
Learning Exponential Families

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Abstract

1 Recently much attention has been paid to implicit probabilistic models – models
2 defined by mapping a simple random variable through a complex transformation,
3 often a deep neural network. These models have been used to great success for
4 variational inference, generation of complex data types, and more. In most all of
5 these settings, the goal has been to find a *particular member* of that model family:
6 optimized parameters index a distribution that is close (via a divergence or clas-
7 sification metric) to a target distribution (such as a posterior or data distribution).
8 Much less attention, however, has been paid to the problem of *learning a model*
9 *itself*. Here we define implicit probabilistic models with specific deep network
10 architecture and optimization procedures in order to learn intractable exponential
11 family models (*not* a single distribution from those models). These exponential
12 families, which are central to some of the most fundamental problems in probabilis-
13 tic inference, are learned accurately, allowing operations like posterior inference
14 to be executed directly and generically by an input choice of natural parameters,
15 rather than performing inference via optimization for each particular realization
16 of a distribution within that model. We demonstrate this ability across a number
17 of non-conjugate exponential families that appear often in the machine learning
18 literature.

1 Introduction

20 Probability models, the fundamental object of Bayesian machine learning, have long challenged
21 researchers with the tradeoff between tractability and expressivity. Though well understood that a
22 model should be chosen to instantiate a set of assumptions and capture existing domain knowledge
23 [1, 2, 3], for many years too-simple models were chosen for their practical advantages (such as
24 conditional conjugacy), which left much to be desired in terms of expressive performance and
25 scalability of these models.

26 More recently the pendulum has swung, via a resurgence in models which map a latent random
27 variable $w \sim q_0$ through a member of a highly expressive function family $\mathcal{G} = \{g_\theta : \theta \in \Theta\}$, the
28 composition resulting in an *implicit probability model* $\mathcal{M} = \{q(g_\theta \circ w) : \theta \in \Theta\}$ (where $q(\cdot)$ is
29 the pushforward density, i.e. the density induced on the image of the random variable w under
30 the function g_θ). Choosing \mathcal{G} to be a parameter-indexed family of neural networks has both a rich
31 history [4, 5], and has recently been used to produce exciting results for density estimation [6, 7, 8],
32 generation of complex data [9], variational inference [10, 11, 12], and more. A noted advantage of
33 these implicit density network models is that in many cases they make minimal assumptions about
34 the data generative (or posterior inference) process. On the other hand, since these models have
35 been chosen to be generic and flexible, they can lack the classic stipulation that a model instantiates
36 existing domain knowledge. The downsides of a too-flexible model with finite data (albeit large)
37 – and the corresponding bias-variance benefit of a restricted model – are textbook knowledge [13,

§7.3], and work on generalization and compressibility in deep networks suggests that this broad class of function families are indeed quite large, perhaps problematically so [14].

Is all the flexibility of an implicit density network model \mathcal{M} always necessary? Consider the case of variational inference, where a generative model $p(z)p_\beta(X|z)$ (latent z , observed data X) is stipulated in the classic sense to embody modeling assumptions (hierarchical model, topic model, Bayesian logistic regression, etc.). When such a model is intractable, it is increasingly common to deploy an implicit “recognition network” model for variational inference [10], which finds a $q_{\theta^*}(z) \in \mathcal{M}$ such that an evidence bound is optimized with respect to the true posterior $p(z|X)$. However, note the widely recognized fact [15] that many such true posteriors $p(z|X)$ belong to models that can be written as exponential families (albeit intractable, due to the choice of sufficient statistics $t(z)$), of the form: $\mathcal{P} = \left\{ \frac{h(z)}{A(\eta)} \exp \{ \eta^\top t(z) \} : \eta \in H \right\}$. Some effort has been made to learn single members of exponential families from their mean parameters [16], but here we are focused on the natural parameterization and the model itself (not simply members thereof).

Should we be able to learn a tractable approximation to this exponential family model, we would in the very least get the bias-variance benefits of an intelligently restricted model space, and at best would get inference “for free” in the sense that we could evaluate approximate posteriors directly without separate optimization for each dataset encountered (a different form of amortized inference [17, 10, 11, 18]). In this paper we aim to learn a restricted model $\mathcal{Q} = \{q(z; \eta : \eta \in H)\}$ that will be a strict subset of the deep implicit model \mathcal{M} and will closely approximate a target exponential family \mathcal{P} . Note the critical difference between this aim and much of the literature that seeks to learn a density $q_\theta^* \in \mathcal{M}$ (we explore this distinction in depth both algorithmically and empirically).

To proceed, we must first specify a set of models $\mathcal{Q} = \{Q_\phi : \phi \in \Phi\}$, from which we can learn a single model Q_{ϕ^*} , and we must second define a sensible parameter space H of each model. To the first, we restrict Θ , the parameter space of \mathcal{M} , to be itself the image of a second deep *parameter network* family $\mathcal{F} = \{f_\phi : \phi \in \Phi\}$, such that $\{f_\phi(\eta) : \eta \in H\} \subset \Theta$. The second part is answered immediately by our choice of target \mathcal{P} , an exponential family which by definition has *natural* parameterization $\eta \in H$. Thus, appealingly, we know that H is precisely the correct parameter space for \mathcal{Q} (as it defines \mathcal{P}), and that the image of H under f_ϕ will be of the correct dimensionality within the codomain Θ ; approximation error between \mathcal{Q} and \mathcal{P} will be caused by the flexibility and learnability of the parameter network f_ϕ and the density network $g_{f_\phi(\eta)}$.

We define this two-network architecture, which we term an *exponential family network* (EFN), and we specify a stochastic optimization procedure over a variant of the typical Kullback-Leibler divergence. We then demonstrate the ability of EFNs to approximately learn exponential families, both known tractable families and well-used intractable families, including hierarchical Dirichlet and truncated normal Poisson families. Finally we demonstrate the utility of this approach in an example inferring the posterior distribution of the latent intensity of a point-process, given neural spike train data. In all, our contributions include:

- a novel implicit model: a two-network deep architecture to learn a probability model along with a doubly stochastic optimization that samples over both natural parameters (the family member to be learned) and data points (observations of the target density);
- analysis of the connections between approximately learning a model and approximate variational inference, and an empirical study that gives insight to possible improvements to variational inference;
- empirical results confirming performance against ground truth in known tractable exponential families and in common intractable exponential families.

2 Exponential family networks

To define exponential family networks (EFN), we begin with relevant context for our modeling choice of exponential families (§2.1). We then describe the primary network architectural constraint and the background we leverage to satisfy that constraint (§2.2). We then introduce EFN in detail, including the optimization algorithm used for learning (§2.3). The similarities with variational inference are then explored in depth in §2.4.



Figure 1: (A) Graphical model for conditionally iid sampling from an exponential family likelihood. (B) Hierarchical Dirichlets – prior $p_0(z)$ (top), three sample conditional Dirichlet datasets X of $N = 2, N = 20, N = 100$ (middle), and three corresponding posteriors that themselves form an exponential family \mathcal{P} (bottom). (C) Architecture for exponential family network (EFN) – density network running top to bottom; parameter network running right to left.

2.1 Exponential families as target model \mathcal{P}

We will focus on a fundamental problem setup in probabilistic inference, that of a latent variable $z \in \mathcal{Z}$ with prior belief $p_0(z)$, and where we observe a dataset $X = \{x_1, \dots, x_N\} \subset \mathcal{X}$ as conditionally independent draws given z . Updating our belief with data produces the posterior $p(z|X) \propto p_0(z) \prod_{i=1}^N p(x_i|z)$. This setup is shown as a graphical model in Figure 1A.

In rare cases these posterior distributions are tractable due to either known conjugacy or to careful historical work (often an inversion, transformation-rejection, or similar custom numerical strategy) that has made these distributions computationally indistinguishable from tractable [19]. It is intriguing then to reflect upon the success that deep networks have offered to function approximation, and ask to what extent we can automate this numerical process, widening the class of effectively tractable distributions.

If we restrict our attention to priors and likelihoods that belong to exponential families $\mathcal{P} = \left\{ \frac{h(\cdot)}{A(\eta)} \exp \{ \eta^\top t(\cdot) \} : \eta \in H \right\}$, the posterior can be also viewed as an exponential family, albeit intractable [15]. For simplicity we will hereafter suppress the base measure $h(\cdot)$. Consider:

$$p_0(z) = \frac{1}{A_0(\alpha)} \exp \{ \alpha^\top t_0(z) \} \quad , \quad p(x_i|z) = \frac{1}{A(z)} \exp \{ \nu(z)^\top t(x_i) \} \quad ,$$

where $t(\cdot)$ is the sufficient statistic vector, and $\nu(z)$ is the natural parameter of the likelihood in natural form [20]. The posterior then has the form:

$$p(z|x_1, \dots, x_N) \propto \exp \left\{ \left[\begin{array}{c} \alpha \\ \sum_i t(x_i) \\ -N \end{array} \right]^\top \left[\begin{array}{c} t_0(z) \\ \nu(z) \\ \log A(z) \end{array} \right] \right\} \quad , \quad (1)$$

which again is an exponential family, albeit intractable.

To give a concrete example, consider the hierarchical Dirichlet – a Dirichlet prior $z \sim \text{Dir}(\alpha)$ (of dimension $|\mathcal{Z}|$) with conditionally iid Dirichlet draws $x_i|z \sim \text{Dir}(\beta z)$, which has been considered historically [21], and is perhaps most notable for its nonparametric extension [22] (and has relevance for multi-corpus extensions of topic models [23, 24]). Figure 1B shows the prior for

110 a given α (top), and three examples of datasets that could arise via this generative model (mid-
 111 dle). A set of basic manipulations shows the hierarchical Dirichlet posterior $p(z|X)$ to be itself an
 112 exponential family with natural parameter $\eta = [\alpha - 1, \sum_i \log(x_i), -N]^\top$ and sufficient statistic
 113 $t(z) = [\log(z), \beta z, \log(B(\beta z))]^\top$.¹ The corresponding posteriors are shown in Figure 1B (bottom).

114 Note importantly that, because the likelihood was chosen to be an exponential family (which is closed
 115 under sampling), this form will not change for any choice of $|Z|$ -dimensional hierarchical Dirichlet
 116 – any draw from the prior, any N , or any particular realization of observed data X (technically the
 117 prior need not be exponential family, but we leave it as such for simplicity). The exponential family
 118 is clearly sufficient for this property, and the Pitman-Koopman Lemma further clarifies that it is also
 119 necessary (under reasonable conditions) [20, §3.3.3].

120 The critical observation here is that, if we can approximately learn an intractable exponential family
 121 (the model itself), then it becomes trivial to perform posterior inference: we simply use the dataset to
 122 index into the natural parameter η of the intractable family, and the posterior distribution is produced.
 123 This is the goal of EFN.

124 2.2 Density networks as generic approximating family \mathcal{M}

Implicit probability models, which we will use for our approximating model family \mathcal{M} , can be
 defined by any base random variable $w \sim p_0$ mapped through any measurable, parameter-indexed
 function family $\mathcal{G} = \{g_\theta : \theta \in \Theta\}$; we denote the induced density on $z = g_\theta(w)$ as $q_\theta(z)$. Though
 trivial to sample from $q_\theta(z)$ for any choice of family \mathcal{G} , we here additionally require that we be able to
 explicitly calculate $q_\theta(z)$. This goal can be readily achieved by designing \mathcal{G} to contain only bijective
 functions, ideally with a Jacobian form that is convenient to compute. Designing that bijective \mathcal{G} as a
 deep neural network family, as we do here, is a well-established idea that has recently seen many
 variants and applications [5, 25, 26, 7, 6, 27, 28, 8, 29]. Specifically, let $z = g_\theta(w) = g_L \circ \dots \circ g_1(w)$
 for bijective vector-valued functions g_ℓ (where for clarity we have suppressed the dependence of each
 on θ), and denote $J_\theta^\ell(z)$ as the Jacobian of the function g_ℓ at the layer activation corresponding to z .
 Then we have:

$$q_\theta(z) = q_0(g_1^{-1} \circ \dots \circ g_L^{-1}(z)) \prod_{\ell=1}^L \frac{1}{|J_\theta^\ell(z)|}.$$

125 The specific form of the layers g_ℓ can be chosen based on empirical considerations; we clarify our
 126 choice in §3. For the remainder (and to avoid confusion when we introduce a second network) we call
 127 this deep bijective neural architecture the *density network*; this network is shown vertically oriented
 128 (flowing from w down to z) in Figure 1C.

129 This density network induces the model $\mathcal{M} = \{q(g_\theta \circ w) : \theta \in \Theta\}$, which previous work has
 130 searched to find a single optimized distribution (such as a posterior or data generative density), on the
 131 assumption and subsequent empirical evidence that the target exponential family member is close to
 132 (or approximately belongs to) \mathcal{M} . We make the same assumption for the exponential family itself
 133 and seek to intelligently restrict \mathcal{M} in order to learn the exponential family.

134 2.3 Exponential family networks as approximating model \mathcal{Q}

135 Having introduced our target model \mathcal{P} , an exponential family with natural parameters $\eta \in H$, and
 136 the density network family \mathcal{M} , we now seek to learn $\mathcal{Q} \approx \mathcal{P}$, where $\mathcal{Q} \subset \mathcal{M}$. To do so we will
 137 parameterize θ , the parameters of the density network, as the image of a second *parameter network*
 138 family $\mathcal{F} = \{f_\phi : H \rightarrow \Theta, \phi \in \Phi\}$. This network is shown flowing from right to left in Figure 1C.
 139 Using a second meta-network to aid or restrict network learning has been used in a variety of settings;
 140 a few examples include parameterizing the optimization algorithm in the so-called “learning to learn”
 141 setting [30], and a more closely related work that used a second network to condition on observations
 142 for local latent variational inference [27], a connection which we explore closely in the following
 143 section.

144 Any choice of parameter network parameters ϕ induces a $|H|$ -dimensional submanifold (the image
 145 $f_\phi(H)$) of the density network parameter space Θ , and as such defines a restricted model $\mathcal{Q}_\phi =$

¹To be clear this model is an exponential family if β is fixed or treated as a latent variable; this fact however
 will not be important for the development of this paper.

146 $\{q_{f_\phi}(z; \eta) : \eta \in H\} \subset \mathcal{M}$; by our choice of H as the natural parameter space of the exponential
 147 family target \mathcal{P} , this model restriction is at least of the correct dimensionality. Our goal then is to
 148 search over the implied set of models $\mathbb{Q} = \{Q_\phi : \phi \in \Phi\}$ to find an optimal ϕ^* such that $Q_{\phi^*} \approx \mathcal{P}$.

Given the connections between the exponential family and Shannon entropy, we will measure the error between Q_ϕ and \mathcal{P} with Kullback-Leibler divergence. Consider for the moment a fixed choice of natural parameter η ; we seek to minimize, over ϕ :

$$D(q_\phi(z; \eta) || p(z; \eta)) \propto \mathbb{E}_{q_\phi} \left(\log q_\phi(z; \eta) - \eta^\top t(z) \right) = \mathbb{E}_{q_\phi} \left(q_0(g_\theta^{-1}(z)) + \sum_{\ell=1}^L \log |J_\theta^\ell(z)| - \eta^\top t(z) \right),$$

149

150 where again we note that $\theta = f_\phi(\eta)$, and thus for a fixed eta, this objective depends only on ϕ . Indeed,
 151 the target $\eta^\top t(z)$ is linear in η (an obvious restatement of the log-linear exponential family form),
 152 giving us some hope that we may be able to learn this model. As a side note, this objective can also
 153 produce approximations of the log partition (as the intercept term implied by this linear target), which
 154 we have found to be reasonably accurate, though nuanced schemes are likely appropriate [31]; we do
 155 not explore that further here.

156 Of course we seek to approximate not just a single target exponential family member ($p(z; \eta)$ for
 157 a fixed η), but rather the entire model $\mathcal{P} = \{p(z; \eta) : \eta \in H\}$. For optimization we thus need to
 158 introduce a distribution $p(\eta)$ (for sampling), leading to the objective:

$$\operatorname{argmin}_{\phi} \mathbb{E}_{p(\eta)} (D(q_\phi(z; \eta) || p(z; \eta))) = \operatorname{argmin}_{\phi} D(q_\phi(z; \eta)p(\eta) || p(z; \eta)p(\eta)).$$

159 Unbiased estimates of this objective are immediate: $q_\phi(z; \eta)$ is sampled by computing calculating
 160 the density network parameters $\theta = f_\phi(\eta)$ (using the parameter network), sampling the latent
 161 $w \sim p_0(w)$, and running that w through the density network; $p(\eta)$ is user defined and thus trivial to
 162 sample. Stochastic optimization can then be carried out on the estimator:

$$\mathbb{L}(\phi) = \frac{1}{K} \frac{1}{M} \sum_{k=1}^K \sum_{m=1}^M \left(q_0(g_{\theta^k}^{-1}(z^m)) + \sum_{\ell=1}^L \log |J_{\theta^k}^\ell(z^m)| - \eta_k^\top t(z^m) \right),$$

163 where $\theta^k = f_\phi(\eta_k)$. Successful optimization over ϕ should thus result in $Q_{\phi^*} \in \mathbb{Q}$ that accurately
 164 approximates the target exponential family; that is, $Q \approx \mathcal{P}$. We call this two-network architecture
 165 and optimization an exponential family network (EFN). What remains for empirical implementation
 166 is to make particular choices of hyperparameters, network layers, and optimization algorithm, which
 167 we specify in §3 below.

168 2.4 Relation to variational inference

169 A tremendous amount of work in recent years has gone into variational inference (VI), and its
 170 similarity to EFN warrants careful attention. In the following, we aim to carefully (and somewhat
 171 pedantically) dissect this question. As such, though EFN can address any target exponential family,
 172 to bring us closest to VI let us here restrict the EFN target model \mathcal{P} to be a family of posterior
 173 distributions.

174 The typical role of variational inference is to infer an approximate posterior $q_\phi(z) \approx p(z|X)$. In this
 175 setting, the difference with EFN is stark, in so much as VI learns this single posterior approximation,
 176 whereas the main goal of the EFN is to approximate the model $\mathcal{P} = p_\eta(z|X) : \eta \in H$: to learn
 177 the family of distributions. More recently, much focus has gone into the particular instance of
 178 VI for local variables z_i , for example $\prod_{i=1}^N p(z_i)p(x_i|z_i)$ (such as a variational autoencoder [10])
 179 or $p(u) \prod_{i=1}^N p(z_i|u)p(x_i|z_i)$ (latent Dirichlet allocation being a canonical example [23, 32]), the
 180 result of which is often an amortized inference/recognition network that produces a local variational
 181 distribution $q_{\phi^*}(z_i|x_i)$. This local variational distribution is typically parameterized explicitly: the
 182 inference network $\mu_\phi(x_i)$ induces a local parametric distribution, often a Gaussian $q(z_i|x_i) \sim$
 183 $\mathcal{N}(z_i; \mu_\phi(x_i))$ [10, for example]. Viewed this way, local-latent-variable VI methods induce a model
 184 $\{q_{\phi^*}(z_i|x_i) : x_i \in X\}$ for a finite dataset X . In that sense, EFN and VI are similar ‘model learning’

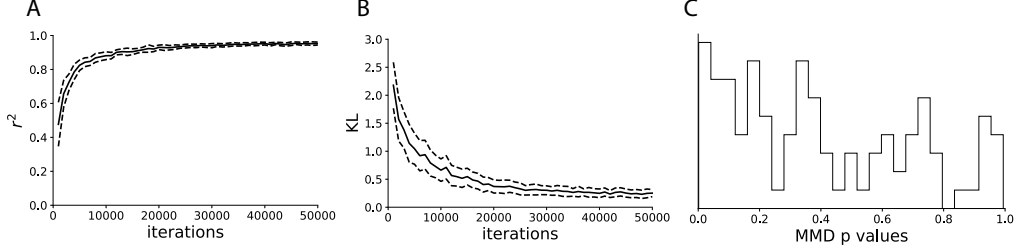


Figure 2: 25-dimensional Dirichlet exponential family network. A.) Distribution of r^2 between the sufficient statistics and log-probability across choices of η throughout optimization. B.) Distribution of KL divergence across choices of η throughout optimization. C.) Distribution of maximum mean discrepancy p-values between EFN samples and ground truth after optimization [37].

185 approaches. Even more closely, as part of a long-standing desire to add structure to VI beyond mean-
 186 field (classically [33, 34]; more recently [35, 36], to name but a few), in several cases a inference
 187 network has been used to parameterize a deep implicit model (in a two-network inference architecture,
 188 to say nothing of whether or not the generative model itself is a deep implicit model); closest to
 189 the EFN architecture is [27] (cf. Figure 2 of [27] with Figure 1C here). Thus EFN (when used for
 190 posterior families) can be seen as a close generalization of VI.

191 However, even accepting this VI-as-a-model view, the difference between the finite dataset X and
 192 the natural parameter space H persists when viewed at a mechanical level; well-known are the
 193 overfitting/generalization issues associated with a finite dataset compared with access to a distribution
 194 $p(\eta)$. Thus one goal of EFN is to allow the model $Q_{\phi^*} \approx \mathcal{P}$ to be learned in the absence of a finite
 195 dataset, such that inference on that dataset can then be executed without concerns of overfitting to
 196 that set (and of course without having to run a VI optimization for every new dataset). Perhaps more
 197 importantly, the “model” implied by VI is parameterized by x_i , and indeed the inference network
 198 takes x_i as input. The EFN on the other hand is considerably more general: as Equation 1 shows, the
 199 posterior includes the natural parameters of the prior, allowing the EFN architecture to learn across a
 200 more general setting that VI can not (since any VI inference network is only parameterized by data).
 201 One final difference made clear by Equation 1 is that the observations are given to the EFN *in natural*
 202 *form* (that is, $t(x_i)$, not x_i) [20]. This choice is a novel insight: by exploiting the known sufficiency
 203 of $t(x_i)$ in the target model \mathcal{P} , some difference in performance for VI may be observed. We explore
 204 this empirically in the following section.

205 Accordingly, while EFN and VI do at a high level bear multiple similarities, the differences are both
 206 material and provoke interesting speculation about means to improve both VI and EFN.

207 3 Results

208 Introductory remarks and then comments about architectural particulars, including planar flow
 209 networks of [27]. Note Number of panar flows is always D (intrinsic dimensionality of flows), units
 210 per layer ramping is always the same function of D . The number of layers in the theta network is
 211 always a function of D - will probably just always use 8 layers. Remember

212 NF1: do full norm flow “variational inference” (explore all of ϕ space with the full flow network
 213 model \mathcal{Q}), which is to say $\arg \min_{\phi} KL(q_{\phi}||p)$.

214 EFN1: be literal to Figure 1C, give the sufficient statistics of that $K=1$ dataset, and learn an EFN from
 215 scratch. This alternative is important because it is the most specific (but kind of annoying, hence
 216 alternative 1) interpretation of norm flow VI paper.

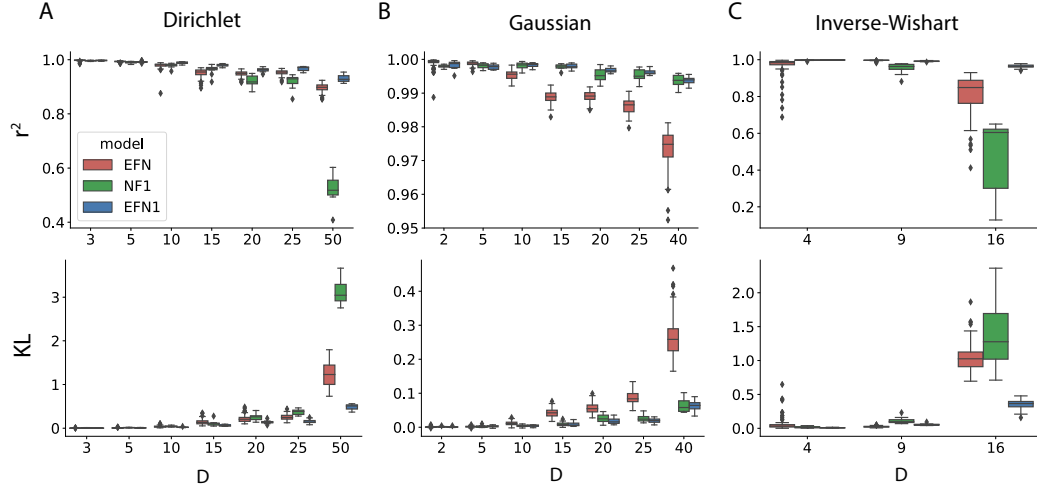


Figure 3: Scaling exponential family networks. A.) Dirichlet. B.) Gaussian C.) Inverse-Wishart

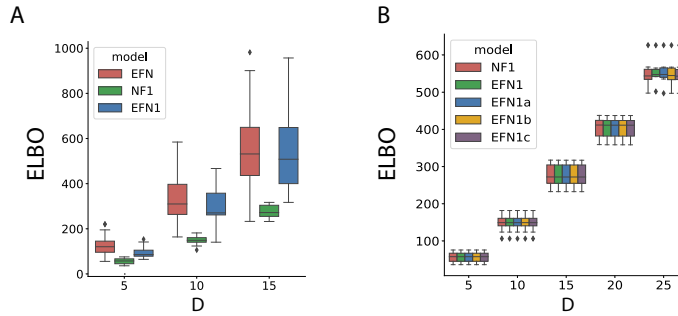


Figure 4: Scaling Dir-Dir

217 3.1 Tractable exponential families

218 3.2 Intractable exponential families

219 **Hierarchical Dirichlets** Hierarchical dirichlets are useful and have some history; most notable is
 220 with the Hierarchical Dirichlet Process [22], but historically this was done in the finite case also [21].
 221 Here is some math. Note that this matters for multi-corpus LDA generally as well [23, 24].

222 **Truncated- and log-normal Poisson** used a lot [38][39][40, 41]

223 Figure 4:

224 EFN in intractable exp fams (connecting to above, but with hard distribs and the ELBO)

225 Panel A: Dir-Dir ELBO by dimensionality for NF1 and EFN and EFN1

226 Panel B: Dir-Dir ELBO by dimensionality for EFN1 vs EFN1a vs 1b vs 1c vs NF1 (with $N = 1$ data
 227 point)

228

229 3.3 Neural spike train analysis

230 Figure 5:

231 Panel A TNP picture example of prior and posterior with a few samples, just for feel good

232 PANEL B: ELBO on held out data as a function of R , for a middle choice of training dataset size N
233 and D .

234 PANEL C: ELBO on held out data as a function of N , for a middle choice of number of samples in
235 the posterior R .

236

237 PANEL D (optional): (ELBO EFN - ELBO NF1) as a surface plot as a function of R, N . That is,
238 positive places is where EFN outperforms, negative NF1.

239 The key point with these is that, while you have the *same exact* flow network architecture, now you
240 have to optimize over ϕ with a limited single dataset. Learning a restricted model space is good for
241 the bias-variance tradeoff! Do this many times so that variance will become clear.

242 —other thoughts— Real data analysis and posterior inference. **Key real data result on TNP.**

243 Get some data from CRCNS that has many spike trains x_i for $i = 1, \dots, N$ (ask Gabriel, as he has
244 done some poking around recently; or look at some of the above TNP/LNP refs).

245 Those spike trains should be conditionally independent draws from the same underlying intensity
246 function z . (for example, trials under the same stimulus)

247 Bin the length of time T into $\approx 20 - 30$ equally spaced time bins. Thus z is now a vector in \mathbb{R}^{20} .

248 Now each spike train x_i is a conditionally independent Poisson vector observation, with rate vector z .

249 Learn the 20 dimensional TNP exp fam, without any regard to this dataset X .

250 No: Panel No: TNP ELBO by dimensionality for NF1 and EFN and EFN1

251 Panel A TNP picture example of prior and posterior with a few samples, just for feel good

252

253 **Now we want to learn the posterior $p(z | \text{some fixed number } R \text{ of data points})$.**

254 To do this for an EFN, just plug in those R points x_{i_1}, \dots, x_{i_R} and the prior as a natural parameter,
255 and job done.

256 To do this for an NF1, train a VI model by taking the log joint with R data points, then go through
257 and resample R points every time from your training dataset with N data points.

258 **PANEL A: ELBO on held out data as a function of R , for a middle choice of training dataset
259 size N .**

260 **PANEL B: ELBO on held out data as a function of N , for a middle choice of number of
261 samples in the posterior R .**

262 **PANEL C: (ELBO EFN - ELBO NF1) as a surface plot as a function of R, N . That is, positive
263 places is where EFN outperforms, negative NF1.**

264 The key point with these is that, while you have the *same exact* flow network architecture, now you
265 have to optimize over ϕ with a limited single dataset. Learning a restricted model space is good
266 for the bias-variance tradeoff! Do this many times so that variance will become clear. **Panel C v2:**

267 **Possibly want to explicitly plot variance of EFN and NF1 to focus on the variance tradeoff**

268 **Panel C v3: change time bin granularity from 10 to 50 to show how this story changes in D .
269 My thought is that all will be exhausted by dimensionality sweeps by this point, so no.**

270 also Notice one pain here is that these panels requires training a new EFN1 at every choice of N and
271 R (but only one EFN). Sorry.

272

273 We hope and expect this will show that when the dataset gets small, this "traditional VI" will get
274 arbitrarily bad (can't learn a network); eventually, there will be so much data that the VI will match
275 or outperform the EFN... outperform because VI can focus specifically on this distribution rather than
276 over the whole family, so the EFN has less effective data for this η (but not because it has a broader
277 range of models, since we believe the EFN contains the closest member). Performance metric should
278 be ELBO on some held out data or something like that (it's a posterior, so log likelihood doesn't
279 really make sense). Test data anyway. Check VI papers for usual metrics. A key point to make
280 here is that one great virtue of EFNs is is learning a restricted model, which should demonstrate the
281 usual bias-variance tradeoff (see for example Hastie and Tibshirani book, Fig 7.2). Or Figure 4 is
282 bias-variance and some sample posteriors in 2-d (showing how nicely it works), and then Fig 5 is the
283 above performance, with both train and test.

284 This will be for one real example X . As such, to get error bars, just take a big dataset and randomly
285 subsample the test set. Then the posterior performance is really for that very dataset, so the sem is
286 coherent and the right thing to calculate/show. Important to clarify that doing so *does not* test how
287 well this does across the entire exp fam, but just this one posterior. ((To test that, we would do it in
288 simulation: generate *many datasets* X , then do the above for every one of them. Same computation
289 for EFN (since its just plugging in a dataset), but VI alternatives 1 and 2 now need to be rerun for
290 every dataset. And it's still simulated data, not really offering something fundamentally more than Fig
291 3 (well ok it's an intractable model, but I'm not sure that offers so much)...let's skip that altogether)).

292 **4 Conclusion**

293 Snappy closing remarks!

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