

Interrogating theoretical models of neural computation with deep inference
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¹ 1 Abstract

² A cornerstone of theoretical neuroscience is the circuit model: a system of equations that captures
³ a hypothesized neural mechanism. Such models are valuable when they give rise to an experimen-
⁴ tally observed phenomenon – whether behavioral or in terms of neural activity – and thus can
⁵ offer insights into neural computation. The operation of these circuits, like all models, critically
⁶ depends on the choices of model parameters. Historically, the gold standard has been to analyt-
⁷ ically derive the relationship between model parameters and computational properties. However,
⁸ this enterprise quickly becomes infeasible as biologically realistic constraints are included into the
⁹ model increasing its complexity, often resulting in *ad hoc* approaches to understanding the relation-
¹⁰ ship between model and computation. We bring recent machine learning techniques – the use of
¹¹ deep generative models for probabilistic inference – to bear on this problem, learning distributions
¹² of parameters that produce the specified properties of computation. Importantly, the techniques
¹³ we introduce offer a principled means to understand the implications of model parameter choices
¹⁴ on computational properties of interest. We motivate this methodology with a worked example
¹⁵ analyzing sensitivity in the stomatogastric ganglion. We then use it to go beyond linear theory
¹⁶ of neuron-type input-responsivity in a model of primary visual cortex, gain a mechanistic under-
¹⁷ standing of rapid task switching in superior colliculus models, and attribute error to connectivity
¹⁸ properties in recurrent neural networks solving a simple mathematical task. More generally, this
¹⁹ work suggests a departure from realism vs tractability considerations, towards the use of modern
²⁰ machine learning for sophisticated interrogation of biologically relevant models.

21 2 Introduction

22 The fundamental practice of theoretical neuroscience is to use a mathematical model to understand
23 neural computation, whether that computation enables perception, action, or some intermediate
24 processing [1]. A neural computation is systematized with a set of equations – the model – and
25 these equations are motivated by biophysics, neurophysiology, and other conceptual considerations.
26 The function of this system is governed by the choice of model parameters, which when configured
27 in a particular way, give rise to a measurable signature of a computation. The work of analyzing a
28 model then requires solving the inverse problem: given a computation of interest, how can we reason
29 about these particular parameter configurations? The inverse problem is crucial for reasoning about
30 likely parameter values, uniquenesses and degeneracies, attractor states and phase transitions, and
31 predictions made by the model.

32 Consider the idealized practice: one carefully designs a model and analytically derives how model
33 parameters govern the computation. Seminal examples of this gold standard (which often adopt
34 approaches from statistical physics) include our field’s understanding of memory capacity in asso-
35 ciative neural networks [2], chaos and autocorrelation timescales in random neural networks [3],
36 the paradoxical effect [4], and decision making [5]. Unfortunately, as circuit models include more
37 biological realism, theory via analytical derivation becomes intractable. This creates an unfavor-
38 able tradeoff. On the one hand, one may tractably analyze systems of equations with unrealistic
39 assumptions (for example symmetry or gaussianity), mathematically formalizing how parameters
40 affect computation in a too-simple model. On the other hand, one may choose a more biologically
41 accurate, scientifically relevant model at the cost of *ad hoc* approaches to analysis (such as sim-
42 ply examining simulated activity), potentially resulting in bad inference of parameters and thus
43 erroneous scientific predictions or conclusions.

44 Of course, this same tradeoff has been confronted in many scientific fields characterized by the
45 need to do inference in complex models. In response, the machine learning community has made
46 remarkable progress in recent years, via the use of deep neural networks as a powerful inference
47 engine: a flexible function family that can map observed phenomena (in this case the measurable
48 signal of some computation) back to probability distributions quantifying the likely parameter
49 configurations. One celebrated example of this approach from machine learning, of which we
50 draw key inspiration for this work, is the variational autoencoder [6, 7], which uses a deep neural
51 network to induce an (approximate) posterior distribution on hidden variables in a latent variable

52 model, given data. Indeed, these tools have been used to great success in neuroscience as well,
53 in particular for interrogating parameters (sometimes treated as hidden states) in models of both
54 cortical population activity [8, 9, 10, 11] and animal behavior [12, 13, 14]. These works have used
55 deep neural networks to expand the expressivity and accuracy of statistical models of neural data
56 [15].

57 However, these inference tools have not significantly influenced the study of theoretical neuroscience
58 models, for at least three reasons. First, at a practical level, the nonlinearities and dynamics of
59 many theoretical models are such that conventional inference tools typically produce a narrow
60 set of insights into these models. Indeed, only in the last few years has deep learning research
61 advanced to a point of relevance to this class of problem. Second, the object of interest from a
62 theoretical model is not typically data itself, but rather a qualitative phenomenon – inspection of
63 model behavior, or better, a measurable signature of some computation – an *emergent property* of
64 the model. Third, because theoreticians work carefully to construct a model that has biological
65 relevance, such a model as a result often does not fit cleanly into the framing of a statistical model.
66 Technically, because many such models stipulate a noisy system of differential equations that can
67 only be sampled or realized through forward simulation, they lack the explicit likelihood and priors
68 central to the probabilistic modeling toolkit.

69 To address these three challenges, we developed an inference methodology – ‘emergent property
70 inference’ – which learns a distribution over parameter configurations in a theoretical model. This
71 distribution has two critical properties: *(i)* it is chosen such that draws from the distribution (pa-
72 rameter configurations) correspond to systems of equations that give rise to a specified emergent
73 property (a set of constraints); and *(ii)* it is chosen to have maximum entropy given those con-
74 straints, such that we identify all likely parameters and can use the distribution to reason about
75 parametric sensitivity and degeneracies [16]. First, we stipulate a bijective deep neural network that
76 induces a flexible family of probability distributions over model parameterizations with a probabil-
77 ity density we can calculate [17, 18, 19]. Second, we quantify the notion of emergent properties as a
78 set of moment constraints on datasets generated by the model. Thus, an emergent property is not a
79 single data realization, but a phenomenon or a feature of the model, which is ultimately the object
80 of interest in theoretical neuroscience. Conditioning on an emergent property requires a variant of
81 deep probabilistic inference methods, which we have previously introduced [20]. Third, because we
82 cannot assume the theoretical model has explicit likelihood on data or the emergent property of
83 interest, we use stochastic gradient techniques in the spirit of likelihood free variational inference

[21]. Taken together, emergent property inference (EPI) provides a methodology for inferring parameter configurations consistent with a particular emergent phenomena in theoretical models. We use a classic example of parametric degeneracy in a biological system, the stomatogastric ganglion [22], to motivate and clarify the technical details of EPI.

Equipped with this methodology, we then investigated three models of current importance in theoretical neuroscience. These models were chosen to demonstrate generality through ranges of biological realism (from conductance-based biophysics to recurrent neural networks), neural system function (from pattern generation to abstract cognitive function), and network scale (from four to infinite neurons). First, we use EPI to produce a set of verifiable hypotheses of input-responsivity in a four neuron-type dynamical model of primary visual cortex; we then validate these hypotheses in the model. Second, we demonstrated how the systematic application of EPI to levels of task performance can generate experimentally testable hypotheses regarding connectivity in superior colliculus. Third, we use EPI to uncover the sources of error in a low-rank recurrent neural network executing a simple mathematical task. The novel scientific insights offered by EPI contextualize and clarify the previous studies exploring these models [23, 24, 25, 26], and more generally, these results point to the value of deep inference for the interrogation of biologically relevant models.

3 Results

3.1 Motivating emergent property inference of theoretical models

Consideration of the typical workflow of theoretical modeling clarifies the need for emergent property inference. First, one designs or chooses an existing model that, it is hypothesized, captures the computation of interest. To ground this process in a well-known example, consider the stomatogastric ganglion (STG) of crustaceans, a small neural circuit which generates multiple rhythmic muscle activation patterns for digestion [27]. Despite full knowledge of STG connectivity and a precise characterization of its rhythmic pattern generation, biophysical models of the STG have complicated relationships between circuit parameters and neural activity [22, 28]. A model of the STG [23] is shown schematically in Figure 1A, and note that the behavior of this model will be critically dependent on its parameterization – the choices of conductance parameters $z = [g_{el}, g_{synA}]$. Specifically, the two fast neurons ($f1$ and $f2$) mutually inhibit one another, and oscillate at a faster frequency than the mutually inhibiting slow neurons ($s1$ and $s2$). The hub neuron (hub) couples with either the fast or slow population or both.

114 Second, once the model is selected, one defines the emergent property, the measurable signal of
115 scientific interest. To continue our running STG example, one such emergent property is the
116 phenomenon of *network syncing* – in certain parameter regimes, the frequency of the hub neuron
117 matches that of the fast and slow populations at an intermediate frequency. This emergent property
118 is shown in Figure 1A at a frequency of 0.53Hz.

119 Third, qualitative parameter analysis ensues: since precise mathematical analysis is intractable in
120 this model, a brute force sweep of parameters is done [23]. Subsequently, a qualitative description
121 is formulated to describe the different parameter configurations that lead to the emergent property.
122 In this last step lies the opportunity for a precise quantification of the emergent property as a
123 statistical feature of the model. Once we have such a methodology, we can infer a probability
124 distribution over parameter configurations that produce this emergent property.

125 Before presenting technical details (in the following section), let us understand emergent property
126 inference schematically: EPI (Fig. 1A gray box) takes, as input, the model and the specified
127 emergent property, and as its output, produces the parameter distribution shown in Figure 1B.
128 This distribution – represented for clarity as samples from the distribution – is then a scientifically
129 meaningful and mathematically tractable object. In the STG model, this distribution can be
130 specifically queried to reveal the prototypical parameter configuration for network syncing (the
131 mode; Figure 1B yellow star), and how network syncing decays based on changes away from the
132 mode. The eigenvectors (of the Hessian of the distribution at the mode) quantitatively formalize
133 the robustness of network syncing (Fig. 1B solid (v_1) and dashed (v_2) black arrows). Indeed,
134 samples equidistant from the mode along these EPI-identified dimensions of sensitivity (v_1) and
135 degeneracy (v_2) agree with error contours (Fig. 1B, contours) and have diminished or preserved
136 network syncing, respectively (Figure 1B inset and activity traces) (see Section 5.2.1).

137 3.2 A deep generative modeling approach to emergent property inference

138 Emergent property inference (EPI) systematizes the three-step procedure of the previous section.
139 First, we consider the model as a coupled set of differential (and potentially stochastic) equations
140 [23]. In the running STG example, the model activity $x = [x_{f1}, x_{f2}, x_{hub}, x_{s1}, x_{s2}]$ is the membrane
141 potential for each neuron, which evolves according to the biophysical conductance-based equation:

$$C_m \frac{dx}{dt} = -h(x; z) = -[h_{leak}(x; z) + h_{Ca}(x; z) + h_K(x; z) + h_{hyp}(x; z) + h_{elec}(x; z) + h_{syn}(x; z)] \quad (1)$$

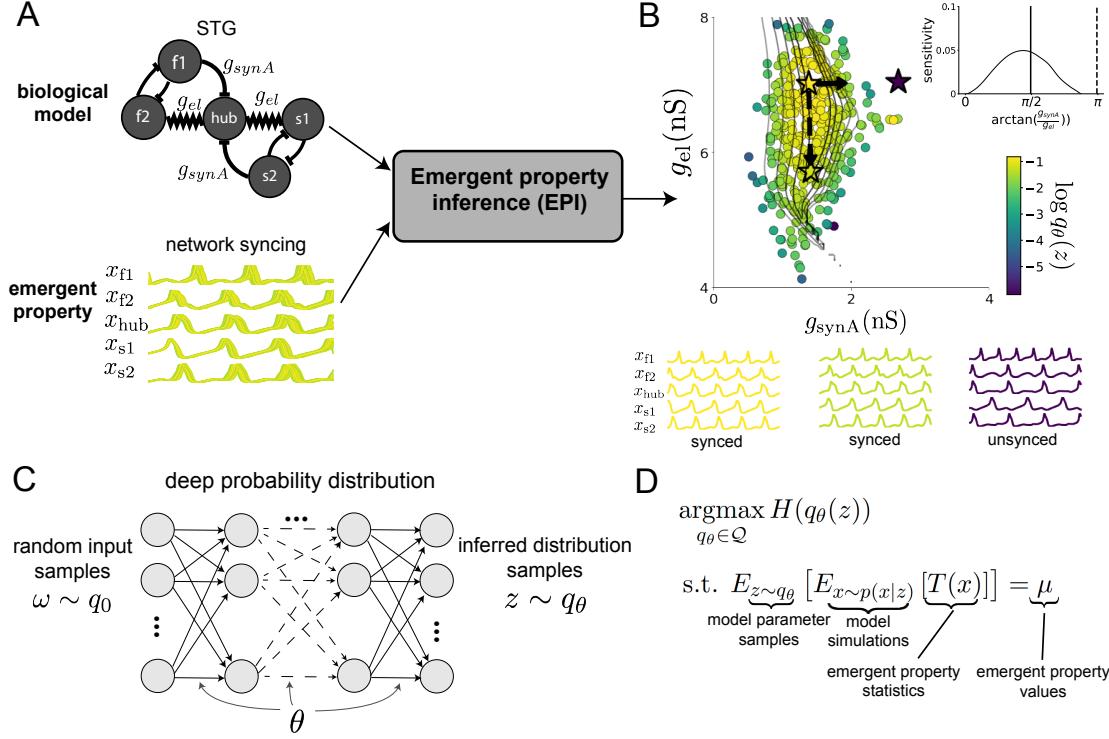


Figure 1: Emergent property inference (EPI) in the stomatogastric ganglion. A. For a choice of model (STG) and emergent property (network syncing), emergent property inference (EPI, gray box) learns a distribution of the model parameters $z = [g_{el}, g_{synA}]$ producing network syncing. In the STG model, jagged connections indicate electrical coupling having electrical conductance g_{el} . Other connections in the diagram are inhibitory synaptic projections having strength g_{synA} onto the hub neuron, and $g_{synB} = 5\text{nS}$ for mutual inhibitory connections. Network syncing traces are colored by log probability density of their generating parameters (stars) in the EPI-inferred distribution. B. The EPI distribution of STG model parameters producing network syncing. Samples are colored by log probability density. Distribution contours of emergent property value error are shown at levels of 2.5×10^{-5} , 5×10^{-5} , 1×10^{-4} , 2×10^{-4} , and 4×10^{-4} (dark to light gray). Eigenvectors of the Hessian at the mode of the inferred distribution are indicated as v_1 (solid) and v_2 (dashed) with lengths scaled by the square root of the absolute value of their eigenvalues. Simulated activity is shown for three samples (stars). (Inset) Sensitivity of the system with respect to network syncing along all dimensions of parameter space away from the mode. v_1 is sensitive to network syncing ($p < 10^{-4}$), while v_2 is not ($p = 0.67$) (see Section 5.2.1). C. Deep probability distributions map a latent random variable w through a deep neural network with weights and biases θ to parameters $z = f_\theta(w)$ distributed as $q_\theta(z)$. D. EPI optimization: To learn the EPI distribution $q_\theta(z)$ of model parameters that produce an emergent property, the emergent property statistics $T(x)$ are set in expectation over model parameter samples $z \sim q_\theta(z)$ and model simulations $x \sim p(x | z)$ to emergent property values μ .

142 where $C_m=1\text{nF}$, and h_{leak} , h_{Ca} , h_K , h_{hyp} , h_{elec} , and h_{syn} are the leak, calcium, potassium, hyper-
 143 polarization, electrical, and synaptic currents, all of which have their own complicated dependence
 144 on x and $z = [g_{\text{el}}, g_{\text{synA}}]$ (see Section 5.2.1).

145 Second, we define the emergent property, which as above is network syncing: oscillation of the
 146 entire population at an intermediate frequency of our choosing (Figure 1A bottom). Quantifying
 147 this phenomenon is straightforward: we define network syncing to be that each neuron’s spiking
 148 frequency – denoted $\omega_{\text{f1}}(x)$, $\omega_{\text{f2}}(x)$, etc. – is close to an intermediate frequency of 0.53Hz. Math-
 149 ematically, we achieve this via constraints on the mean and variance of $\omega_{\alpha}(x)$ for each neuron
 150 $\alpha \in \{\text{f1}, \text{f2}, \text{hub}, \text{s1}, \text{s2}\}$:

$$\mathbb{E}[T(x)] \triangleq \mathbb{E} \begin{bmatrix} \omega_{\text{f1}}(x) \\ \vdots \\ (\omega_{\text{f1}}(x) - 0.53)^2 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0.53 \\ \vdots \\ 0.025^2 \\ \vdots \end{bmatrix} \triangleq \mu, \quad (2)$$

151 which completes the quantification of the emergent property.

152 Third, we perform emergent property inference: we find a distribution over parameter configura-
 153 tions z , and insist that samples from this distribution produce the emergent property; in other
 154 words, they obey the constraints introduced in Equation 2. This distribution will be chosen from
 155 a family of probability distributions $\mathcal{Q} = \{q_{\theta}(z) : \theta \in \Theta\}$, defined by a deep generative distribution
 156 of the normalizing flow class [17, 18, 19] – neural networks which transform a simple distribution
 157 into a suitably complicated distribution (as is needed here). This deep distribution is represented
 158 in Figure 1C (see Section 5.1). Then, mathematically, we must solve the following optimization
 159 program:

$$\begin{aligned} & \underset{q_{\theta} \in \mathcal{Q}}{\operatorname{argmax}} H(q_{\theta}(z)) \\ & \text{s.t. } \mathbb{E}_{z \sim q_{\theta}} [\mathbb{E}_{x \sim p(x|z)} [T(x)]] = \mu, \end{aligned} \quad (3)$$

160 where $T(x), \mu$ are defined as in Equation 2, and $p(x|z)$ is the intractable distribution of data from
 161 the model, x , given that model’s parameters z (we access samples from this distribution by running
 162 the model forward). The purpose of each element in this program is detailed in Figure 1D. Finally,
 163 we recognize that many distributions in \mathcal{Q} will respect the emergent property constraints, so we
 164 select that which has maximum entropy. This principle, captured in Equation 3 by the primal
 165 objective H , identifies parameter distributions with minimal assumptions beyond some chosen
 166 structure [29, 30, 20, 31]. Such a normative principle of maximum entropy, which is also that

167 of Bayesian inference, naturally fits with our scientific objective of reasoning about parametric
168 sensitivity and robustness. The recovered distribution of EPI is as variable as possible along each
169 parametric manifold such that it produces the emergent property.

170 EPI optimizes the weights and biases θ of the deep neural network (which induces the probability
171 distribution) by iteratively solving Equation 3. The optimization is complete when the sampled
172 models with parameters $z \sim q_\theta$ produce activity consistent with the specified emergent property
173 (Fig. S4). Such convergence is evaluated with a hypothesis test that the mean of each emergent
174 property statistic is not different than its emergent property value (see Section 5.1.2). Further
175 validation of EPI is available in the supplementary materials, where we analyze a simpler model
176 for which ground-truth statements can be made (Section 5.1.1).

177 In relation to broader methodology, inspection of the EPI objective reveals a natural relationship to
178 posterior inference. Specifically, EPI executes variational inference in an exponential family model,
179 the sufficient statistics and mean parameter of which are defined by the emergent property statistics
180 and values, respectively (see Section 5.1.4). Equipped with this method, we may examine structure
181 in such inferred distributions or make comparisons between inferred distributions at graded values
182 of emergent property statistics. We now prove out the value of EPI by using it to investigate and
183 produce novel insights about three prominent models in neuroscience.

184 3.3 Comprehensive input-responsivity in a nonlinear sensory system

185 Dynamical models of excitatory (E) and inhibitory (I) populations with supralinear input-output
186 function have succeeded in explaining a host of experimentally documented phenomena. In a regime
187 characterized by inhibitory stabilization of strong recurrent excitation, these models give rise to
188 paradoxical responses [4], selective amplification [32], surround suppression [33] and normalization
189 [34]. Despite their strong predictive power, E-I circuit models rely on the assumption that inhibi-
190 tion can be studied as an indivisible unit. However, experimental evidence shows that inhibition
191 is composed of distinct elements – parvalbumin (P), somatostatin (S), VIP (V) – composing 80%
192 of GABAergic interneurons in V1 [35, 36, 37], and that these inhibitory cell types follow specific
193 connectivity patterns (Fig. 2A) [38]. Recent theoretical advances [24, 39, 40], have only started
194 to address the consequences of this multiplicity in the dynamics of V1, strongly relying on linear
195 theoretical tools. Here, we go beyond linear theory by systematically generating and evaluating hy-
196 potheses of circuit model function using EPI distributions of neuron-type inputs producing various
197 neuron-type population responses.

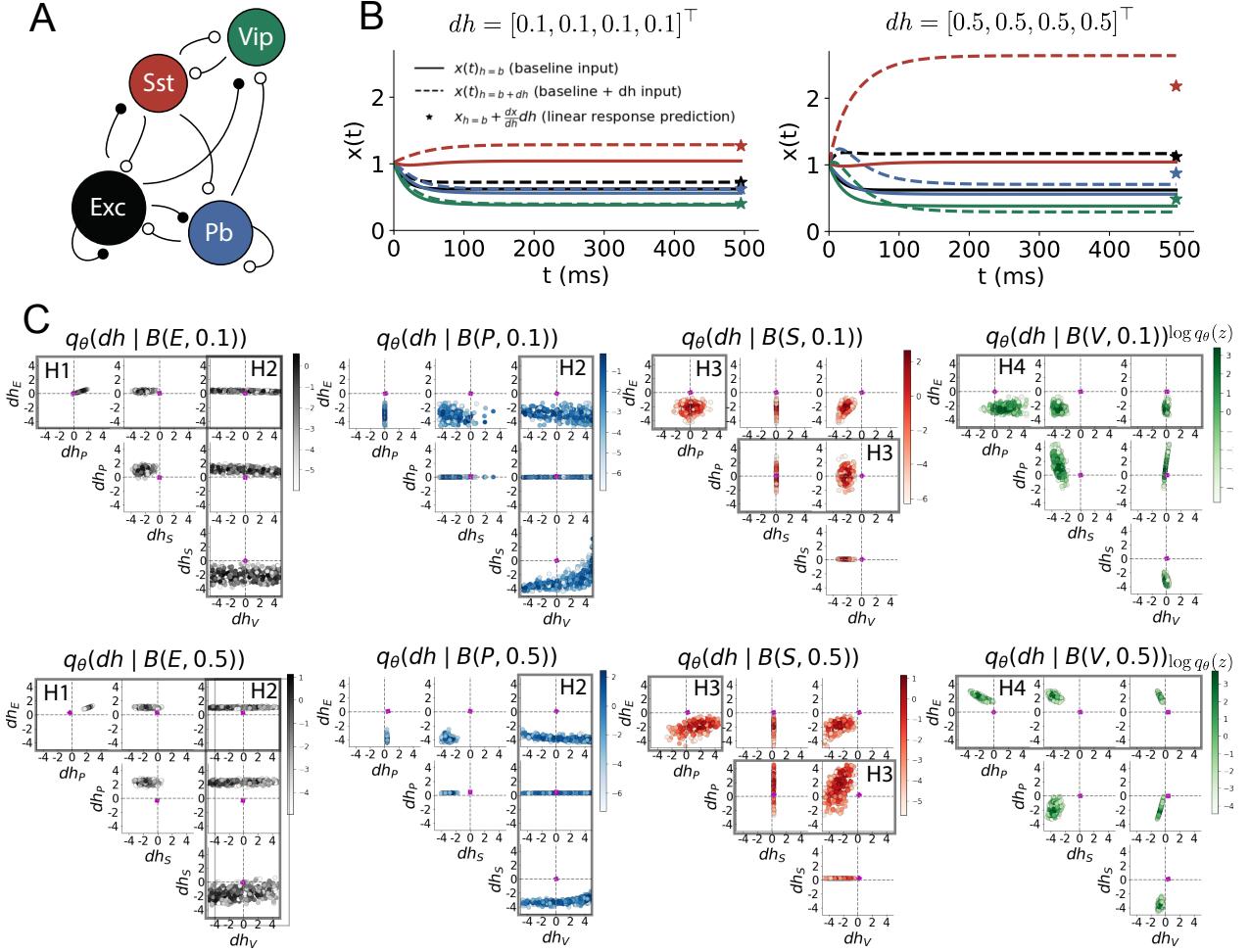


Figure 2: Hypothesis generation through EPI in a V1 model. A. Four-population model of primary visual cortex with excitatory (black), parvalbumin (blue), somatostatin (red), and VIP (green) neurons. Some neuron-types largely do not form synaptic projections to others (excitatory and inhibitory projections filled and unfilled, respectively). B. Linear response predictions become inaccurate with greater input strength. V1 model simulations for input (solid) $h = b$ and (dashed) $h = b + dh$. Stars indicate the linear response prediction. C. EPI distributions on differential input dh conditioned on differential response $\mathcal{B}(\alpha, y)$. Supporting evidence for the four generated hypotheses are indicated by gray boxes with labels H1, H2, H3, and H4. The linear prediction from two standard deviations away from y (from negative to positive) is overlaid in magenta (very small, near origin).

198 Specifically, we consider a four-dimensional circuit model with dynamical state given by the firing
 199 rate x of each neuron-type population $x = [x_E, x_P, x_S, x_V]^\top$. Given a time constant of $\tau = 20$ ms
 200 and a power $n = 2$, the dynamics are driven by the rectified and exponentiated sum of recurrent
 201 (Wx) and external h inputs:

$$\tau \frac{dx}{dt} = -x + [Wx + h]_+^n. \quad (4)$$

202 We considered fixed effective connectivity weights W approximated from experimental recordings of
 203 publicly available datasets of mouse V1 [41, 42] (see Section 5.2.2). The input $h = b + dh$ is comprised
 204 of a baseline input $b = [b_E, b_P, b_S, b_V]^\top$ and a differential input $dh = [dh_E, dh_P, dh_S, dh_V]^\top$ to each
 205 neuron-type population. Throughout subsequent analyses, the baseline input is $b = [1, 1, 1, 1]^\top$.

206 With this model, we are interested in the differential responses of each neuron-type population to
 207 changes in input dh . Initially, we studied the linearized response of the system to input $\frac{dx_{ss}}{dh}$ at the
 208 steady state response x_{ss} , i.e. a fixed point. All analyses of this model consider the steady state
 209 response, so we drop the notation ss from here on. While this linearization accurately predicts
 210 differential responses $dx = [dx_E, dx_P, dx_S, dx_V]^\top$ for small differential inputs to each population
 211 $dh = [0.1, 0.1, 0.1, 0.1]^\top$ (Fig 2B left), the linearization is a poor predictor in this nonlinear model
 212 more generally (Fig. 2B right). Currently available approaches to deriving the steady state response
 213 of the system are limited.

214 To get a more comprehensive picture of the input-responsivity of each neuron-type beyond linear
 215 theory, we used EPI to learn a distribution of the differential inputs to each population dh that
 216 produce an increase of y in the rate of each neuron-type population $\alpha \in \{E, P, S, V\}$. We want
 217 to know the differential inputs dh that result in a differential steady state dx_α (the change in x_α
 218 when receiving input $h = b + dh$ with respect to the baseline $h = b$) of value y with some small,
 219 arbitrarily chosen amount of variance 0.01^2 . These statements amount to the emergent property

$$\mathcal{B}(\alpha, y) \triangleq \mathbb{E} \begin{bmatrix} dx_\alpha \\ (dx_\alpha - y)^2 \end{bmatrix} = \begin{bmatrix} y \\ 0.01^2 \end{bmatrix}. \quad (5)$$

220 We maintain the notation $\mathcal{B}(\cdot)$ throughout the rest of the study as short hand for emergent property,
 221 which represents a different signature of computation in each application.

222 Using EPI, we inferred the distribution of dh shown in Figure 2C producing $\mathcal{B}(\alpha, y)$. Columns
 223 correspond to inferred distributions of excitatory ($\alpha = E$, red), parvalbumin ($\alpha = P$, blue), so-
 224 matostatin ($\alpha = S$, red) and VIP ($\alpha = V$, green) neuron-type response increases, while each

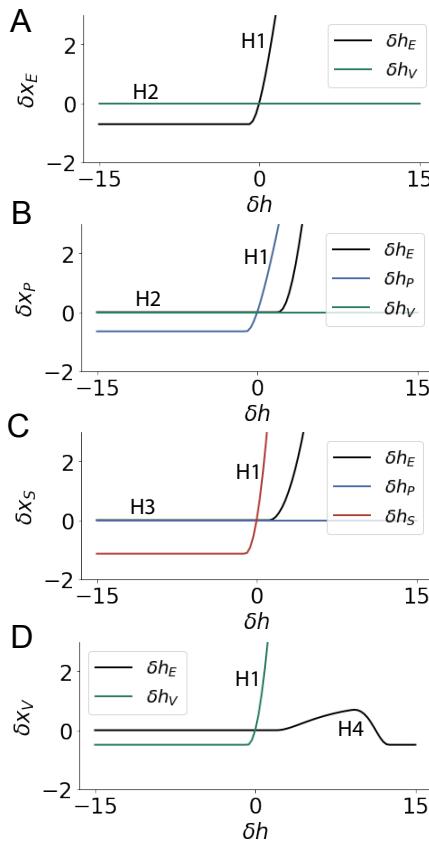


Figure 3: Confirming EPI generated hypotheses in V1. A. Differential responses δx_E by the E-population to changes in individual input $\delta h_\alpha \hat{u}_\alpha$ away from the mode of the EPI distribution dh^* . B-D Same plots for the P-, S-, and V-populations. Labels H1, H2, H3, and H4 indicate which curves confirm which hypotheses.

225 row corresponds to increase amounts of $y \in \{0.1, 0.5\}$. For each pair of parameters, we show the
 226 two-dimensional marginal distribution of samples colored by $\log q_\theta(dh \mid \mathcal{B}(\alpha, y))$. The inferred dis-
 227 tributions immediately suggest four hypotheses:

228

- 229 H1: as is intuitive, each neuron-type's firing rate should be sensitive to that neuron-type's
 230 direct input (e.g. Fig. 2C H1 gray boxes indicate low variance in dh_E when $\alpha = E$. Same
 231 observation in all inferred distributions);
 232 H2: the E- and P-populations should be largely unaffected by input to the V-population (Fig.
 233 2C H2 gray boxes indicate high variance in dh_V when $\alpha \in \{E, P\}$);
 234 H3: the S-population should be largely unaffected by input to the P-population (Fig. 2C H3
 235 gray boxes indicate high variance in dh_P when $\alpha = S$);
 236 H4: there should be a nonmonotonic response of the V-population with input to the E-
 237 population (Fig. 2C H4 gray boxes indicate that negative dh_E should result in small dx_V ,
 238 but positive dh_E should elicit a larger dx_V);

239 We evaluate these hypotheses by taking perturbations in individual neuron-type input δh_α away

240 from the modes of the inferred distributions at $y = 0.1$

$$dh^* = z^* = \operatorname{argmax}_z \log q_\theta(z | \mathcal{B}(\alpha, 0.1)). \quad (6)$$

241 Here δx_α is the change in steady state response of the system with input $h = b + dh^* + \delta h_\alpha \hat{u}_\alpha$
242 compared to $h = b + dh^*$, where \hat{u}_α is a unit vector in the dimension of α . The EPI-generated
243 hypotheses are confirmed (for details, see Section 5.2.2):

244 H1: the neuron-type responses are sensitive to their direct inputs (Fig. 3A black, 3B blue,
245 3C red, 3D green);

246 H2: the E- and P-populations are not affected by δh_V (Fig. 3A green, 3B green);

247 H3: the S-population is not affected by δh_P (Fig. 3C blue);

248 H4: the V-population exhibits a nonmonotonic response to δh_E (Fig. 3D black), and is in
249 fact the only population to do so (Fig. 3A-C black).

250 These hypotheses were in stark contrast to what was available to us via traditional analytical linear
251 prediction (Fig. 2C, magenta, see Section 5.2.2).

252 Here, we examined the neuron-type responsivity of this model of V1 with scientifically motivated
253 choice of connectivity W . With EPI, we could just as easily have examined the distribution of such
254 W 's consistent with some response characteristics for a fixed input h or another emergent property
255 such as inhibition stabilization. Most importantly, this analysis is a proof-of-concept demonstrating
256 the valuable ability to condition parameters of interest of a neural circuit model on some chosen
257 emergent property. To this point, we have shown the utility of EPI on relatively low-level emergent
258 properties like network syncing and differential neuron-type population responses. In the remainder
259 of the study, we focus on using EPI to understand models of more abstract cognitive function.

260 3.4 Identifying neural mechanisms of flexible task switching

261 In a rapid task switching experiment [43], rats were explicitly cued on each trial to either orient
262 towards a visual stimulus in the Pro (P) task or orient away from a visual stimulus in the Anti
263 (A) task (Fig. 4a). Neural recordings in the midbrain superior colliculus (SC) exhibited two
264 populations of neurons that simultaneously represented both task context (Pro or Anti) and motor
265 response (contralateral or ipsilateral to the recorded side): the Pro/Contra and Anti/Ipsi neurons
266 [25]. Duan et al. proposed a model of SC that, like the V1 model analyzed in the previous section, is
267 a four-population dynamical system. We analyzed this model, where the neuron-type populations
268 are functionally-defined as the Pro- and Anti-populations in each hemisphere (left (L) and right

269 (R)), their connectivity is parameterized geometrically (Fig. 4B). The input-output function of
 270 this model is chosen such that the population responses $x = [x_{LP}, x_{LA}, x_{RP}, x_{RA}]^\top$ are bounded
 271 from 0 to 1 giving rise to high (1) or low (0) responses at the end of the trial:

$$x_\alpha = \left(\frac{1}{2} \tanh \left(\frac{u_\alpha - \epsilon}{\zeta} \right) + \frac{1}{2} \right) \quad (7)$$

272 where $\epsilon = 0.05$ and $\zeta = 0.5$. The dynamics evolve with timescale $\tau = 0.09$ via an internal variable
 273 u governed by connectivity weights W

$$\tau \frac{du}{dt} = -u + Wx + h + \sigma dB \quad (8)$$

274 with gaussian noise of variance $\sigma^2 = 1$. The input h is comprised of a cue-dependent input to the
 275 Pro or Anti populations, a stimulus orientation input to either the Left or Right populations, and
 276 a choice-period input to the entire network (see Section 5.2.3). Here, we use EPI to determine the
 277 changes in network connectivity $z = [sW_P, sW_A, vW_{PA}, vW_{AP}, dW_{PA}, dW_{AP}, hW_P, hW_A]$ resulting
 278 in greater levels of rapid task switching accuracy.

279 To quantify the emergent property of rapid task switching at various levels of accuracy, we consid-
 280 ered the requirements of this model in this behavioral paradigm. At the end of successful trials,
 281 the response of the Pro population in the hemisphere of the correct choice must have a value near
 282 1, while the Pro population in the opposite hemisphere must have a value near 0. Constraining a
 283 population response $x_\alpha \in [0, 1]$ to be either 0 or 1 can be achieved by requiring that it has Bernoulli
 284 variance (see Section 5.2.3). Thus, we can formulate rapid task switching at a level of accuracy
 285 $p \in [0, 1]$ in both tasks in terms of the average steady response of the Pro population \hat{p} of the
 286 correct choice, the error in Bernoulli variance of that Pro neuron σ_{err}^2 , and the average difference
 287 in Pro neuron responses d in both Pro and Anti trials:

$$\mathcal{B}(p) \triangleq \mathbb{E} \begin{bmatrix} \hat{p}_P \\ \hat{p}_A \\ (\hat{p}_P - p)^2 \\ (\hat{p}_A - p)^2 \\ \sigma_{P,err}^2 \\ \sigma_{A,err}^2 \\ d_P \\ d_A \end{bmatrix} = \begin{bmatrix} p \\ p \\ 0.15^2 \\ 0.15^2 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}. \quad (9)$$

288 Thus, $\mathcal{B}(p)$ denotes Bernoulli, winner-take-all responses between Pro neurons in a model executing
 289 rapid task switching near accuracy level p .

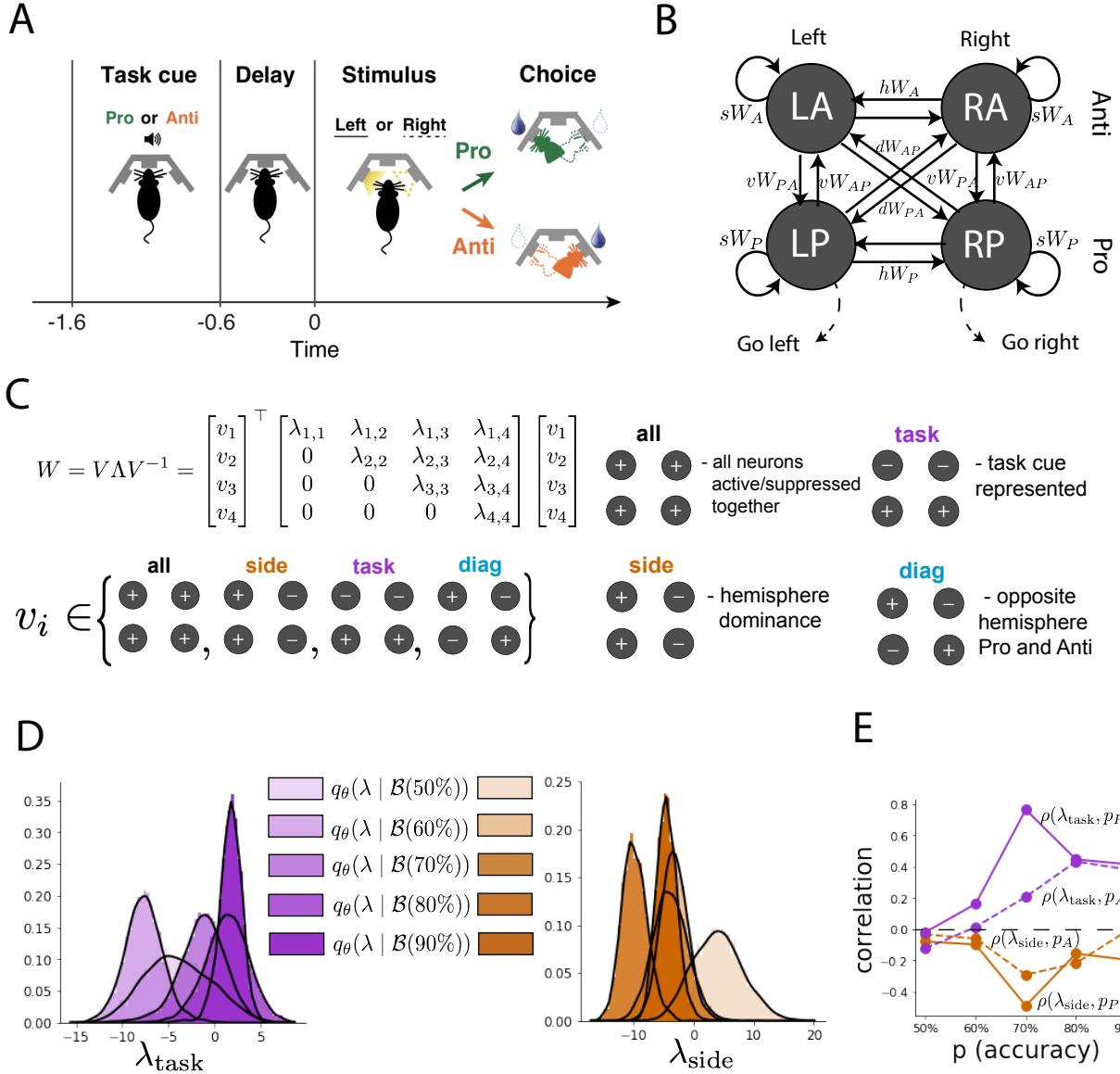


Figure 4: EPI reveals changes in SC [25] connectivity that control task accuracy. A. Rapid task switching behavioral paradigm (see text). B. Model of superior colliculus (SC). Neurons: LP - left pro, RP - right pro, LA - left anti, RA - right anti. Parameters: sW - self, hW - horizontal, vW - vertical, dW - diagonal weights. Subscripts P and A of connectivity weights indicate Pro or Anti populations, and e.g. vW_{PA} is a vertical weight from an Anti to a Pro population. C. The Schur decomposition of the weight matrix $W = V \Lambda V^{-1}$ is a unique decomposition with orthogonal V and upper triangular Λ . Schur modes: v_{all} , v_{task} , v_{side} , and v_{diag} . D. The marginal EPI distributions of the Schur eigenvalues at each level of task accuracy. E. The correlation of Schur eigenvalue with task performance in each learned EPI distribution.

290 We used EPI to learn distributions of the SC weight matrix parameters z conditioned on of various
 291 levels of rapid task switching accuracy $\mathcal{B}(p)$ for $p \in \{50\%, 60\%, 70\%, 80\%, 90\%\}$. To make sense
 292 of these inferred distributions, we followed the approach of Duan et al. by decomposing the con-
 293 nectivity matrix $W = V\Lambda V^{-1}$ in such a way (the Schur decomposition) that the basis vectors v_i
 294 are the same for all W (Fig. 4C). These basis vectors have intuitive roles in processing for this
 295 task, and are accordingly named the *all* mode - all neurons co-fluctuate, *side* mode - one side
 296 dominates the other, *task* mode - the Pro or Anti populations dominate the other, and *diag* mode -
 297 Pro- and Anti-populations of opposite hemispheres dominate the opposite pair. The corresponding
 298 eigenvalues (e.g. λ_{task} , which change according to W) indicate the degree to which activity along
 299 that mode is increased or decreased by W .

300 We found that for greater task accuracies, the task mode eigenvalue increases, indicating the
 301 importance of W to the task representation (Fig. 4D, purple; adjacent distributions from 60%
 302 to 90% have $p < 10^{-4}$, Mann-Whitney test with 50 estimates and 100 samples). Stepping from
 303 random chance (50%) networks to marginally task-performing (60%) networks, there is a marked
 304 decrease of the side mode eigenvalues (Fig. 4D, orange; $p < 10^{-4}$). Such side mode suppression
 305 relative to 50% remains in the models achieving greater accuracy, revealing its importance towards
 306 task performance. There were no interesting trends with task accuracy in the all or diag mode
 307 (hence not shown in Fig. 4). Importantly, we can conclude from our methodology that side
 308 mode suppression in W allows rapid task switching, and that greater task-mode representations
 309 in W increase accuracy. These hypotheses are confirmed by forward simulation of the SC model
 310 (Fig. 4E, see Section 5.2.3) suggesting experimentally testable predictions: increase in rapid task
 311 switching performance should be correlated with changes in effective connectivity corresponding to
 312 an increase in task mode and decrease in side mode eigenvalues.

313 3.5 Linking RNN connectivity to error

314 So far, each model we have studied was designed from fundamental biophysical principles, genetically-
 315 or functionally-defined neuron types. At a more abstract level of modeling, recurrent neural net-
 316 works (RNNs) are high-dimensional dynamical models of computation that are becoming increas-
 317 ingly popular in neuroscience research [44]. In theoretical neuroscience, RNN dynamics usually
 318 follow the equation

$$\frac{dx}{dt} = -x + W\phi(x) + h, \quad (10)$$

319 where x is the network activity, W is the network connectivity, $\phi(\cdot) = \tanh(\cdot)$, and h is the input to
 320 the system. Such RNNs are trained to do a task from a systems neuroscience experiment, and then
 321 the unit activations of the trained RNN are compared to recorded neural activity. Fully-connected
 322 RNNs with tens of thousands of parameters are challenging to characterize [45], especially making
 323 statistical inferences about their parameterization. Alternatively, we considered a rank-1, N -neuron
 324 RNN with connectivity consisting of the sum of a random and a structured component:

$$W = g\chi + \frac{1}{N}mn^\top. \quad (11)$$

325 The random component $g\chi$ has strength g , and random component weights are Gaussian dis-
 326 tributed $\chi_{i,j} \sim \mathcal{N}(0, \frac{1}{N})$. The structured component $\frac{1}{N}mn^\top$ has entries of m and n drawn from
 327 Gaussian distributions $m_i \sim \mathcal{N}(M_m, 1)$ and $n_i \sim \mathcal{N}(M_n, 1)$. Recent theoretical work derives the
 328 low-dimensional response properties of low-rank networks from statistical parameterizations of their
 329 connectivity, such as $z = [g, M_m, M_n]$ [26]. We used EPI to infer the parameterizations of rank-
 330 1 RNNs solving an example task, enabling discovery of properties of connectivity that result in
 331 different types of error in the computation.

332 The task we consider is Gaussian posterior conditioning: calculate the parameters of a posterior
 333 distribution induced by a prior $p(\mu_y) = \mathcal{N}(\mu_0 = 4, \sigma_0^2 = 1)$ and a likelihood $p(y|\mu_y) = \mathcal{N}(\mu_y, \sigma_y^2 =$
 334 1), given a single observation y . Conjugacy offers the result analytically; $p(\mu_y|y) = \mathcal{N}(\mu_{post}, \sigma_{post}^2)$,
 335 where:

$$\mu_{post} = \frac{\frac{\mu_0}{\sigma_0^2} + \frac{y}{\sigma_y^2}}{\frac{1}{\sigma_0^2} + \frac{1}{\sigma_y^2}} \quad \sigma_{post}^2 = \frac{1}{\frac{1}{\sigma_0^2} + \frac{1}{\sigma_y^2}}. \quad (12)$$

336 To solve this Gaussian posterior conditioning task, the RNN response to a constant input $h =$
 337 $yr + (n - M_n)$ must equal the posterior mean along readout vector r , where

$$\kappa_r = \frac{1}{N} \sum_{j=1}^N r_j \phi(x_j). \quad (13)$$

338 Additionally, the amount of chaotic variance Δ_T must equal the posterior variance. Theory for
 339 low-rank RNNs allows us to express κ_r and Δ_T in terms of each other through a solvable system of
 340 nonlinear equations (see Section 5.2.4) [26]. This theory facilitates the mathematical formalization
 341 of task execution into an emergent property, where the emergent property statistics of the RNN
 342 activity are κ_r and Δ_T , and the emergent property values are the ground truth posterior mean

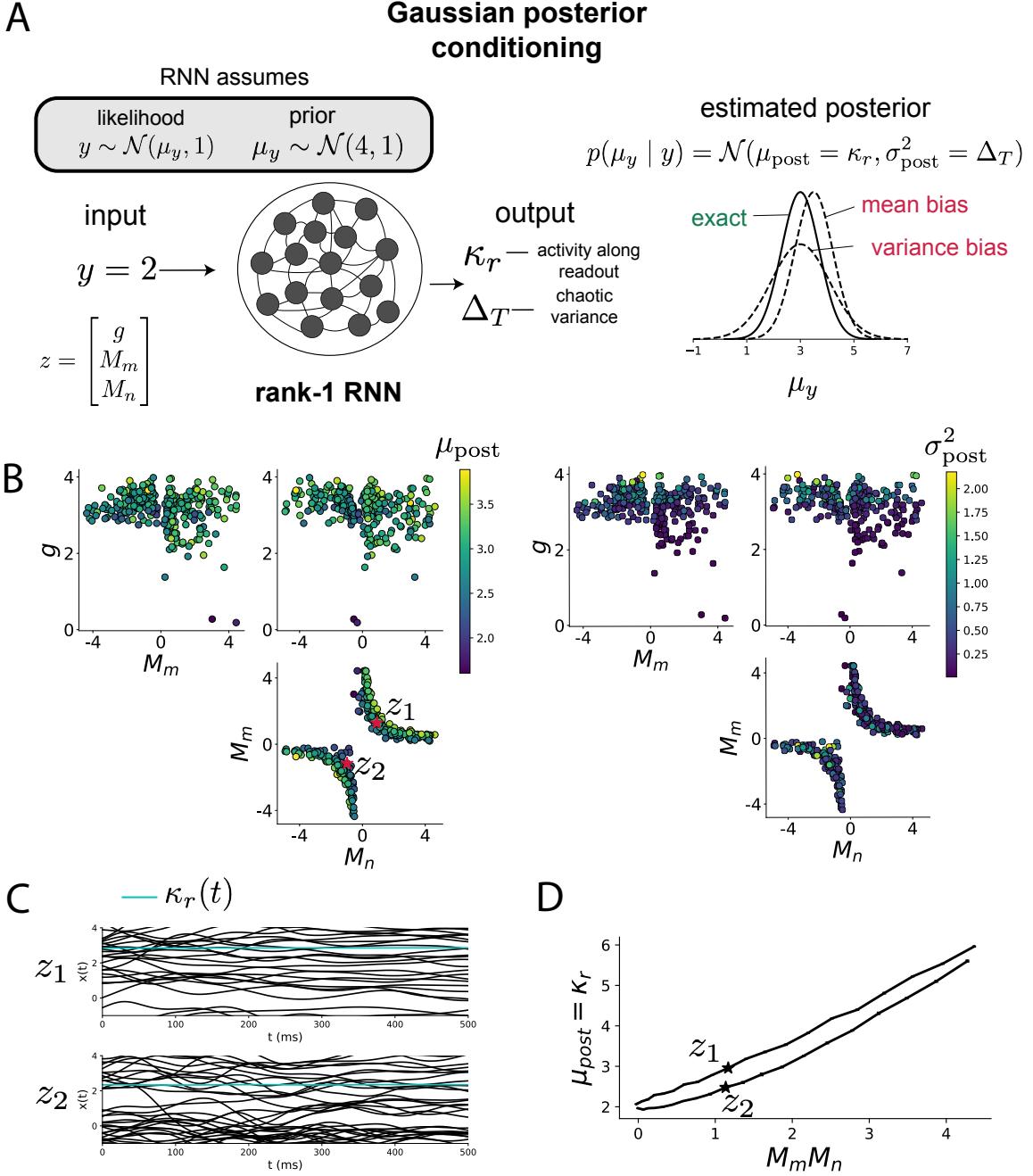


Figure 5: Sources of error in an RNN solving a simple task. A. (left) A rank-1 RNN executing a Gaussian posterior conditioning computation on μ_y . (right) Error in this computation can come from over- or underestimating the posterior mean or variance. B. EPI distribution of rank-1 RNNs executing Gaussian posterior conditioning. Samples are colored by (left) posterior mean $\mu_{\text{post}} = \kappa_r$ and (right) posterior variance $\sigma_{\text{post}}^2 = \Delta_T$. C. Finite-size network simulations of 2,000 neurons with parameters z_1 and z_2 sampled from the inferred distribution. Activity along readout κ_r (cyan) is stable despite chaotic fluctuations. D. The posterior mean computed by RNNs parameterized by z_1 and z_2 perturbed in the dimension of the product of M_m and M_n . Means and standard errors are shown across 10 realizations of 2,000-neuron networks.

³⁴³ μ_{post} and variance σ_{post}^2 :

$$\mathbb{E} \begin{bmatrix} \kappa_r \\ \Delta_T \\ (\kappa_r - \mu_{\text{post}})^2 \\ (\Delta_T^2 - \sigma_{\text{post}}^2)^2 \end{bmatrix} = \begin{bmatrix} \mu_{\text{post}} \\ \sigma_{\text{post}}^2 \\ 0.1 \\ 0.1 \end{bmatrix}. \quad (14)$$

³⁴⁴ We chose a substantial amount of variance in these emergent property statistics, so that the inferred
³⁴⁵ distribution resulted in RNNs with a variety of errors in their solutions to the gaussian posterior
³⁴⁶ conditioning problem.

³⁴⁷ EPI was used to learn distributions of RNN connectivity properties $z = [g, M_m, M_n]$ executing
³⁴⁸ Gaussian posterior conditioning given an input of $y = 2$, where the true posterior is $\mu_{\text{post}} = 3$ and
³⁴⁹ $\sigma_{\text{post}} = 0.5$ (Fig. 5A). We examined the nature of the over- and under-estimation of the posterior
³⁵⁰ means (Fig. 5B left) and variances (Fig. 5B right) in the inferred distributions (300 samples).
³⁵¹ The symmetry in the M_m - M_n plane, suggests a degeneracy in the product of M_m and M_n (Fig.
³⁵² 5B). Indeed, $M_m M_n$ strongly determines the posterior mean ($r = 0.62, p < 10^{-4}$). Furthermore,
³⁵³ the random strength g strongly determines the chaotic variance ($r = 0.56, p < 10^{-4}$). Neither of
³⁵⁴ these observations were obvious from what mathematical analysis is available in networks of this
³⁵⁵ type (see Section 5.2.4). While the link between random strength g and chaotic variance Δ_T (and
³⁵⁶ resultingly posterior variance in this problem) is well-known [3], the distribution admits a novel
³⁵⁷ hypothesis: the estimation of the posterior mean by the RNN increases with $M_m M_n$.

³⁵⁸ We tested this prediction by taking parameters z_1 and z_2 as representative samples from the positive
³⁵⁹ and negative M_m - M_n quadrants, respectively. Instead of using the theoretical predictions shown in
³⁶⁰ Figure 5B, we simulated finite-size realizations of these networks with 2,000 neurons (e.g. Fig. 5C).
³⁶¹ We perturbed these parameter choices by $M_m M_n$ clarifying that the posterior mean can be directly
³⁶² controlled in this way (Fig. 5D; $p < 10^{-4}$), see Section 5.2.4). Thus, EPI confers a clear picture
³⁶³ of error in this computation: the product of the low rank vector means M_m and M_n modulates
³⁶⁴ the estimated posterior mean while the random strength g modulates the estimated posterior
³⁶⁵ variance. This novel procedure of inference on reduced parameterizations of RNNs conditioned on
³⁶⁶ the emergent property of task execution is generalizable to other settings modeled in [26] like noisy
³⁶⁷ integration and context-dependent decision making (Fig. S5).

368 **4 Discussion**

369 **4.1 EPI is a general tool for theoretical neuroscience**

370 Biologically realistic models of neural circuits are comprised of complex nonlinear differential equa-
371 tions, making traditional theoretical analysis and statistical inference intractable. We advance the
372 capabilities of statistical inference in theoretical neuroscience by presenting EPI, a deep inference
373 methodology for learning parameter distributions of theoretical models performing neural compu-
374 tation. We have demonstrated the utility of EPI on biological models (STG), intermediate-level
375 models of interacting genetically- and functionally-defined neuron-types (V1, SC), and the most
376 abstract of models (RNNs). We are able to condition both deterministic and stochastic models on
377 low-level emergent properties like spiking frequency of membrane potentials, as well as high-level
378 cognitive function like posterior conditioning. Technically, EPI is tractable when the emergent
379 property statistics are continuously differentiable with respect to the model parameters, which is
380 very often the case; this emphasizes the general applicability of EPI.

381 In this study, we have focused on applying EPI to low dimensional parameter spaces of models
382 with low dimensional dynamical states. These choices were made to present the reader with a
383 series of interpretable conclusions, which is more challenging in high dimensional spaces. In fact,
384 EPI should scale reasonably to high dimensional parameter spaces, as the underlying technology has
385 produced state-of-the-art performance on high-dimensional tasks such as texture generation [20]. Of
386 course, increasing the dimensionality of the dynamical state of the model makes optimization more
387 expensive, and there is a practical limit there as with any machine learning approach. Although,
388 theoretical approaches (e.g. [26]) can be used to reason about the wholistic activity of such high
389 dimensional systems by introducing some degree of additional structure into the model.

390 **4.2 Novel hypotheses from EPI**

391 In neuroscience, machine learning has primarily been used to reveal structure in large-scale neural
392 datasets [46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56] (see review, [15]). Such careful inference procedures
393 are developed for these statistical models allowing precise, quantitative reasoning, which clarifies
394 the way data informs beliefs about the model parameters. However, these statistical models lack
395 resemblance to the underlying biology, making it unclear how to go from the structure revealed by
396 these methods, to the neural mechanisms giving rise to it. In contrast, theoretical neuroscience has
397 focused on careful mechanistic modeling and the production of emergent properties of computation.

398 The careful steps of *i.*) model design and *ii.*) emergent property definition, are followed by *iii.)*
399 practical inference methods resulting in an opaque characterization of the way model parameters
400 govern computation. In this work, we replaced this opaque procedure of parameter identification
401 in theoretical neuroscience with emergent property inference, opening the door to careful inference
402 in careful models of neural computation.

403 Biologically realistic models of neural circuits often prove formidable to analyze. Two main factors
404 contribute to the difficulty of this endeavor. First, in most neural circuit models, the number
405 of parameters scales quadratically with the number of neurons, limiting analysis of its parameter
406 space. Second, even in low dimensional circuits, the structure of the parametric regimes governing
407 emergent properties is intricate. For example, these circuit models can support more than one
408 steady state [57] and non-trivial dynamics on strange attractors [58].

409 In Section 3.3, we advanced the tractability of low-dimensional neural circuit models by showing
410 that EPI offers insights about cell-type specific input-responsivity that cannot be afforded through
411 the available linear analytical methods [24, 39, 40]. By flexibly conditioning this V1 model on
412 different emergent properties, we performed an exploratory analysis of a *model* rather than a
413 dataset, generating a set of testable hypotheses, which were proved out. Furthermore, exploratory
414 analyses can be directed towards formulating hypotheses of a specific form. For example, model
415 parameter dependencies on behavioral performance can be assessed by using EPI to condition on
416 various levels of task accuracy (See Section 3.4). This analysis identified experimentally testable
417 predictions (proved out *in-silico*) of patterns of effective connectivity in SC that should be correlated
418 with increased performance.

419 In our final analysis, we presented a novel procedure for doing statistical inference on interpretable
420 parameterizations of RNNs executing simple tasks. Specifically, we analyzed RNNs solving a pos-
421 terior conditioning problem in the spirit of [59, 60]. This methodology relies on recently extended
422 theory of responses in random neural networks with low-rank structure [26]. While we focused
423 on rank-1 RNNs, which were sufficient for solving this task, this inference procedure generalizes
424 to RNNs of greater rank necessary for more complex tasks. The ability to apply the probabilistic
425 model selection toolkit to RNNs should prove invaluable as their use in neuroscience increases.

426 EPI leverages deep learning technology for neuroscientific inquiry in a categorically different way
427 than approaches focused on training neural networks to execute behavioral tasks [61]. These works
428 focus on examining optimized deep neural networks while considering the objective function, learn-
429 ing rule, and architecture used. This endeavor efficiently obtains sets of parameters that can be

430 reasoned about with respect to such considerations, but lacks the careful probabilistic treatment of
431 parameter inference in EPI. These approaches can be used complementarily to enhance the practice
432 of theoretical neuroscience.

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440 **Data availability statement:**

441 The datasets generated during and/or analysed during the current study are available from the
442 corresponding author upon reasonable request.

443 **Code availability statement:**

444 The software written for the current study is available from the corresponding author upon rea-
445 sonable request.

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639 **5 Methods**

640 **5.1 Emergent property inference (EPI)**

641 Emergent property inference (EPI) learns distributions of theoretical model parameters that pro-
 642 duce emergent properties of interest by combining ideas from maximum entropy flow networks
 643 (MEFNs) [20] and likelihood-free variational inference (LFVI) [21]. Consider model parameteri-
 644 zation z and data x which has an intractable likelihood $p(x | z)$ defined by a model simulator of
 645 which samples are available $x \sim p(x | z)$. EPI optimizes a distribution $q_\theta(z)$ (itself parameterized
 646 by θ) of model parameters z to produce an emergent property of interest \mathcal{B} ,

$$\mathcal{B} \triangleq \mathbb{E}_{z \sim q_\theta} [\mathbb{E}_{x \sim p(x|z)} [T(x)]] = \mu. \quad (15)$$

647 Precisely, the emergent property statistics $T(x)$ must equal the emergent property values μ , in
 648 expectation over the EPI distribution of parameters $q_\theta(z)$ and the distribution of simulated activity
 649 $p(x | z)$. This is a viable way to represent emergent properties in theoretical models, as we have
 650 demonstrated in the main text, and enables the EPI optimization.

651 With EPI, we use deep probability distributions to learn flexible approximations to model parameter
 652 distributions $q_\theta(z)$. In deep probability distributions, a simple random variable $w \sim q_0(w)$ is
 653 mapped deterministically via a sequence of deep neural network layers (f_1, \dots, f_l) parameterized by
 654 weights and biases θ to the support of the distribution of interest:

$$z = f_\theta(\omega) = f_l(\dots f_1(w)). \quad (16)$$

655 Given a simulator defined by a theoretical model $x \sim p(x | z)$ and some emergent property of
 656 interest \mathcal{B} , $q_\theta(z)$ is optimized via the neural network parameters θ to find a maximally entropic
 657 distribution q_θ^* within the deep variational family \mathcal{Q} producing the emergent property:

$$\begin{aligned} q_\theta^*(z) &= \operatorname{argmax}_{q_\theta \in \mathcal{Q}} H(q_\theta(z)) \\ &\text{s.t. } \mathbb{E}_{z \sim q_\theta} [\mathbb{E}_{x \sim p(x|z)} [T(x)]] = \mu. \end{aligned} \quad (17)$$

658 Since we are optimizing parameters θ of our deep probability distribution with respect to the
 659 entropy $H(q_\theta(z))$, we must take gradients with respect to the log probability density of samples
 660 from the deep probability distribution. Entropy of $q_\theta(z)$ can be expressed as an expectation of
 661 the negative log density of parameter samples z over the randomness in the parameterless initial
 662 distribution q_0 :

$$H(q_\theta(z)) = \int -q_\theta(z) \log(q_\theta(z)) dz = \mathbb{E}_{z \sim q_\theta} [-\log(q_\theta(z))] = \mathbb{E}_{w \sim q_0} [-\log(q_\theta(f_\theta(w)))]. \quad (18)$$

663 Thus, the gradient of the entropy of the deep probability distribution can be estimated as an
 664 average of gradients of the log density of samples z :

$$\nabla_{\theta} H(q_{\theta}(z)) = \mathbb{E}_{w \sim q_0} [-\nabla_{\theta} \log(q_{\theta}(f_{\theta}(w)))]. \quad (19)$$

665 In EPI, MEFNs are purposed towards variational learning of model parameter distributions. A
 666 closely related methodology, variational inference, uses optimization to approximate posterior dis-
 667 tributions [62]. Standard methods like stochastic gradient variational Bayes [6] or black box varia-
 668 tional inference [63] simply do not work for inference in theoretical models of neural circuits, since
 669 they require tractable likelihoods $p(x | z)$. Work on likelihood-free variational inference (LFVI) [21],
 670 which like EPI seeks to do inference in models with intractable likelihoods, employs an additional
 671 deep neural network as a ratio estimator, enabling an estimation of the optimization objective for
 672 variational inference. Like LFVI, EPI can be framed as variational inference (see Section 5.1.4).
 673 But, unlike LFVI, EPI uses a single deep network to learn a distribution and is optimized to pro-
 674 duce an emergent property, rather than condition on data points. Optimizing the EPI objective is
 675 a technological challenge, the details of which we elaborate in Section 5.1.2. Before going through
 676 those details, we ground this optimization in a toy example.

677 We note that, during our preparation and early presentation of this work [64, 65], another work
 678 has arisen with broadly similar goals: bringing statistical inference to mechanistic models of neural
 679 circuits ([66, 67, 68], preprint posted simultaneously with this preprint). We are encouraged by
 680 this general problem being recognized by others in the community, and we emphasize that these
 681 works offer complementary neuroscientific contributions (different theoretical models of focus) and
 682 use different technical methodologies (ours is built on our prior work [20], theirs similarly [69]).
 683 These distinct methodologies and scientific investigations emphasize the increased importance and
 684 timeliness of both works.

685 5.1.1 Example: 2D LDS

686 To gain intuition for EPI, consider a two-dimensional linear dynamical system (2D LDS) model
 687 (Fig. S1A):

$$\tau \frac{dx}{dt} = Ax \quad (20)$$

688 with

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}. \quad (21)$$

689 To run EPI with the dynamics matrix elements as the free parameters $z = [a_1, a_2, a_3, a_4]$ (fixing
 690 $\tau = 1$), the emergent property statistics $T(x)$ were chosen to contain the first and second moments of
 691 the oscillatory frequency, $\frac{\text{imag}(\lambda_1)}{2\pi}$, and the growth/decay factor, $\text{real}(\lambda_1)$, of the oscillating system.
 692 λ_1 is the eigenvalue of greatest real part when the imaginary component is zero, and alternatively
 693 of positive imaginary component when the eigenvalues are complex conjugate pairs. To learn the
 694 distribution of real entries of A that produce a band of oscillating systems around 1Hz, we formal-
 695 ized this emergent property as $\text{real}(\lambda_1)$ having mean zero with variance 0.25^2 , and the oscillation
 696 frequency $2\pi\text{imag}(\lambda_1)$ having mean $\omega = 1$ Hz with variance $(0.1\text{Hz})^2$:

$$\mathbb{E}[T(x)] \triangleq \mathbb{E} \begin{bmatrix} \text{real}(\lambda_1) \\ \text{imag}(\lambda_1) \\ (\text{real}(\lambda_1) - 0)^2 \\ (\text{imag}(\lambda_1) - 2\pi\omega)^2 \end{bmatrix} = \begin{bmatrix} 0.0 \\ 2\pi\omega \\ 0.25^2 \\ (2\pi 0.1)^2 \end{bmatrix} \triangleq \mu. \quad (22)$$

697

698 Unlike the models we presented in the main text, this model admits an analytical form for the
 699 mean emergent property statistics given parameter z , since the eigenvalues can be calculated using
 700 the quadratic formula:

$$\lambda = \frac{\left(\frac{a_1+a_4}{\tau}\right) \pm \sqrt{\left(\frac{a_1+a_4}{\tau}\right)^2 + 4\left(\frac{a_2a_3-a_1a_4}{\tau}\right)}}{2}. \quad (23)$$

701 Importantly, even though $\mathbb{E}_{x \sim p(x|z)}[T(x)]$ is calculable directly via a closed form function and
 702 does not require simulation, we cannot derive the distribution q_θ^* directly. This fact is due to the
 703 formally hard problem of the backward mapping: finding the natural parameters η from the mean
 704 parameters μ of an exponential family distribution [70]. Instead, we used EPI to approximate this
 705 distribution (Fig. S1B). We used a real-NVP normalizing flow architecture with four masks, two
 706 neural network layers of 15 units per mask, with batch normalization momentum 0.99, mapped
 707 onto a support of $z_i \in [-10, 10]$. (see Section 5.1.3).

708 Even this relatively simple system has nontrivial (though intuitively sensible) structure in the
 709 parameter distribution. To validate our method, we analytically derived the contours of the prob-
 710 ability density from the emergent property statistics and values. In the a_1 - a_4 plane, the black
 711 line at $\text{real}(\lambda_1) = \frac{a_1+a_4}{2} = 0$, dotted black line at the standard deviation $\text{real}(\lambda_1) = \frac{a_1+a_4}{2} \pm 0.25$,
 712 and the dotted gray line at twice the standard deviation $\text{real}(\lambda_1) = \frac{a_1+a_4}{2} \pm 0.5$ follow the contour
 713 of probability density of the samples (Fig. S2A). The distribution precisely reflects the desired
 714 statistical constraints and model degeneracy in the sum of a_1 and a_4 . Intuitively, the parameters
 715 equivalent with respect to emergent property statistic $\text{real}(\lambda_1)$ have similar log densities.

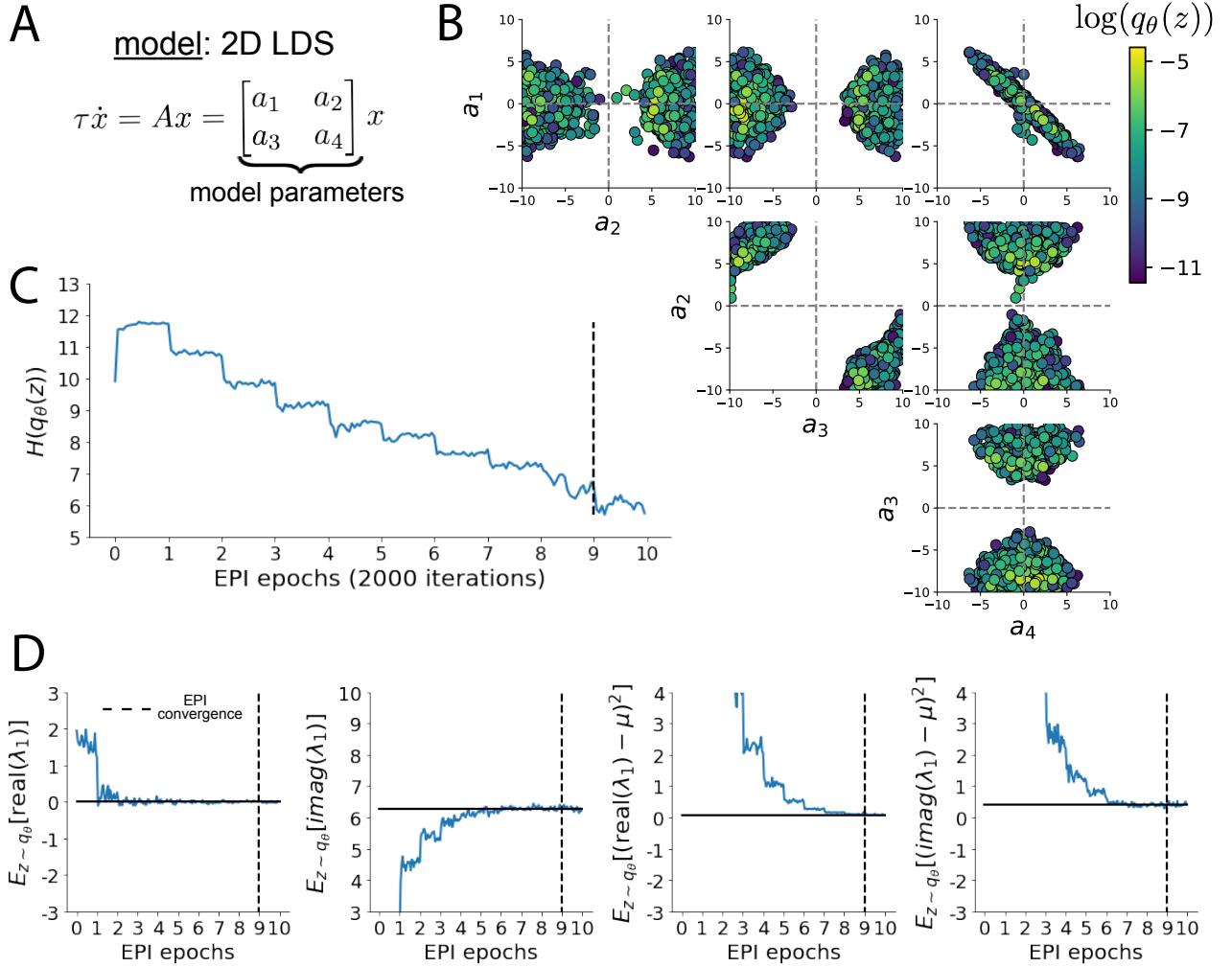


Fig. S1: A. Two-dimensional linear dynamical system model, where real entries of the dynamics matrix A are the parameters. B. The EPI distribution for a two-dimensional linear dynamical system with $\tau = 1$ that produces an average of 1Hz oscillations with some small amount of variance. Dashed lines indicate the parameter axes. C. Entropy throughout the optimization. At the beginning of each augmented Lagrangian epoch (2,000 iterations), the entropy dipped due to the shifted optimization manifold where emergent property constraint satisfaction is increasingly weighted. D. Emergent property moments throughout optimization. At the beginning of each augmented Lagrangian epoch, the emergent property moments adjust closer to their constraints.

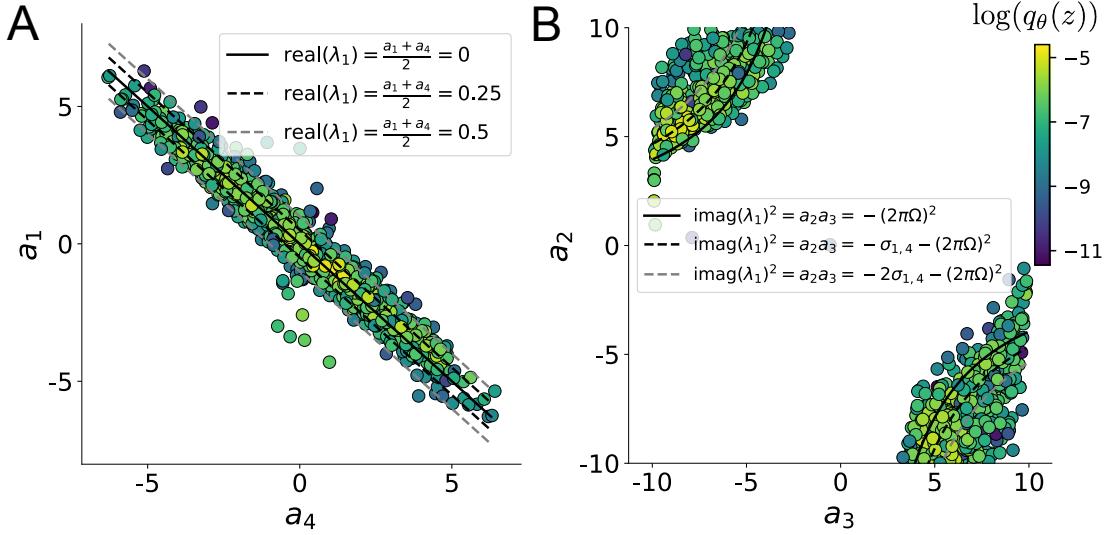


Fig. S2: A. Probability contours in the a_1 - a_4 plane were derived from the relationship to emergent property statistic of growth/decay factor $\text{real}(\lambda_1)$. B. Probability contours in the a_2 - a_3 plane were derived from the emergent property statistic of oscillation frequency $2\pi\text{imag}(\lambda_1)$.

716 To explain the bimodality of the EPI distribution, we examined the imaginary component of λ_1 .
 717 When $\text{real}(\lambda_1) = \frac{a_1 + a_4}{2} = 0$, we have

$$\text{imag}(\lambda_1) = \begin{cases} \sqrt{\frac{a_1 a_4 - a_2 a_3}{\tau}}, & \text{if } a_1 a_4 < a_2 a_3 \\ 0 & \text{otherwise} \end{cases}. \quad (24)$$

718 When $\tau = 1$ and $a_1 a_4 > a_2 a_3$ (center of distribution above), we have the following equation for the
 719 other two dimensions:

$$\text{imag}(\lambda_1)^2 = a_1 a_4 - a_2 a_3 \quad (25)$$

720 Since we constrained $\mathbb{E}_{z \sim q_\theta} [\text{imag}(\lambda)] = 2\pi$ (with $\omega = 1$), we can plot contours of the equation
 721 $\text{imag}(\lambda_1)^2 = a_1 a_4 - a_2 a_3 = (2\pi)^2$ for various $a_1 a_4$ (Fig. S2B). With $\sigma_{1,4} = \mathbb{E}_{z \sim q_\theta} (|a_1 a_4 - E_{q_\theta}[a_1 a_4]|)$,
 722 we show the contours as $a_1 a_4 = 0$ (black), $a_1 a_4 = -\sigma_{1,4}$ (black dotted), and $a_1 a_4 = -2\sigma_{1,4}$ (grey
 723 dotted). This validates the curved structure of the inferred distribution learned through EPI. We
 724 took steps in negative standard deviation of $a_1 a_4$ (dotted and gray lines), since there are few positive
 725 values $a_1 a_4$ in the learned distribution. Subtler combinations of model and emergent property will
 726 have more complexity, further motivating the use of EPI for understanding these systems. As we
 727 expect, the distribution results in samples of two-dimensional linear systems oscillating near 1Hz
 728 (Fig. S3).

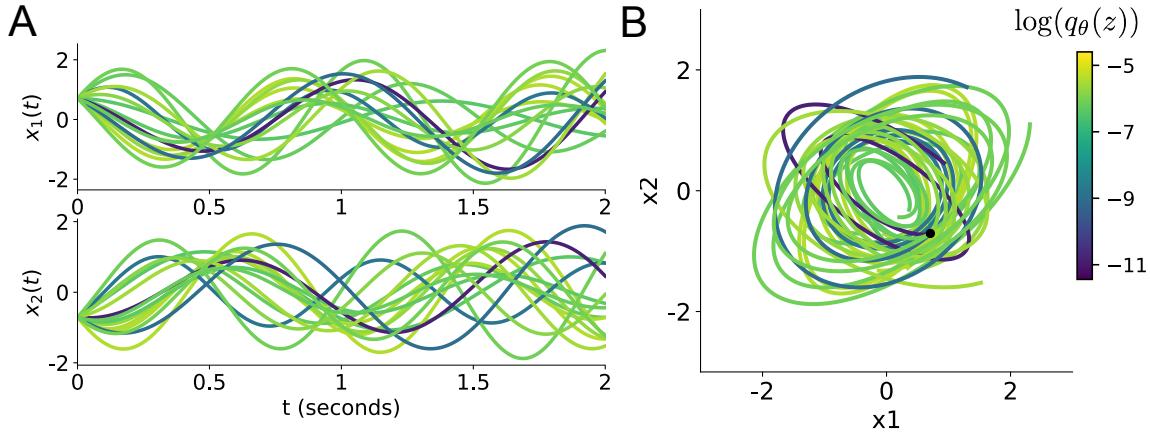


Fig. S3: Sampled dynamical systems $z \sim q_\theta(z)$ and their simulated activity from $x(0) = [\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}]$ colored by log probability. A. Each dimension of the simulated trajectories throughout time. B. The simulated trajectories in phase space.

729 5.1.2 Augmented Lagrangian optimization

730 To optimize $q_\theta(z)$ in Equation 17, the constrained optimization is executed using the augmented
 731 Lagrangian method. The following objective is minimized:

$$L(\theta; \eta, c) = -H(q_\theta) + \eta^\top R(\theta) + \frac{c}{2} \|R(\theta)\|^2 \quad (26)$$

732 where $R(\theta) = \mathbb{E}_{z \sim q_\theta} [\mathbb{E}_{x \sim p(x|z)} [T(x) - \mu]]$, $\eta \in \mathbb{R}^m$ are the Lagrange multipliers where $m = |\mu| =$
 733 $|T(x)|$, and c is the penalty coefficient. These Lagrange multipliers are closely related to the natural
 734 parameters of exponential families (see Section 5.1.4). Deep neural network weights and biases θ of
 735 the deep probability distribution are optimized according to Equation 26 using the Adam optimizer
 736 with its standard parameterization [71]. η is initialized to the zero vector and adapted following
 737 each augmented Lagrangian epoch, which is a period of optimization with fixed (η, c) for a given
 738 number of stochastic optimization iterations. A low value of c is used initially, and conditionally
 739 increased after each epoch based on constraint error reduction. For example, the initial value of
 740 c was $c_0 = 10^{-3}$ during EPI with the oscillating 2D LDS (Fig. S1C). The penalty coefficient is
 741 updated based on the result of a hypothesis test regarding the reduction in constraint violation. The
 742 p-value of $\mathbb{E}[|R(\theta_{k+1})|] > \gamma \mathbb{E}[|R(\theta_k)|]$ is computed, and c_{k+1} is updated to βc_k with probability
 743 $1-p$. The other update rule is $\eta_{k+1} = \eta_k + c_k \frac{1}{n} \sum_{i=1}^n (T(x^{(i)}) - \mu)$ given a batch size n . Throughout
 744 the study, $\beta = 4.0$, $\gamma = 0.25$, and the batch size was a hyperparameter, which varied according to
 745 the application of EPI.

746 The intention is that c and η start at values encouraging entropic growth early in optimization.
747 With each training epoch in which the update rule for c is invoked by unsatisfactory constraint
748 error reduction, the constraint satisfaction terms are increasingly weighted, resulting in a decreased
749 entropy. This encourages the discovery of suitable regions of parameter space, and the subsequent
750 refinement of the distribution to produce the emergent property. In the oscillating 2D LDS example,
751 each augmented Lagrangian epoch ran for 2,000 iterations (Fig. S1C-D). Notice the initial entropic
752 growth, and subsequent reduction upon each update of η and c . The momentum parameters of the
753 Adam optimizer were reset at the end of each augmented Lagrangian epoch.

754 Rather than starting optimization from some θ drawn from a randomized distribution, we found
755 that initializing $q_\theta(z)$ to approximate an isotropic Gaussian distribution conferred more stable, con-
756 sistent optimization. The parameters of the Gaussian initialization were chosen on an application-
757 specific basis. Throughout the study, we chose isotropic Gaussian initializations with mean μ_{init} at
758 the center of the distribution support and some standard deviation σ_{init} , except for one case, where
759 an initialization informed by random search was used (see Section 5.2.2).

760 To assess whether EPI distribution $q_\theta(z)$ produces the emergent property, we defined a hypothesis
761 testing convergence criteria. The algorithm has converged when a null hypothesis test of constraint
762 violations $R(\theta)_i$ being zero is accepted for all constraints $i \in \{1, \dots, m\}$ at a significance threshold
763 $\alpha = 0.05$. This significance threshold is adjusted through Bonferroni correction according to the
764 number of constraints m . The p-values for each constraint are calculated according to a two-tailed
765 nonparametric test, where 200 estimations of the sample mean $R(\theta)^i$ are made from k resamplings
766 of z from a finite sample of size n taken at the end of the augmented Lagrangian epoch. k is
767 determined by a fraction of the batch size ν , which varies according to the application. In the
768 linear two-dimensional system example, we used a batch size of $n = 1000$ and set $\nu = 0.1$ resulting
769 in convergence after the ninth epoch of optimization. (Fig. S1C-D black dotted line).

770 When assessing the suitability of EPI for a particular modeling question, there are some important
771 technical considerations. First and foremost, as in any optimization problem, the defined emergent
772 property should always be appropriately conditioned (constraints should not have wildly different
773 units). Furthermore, if the program is underconstrained (not enough constraints), the distribution
774 grows (in entropy) unstably unless mapped to a finite support. If overconstrained, there is no pa-
775 rameter set producing the emergent property, and EPI optimization will fail (appropriately). Next,
776 one should consider the computational cost of the gradient calculations. In the best circumstance,
777 there is a simple, closed form expression (e.g. Section 5.1.1) for the emergent property statistic

778 given the model parameters. On the other end of the spectrum, many forward simulation iterations
 779 may be required before a high quality measurement of the emergent property statistic is available
 780 (e.g. Section 5.2.1). In such cases, optimization will be expensive.

781 **5.1.3 Normalizing flows**

782 Deep probability models typically consist of several layers of fully connected neural networks.
 783 When each neural network layer is restricted to be a bijective function, the sample density can be
 784 calculated using the change of variables formula at each layer of the network. For $z' = f(z)$,

$$q(z') = q(f^{-1}(z')) \left| \det \frac{\partial f^{-1}(z')}{\partial z'} \right| = q(z) \left| \det \frac{\partial f(z)}{\partial z} \right|^{-1}. \quad (27)$$

785 However, this computation has cubic complexity in dimensionality for fully connected layers. By
 786 restricting our layers to normalizing flows [17] – bijective functions with fast log determinant Ja-
 787 cobian computations, we can tractably optimize deep generative models with objectives that are a
 788 function of sample density, like entropy. Most of our analyses use either a planar flow [17] or real
 789 NVP [72], which have proven effective in our architecture searches. Planar flow architectures are
 790 specified by the number of planar bijection layers used, while real NVP architectures are specified
 791 by the number of masks, neural network layers per mask, units per layer, and batch normalization
 792 momentum parameter.

793 **5.1.4 Emergent property inference as variational inference in an exponential family**

794 Now that we have fully described the EPI method, we consider its broader contextualization as a
 795 statistical method and its relation to Bayesian inference. In Bayesian inference a prior belief about
 796 model parameters z is formalized into a prior distribution $p(z)$, and the statistical model capturing
 797 the effect of z on observed data points x is formalized in the likelihood distribution $p(x | z)$. In
 798 Bayesian inference, we obtain a posterior distribution $p(z | x)$, which captures how the data inform
 799 our knowledge of model parameters using Bayes’ rule:

$$p(z | x) = \frac{p(x | z)p(z)}{p(x)}. \quad (28)$$

800 The posterior distribution is analytically available when the prior is conjugate with the likelihood.
 801 However, conjugacy is rare in practice, and alternative methods, such as variational inference [62],
 802 are utilized.

803 As we compare EPI to variational inference, it is important to consider that EPI is a maximum
 804 entropy method, and that maximum entropy methods have a fundamental relationship with expo-
 805 nential family distributions. A maximum entropy distribution of form:

$$\begin{aligned} p^*(z) &= \operatorname{argmax}_{p \in \mathcal{P}} H(p(z)) \\ \text{s.t. } \mathbb{E}_{z \sim p}[T(z)] &= \mu. \end{aligned} \quad (29)$$

806 will have probability density in the exponential family:

$$p^*(z) \propto \exp(\eta^\top T(z)). \quad (30)$$

807 The mappings between the mean parameterization μ and the natural parameterization η are for-
 808 mally hard to identify [70].

809 Now, consider the goal of doing variational inference with an exponential family posterior dis-
 810 tribution $p(z | x)$. We use the following abbreviated notation to collect the base measure $b(z)$
 811 and sufficient statistics $T(z)$ into $\tilde{T}(z)$ and likewise concatenate a 1 onto the end of the natural
 812 parameter $\tilde{\eta}(x)$. The log normalizing constant $A(\eta(x))$ remains unchanged:

$$\begin{aligned} p(z | x) &= b(z) \exp\left(\eta(x)^\top T(z) - A(\eta(x))\right) = \exp\left(\begin{bmatrix} \eta(x) \\ 1 \end{bmatrix}^\top \begin{bmatrix} T(z) \\ b(z) \end{bmatrix} - A(\eta(x))\right). \\ &= \exp\left(\tilde{\eta}(x)^\top \tilde{T}(z) - A(\eta(x))\right) \end{aligned} \quad (31)$$

813 Variational inference with an exponential family posterior distribution uses optimization to mini-
 814 mize the following divergence [62]:

$$q_\theta^* = \operatorname{argmin}_{q_\theta \in Q} KL(q_\theta || p(z | x)). \quad (32)$$

815 $q_\theta(z)$ is the variational approximation to the posterior with variational parameters θ . We can write
 816 this KL divergence in terms of entropy of the variational approximation:

$$KL(q_\theta || p(z | x)) = \mathbb{E}_{z \sim q_\theta} [\log(q_\theta(z))] - \mathbb{E}_{z \sim q_\theta} [\log(p(z | x))] \quad (33)$$

817

$$= -H(q_\theta) - \mathbb{E}_{z \sim q_\theta} [\tilde{\eta}(x)^\top \tilde{T}(z) - A(\eta(x))]. \quad (34)$$

818 As far as the variational optimization is concerned, the log normalizing constant is independent of
 819 $q_\theta(z)$, so it can be dropped

$$\operatorname{argmin}_{q_\theta \in Q} KL(q_\theta || p(z | x)) = \operatorname{argmin}_{q_\theta \in Q} -H(q_\theta) - \mathbb{E}_{z \sim q_\theta} [\tilde{\eta}(x)^\top \tilde{T}(z)]. \quad (35)$$

820 Further, we can write the objective in terms of the first moment of the sufficient statistics $\mu =$
 821 $\mathbb{E}_{z \sim p(z|x)} [T(z)]$:

$$= \underset{q_\theta \in Q}{\operatorname{argmin}} -H(q_\theta) - \mathbb{E}_{z \sim q_\theta} \left[\tilde{\eta}(x)^\top (\tilde{T}(z) - \mu) \right] + \tilde{\eta}(x)^\top \mu, \quad (36)$$

822 which simplifies to

$$= \underset{q_\theta \in Q}{\operatorname{argmin}} -H(q_\theta) - \mathbb{E}_{z \sim q_\theta} \left[\tilde{\eta}(x)^\top (\tilde{T}(z) - \mu) \right]. \quad (37)$$

823 .

824 In comparison, in emergent property inference (EPI), we solve the following problem:

$$q_\theta^*(z) = \underset{q_\theta \in Q}{\operatorname{argmax}} H(q_\theta(z)), \text{ s.t. } \mathbb{E}_{z \sim q_\theta} [\mathbb{E}_{x \sim p(x|z)} [T(x)]] = \mu. \quad (38)$$

825 The Lagrangian objective (without augmentation) is

$$q_\theta^* = \underset{q_\theta \in Q}{\operatorname{argmin}} -H(q_\theta) + \eta_{\text{opt}}^\top \left(\mathbb{E}_{z \sim q_\theta} [\tilde{T}(z)] - \mu \right). \quad (39)$$

826 Thus, as the optimization proceeds, η_{opt}^\top should converge to the natural parameter $\tilde{\eta}(x)$ through
 827 its adaptations in each epoch (see Section 5.1.2).

828 We have shown that there is indeed a clear relationship between Bayesian inference and EPI.
 829 Specifically, EPI is executing variational inference in an exponential family posterior, whose suffi-
 830 cient statistics are the emergent property statistics and mean parameterization are the emergent
 831 property values. However, in EPI we have not specified a prior distribution, or collected data,
 832 which can inform us about model parameters. Instead we have a mathematical specification of
 833 an emergent property, which the model must produce, and a maximum entropy selection prin-
 834 ciple. Accordingly, we replace the notation of $p(z | x)$ with $p(z | \mathcal{B})$ conceptualizing an inferred
 835 distribution that obeys emergent property \mathcal{B} (see Section 5.1).

836 5.2 Theoretical models

837 In this study, we used emergent property inference to examine several models relevant to theoretical
 838 neuroscience. Here, we provide the details of each model and the related analyses.

839 5.2.1 Stomatogastric ganglion

840 We analyze how the parameters $z = [g_{\text{el}}, g_{\text{synA}}]$ govern the emergent phenomena of network syncing
 841 in a model of the stomatogastric ganglion (STG) [23] shown in Figure 1A with activity $x =$
 842 $[x_{\text{f1}}, x_{\text{f2}}, x_{\text{hub}}, x_{\text{s1}}, x_{\text{s2}}]$, using the same hyperparameter choices as Gutierrez et al. Each neuron's

843 membrane potential $x_\alpha(t)$ for $\alpha \in \{\text{f1, f2, hub, s1, s2}\}$ is the solution of the following differential
 844 equation:

$$C_m \frac{dx_\alpha}{dt} = -[h_{\text{leak}}(x; z) + h_{Ca}(x; z) + h_K(x; z) + h_{hyp}(x; z) + h_{elec}(x; z) + h_{syn}(x; z)]. \quad (40)$$

845 The membrane potential of each neuron is affected by the leak, calcium, potassium, hyperpolariza-
 846 tion, electrical and synaptic currents, respectively, which are functions of all membrane potentials
 847 and the conductance parameters z . The capacitance of the cell membrane was set to $C_m = 1nF$.
 848 Specifically, the currents are the difference in the neuron's membrane potential and that current
 849 type's reversal potential multiplied by a conductance:

$$h_{\text{leak}}(x; z) = g_{\text{leak}}(x_\alpha - V_{\text{leak}}) \quad (41)$$

$$h_{elec}(x; z) = g_{\text{el}}(x_\alpha^{\text{post}} - x_\alpha^{\text{pre}}) \quad (42)$$

$$h_{syn}(x; z) = g_{\text{syn}}S_\infty^{\text{pre}}(x_\alpha^{\text{post}} - V_{\text{syn}}) \quad (43)$$

$$h_{Ca}(x; z) = g_{Ca}M_\infty(x_\alpha - V_{Ca}) \quad (44)$$

$$h_K(x; z) = g_KN(x_\alpha - V_K) \quad (45)$$

$$h_{hyp}(x; z) = g_hH(x_\alpha - V_{hyp}). \quad (46)$$

850 The reversal potentials were set to $V_{\text{leak}} = -40mV$, $V_{Ca} = 100mV$, $V_K = -80mV$, $V_{hyp} = -20mV$,
 851 and $V_{syn} = -75mV$. The other conductance parameters were fixed to $g_{\text{leak}} = 1 \times 10^{-4}\mu S$. g_{Ca} ,
 852 g_K , and g_{hyp} had different values based on fast, intermediate (hub) or slow neuron. The fast
 853 conductances had values $g_{Ca} = 1.9 \times 10^{-2}$, $g_K = 3.9 \times 10^{-2}$, and $g_{hyp} = 2.5 \times 10^{-2}$. The intermediate
 854 conductances had values $g_{Ca} = 1.7 \times 10^{-2}$, $g_K = 1.9 \times 10^{-2}$, and $g_{hyp} = 8.0 \times 10^{-3}$. Finally, the
 855 slow conductances had values $g_{Ca} = 8.5 \times 10^{-3}$, $g_K = 1.5 \times 10^{-2}$, and $g_{hyp} = 1.0 \times 10^{-2}$.

856 Furthermore, the Calcium, Potassium, and hyperpolarization channels have time-dependent gating
 857 dynamics dependent on steady-state gating variables M_∞ , N_∞ and H_∞ , respectively:

$$M_\infty = 0.5 \left(1 + \tanh \left(\frac{x_\alpha - v_1}{v_2} \right) \right) \quad (47)$$

$$\frac{dN}{dt} = \lambda_N(N_\infty - N) \quad (48)$$

$$N_\infty = 0.5 \left(1 + \tanh \left(\frac{x_\alpha - v_3}{v_4} \right) \right) \quad (49)$$

$$\lambda_N = \phi_N \cosh \left(\frac{x_\alpha - v_3}{2v_4} \right) \quad (50)$$

866

$$\frac{dH}{dt} = \frac{(H_\infty - H)}{\tau_h} \quad (51)$$

867

$$H_\infty = \frac{1}{1 + \exp\left(\frac{x_\alpha + v_5}{v_6}\right)} \quad (52)$$

868

$$\tau_h = 272 - \left(\frac{-1499}{1 + \exp\left(\frac{-x_\alpha + v_7}{v_8}\right)} \right). \quad (53)$$

869 where we set $v_1 = 0mV$, $v_2 = 20mV$, $v_3 = 0mV$, $v_4 = 15mV$, $v_5 = 78.3mV$, $v_6 = 10.5mV$,
 870 $v_7 = -42.2mV$, $v_8 = 87.3mV$, $v_9 = 5mV$, and $v_{th} = -25mV$.

871 Finally, there is a synaptic gating variable as well:

$$S_\infty = \frac{1}{1 + \exp\left(\frac{v_{th} - x_\alpha}{v_9}\right)}. \quad (54)$$

872 When the dynamic gating variables are considered, this is actually a 15-dimensional nonlinear
 873 dynamical system.

874 In order to measure the frequency of the hub neuron during EPI, the STG model was simulated
 875 for $T = 200$ time steps of $dt = 25ms$. In EPI, since gradients are taken through the simulation
 876 process, the number of time steps are kept modest if possible. The chosen dt and T were the
 877 most computationally convenient choices yielding accurate frequency measurement. Poor resolution
 878 afforded by the discrete Fourier transform motivated the use of an alternative basis of complex
 879 exponentials to measure spiking frequency. Instead, we used a basis of complex exponentials with
 880 frequencies from 0.0-1.0 Hz at 0.01Hz resolution, $\Phi = [0.0, 0.01, \dots, 1.0]^\top$

881 Another consideration was that the frequency spectra of the neuron membrane potentials had sev-
 882 eral peaks. High-frequency sub-threshold activity obscured the maximum frequency measurement
 883 in the complex exponential basis. Accordingly, subthreshold activity was set to zero, and the
 884 whole signal was low-pass filtered with a moving average window of length 20. The signal was
 885 subsequently mean centered. After this preprocessing, the maximum frequency in the filter bank
 886 accurately reflected the firing frequency.

887 Finally, to differentiate through the maximum frequency identification, we used a sum-of-powers
 888 normalization. Let $\mathcal{X}_\alpha \in \mathcal{C}^{|\Phi|}$ be the complex exponential filter bank dot products with the signal
 889 $x_\alpha \in \mathbb{R}^N$, where $\alpha \in \{f1, f2, \text{hub}, s1, s2\}$. The “frequency identification” vector is

$$v_\alpha = \frac{|\mathcal{X}_\alpha|^\beta}{\sum_{k=1}^N |\mathcal{X}_\alpha(k)|^\beta}. \quad (55)$$

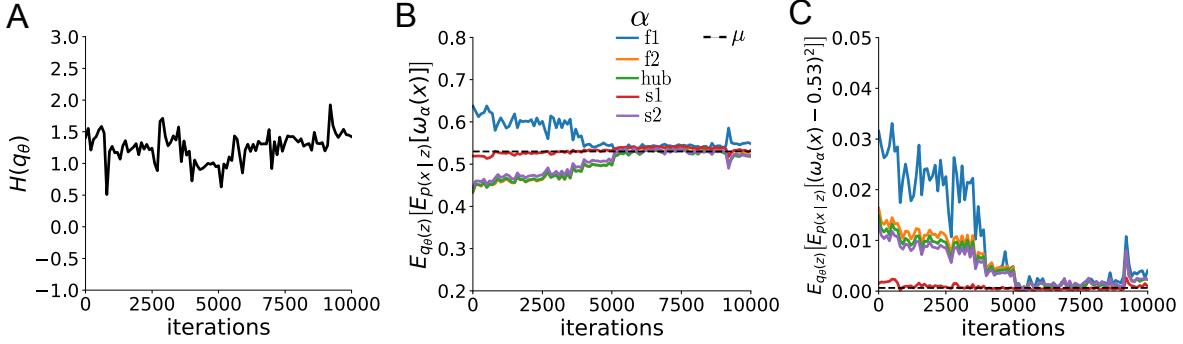


Fig. S4: EPI optimization of the STG model producing network syncing. A. Entropy throughout optimization. B. The first moment emergent property statistics converge to the emergent property values at 10,000 iterations, following the fourth augmented Lagrangian epoch of 2,500 iterations. Since $q_\theta(z)$ failed to produce enough samples yielding $\omega_{f1}(x)$ less than 0.53Hz, the convergence criteria were not satisfied after the third epoch at 7,500 iterations. C. The second moment emergent property statistics converge to the emergent property values.

- 890 The frequency is then calculated as $\omega_\alpha = v_\alpha^\top \Phi$ with $\beta = 100$.
 891 Network syncing, like all other emergent properties in this work, are defined by the emergent
 892 property statistics and values. The emergent property statistics are the first and second moments
 893 of the firing frequencies. The first moments were set to 0.53Hz, and the second moments were set
 894 to 0.025Hz²:

$$E \begin{bmatrix} \omega_{f1} \\ \omega_{f2} \\ \omega_{\text{hub}} \\ \omega_{s1} \\ \omega_{s2} \\ (\omega_{f1} - 0.53)^2 \\ (\omega_{f2} - 0.53)^2 \\ (\omega_{\text{hub}} - 0.53)^2 \\ (\omega_{s1} - 0.53)^2 \\ (\omega_{s2} - 0.53)^2 \end{bmatrix} = \begin{bmatrix} 0.53 \\ 0.53 \\ 0.53 \\ 0.53 \\ 0.53 \\ 0.025^2 \\ 0.025^2 \\ 0.025^2 \\ 0.025^2 \\ 0.025^2 \end{bmatrix} \quad (56)$$

- 895 for the EPI distribution shown in Fig. 1B. Throughout optimization, the augmented Lagrangian
 896 parameters η and c , were updated after each epoch of 2,500 iterations (see Section 5.1.2). The
 897 optimization converged after four epochs (Fig. S4).

998 For EPI in Fig 2C, we used a real NVP architecture with four masks and two layers of 10 units
 899 per mask, and batch normalization momentum of 0.99 mapped onto a support of $z = [g_{\text{el}}, g_{\text{synA}}] \in$
 900 $[4, 8] \times [0, 4]$. We used an augmented Lagrangian coefficient of $c_0 = 10^2$, a batch size $n = 300$,
 901 set $\nu = 0.1$, and initialized $q_\theta(z)$ to produce an isotropic Gaussian with mean $\mu_{\text{init}} = [6, 2]$ with
 902 standard deviation $\sigma_{\text{init}} = 0.5$.

903 We calculated the Hessian at the mode of the inferred EPI distribution. The Hessian of a probability
 904 model is the second order gradient of the log probability density $\log q_\theta(z)$ with respect to the
 905 parameters z : $\frac{\partial^2 \log q_\theta(z)}{\partial z \partial z^\top}$. With EPI, we can examine the Hessian, which is analytically available
 906 throughout distribution, to indicate the dimensions of parameter space that are sensitive (high
 907 magnitude eigenvalue), and which are degenerate (low magnitude eigenvalue) with respect to the
 908 emergent property produced. In Figure 1B, the eigenvectors of the Hessian v_1 and v_2 are shown
 909 evaluated at the mode of the distribution. The length of the arrows is inversely proportional to the
 910 square root of absolute value of their eigenvalues $\lambda_1 = -10.8$ and $\lambda_2 = -2.27$. We quantitatively
 911 measured the sensitivity of the model with respect to network syncing along the eigenvectors of the
 912 Hessian (Fig. 1B, inset). Sensitivity was measured as the slope coefficient of linear regression fit
 913 to network syncing error (the sum of squared differences of each neuron's frequency from 0.53Hz)
 914 as a function of parametric perturbation magnitude (maximum 0.25) away from the mode along
 915 both orientations indicated by the eigenvector with 100 equally spaced samples. The sensitivity
 916 slope coefficient of eigenvector v_1 with respect to network syncing was significant ($\beta = 4.82 \times 10^{-2}$,
 917 $p < 10^{-4}$). In contrast, eigenvector v_2 did not identify a dimension of parameter space significantly
 918 sensitive to network syncing ($\beta = 8.65 \times 10^{-4}$ with $p = .67$). These sensitivities were compared to
 919 all other dimensions of parameter space (100 equally spaced angles from 0 to π), revealing that the
 920 Hessian eigenvectors indeed identified the directions of greatest sensitivity and degeneracy (Fig.
 921 1B, inset). The contours of Figure 1 were calculated as error in $T(x)$ from μ in both the first and
 922 second moment emergent property statistics.

923 **5.2.2 Primary visual cortex**

924 The dynamics of each neural populations average rate $x = [x_E, x_P, x_S, x_V]^\top$ are given by:

$$\tau \frac{dx}{dt} = -x + [Wx + h]_+^n. \quad (57)$$

925 By consolidating information from many experimental datasets, Billeh et al. [42] produce estimates

926 of the synaptic strength (in mV)

$$M = \begin{bmatrix} 0.36 & 0.48 & 0.31 & 0.28 \\ 1.49 & 0.68 & 0.50 & 0.18 \\ 0.86 & 0.42 & 0.15 & 0.32 \\ 1.31 & 0.41 & 0.52 & 0.37 \end{bmatrix} \quad (58)$$

927 and connection probability

$$C = \begin{bmatrix} 0.16 & 0.411 & 0.424 & 0.087 \\ 0.395 & .451 & 0.857 & 0.02 \\ 0.182 & 0.03 & 0.082 & 0.625 \\ 0.105 & 0.22 & 0.77 & 0.028 \end{bmatrix}. \quad (59)$$

928 Multiplying these connection probabilities and synaptic efficacies gives us an effective connectivity

929 matrix:

$$W_{\text{full}} = C \odot M = \begin{bmatrix} 0.16 & 0.411 & 0.424 & 0.087 \\ 0.395 & .451 & 0.857 & 0.02 \\ 0.182 & 0.03 & 0.082 & 0.625 \\ 0.105 & 0.22 & 0.77 & 0.028 \end{bmatrix}. \quad (60)$$

930 Theoretical work on these systems considers a subset of the effective connectivities [24, 39, 40]

$$W = \begin{bmatrix} W_{EE} & W_{EP} & W_{ES} & 0 \\ W_{PE} & W_{PP} & W_{PS} & 0 \\ W_{SE} & 0 & 0 & W_{SV} \\ W_{VE} & W_{VP} & W_{VS} & 0 \end{bmatrix}. \quad (61)$$

931 In coherence with this work, we only keep the entries of W_{full} corresponding to parameters in
932 Equation 61.

933 We look at how this four-dimensional nonlinear dynamical model of V1 responds to different inputs,
934 and compare the predictions of the linear response to the approximate posteriors obtained through
935 EPI. The input to the system is the sum of a baseline input $b = [1, 1, 1, 1]^\top$ and a differential input
936 dh :

$$h = b + dh. \quad (62)$$

937 All simulations of this system had $T = 100$ time points, a time step $dt = 5\text{ms}$, and time constant
938 $\tau = 20\text{ms}$. The system was initialized to a random draw $x(0)_i \sim \mathcal{N}(1, 0.01)$.

939 We can describe the dynamics of this system more generally by

$$\dot{x}_i = -x_i + f(u_i) \quad (63)$$

940 where the input to each neuron is

$$u_i = \sum_j W_{ij} x_j + h_i. \quad (64)$$

941 Let $F_{ij} = \gamma_i \delta(i, j)$, where $\gamma_i = f'(u_i)$. Then, the linear response is

$$\frac{dx_{ss}}{dh} = F(W \frac{dx_{ss}}{dh} + I) \quad (65)$$

942 which is calculable by

$$\frac{dx_{ss}}{dh} = (F^{-1} - W)^{-1}. \quad (66)$$

943 This calculation is used to produce the magenta lines in Figure 2C, which show the linearly predicted
944 inputs that generate a response from two standard deviations (of \mathcal{B}) below and above y .

945 The emergent property we considered was the first and second moments of the change in steady
946 state rate dx_{ss} between the baseline input $h = b$ and $h = b + dh$. We use the following notation to
947 indicate that the emergent property statistics were set to the following values:

$$\mathcal{B}(\alpha, y) \triangleq \mathbb{E} \begin{bmatrix} dx_{\alpha,ss} \\ (dx_{\alpha,ss} - y)^2 \end{bmatrix} = \begin{bmatrix} y \\ 0.01^2 \end{bmatrix}. \quad (67)$$

948 In the final analysis for this model, we sweep the input one neuron at a time away from the mode
949 of each inferred distributions $dh^* = z^* = \text{argmax}_z \log q_\theta(z | \mathcal{B}(\alpha, 0.1))$. The differential responses
950 $\delta x_{\alpha,ss}$ are examined at perturbed inputs $h = b + dh^* + \delta h_\alpha \hat{u}_\alpha$ where \hat{u}_α is a unit vector in the
951 dimension of α and δx is evaluated at 101 equally spaced samples of δh_α from -15 to 15.

952 We measured the linear regression slope between neuron-types of δx and δh to confirm the hy-
953 potheses H1-H3 (H4 is simply observing the nonmonotonicity) and report the p values for tests of
954 non-zero slope.

955 H1: the neuron-type responses are sensitive to their direct inputs. E-population: $\beta = 1.62$,
956 $p < 10^{-4}$ (Fig. 3A black), P-population: $\beta = 1.06$, $p < 10^{-4}$ (Fig. 3B blue), S-population:
957 $\beta = 6.80$, $p < 10^{-4}$ (Fig. 3C red), V-population: $\beta = 6.41$, $p < 10^{-4}$ (Fig. 3D green).

958 H2: the E-population ($\beta = 0$, $p = 1$) and P-populations ($\beta = 0$, $p = 1$) are not affected by
959 δh_V (Fig. 3A green, 3B green);

960 H3: the S-population is not affected by δh_P ($\beta = 0$, $p = 1$) (Fig. 3C blue);

961

962 For each $\mathcal{B}(\alpha, y)$ with $\alpha \in \{E, P, S, V\}$ and $y \in \{0.1, 0.5\}$, we ran EPI using a real NVP architecture
963 of four masks layers with two hidden layers of 10 units, mapped to a support of $z_i \in [-5, 5]$ with
964 no batch normalization. We used an augmented Lagrangian coefficient of $c_0 = 10^5$, a batch size
965 $n = 1000$, set $\nu = 0.5$. The EPI distributions shown in Fig. 2 are the converged distributions with
966 maximum entropy across random seeds.

967 We set the parameters of the Gaussian initialization μ_{init} and Σ_{init} to the mean and covariance of
968 random samples $z^{(i)} \sim \mathcal{U}(-5, 5)$ that produced emergent property statistic $dx_{\alpha,ss}$ within a bound
969 ϵ of the emergent property value y . $\epsilon = 0.01$ was set to be one standard deviation of the emergent
970 property value according to the emergent property value 0.01^2 of the variance emergent property
971 statistic.

972 5.2.3 Superior colliculus

973 In the model of Duan et al [25], there are four total units: two in each hemisphere corresponding to
974 the Pro/Contra and Anti/Ipsi populations. They are denoted as left Pro (LP), left Anti (LA), right
975 Pro (RP) and right Anti (RA). Each unit has an activity (x_α) and internal variable (u_α) related
976 by

$$x_\alpha = \left(\frac{1}{2} \tanh \left(\frac{u_\alpha - \epsilon}{\zeta} \right) + \frac{1}{2} \right) \quad (68)$$

977 where $\alpha \in \{LP, LA, RA, RP\}$ $\epsilon = 0.05$ and $\zeta = 0.5$ control the position and shape of the nonlin-
978 earity, respectively.

979 We order the elements of x and u in the following manner

$$x = \begin{bmatrix} x_{LP} \\ x_{LA} \\ x_{RP} \\ x_{RA} \end{bmatrix} \quad u = \begin{bmatrix} u_{LP} \\ u_{LA} \\ u_{RP} \\ u_{RA} \end{bmatrix}. \quad (69)$$

980 The internal variables follow dynamics:

$$\tau \frac{du}{dt} = -u + Wx + h + \sigma dB \quad (70)$$

981 with time constant $\tau = 0.09s$ and Gaussian noise σdB controlled by the magnitude of $\sigma = 1.0$. The
982 weight matrix has 8 parameters sW_P , sW_A , vW_{PA} , vW_{AP} , hW_P , hW_A , dW_{PA} , and dW_{AP} (Fig.

983 4B):

$$W = \begin{bmatrix} sW_P & vW_{PA} & hW_P & dW_{PA} \\ vW_{AP} & sW_A & dW_{AP} & hW_A \\ hW_P & dW_{PA} & sW_P & vW_{PA} \\ dW_{AP} & hW_A & vW_{AP} & sW_A \end{bmatrix}. \quad (71)$$

984 The system receives five inputs throughout each trial, which has a total length of 1.8s.

$$h = h_{\text{rule}} + h_{\text{choice-period}} + h_{\text{light}}. \quad (72)$$

985 There are rule-based inputs depending on the condition,

$$h_{P,\text{rule}}(t) = \begin{cases} I_{P,\text{rule}}[1, 0, 1, 0]^\top, & \text{if } t \leq 1.2s \\ 0, & \text{otherwise} \end{cases} \quad (73)$$

986

$$h_{A,\text{rule}}(t) = \begin{cases} I_{A,\text{rule}}[0, 1, 0, 1]^\top, & \text{if } t \leq 1.2s \\ 0, & \text{otherwise} \end{cases} \quad (74)$$

987 a choice-period input,

$$h_{\text{choice}}(t) = \begin{cases} I_{\text{choice}}[1, 1, 1, 1]^\top, & \text{if } t > 1.2s \\ 0, & \text{otherwise} \end{cases} \quad (75)$$

988 and an input to the right or left-side depending on where the light stimulus is delivered.

$$h_{\text{light}}(t) = \begin{cases} I_{\text{light}}[1, 1, 0, 0]^\top, & \text{if } t > 1.2s \text{ and Left} \\ I_{\text{light}}[0, 0, 1, 1]^\top, & \text{if } t > 1.2s \text{ and Right} \\ 0, & t \leq 1.2s \end{cases}. \quad (76)$$

989 The input parameterization was fixed to $I_{P,\text{rule}} = 10$, $I_{A,\text{rule}} = 10$, $I_{\text{choice}} = 2$, and $I_{\text{light}} = 1$.

990 To produce an accuracy rate of p_{LP} in the Left, Pro condition, let \hat{p}_i be the empirical average
 991 steady state response (final x_{LP} at end of task) over M=500 Gaussian noise draws for a given SC
 992 model parameterization z_i :

$$\hat{p}_i = \mathbb{E}_{\sigma dB} [x_{LP} | s = L, c = P, z = z_i] = \frac{1}{M} \sum_{j=1}^M x_{LP}(s = L, c = P, z = z_i, \sigma dB_j) \quad (77)$$

993 where stimulus $s \in \{L, R\}$, cue $c \in \{P, A\}$, and σdB_j is the Gaussian noise on trial j . As with the
 994 V1 model, we only consider steady state responses of x , so x_α is used from here on to denote the

995 steady state activity at the end of the trial. For the first emergent property statistic, the average
 996 over EPI samples (from $q_\theta(z)$) is set to the desired value p_{LP} :

$$\mathbb{E}_{z_i \sim q_\phi} [\mathbb{E}_{\sigma dB} [x_{LP,ss} | s = L, c = P, z = z_i]] = \mathbb{E}_{z_i \sim q_\phi} [\hat{p}_i] = p_{LP}. \quad (78)$$

997 For the next emergent property statistic, we ask that the variance of the steady state responses
 998 across Gaussian draws, is the Bernoulli variance for the empirical rate \hat{p}_i :

$$\mathbb{E}_{z \sim q_\phi} [\sigma_{err}^2] = 0 \quad (79)$$

999 where the Bernoulli variance error σ_{err}^2 for the Pro task, left condition is

$$\sigma_{err}^2 = Var_{\sigma dB} [x_{LP} | s = L, c = P, z = z_i] - \hat{p}_i(1 - \hat{p}_i). \quad (80)$$

1000 We have an additional constraint that the Pro neuron on the opposite hemisphere should have the
 1001 opposite value (0 and 1). We can enforce this with another constraint:

$$\mathbb{E}_{z \sim q_\phi} [d_P] = 1, \quad (81)$$

1002 where the distance between Pro neuron steady states d_P in the Pro condition is

$$d_P = \mathbb{E}_{\sigma dB} [(x_{LP} - x_{RP})^2 | s = L, c = P, z = z_i] \quad (82)$$

1003 The emergent property statistics only need to be measured during the Left stimulus condition of
 1004 the Pro and Anti tasks, since the network is symmetrically parameterized. In total, the emergent
 1005 property of rapid task switching at accuracy level p was defined as

$$\mathcal{B}(p) \triangleq \mathbb{E} \begin{bmatrix} \hat{p}_P \\ \hat{p}_A \\ (\hat{p}_P - p)^2 \\ (\hat{p}_A - p)^2 \\ \sigma_{P,err}^2 \\ \sigma_{A,err}^2 \\ d_P \\ d_A \end{bmatrix} = \begin{bmatrix} p \\ p \\ 0.15^2 \\ 0.15^2 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}. \quad (83)$$

1006 Since the maximum variance of a random variable bounded from 0 to 1 is the Bernoulli variance
 1007 $\hat{p}(1 - \hat{p})$, and the maximum squared difference between two variables bounded from 0 to 1 is 1, we
 1008 do not need to control the second moment of these test statistics. These variables are dynamical

1009 system states and can only exponentially decay (or saturate) to 0 (or 1), so the Bernoulli variance
 1010 error and squared difference constraints cannot be satisfied exactly in simulation. This is important
 1011 to be mindful of when evaluating the convergence criteria. Instead of using our usual hypothesis
 1012 testing criteria for convergence to the emergent property, we set a slack variable threshold only for
 1013 these technically infeasible emergent property values to 0.05.

1014 Using EPI to learn distributions of dynamical systems producing Bernoulli responses at a given rate
 1015 (with small variance around that rate) was more challenging than expected. There is a pathology in
 1016 this optimization setup, where the learned distribution of weights is bimodal attributing a fraction
 1017 p of the samples to an expansive mode (which always sends x_{LP} to 1), and a fraction $1 - p$ to a
 1018 decaying mode (which always sends x_{LP} to 0). This pathology was avoided using an inequality
 1019 constraint prohibiting parameter samples that resulted in low variance of responses across noise.

λ	\hat{p}	$q_\theta(z)$	r	p-value
λ_{task}	\hat{p}_P	$q(z \mid \mathcal{B}(60\%))$	1.24×10^{-01}	$p < 10^{-4}$
λ_{task}	\hat{p}_P	$q(z \mid \mathcal{B}(70\%))$	7.56×10^{-01}	$p < 10^{-4}$
λ_{task}	\hat{p}_P	$q(z \mid \mathcal{B}(80\%))$	4.59×10^{-01}	$p < 10^{-4}$
λ_{task}	\hat{p}_P	$q(z \mid \mathcal{B}(90\%))$	3.76×10^{-01}	$p < 10^{-4}$
λ_{task}	\hat{p}_A	$q(z \mid \mathcal{B}(60\%))$	4.80×10^{-02}	$p < .01$
λ_{task}	\hat{p}_A	$q(z \mid \mathcal{B}(70\%))$	2.08×10^{-01}	$p < 10^{-4}$
λ_{task}	\hat{p}_A	$q(z \mid \mathcal{B}(80\%))$	4.84×10^{-01}	$p < 10^{-4}$
λ_{task}	\hat{p}_A	$q(z \mid \mathcal{B}(90\%))$	4.25×10^{-01}	$p < 10^{-4}$
λ_{side}	\hat{p}_P	$q(z \mid \mathcal{B}(50\%))$	-7.57×10^{-02}	$p < 10^{-4}$
λ_{side}	\hat{p}_P	$q(z \mid \mathcal{B}(60\%))$	-6.73×10^{-02}	$p < 10^{-4}$
λ_{side}	\hat{p}_P	$q(z \mid \mathcal{B}(70\%))$	-4.86×10^{-01}	$p < 10^{-4}$
λ_{side}	\hat{p}_P	$q(z \mid \mathcal{B}(80\%))$	-1.43×10^{-01}	$p < 10^{-4}$
λ_{side}	\hat{p}_P	$q(z \mid \mathcal{B}(90\%))$	-1.93×10^{-01}	$p < 10^{-4}$
λ_{side}	\hat{p}_A	$q(z \mid \mathcal{B}(60\%))$	-7.60×10^{-02}	$p < 10^{-4}$
λ_{side}	\hat{p}_A	$q(z \mid \mathcal{B}(70\%))$	-2.73×10^{-01}	$p < 10^{-4}$
λ_{side}	\hat{p}_A	$q(z \mid \mathcal{B}(80\%))$	-2.74×10^{-01}	$p < 10^{-4}$

Table 1: Table of significant correlation values from Fig. 4E.

1020 For each accuracy level p , we ran EPI for 10 different random seeds using an architecture of 10
 1021 planar flows with a support of $z \in \mathbb{R}^8$. We used an augmented Lagrangian coefficient of $c_0 = 10^2$, a

batch size $n = 300$, and set $\nu = 0.5$, and initialized $q_\theta(z)$ to produce an isotropic Gaussian of zero mean with standard deviation $\sigma_{\text{init}} = 1$. The EPI distributions shown in Fig. 4 are the converged distributions with maximum entropy across random seeds.

We report significant correlations r and their p-values from Figure 4E in Table 1. Correlations were measured from 5,000 samples of $q_\theta(z | \mathcal{B}(p))$ and p-values are reported for one-tailed tests, since we hypothesized a positive correlation between task accuracies p_P or p_A and λ_{task} , and a negative correlation between task accuracies p_P and p_A and λ_{side} .

5.2.4 Rank-1 RNN

Extensive research on random fully-connected recurrent neural networks has resulted in foundational theories of their activity [3, 73]. Furthermore, independent research on training these models to perform computations suggests that learning occurs through low-rank perturbations to the connectivity (e.g. [74, 75]). Recent theoretical work extends theory for random neural networks [3] to those with added low-rank structure [26]. In Section 3.5, we used this theory to enable EPI on RNN parameters conditioned on the emergent property of task execution.

Such RNNs have the following dynamics:

$$\frac{dx}{dt} = -x + W\phi(x) + h, \quad (84)$$

where x is network activity, W is the connectivity weight matrix, $\phi(\cdot) = \tanh(\cdot)$ is the input-output function, and h is the input to the system. In a rank-1 RNN (which was sufficiently complex for the Gaussian posterior conditioning task), W is the sum of a random component with strength g and a structured component determined by the outer product of vectors m and n :

$$W = g\chi + \frac{1}{N}mn^\top, \quad (85)$$

where $\chi_{ij} \sim \mathcal{N}(0, \frac{1}{N})$, and the entries of m and n are distributed as $m_i \sim \mathcal{N}(M_m, 1)$ and $n_i \sim \mathcal{N}(M_n, 1)$. For EPI, we consider $z = [g, M_m, M_n]$, which are the parameters governing the connectivity properties of the RNN.

From such a parameterization z , the theory of Mastrogiovanni et al. produces solutions for variables describing the low dimensional response properties of the RNN. These “dynamic mean field” (DMF) variables (e.g. the activity along a vector κ_v , the total variance Δ_0 , structured variance Δ_∞ , and the chaotic variance Δ_T) are derived to be functions of one another and connectivity parameters z . The collection of these derived functions results in a system of equations, whose solution must

1049 be obtained through a nonlinear system of equations solver. The iterative steps of this system
 1050 of equations solver are differentiable, so we take gradients through this solve process. The DMF
 1051 variables provide task-relevant information about the RNN's response to task inputs.

1052 In the Gaussian posterior conditioning example, κ_r and Δ_T are DMF variables used as task-relevant
 1053 emergent property statistics μ_{post} and σ_{post}^2 . Specifically, we solve for the DMF variables κ_r , κ_n ,
 1054 Δ_0 and Δ_∞ , where the readout is nominally chosen to point in the unit orthant $r = [1, \dots, 1]^\top$. The
 1055 consistency equations for these variables in the presence of a constant input $h = yr - (n - M_n)$ can
 1056 be derived following [26]:

$$\begin{aligned} \kappa_r &= G_1(\kappa_r, \kappa_n, \Delta_0, \Delta_\infty) = M_m \kappa_n + y \\ \kappa_n &= G_2(\kappa_r, \kappa_n, \Delta_0, \Delta_\infty) = M_n \langle [\phi_i] \rangle + \langle [\phi'_i] \rangle \\ \frac{\Delta_0^2 - \Delta_\infty^2}{2} &= G_3(\kappa_r, \kappa_n, \Delta_0, \Delta_\infty) = g^2 \left(\int \mathcal{D}z \Phi^2(\kappa_r + \sqrt{\Delta_0} z) - \int \mathcal{D}z \int \mathcal{D}x \Phi(\kappa_r + \sqrt{\Delta_0 - \Delta_\infty} x + \sqrt{\Delta_\infty} z) \right) \\ &\quad + (\kappa_n^2 + 1)(\Delta_0 - \Delta_\infty) \\ \Delta_\infty &= G_4(\kappa_r, \kappa_n, \Delta_0, \Delta_\infty) = g^2 \int \mathcal{D}z \left[\int \mathcal{D}x \phi(\kappa_r + \sqrt{\Delta_0 - \Delta_\infty} x + \sqrt{\Delta_\infty} z) \right]^2 + \kappa_n^2 + 1 \end{aligned} \quad (86)$$

1057 where here z is a gaussian integration variable. We can solve these equations by simulating the
 1058 following Langevin dynamical system to a steady state:

$$\begin{aligned} l(t) &= \frac{\Delta_0(t)^2 - \Delta_\infty(t)^2}{2} \\ \Delta_0(t) &= \sqrt{2l(t) + \Delta_\infty(t)^2} \\ \frac{d\kappa_r(t)}{dt} &= -\kappa_r(t) + G_1(\kappa_r(t), \kappa_n(t), \Delta_0(t), \Delta_\infty(t)) \\ \frac{d\kappa_n(t)}{dt} &= -\kappa_n(t) + G_2(\kappa_r(t), \kappa_n(t), \Delta_0(t), \Delta_\infty(t)) \\ \frac{dl(t)}{dt} &= -l(t) + G_3(\kappa_r(t), \kappa_n(t), \Delta_0(t), \Delta_\infty(t)) \\ \frac{d\Delta_\infty(t)}{dt} &= -\Delta_\infty(t) + G_4(\kappa_r(t), \kappa_n(t), \Delta_0(t), \Delta_\infty(t)) \end{aligned} \quad (87)$$

1059 Then, the chaotic variance, which is necessary for the Gaussian posterior conditioning example, is
 1060 simply calculated via $\Delta_T = \Delta_0 - \Delta_\infty$.

1061 We ran EPI using a real NVP architecture of two masks and two layers per mask with 10 units
 1062 mapped to a support of $z = [g, M_m, M_n] \in [0, 5] \times [-5, 5] \times [-5, 5]$ with no batch normalization.
 1063 We used an augmented Lagrangian coefficient of $c_0 = 1$, a batch size $n = 300$, set $\nu = 0.15$,
 1064 and initialized $q_\theta(z)$ to produce an isotropic Gaussian with mean $\mu_{\text{init}} = [2.5, 0, 0]$ with standard

1065 deviation $\sigma_{\text{init}} = 2.0$. The EPI distribution shown in Fig. 5 is the converged distributions with
1066 maximum entropy across five random seeds.

1067 To examine the effect of product $M_m M_n$ on the posterior mean, μ_{post} we took perturbations in
1068 $M_m M_n$ away from two representative parameters z_1 and z_2 in 21 equally space increments from
1069 -1 to 1. For each perturbation, we sampled 10 2,000-neuron RNNs and measure the calculated
1070 posterior means. In Fig. 5D, we plot the product of $M_m M_n$ in the perturbation versus the average
1071 posterior mean across 10 network realizations with standard error bars. The correlation between
1072 perturbation product $M_m M_n$ and μ_{post} was measured over all simulations. For perturbations away
1073 from z_1 the correlation was 0.995 with $p < 10^{-4}$, and for perturbations away from z_2 the correlation
1074 was 0.983 with $p < 10^{-4}$.

1075 In addition to the Gaussian posterior conditioning example in Section 3.5, we modeled two tasks
1076 from Mastrogiuseppe et al.: noisy detection and context-dependent discrimination. We used the
1077 same theoretical equations and task setups described in their study.

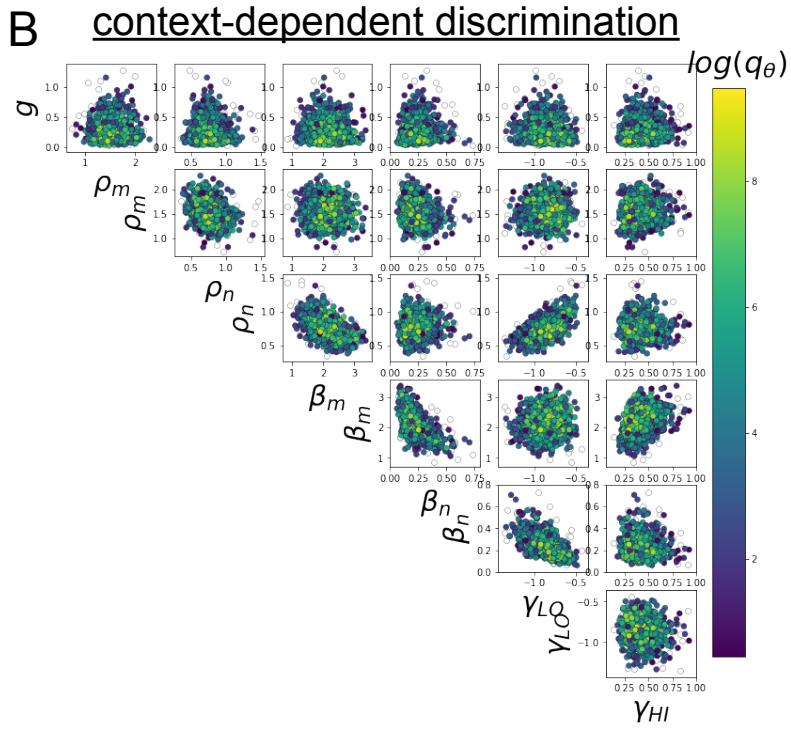
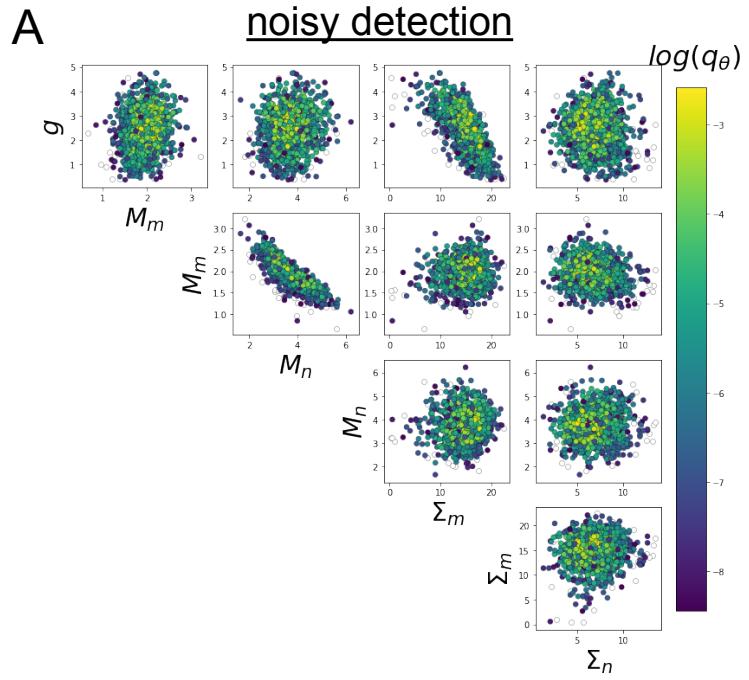


Fig. S5: A. EPI for rank-1 networks doing noisy discrimination. B. EPI for rank-2 networks doing context-dependent discrimination. See [26] for theoretical equations and task description.