Mathematical concept behind raytracing a sphere or circle

Introduction

Every time I'm gonna talk about raytracing it's referred to a sphere or circle if not mentioned otherwise.
When talking about raytracing we talk about having a circle and a single point in our coordinate system that's emitting lines to which we'll refer as rays.
So now we need a point in space, a direction and a circle. What we're gonna calculate is the scalar for our line.
Although we're gonna go through the process with a circle, tracing a sphere works exactly the same way but you have x, y, z coordinates instead of just x, y what leads to solving more parenthesis. A sphere is being mentioned down below but without in depth explanations.

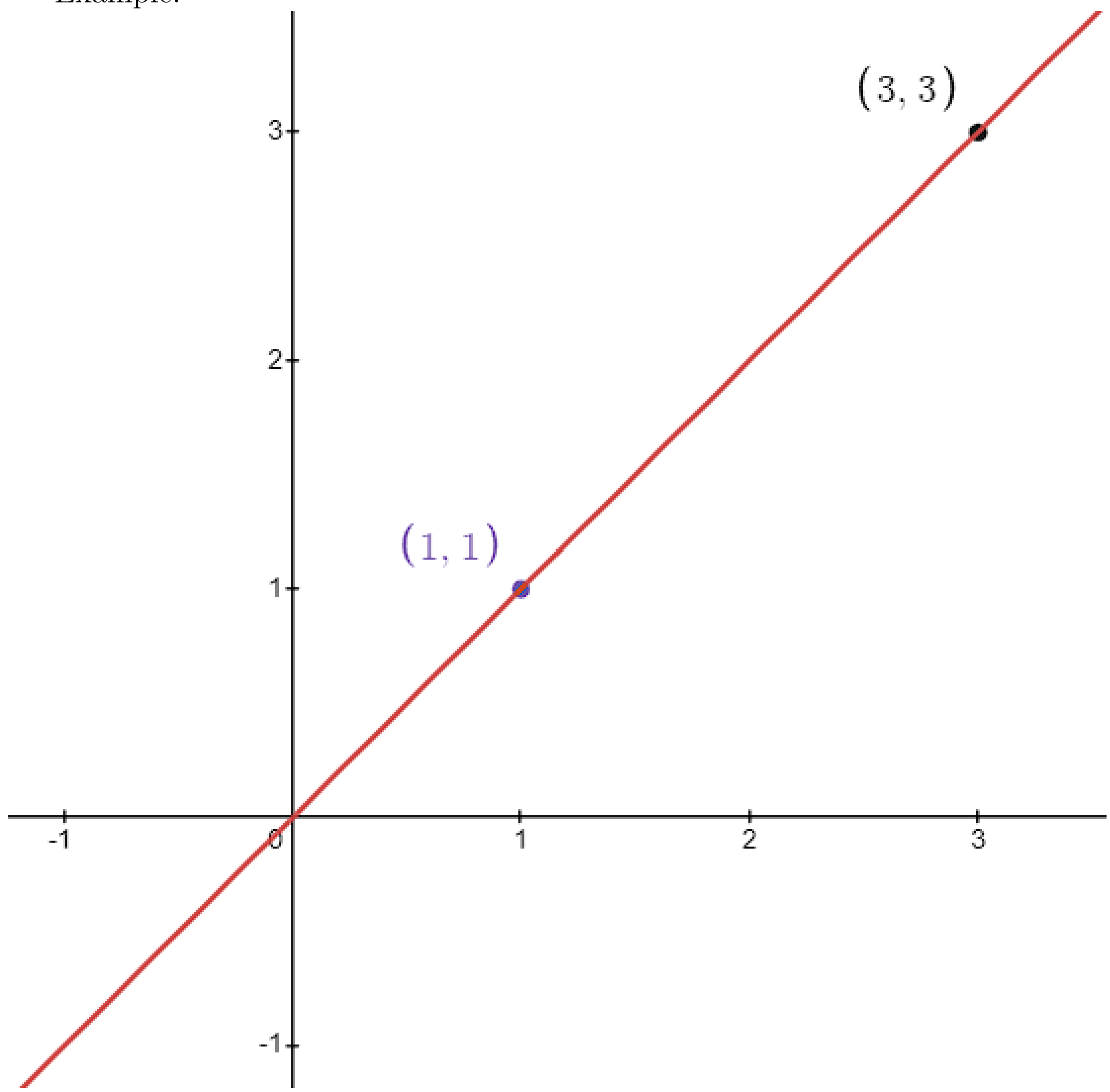
Line equation

We have a line that's defined with $y = mx + c$. But we don't have this equation just yet since we only have an origin and a direction. Now we want to "emit" a line from our origin and see if it will ever intersect with our circle. If it does our intersection is defined with $\vec{c} = \vec{a} + \vec{b}t$ $(a = origin, b = direction, t = scalar, c = intersection)$
We can even split this equation into solving x and y separately.

$$\vec{c} = \vec{a} + \vec{b}t \begin{cases} c_x = a_x + b_x t \\ c_y = a_y + b_y t \end{cases}$$

The scalar describes how many times we have to go in our direction from our origin.

Example:



Here the origin is $(1, 1)$ and the destination is $(3, 3)$. That means the direction is $(1, 1)$ and the scalar $t = 2$.

$$\vec{c} = \vec{a} + \vec{b}t \begin{cases} c_x = a_x + b_x t \\ c_y = a_y + b_y t \end{cases}$$

$$c_x : 1 + 1 * 2 = 3$$

$$c_y : 1 + 1 * 2 = 3$$

$$=> c(3, 3)$$

Circular equation

$$(x - c_x)^2 + (y - c_y)^2 - r^2 = 0$$

c = center of circle, r = radius, (x, y) is the intersection with a line.

Find the intersection

To find the intersection of a point with a direction and a circle we insert the equation for the intersection point of the line into the circular equation.

$$(a_x + b_x t - c_x)^2 + (a_y + b_y t - c_y)^2 - r^2 = 0$$

Next step is to bring it in the form of a quadratic equation

$$ax^2 + bx + c = 0$$

Let's use t instead of x just make clear that we want to find the scalar

$$at^2 + bt + c = 0$$

1. Step: Solve the parentheses

$$(a_x + b_x t - c_x)^2 = (a_x + b_x t - c_x) * (a_x + b_x t - c_x)$$
$$(a_x + b_x t - c_x) * (a_x + b_x t - c_x)$$
$$= a_x^2 + a_x b_x t - a_x c_x + a_x b_x t + b_x^2 t^2 - b_x t c_x - a_x c_x - b_x t c_x + c_x^2$$
$$= a_x^2 + 2a_x b_x t - 2a_x c_x + b_x^2 t^2 - 2b_x t c_x + c_x^2$$

Solving for y works the exact same way:

$$a_y^2 + 2a_y b_y t - 2a_y c_y + b_y^2 t^2 - 2b_y t c_y + c_y^2$$

Insert it back into the quadratic equation

$$a_x^2 + 2a_x b_x t - 2a_x c_x + b_x^2 t^2 - 2b_x t c_x + c_x^2 + a_y^2 + 2a_y b_y t - 2a_y c_y + b_y^2 t^2 - 2b_y t c_y + c_y^2 - r^2 = 0$$

2. Step: Simplify it by bringing same terms together

$$(b_x^2 + b_y^2)t^2 + (2a_x b_x - 2b_x c_x + 2a_y b_y - 2b_y c_y)t + (a_x^2 - 2a_x c_x + c_x^2 + a_y^2 - 2a_y c_y + c_y^2 - r^2) = 0$$

Now it's in the proper form to use the quadratic formula

3. Step: Solve the equation using the quadratic formula
Quadratic formula:

$$\frac{-b + -\sqrt{b^2 - 4ac}}{2a}$$

First of all check the discriminant D

$$D = b^2 - 4ac$$

If $D < 0$ no intersections
If $D = 0$ one intersection
If $D > 0$ two intersections

If there are no intersections you can stop the calculations right there.

4. Step: Calculate t values
$D > 0$ (two intersections)

$$t_{1,2} = \frac{-b + -\sqrt{D}}{2a} \begin{cases} t_1 = \frac{-b+\sqrt{D}}{2a} \\ t_2 = \frac{-b-\sqrt{D}}{2a} \end{cases}$$

$D = 0$ (one intersection)

$$t = \frac{-b}{2a}$$

5. Step: Get correct t value (the closer one, only if you have 2 intersections)
Get both intersections:

$$\vec{c_1} = \vec{a} + \vec{b}t_1$$

$$\vec{c_2} = \vec{a} + \vec{b}t_2$$

Calculate the vectors:

$$\vec{v_1} = \vec{c_1} - \vec{a}$$

$$\vec{v_2} = \vec{c_2} - \vec{a}$$

Calculate the distance origin - intersection (pythagorean theorem):

$$d_1 = {v_1}_x^2 + {v_1}_y^2$$
$$d_2 = {v_2}_x^2 + {v_2}_y^2$$

Check which one is closer to the origin:
If $d_1 < d_2$: t_1 is the correct t value
If $d_2 < d_1$: t_2 is the correct t value

Now the intersection point can be calculated:

$$\vec{c} = \vec{a} + \vec{b}t$$

The vector from the origin to the intersection is:

$$\vec{v} = \vec{c} - \vec{a}$$

Example:
Origin $a = (-300, 0)$
Direction $b = (1, 0)$
Radius $r = 100$
Circle $c = (0, 0)$

1. Step: insert it into the quadratic equation

$$(b_x^2+b_y^2)t^2+(2a_xb_x-2b_xc_x+2a_yb_y-2b_yc_y)t+(a_x^2-2a_xc_x+c_x^2+a_y^2-2a_yc_y+c_y^2-r^2) = 0$$

$$=> (1^2 + 0^2)t^2+$$
$$(2 * (-300) * 1 - 2 * 1 * 0 + 2 * 0 * 0 - 2 * 0 * 0)t+$$
$$((-300)^2 - 2 * (-300) * 0 + 0^2 + 0^2 - 2 * 0 * 0 + 0^2 - 100^2) = 0$$
$$=> t^2 - 600t + 80000 = 0$$

2. Step: Solve the quadratic equation
Discriminant $D = b^2 - 4ac => D = (-600)^2 - 4 * 1 * 80000 = 40000$
$=> D > 0$: two intersections

Insert it into the quadratic formula:

$$t_1 = \frac{-(-600) + \sqrt{40000}}{2 * 1} = 400$$

$$t_2 = \frac{-(-600) - \sqrt{40000}}{2 * 1} = 200$$

Get the proper t value by calculating both intersections:

$$\vec{c_1} = \vec{a} + \vec{b}t_1 = \begin{pmatrix} -300 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} * 400 = \begin{pmatrix} 100 \\ 0 \end{pmatrix}$$

$$\vec{c_2} = \vec{a} + \vec{b}t_2 = \begin{pmatrix} -300 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} * 200 = \begin{pmatrix} -100 \\ 0 \end{pmatrix}$$

Calculate the origin - intersection vector:

$$\vec{v_1} = \vec{c_1} - \vec{a} = \begin{pmatrix} 100 \\ 0 \end{pmatrix} - \begin{pmatrix} -300 \\ 0 \end{pmatrix} = \begin{pmatrix} 400 \\ 0 \end{pmatrix}$$

$$\vec{v_2} = \vec{c_2} - \vec{a} = \begin{pmatrix} -100 \\ 0 \end{pmatrix} - \begin{pmatrix} -300 \\ 0 \end{pmatrix} = \begin{pmatrix} 200 \\ 0 \end{pmatrix}$$

Calculate the distance between the origin and the intersection using the pythagorean theorem

$$d_1 = {v_1}_x^2 + {v_1}_y^2 = 400^2 + 0^2 = 160000$$

$$d_2 = {v_2}_x^2 + {v_2}_y^2 = 200^2 + 0^2 = 40000$$

Check which one is shorter, in this case $d_s < d_1 => t_2$ is the correct t value
Now you know c_2 is your intersection and $\vec{v_2}$ is the proper vector.

Line equation 3D

$$\vec{c} = \vec{a} + \vec{b}t \begin{cases} c_x = a_x + b_x t \\ c_y = a_y + b_y t \\ c_z = a_z + b_z t \end{cases}$$

Sphere equation

$$(x - c_x)^2 + (y - c_y)^2 + (z - c_z)^2 - r^2 = 0$$

c = center of sphere, r = radius, (x, y, z) is the intersection with a line.

Find the intersection 3D

$$(a_x + b_x t - c_x)^2 + (a_y + b_y t - c_y)^2 + (a_z + b_z t)^2 - r^2 = 0$$

Same as before, just bring it in the form of a quadratic equation

$$at^2 + bt + c = 0$$

Solved parentheses:

$$a_x^2 + 2a_x b_x t - 2a_x c_x + b_x^2 t^2 - 2b_x t c_x + c_x^2$$

$$a_y^2 + 2a_y b_y t - 2a_y c_y + b_y^2 t^2 - 2b_y t c_y + c_y^2$$

$$a_z^2 + 2a_z b_z t - 2a_z c_z + b_z^2 t^2 - 2b_z t c_z + c_z^2$$

Insert it into the quadratic equation:

$$a_x^2 + 2a_x b_x t - 2a_x c_x + b_x^2 t^2 - 2b_x t c_x + c_x^2$$

$$+a_y^2 + 2a_y b_y t - 2a_y c_y + b_y^2 t^2 - 2b_y t c_y + c_y^2$$

$$+a_z^2 + 2a_z b_z t - 2a_z c_z + b_z^2 t^2 - 2b_z t c_z + c_z^2 - r^2 = 0$$

Simplify:

$$
\begin{gathered}
(b_x^2 + b_y^2 + b_z^2)t^2 \\
+(2a_xb_x - 2b_xc_x + 2a_yb_y - 2b_yc_y + 2a_zb_z - 2b_zc_z)t \\
+(a_x^2 - 2a_xc_x + c_x^2 + a_y^2 - 2a_yc_y + c_y^2 + a_z^2 - 2a_zc_z + c_z^2 - r^2) = 0
\end{gathered}
$$

From now on it works the exact same way as if you'd use a circle. You can simply continue from step 3. in "Find the intersection". The only noticeable difference is calculating the distance from origin - intersection

$$d_1 = {v_1}_x^2 + {v_1}_y^2 + {v_1}_z^2$$

$$d_2 = {v_2}_x^2 + {v_2}_y^2 + {v_2}_z^2$$