – the
²⁵ model – and these equations are motivated by biophysics, neurophysiology, and other conceptual
²⁶ considerations. The function of this system is governed by the choice of model parameters, which

when configured appropriately, give rise to a measurable signature of a computation. The work of analyzing a model then requires solving the inverse problem: given a computation of interest, how can we reason about these suitable parameter configurations? The inverse problem is crucial for reasoning about likely parameter values, uniquenesses and degeneracies, attractor states and phase transitions, and predictions made by the model.

Consider the idealized practice: one carefully designs a model and analytically derives how model parameters govern the computation. Seminal examples of this gold standard include our field's understanding of memory capacity in associative neural networks [2] and chaos and autocorrelation timescales in random neural networks [3] (which use models and analyses originating in physics), as well as the paradoxical effect in excitatory/inhibitory networks [4], we need [?], more examples [?]. Unfortunately, as circuit models include more biological realism, theory via analytic derivation becomes intractable. This creates an unfavorable tradeoff. On the one hand, one may tractably analyze systems of equations with unrealistic assumptions (for example symmetry or gaussianity), producing accurate inferences about parameters of a too-simple model. On the other hand, one may choose a more biologically accurate, scientifically relevant model at the cost of *ad hoc* approaches to analysis (simply examining simulated activity), potentially resulting in bad inferences and thus erroneous scientific predictions and conclusions.

Of course, this same tradeoff has been confronted in many scientific fields and engineering problems characterized by the need to do inference in complex models. In response, the machine learning community has made remarkable progress in recent years, via the use of deep neural networks as a powerful inference engine: a flexible function family that can map observed phenomena (in this case the measurable signal of some computation) back to probability distributions quantifying the likely parameter configurations. One celebrated example of this approach from machine learning, of which we draw key inspiration for this work, is the variational autoencoder [5, 6], which uses a deep neural network to induce an (approximate) posterior distribution on hidden variables in a latent variable model, given data. Indeed, these tools have been used to great success in neuroscience as well, in particular for interrogating parameters (sometimes treated as hidden states) in models of both cortical population activity [7, 8, 9, 10] and animal behavior [11, 12, 13]. These works have used deep neural networks to expand the expressivity and accuracy of statistical models of neural data [14].

However, these inference tools have not significantly influenced the study of theoretical neuroscience models, for at least three reasons. First, at a practical level, the nonlinearities and dynamics of

many theoretical models are such that conventional inference tools typically produce a narrow set of insights into these models. Indeed, only in the last few years has deep learning research advanced to a point of relevance to this class of problem. Second, the object of interest from a theoretical model is not typically data itself, but rather a qualitative phenomenon – inspection of model behavior, or better, a measurable signature of some computation – an *emergent property* of the model. Third, because carefully constructed biological models do not fit cleanly into the framing of a statistical model. Technically, because many such models stipulate a noisy system of differential equations that can only be sampled or realized through forward simulation, they lack the explicit likelihood and priors central to the probabilistic modeling toolkit.

To address these three challenges, we developed an inference methodology – ‘emergent property inference’ – which learns a distribution over parameter configurations in a theoretical model. This distribution has two critical properties: *(i)* it is chosen such that draws from the distribution (parameter configurations) correspond to systems of equations that give rise to a specified emergent property (a set of constraints); and *(ii)* it is chosen to have maximum entropy given those constraints, such that we identify all likely parameters and can use the distribution to reason about parametric sensitivity and degeneracies [15]. First, we stipulate a bijective deep neural network that induces a flexible family of probability distributions over model parameterizations with a probability density we can calculate [16, 17, 18]. Second, we quantify the notion of emergent properties as a set of moment constraints on datasets generated by the model. Thus, an emergent property is not a single data realization, but a phenomenon or a feature of the model, which is ultimately the object of interest in theoretical neuroscience. Conditioning on an emergent property requires a variant of deep probabilistic inference methods, which we have previously introduced [19]. Third, because we cannot assume the theoretical model has explicit likelihood on data or the emergent property of interest, we use stochastic gradient techniques in the spirit of likelihood free variational inference [20]. Taken together, emergent property inference (EPI) provides a methodology for inferring parameter configurations consistent with a particular emergent phenomena in theoretical models. We use a classic example of parametric degeneracy in a biological system, the stomatogastric ganglion [21], to motivate and clarify the technical details of EPI.

Equipped with this methodology, we then investigated three models of current importance in theoretical neuroscience. These models were chosen to demonstrate generality through ranges of biological realism (from conductance-based biophysics to recurrent neural networks), neural system function (from pattern generation to abstract cognitive function), and network scale (from four to

91 infinite neurons). First, we use EPI to produce a set of verifiable hypotheses of input-responsivity
 92 in a four neuron-type dynamical model of primary visual cortex; we then validate these hypotheses
 93 in the model. Second, we demonstrated how the systematic application of EPI to levels of task
 94 performance can generate experimentally testable hypotheses regarding connectivity in superior
 95 colliculus. Third, we use EPI to uncover the sources of bias in a low-rank recurrent neural network
 96 executing a toy mathematical computation. The novel scientific insights offered by EPI contextualize
 97 and clarify the previous studies exploring these models [22, 23, 24, 25] and more generally,
 98 suggests a departure from realism vs tractability considerations towards the use of modern machine
 99 learning for sophisticated interrogation of biologically relevant models.

100 We note that, during our preparation and early presentation of this work [26, 27], another work
 101 has arisen with broadly similar goals: bringing statistical inference to mechanistic models of neural
 102 circuits [28]. We are excited by this broad problem being recognized by the community, and we
 103 emphasize that these works offer complementary neuroscientific contributions and use different
 104 technical methodologies. While we have advanced our research on deep generative modeling [19]
 105 to a point of significant relevance to statistical inference in theoretical neuroscience, they have also
 106 furthered their research on approximate Bayesian inference in such models [29]. The existence of
 107 these complementary methodologies emphasizes the increased importance and timeliness of both
 108 works.

109 3 Results

110 3.1 Motivating emergent property inference of theoretical models

111 Consideration of the typical workflow of theoretical modeling clarifies the need for emergent prop-
 112 erty inference. First, one designs or chooses an existing model that, it is hypothesized, captures
 113 the computation of interest. To ground this process in a well-known example, consider the stom-
 114 atogastric ganglion (STG) of crustaceans, a small neural circuit which generates multiple rhythmic
 115 muscle activation patterns for digestion [30]. Despite full knowledge of STG connectivity and a
 116 precise characterization of its rhythmic pattern generation, biophysical models of the STG have
 117 complicated relationships between circuit parameters and neural activity [21, 31]. A model of the
 118 STG [22] is shown schematically in Figure 1A, and note that the behavior of this model will be crit-
 119 ically dependent on its parameterization – the choices of conductance parameters $z = [g_{el}, g_{synA}]$.
 120 Specifically, the two fast neurons ($f1$ and $f2$) mutually inhibit one another, and oscillate at a faster

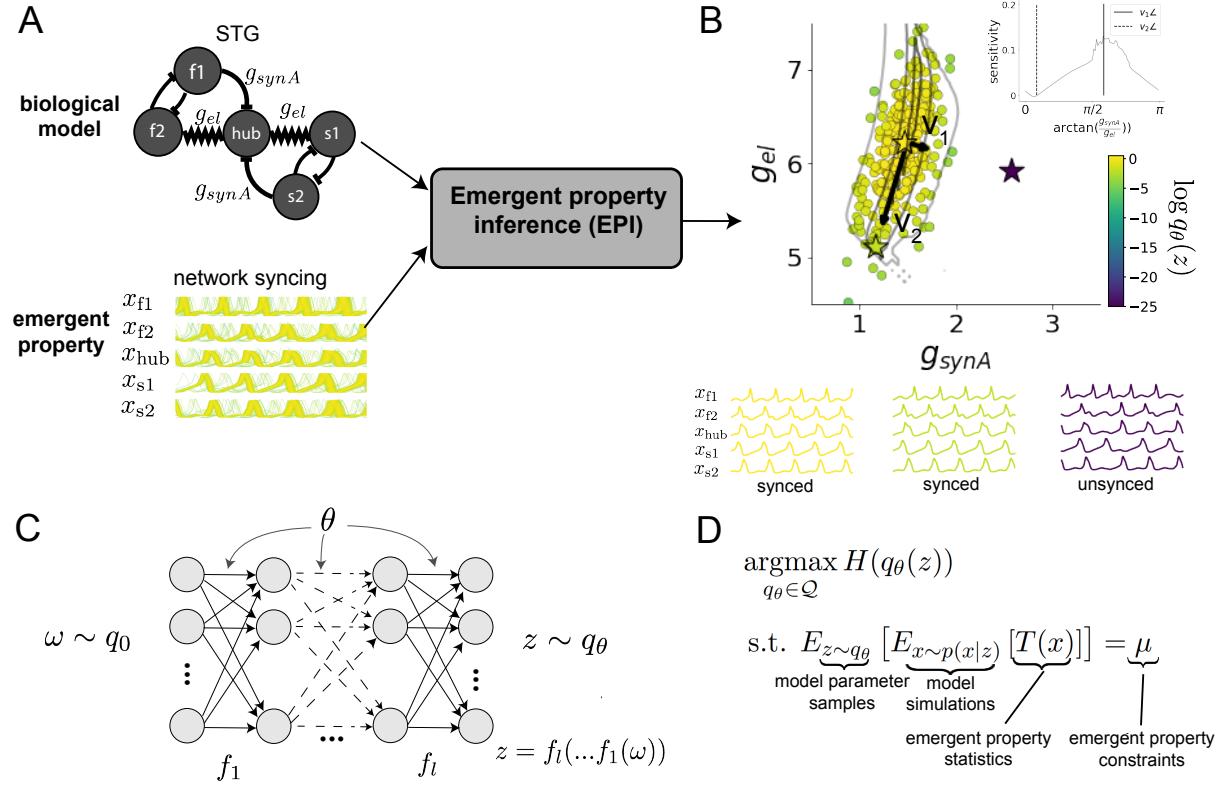


Figure 1: Emergent property inference (EPI) in the stomatogastric ganglion. A. For a choice of model (STG) and emergent property (network syncing), emergent property inference (EPI) learns a posterior distribution of the model parameters $z = [g_{el}, g_{synA}]^\top$ conditioned on network syncing. B. An EPI distribution of STG model parameters producing network syncing. Samples are colored by log density. Distribution contours of emergent property value error are shown at levels of 2×10^{-6} , 2×10^{-5} , and 2×10^{-4} . The eigenvectors of the Hessian at the mode of the inferred distribution are indicated as v_1 and v_2 . Simulated activity is shown for three samples (stars). (Inset) Sensitivity of the system with respect to network syncing along all dimensions of parameter space away from the mode. (see Section A.2.1). C. Deep probability distributions map a latent random variable w through a deep neural network with weights and biases θ to parameters z distributed as $q_\theta(z)$. D. EPI learns a distribution $q_\theta(z)$ of model parameters that produce an emergent property: the emergent property statistics $T(x)$ are fixed in expectation over parameter distribution samples $z \sim q_\theta(z)$ to particular values μ .

frequency than the mutually inhibiting slow neurons (s_1 and s_2), and the hub neuron (hub) couples with the fast or slow population or both.

Second, once the model is selected, one defines the emergent property, the measurable signal of scientific interest. To continue our running STG example, one such emergent property is the phenomenon of *network syncing* – in certain parameter regimes, the frequency of the hub neuron matches that of the fast and slow populations at an intermediate frequency. This emergent property is shown in Figure 1A at a frequency of 0.54Hz.

Third, qualitative parameter analysis ensues: since precise mathematical analysis is intractable in this model, a brute force sweep of parameters is done [22]. Subsequently, a qualitative description is formulated to describe the different parameter configurations that lead to the emergent property. In this last step lies the opportunity for a precise quantification of the emergent property as a statistical feature of the model. Once we have such a methodology, we can infer a probability distribution over parameter configurations that produce this emergent property.

Before presenting technical details (in the following section), let us understand emergent property inference schematically: EPI (Fig. 1A gray box) takes, as input, the model and the specified emergent property, and as its output, produces the parameter distribution shown in Figure 1B. This distribution – represented for clarity as samples from the distribution – is then a scientifically meaningful and mathematically tractable object. In the STG model, this distribution can be specifically queried to reveal the prototypical parameter configuration for network syncing (the mode; Figure 1B yellow star), and how network syncing decays based on changes away from the mode. Intuitively, the probability density of the samples is in agreement with the emergent property value error (Fig. 1B contours). Furthermore, the eigenvectors of the distribution Hessian at the mode can be queried to quantitatively formalize the robustness of network syncing (Fig. 1B v_1, v_2). Indeed, samples equidistant from the mode along these EPI-identified dimensions of sensitivity (v_1) and degeneracy (v_2) have diminished or preserved network syncing, respectively (Figure 1B inset and activity traces). Further validation of EPI is available in the supplementary materials, where we analyze a simpler model for which ground-truth statements can be made (Section A.1.1).

3.2 A deep generative modeling approach to emergent property inference

Emergent property inference (EPI) systematizes the three-step procedure of the previous section. First, we consider the model as a coupled set of differential (and potentially stochastic) equations

[22]. In the running STG example, the dynamical state $x = [x_{f1}, x_{f2}, x_{hub}, x_{s1}, x_{s2}]$ is the membrane potential for each neuron, which evolves according to the biophysical conductance-based equation:

$$C_m \frac{dx}{dt} = -h(x; z) = -[h_{leak}(x; z) + h_{Ca}(x; z) + h_K(x; z) + h_{hyp}(x; z) + h_{elec}(x; z) + h_{syn}(x; z)] \quad (1)$$

where $C_m=1\text{nF}$, and h_{leak} , h_{Ca} , h_K , h_{hyp} , h_{elec} , h_{syn} are the leak, calcium, potassium, hyperpolarization, electrical, and synaptic currents, all of which have their own complicated dependence on x and $z = [g_{el}, g_{synA}]$ (see Section A.2.1).

Second, we define the emergent property, which as above is network syncing: oscillation of the entire population at an intermediate frequency of our choosing (Figure 1A bottom). Quantifying this phenomenon is straightforward: we define network syncing to be that each neuron’s spiking frequency – denoted $\omega_{f1}(x)$, $\omega_{f2}(x)$, etc. – is close to an intermediate frequency of 0.55Hz. Mathematically, we achieve this via constraints on the mean and variance of $\omega_i(x)$ for each neuron $i \in \{f1, f2, hub, s1, s2\}$, and thus:

$$E[T(x)] \triangleq E \begin{bmatrix} \omega_{f1}(x) \\ \vdots \\ (\omega_{f1}(x) - 0.55)^2 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0.55 \\ \vdots \\ 0.025^2 \\ \vdots \end{bmatrix} \triangleq \mu, \quad (2)$$

which completes the quantification of the emergent property.

Third, we perform emergent property inference: we find a distribution over parameter configurations z , and insist that samples from this distribution produce the emergent property; in other words, they obey the constraints introduced in Equation 14. This distribution will be chosen from a family of probability distributions $\mathcal{Q} = \{q_\theta(z) : \theta \in \Theta\}$, defined by a deep generative distribution of the normalizing flow class [16, 17, 18] – neural networks which transform a simple distribution into a suitably complicated distribution (as is needed here). This deep distribution is represented in Figure 1C (and see Methods for more detail). Then, mathematically, we must solve the following optimization program:

$$\begin{aligned} & \underset{q_\theta \in \mathcal{Q}}{\operatorname{argmax}} H(q_\theta(z)) \\ & \text{s.t. } E_{z \sim q_\theta} [E_{x \sim p(x|z)} [T(x)]] = \mu, \end{aligned} \quad (3)$$

where $T(x), \mu$ are defined as in Equation 14, and $p(x|z)$ is the intractable distribution of data from the model (x), given that model’s parameters z (we access samples from this distribution by running

the model forward). The purpose of each element in this program is detailed in Figure 1D. Finally, we recognize that many distributions in \mathcal{Q} will respect the emergent property constraints, so we require a normative principle to select amongst them. This principle is captured in Equation 3 by the primal objective H . Here we chose Shannon entropy as a means to find parameter distributions with minimal assumptions beyond some chosen structure [32, 33, 19, 34], but we emphasize that the EPI method is unaffected by this choice (but the results of course will depend on the primal objective chosen).

EPI optimizes the weights and biases θ of the deep neural network (which induces the probability distribution) by iteratively solving Equation 3. The optimization is complete when the sampled models with parameters $z \sim q_\theta$ produce activity consistent with the specified emergent property. Such convergence is evaluated with a hypothesis test that the mean of each emergent property statistic is not different than its emergent property value (see Section A.1.2). Equipped with this method, we now prove out the value of EPI by using it to investigate three prominent models in neuroscience, using EPI to produce new insights about these models.

3.3 Comprehensive input-responsivity in a nonlinear sensory system

In studies of primary visual cortex (V1), theoretical models with excitatory (E) and inhibitory (I) populations have reproduced a host of experimentally documented phenomena. In particular regimes of excitation and inhibition, these E/I models exhibit the paradoxical effect [4], selective amplification [35], surround suppression [36], and sensory integrative properties [37]. Extending this model using experimental evidence of three genetically-defined classes of inhibitory neurons [38, 39], recent work [23] has investigated a four-population model – excitatory (E), parvalbumin (P), somatostatin (S), and vasointestinal peptide (V) neurons – as shown in Fig. 2A. The dynamical state of this model is the firing rate of each neuron-type population $x = [x_E, x_P, x_S, x_V]^\top$, which evolves according to rectified ($\llbracket \cdot \rrbracket_+$) and exponentiated dynamics:

$$\tau \frac{dx}{dt} = -x + [Wx + h]_+^n \quad (4)$$

with effective connectivity weights W and input h . In our analysis, we set the time constant $\tau = 20\text{ms}$ and dynamics coefficient $n = 2$. Also, as is fairly standard, we obtain an informative estimate of the effective connectivities between these neuron-types W in mice by multiplying their probability of connection with their average synaptic strength [40, 41] (see Section A.2.2). Given

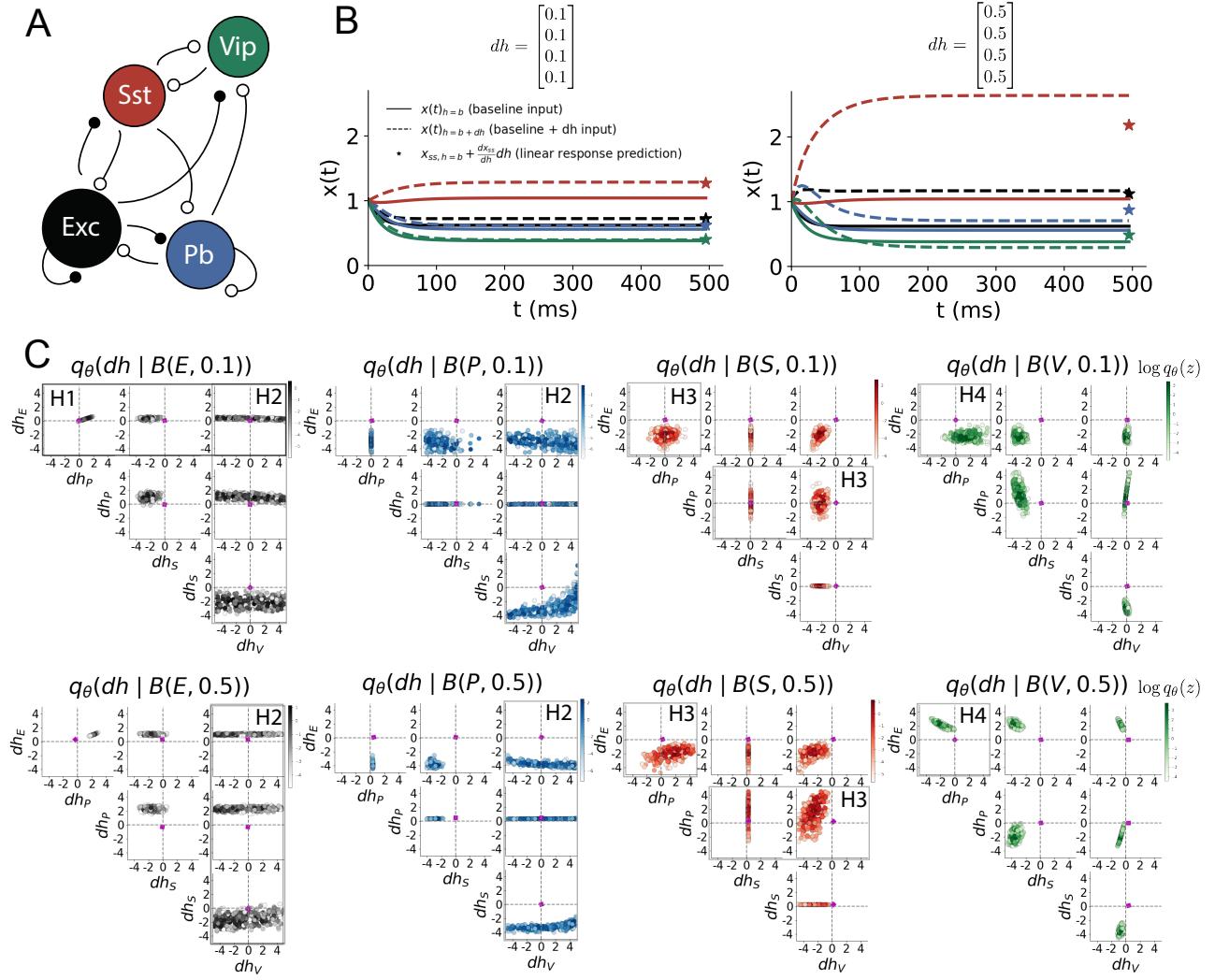


Figure 2: Hypothesis generation through EPI in a V1 model. A. Four-population model of primary visual cortex with excitatory (black), parvalbumin (blue), somatostatin (red), and vip (green) neurons. Some neuron-types largely do not form synaptic projections to others (excitatory and inhibitory projections filled and unfilled, respectively). B. Linear response predictions become inaccurate with greater input strength. V1 model simulations for input ($h = b$) and ($h = b + dh$) with $b = [1, 1, 1, 1]^T$ and (left) $dh = [0.1, 0.1, 0.1, 0.1]^T$ (right) $dh = [0.5, 0.5, 0.5, 0.5]^T$. Stars indicate the linear response prediction. C. EPI distributions on differential input dh conditioned on differential response $\mathcal{B}(\alpha, y)$. Supporting evidence for the four generated hypotheses are indicated by gray boxes with labels H1, H2, H3, and H4. The linear prediction from two standard deviations away from y (from negative to positive) is overlaid in magenta (very small, near origin).

201 these fixed choices of W , n , and τ , we studied the system's response to input

$$h = b + dh, \quad (5)$$

202 where the input h is comprised of a baseline input $b = [b_E, b_P, b_S, b_V]^\top$ and a differential input
 203 $dh = [dh_E, dh_P, dh_S, dh_V]^\top$ to each neuron-type population. Throughout subsequent analyses, the
 204 baseline input is $b = [1, 1, 1, 1]^\top$.

205 Having established our model, we now define the emergent property. We begin with the linearized
 206 response of the system to input $\frac{dx_{ss}}{dh}$ at the steady state x_{ss} , i.e. a fixed point. While this lin-
 207 earization accurately predicts differential responses $dx_{ss} = [dx_{E,ss}, dx_{P,ss}, dx_{S,ss}, dx_{V,ss}]$ for small
 208 differential inputs to each population $dh = [0.1, 0.1, 0.1, 0.1]$ (Fig. 2B, left), linearization is a poor
 209 predictor in this nonlinear model more generally (Fig. 3B, right). Currently available approaches
 210 to deriving the steady state response of this system are limited.

211 To get a more comprehensive picture of the input-responsivity of each neuron-type, we used EPI
 212 to learn a distribution of the differential inputs to each population dh that produce an increase
 213 of $y \in \{0.1, 0.5\}$ in the rate of each neuron-type population $\alpha \in \{E, P, S, V\}$. We want to know
 214 the differential inputs dh that result in a differential steady state $dx_{\alpha,ss}$ (the change in $x_{\alpha,ss}$ when
 215 receiving input $h = b + dh$ with respect to the baseline $h = b$) of value y with some small, arbitrarily
 216 chosen amount of variance 0.01^2 . These statements amount to the emergent property

$$\mathcal{B}(\alpha, y) \triangleq E \begin{bmatrix} dx_{\alpha,ss} \\ (dx_{\alpha,ss} - y)^2 \end{bmatrix} = \begin{bmatrix} y \\ 0.01^2 \end{bmatrix} \quad (6)$$

217 We continue to use $\mathcal{B}(\cdot)$ throughout the rest of the study as short hand for emergent property, which
 218 represents a different signature of computation in each application. In Each column of Figure 2C
 219 visualizes the inferred distribution of dh corresponding to a excitatory (red), parvalbumin (blue),
 220 somatostatin (red) and vip (green) neuron-type increase, while each row corresponds to amounts of
 221 increase 0.1 and 0.5. These distributions conditioned on such emergent properties are now available
 222 through EPI. For each pair of parameters we show the two-dimensional marginal distribution of
 223 samples colored by $\log q_\theta(dh | \mathcal{B}(\alpha, y))$. The inferred distributions immediately suggest four hy-
 224 potheses:

225

226 H1: as is intuitive, each neuron-type's firing rate should be sensitive to that neuron-type's direct
 227 input (e.g. Fig. 2C H1 indicates low variance in dh_E when $\alpha = E$. Same observation in all inferred
 228 distributions);

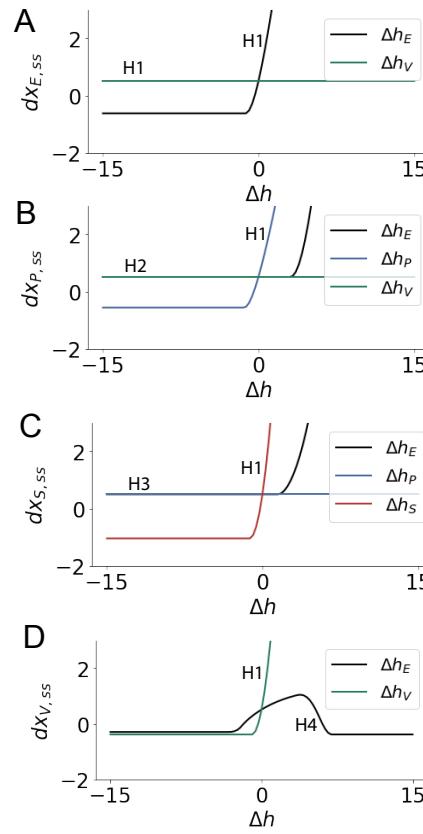


Figure 3: Confirming EPI generated hypotheses in V1. A. Differential responses by the E-population to changes in individual input $\Delta h_\alpha u_\alpha$ away from the mode of the EPI distribution dh^* . B-D Same plots for the P-, S-, and V-populations. Labels H1, H2, H3, and H4 indicate which curves confirm which hypotheses.

229 H2: the E- and P-populations should be largely unaffected by dh_V (Fig. 2C H2 indicates high
 230 variance in dh_V when $\alpha \in \{E, P\}$);
 231 H3: the S-population should be largely unaffected by dh_P (Fig. 2C H3 indicate high variance in
 232 dh_P when $\alpha = S$);
 233 H4: there should be a nonmonotonic response of $dx_{V,ss}$ with dh_E (Fig. 2C H4 indicates that
 234 negative dh_E should result in small $dx_{V,ss}$, but positive dh_E should elicit a larger $dx_{V,ss}$);
 235 We evaluate these hypotheses by taking steps in individual neuron-type input Δh_α away from the
 236 modes of the inferred distributions at $y = 0.1$.

$$dh^* = z^* = \underset{z}{\operatorname{argmax}} \log q_\theta(z \mid \mathcal{B}(\alpha, 0.1)) \quad (7)$$

237 Now, $dx_{\alpha,ss}$ is the steady state response to the system with input $h = b + dh^* + \Delta h_\alpha u_\alpha$ where u_α
 238 is a unit vector in the dimension of α . The EPI-generated hypotheses are confirmed.

- 239 • the neuron-type responses are sensitive to their direct inputs (Fig. 3A black, 3B blue, 3C
 240 red, 3D green);
 241 • the E- and P-populations are not affected by dh_V (Fig. 3A green, 3B green);

- 242 • the S-population is not affected by dh_P (Fig. 3C blue);
 243 • the V-population exhibits a nonmonotonic response to dh_E (Fig. 3D black), and is in fact
 244 the on population to do so (Fig. 3A-C black).

245 These hypotheses were in stark contrast to what was available to us via traditional analytical linear
 246 prediction (Fig. 2C, magenta). To this point, we have shown the utility of EPI on relatively low-
 247 level emergent properties like network syncing and differential neuron-type population responses.
 248 In the remainder of the study, we focus on using EPI to understand models of more abstract
 249 cognitive function.

250 **3.4 Identifying neural mechanisms of behavioral learning.**

251 Identifying measurable biological changes that result in improved behavior is important for neuro-
 252 science, since they may indicate how the learning brain adapts. In a rapid task switching experiment
 253 [42], rats were explicitly cued on each trial to either orient towards a visual stimulus in the Pro
 254 (P) task or orient away from a visual stimulus in the Anti (A) task (Fig. 3a). Neural recordings
 255 in the midbrain superior colliculus (SC) exhibited two populations of neurons that simultaneously
 256 represented both task context (Pro or Anti) and motor response (contralateral or ipsilateral to the
 257 recorded side): the Pro/Contra and Anti/Ipsi neurons [24]. Duan et al. proposed a model of SC
 258 that, like the V1 model analyzed in the previous section, is a four-population dynamical system.
 259 Here, the neuron-type populations are functionally-defined as the Pro- and Anti-populations in each
 260 hemisphere (left (L) and right (R)). The Pro- or Anti-populations receive an input determined by
 261 the cue, and then the left and right populations receive an input based on the side of the light
 262 stimulus. Activities were bounded between 0 and 1, so that a high output of the Pro population
 263 in a given hemisphere corresponds to the contralateral response. An additional stipulation is that
 264 when one Pro population responds with a high-output, the opposite Pro population must respond
 265 with a low output. Finally, this circuit operates in the presence of Gaussian noise resulting in trial-
 266 to-trial variability (see Section A.2.3). The connectivity matrix is parameterized by the geometry
 267 of the population arrangement (Fig. 3B).

268 Here, we used EPI to learn distributions of the SC weight matrix parameters $z = W$ conditioned
 269 on various levels of rapid task switching accuracy $\mathcal{B}(p)$ for $p \in \{50\%, 60\%, 70\%, 80\%, 90\%\}$ (see
 270 Section A.2.3). Following the approach in Duan et al., we decomposed the connectivity matrix
 271 $W = QAQ^{-1}$ in such a way (the Schur decomposition) that the basis vectors q_i are the same for all

²⁷² W (Fig. 3C). These basis vectors have intuitive roles in processing for this task, and are accordingly
²⁷³ named the *all* mode - all neurons co-fluctuate, *side* mode - one side dominates the other, *task* mode
²⁷⁴ - the Pro or Anti populations dominate the other, and *diag* mode - Pro- and Anti-populations of
²⁷⁵ opposite hemispheres dominate the opposite pair. The corresponding eigenvalues (e.g. a_{task} , which
²⁷⁶ change according to W) indicate the degree to which activity along that mode is increased or
²⁷⁷ decreased by W .

²⁷⁸ EPI demonstrates that, for greater task accuracies, the task mode eigenvalue increases, indicating
²⁷⁹ the importance of W to the task representation (Fig. 4D, purple). Stepping from random chance
²⁸⁰ (50%) networks to marginally task-performing (60%) networks, there is a marked decrease of the
²⁸¹ side mode eigenvalues (Fig. 3D, orange). Such side mode suppression remains in the models
²⁸² achieving greater accuracy, revealing its importance towards task performance. There were no
²⁸³ interesting trends with learning in the all or diag mode (hence not shown in Fig. 3). Importantly,
²⁸⁴ we can conclude from our methodology that side mode suppression in W allows rapid task switching,
²⁸⁵ and that greater task-mode representations in W increase accuracy. These hypotheses are confirmed
²⁸⁶ by forward simulation of the SC model (Fig. 3E). Thus, EPI produces novel, experimentally testable
²⁸⁷ predictions: effective connectivity between these populations changes throughout learning, in a way
²⁸⁸ that increases its task mode and decreases its side mode eigenvalues.

²⁸⁹ 3.5 Linking RNN connectivity to computational error

²⁹⁰ So far, each model we have studied was designed from fundamental biophysical principles, genetically-
²⁹¹ or functionally-defined neuron types. At a more abstract level of modeling, recurrent neural net-
²⁹² works (RNNs) are high-dimensional dynamical models of computation that are becoming increas-
²⁹³ ingly popular in neuroscience research [43]. In theoretical neuroscience, RNN dynamics usually
²⁹⁴ follow the equation

$$\frac{dx}{dt} = -x(t) + W\phi(x(t)) + I(t), \quad (8)$$

²⁹⁵ where $x(t)$ is the network activity, W is the network connectivity, $\phi(\cdot) = \tanh(\cdot)$, and $I(t)$ is the
²⁹⁶ input to the system. Such RNNs are trained to do a task from a systems neuroscience experiment,
²⁹⁷ and then the unit activations of the trained RNN are compared to recorded neural activity. Fully-
²⁹⁸ connected RNNs with tens of thousands of parameters are challenging to characterize [44], especially
²⁹⁹ making statistical inferences about their parameterization. Alternatively, we consider a rank-1, N -
³⁰⁰ neuron RNN with connectivity

$$W = g\chi + \frac{1}{N}mn^\top, \quad (9)$$

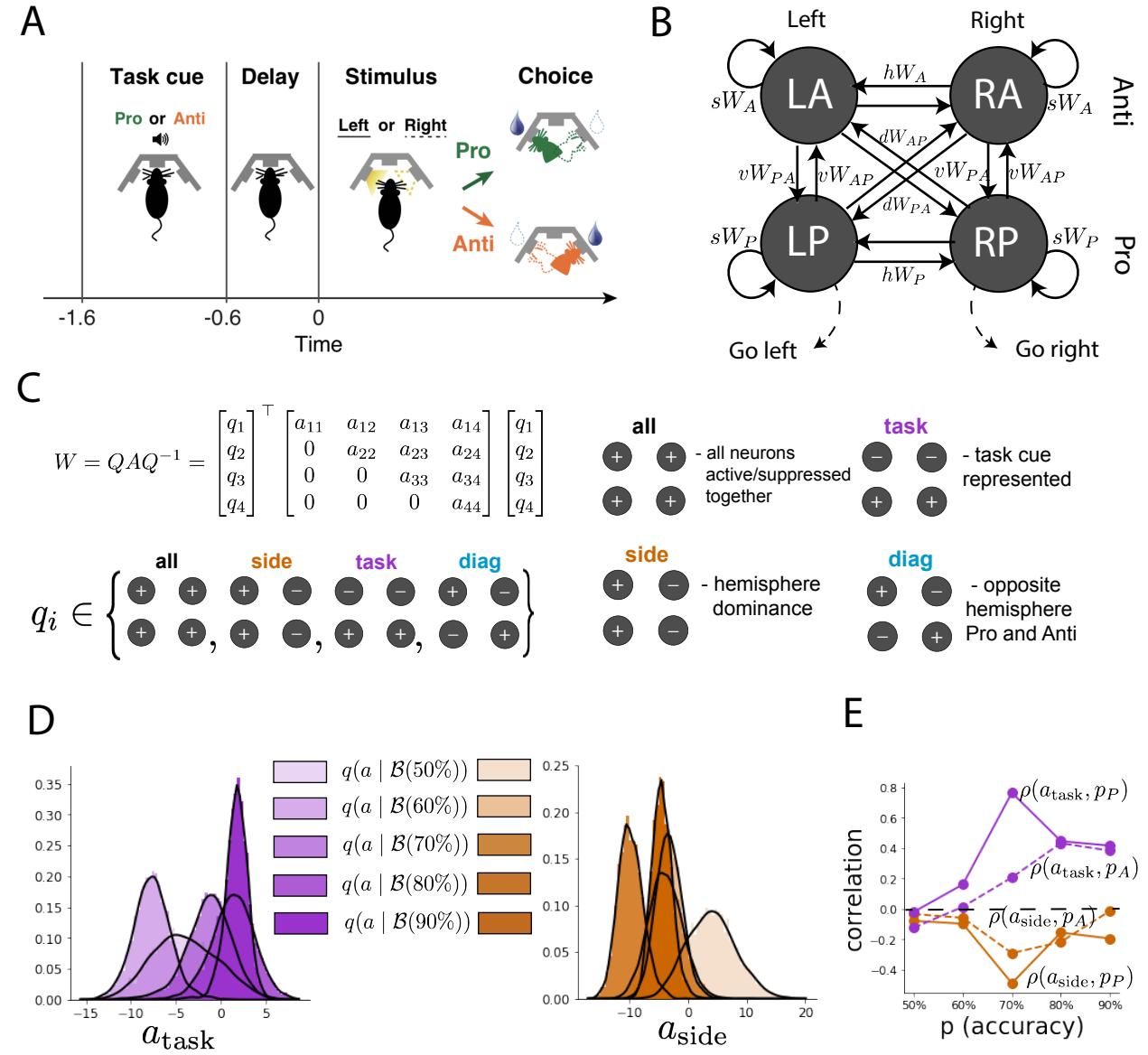


Figure 4: EPI reveals changes in SC [24] connectivity that control task accuracy. A. Rapid task switching behavioral paradigm (see text). B. Model of superior colliculus (SC). Neurons: LP - left pro, RP - right pro, LA - left anti, RA - right anti. Parameters: sW - self, hW - horizontal, vW - vertical, dW - diagonal weights. C. The Schur decomposition of the weight matrix $W = QAQ^{-1}$ is a unique decomposition with orthogonal Q and upper triangular A . Schur modes: q_{all} , q_{task} , q_{side} , and q_{diag} . D. The marginal EPI distributions of the Schur eigenvalues at each level of task accuracy. E. The correlation of Schur eigenvalue with task performance in each learned EPI distribution.

301 where $\chi_{ij} \sim \mathcal{N}(0, \frac{1}{N})$, g is the random strength, and the entries of m and n are drawn from Gaussian
 302 distributions $m_i \sim \mathcal{N}(M_m, 1)$ and $n_i \sim \mathcal{N}(M_n, 1)$. We use EPI to infer the parameterizations of
 303 rank-1 RNNs solving an example task, enabling discovery of properties of connectivity that result
 304 in different types of computational errors.

305 The task we consider is Gaussian posterior conditioning: calculate the parameters of a posterior
 306 distribution induced by a prior $p(\mu_y) = \mathcal{N}(\mu_0 = 4, \sigma_0^2 = 1)$ and a likelihood $p(y|\mu_y) = \mathcal{N}(\mu_y, \sigma_y^2 =$
 307 $1)$, given a single observation y . Conjugacy offers the result analytically; $p(\mu_y|y) = \mathcal{N}(\mu_{post}, \sigma_{post}^2)$,
 308 where:

$$\mu_{post} = \frac{\frac{\mu_0}{\sigma_0^2} + \frac{y}{\sigma_y^2}}{\frac{1}{\sigma_0^2} + \frac{1}{\sigma_y^2}} \quad \sigma_{post}^2 = \frac{1}{\frac{1}{\sigma_0^2} + \frac{1}{\sigma_y^2}}. \quad (10)$$

309 The RNN is trained to solve this task by producing readout activity that is on average the posterior
 310 mean μ_{post} , and activity whose variability is the posterior variance σ_{post}^2 (a setup inspired by
 311 [45]). To solve this Gaussian posterior conditioning task, the RNN response to a constant input
 312 $I(t) = yw + (n - M_n)$ must equal the posterior mean along readout vector w , where

$$\kappa_w = \frac{1}{N} \sum_{j=1}^N w_j \phi(x_j) \quad (11)$$

313 Additionally, the amount of chaotic variance Δ_T must equal the posterior variance. κ_w and Δ_T can
 314 be expressed in terms of each other through a solvable system of nonlinear equations (see Section
 315 A.2.4) [25]. This theory allows us to mathematically formalize the execution of this task into an
 316 emergent property, where the emergent property statistics of the RNN activity are k_w and Δ_T and
 317 the emergent property values are the ground truth posterior mean μ_{post} and variance σ_{post}^2 :

$$E \begin{bmatrix} \kappa_w \\ \Delta_T \\ (\kappa_w - \mu_{post})^2 \\ (\Delta_T^2 - \sigma_{post}^2)^2 \end{bmatrix} = \begin{bmatrix} \mu_{post} \\ \sigma_{post}^2 \\ 0.1 \\ 0.1 \end{bmatrix} \quad (12)$$

318 We specify a substantial amount of variability in the variance constraints so that the inferred
 319 distribution results in RNNs with a variety biases in their solutions to the gaussian posterior
 320 conditioning problem.

321 We used EPI to learn distributions of RNN connectivity properties $z = [g \ M_m \ M_n]$ executing
 322 Gaussian posterior conditioning given an input of $y = 2$. (see Section A.2.4) (Fig. 5B). The true
 323 Gaussian conditioning posterior for an input of $y = 2$ is $\mu_{post} = 3$ and $\sigma_{post} = 0.5$. We examined
 324 the nature of the over- and under-estimation of the posterior means (Fig. 5B, left) and variances

(Fig. 5B, right) in the inferred distributions. There is rough symmetry in the M_m - M_n plane, suggesting a degeneracy in the product of M_m and M_n (Fig. 5B). The product of M_m and M_n almost completely determines the posterior mean (Fig. 5B, left), and the random strength g is the most influential variable on the temporal variance (Fig. 5B, right). Neither of these observations were obvious from what mathematical analysis is available in networks of this type (see Section A.2.4). They lead to the following hypotheses:

H1: The posterior mean of the RNN increases with the product of M_m and M_n ;

H2: The posterior variance increases with g ;

Testing these now in finite-size networks. Will write end of this later.

This novel procedure of doing inference in interpretable parameterizations of RNNs conditioned on the emergent property of task execution is straightforwardly generalizable to other tasks like noisy integration and context-dependent decision making (Fig. S1).

4 Discussion

4.1 EPI is a general tool for theoretical neuroscience

Models of biological systems are often comprised of complex nonlinear differential equations, making traditional theoretical analysis and statistical inference intractable. In contrast, EPI is capable of learning distributions of parameters in such models producing measurable signatures of computation. We have demonstrated its utility on biological models (STG), intermediate-level models of interacting genetically- and functionally-defined neuron-types (V1, SC), and the most abstract of models (RNNs). We are able to condition both deterministic and stochastic models on low-level emergent properties like firing rates of membrane potentials, as well as high-level cognitive function like Gaussian posterior conditioning. Technically, EPI is tractable when the emergent property statistics are continuously differentiable with respect to the model parameters, which is very often the case; this emphasizes the general utility of EPI.

In this study, we have focused on applying EPI to low dimensional parameter spaces of models with low dimensional dynamical state. These choices were made to present the reader with a series of interpretable conclusions, which is more challenging in high dimensional spaces. In fact, EPI should scale reasonably to high dimensional parameter spaces, as the underlying technology has

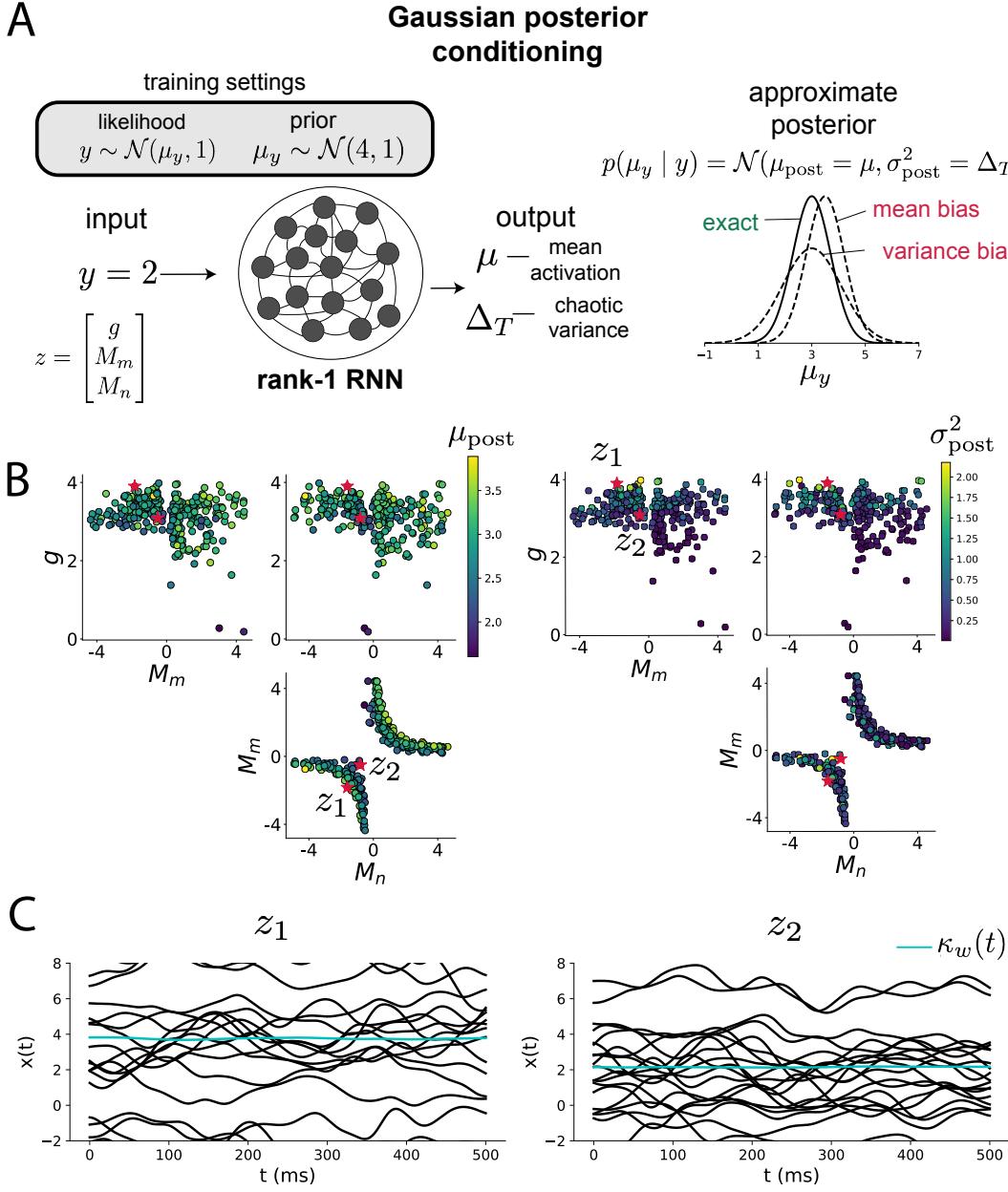


Figure 5: Sources of solution bias in an RNN computation. A. (left) A rank-1 RNN executing a Gaussian posterior conditioning computation on μ_y . (right) Bias in this computation can come from over- or under-estimating the posterior mean or variance. B. EPI distribution of rank-1 RNNs executing Gaussian posterior conditioning. Samples are colored by (left) posterior mean $\mu_{\text{post}} = \kappa_w$ and (right) posterior variance $\sigma_{\text{post}}^2 = \Delta_T$. C. Finite-size networks sampled from the distribution perform the calculation and have the computational biases expected from their parameter values. Activity along readout κ_w (cyan).

354 produced state-of-the-art performance on high-dimensional tasks such as texture generation [19].
355 Of course, increasing the dimensionality of the dynamical state of the model makes optimization
356 more expensive, and there is a practical limit there as with any machine learning approach. For
357 systems with high dimensional state, we recommend using theoretical approaches (e.g. [25]) to
358 reason about reduced parameterizations of such high-dimensional systems.

359 There are additional technical considerations when assessing the suitability of EPI for a particu-
360 lar modeling question. First and foremost, as in any optimization problem, the defined emergent
361 property should always be appropriately conditioned (constraints should not have wildly different
362 units). Furthermore, if the program is underconstrained (not enough constraints), the distribution
363 grows (in entropy) unstably unless mapped to a finite support. If overconstrained, there is no pa-
364 rameter set producing the emergent property, and EPI optimization will fail (appropriately). Next,
365 one should consider the computational cost of the gradient calculations. In the best circumstance,
366 there is a simple, closed form expression (e.g. Section A.1.1) for the emergent property statistic
367 given the model parameters. On the other end of the spectrum, many forward simulation iterations
368 may be required before a high quality measurement of the emergent property statistic is available
369 (e.g. Section A.2.1). In such cases, optimization will be expensive.

370 **4.2 Novel hypotheses from EPI**

371 Machine learning has played an effective, multifaceted role in neuroscientific progress. Primarily,
372 it has revealed structure in large-scale neural datasets [46, 47, 48, 49, 50, 51] (see review, [14]).
373 Secondarily, trained algorithms of varying degrees of biological relevance are beginning to be viewed
374 as fully-observable computational systems comparable to the brain [44, 52].

375 For example, consider the fact that we do not fully understand the four-dimensional models of V1
376 [23]. Because analytical approaches to studying nonlinear dynamical systems become increasingly
377 complicated when stepping from two-dimensional to three- or four-dimensional systems in the
378 absence of restrictive simplifying assumptions [53], it is unsurprising that this model has been a
379 challenge. In Section 3.3, we showed that EPI was far more informative about neuron-type input
380 responsibility than the predictions afforded through analysis. By flexibly conditioning this V1 model
381 on different emergent properties, we performed an exploratory analysis of a *model* rather than a
382 dataset, which generated and proved out a set of testable predictions.

383 Of course, exploratory analyses can also be directed. For example, when interested in model

384 changes during learning, one can use EPI to condition as we did in Section 3.4. This analysis
385 identified experimentally testable predictions (proved out *in-silico*) of changes in connectivity in
386 SC throughout learning. Precisely, we predict that an initial reduction in side mode eigenvalue,
387 and a steady increase in task mode eigenvalue will take place, during learning, in the effective
388 connectivity matrices of learning rats.

389 In our final analysis, we present a novel procedure for doing statistical inference on interpretable
390 parameterizations of RNNs executing simple tasks . This methodology relies on recently extended
391 theory of responses in random neural networks with minimal structure [25]. With this methodology,
392 we can finally open the probabilistic model selection toolkit reasoning about the connectivity of
393 RNNs solving tasks.

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542 **A Methods**

543 **A.1 Emergent property inference (EPI)**

544 Emergent property inference (EPI) learns distributions of theoretical model parameters that pro-
 545 duce emergent properties of interest. EPI combines ideas from likelihood-free variational inference
 546 [20] and maximum entropy flow networks [19]. A maximum entropy flow network is used as a deep
 547 probability distribution for the parameters, while these samples often parameterize a differentiable
 548 model simulator, which may lack a tractable likelihood function.

549 Consider model parameterization z and data x generated from some theoretical model simulator
 550 represented as $p(x | z)$, which may be deterministic or stochastic. Theoretical models usually have
 551 known sampling procedures for simulating activity given a circuit parameterization, yet often lack
 552 an explicit likelihood function due to the nonlinearities and dynamics. With EPI, a distribution
 553 on parameters z is learned, that yields an emergent property of interest \mathcal{B} ,

$$\mathcal{B} \leftrightarrow E_{z \sim q_\theta} [E_{x \sim p(x|z)} [T(x)]] = \mu \quad (13)$$

554 by making an approximation $q_\theta(z)$ to $p(z | \mathcal{B})$ (see Section A.1.5). So, over the DSN distribution
 555 $q_\theta(z)$ of model $p(x | z)$ for behavior \mathcal{B} , the emergent properties $T(x)$ are constrained in expectation
 556 to μ .

557 In deep probability distributions, a simple random variable $w \sim p_0$ is mapped deterministically
 558 via a function f_θ parameterized by a neural network to the support of the distribution of interest
 559 where $z = f_\theta(w) = f_l(\dots f_1(w))$. Given a theoretical model $p(x | z)$ and some behavior of interest
 560 \mathcal{B} , the deep probability distributions are trained by optimizing the neural network parameters θ to
 561 find a good approximation q_θ^* within the deep variational family Q to $p(z | \mathcal{B})$.

562 In most settings (especially those relevant to theoretical neuroscience) the likelihood of the behavior
 563 with respect to the model parameters $p(T(x) | z)$ is unknown or intractable, requiring an alternative
 564 to stochastic gradient variational Bayes [5] or black box variational inference[54]. These types
 565 of methods called likelihood-free variational inference (LFVI, [20]) skate around the intractable
 566 likelihood function in situations where there is a differentiable simulator. Akin to LFVI, DSNs are
 567 optimized with the following objective for a given theoretical model, emergent property statistics
 568 $T(x)$, and emergent property constraints μ :

$$\begin{aligned} q_\theta^*(z) &= \operatorname{argmax}_{q_\theta \in Q} H(q_\theta(z)) \\ \text{s.t. } E_{z \sim q_\theta} [E_{x \sim p(x|z)} [T(x)]] &= \mu \end{aligned} \quad (14)$$

569 Optimizing this objective is a technological accomplishment in its own right, the details of which
 570 we elaborate in Section A.1.2. Before going through those details, we ground this optimization in
 571 a toy example.

572 **A.1.1 Example: 2D LDS**

573 To gain intuition for EPI, consider two-dimensional linear dynamical systems, $\tau \dot{x} = Ax$ with

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$$

574 that produce a band of oscillations. To do EPI with the dynamics matrix elements as the free
 575 parameters $z = [a_1, a_2, a_3, a_4]$, and fixing $\tau = 1$, such that the posterior yields a band of oscillations,
 576 the emergent property statistics $T(x)$ are chosen to contain the first- and second-moments of the
 577 oscillatory frequency Ω and the growth/decay factor d of the oscillating system. To learn the
 578 distribution of real entries of A that yield a distribution of d with mean zero with variance 0.25^2 ,
 579 and oscillation frequency Ω with mean 1 Hz with variance $(0.1\text{Hz})^2$, then we would select the real
 580 part of the complex conjugate eigenvalues $\text{real}(\lambda_1) = d$ (via an arbitrary choice of eigenvalue of the
 581 dynamics matrix λ_1) and the positive imaginary component of one of the eigenvalues $\text{imag}(\lambda_1) =$
 582 $2\pi\Omega$ as the emergent property statistics. Those emergent property statistics are then constrained
 583 to

$$\mu = E \begin{bmatrix} \text{real}(\lambda_1) \\ \text{imag}(\lambda_1) \\ (\text{real}(\lambda_1) - 0)^2 \\ (\text{imag}(\lambda_1) - 2\pi\Omega)^2 \end{bmatrix} = \begin{bmatrix} 0.0 \\ 2\pi\Omega \\ 0.25^2 \\ (2\pi 0.1)^2 \end{bmatrix} \quad (15)$$

584 where $\Omega = 1\text{Hz}$. Unlike the models we study in the paper which calculate $E_{x \sim p(x|z)} [T(x)]$ via
 585 forward simulation, we have a closed form for the eigenvalues of the dynamics matrix. λ can be
 586 calculated using the quadratic formula:

$$\lambda = \frac{\left(\frac{a_1+a_4}{\tau}\right) \pm \sqrt{\left(\frac{a_1+a_4}{\tau}\right)^2 + 4\left(\frac{a_2a_3-a_1a_4}{\tau}\right)}}{2} \quad (16)$$

587 where λ_1 is the eigenvalue of $\frac{1}{\tau}A$ with greatest real part. Even though $E_{x \sim p(x|z)} [T(x)]$ is calculable
 588 directly via a closed form function and does not require simulation, we cannot derive the distribution
 589 q_θ^* directly. This is due to the formally hard problem of the backward mapping: finding the natural
 590 parameters η from the mean parameters μ of an exponential family distribution [55]. Instead, we
 591 can use EPI to learn the linear system parameters producing such a band of oscillations (Fig. S2B).

592 Even this relatively simple system has nontrivial (though intuitively sensible) structure in the
 593 parameter distribution. To validate our method (further than that of the underlying technology
 594 on a ground truth solution [19]) we can analytically derive the contours of the probability density

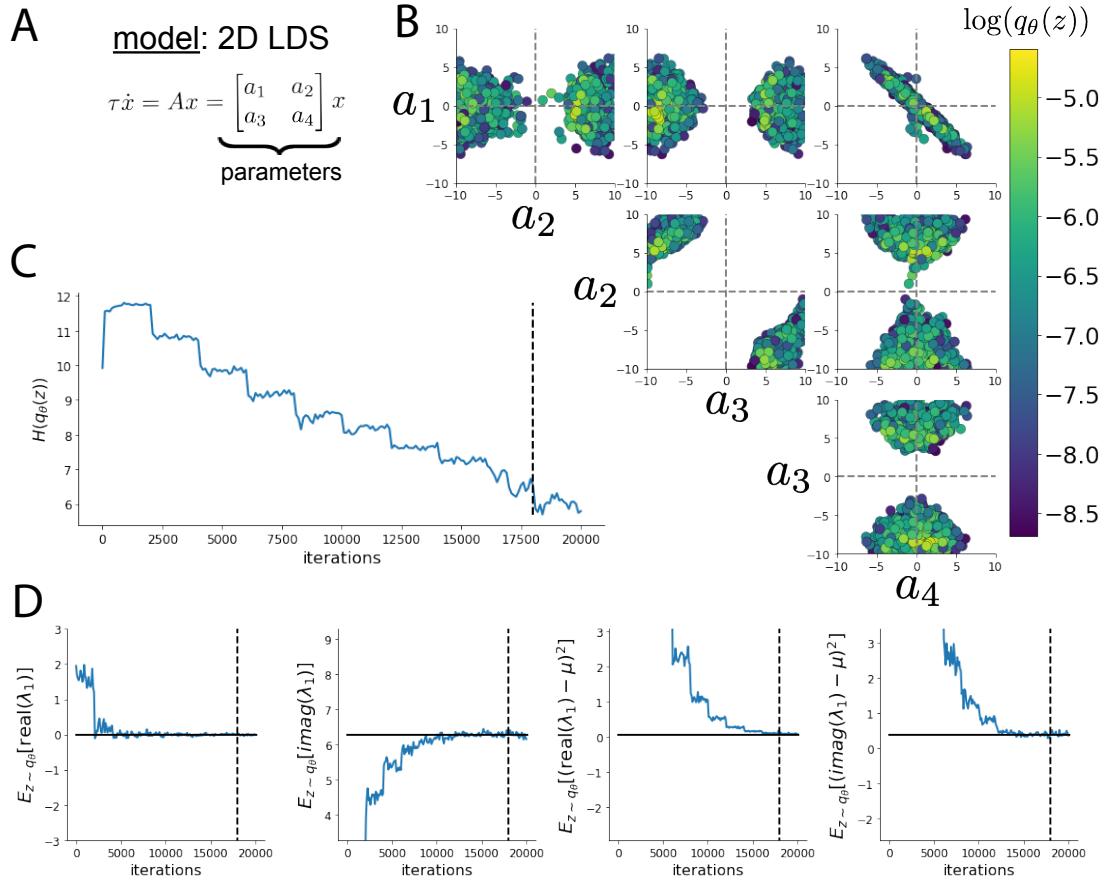


Fig. S2: A. Two-dimensional linear dynamical system model, where real entries of the dynamics matrix A are the parameters. B. The DSN distribution for a 2D LDS with $\tau = 1$ that produces an average of 1Hz oscillations with some small amount of variance. C. Entropy throughout the optimization. At the beginning of each augmented Lagrangian epoch (5,000 iterations), the entropy dips due to the shifted optimization manifold where emergent property constraint satisfaction is increasingly weighted. D. Emergent property moments throughout optimization. At the beginning of each augmented Lagrangian epoch, the emergent property moments move closer to their constraints.

595 from the emergent property statistics and values (Fig. S3). In the $a_1 - a_4$ plane, is a black line
 596 at $\text{real}(\lambda_1) = \frac{a_1 + a_4}{2} = 0$, a dotted black line at the standard deviation $\text{real}(\lambda_1) = \frac{a_1 + a_4}{2} \pm 1$, and a
 597 grey line at twice the standard deviation $\text{real}(\lambda_1) = \frac{a_1 + a_4}{2} \pm 2$ (Fig. S3A). Here the lines denote the
 598 set of solutions at fixed behaviors, which overlay the posterior obtained through EPI. The learned
 599 DSN distribution precisely reflects the desired statistical constraints and model degeneracy in the
 600 sum of a_1 and a_4 . Intuitively, the parameters equivalent with respect to emergent property statistic
 601 $\text{real}(\lambda_1)$ have similar log densities.

602 To explain the structure in the bimodality of the DSN posterior, we can look at the imaginary
 603 component of λ_1 . When $\text{real}(\lambda_1) = \frac{a_1 + a_4}{2} = 0$, we have

$$\text{imag}(\lambda_1) = \begin{cases} \sqrt{\frac{a_1 a_4 - a_2 a_3}{\tau}}, & \text{if } a_1 a_4 < a_2 a_3 \\ 0 & \text{otherwise} \end{cases} \quad (17)$$

604 When $\tau = 1$ and $a_1 a_4 > a_2 a_3$ (center of distribution above), we have the following equation for the
 605 other two dimensions:

$$\text{imag}(\lambda_1)^2 = a_1 a_4 - a_2 a_3 \quad (18)$$

606 Since we constrained $E_{q_\theta}[\text{imag}(\lambda)] = 2\pi$ (with $\omega = 1$), we can plot contours of the equation
 607 $\text{imag}(\lambda_1)^2 = a_1 a_4 - a_2 a_3 = (2\pi)^2$ for various $a_1 a_4$ (Fig. S3A). If $\sigma_{1,4} = E_{q_\theta}(|a_1 a_4 - E_{q_\theta}[a_1 a_4]|)$,
 608 then we plot the contours as $a_1 a_4 = 0$ (black), $a_1 a_4 = -\sigma_{1,4}$ (black dotted), and $a_1 a_4 = -2\sigma_{1,4}$
 609 (grey dotted) (Fig. S3B). This validates the curved structure of the inferred distribution learned
 610 through EPI. We take steps in negative standard deviation of $a_1 a_4$ (dotted and gray lines), since
 611 there are few positive values $a_1 a_4$ in the posterior. Subtler model-behavior combinations will have
 612 even more complexity, further motivating the use of EPI for understanding these systems. Indeed,
 613 we sample a distribution of systems oscillating near 1Hz (Fig. S4).

614 A.1.2 Augmented Lagrangian optimization

615 To optimize $q_\theta(z)$ in Equation ??, the constrained optimization is performed using the augmented
 616 Lagrangian method. The following objective is minimized:

$$L(\theta; \alpha, c) = -H(q_\theta) + \alpha^\top \delta(\theta) + \frac{c}{2} \|\delta(\theta)\|^2 \quad (19)$$

617 where $\delta(\theta) = E_{z \sim q_\theta} [E_{x \sim p(x|z)} [T(x) - \mu]]$, $\alpha \in \mathcal{R}^m$ are the Lagrange multipliers and c is the penalty
 618 coefficient. For a fixed (α, c) , θ is optimized with stochastic gradient descent. A low value of c is

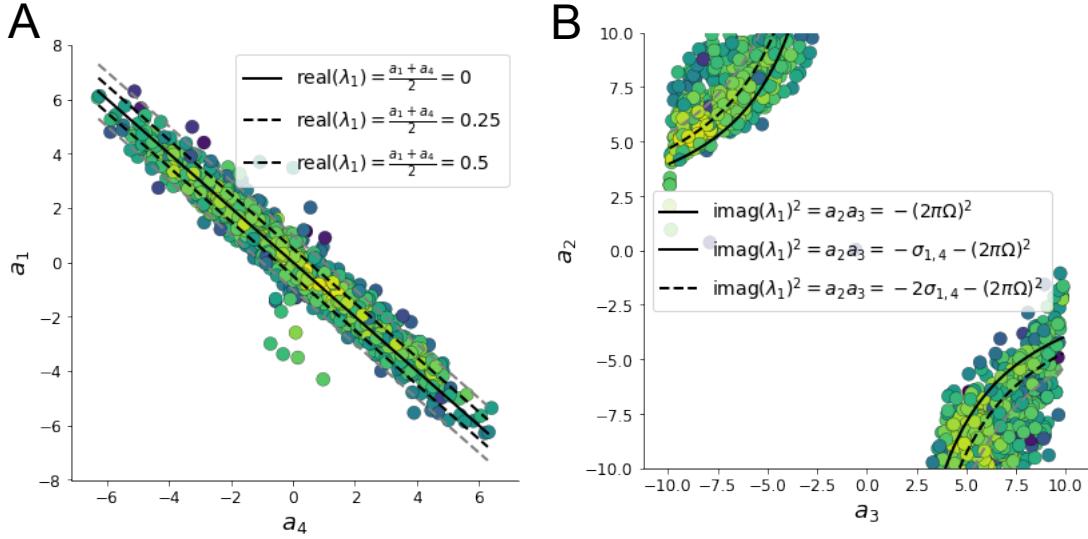


Fig. S3: A. Probability contours in the $a_1 - a_4$ plane can be derived from the relationship to emergent property statistic of growth/decay factor. B. Probability contours in the $a_2 - a_3$ plane can be derived from relationship to the emergent property statistic of oscillation frequency.

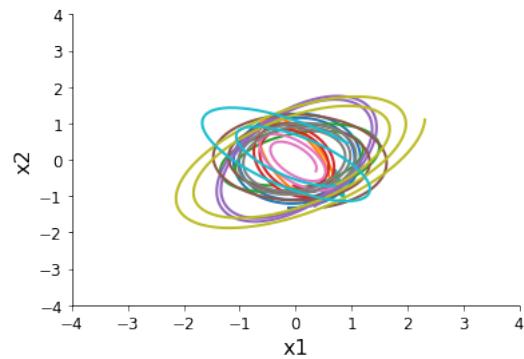


Fig. S4: Sampled dynamical system trajectories from the EPI distribution. Each trajectory is initialized at $x(0) = \left[\frac{\sqrt{2}}{2} \quad -\frac{\sqrt{2}}{2} \right]$.

used initially, and increased during each augmented Lagrangian epoch – a period of optimization with fixed α and c for a given number of stochastic optimization iterations. Similarly, α is tuned each epoch based on the constraint violations. For the linear 2-dimensional system (Fig. S2C) optimization hyperparameters are initialized to $c_1 = 10^{-4}$ and $\alpha_1 = 0$. The penalty coefficient is updated based on a hypothesis test regarding the reduction in constraint violation. The p-value of $E[\|\delta(\theta_{k+1})\|] > \gamma E[\|\delta(\theta_k)\|]$ is computed, and c_{k+1} is updated to βc_k with probability $1 - p$. Throughout the project, $\beta = 4.0$ and $\gamma = 0.25$ is used. The other update rule is $\alpha_{k+1} = \alpha_k + c_k \frac{1}{n} \sum_{i=1}^n (T(x^{(i)}) - \mu)$. In this example, each augmented Lagrangian epoch ran for 2,000 iterations. We consider the optimization to have converged when a null hypothesis test of constraint violations being zero is accepted for all constraints at a significance threshold 0.05. This is the dotted line on the plots below depicting the optimization cutoff of EPI optimization for the 2-dimensional linear system. If the optimization is left to continue running, entropy usually decreases, and structural pathologies in the distribution may be introduced.

The intention is that c and α start at values encouraging entropic growth early in optimization. Then, as they increase in magnitude with each training epoch, the constraint satisfaction terms are increasingly weighted, resulting in a decrease in entropy. Rather than using a naive initialization, before EPI, we optimize the deep probability distribution parameters to generate samples of an isotropic Gaussian of a selected variance, such as 1.0 for the 2D LDS example. This provides a convenient starting point, whose level of entropy is controlled by the user.

638 A.1.3 Normalizing flows

Since we are optimizing parameters θ of our deep probability distribution with respect to the entropy, we will need to take gradients with respect to the log-density of samples from the deep probability distribution.

$$H(q_\theta(z)) = \int -q_\theta(z) \log(q_\theta(z)) dz = E_{z \sim q_\theta} [-\log(q_\theta(z))] = E_{\omega \sim q_0} [-\log(q_\theta(f_\theta(\omega)))] \quad (20)$$

$$\nabla_\theta H(q_\theta(z)) = E_{\omega \sim q_0} [-\nabla_\theta \log(q_\theta(f_\theta(\omega)))] \quad (21)$$

Deep probability models typically consist of several layers of fully connected neural networks. When each neural network layer is restricted to be a bijective function, the sample density can be calculated using the change of variables formula at each layer of the network. For $z' = f(z)$,

$$q(z') = q(f^{-1}(z')) \left| \det \frac{\partial f^{-1}(z')}{\partial z'} \right| = q(z) \left| \det \frac{\partial f(z)}{\partial z} \right|^{-1} \quad (22)$$

646 However, this computation has cubic complexity in dimensionality for fully connected layers. By
 647 restricting our layers to normalizing flows [16] – bijective functions with fast log determinant ja-
 648 cobian computations, we can tractably optimize deep generative models with objectives that are a
 649 function of sample density, like entropy. Most of our analyses use real NVP [56], which have proven
 650 effective in our architecture searches, and have the advantageous features of fast sampling and fast
 651 density evaluation.

652 **A.1.4 Related work**

653 (To come)

654

655 **A.1.5 Emergent property inference as variational inference in an exponential family**

656 (To come)

657

658 **A.2 Theoretical models**

659 In this study, we used emergent property inference to examine several models relevant to theoretical
 660 neuroscience. Here, we provide the details of each model and the related analyses.

661 **A.2.1 Stomatogastric ganglion**

662 Each neuron's membrane potential $x_m(t)$ is the solution of the following differential equation.

$$C_m \frac{dx_m}{dt} = -[h_{leak}(x; z) + h_{Ca}(x; z) + h_K(x; z) + h_{hyp}(x; z) + h_{elec}(x; z) + h_{syn}(x; z)] \quad (23)$$

663 The membrane potential of each neuron is affected by the leak, calcium, potassium, hyperpolariza-
 664 tion, electrical and synaptic currents, respectively. The capacitance of the cell membrane was set to
 665 $C_m = 1nF$. Each current is a function of the neuron's membrane potential x_m and the parameters
 666 of the circuit such as g_{el} and g_{syn} , whose effect on the circuit is considered in the motivational

example of EPI in Fig. 1. Specifically, the currents are the difference in the neuron’s membrane potential and that current type’s reversal potential multiplied by a conductance:

$$h_{leak}(x; z) = g_{leak}(x_m - V_{leak}) \quad (24)$$

$$h_{elec}(x; z) = g_{el}(x_m^{post} - x_m^{pre}) \quad (25)$$

$$h_{syn}(x; z) = g_{syn}S_\infty^{pre}(x_m^{post} - V_{syn}) \quad (26)$$

$$h_{Ca}(x; z) = g_{Ca}M_\infty(x_m - V_{Ca}) \quad (27)$$

$$h_K(x; z) = g_KN(x_m - V_K) \quad (28)$$

$$h_{hyp}(x; z) = g_hH(x_m - V_{hyp}) \quad (29)$$

The reversal potentials were set to $V_{leak} = -40mV$, $V_{Ca} = 100mV$, $V_K = -80mV$, $V_{hyp} = -20mV$, and $V_{syn} = -75mV$. The other conductance parameters were fixed to $g_{leak} = 1 \times 10^{-4}\mu S$. g_{Ca} , g_K , and g_{hyp} had different values based on fast, intermediate (hub) or slow neuron. Fast: $g_{Ca} = 1.9 \times 10^{-2}$, $g_K = 3.9 \times 10^{-2}$, and $g_{hyp} = 2.5 \times 10^{-2}$. Intermediate: $g_{Ca} = 1.7 \times 10^{-2}$, $g_K = 1.9 \times 10^{-2}$, and $g_{hyp} = 8.0 \times 10^{-3}$. Intermediate: $g_{Ca} = 8.5 \times 10^{-3}$, $g_K = 1.5 \times 10^{-2}$, and $g_{hyp} = 1.0 \times 10^{-2}$.

Furthermore, the Calcium, Potassium, and hyperpolarization channels have time-dependent gating dynamics dependent on steady-state gating variables M_∞ , N_∞ and H_∞ , respectively.

$$M_\infty = 0.5 \left(1 + \tanh \left(\frac{x_m - v_1}{v_2} \right) \right) \quad (30)$$

$$\frac{dN}{dt} = \lambda_N(N_\infty - N) \quad (31)$$

$$N_\infty = 0.5 \left(1 + \tanh \left(\frac{x_m - v_3}{v_4} \right) \right) \quad (32)$$

$$\lambda_N = \phi_N \cosh \left(\frac{x_m - v_3}{2v_4} \right) \quad (33)$$

$$\frac{dH}{dt} = \frac{(H_\infty - H)}{\tau_h} \quad (34)$$

$$H_\infty = \frac{1}{1 + \exp \left(\frac{x_m + v_5}{v_6} \right)} \quad (35)$$

$$\tau_h = 272 - \left(\frac{-1499}{1 + \exp \left(\frac{-x_m + v_7}{v_8} \right)} \right) \quad (36)$$

where we set $v_1 = 0mV$, $v_2 = 20mV$, $v_3 = 0mV$, $v_4 = 15mV$, $v_5 = 78.3mV$, $v_6 = 10.5mV$, $v_7 = -42.2mV$, $v_8 = 87.3mV$, $v_9 = 5mV$, and $v_{th} = -25mV$. These are the same parameter values used in [22].

690 Finally, there is a synaptic gating variable as well:

$$S_\infty = \frac{1}{1 + \exp\left(\frac{v_{th} - x_m}{v_0}\right)} \quad (37)$$

691 When the dynamic gating variables are considered, this is actually a 15-dimensional nonlinear
692 dynamical system.

693 In order to measure the frequency of the hub neuron during EPI, the STG model was simulated
694 for $T = 500$ time steps of $dt = 25ms$. In EPI, since gradients are taken through the simulation
695 process, the number of time steps are kept as modest if possible. The chosen dt and T were the
696 most computationally convenient choices yielding accurate frequency measurement.

697 Our original approach to measuring frequency was to take the max of the fast Fourier transform
698 (FFT) of the simulated time series. There are a few key considerations here. One is resolution
699 in frequency space. Each FFT entry will correspond to a signal frequency of $\frac{F_s k}{N}$, where N is
700 the number of samples used for the FFT, $F_s = \frac{1}{dt}$, and $k \in [0, 1, \dots, N - 1]$. Our resolution is
701 improved by increasing N and decreasing dt . Increasing $N = T - b$, where b is some fixed number
702 of buffer burn-in initialization samples, necessitates an increase in simulation time steps T , which
703 directly increases computational cost. Increasing F_s (decreasing dt) increases system approximation
704 accuracy, but requires more time steps before a full cycle is observed. At the level of $dt = 0.025$,
705 thousands of temporal samples were required for resolution of .01Hz. These challenges in frequency
706 resolution with the discrete Fourier transform motivated the use of an alternative basis of complex
707 exponentials. Instead, we used a basis of complex exponentials with frequencies from 0.0-1.0 Hz at
708 0.01Hz resolution, $\Phi = [0.0, 0.01, \dots, 1.0]^\top$

709 Another consideration was that the frequency spectra of the hub neuron has several peaks. This
710 was due to high-frequency sub-threshold activity. The maximum frequency was often not the firing
711 frequency. Accordingly, subthreshold activity was set to zero, and the whole signal was low-pass
712 filtered with a moving average window of length 20. The signal was subsequently mean centered.
713 After this pre-processing, the maximum frequency in the filter bank accurately reflected the firing
714 frequency.

715 Finally, to differentiate through the maximum frequency identification step, we used a sum-of-
716 powers normalization strategy: Let $\mathcal{X}_i \in \mathcal{C}^{|\Phi|}$ be the complex exponential filter bank dot products
717 with the signal $x_i \in \mathcal{R}^N$, where $i \in \{\text{f1}, \text{f2}, \text{hub}, \text{s1}, \text{s2}\}$. The “frequency identification” vector is

$$u_i = \frac{|\mathcal{X}_i|^\alpha}{\sum_{k=1}^N |\mathcal{X}_i(k)|^\alpha} \quad (38)$$

- 718 The frequency is then calculated as $\Omega_i = u_i^\top \Phi$ with $\alpha = 100$.
- 719 Network syncing, like all other emergent properties in this work, are defined by the emergent
720 property statistics and values. The emergent property statistics are the first- and second-moments
721 of the firing frequencies. The first moments are set to 0.55Hz, while the second moments are set to
722 0.025Hz².

$$E \begin{bmatrix} \Omega_{f1} \\ \Omega_{f2} \\ \Omega_{hub} \\ \Omega_{s1} \\ \Omega_{s2} \\ (\Omega_{f1} - 0.55)^2 \\ (\Omega_{f2} - 0.55)^2 \\ (\Omega_{hub} - 0.55)^2 \\ (\Omega_{s1} - 0.55)^2 \\ (\Omega_{s2} - 0.55)^2 \end{bmatrix} = \begin{bmatrix} 0.55 \\ 0.55 \\ 0.55 \\ 0.55 \\ 0.55 \\ 0.025^2 \\ 0.025^2 \\ 0.025^2 \\ 0.025^2 \\ 0.025^2 \end{bmatrix} \quad (39)$$

- 723 For EPI in Fig 2C, we used a real NVP architecture with two coupling layers. Each coupling layer
724 had two hidden layers of 10 units each, and we mapped onto a support of $z \in \left[\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 10 \\ 8 \end{bmatrix} \right]$. We
725 have shown the EPI optimization that converged with maximum entropy across 2 random seeds
726 and augmented Lagrangian coefficient initializations of $c_0=0$, 2, and 5.

727 **A.2.2 Primary visual cortex**

- 728 The dynamics of each neural populations average rate $x = \begin{bmatrix} x_E \\ x_P \\ x_S \\ x_V \end{bmatrix}$ are given by:
- $$\tau \frac{dx}{dt} = -x + [Wx + h]_+^n \quad (40)$$

- 729 Some neuron-types largely lack synaptic projections to other neuron-types [57], and it is popular

⁷³⁰ to only consider a subset of the effective connectivities [23].

$$W = \begin{bmatrix} W_{EE} & W_{EP} & W_{ES} & 0 \\ W_{PE} & W_{PP} & W_{PS} & 0 \\ W_{SE} & 0 & 0 & W_{SV} \\ W_{VE} & W_{VP} & W_{VS} & 0 \end{bmatrix} \quad (41)$$

⁷³¹ By consolidating information from many experimental datasets, Billeh et al. [41] produce estimates
⁷³² of the synaptic strength (in mV)

$$M = \begin{bmatrix} 0.36 & 0.48 & 0.31 & 0.28 \\ 1.49 & 0.68 & 0.50 & 0.18 \\ 0.86 & 0.42 & 0.15 & 0.32 \\ 1.31 & 0.41 & 0.52 & 0.37 \end{bmatrix} \quad (42)$$

⁷³³ and connection probability

$$C = \begin{bmatrix} 0.16 & 0.411 & 0.424 & 0.087 \\ 0.395 & .451 & 0.857 & 0.02 \\ 0.182 & 0.03 & 0.082 & 0.625 \\ 0.105 & 0.22 & 0.77 & 0.028 \end{bmatrix} \quad (43)$$

⁷³⁴ Multiplying these connection probabilities and synaptic efficacies gives us an effective connectivity
⁷³⁵ matrix:

$$W_{\text{full}} = C \odot M = \begin{bmatrix} 0.16 & 0.411 & 0.424 & 0.087 \\ 0.395 & .451 & 0.857 & 0.02 \\ 0.182 & 0.03 & 0.082 & 0.625 \\ 0.105 & 0.22 & 0.77 & 0.028 \end{bmatrix} \quad (44)$$

⁷³⁶ From use the entries of this full effective connectivity matrix that are not considered to be ineffectual.
⁷³⁷

⁷³⁸ We look at how this four-dimensional nonlinear dynamical model of V1 responds to different inputs,
⁷³⁹ and compare the predictions of the linear response to the approximate posteriors obtained through
⁷⁴⁰ EPI. The input to the system is the sum of a baseline input $b = [1 \ 1 \ 1 \ 1]^T$ and a differential
⁷⁴¹ input dh :

$$h = b + dh \quad (45)$$

⁷⁴² All simulations of this system had $T = 100$ time points, a time step $dt = 5\text{ms}$, and time constant
⁷⁴³ $\tau = 20\text{ms}$. And the system was initialized to a random draw $x(0)_i \sim \mathcal{N}(1, 0.01)$.

⁷⁴⁴ We can describe the dynamics of this system more generally by

$$\dot{x}_i = -x_i + f(u_i) \quad (46)$$

⁷⁴⁵ where the input to each neuron is

$$u_i = \sum_j W_{ij} x_j + h_i \quad (47)$$

⁷⁴⁶ Let $F_{ij} = \gamma_i \delta(i, j)$, where $\gamma_i = f'(u_i)$. Then, the linear response is

$$\frac{dx_{ss}}{dh} = F(W \frac{dx_{ss}}{dh} + I) \quad (48)$$

⁷⁴⁷ which is calculable by

$$\frac{dx_{ss}}{dh} = (F^{-1} - W)^{-1} \quad (49)$$

⁷⁴⁸ The emergent property we considered was the first and second moments of the change in rate dx

⁷⁴⁹ between the baseline input $h = b$ and $h = b + dh$. We use the following notation to indicate that

⁷⁵⁰ the emergent property statistics were set to the following values:

$$\mathcal{B}(\alpha, y) \leftrightarrow E \begin{bmatrix} dx_{\alpha,ss} \\ (dx_{\alpha,ss} - y)^2 \end{bmatrix} = \begin{bmatrix} y \\ 0.01^2 \end{bmatrix} \quad (50)$$

⁷⁵¹ In the final analysis for this model, we sweep the input one neuron at a time away from the mode

⁷⁵² of each inferred distributions $dh^* = z^* = \text{argmax}_z \log q_\theta(z \mid \mathcal{B}(\alpha, 0.1))$. The differential responses

⁷⁵³ $dx_{\alpha,ss}$ are examined at perturbed inputs $h = b + dh^* + \Delta h_\alpha u_\alpha$ where u_α is a unit vector in the

⁷⁵⁴ dimension of α and $\Delta h_\alpha \in [-15, 15]$.

⁷⁵⁵ For each $\mathcal{B}(\alpha, y)$ with $\alpha \in \{E, P, S, V\}$ and $y \in \{0.1, 0.5\}$, we ran EPI with five different random

⁷⁵⁶ initial seeds using an architecture of four coupling layers, each with two hidden layers of 10 units.

⁷⁵⁷ We set $c_0 = 10^5$. The support of the learned distribution was restricted to $z_i \in [-5, 5]$.

⁷⁵⁸ A.2.3 Superior colliculus

⁷⁵⁹ There are four total units: two in each hemisphere corresponding to the Pro/Contra and Anti/Ipsi

⁷⁶⁰ populations. Each unit has an activity (x_i) and internal variable (u_i) related by

$$x_i(t) = \left(\frac{1}{2} \tanh \left(\frac{v_i(t) - \epsilon}{\zeta} \right) + \frac{1}{2} \right) \quad (51)$$

⁷⁶¹ $\epsilon = 0.05$ and $\zeta = 0.5$ control the position and shape of the nonlinearity, respectively.

⁷⁶² We can order the elements of x_i and v_i into vectors x and v with elements

$$x = \begin{bmatrix} x_{LP} \\ x_{LA} \\ x_{RP} \\ x_{RA} \end{bmatrix} \quad v = \begin{bmatrix} v_{LP} \\ v_{LA} \\ v_{RP} \\ v_{RA} \end{bmatrix} \quad (52)$$

⁷⁶³ The internal variables follow dynamics:

$$\tau \frac{dv}{dt} = -v + Wx + h + \sigma dB \quad (53)$$

⁷⁶⁴ with time constant $\tau = 0.09s$ and Gaussian noise σdB controlled by the magnitude of $\sigma = 1.0$. The
⁷⁶⁵ weight matrix has 8 parameters sW_P , sW_A , vW_{PA} , vW_{AP} , hW_P , hW_A , dW_{PA} , and dW_{AP} (Fig.
⁷⁶⁶ 4B).

$$W = \begin{bmatrix} sW_P & vW_{PA} & hW_P & dW_{PA} \\ vW_{AP} & sW_A & dW_{AP} & hW_A \\ hW_P & dW_{PA} & sW_P & vW_{PA} \\ dW_{AP} & hW_A & vW_{AP} & sW_A \end{bmatrix} \quad (54)$$

⁷⁶⁷ The system receives five inputs throughout each trial, which has a total length of 1.8s.

$$h = h_{\text{rule}} + h_{\text{choice-period}} + h_{\text{light}} \quad (55)$$

⁷⁶⁸ There are rule-based inputs depending on the condition,

$$h_{P,\text{rule}}(t) = \begin{cases} I_{P,\text{rule}} \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^\top, & \text{if } t \leq 1.2s \\ 0, & \text{otherwise} \end{cases} \quad (56)$$

$$h_{A,\text{rule}}(t) = \begin{cases} I_{A,\text{rule}} \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^\top, & \text{if } t \leq 1.2s \\ 0, & \text{otherwise} \end{cases} \quad (57)$$

⁷⁷⁰ a choice-period input,

$$h_{\text{choice}}(t) = \begin{cases} I_{\text{choice}} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^\top, & \text{if } t > 1.2s \\ 0, & \text{otherwise} \end{cases} \quad (58)$$

⁷⁷¹ and an input to the right or left-side depending on where the light stimulus is delivered.

$$h_{\text{light}}(t) = \begin{cases} I_{\text{light}} \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^\top, & \text{if } t > 1.2s \text{ and Left} \\ I_{\text{light}} \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}^\top, & \text{if } t > 1.2s \text{ and Right} \\ 0, & t \leq 1.2s \end{cases} \quad (59)$$

772 The input parameterization was fixed to $I_{P,\text{rule}} = 10$, $I_{A,\text{rule}} = 10$, $I_{\text{choice}} = 2$, and $I_{\text{light}} = 1$
 773 To produce a Bernoulli rate of p_{LP} in the Left, Pro condition (we can generalize this to either cue,
 774 or stimulus condition), let \hat{p}_i be the empirical average steady state (ss) response (final x_{LP} at end
 775 of task) over $M=500$ Gaussian noise draws for a given SC model parameterization z_i :

$$\hat{p}_i = E_{\sigma dB} [x_{LP,ss} | s = L, c = P, z_i] = \frac{1}{M} \sum_{j=1}^M x_{LP,ss}(s = L, c = P, z_i, \sigma dB_j) \quad (60)$$

776 For the first constraint, the average over posterior samples (from $q_\theta(z)$) to be p_{LP} :

$$E_{z_i \sim q_\phi} [E_{\sigma dB} [x_{LP,ss} | s = L, c = P, z_i]] = E_{z_i \sim q_\phi} [\hat{p}_i] = p_{LP} \quad (61)$$

777 We can then ask that the variance of the steady state responses across Gaussian draws, is the
 778 Bernoulli variance for the empirical rate \hat{p}_i .

$$E_{z \sim q_\phi} [\sigma_{err}^2] = 0 \quad (62)$$

779

$$\sigma_{err}^2 = Var_{\sigma dB} [x_{LP,ss} | s = L, c = P, z_i] - \hat{p}_i(1 - \hat{p}_i) \quad (63)$$

780 We have an additional constraint that the Pro neuron on the opposite hemisphere should have the
 781 opposite value. We can enforce this with a final constraint:

$$E_{z \sim q_\phi} [d_P] = 1 \quad (64)$$

782

$$E_{\sigma dB} [(x_{LP,ss} - x_{RP,ss})^2 | s = L, c = P, z_i] \quad (65)$$

783 We refer to networks obeying these constraints as Bernoulli, winner-take-all networks. Since the
 784 maximum variance of a random variable bounded from 0 to 1 is the Bernoulli variance ($\hat{p}(1 - \hat{p})$),
 785 and the maximum squared difference between two variables bounded from 0 to 1 is 1, we do not
 786 need to control the second moment of these test statistics. In reality, these variables are dynamical
 787 system states and can only exponentially decay (or saturate) to 0 (or 1), so the Bernoulli variance
 788 error and squared difference constraints can only be undershot. This is important to be mindful
 789 of when evaluating the convergence criteria. Instead of using our usual hypothesis testing criteria
 790 for convergence to the emergent property, we set a slack variable threshold for these technically
 791 infeasible constraints to 0.05.

792 Training DSNs to learn distributions of dynamical system parameterizations that produce Bernoulli
 793 responses at a given rate (with small variance around that rate) was harder to do than expected.

794 There is a pathology in this optimization setup, where the learned distribution of weights is bimodal
 795 attributing a fraction p of the samples to an expansive mode (which always sends x_{LP} to 1), and a
 796 fraction $1 - p$ to a decaying mode (which always sends x_{LP} to 0). This pathology was avoided using
 797 an inequality constraint prohibiting parameter samples that resulted in low variance of responses
 798 across noise.

799 In total, the emergent property of rapid task switching accuracy at level p was defined as

$$\mathcal{B}(p) \leftrightarrow \begin{bmatrix} \hat{p}_P \\ \hat{p}_A \\ (\hat{p}_P - p)^2 \\ (\hat{p}_A - p)^2 \\ \sigma_{P,err}^2 \\ \sigma_{A,err}^2 \\ d_P \\ d_A \end{bmatrix} = \begin{bmatrix} p \\ p \\ 0.15^2 \\ 0.15^2 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad (66)$$

800 For each accuracy level p , we ran EPI for 10 different random seeds and selected the maximum
 801 entropy solution using an architecture of 10 planar flows with $c_0 = 2$. The support of z was \mathcal{R}^8 .

802 A.2.4 Rank-1 RNN

803 Recent work establishes a link between RNN connectivity weights and the resulting dynamical
 804 responses of the network, using dynamic mean field theory (DMFT) [25]. Specifically, DMFT
 805 describes the properties of activity in infinite-size neural networks given a distribution on the
 806 connectivity weights. In such a model, the connectivity of a rank-1 RNN (which was sufficient for
 807 our task), has weight matrix W , which is the sum of a random component with strength determined
 808 by g and a structured component determined by the outer product of vectors m and n :

$$W = g\chi + \frac{1}{N}mn^\top, \quad (67)$$

809 where the activity x evolves as and $I(t)$ is some input, ϕ is the tanh nonlinearity, and $\chi_{ij} \sim \mathcal{N}(0, \frac{1}{N})$.
 810 The entries of m and n are drawn from Gaussian distributions $m_i \sim \mathcal{N}(M_m, 1)$ and $n_i \sim \mathcal{N}(M_n, 1)$.
 811 From such a parameterization, this theory produces consistency equations for the dynamic mean
 812 field variables in terms of parameters like g , M_m , and M_n , which we study in Section 3.5. That
 813 is the dynamic mean field variables (e.g. the activity along a vector κ_v , the total variance

814 Δ_0 , structured variance Δ_∞ , and the chaotic variance Δ_T) are written as functions of one another
815 in terms of connectivity parameters. The values of these variables can be used obtained using a
816 nonlinear system of equations solver. These dynamic mean field variables are then cast as task-
817 relevant variables with respect to the context of the provided inputs. Mastrogiuseppe et al. designed
818 low-rank RNN connectivities via minimalist connectivity parameters to solve canonical tasks from
819 behavioral neuroscience.

820 We consider the DMFT equation solver as a black box that takes in a low-rank parameterization z
821 (e.g. $z = [g \quad M_m \quad M_n]$) and outputs the values of the dynamic mean field variables, of which we
822 cast κ_w and Δ_T as task-relevant variables μ_{post} and σ_{post}^2 in the Gaussian posterior conditioning
823 toy example. Importantly, the solution produced by the solver is differentiable with respect to the
824 input parameters, allowing us to use DMFT to calculate the emergent property statistics in EPI
825 to learn distributions on such connectivity parameters of RNNs that execute tasks.

826 Specifically, we solve for the mean field variables κ_w , κ_n , Δ_0 and Δ_∞ , where the readout is nominally
827 chosen to point in the unit orthant $w = [1 \quad \dots \quad 1]^\top$. The consistency equations for these variables
828 in the presence of an constant input $I(t) = y - (n - M_n)$ can be derived following [25] are

$$\begin{aligned} \kappa_w &= F(\kappa_w, \kappa_n, \Delta_0, \Delta_\infty) = M_m \kappa_n + y \\ \kappa_n &= G(\kappa_w, \kappa_n, \Delta_0, \Delta_\infty) = M_n \langle [\phi_i] \rangle + \langle [\phi'_i] \rangle \\ \frac{\Delta_0^2 - \Delta_\infty^2}{2} &= H(\kappa_w, \kappa_n, \Delta_0, \Delta_\infty) = g^2 \left(\int \mathcal{D}z \Phi^2(\kappa_w + \sqrt{\Delta_0} z) - \int \mathcal{D}z \int \mathcal{D}x \Phi(\kappa_w + \sqrt{\Delta_0 - \Delta_\infty} x + \sqrt{\Delta_\infty} z) \right) \\ &\quad + (\kappa_n^2 + 1)(\Delta_0 - \Delta_\infty) \\ \Delta_\infty &= L(\kappa_w, \kappa_n, \Delta_0, \Delta_\infty) = g^2 \int \mathcal{D}z \left[\int \mathcal{D}x \phi(\kappa_w + \sqrt{\Delta_0 - \Delta_\infty} x + \sqrt{\Delta_\infty} z) \right]^2 + \kappa_n^2 + 1 \end{aligned} \tag{68}$$

829 where z here is a gaussian integration variable. We can solve these equations by simulating the
830 following Langevin dynamical system.

$$\begin{aligned} x(t) &= \frac{\Delta_0(t)^2 - \Delta_\infty(t)^2}{2} \\ \Delta_0(t) &= \sqrt{2x(t) + \Delta_\infty(t)^2} \\ \dot{\kappa}_w(t) &= -\kappa_w(t) + F(\kappa_w(t), \kappa_n(t), \Delta_0(t), \Delta_\infty(t)) \\ \dot{\kappa}_n(t) &= -\kappa_n + G(\kappa_w(t), \kappa_n(t), \Delta_0(t), \Delta_\infty(t)) \\ \dot{x}(t) &= -x(t) + H(\kappa_w(t), \kappa_n(t), \Delta_0(t), \Delta_\infty(t)) \\ \dot{\Delta_\infty}(t) &= -\Delta_\infty(t) + L(\kappa_w(t), \kappa_n(t), \Delta_0(t), \Delta_\infty(t)) \end{aligned} \tag{69}$$

831 Then, the temporal variance, which is necessary for the Gaussian posterior conditioning example,
832 is simply calculated via

$$\Delta_T = \Delta_0 - \Delta_\infty \quad (70)$$

833 **A.3 Supplementary Figures**

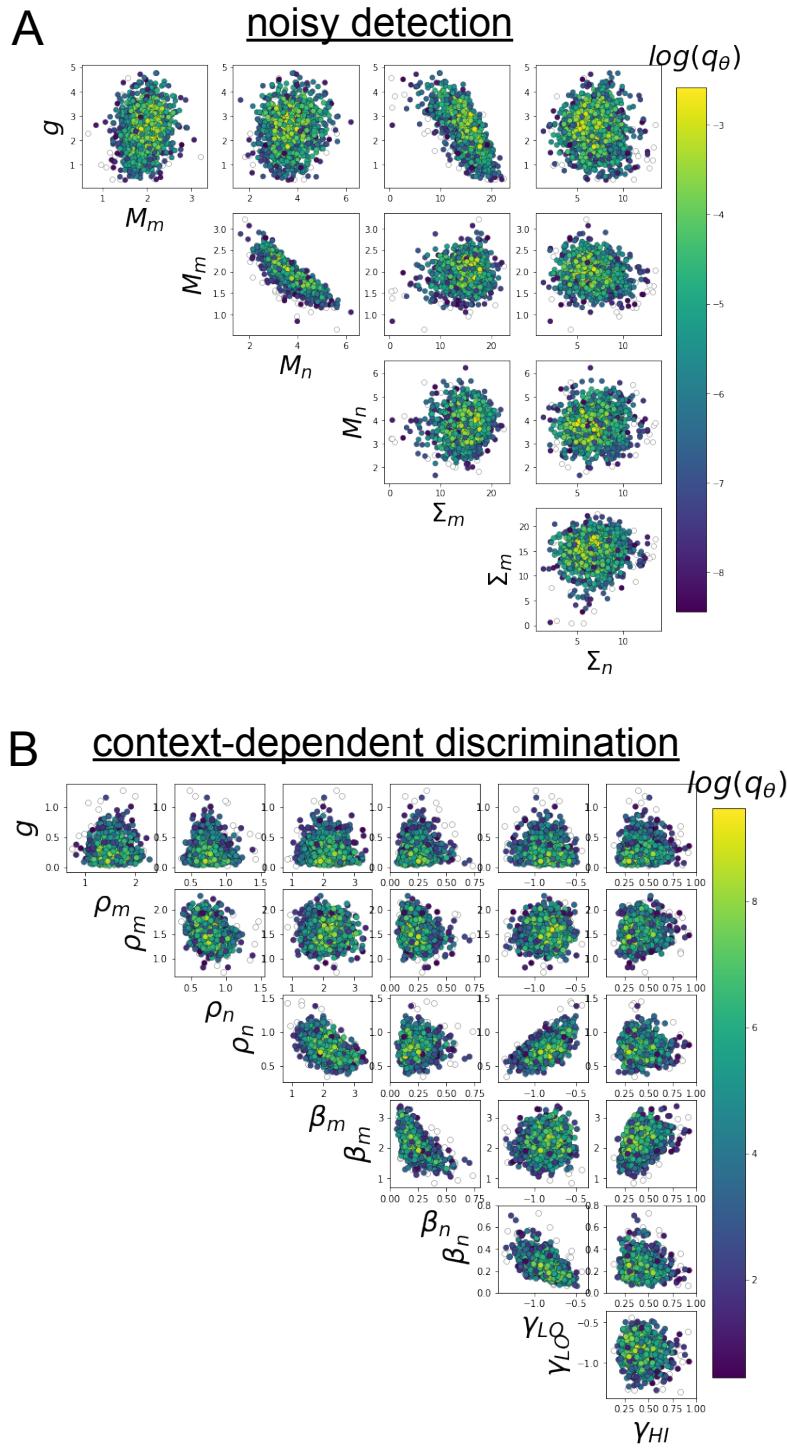


Fig. S1: A. EPI for rank-1 networks doing discrimination. B. EPI for rank-2 networks doing context-dependent discrimination.