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# Learning Exponential Families

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Anonymous Author(s)

Affiliation

Address

email

## Abstract

1 Recently much attention has been paid to implicit probabilistic models – models  
2 defined by mapping a simple random variable through a complex transformation,  
3 often a deep neural network. These models have been used to great success for  
4 variational inference, generation of complex data types, and more. In most all of  
5 these settings, the goal has been to find a *particular member* of that model family:  
6 optimized parameters index a distribution that is close (via a divergence or clas-  
7 sification metric) to a target distribution (such as a posterior or data distribution).  
8 Much less attention, however, has been paid to the problem of *learning a model*  
9 *itself*. Here we define implicit probabilistic models with specific deep network  
10 architecture and optimization procedures in order to learn intractable exponential  
11 family models (*not* a single distribution from those models). These exponential  
12 families, which are central to some of the most fundamental problems in probabilis-  
13 tic inference, are learned accurately and scalably, allowing operations like posterior  
14 inference to be executed directly and generically by an input choice of natural  
15 parameters, rather than performing inference via optimization for each particular  
16 realization of a distribution within that model. We demonstrate this ability across  
17 a number of non-conjugate exponential families that appear often in the machine  
18 learning literature.

## 1 Introduction

19  
20 Probability models, the fundamental object of Bayesian machine learning, have long challenged  
21 researchers with the tradeoff between tractability and expressivity. Though well understood that a  
22 model should be chosen to instantiate a set of assumptions and capture existing domain knowledge  
23 [1, 2, 3], for many years too-simple models were chosen for their practical advantages (such as  
24 conditional conjugacy), which left much to be desired in terms of expressive performance and  
25 scalability of these models.

26 More recently the pendulum has swung in the other direction, via models which map a latent random  
27 variable  $z \sim \mathbb{P}_0$  through a member of a highly expressive function family  $\mathcal{G} = \{g_\theta : \theta \in \Theta\}$ , the  
28 composition resulting in an *implicit probability model*  $\mathcal{M} = \{\mathbb{P}(g_\theta \circ z) : \theta \in \Theta\}$ . Choosing  $\mathcal{G}$  to be  
29 a parameter-indexed family of deep neural networks has both a rich history [4, 5, for example], and  
30 has recently been used to produce state-of-the-art results for density estimation [6, 7, 8], generation of  
31 complex data [9], variational inference [10, 11, 12], and more. A noted advantage of these models is  
32 that in many cases they make minimal assumptions about the data generative (or posterior inference)  
33 process. This blessing is of course also a curse: since these models have been chosen to be generic and  
34 flexible, they can lack the classic stipulation that a model instantiates existing domain knowledge. The  
35 downsides of a too-flexible model with finite data (albeit large) – and the corresponding bias-variance  
36 benefit of a restricted model – are textbook knowledge [13, §7.3], and work on generalization and  
37 compressibility in deep networks suggests that these function families  $\mathcal{G}$  are indeed quite large,  
38 perhaps larger than needs be [14].

Need to get to the meat now. Here we consider the problem of learning a model, not a distribution from within that model. Specifically, noting that many of our most common problems in probabilistic inference in fact have the form of an intractable exponential family [15], we treat the problem of learning models of the form.... most fundamental problem in probabilistic inference inference of a latent parameter  $z$  given some conditionally iid observations  $x_i|z$ .

Consider specifically the popular problem of variational inference with an approximate posterior model defined by a deep recognition (inference) network [4, 17, 10].

with deep neural networks forming the , where implicit probability models are as recognition inference for variational inference.

- while offering many advantages, two shortcomings: represent a potentially too-flexible model, and are used to find single posterior distributions (often on local variables).
- VI has to re-learn on every dataset; yes it can amortize across points from the same dataset, but not across datasets in the same model. Given the frequency of certain non-conjugate models appearing – hierarchies of Dirichlet distributions, log Gaussian Poisson models, etc – this seems needless to continue considering this as an “intractable” exp fam.

Here we learn an exp fam *model*:

- We investigate the problem of learning exp fams, not individual distributions. Inherent in all the above approaches is an algorithmic procedure to select a *single* distribution  $q_\theta(z)$  from among the *model*  $\mathcal{Q}$ . Implicit in this effort is the belief that  $\mathcal{Q}$  is suitably general to contain the true distribution of interest, or at least an adequately close approximation.
- Many models are exp fams, though intractable. [15]. It is worth revisiting whence that intractability arises, often just because hard work has not yet been put into deriving transformation samplers. Many intractable distributions encountered in machine learning belong to exponential families. In rare cases these distributions are tractable due to either known conjugacy in the problem setup (such as the normal-inverse-Wishart), or due to careful numerical work historically that has made these distributions computationally indistinguishable from tractable (eg the Dirichlet). [16]. not a known mapping from other simpler distributions (eg the Wishart via the Bartlett decomposition), an inversion, transformation-rejection algorithm, or similar custom numerical solution [16]. It is intriguing then to reflect upon the success that deep neural networks have offered to function approximation, and ask to what extent we can automate this numerical process, widening the class of effectively tractable exponential family distributions. Also we always sample from intractable families via some transformations [16]; the fact that some have known constructions (ratio of gammas, Bartlett decomposition, etc) should not distract from the fundamental nature of this process.
- We leverage old and new work recently much attention has been paid to bijective neural networks, networks that admit tractable density calculations. An old idea with new options.
- EFNs allow the embodiment of modeling assumptions without sacrificing expressivity
- concept here is to learn something we care about already and get the usual benefits of learning a restricted model space [13, §7, for example]
- EFNs include neural net observation models in many cases, so don’t despair. (like a VAE generator)

Mechanically:

- we parameterize a network whose input is the natural parameters of the exponential family being learned
- the output of this *parameter* network is the parameters  $\phi$  of a bijective neural network that allows density to be calculated.
- Can use this as an initializer if more specific training is required.

Our contributions include:

- novel architecture to learn a model, not a particular member

- stochastic optimization that samples over the model space: sampling both natural parameters (the family member to be learned) and data points (the observed density points)
- our choice of exp fam produces a linear regression type problem in KL divergence. We leverage the natural parameterization of exponential families to derive a novel objective that is amenable to stochastic optimization.
- empirical results confirming against ground truth in known “tractable” families like the Dirichlet, inverse Wishart, and Gaussian.
- empirical results demonstrating inference performance in common “intractable” families including the hierarchical dirichlet, the log Gaussian Poisson.
- Demonstration that there is surprisingly little performance loss training a single posterior vs an entire model, advocating its broader use, at least as an initializer if not as an amortizer. Here we offer what can be seen as a different sort of amortization, over datasets themselves. The exp fam may be challenging to learn, but then it can be used at trivial cost. We will focus more on the distinction with variational inference later. We use IPM vs generative to clarify that we are not simply dealing in the inference case, but the more general problem of learning probabilistic models (nor just single members of these models).
- offer insights into parameterizing existing VI and similar to increase performance (Fig 4)
- careful treatment distinguishing this from VI. The similarities to VI are clear.s

## 2 Exponential family networks

### 2.1 Implicit probability models via density networks

bants. defines a  $Q$ . *Why this is coherent*  $\Theta$  defines quite a big  $Q$ , and indeed the subject of compressibility, generalization, etc is of keen interest to many [14]. So actually the space of distributions is quite large, and in many cases certainly larger than it needs be. Why? Well, we know precisely the parameter space of the exponential family; it is defined by the *natural* parameters  $\eta \in \mathbb{R}^p$  (or whatever we choose there).

Density networks are an old idea [5], as are neural networks to fit a probability model to data [4, 17].

We choose flow networks [18]. And “implicit generative models aka density networks” (or rather, density networks are the instantiation of an IGM with deep nets, which is effectively synonymous these days. And invertible networks In that vein probably definitely cite invertible/bijective deep nets in general [19, 7, 6, 18, 20, 8, 21]. Note that what norm flows [18] did is make it tractable and scalable and in the modern VAE style, and even that is probably overstating the case. That makes these comparisons legitimate and apples to apples. Gaussianization is an old idea that this is basically the inverse of [22]; same idea in more depth and that argues for the normal prior in [23]. Really the norm flow is not so special as this is a well established classic idea.

More generally there has been a lot of attention to making these more flexible in structured variational inference. Any generalization of this is also dandy though, so could use a mean field approach (standard) or any of the things that go beyond mean field, either classically [24, 25]; this is called structured variational inference or newer stuff [26] [27], to name but a few.

### 2.2 Exponential families

bants. Pitman-Koopman Lemma [28, §3.3.3] Defines an  $M$ .

*Why this is important.* Exp fams are awesome and fundamental. Also [15] rightly point out that many many inference problems can be cast as exponential families. Can we cast the VAE encoder network as a suitable exp fam... sure I think that’s right; the network parameters of  $z$  form the statistics, and then the observations are  $\eta$ ’s.

Common examples in the ML community include hierarchical Dirichlet and log Gaussian Poisson.

Note briefly that one common model that this does not conveniently include is local latent variable models like LDA and logistic regression, as they define larger and larger exp fams as they go (yes they are exp fams, but not of a fixed parameterization under sampling).

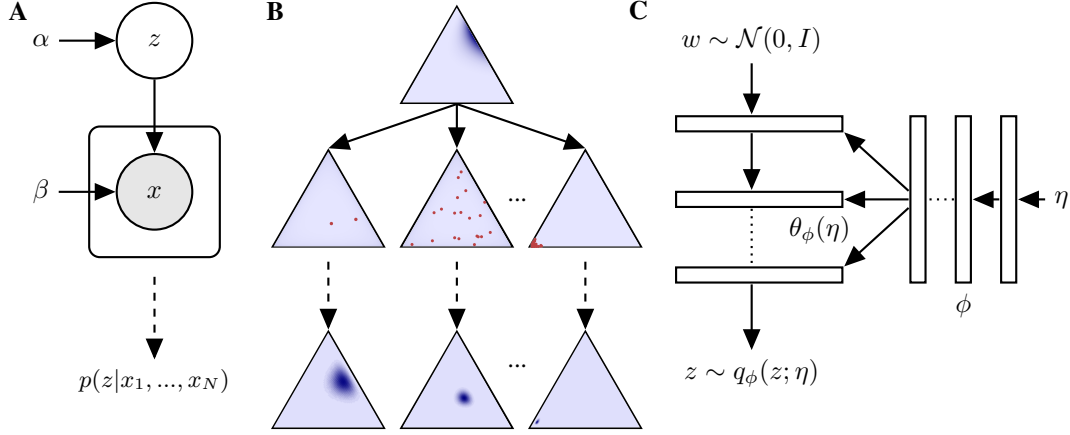


Figure 1: Learning exponential families. A shows the graphical model, emphasizing conditional iid sampling. B shows Dirichlet prior (a density), conditional Dirichlet observations (some observed points in the simplex), and then the posteriors learned by an EFN. SRB to fill in these triangles. C shows the EFN network schematic.

136 Note somewhere that the natural parameter space needs to be considered in general. That is, not all  $\eta$   
 137 lead to a valid distribution (standard fact, see for example [15]). In practice that's not often a problem,  
 138 as the space is known for most distributions one uses, and when one composes them in a posterior  
 139 scheme (for example), this is inherited (eg the normal covariance...). So we skip that here. But yes in  
 140 general that needs to be considered.

### 141 2.3 Exponential family networks

142 includes the network definition of Fig 1c, the objective, and the optimization algorithm.

143 This should not be confused with "Learning to learn by gradient descent by gradient descent" [29]

144 Another related work is that this is somehow the dual of MEFN [30], or a generalization of the dual  
 145 problem. In the wainwright and jordan sense of forward and backward mappings. Stuff on sampling  
 146 from Gibbs distributions (max ent models), and sampling from exp fams generally, with MCMC and  
 147 such.

148 Note that this objective can also produce approximations of the log partition, via essentially linear  
 149 regression; more nuanced schemes are recommended [31]. We don't explore that here.

### 150 2.4 Relation to variational inference

151 We have already covered related work; here we scrutinize EFNs in terms of VI.

152 We are interested in perhaps the most classic inference problem:

$$p(z|x) \propto p(z) \prod_{i=1}^n p(x_i|z)$$

153 shown with the attached plate model (not local latents). Supposing as is often the case that the  
 154 likelihood is a member of the  $s$  exp fam, we have:

$$p(z|x) \propto \exp \left\{ \left[ \sum_{i=1}^n s(x_i) \right]^\top [t(z)] + g_0(\alpha, z) \right\}$$

Important to distinguish carefully from VI. In a sense VI does parameterize a family: given data, you get local variational parameters and that parameterizes a density (like a regular VAE). Inference networks are exclusively used to data to amortize with a global set of parameters a variational distribution, not a model. Of course it is in a sense a model, but that's a bunch of normals. The sampling mechanism is easy (Gaussian).

where the natural parameters of the sampling distribution are indexed by the latent parameter on which we want to inference ( $z$ ). Here I've written the prior as arbitrary, and possibly not exp fam, which is fine, since this is still an exp fam in the sense of, for a fixed  $\alpha$ , the function  $g_0$  can just be viewed as a sufficient statistic. Even if  $\alpha$  is not fixed though, we can sample over that too to learn the whole fam (but maybe not if we want to infer it?). Regardless, life is simpler to make sense of if we take an exp fam prior  $g_0(\alpha, z) = \alpha^\top t_0(z)$ , and then the desired posterior is an intractable exp fam, but still just an exp fam.

Note: consider changing all  $z$  to  $\theta$  to remind the average reader that we're doing real bayesian inference and not just run of the mill VI with local latents in a nonlinear dimension reduction setting. Perhaps an important reminder that most all of VAE and such are for inference of local latents, and that's a little bit too bad. We fix that.

Another key idea that EFNs enable is to ask if learning the  $\theta(\eta)$  network leads to better VI in terms of inference networks, since it is apparently appropriately regularized and can just take suff stats. That's testable if we have time.

In a restricted technical sense, rather close: VAE and other black box VI that uses reparameterization results in a conditional density  $q_\phi(z|x)$ . If we consider  $\eta$  as  $x$ , then sure yes the previous stuff specifies a model  $\mathcal{Q}_{VAE} = \{q_\phi(z|x) : x \in X\}$ . But that's a little silly, and any way that is very often a normal family with variational parameters specified by (a deep function of)  $x$ . Much closer is Figure 2 in Rezende and Mohamed, where like here they use a network to index the *parameters* of the normalizing flow. In that case it's a function of  $x$  the observation, and as such that network is an inference network; here it's a function of  $\eta$  and as such is a parameter network. That's just nomenclature, so naturally the next question is do they differ at some other level. Yes, distinctly. The other term implied in a VI (or norm flow VAE style as they use) is the expected log joint  $E_{q_\phi(x)}(\log p_\theta(x, z))$ . Now sure that's a loss function on  $x, z$ , so then when we look at that same term in EFN we see  $E_{q_\phi(\eta)}(\eta^\top t(z))$ , which sure also looks like a loss function on  $\eta, z$ . And yes, they are both unnormalized (in the sense that VI is an ELBO / joint  $p(x, z)$  and EFN lacks the normalizer because it's constant, so we're not getting a KL estimate). A picky difference is that the exp family doesn't really correspond to a proper unnormalized log joint (though I suppose it could), as there is not a prior on  $\eta$  in the objective (but is that just ignoring  $p(\eta)$  in our sampling scheme?). But yes if we want to be reductionist and pedantic [use nicer words] in general we could see this as a specific case where  $x = \eta$  and thus we are learning a family just as in the inference case. Or rather, we are putting the data in as sufficient stat (computation of natural parameters), but that's nonobvious. And for example we are giving in the bayesian logistic regression example full datasets for inference instead of single data points. To make this as close as possible, we write  $p(\eta|z) = \frac{1}{A(t(z))} \exp\{\eta^\top t(z)\}$ . That's the "likelihood" of an EFN in some wonky sense. So this reveals the mechanical differences: first,  $t(z)$  is not a deep generative model with parameters  $\theta$ , but rather it is a fixed set of sufficient statistics that define the exp fam. Next, there is no clear prior  $p(z)$ , which is critical to understanding how VI behaves (see Hoffman and Johnson ELBO surgery paper, also Duvenaud's <https://arxiv.org/pdf/1801.03558.pdf>). So yes there is a hand wavy sense in which EFN is a specific case of norm flow, but of course it is. And anyway norm flow is a specific case of a DNN architecture or Helmholtz machine or deep density network (Ripple and Adams). This is just rambling but good to have all perspective here. Ok so what to do? First, then we need to produce really compelling results focusing on when learning an exp fam is key. Second we need some very tight language to draw this distinction without seeming a small tweak on normalizing flows. One way to do this is the restricted model class argument, a la Fig 7.2 in Hastie and Tibshirani. Another is to actually produce a conditional exp fam, as in something indexed on both  $x$  and  $\eta$ . Third, possible novelties in norm flows, like triple spinners or other better choices than planar flows (yuck).

Another point is that it's unknown if posterior contraction can be well modeled. As in, we know that most VI NF type things are conditioned on a single data point, so the posterior variance can tend to be rather homogenous. One more contribution is to offer that contraction study; as we get more data points we will get more posterior contraction, so this tests the ability of this model to learn that.

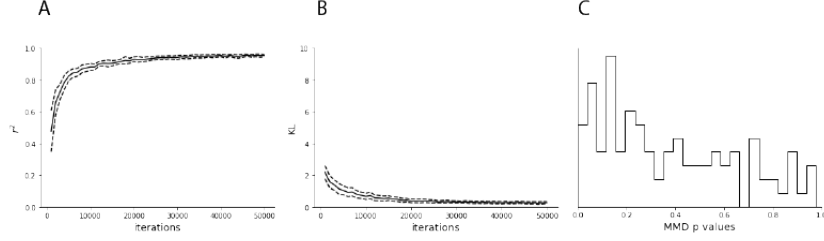


Figure 2: 25-dimensional Dirichlet exponential family network. A.) Distribution of  $r^2$  between the sufficient statistics and log-probability across choices of  $\eta$  throughout optimization. B.) Distribution of KL divergence across choices of  $\eta$  throughout optimization. C.) Distribution of maximum mean discrepancy p-values between EFN samples and ground truth after optimization [34].

211 Key distinctions:

- 212 • narrow mechanical sense this is VI with an observation of the natural parameters, namely
- 213 the sample exp fam over all data. but that’s pedantic.
- 214 • no generative model in the usual sense: yes, we can consider a prior and then some
- 215 observation model as the genrative model, but in any event it’s not a neural net.
- 216 • we lack a finite data set  $X$ , so the objective is technically different. We stipulate a distribution
- 217 and then this is expectation over that model space, a KL or a KL to the broader joint with  $\eta$ .
- 218 This is concretely different, as we typically use a fixed size dataset  $X$  so we can calculate
- 219 the ELBO over the

220 Latest key distinctions:

- 221 • prior is in parameter network, unlike essentially all others, even if you take a narrow view
- 222 that  $\sum_i t(x_i)$  is a single data point. Prior has been recognized for mattering in the ELBO,
- 223 though this sentence is a dubious distinction [32, 33] (dubious need for these refs)
- 224 • data is given via an assumption of sufficiency, namely in natural form [28], not in  $x$  form. Of
- 225 course this is sensible as in some settings we don’t know the natural form of the generative
- 226 model, but that’s a key difference with SVI; plenty of those models are not deep nets (and
- 227 shouldn’t be, if there is an intent of statistical inference rather than nonlinear dimensionality
- 228 reduction / autoencoding ) and there we *do* know the natural parameters.

## 229 3 Results

230 Introductory remarks and then comments about architectural particulars, including planar flow  
 231 networks of [18]. Note Number of panar flows is always  $D$  (intrinsic dimensionality of flows), units  
 232 per layer ramping is always the same function of  $D$ . The number of layers in the theta network is  
 233 always a function of  $D$  - will probably just always use 8 layers. Remember

234 NF1: do full norm flow “variational inference” (explore all of  $\phi$  space with the full flow network  
 235 model  $\mathcal{Q}$ ), which is to say  $\arg \min_{\phi} KL(q_{\phi}||p)$ .

236 EFN1: be literal to Figure 1C, give the sufficient statistics of that  $K=1$  dataset, and learn an EFN from  
 237 scratch. This alternative is important because it is the most specific (but kind of annoying, hence  
 238 alternative 1) interpretation of norm flow VI paper.

### 239 3.1 Tractable exponential families

### 240 3.2 Intractable exponential families

241 **Hierarchical Dirichlets** Hierarchical dirichlets are useful and have some history; most notable is  
 242 with the Hierarchical Dirichlet Process [35], but historically this was done in the finite case also [36].  
 243 Here is some math. Note that this matters for multi-corpus LDA generally as well [37, 38].

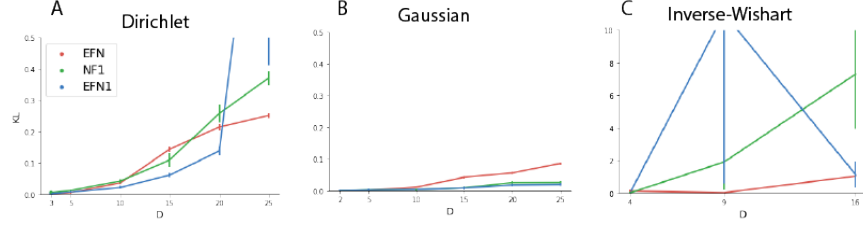


Figure 3: Scaling exponential family networks. A.) Dirichlet. B.) Gaussian C.) Inverse-Wishart

**Truncated- and log-normal Poisson** used a lot [39][40][41, 42]

Figure 4:

EFN in intractable exp fams (connecting to above, but with hard distribs and the ELBO)

Panel A: Dir-Dir ELBO by dimensionality for NF1 and EFN and EFN1

Panel B: Dir-Dir ELBO by dimensionality for EFN1 vs EFN1a vs 1b vs 1c vs NF1 (with  $N = 1$  data point)

### 3.3 Neural spike train analysis

Figure 5:

Panel A TNP picture example of prior and posterior with a few samples, just for feel good

PANEL B: ELBO on held out data as a function of  $R$ , for a middle choice of training dataset size  $N$  and  $D$ .

PANEL C: ELBO on held out data as a function of  $N$ , for a middle choice of number of samples in the posterior  $R$ .

PANEL D (optional): (ELBO EFN - ELBO NF1) as a surface plot as a function of  $R, N$ . That is, positive places is where EFN outperforms, negative NF1.

The key point with these is that, while you have the *same exact* flow network architecture, now you have to optimize over  $\phi$  with a limited single dataset. Learning a restricted model space is good for the bias-variance tradeoff! Do this many times so that variance will become clear.

—other thoughts— Real data analysis and posterior inference. **Key real data result on TNP.**

Get some data from CRCNS that has many spike trains  $x_i$  for  $i = 1, \dots, N$  (ask Gabriel, as he has done some poking around recently; or look at some of the above TNP/LNP refs).

Those spike trains should be conditionally independent draws from the same underlying intensity function  $z$ . (for example, trials under the same stimulus)

Bin the length of time  $T$  into  $\approx 20 - 30$  equally spaced time bins. Thus  $z$  is now a vector in  $\mathbb{R}^{20}$ .

Now each spike train  $x_i$  is a conditionally independent Poisson vector observation, with rate vector  $z$ . Learn the 20 dimensional TNP exp fam, without any regard to this dataset  $X$ .

No: Panel No: TNP ELBO by dimensionality for NF1 and EFN and EFN1

Panel A TNP picture example of prior and posterior with a few samples, just for feel good

**Now we want to learn the posterior  $p(z | \text{some fixed number } R \text{ of data points})$ .**

To do this for an EFN, just plug in those  $R$  points  $x_{i_1}, \dots, x_{i_R}$  and the prior as a natural parameter, and job done.

To do this for an NF1, train a VI model by taking the log joint with  $R$  data points, then go through and resample  $R$  points every time from your training dataset with  $N$  data points.

**PANEL A: ELBO on held out data as a function of  $R$ , for a middle choice of training dataset size  $N$ .**

**PANEL B: ELBO on held out data as a function of  $N$ , for a middle choice of number of samples in the posterior  $R$ .**

**PANEL C: (ELBO EFN - ELBO NF1) as a surface plot as a function of  $R, N$ . That is, positive places is where EFN outperforms, negative NF1.**

The key point with these is that, while you have the *same exact* flow network architecture, now you have to optimize over  $\phi$  with a limited single dataset. Learning a restricted model space is good

288 for the bias-variance tradeoff! Do this many times so that variance will become clear. **Panel C v2:**  
289 **Possibly want to explicitly plot variance of EFN and NF1 to focus on the variance tradeoff**  
290 **Panel C v3: change time bin granularity from 10 to 50 to show how this story changes in  $D$ .**  
291 **My thought is that all will be exhausted by dimensionality sweeps by this point, so no.**  
292 also Notice one pain here is that these panels requires training a new EFN1 at every choice of  $N$  and  
293  $R$  (but only one EFN). Sorry.

294  
295 We hope and expect this will show that when the dataset gets small, this "traditional VI" will get  
296 arbitrarily bad (can't learn a network); eventually, there will be so much data that the VI will match  
297 or outperform the EFN... outperform because VI can focus specifically on this distribution rather than  
298 over the whole family, so the EFN has less effective data for this  $\eta$  (but not because it has a broader  
299 range of models, since we believe the EFN contains the closest member). Performance metric should  
300 be ELBO on some held out data or something like that (it's a posterior, so log likelihood doesn't  
301 really make sense). Test data anyway. Check VI papers for usual metrics. A key point to make  
302 here is that one great virtue of EFNs is learning a restricted model, which should demonstrate the  
303 usual bias-variance tradeoff (see for example Hastie and Tibshirani book, Fig 7.2). Or Figure 4 is  
304 bias-variance and some sample posteriors in 2-d (showing how nicely it works), and then Fig 5 is the  
305 above performance, with both train and test.

306 This will be for one real example  $X$ . As such, to get error bars, just take a big dataset and randomly  
307 subsample the test set. Then the posterior performance is really for that very dataset, so the sem is  
308 coherent and the right thing to calculate/show. Important to clarify that doing so *does not* test how  
309 well this does across the entire exp fam, but just this one posterior. ((To test that, we would do it in  
310 simulation: generate *many datasets*  $X$ , then do the above for every one of them. Same computation  
311 for EFN (since its just plugging in a dataset), but VI alternatives 1 and 2 now need to be rerun for  
312 every dataset. And it's still simulated data, not really offering something fundamentally more than Fig  
313 3 (well ok it's an intractable model, but I'm not sure that offers so much)...let's skip that altogether)).

## 314 4 Conclusion

315 Snappy closing remarks!



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410 *processing systems*, pages 329–336, 2008.

## 411 5 Appendix

412 Exponential form of posterior for Dirichlet-Dirichlet

413  $\mathbf{z} \sim \text{Dir}(\boldsymbol{\alpha}_0)$

414  $\mathbf{x}_i \sim \text{Dir}(\beta \mathbf{z})$

415  $p(\mathbf{z}) \propto \exp(\boldsymbol{\alpha}_0^T \log(\mathbf{z}) - \sum_{d=1}^D \log(z_d))$

416  $p(\mathbf{x}_i | \mathbf{z}) \propto \exp(\beta \mathbf{z}^T \log(\mathbf{x}_i) - \sum_{d=1}^D \log(x_{i,d}) - (\sum_{d=1}^D \log(\Gamma(\beta z_d)) - \log(\Gamma(\beta \sum_{d=1}^D z_d))))$

417  $p(X | \mathbf{z}) \propto \exp(\beta \mathbf{z}^T [\sum_{i=1}^N \log(\mathbf{x}_i)] - \sum_{i,d=1}^{N,D} \log(x_{i,d}) - N(\sum_{d=1}^D \log(\Gamma(\beta z_d)) - \log(\Gamma(\beta \sum_{d=1}^D z_d))))$

418

419  $p(\mathbf{z} | X) \propto p(\mathbf{z})p(X | \mathbf{z})$

420  $\propto \exp(\boldsymbol{\alpha}_0^T \log(\mathbf{z}) - \sum_{d=1}^D \log(z_d))$

421  $\exp(\beta \mathbf{z}^T [\sum_{i=1}^N \log(\mathbf{x}_i)] - \sum_{i,d=1}^{N,D} \log(x_{i,d}) - N(\sum_{d=1}^D \log(\Gamma(\beta z_d)) - \log(\Gamma(\beta \sum_{d=1}^D z_d))))$

422 We don't care about the term that just has  $x$  in it.

423  $p(\mathbf{z} | X) \propto \exp(\boldsymbol{\alpha}_0^T \log(\mathbf{z}) + \beta [\sum_{i=1}^N \log(\mathbf{x}_i)]^T \mathbf{z} - \sum_{d=1}^D \log(z_d) - N(\sum_{d=1}^D \log(\Gamma(\beta z_d)) - \log(\Gamma(\beta \sum_{d=1}^D z_d))))$

424  $p(\mathbf{z} | X) \propto \exp\left(\begin{pmatrix} \boldsymbol{\alpha}_0 - \mathbf{1} \\ \sum_{i=1}^N \log(\mathbf{x}_i) \\ -N\mathbf{1} \\ -N \end{pmatrix}^T \begin{pmatrix} \log(\mathbf{z}) \\ \beta \mathbf{z} \\ \log(\Gamma(\beta \mathbf{z})) \\ \log(\Gamma(\beta \sum_{d=1}^D z_d)) \end{pmatrix}\right)$

425 This seems right to me. I moved  $\beta$  for the second element of the natural parameters to be over with

426 his other  $\beta$ -friends in the sufficient statistics.

427 Here's a more cleaned up version:

$$p(\mathbf{z} | X) \propto \exp\left\{\left[\begin{pmatrix} \boldsymbol{\alpha}_0 - \mathbf{1} \\ \sum_{i=1}^N \log(\mathbf{x}_i) \\ -N\mathbf{1} \\ -N \end{pmatrix}\right]^\top \left[\begin{pmatrix} \log(\mathbf{z}) \\ \beta \mathbf{z} \\ \log(\Gamma(\beta \mathbf{z})) \\ \log(\Gamma(\beta \mathbf{1}^\top \mathbf{z})) \end{pmatrix}\right]\right\} \triangleq \exp\{\boldsymbol{\eta}^\top t(\mathbf{z})\}$$

428 or just using the Beta function:

$$p(\mathbf{z} | X) \propto \exp\left\{\left[\begin{pmatrix} \boldsymbol{\alpha}_0 - \mathbf{1} \\ \sum_{i=1}^N \log(\mathbf{x}_i) \\ -N \end{pmatrix}\right]^\top \left[\begin{pmatrix} \log(\mathbf{z}) \\ \beta \mathbf{z} \\ \log(B(\beta \mathbf{z})) \end{pmatrix}\right]\right\} \triangleq \exp\{\boldsymbol{\eta}^\top t(\mathbf{z})\}$$