

Interrogating theoretical models of neural computation with deep learning

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¹ 1 Abstract

² The cornerstone of theoretical neuroscience is the circuit model: a system of equations that captures
³ a hypothesized neural mechanism of scientific importance. Such models are valuable when they give
⁴ rise to an experimentally observed phenomenon – whether behavioral or in terms of neural activity –
⁵ and thus can offer insight into neural computation. The operation of these circuits, like all models,
⁶ critically depends on the choices of model parameters. Historically, the gold standard has been
⁷ to analytically derive the relationship between model parameters and computational properties.
⁸ However, this enterprise quickly becomes infeasible as biologically realistic constraints are included
⁹ into the model, often resulting in *ad hoc* approaches to understanding the relationship between
¹⁰ model and computation. We bring recent machine learning techniques – the use of deep generative
¹¹ models for probabilistic inference – to bear on this problem, learning distributions of parameters
¹² that produce the specified properties of computation. Importantly, the techniques we introduce offer
¹³ a principled means to understand the implications of model parameter choices on computational
¹⁴ properties of interest. We motivate this methodology with a worked example analyzing sensitivity in
¹⁵ the stomatogastric ganglion. We then use it to generate insights into neuron-type input-responsivity
¹⁶ in a model of primary visual cortex, a new understanding of rapid task switching in superior
¹⁷ colliculus models, and attribution of bias in recurrent neural networks solving a toy mathematical
¹⁸ problem. More generally, this work suggests a departure from realism vs tractability considerations
¹⁹ towards the use of modern machine learning for sophisticated interrogation of biologically relevant
²⁰ models.

²¹ 2 Introduction

²² The fundamental practice of theoretical neuroscience is to use a mathematical model to understand
²³ neural computation, whether that computation enables perception, action, or some intermediate
²⁴ processing [1]. In this field, a neural computation is systematized with a set of equations – the
²⁵ model – and these equations are motivated by biophysics, neurophysiology, and other conceptual
²⁶ considerations. The function of this system is governed by the choice of model parameters, which

27 when configured appropriately, give rise to a measurable signature of a computation. The work of
28 analyzing a model then becomes the inverse problem: given a computation of interest, how can we
29 reason about these suitable parameter configurations – their likely values, their uniquenesses and
30 degeneracies, their attractor states and phase transitions, and more?

31 Consider the idealized practice: a theorist considers a model carefully and analytically derives how
32 model parameters govern the computation. Seminal examples of this gold standard include our
33 field’s understanding of memory capacity in associative neural networks [2], chaos and autocorrela-
34 tion timescales in random neural networks [3], and the paradoxical effect in excitatory/inhibitory
35 networks [4]. Unfortunately, as circuit models include more biological realism, theory via analytic
36 derivation becomes intractable. This fact creates an unfavorable tradeoff for the theorist. On the
37 one hand, one may tractably analyze systems of equations with unrealistic assumptions (for ex-
38 ample symmetry or gaussianity), producing accurate inferences about parameters of a too-simple
39 model. On the other hand, one may choose a more biologically relevant model at the cost of *ad hoc*
40 approaches to analysis (simply examining simulated activity), producing questionable or partial
41 inferences about parameters of an appropriately complex, scientifically relevant model.

42 Of course, this same tradeoff has been confronted in many scientific fields and engineering problems
43 characterized by the need to do inference in complex models. In response, the machine learning
44 community has made remarkable progress in recent years, via the use of deep neural networks as a
45 powerful inference engine: a flexible function family that can map observed phenomena (in this case
46 the measurable signal of some computation) back to probability distributions quantifying the likely
47 parameter configurations. One celebrated example of this approach from the machine learning
48 community, from which we draw key inspiration for this work, is the variational autoencoder [5, 6],
49 which uses a deep neural network to induce an (approximate) posterior distribution on hidden
50 variables in a latent variable model, given data. Indeed, these tools have been used to great success
51 in neuroscience as well, in particular for interrogating parameters (sometimes treated as hidden
52 states) in models of both cortical population activity [7, 8, 9, 10] and animal behavior [11, 12, 13].
53 These works have used deep neural networks to expand the expressivity and accuracy of statistical
54 models of neural data [14].

55 However, these inference tools have not significantly influenced the study of theoretical neuroscience
56 models, for at least three reasons. First, at a practical level, the nonlinearities and dynamics of
57 many theoretical models are such that conventional inference tools typically produce a narrow set
58 of insights into these models. Indeed, only in the last few years has the deep learning toolkit

59 expanded to a point of relevance to this class of problem. Second, the object of interest from a
60 theoretical model is not typically data itself, but rather a qualitative phenomenon – inspection of
61 model behavior, or better, a measurable signature of some computation – an *emergent property* of
62 the model. Third, because theoreticians work carefully to construct a model that has biological
63 relevance, such a model as a result often does not fit cleanly into the framing of a statistical model.
64 Technically, because many such models stipulate a noisy system of differential equations that can
65 only be sampled or realized through forward simulation, they lack the explicit likelihood and priors
66 central to the probabilistic modeling toolkit.

67 To address these three challenges, we developed an inference methodology – ‘emergent property
68 inference’ – which learns a distribution over parameter configurations in a theoretical model. Crit-
69 ically, this distribution is such that draws from the distribution (parameter configurations) corre-
70 spond to systems of equations that give rise to a specified emergent property. First, we stipulate a
71 bijective deep neural network that induces a flexible family of probability distributions over model
72 parameterizations with a probability density we can calculate [15, ?, ?]. Second, we quantify the
73 notion of emergent properties as a set of moment constraints on datasets generated by the model.
74 Thus an emergent property is not a single data realization, but a phenomenon or a feature of the
75 model, which is the central object of interest to the theorist (unlike say the statistical neuroscien-
76 tist). Conditioning on an emergent property requires an extension of deep probabilistic inference
77 methods, which we have produced [16]. Third, because we can not assume the theoretical model
78 has explicit likelihood on data or the emergent property of interest, we use stochastic gradient
79 techniques in the spirit of likelihood free variational inference [17]. Taken together, emergent prop-
80 erty inference (EPI) provides a methodology for inferring and then reasoning about parameter
81 configurations that give rise to particular emergent phenomena in theoretical models.

82 Equipped with this methodology, we investigated three models of current importance in theoretical
83 neuroscience. These models were chosen to demonstrate generality through ranges of biological
84 realism (conductance-based biophysics to recurrent neural networks), neural system function (pat-
85 tern generation to abstract cognitive function), and network scale (four to infinite neurons). First,
86 to motivate the contribution of emergent property inference, we investigated network syncing in
87 a classic model of the stomatogastric ganglion [18]. Second, we generated then evaluated a set
88 of verifiable hypotheses of input-responsibility in a four neuron-type dynamical model of primary
89 visual cortex. Third, we demonstrated how the systematic application of EPI to levels of task
90 performance can generate experimentally testable hypotheses regarding connectivity in superior

91 colliculus. Fourth, we leveraged the flexibility of EPI to uncover the sources of bias in a low-rank
 92 recurrent neural network executing a toy computation. The novel scientific insights offered by EPI
 93 contextualize and clarify the previous studies exploring these models [18, 19, 20, 21] and more
 94 generally offer a quantitative grounding for theoretical models going forward, pointing a way to
 95 how rigorous statistical inference can enhance theoretical neuroscience at large.

96 We note that, during our preparation and early presentation of this work [22, 23], another work
 97 has arisen with broadly similar goals: bringing statistical inference to mechanistic models of neural
 98 circuits [24]. We are excited by this broad problem being recognized by the community, and we
 99 emphasize that these works offer complementary neuroscientific contributions and use different
 100 technical methodologies. Scientifically, our work has focused primarily on systems-level theoretical
 101 models, while their focus has been on lower-level cellular models. Secondly, there are several key
 102 technical differences in the approaches (see Section A.1.4) perhaps most notably is our focus on
 103 the emergent property – the measurable signal of the computation in question, vs their focus
 104 on observed datasets; both certainly are worthy pursuits. The existence of these complementary
 105 methodologies emphasizes the increased importance and timeliness of both works.

106 3 Results

107 3.1 Motivating emergent property inference of theoretical models

108 Consideration of the typical workflow of theoretical modeling clarifies the need for emergent prop-
 109 erty inference. First, the theorist designs or chooses an existing model that, it is hypothesized,
 110 captures the computation of interest. To ground this process in a well-known example, consider
 111 the stomatogastric ganglion (STG) of crustaceans, a small neural circuit which generates multiple
 112 rhythmic muscle activation patterns for digestion [25]. A model of the STG [18] is shown schemat-
 113 ically in Figure 1A, and note that the behavior of this model will be critically dependent on its
 114 parameterization – the choices of conductance parameters $z = [g_{el}, g_{synA}]$. Specifically, the two
 115 fast neurons (f_1 and f_2) mutually inhibit one another, and oscillate at a faster frequency than
 116 the mutually inhibiting slow neurons (s_1 and s_2), and the hub neuron (hub) couples with the fast
 117 or slow population or both. Second, once the model is selected, the theorist defines the emergent
 118 property, the measurable signal of scientific interest. To continue our running STG example, one
 119 such emergent property is the phenomenon of *network syncing* – in certain parameter regimes,
 120 the frequency of the hub neuron matches that of the fast and slow populations at an intermediate

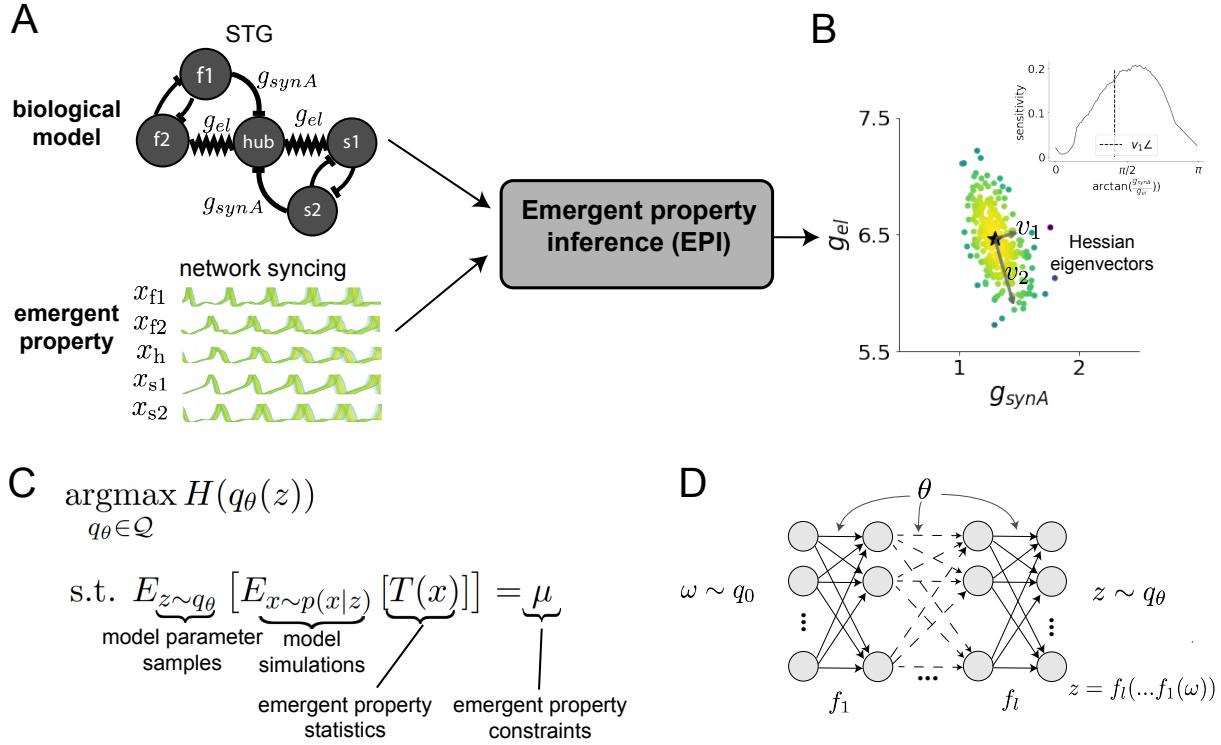


Figure 1: Emergent property inference (EPI) in the stomatogastric ganglion. A. For a choice of model (STG) and emergent property (network syncing), emergent property inference (EPI) learns a posterior distribution of the model parameters $z = [g_{el}, g_{synA}]^\top$ conditioned on network syncing. B. An EPI distribution of STG model parameters producing network syncing. The eigenvectors of the Hessian at the mode of the inferred distribution are indicated as v_1 and v_2 . (Inset) Sensitivity of the system with respect to network syncing along all dimensions of parameter space away from the mode. (see Section A.2.1). C. EPI learns a distribution $q_\theta(z)$ of model parameters that produce an emergent property: the emergent property statistics $T(x)$ are fixed in expectation over parameter distribution samples $z \sim q_\theta(z)$ to particular values μ . EPI distributions maximize randomness via entropy, although other measures are sensible. D. Deep probability distributions map a latent random variable $\omega \sim q_0$, where q_0 is chosen to be simple distribution such as an isotropic gaussian, through a highly expressive function family $f_\theta(\omega) = f_l(\dots f_1(\omega))$ parameterized by the neural network weights and biases $\theta \in \Theta$. This mapping induces an implicit probability model $q(g_\theta(\omega)) \in \mathcal{Q}$

frequency. This emergent property is shown in Figure 1A at a frequency of 0.55Hz. Third, qualitative parameter analysis ensues: since precise mathematical analysis is intractable in this model, a brute force sweep of parameters is done. Subsequently, a qualitative description is formulated to describe of the different parameter configurations that lead to the emergent property. In this last step lies the opportunity for a precise quantification of the emergent property as a statistical feature of the model. Equipped with this methodology, we can infer a probability distribution over parameter configurations that produce this quantified emergent property.

Before presenting technical details (in the following section), let us understand emergent property inference schematically: the black box in Figure 1A takes, as input, the model and the specified emergent property, and produces as output the parameter distribution shown in Figure 1B. This distribution – represented for clarity as samples from the distribution – is then a scientifically meaningful and mathematically tractable object. It conveys parameter regions critical to the emergent property, directions in parameter space that will be invariant (or not) to that property. In the STG model, this distribution can be specifically queried to determine the prototypical parameter configuration for network syncing (the mode; Figure 1B star), and then how quickly network syncing will decay based on changes away that mode (Figure 1B, inset). While it is impossible to determine whether we have converged to the maximum entropy distribution, the emergent property statistics have converged to the emergent property values. For further validation, we apply EPI to condition a two-dimensional linear dynamical system model on a band of oscillations around 1Hz, from which we can analytically derive the contours of the inferred distribution (see Section A.1.1). Taken together, bringing careful inference to theoretical models offers deeper insight into the behavior of these models, and the opportunity to make rigorous this last step in the practice of theoretical neuroscience.

3.2 A deep generative modeling approach to emergent property inference

Emergent property inference (EPI) systematizes the three-step procedure of the previous section. First, we consider the model as a coupled set of differential (and potentially stochastic) equations [18]. In the running STG example the dynamical state $x = [x_{f1}, x_{f2}, x_{hub}, x_{s1}, x_{s2}]$ is the membrane potential for each neuron, which evolves according to the biophysical conductance-based equation:

$$C_m \frac{\partial x}{\partial t} = -h(x; z) = -[h_{leak}(x; z) + h_{Ca}(x; z) + h_K(x; z) + h_{hyp}(x; z) + h_{elec}(x; z) + h_{syn}(x; z)] \quad (1)$$

where $C_m = 1\text{nF}$, and h_{leak} , h_{Ca} , h_K , h_{hyp} , h_{elec} , h_{syn} are the leak, calcium, potassium, hyperpolarization, electrical, and synaptic currents, all of which have their own complicated dependence on x and $z = [g_{\text{el}}, g_{\text{synA}}]$ (see Section A.2.1).

Second, we define the emergent property, which as above is network syncing: the phase locking of the population and its oscillation at an intermediate frequency of our choosing (Figure 1A bottom). Quantifying this phenomenon is straightforward: we define network syncing to be that the spiking frequency of each neuron is close to an intermediate frequency of 0.55Hz. Thus, our measurable signature of computation – the firing frequencies of each neuron $\omega_{f1}(x)$, $\omega_{f2}(x)$, etc.– are statistics of the membrane potential activity x which we insist be near a particular value 0.55Hz. Mathematically, we achieve this via constraints on the mean and variance of $\omega_i(x)$ for each neuron $i \in \{f1, f2, \text{hub}, s1, s2\}$, and thus:

$$E[T(x)] \triangleq E \begin{bmatrix} \omega_{f1}(x) \\ \vdots \\ (\omega_{f1}(x) - 0.55)^2 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0.55 \\ \vdots \\ 0.025^2 \\ \vdots \end{bmatrix} \triangleq \mu, \quad (2)$$

which completes the quantification of the emergent property.

Third, we perform emergent property inference: we find a distribution over parameter configurations z , and insist that samples from this distribution produce the emergent property; in other words, they obey the constraints introduced in Equation 2. This distribution will be chosen from a family of probability distributions $\mathcal{Q} = \{q_\theta(z) : \theta \in \Theta\}$, defined by a deep generative model of the normalizing flow class [15, ?, ?] – neural networks which transform a simple distribution into a suitably complicated distribution (as is needed here). This deep model is represented in Figure 1E (and see Methods for more detail). Then, mathematically, we must solve the following optimization program:

$$\begin{aligned} & \underset{q_\theta \in \mathcal{Q}}{\operatorname{argmax}} H(q_\theta(z)) \\ & \text{s.t. } E_{z \sim q_\theta} [E_{x \sim p(x|z)} [T(x)]] = \mu, \end{aligned} \quad (3)$$

where $T(x), \mu$ are defined as in Equation 3. The purpose of each element in this program is detailed in Figure 1D. Finally, we recognize that many distributions in \mathcal{Q} will respect the emergent property constraints, so we require a normative principle to select amongst them. This principle is captured in Equation 3 by the primal objective H . Here we chose Shannon entropy to model parameter distributions with minimal assumptions beyond some chosen structure [26, 27, 16, 28], but we

¹⁷⁴ emphasize that the EPI method is unaffected by this choice (the results of course will depend on
¹⁷⁵ this choice). Stating such a problem is easy enough; finding a tractable and suitably flexible family
¹⁷⁶ of probability distributions (\mathcal{Q}) is hard.

¹⁷⁷ EPI optimizes the weights and biases θ of the deep neural network (which induces the probability
¹⁷⁸ distribution) by iteratively solving Equation 3. The optimization is complete when the sampled
¹⁷⁹ models with parameters $z \sim q_\theta$ produce activity consistent with the specified emergent property.
¹⁸⁰ Such convergence is evaluated with a hypothesis test that the mean of each emergent property
¹⁸¹ statistic is no different than its emergent property value (see Section A.1.2). Armed with this
¹⁸² method, we now prove out the value of this technology by investigating a range of models and
¹⁸³ using EPI to produce novel scientific insights.

¹⁸⁴ **3.3 Comprehensive input-responsivity in a nonlinear sensory system**

¹⁸⁵ The first model we focus on is a nonlinear dynamical model of sensory processing in primary visual
¹⁸⁶ cortex (V1). Dynamical models with two populations (excitatory (E) and inhibitory (I) neurons)
¹⁸⁷ have already been used to reproduce a host of experimentally documented phenomena in primary
¹⁸⁸ visual cortex (V1). In particular regimes of excitation and inhibition, these models exhibit the
¹⁸⁹ paradoxical effect [4], selective amplification [29], surround suppression [30], and sensory integrative
¹⁹⁰ properties [31]. Experimental evidence shows that inhibitory neurons mostly belong to one of three
¹⁹¹ genetically-defined classes [32, 33].

¹⁹² An important extension of these E/I models is to break the inhibitory population into its subtypes,
¹⁹³ leading to a four-neuron population: excitatory (E), parvalbumin (P), somatostatin (S), and va-
¹⁹⁴ sointestinal peptide (V) neurons [19] (Fig. 2A). The dynamical state of this model is the firing
¹⁹⁵ rate of each neuron-type population $x = [x_E \ x_P \ x_S \ x_V]^\top$, which evolves according to rectified,
¹⁹⁶ exponentiated dynamics:

$$\tau \frac{dx}{dt} = -x + [Wx + h]_+^n \quad (4)$$

¹⁹⁷ with effective connectivity weights W and input h . In our analysis, we set the time constant
¹⁹⁸ $\tau = 20\text{ms}$ and dynamics coefficient $n = 2$. Also, as is fairly standard, we obtain an informative
¹⁹⁹ estimate of the effective connectivities between these neuron-types W in mice by multiplying their
²⁰⁰ probability of connection with their average synaptic strength [?] (citation on way) (see Section
²⁰¹ A.2.2). Given these fixed choices of W , n , and τ , we studied the system's response to input

$$h = b + dh \quad (5)$$

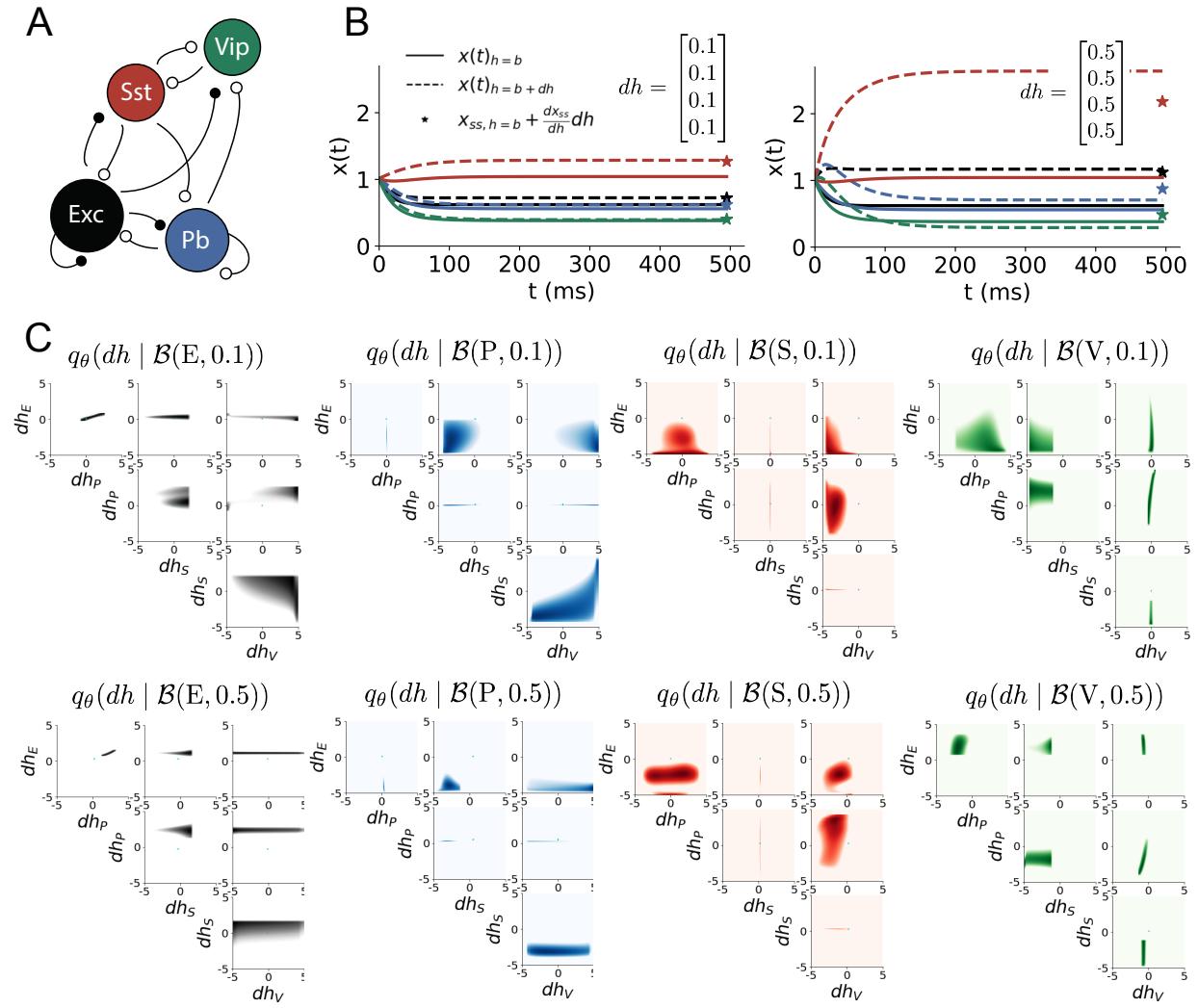


Figure 2: Exploring neuron-type responsivity in V1. A. Four-population model of primary visual cortex with excitatory (black), parvalbumin (blue), somatostatin (red), and vip (green) neurons. Some neuron-types largely do not form synaptic projections to others (excitatory and inhibitory projections filled and unfilled, respectively). B. Linear response predictions become inaccurate with greater input strength. V1 model simulations for input (solid) $h = b$ and (dashed) $h = b + dh$ with $b = [1, 1, 1, 1]^\top$ and (left) $dh = [0.1, 0.1, 0.1, 0.1]^\top$ (right) $dh = [0.5, 0.5, 0.5, 0.5]^\top$. Stars indicate the linear response prediction. C. EPI distributions on differential input dh conditioned on differential response $\mathcal{B}(\alpha, y)$ (see text). The linear prediction from two standard deviations away from y (from negative to positive) is overlaid in cyan (very small, near origin).

202 where the input h is comprised of a baseline input $b = [b_E, b_P, b_S, b_V]$ and a differential input
 203 $dh = [dh_E, dh_P, dh_S, dh_V]$ to each neuron-type population. Throughout subsequent analyses, the
 204 baseline input is $b = [1, 1, 1, 1]$.

205 Having established our model, we now turn to defining the emergent property. We begin with
 206 the linearized response of the system $\frac{dx_{ss}}{dh}$ at fixed points x_{ss} . While this linearization accurately
 207 predicts differential responses $dx_{ss} = \begin{bmatrix} dx_{E,ss} & dx_{P,ss} & dx_{S,ss} & dx_{V,ss} \end{bmatrix}$ for small differential inputs
 208 to each population $dh = \begin{bmatrix} 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix}$ (Fig. 2B, left), it can be misleading in such a nonlinear
 209 model for a greater differential input strength $dh = \begin{bmatrix} 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix}$ (Fig. 3B, right). In fact,
 210 the linearly predicted response for the V-population to $dh = \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \end{bmatrix}$ was actually
 211 in the opposite direction of the true response (Fig. 2B, right, green). This shows that currently
 212 available approaches to deriving the steady state response of this system are limited.

213 To get a more comprehensive picture of the input-responsivity of each neuron-type, we used EPI
 214 to learn a distribution of differential inputs dh that cause the rate of each neuron-type population
 215 $\alpha \in \{E, P, S, V\}$ to increase by a value $y \in 0.1, 0.5$ denoted by the emergent property

$$\mathcal{B}(\alpha, y) \leftrightarrow E \begin{bmatrix} dx_{\alpha,ss} \\ (dx_{\alpha,ss} - y)^2 \end{bmatrix} = \begin{bmatrix} y \\ 0.01^2 \end{bmatrix} \quad (6)$$

216 Note that we restrict the variance of the emergent property statistic $dx_{\alpha,ss}$ by setting its second
 217 moment to a small value. In Fig. 2C, each column visualizes the inferred distribution of dh
 218 corresponding to a specific neuron-type increase, while each row corresponds to amounts of increase
 219 0.1 and 0.5. For visualization of this four-dimensional distribution, we show the two-dimensional
 220 marginal densities. The inferred distributions suggest a slate of testable hypotheses. 1. As expected,
 221 each neuron-type's rate is sensitive to its direct input. 2. The E- and P-populations are largely
 222 unaffected by dh_V . 3. Similarly, The S-population is largely unaffected by dh_P . 4. Since EPI
 223 showed that negative dh_E results in small $dx_{V,ss}$, but positive dh_E elicited a larger $dx_{V,ss}$ we predict
 224 that there is a nonmonotonic response of $dx_{V,ss}$ with dh_E .

225 We evaluate these hypotheses by taking steps in individual neuron-type input Δh_α away from the
 226 modes of the inferred distributions

$$dh^* = z^* = \underset{z}{\operatorname{argmax}} \log q_\theta(z | \mathcal{B}(\alpha, 0.1)) \quad (7)$$

227 Now, $dx_{\alpha,ss}$ is the steady state response to the system with input $h = b + dh^* + \Delta h_\alpha u_\alpha$ where
 228 u_α is a unit vector in the dimension of α . Our hypotheses suggested by EPI are confirmed. 1.
 229 the neuron-type responses are sensitive to their direct inputs (Fig. 3A black, 3B blue, 3C red, 3D

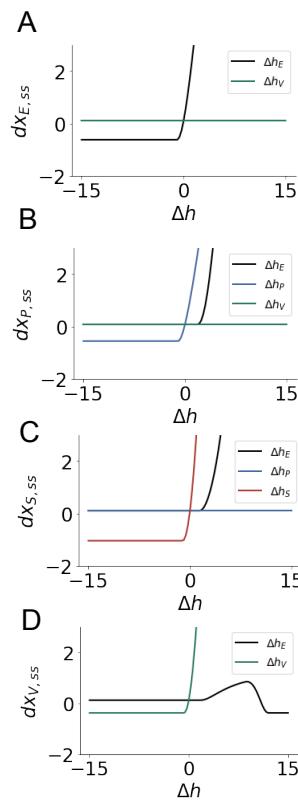


Figure 3: Confirming EPI generated hypotheses in V1. A. Differential responses by the E-population to changes in individual input $\Delta h_\alpha u_\alpha$ away from the mode of the EPI distribution dh^* . B-D Same plots for the P-, S-, and V-populations for the inputs for which hypotheses were formulated.

230 green), 2. the E- and P-populations are not affected by dh_V (Fig. 3A green, 3B green), 3. the
 231 S-population is not affected by dh_P (Fig. 3C blue), and 4. the V-population has a nonmonotonic
 232 response to dh_E (Fig. 3D black). All of this validated insight gained beyond what the analytic
 233 linear prediction told us (Fig. 2C, cyan).

234 To this point, we have shown the utility of EPI on relatively low-level emergent properties like
 235 network syncing and differential neuron-type population responses. In the remainder of the study,
 236 we focus on using EPI to understand models of more abstract cognitive function.

237 3.4 Identifying neural mechanisms of behavioral learning.

238 Identifying measurable biological changes that result in improved behavior is important for neuro-
 239 science, since they may indicate how the learning brain adapts. In a rapid task switching exper-
 240 iment [?], where rats were to respond right (R) or left (L) to the side of a light stimulus in the
 241 pro (P) task, and oppositely in the anti (A) task predicated by an auditory cue (Fig. 3A), neural
 242 recordings exhibited two population of neurons in each hemisphere of superior colliculus (SC) that
 243 simultaneously represented both task condition and motor response: the Pro/contralateral and
 244 Anti/ipsilateral neurons [20]. Duan et al. proposed a model of SC that, like the V1 model analyzed

in the previous section, is a four-population dynamical system. Here, the neuron-type populations are functionally-defined as the Pro- and Anti-populations in each hemisphere (left (L) and right (R)). The Pro- or Anti-populations receive an input determined by the cue, and then the left and right populations receive an input based on the side of the light stimulus. Activities were bounded from 0-1, so that a high output (1) of the Pro population in a given hemisphere corresponds to the contralateral response. An additional stipulation is that when one Pro population responds with a high-output, the opposite Pro population must respond with a low output (0). Finally, this circuit operates in the presence of gaussian noise resulting in trial-to-trial variability (see Section A.2.3). The connectivity matrix is parameterized by the geometry of the population arrangement (Fig. 3B).

Here, we used EPI to learn connectivity distributions consistent with various levels of accuracy in the rapid task switching behavioral paradigm. EPI was used to learn distributions of the SC weight matrix parameters $z = W$ conditioned on of various levels of rapid task switching accuracy $\mathcal{B}(p)$ for $p \in \{50\%, 60\%, 70\%, 80\%, 90\%\}$ (see Section A.2.3). There is a decomposition for of the connectivity matrix $W = QAQ^{-1}$, in which the eigenvectors q_i are the same for all W (Fig. 3C). These consistent eigenvectors have intuitive roles in processing for this task, and are accordingly named the *all* - all neurons co-fluctuate, *side* - one side dominates the other, *task* - the Pro or Anti populations dominate the other, and *diag* - Pro- and Anti-populations of opposite hemispheres dominate the opposite pair. The corresponding eigenvalues (e.g. a_{task} , which change according to W) indicate the degree to which activity along that mode is increased or decreased by W .

For greater task accuracies, the task mode eigenvalue increases, indicating the criticality of supporting the task representation in the connectivity of W , (Fig. 4D, purple). Stepping from random chance (50%) networks to marginally task-performing (60%) networks, there is a marked decrease of the side mode eigenvalues (Fig. 3D, orange). Such side mode suppression remains in the models achieving greater accuracy, revealing its importance towards task performance. There were no interesting trends with learning in the all or diag mode. Significantly, we can conclude from our methodology optimized to find all connectivities consistent with a level of accuracy, that side mode suppression in W allows rapid task switching, and that greater task-mode representations in W increase accuracy. These hypotheses are proved out in the model (Fig. 3E). Thus, our EPI-enabled analyses produce novel, experimentally testable predictions that effective connectvity between these populations changes throughout learning in a way that increases its task mode and decreses its side mode eigenvalues.

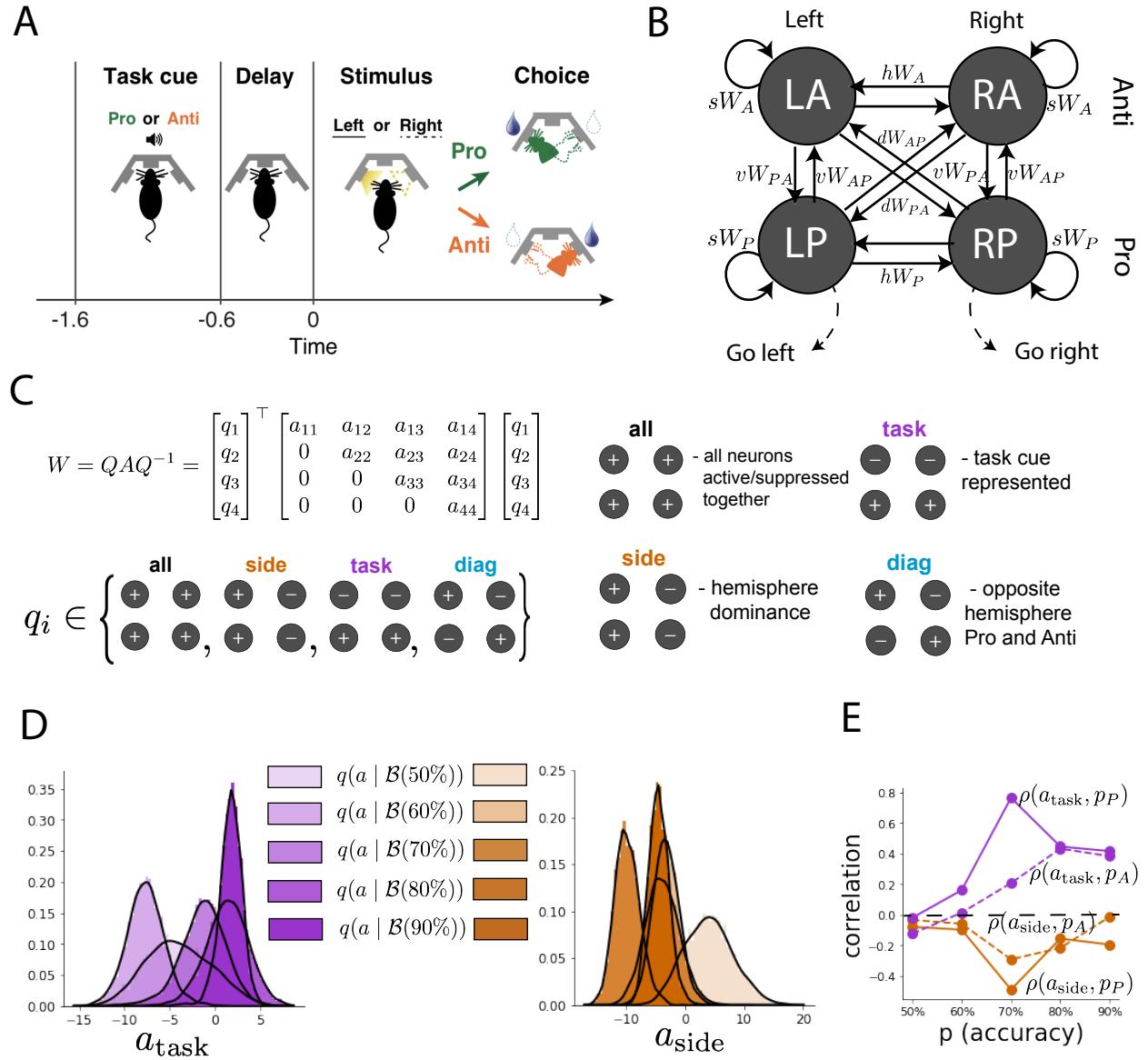


Figure 4: EPI reveals changes in SC [20] connectivity that result in greater task accuracy. A. Rapid task switching behavioral paradigm. In the Pro (Anti) condition indicated by an auditory cue, rats respond by poking into a side port to the same (opposite) side as the light stimulus that is provided after a delay to receive a reward. B. Model of superior colliculus (SC). Neurons: LP - left pro, RP - right pro, LA - left anti, RA - right anti. Parameters: sW - self, hW - horizontal, vW - vertical, dW - diagonal weights. C. The Schur decomposition of the weight matrix $W = QAQ^{-1}$ is a unique decomposition with orthogonal Q and upper triangular A . The invariant Schur eigenmodes (symmetry of W) are labeled by their hypothesized role in computation: q_{all} , q_{task} , q_{side} , and q_{diag} . The values of A are what change for different realizations of W . D. The marginal EPI distributions of the Schur eigenvalues at each level of task accuracy. E. The correlation of Schur eigenvalue with task performance in each learned EPI distribution.

²⁷⁷ **3.5 Characterizing the sources of bias in RNN computation**

²⁷⁸ So far, each biologically realistic model we have studied was designed from fundamental biophysical
²⁷⁹ principles, genetically- or functionally-defined neuron types. At a more abstract level of modeling,
²⁸⁰ recurrent neural networks (RNNs) are high-dimensional models of computation, which have become
²⁸¹ increasingly popular in neuroscience research [34]. Typically, RNNs are trained to do a task from a
²⁸² systems neuroscience experiment, and then the unit activations of the trained RNN are compared
²⁸³ to recorded neural activity. A monumental challenge for this line of work is to link findings at
²⁸⁴ this level of abstraction with interpretable biophysical mechanisms in the brain. Here we leverage
²⁸⁵ recent theoretical work to run EPI on interpretable parameterizations of RNN connectivity solving
²⁸⁶ a toy problem.

²⁸⁷ Importantly, recent work establishes such a link between RNN connectivity weights and the re-
²⁸⁸ sulting dynamical responses of the network using dynamic mean field theory (DMFT) for neural
²⁸⁹ networks [3]. Specifically, DMFT describes the properties of activity in infinite-size neural networks
²⁹⁰ given a distribution on the connectivity weights. This theory has been extended from random neural
²⁹¹ networks to low rank RNNs, which have low-dimensional parameterizations of RNN connectivity
²⁹² via the pairwise correlations of the low-rank vectors (i.e. the low-rank “geometry”) [21]. For
²⁹³ example, the connectivity of a rank-1 RNN J is the sum of a random component with strength
²⁹⁴ determined by g and a structured component determined by the outer product of vectors m and n :

$$J = g\chi + \frac{1}{N}mn^\top \quad (8)$$

²⁹⁵ where the activity x evolves as

$$\frac{\partial x}{\partial t} = -x(t) + J\phi(x(t)) + I(t) \quad (9)$$

²⁹⁶ $I(t)$ is some input, ϕ is the tanh nonlinearity, and $\chi_{ij} \sim \mathcal{N}(0, \frac{1}{N})$. The entries of m and n are
²⁹⁷ drawn from gaussian distributions $m_i \sim \mathcal{N}(M_m, 1)$ and $n_i \sim \mathcal{N}(M_n, 1)$, whose parameters M_m and
²⁹⁸ M_n determine their degree of correlation.

²⁹⁹ Mastrogiovisepppe et al. are able to design low-rank connectivities via the pairwise correlations of such
³⁰⁰ low-rank vectors that solve tasks from behavioral neuroscience. An important detail is that a non-
³⁰¹ linear system of equations solver must be used to obtain the task-relevant variables of interest from
³⁰² the derived consistency equations (see Section A.2.4). However, we can consider the DMFT equa-
³⁰³ tion solver as a black box that takes in a low-rank parameterization z (e.g. $z = [g \ M_m \ M_n]$) and
³⁰⁴ outputs task-relevant response variables (e.g. average network activity μ , the temporal variability

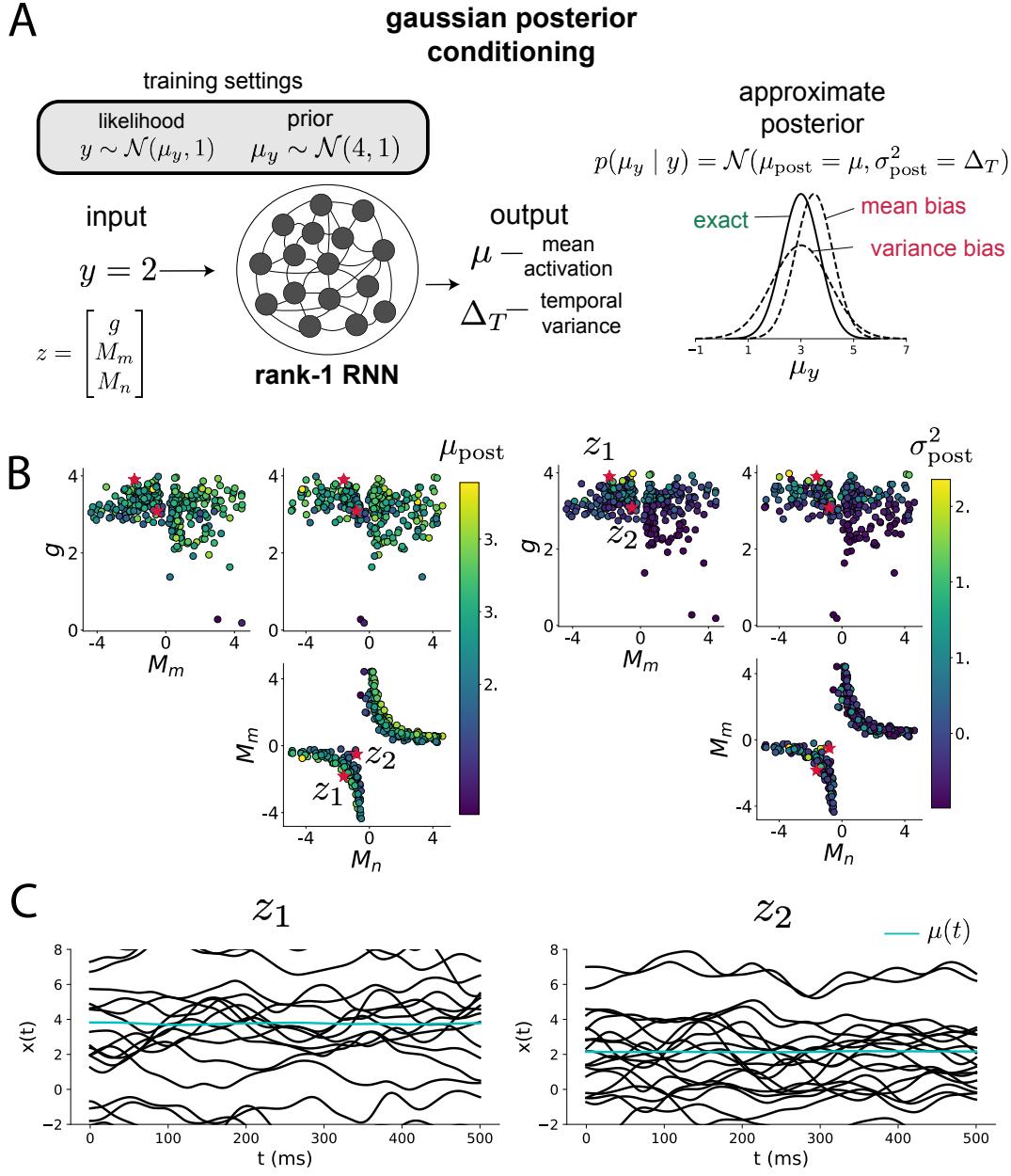


Figure 5: Sources of bias in RNN computation. A. (left) A rank-1 RNN running approximate Bayesian inference on μ_y assuming a gaussian likelihood variance of 1 and a prior of $\mathcal{N}(4, 1)$. (center) The rank-1 RNN represents the computed gaussian posterior mean μ_{post} and variance σ_{post}^2 in its mean activity μ and its temporal variance Δ_T . (right) Bias in this computation can come from over- or under-estimating the posterior mean or variance. B. Distribution of rank-1 RNNs executing approximate Bayesian inference. Samples are colored by (left) posterior mean $\mu_{\text{post}} = \mu$ and (right) posterior variance $\sigma_{\text{post}}^2 = \Delta_T$. C. Finite size realizations agree with the DMFT theory.

305 in the network Δ_T , or network activity along a given dimension κ). Furthermore, we recognize that
 306 the solution produced by the solver is differentiable with respect to the input parameters. Thus, we
 307 are able to combine this DMFT with EPI to learn distributions on such connectivity parameters
 308 of RNNs that execute neuroscientific tasks via an emergent property defined on the task-relevant
 309 responses produced by DMFT.

310 For our toy problem, we consider the emergent property of gaussian posterior conditioning. Specif-
 311 ically, we ask the RNN to calculate the parameters of a gaussian posterior distribution on the mean
 312 of a gaussian likelihood μ_y given a single observation of y and a gaussian prior $p(\mu_y) = \mathcal{N}(4, 1)$
 313 (Fig. 5A). Assuming the variance of the gaussian likelihood is 1, the true posterior for an input of
 314 $y = 2$ is $p(\mu_y | y) = \mathcal{N}(3, 0.5)$. We used EPI to learn distributions of RNNs producing the correct
 315 posterior mean and variance in their mean activity $\mu = \mu_{\text{post}}$ and temporal variance $\Delta_T = \sigma_{\text{post}}^2$
 316 given an input of $y = 2$. (see Section A.2.4) (Fig. 5B).

317 When specifying the emergent property of gaussian posterior conditioning, we allowed a substantial
 318 amount of variability in the second moment constraints of the network mean μ and temporal
 319 variance Δ_T . This resulted in a distribution of rank-1 RNN parameterizations having a wide
 320 variety biases in the resulting μ_{post} and σ_{post}^2 (under- or over-estimations of the posterior means
 321 and variances). We can examine the nature of the biases in this toy computation by visualizing the
 322 produced posterior means (Fig. 5B, left) and variances (Fig. 5B, right) in the inferred distribution.
 323 The infererred distribution has roughly symmetric in the M_m - M_n plane, suggesting there is a
 324 degeneracy in the product of M_m and M_n (Fig. 5B). The product of M_m and M_n almost completely
 325 determines the posterior mean (Fig. 5B, left), and the random strength g is the most influential
 326 variable on the temporal variance (Fig. 5B, right). Neither of these observations were obvious from
 327 the consistency equations afforded by DMFT (see Section A.2.4).

328 When working with DMFT, it's important to check that finite-size realizations of these infinite-
 329 size networks match the theoretical predictions. We check 2,000-neuron realizations of drawn
 330 parameters z_1 and z_2 from the inferred distribution. z_1 has relatively high g and high $M_m M_n$,
 331 whereas z_2 has relatively low g and low $M_m M_n$. Confirming our intuition, z_1 overestimates the
 332 posteior mean, since mean activity $\mu(t) > 3$ (Fig. 5C, left cyan). In turn, z_2 underestimates the
 333 posterior mean, since $\mu(t) < 3$ (Fig. 5C, right cyan). Finally, z_1 results in evidently greater temporal
 334 variance than z_2 . This novel procedure of doing inference in interpretable parameterizations of
 335 RNNs conditioned on task execution is straightforwardly generalizable to other tasks like noisy
 336 integration and context-dependent decision making (Fig. S1).

337 4 Discussion**338 4.1 EPI is a general tool for theoretical neuroscience.**

339 Models of biological systems often have complex nonlinear differential equations, making traditional
340 statistical inference intractable. In contrast, EPI is capable of learning distributions of parameters
341 in such models producing measurable signatures of computation. We have demonstrated its utility
342 on biological models (STG), intermediate-level models of interacting genetically- and functionally-
343 defined neuron-types (V1, SC), and the most abstract of models (RNNs). We are able to condi-
344 tion both deterministic and stochastic models on low-level emergent properties like firing rates of
345 membrane potentials, as well as high-level cognitive function like approximate Bayesian inference.
346 Technically, EPI is tractable when the emergent property statistics are continuously differentiable
347 with respect to the model parameters, which is very often the case; this emphasizes the general
348 utility of EPI.

349 In this study, we have focused on applying EPI to low dimensional parameter spaces of models
350 with low dimensional dynamical state. These choices were made to present the reader with a series
351 of interpretable conclusions, which is more challenging in high dimensional spaces. In fact, EPI
352 should scale reasonably to high dimensional parameter spaces, as the underlying technology has
353 produced state-of-the-art performance on high-dimensional tasks such as texture generation [16].
354 Of course, increasing the dimensionality of the dynamical state of the model makes optimization
355 more expensive, and there is a practical limit there as with any machine learning approach. For
356 systems with high dimensional state, we recommend using theoretical approaches (e.g. [21]) to
357 reason about reduced parameterizations of such high-dimensional systems.

358 There are additional technical considerations when assessing the suitability of EPI for a particu-
359 lar modeling question. First and foremost, as in any optimization problem, the defined emergent
360 property should always be appropriately conditioned (constraints should not have wildly different
361 units). Furthermore, if the program is underconstrained (not enough constraints), the distribution
362 grows (in entropy) unstably unless mapped to a finite support. If overconstrained, there is no pa-
363 rameter set producing the emergent property, and EPI optimization will fail (appropriately). Next,
364 one should consider the computational cost of the gradient calculations. In the best circumstance,
365 there is a simple, closed form expression (e.g. Section A.1.1) for the emergent property statistic
366 given the model parameters. On the other end of the spectrum, many forward simulation iterations
367 may be required before a high quality measurement of the emergent property statistic is available

368 (e.g. Section A.2.1). In such cases, optimization will be expensive.

369 **4.2 Novel hypotheses from EPI**

370 Machine learning has played an effective, multifaceted role in neuroscientific progress. Primarily,
371 it has revealed structure in large-scale neural datasets [35, 36, 37, 38, 39, 40] (see review, [14]).
372 Secondarily, trained algorithms of varying degrees of biological relevance are beginning to be viewed
373 as fully-observable computational systems comparable to the brain [41, 42]. Theorists can use deep
374 learning for probabilistic inference to understand their models and their behavior.

375 For example, consider the fact that we do not yet understand just a four-dimensional, deterministic
376 model of V1 [19]. This should not be surprising, since analytic approaches to studying nonlinear
377 dynamical systems become increasingly complicated when stepping from two-dimensional to three-
378 or four-dimensional systems in the absence of restrictive simplifying assumptions [43]. We promote
379 the recognition of analytic difficulty, and alternatively the use of EPI to gain the desired model
380 insights. In Section 3.3, we showed that EPI was far more informative about neuron-type input
381 responsivity than the predictions afforded through analysis. By flexibly conditioning this V1 model
382 on different emergent properties, we performed an exploratory analysis of a *model* rather than a
383 dataset, which generated and proved out a set of testable predictions.

384 Exploratory analyses can be directed. For example, when interested in model changes during learn-
385 ing, one can use EPI to condition on various levels of an emergent property statistic indicative of
386 performance like task accuracy in a behavioral paradigm (see Section 3.4). This analysis iden-
387 tified experimentally testable predictions (proved out *in-silico*) of changes in connectivity in SC
388 throughout learning of a rapid task switching behavior. Precisely, we predict an initial reduction
389 in side mode eigenvalue, and a steady increase in task mode eigenvalue in the effective connectivity
390 matrices of learning rats.

391 In our final analysis, we present a novel procedure for doing statistical inference on interpretable
392 parameterizations of RNNs executing tasks from behavioral neuroscience. This methodology relies
393 on recently extended theory of responses in random neural networks with minimal structure [21].
394 With this methodology, we can finally open the probabilistic model selection toolkit reasoning
395 about the connectivity of RNNs solving tasks.

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520 **A Methods**

521 **A.1 Emergent property inference (EPI)**

522 Emergent property inference (EPI) learns distributions of theoretical model parameters that pro-
523 duce emergent properties of interest. EPI combines ideas from likelihood-free variational inference
524 [17] and maximum entropy flow networks [16]. A maximum entropy flow network is used as a deep
525 probability distribution for the parameters, while these samples often parameterize a differentiable
526 model simulator, which may lack a tractable likelihood function.

527 Consider model parameterization z and data x generated from some theoretical model simulator
528 represented as $p(x | z)$, which may be deterministic or stochastic. Theoretical models usually have
529 known sampling procedures for simulating activity given a circuit parameterization, yet often lack
530 an explicit likelihood function due to the nonlinearities and dynamics. With EPI, a distribution
531 on parameters z is learned, that yields an emergent property of interest \mathcal{B} ,

$$\mathcal{B} \leftrightarrow E_{z \sim q_\theta} [E_{x \sim p(x|z)} [T(x)]] = \mu \quad (10)$$

532 by making an approximation $q_\theta(z)$ to $p(z | \mathcal{B})$ (see Section A.1.5). So, over the DSN distribution
533 $q_\theta(z)$ of model $p(x | z)$ for behavior \mathcal{B} , the emergent properties $T(x)$ are constrained in expectation
534 to μ .

535 In deep probability distributions, a simple random variable $w \sim p_0$ is mapped deterministically
536 via a function f_θ parameterized by a neural network to the support of the distribution of interest

537 where $z = f_\theta(\omega) = f_l(\dots f_1(\omega))$. Given a theoretical model $p(x | z)$ and some behavior of interest
 538 \mathcal{B} , the deep probability distributions are trained by optimizing the neural network parameters θ to
 539 find a good approximation q_θ^* within the deep variational family Q to $p(z | \mathcal{B})$.

540 In most settings (especially those relevant to theoretical neuroscience) the likelihood of the behavior
 541 with respect to the model parameters $p(T(x) | z)$ is unknown or intractable, requiring an alternative
 542 to stochastic gradient variational bayes [5] or black box variational inference[44]. These types
 543 of methods called likelihood-free variational inference (LFVI, [17]) skate around the intractable
 544 likelihood function in situations where there is a differentiable simulator. Akin to LFVI, DSNs are
 545 optimized with the following objective for a given theoretical model, emergent property statistics
 546 $T(x)$, and emergent property constraints μ :

$$\begin{aligned} q_\theta^*(z) &= \underset{q_\theta \in Q}{\operatorname{argmax}} H(q_\theta(z)) \\ \text{s.t. } E_{z \sim q_\theta} [E_{x \sim p(x|z)} [T(x)]] &= \mu \end{aligned} \tag{11}$$

547 Optimizing this objective is a technological accomplishment in its own right, the details of which
 548 we elaborate in Section A.1.2. Before going through those details, we ground this optimization in
 549 a toy example.

550 A.1.1 Example: 2D LDS

551 To gain intuition for EPI, consider two-dimensional linear dynamical systems, $\tau \dot{x} = Ax$ with

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$$

552 that produce a band of oscillations. To do EPI with the dynamics matrix elements as the free
 553 parameters $z = [a_1, a_2, a_3, a_4]$, and fixing $\tau = 1$, such that the posterior yields a band of oscillations,
 554 the emergent property statistics $T(x)$ are chosen to contain the first- and second-moments of the
 555 oscillatory frequency Ω and the growth/decay factor d of the oscillating system. To learn the
 556 distribution of real entries of A that yield a distribution of d with mean zero with variance 0.25^2 ,
 557 and oscillation frequency Ω with mean 1 Hz with variance $(0.1\text{Hz})^2$, then we would select the real
 558 part of the complex conjugate eigenvalues $\text{real}(\lambda_1) = d$ (via an arbitrary choice of eigenvalue of the
 559 dynamics matrix λ_1) and the positive imaginary component of one of the eigenvalues $\text{imag}(\lambda_1) =$
 560 $2\pi\Omega$ as the emergent property statistics. Those emergent property statistics are then constrained

561 to

$$\mu = E \begin{bmatrix} \text{real}(\lambda_1) \\ \text{imag}(\lambda_1) \\ (\text{real}(\lambda_1) - 0)^2 \\ (\text{imag}(\lambda_1) - 2\pi\Omega)^2 \end{bmatrix} = \begin{bmatrix} 0.0 \\ 2\pi\Omega \\ 0.25^2 \\ (2\pi 0.1)^2 \end{bmatrix} \quad (12)$$

562 where $\Omega = 1\text{Hz}$. Unlike the models we study in the paper which calculate $E_{x \sim p(x|z)} [T(x)]$ via
 563 forward simulation, we have a closed form for the eigenvalues of the dynamics matrix. λ can be
 564 calculated using the quadratic formula:

$$\lambda = \frac{\left(\frac{a_1+a_4}{\tau}\right) \pm \sqrt{\left(\frac{a_1+a_4}{\tau}\right)^2 + 4\left(\frac{a_2a_3-a_1a_4}{\tau}\right)}}{2} \quad (13)$$

565 where λ_1 is the eigenvalue of $\frac{1}{\tau}A$ with greatest real part. Even though $E_{x \sim p(x|z)} [T(x)]$ is calculable
 566 directly via a closed form function and does not require simulation, we cannot derive the distribution
 567 q_θ^* directly. This is due to the formally hard problem of the backward mapping: finding the natural
 568 parameters η from the mean parameters μ of an exponential family distribution [45]. Instead, we
 569 can use EPI to learn the linear system parameters producing such a band of oscillations (Fig. S2B).

570 Even this relatively simple system has nontrivial (though intuitively sensible) structure in the
 571 parameter distribution. To validate our method (further than that of the underlying technology
 572 on a ground truth solution [16]) we can analytically derive the contours of the probability density
 573 from the emergent property statistics and values (Fig. S3). In the $a_1 - a_4$ plane, is a black line
 574 at $\text{real}(\lambda_1) = \frac{a_1+a_4}{2} = 0$, a dotted black line at the standard deviation $\text{real}(\lambda_1) = \frac{a_1+a_4}{2} \pm 1$, and a
 575 grey line at twice the standard deviation $\text{real}(\lambda_1) = \frac{a_1+a_4}{2} \pm 2$ (Fig. S3A). Here the lines denote the
 576 set of solutions at fixed behaviors, which overlay the posterior obtained through EPI. The learned
 577 DSN distribution precisely reflects the desired statistical constraints and model degeneracy in the
 578 sum of a_1 and a_4 . Intuitively, the parameters equivalent with respect to emergent property statistic
 579 $\text{real}(\lambda_1)$ have similar log densities.

580 To explain the structure in the bimodality of the DSN posterior, we can look at the imaginary
 581 component of λ_1 . When $\text{real}(\lambda_1) = \frac{a_1+a_4}{2} = 0$, we have

$$\text{imag}(\lambda_1) = \begin{cases} \sqrt{\frac{a_1a_4-a_2a_3}{\tau}}, & \text{if } a_1a_4 < a_2a_3 \\ 0 & \text{otherwise} \end{cases} \quad (14)$$

582 When $\tau = 1$ and $a_1a_4 > a_2a_3$ (center of distribution above), we have the following equation for the
 583 the other two dimensions:

$$\text{imag}(\lambda_1)^2 = a_1 a_4 - a_2 a_3 \quad (15)$$

584 Since we constrained $E_{q_\theta}[\text{imag}(\lambda)] = 2\pi$ (with $\omega = 1$), we can plot contours of the equation
 585 $\text{imag}(\lambda_1)^2 = a_1 a_4 - a_2 a_3 = (2\pi)^2$ for various $a_1 a_4$ (Fig. S3A). If $\sigma_{1,4} = E_{q_\theta}(|a_1 a_4 - E_{q_\theta}[a_1 a_4]|)$,
 586 then we plot the contours as $a_1 a_4 = 0$ (black), $a_1 a_4 = -\sigma_{1,4}$ (black dotted), and $a_1 a_4 = -2\sigma_{1,4}$
 587 (grey dotted) (Fig. S3B). This validates the curved structure of the inferred distribution learned
 588 through EPI. We take steps in negative standard deviation of $a_1 a_4$ (dotted and gray lines), since
 589 there are few positive values $a_1 a_4$ in the posterior. More subtle model-behavior combinations will
 590 have even more complexity, further motivating the use of EPI for understanding these systems.

591 A.1.2 Augmented Lagrangian optimization

592 To optimize $q_\theta(z)$ in equation 1, the constrained optimization is performed using the augmented
 593 Lagrangian method. The following objective is minimized:

$$L(\theta; \alpha, c) = -H(q_\theta) + \alpha^\top \delta(\theta) + \frac{c}{2} \|\delta(\theta)\|^2 \quad (16)$$

594 where $\delta(\theta) = E_{z \sim q_\theta} [E_{x \sim p(x|z)} [T(x) - \mu]]$, $\alpha \in \mathcal{R}^m$ are the Lagrange multipliers and c is the penalty
 595 coefficient. For a fixed (α, c) , θ is optimized with stochastic gradient descent. A low value of c is
 596 used initially, and increased during each augmented Lagrangian epoch – a period of optimization
 597 with fixed α and c for a given number of stochastic optimziation iterations. Similarly, α is tuned
 598 each epoch based on the constraint violations. For the linear 2-dimensional system (Fig. S2C)
 599 optimization hyperparameters are initialized to $c_1 = 10^{-4}$ and $\alpha_1 = 0$. The penalty coefficient
 600 is updated based on a hypothesis test regarding the reduction in constraint violation. The p-
 601 value of $E[\|\delta(\theta_{k+1})\|] > \gamma E[\|\delta(\theta_k)\|]$ is computed, and c_{k+1} is updated to βc_k with probability
 602 $1 - p$. Throughout the project, $\beta = 4.0$ and $\gamma = 0.25$ is used. The other update rule is $\alpha_{k+1} =$
 603 $\alpha_k + c_k \frac{1}{n} \sum_{i=1}^n (T(x^{(i)}) - \mu)$. In this example, each augmented Lagrangian epoch ran for 2,000
 604 iterations. We consider the optimization to have converged when a null hypothesis test of constraint
 605 violations being zero is accepted for all constraints at a significance threshold 0.05. This is the dotted
 606 line on the plots below depicting the optimization cutoff of EPI optimization for the 2-dimensional
 607 linear system. If the optimization is left to continue running, entropy usually decreases, and
 608 structural pathologies in the distribution may be introduced.

609 The intention is that c and α start at values encouraging entropic growth early in optimization.
 610 Then, as they increase in magnitude with each training epoch, the constraint satisfaction terms are

611 increasingly weighted, resulting in a decrease in entropy. Rather than using a naive initialization,
 612 before EPI, we optimize the deep probability distribution parameters to generate samples of an
 613 isotropic gaussian of a selected variance, such as 1.0 for the 2D LDS example. This provides a
 614 convenient starting point, whose level of entropy is controlled by the user.

615 **A.1.3 Normalizing flows**

616 Since we are optimizing parameters θ of our deep probability distribution with respect to the
 617 entropy, we will need to take gradients with respect to the log-density of samples from the deep
 618 probability distribution.

$$H(q_\theta(z)) = \int -q_\theta(z) \log(q_\theta(z)) dz = E_{z \sim q_\theta} [-\log(q_\theta(z))] = E_{\omega \sim q_0} [-\log(q_\theta(f_\theta(\omega)))] \quad (17)$$

$$\nabla_\theta H(q_\theta(z)) = E_{\omega \sim q_0} [-\nabla_\theta \log(q_\theta(f_\theta(\omega)))] \quad (18)$$

620 Deep probability models typically consist of several layers of fully connected neural networks.
 621 When each neural network layer is restricted to be a bijective function, the sample density can be
 622 calculated using the change of variables formula at each layer of the network. For $z' = f(z)$,

$$q(z') = q(f^{-1}(z')) \left| \det \frac{\partial f^{-1}(z')}{\partial z'} \right| = q(z) \left| \det \frac{\partial f(z)}{\partial z} \right|^{-1} \quad (19)$$

623 However, this computation has cubic complexity in dimensionality for fully connected layers. By
 624 restricting our layers to normalizing flows [15] – bijective functions with fast log determinant
 625 jacobian computations, we can tractably optimize deep generative models with objectives that are
 626 a function of sample density, like entropy. Most of our analyses use real NVP [46], which have
 627 proven effective in our architecture searches, and have the advantageous features of fast sampling
 628 and fast density evaluation.

629 **A.1.4 Related work**

630 (To come)

631

632 **A.1.5 Emergent property inference as variational inference in an exponential family**

633 (To come)

634

635 **A.2 Theoretical models**

636 In this study, we used emergent property inference to examine several models relevant to theoretical
637 neuroscience. Here, we provide the details of each model and the related analyses.

638 **A.2.1 Stomatogastric ganglion**

639 Each neuron's membrane potential $x_m(t)$ is the solution of the following differential equation.

$$C_m \frac{\partial x_m}{\partial t} = -[h_{leak} + h_{Ca} + h_K + h_{hyp} + h_{elec} + h_{syn}] \quad (20)$$

640 The membrane potential of each neuron is affected by the leak, calcium, potassium, hyperpolar-
641 ization, electrical and synaptic currents, respectively. The capacitance of the cell membrane was
642 set to $C_m = 1nF$. Each current is a function of the the neuron's membrane potential x_m and
643 the parameters of the circuit such as g_{el} and g_{syn} , whose affect on the circuit is considered in the
644 motivational example of EPI in Fig. 1. Specifically, the currents are the difference in the neuron's
645 membrane potential and that current type's reversal potential multiplied by a conductance:

$$h_{leak} = g_{leak}(x_m - V_{leak}) \quad (21)$$

646

$$h_{elec} = g_{el}(x_m^{post} - x_m^{pre}) \quad (22)$$

647

$$h_{syn} = g_{syn}S_\infty^{pre}(x_m^{post} - V_{syn}) \quad (23)$$

648

$$h_{Ca} = g_{Ca}M_\infty(x_m - V_{Ca}) \quad (24)$$

649

$$h_K = g_KN(x_m - V_K) \quad (25)$$

650

$$h_{hyp} = g_hH(x_m - V_{hyp}) \quad (26)$$

651 The reversal potentials were set to $V_{leak} = -40mV$, $V_{Ca} = 100mV$, $V_K = -80mV$, $V_{hyp} = -20mV$,
652 and $V_{syn} = -75mV$. The other conductance parameters were fixed to $g_{leak} = 1 \times 10^{-4}\mu S$. g_{Ca} ,
653 g_K , and g_{hyp} had different values based on fast, intermediate (hub) or slow neuron. Fast: $g_{Ca} =$
654 1.9×10^{-2} , $g_K = 3.9 \times 10^{-2}$, and $g_{hyp} = 2.5 \times 10^{-2}$. Intermediate: $g_{Ca} = 1.7 \times 10^{-2}$, $g_K = 1.9 \times 10^{-2}$,
655 and $g_{hyp} = 8.0 \times 10^{-3}$. Intermediate: $g_{Ca} = 8.5 \times 10^{-3}$, $g_K = 1.5 \times 10^{-2}$, and $g_{hyp} = 1.0 \times 10^{-2}$.

656 Furthermore, the Calcium, Potassium, and hyperpolarization channels have time-dependent gating
 657 dynamics dependent on steady-state gating variables M_∞ , N_∞ and H_∞ , respectively.

$$M_\infty = 0.5 \left(1 + \tanh \left(\frac{x_m - v_1}{v_2} \right) \right) \quad (27)$$

$$\frac{\partial N}{\partial t} = \lambda_N (N_\infty - N) \quad (28)$$

$$N_\infty = 0.5 \left(1 + \tanh \left(\frac{x_m - v_3}{v_4} \right) \right) \quad (29)$$

$$\lambda_N = \phi_N \cosh \left(\frac{x_m - v_3}{2v_4} \right) \quad (30)$$

$$\frac{\partial H}{\partial t} = \frac{(H_\infty - H)}{\tau_h} \quad (31)$$

$$H_\infty = \frac{1}{1 + \exp \left(\frac{x_m + v_5}{v_6} \right)} \quad (32)$$

$$\tau_h = 272 - \left(\frac{-1499}{1 + \exp \left(\frac{-x_m + v_7}{v_8} \right)} \right) \quad (33)$$

664 where we set $v_1 = 0mV$, $v_2 = 20mV$, $v_3 = 0mV$, $v_4 = 15mV$, $v_5 = 78.3mV$, $v_6 = 10.5mV$,
 665 $v_7 = -42.2mV$, $v_8 = 87.3mV$, $v_9 = 5mV$, and $v_{th} = -25mV$. These are the same parameter
 666 values used in [18].

667 Finally, there is a synaptic gating variable as well:

$$S_\infty = \frac{1}{1 + \exp \left(\frac{v_{th} - x_m}{v_9} \right)} \quad (34)$$

668 When the dynamic gating variables are considered, this is actually a 15-dimensional nonlinear
 669 dynamical system.

670 In order to measure the frequency of the hub neuron during EPI, the STG model was simulated
 671 for $T = 500$ time steps of $dt = 25ms$. In EPI, since gradients are taken through the simulation
 672 process, the number of time steps are kept as modest if possible. The chosen dt and T were the
 673 most computationally convenient choices yielding accurate frequency measurement.

674 Our original approach to measuring frequency was to take the max of the fast Fourier transform
 675 (FFT) of the simulated time series. There are a few key considerations here. One is resolution
 676 in frequency space. Each FFT entry will correspond to a signal frequency of $\frac{F_s k}{N}$, where N is
 677 the number of samples used for the FFT, $F_s = \frac{1}{dt}$, and $k \in [0, 1, \dots, N - 1]$. Our resolution is
 678 improved by increasing N and decreasing dt . Increasing $N = T - b$, where b is some fixed number of

679 buffer burn-in initialization samples, necessitates an increase in simulation time steps T , which
 680 directly increases computational cost. Increasing F_s (decreasing dt) increases system approximation
 681 accuracy, but requires more time steps before a full cycle is observed. At the level of $dt = 0.025$,
 682 thousands of temporal samples were required for resolution of .01Hz. These challenges in frequency
 683 resolution with the discrete Fourier transform motivated the use of an alternative basis of complex
 684 exponentials. Instead, we used a basis of complex exponentials with frequencies from 0.0-1.0 Hz at
 685 0.01Hz resolution, $\Phi = [0.0, 0.01, \dots, 1.0]^\top$

686 Another consideration was that the frequency spectra of the hub neuron has several peaks. This
 687 was due to high-frequency sub-threshold activity. The maximum frequency was often not the firing
 688 frequency. Accordingly, subthreshold activity was set to zero, and the whole signal was low-pass
 689 filtered with a moving average window of length 20. The signal was subsequently mean centered.
 690 After this pre-processing, the maximum frequency in the filter bank accurately reflected the firing
 691 frequency.

692 Finally, to differentiate through the maximum frequency identification step, we used a sum-of-
 693 powers normalization strategy: Let $\mathcal{X}_i \in \mathcal{C}^{|\Phi|}$ be the complex exponential filter bank dot products
 694 with the signal $x_i \in \mathcal{R}^N$, where $i \in \{f1, f2, \text{hub}, s1, s2\}$. The “frequency identification” vector is

$$u_i = \frac{|\mathcal{X}_i|^\alpha}{\sum_{k=1}^N |\mathcal{X}_i(k)|^\alpha}$$

695 . The frequency is then calculated as $\Omega_i = u_i^\top \Phi$ with $\alpha = 100$.

696 Network syncing, like all other emergent properties in this work, are defined by the emergent
 697 property statistics and values. The emergent property statistics are the first- and second-moments
 698 of the firing frequencies. The first moments are set to 0.55Hz, while the second moments are set to
 699 0.025Hz^2 .

$$E \begin{bmatrix} \Omega_{f1} \\ \Omega_{f2} \\ \Omega_{\text{hub}} \\ \Omega_{s1} \\ \Omega_{s2} \\ (\Omega_{f1} - 0.55)^2 \\ (\Omega_{f2} - 0.55)^2 \\ (\Omega_{\text{hub}} - 0.55)^2 \\ (\Omega_{s1} - 0.55)^2 \\ (\Omega_{s2} - 0.55)^2 \end{bmatrix} = \begin{bmatrix} 0.55 \\ 0.55 \\ 0.55 \\ 0.55 \\ 0.55 \\ 0.025^2 \\ 0.025^2 \\ 0.025^2 \\ 0.025^2 \\ 0.025^2 \end{bmatrix} \quad (35)$$

700 For EPI in Fig 2C, we used a real NVP architecture with two coupling layers. Each coupling layer
 701 had two hidden layers of 10 units each, and we mapped onto a support of $z \in \left[\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 10 \\ 8 \end{bmatrix} \right]$. We
 702 have shown the EPI optimization that converged with maximum entropy across 2 random seeds
 703 and augmented Lagrangian coefficient initializations of $c_0=0, 2$, and 5.

704 **A.2.2 Primary visual cortex**

705 **Draft in progress:**

706 The dynamics of each neural populations average rate $x = \begin{bmatrix} x_E \\ x_P \\ x_S \\ x_V \end{bmatrix}$ are given by:

$$\tau \frac{dx}{dt} = -x + [Wx + h]_+^n \quad (36)$$

707 Some neuron-types largely lack synaptic projections to other neuron-types [47], and it is popular
 708 to only consider a subset of the effective connectivities [19].

$$W = \begin{bmatrix} W_{EE} & W_{EP} & W_{ES} & 0 \\ W_{PE} & W_{PP} & W_{PS} & 0 \\ W_{SE} & 0 & 0 & W_{SV} \\ W_{VE} & W_{VP} & W_{VS} & 0 \end{bmatrix} \quad (37)$$

709 (TODO: ask Ken about how to introduce these values and what to ref). Estimates of the of the
 710 probability of connection and strength of connection from the Allen institute result in an estimate
 711 of the effective connectivity:

$$W = \begin{bmatrix} 0.0576 & 0.19728 & 0.13144 & 0 \\ 0.58855 & 0.30668 & 0.4285 & 0 \\ 0.15652 & 0 & 0 & 0.2 \\ 0.13755 & 0.0902 & 0.4004 & 0 \end{bmatrix} \quad (38)$$

712 We look at how this four-dimensional nonlinear dynamical model of V1 responds to different inputs,
 713 and compare the predictions of the linear response to the approximate posteriors obtained through
 714 EPI. The input to the system is the sum of a baseline input $b = [1 \ 1 \ 1]^T$ and a differential
 715 input dh :

$$h = b + dh \quad (39)$$

- 716 All simulations of this system had $T = 100$ time points, a time step $dt = 5\text{ms}$, and time constant
 717 $\tau = 20\text{ms}$. And the system was initialized to a random draw $x(0)_i \sim \mathcal{N}(1, 0.01)$.
 718 We can describe the dynamics of this system more generally by

$$\dot{x}_i = -x_i + f(u_i) \quad (40)$$

719 where the input to each neuron is

$$u_i = \sum_j W_{ij} x_j + h_i \quad (41)$$

720 Let $F_{ij} = \gamma_i \delta(i, j)$, where $\gamma_i = f'(u_i)$. Then, the linear response is

$$\frac{\partial x_{ss}}{\partial h} = F(W \frac{\partial x_{ss}}{\partial h} + I) \quad (42)$$

721 which is calculable by

$$\frac{\partial x_{ss}}{\partial h} = (F^{-1} - W)^{-1} \quad (43)$$

722 The emergent property we considered was the first and second moments of the change in rate dr
 723 between the baseline input $h = b$ and $h = b + dh$. We use the following notation to indicate that
 724 the emergent property statistics were set to the following values:

$$\mathcal{B}(\alpha, y) \leftrightarrow E \begin{bmatrix} dx_{\alpha,ss} \\ (dx_{\alpha,ss} - y)^2 \end{bmatrix} = \begin{bmatrix} y \\ 0.01^2 \end{bmatrix} \quad (44)$$

- 725 For each $\mathcal{B}(\alpha, y)$ with $\alpha \in \{E, P, S, V\}$ and $y \in \{0.1, 0.5\}$, we ran EPI with five different random
 726 initial seeds using an architecture of four coupling layers, each with two hidden layers of 10 units.
 727 We set $c_0 = 10^5$.

728 A.2.3 Superior colliculus

729 Draft in progress:

730 There are four total units: two in each hemisphere corresponding to the PRO/CONTRA and
 731 ANTI/IPSI populations. Each unit has an activity (x_i) and internal variable (u_i) related by

$$x_i(t) = \left(\frac{1}{2} \tanh \left(\frac{u_i(t) - \epsilon}{\zeta} \right) + \frac{1}{2} \right) \quad (45)$$

732 $\epsilon = 0.05$ and $\zeta = 0.5$ control the position and shape of the nonlinearity, respectively.

⁷³³ We can order the elements of x_i and u_i into vectors x and u with elements

$$x = \begin{bmatrix} x_{LP} \\ x_{LA} \\ x_{RP} \\ x_{RA} \end{bmatrix} \quad u = \begin{bmatrix} u_{LP} \\ u_{LA} \\ u_{RP} \\ u_{RA} \end{bmatrix} \quad (46)$$

⁷³⁴ The internal variables follow dynamics:

$$\tau \frac{\partial u}{\partial t} = -u + Wx + h + \sigma \partial B \quad (47)$$

⁷³⁵ with time constant $\tau = 0.09s$ and gaussian noise $\sigma \partial B$ controlled by the magnitude of $\sigma = 1.0$. The
⁷³⁶ weight matrix has 8 parameters sW_P , sW_A , vW_{PA} , vW_{AP} , hW_P , hW_A , dW_{PA} , and dW_{AP} (Fig.
⁷³⁷ 4B).

$$W = \begin{bmatrix} sW_P & vW_{PA} & hW_P & dW_{PA} \\ vW_{AP} & sW_A & dW_{AP} & hW_A \\ hW_P & dW_{PA} & sW_P & vW_{PA} \\ dW_{AP} & hW_A & vW_{AP} & sW_A \end{bmatrix} \quad (48)$$

⁷³⁸ The system receives five inputs throughout each trial, which has a total length of 1.8s.

$$h = h_{\text{rule}} + h_{\text{choice-period}} + h_{\text{light}} \quad (49)$$

⁷³⁹ There are rule-based inputs depending on the condition,

$$h_{P,\text{rule}}(t) = \begin{cases} I_{P,\text{rule}} \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^\top, & \text{if } t \leq 1.2s \\ 0, & \text{otherwise} \end{cases} \quad (50)$$

$$h_{A,\text{rule}}(t) = \begin{cases} I_{A,\text{rule}} \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^\top, & \text{if } t \leq 1.2s \\ 0, & \text{otherwise} \end{cases} \quad (51)$$

⁷⁴¹ a choice-period input,

$$h_{\text{choice}}(t) = \begin{cases} I_{\text{choice}} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^\top, & \text{if } t > 1.2s \\ 0, & \text{otherwise} \end{cases} \quad (52)$$

⁷⁴² and an input to the right or left-side depending on where the light stimulus is delivered.

$$h_{\text{light}}(t) = \begin{cases} I_{\text{light}} \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^\top, & \text{if } t > 1.2s \text{ and Left} \\ I_{\text{light}} \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}^\top, & \text{if } t > 1.2s \text{ and Right} \\ 0, & t \leq 1.2s \end{cases} \quad (53)$$

- 743 The input parameterization was fixed to $I_{P,\text{rule}} = 10$, $I_{A,\text{rule}} = 10$, $I_{\text{choice}} = 2$, and $I_{\text{light}} = 1$
- 744 TODO: this is probably a good place to explain the intuition behind the naming of the Schur
745 eigenmodes.
- 746 To produce a Bernoulli rate of p_{LP} in the Left, Pro condition (we can generalize this to either cue,
747 or stimulus condition), let \hat{p}_i be the empirical average steady state (ss) response (final V_{LP} at end
748 of task) over M=500 gaussian noise draws for a given SC model parameterization z_i :

$$\hat{p}_i = E_{\sigma \partial B} [x_{LP,ss} | s = L, c = P, z_i] = \frac{1}{M} \sum_{j=1}^M x_{LP,ss}(s = L, c = P, z_i, \sigma \partial B_j) \quad (54)$$

- 749 For the first constraint, the average over posterior samples (from $q_\theta(z)$) to be p_{LP} :

$$E_{z_i \sim q_\phi} [E_{\sigma \partial B} [x_{LP,ss} | s = L, c = P, z_i]] = E_{z_i \sim q_\phi} [\hat{p}_i] = p_{LP} \quad (55)$$

- 750 We can then ask that the variance of the steady state responses across gaussian draws, is the
751 Bernoulli variance for the empirical rate \hat{p}_i .

$$E_{z \sim q_\phi} [\sigma_{err}^2] = 0 \quad (56)$$

- 752
- $$\sigma_{err}^2 = Var_{\sigma \partial B} [x_{LP,ss} | s = L, c = P, z_i] - \hat{p}_i(1 - \hat{p}_i) \quad (57)$$

- 753 We have an additional constraint that the Pro neuron on the opposite hemisphere should have the
754 opposite value. We can enforce this with a final constraint:

$$E_{z \sim q_\phi} [d_P] = 1 \quad (58)$$

- 755
- $$E_{\sigma \partial W} [(x_{LP,ss} - x_{RP,ss})^2 | s = L, c = P, z_i] \quad (59)$$

- 756 We refer to networks obeying these constraints as Bernoulli, winner-take-all networks. Since the
757 maximum variance of a random variable bounded from 0 to 1 is the Bernoulli variance ($\hat{p}(1 - \hat{p})$),
758 and the maximum squared difference between two variables bounded from 0 to 1 is 1, we do not
759 need to control the second moment of these test statistics. In reality, these variables are dynamical
760 system states and can only exponentially decay (or saturate) to 0 (or 1), so the Bernoulli variance
761 error and squared difference constraints can only be undershot. This is important to be mindful
762 of when evaluating the convergence criteria. Instead of using our usual hypothesis testing criteria
763 for convergence to the emergent property, we set a slack variable threshold for these technically
764 infeasible constraints to 0.05.

765 Training DSNs to learn distributions of dynamical system parameterizations that produce Bernoulli
 766 responses at a given rate (with small variance around that rate) was harder to do than expected.
 767 There is a pathology in this optimization setup, where the learned distribution of weights is bimodal
 768 attributing a fraction p of the samples to an expansive mode (which always sends x_{LP} to 1), and a
 769 fraction $1 - p$ to a decaying mode (which always sends x_{LP} to 0). This pathology was avoided using
 770 an inequality constraint prohibiting parameter samples that resulted in low variance of responses
 771 across noise.

772 In total, the emergent property of rapid task switching accuracy at level p was defined as

$$\mathcal{B}(p) \leftrightarrow \begin{bmatrix} \hat{p}_P \\ \hat{p}_A \\ (\hat{p}_P - p)^2 \\ (\hat{p}_A - p)^2 \\ \sigma_{P,err}^2 \\ \sigma_{A,err}^2 \\ d_P \\ d_A \end{bmatrix} = \begin{bmatrix} p \\ p \\ 0.15^2 \\ 0.15^2 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad (60)$$

773 For each accuracy level p , we ran EPI for 10 different random seeds and selected the maximum
 774 entropy solution using an architecture of 10 planar flows with $c_0 = 2$.

775 **A.2.4 Rank-1 RNN**

776 **Draft in progress:**

777 The network dynamics of neuron i 's rate x evolve according to:

$$\dot{x}_i(t) = -x_i(t) + \sum_{j=1}^N J_{ij} \phi(x_j(t)) + I_i \quad (61)$$

778 where the connectivity is comprised of a random and structured component:

$$J_{ij} = g\chi_{ij} + P_{ij} \quad (62)$$

779 The random all-to-all component has elements drawn from $\chi_{ij} \sim \mathcal{N}(0, \frac{1}{N})$, and the structured
 780 component is a sum of r unit rank terms:

$$P_{ij} = \sum_{k=1}^r \frac{m_i^{(k)} n_j^{(k)}}{N} \quad (63)$$

⁷⁸¹ We use this theory to compute $T(x)$ when running EPI.

⁷⁸² Rank-1 vectors m and n have elements drawn

$$m_i \sim \mathcal{N}(M_m, \Sigma_m)$$

⁷⁸³

$$n_i \sim \mathcal{N}(M_n, \Sigma_n)$$

⁷⁸⁴ The current has the following statistics:

$$I = M_I + \frac{\Sigma_{mI}}{\Sigma_m} x_1 + \frac{\Sigma_{nI}}{\Sigma_n} x_2 + \Sigma_\perp h$$

⁷⁸⁵ where x_1 , x_2 , and h are standard normal random variables.

⁷⁸⁶ The $\ddot{\Delta}$ equation is broken into the equation for Δ_0 and Δ_∞ by the autocorrelation dynamics
⁷⁸⁷ assertions.

$$\ddot{\Delta}(\tau) = -\frac{\partial V}{\partial \Delta}$$

⁷⁸⁸

$$\ddot{\Delta} = \Delta - \{g^2 \langle [\phi_i(t)\phi_i(t+\tau)] \rangle + \Sigma_m^2 \kappa^2 + 2\Sigma_{mI}\kappa + \Sigma_I^2\}$$

⁷⁸⁹ We can write out the potential function by integrating the negated RHS.

$$V(\Delta, \Delta_0) = \int \mathcal{D}\Delta \frac{\partial V(\Delta, \Delta_0)}{\partial \Delta}$$

⁷⁹⁰

$$V(\Delta, \Delta_0) = -\frac{\Delta^2}{2} + g^2 \langle [\Phi_i(t)\Phi_i(t+\tau)] \rangle + (\Sigma_m^2 \kappa^2 + 2\Sigma_{mI}\kappa + \Sigma_I^2)\Delta + C$$

⁷⁹¹ We assume that as time goes to infinity, the potential relaxes to a steady state.

$$\frac{\partial V(\Delta_\infty, \Delta_0)}{\partial \Delta} = 0$$

⁷⁹²

$$\frac{\partial V(\Delta_\infty, \Delta_0)}{\partial \Delta} = -\Delta + \{g^2 \langle [\phi_i(t)\phi_i(t+\infty)] \rangle + \Sigma_m^2 \kappa^2 + 2\Sigma_{mI}\kappa + \Sigma_I^2\} = 0$$

⁷⁹³

$$\Delta_\infty = g^2 \langle [\phi_i(t)\phi_i(t+\infty)] \rangle + \Sigma_m^2 \kappa^2 + 2\Sigma_{mI}\kappa + \Sigma_I^2$$

⁷⁹⁴

$$\Delta_\infty = g^2 \int \mathcal{D}z \left[\int \mathcal{D}x \phi(\mu + \sqrt{\Delta_0 - \Delta_\infty}x + \sqrt{\Delta_\infty}z) \right]^2 + \Sigma_m^2 \kappa^2 + 2\Sigma_{mI}\kappa + \Sigma_I^2$$

⁷⁹⁵ Also, we assume that the energy of the system is perserved throughout the entirety of its evolution.

$$V(\Delta_0, \Delta_0) = V(\Delta_\infty, \Delta_0)$$

⁷⁹⁶

$$-\frac{\Delta_0^2}{2} + g^2 \langle [\Phi_i(t)\Phi_i(t)] \rangle + (\Sigma_m^2 \kappa^2 + 2\Sigma_{mI}\kappa + \Sigma_I^2)\Delta_0 + C = -\frac{\Delta_\infty^2}{2} + g^2 \langle [\Phi_i(t)\Phi_i(t)] \rangle + (\Sigma_m^2 \kappa^2 + 2\Sigma_{mI}\kappa + \Sigma_I^2)\Delta_\infty + C$$

⁷⁹⁷

$$\frac{\Delta_0^2 - \Delta_\infty^2}{2} = g^2 (\langle [\Phi_i(t)\Phi_i(t)] \rangle - \langle [\Phi_i(t)\Phi_i(t)] \rangle) + (\Sigma_m^2 \kappa^2 + 2\Sigma_{mI}\kappa + \Sigma_I^2)(\Delta_0 - \Delta_\infty)$$

$$\frac{\Delta_0^2 - \Delta_\infty^2}{2} = g^2 \left(\int \mathcal{D}z \Phi^2(\mu + \sqrt{\Delta_0}z) - \int \mathcal{D}z \int \mathcal{D}x \Phi(\mu + \sqrt{\Delta_0 - \Delta_\infty}x + \sqrt{\Delta_\infty}z) \right) \\ + (\Sigma_m^2 \kappa^2 + 2\Sigma_{mI}\kappa + \Sigma_I^2)(\Delta_0 - \Delta_\infty)$$

Consistency equations:

800

$$\mu = F(\mu, \kappa, \Delta_0, \Delta_\infty) = M_m \kappa + M_I \\ \kappa = G(\mu, \kappa, \Delta_0, \Delta_\infty) = M_n \langle [\phi_i] \rangle + \Sigma_{nI} \langle [\phi'_i] \rangle \\ \frac{\Delta_0^2 - \Delta_\infty^2}{2} = H(\mu, \kappa, \Delta_0, \Delta_\infty) = g^2 \left(\int \mathcal{D}z \Phi^2(\mu + \sqrt{\Delta_0}z) - \int \mathcal{D}z \int \mathcal{D}x \Phi(\mu + \sqrt{\Delta_0 - \Delta_\infty}x + \sqrt{\Delta_\infty}z) \right) \\ + (\Sigma_m^2 \kappa^2 + 2\Sigma_{mI}\kappa + \Sigma_I^2)(\Delta_0 - \Delta_\infty) \\ \Delta_\infty = L(\mu, \kappa, \Delta_0, \Delta_\infty) = g^2 \int \mathcal{D}z \left[\int \mathcal{D}x \phi(\mu + \sqrt{\Delta_0 - \Delta_\infty}x + \sqrt{\Delta_\infty}z) \right]^2 + \Sigma_m^2 \kappa^2 + 2\Sigma_{mI}\kappa + \Sigma_I^2 \\ (64)$$

801 We can solve these equations by simulating the following Langevin dynamical system.

$$x(t) = \frac{\Delta_0(t)^2 - \Delta_\infty(t)^2}{2} \\ \Delta_0(t) = \sqrt{2x(t) + \Delta_\infty(t)^2} \\ \dot{\mu}(t) = -\mu(t) + F(\mu(t), \kappa(t), \Delta_0(t), \Delta_\infty(t)) \\ \dot{\kappa}(t) = -\kappa + G(\mu(t), \kappa(t), \Delta_0(t), \Delta_\infty(t)) \\ \dot{x}(t) = -x(t) + H(\mu(t), \kappa(t), \Delta_0(t), \Delta_\infty(t)) \\ \dot{\Delta}_\infty(t) = -\Delta_\infty(t) + L(\mu(t), \kappa(t), \Delta_0(t), \Delta_\infty(t)) \\ (65)$$

802 Then, the temporal variance is simply calculated via

$$\Delta_T = \Delta_0 - \Delta_\infty \\ (66)$$

803 TODO Need to explain the warm starting for the aficionados.

804 TODO explain the density network architectures used.

805 A.3 Supplementary Figures

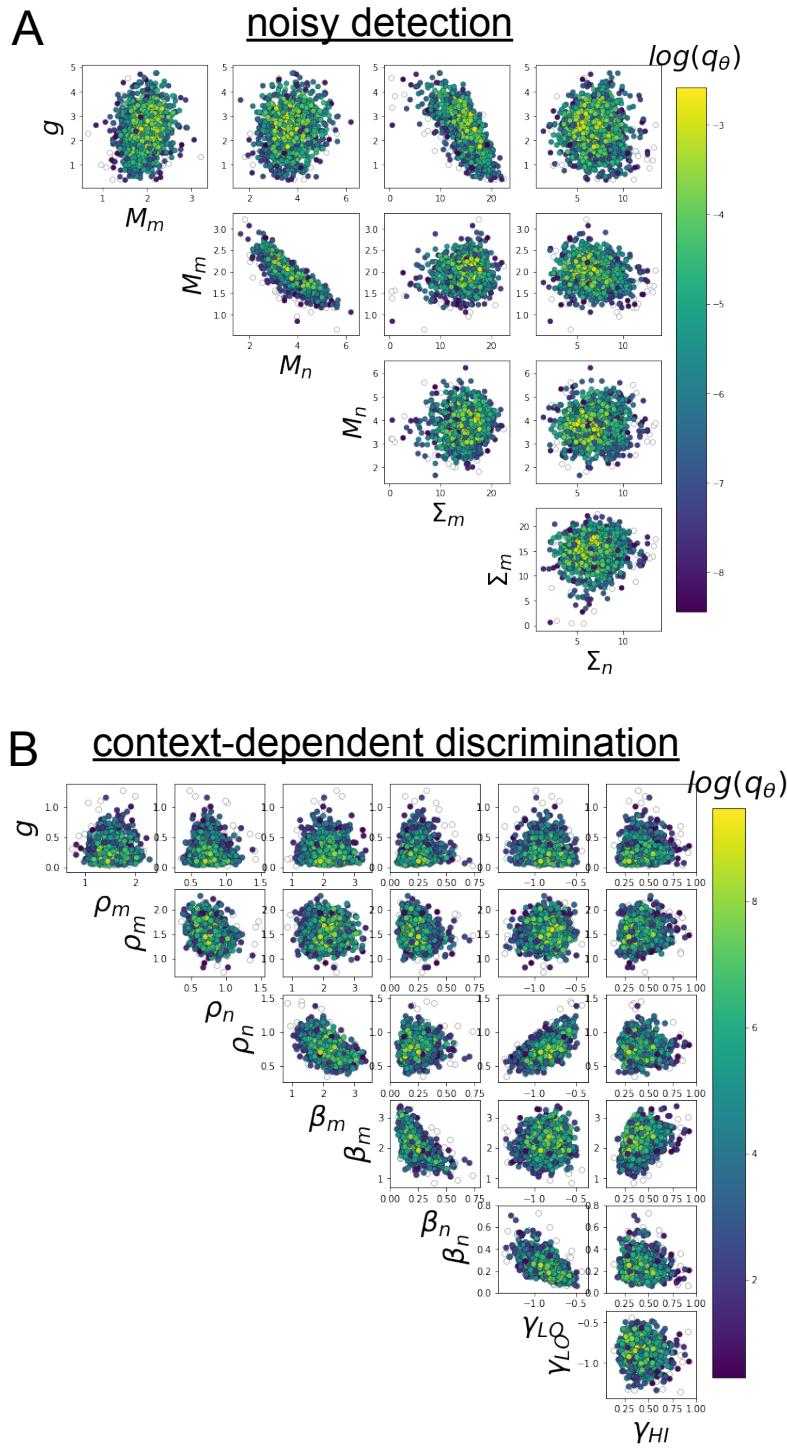


Fig. S1: A. EPI for rank-1 networks doing discrimination. B. EPI for rank-2 networks doing context-dependent discrimination.

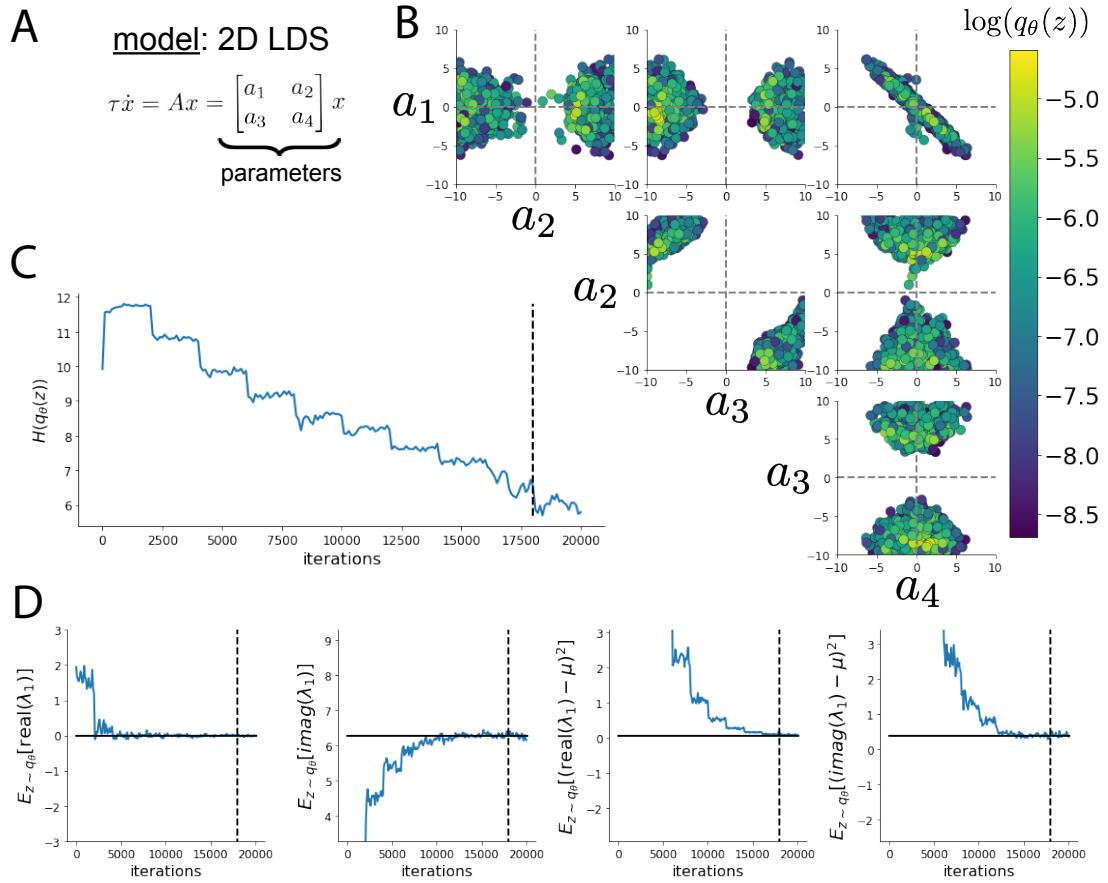


Fig. S2: A. Two-dimensional linear dynamical system model, where real entries of the dynamics matrix A are the parameters. B. The DSN distribution for a 2D LDS with $\tau = 1$ that produces an average of 1Hz oscillations with some small amount of variance. C. Entropy throughout the optimization. At the beginning of each augmented lagrangian epoch (5,000 iterations), the entropy dips due to the shifted optimization manifold where emergent property constraint satisfaction is increasingly weighted. D. Emergent property moments throughout optimization. At the beginning of each augmented lagrangian epoch, the emergent property moments move closer to their constraints.

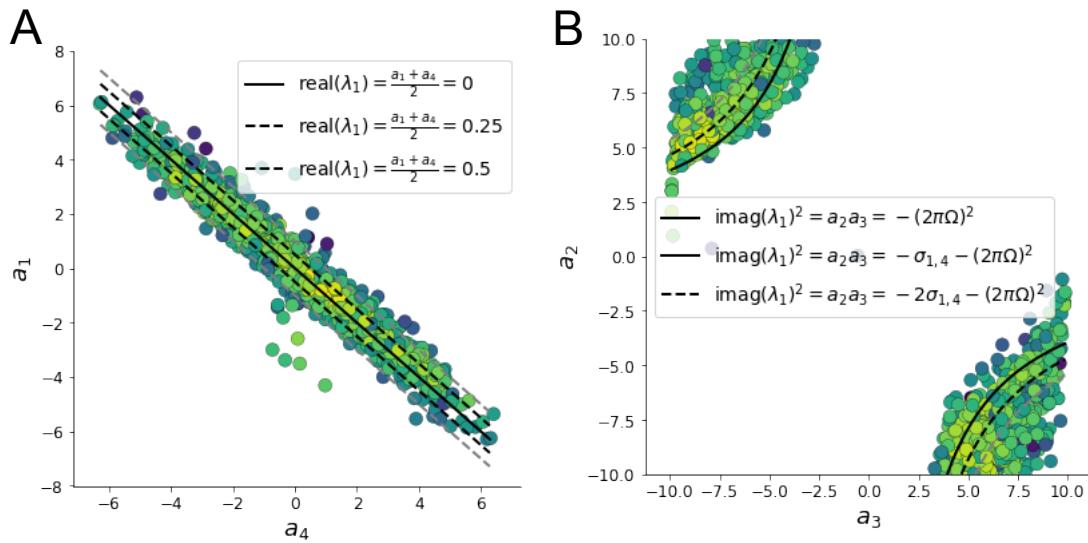


Fig. S3: A. Probability contours in the $a_1 - a_4$ plane can be derived from the relationship to emergent property statistic of growth/decay factor. B. Probability contours in the $a_2 - a_3$ plane can be derived from relationship to the emergent property statistic of oscillation frequency.