Mathematical concept behind raytracing a sphere or circle

Introduction

Every time we'll talk about raytracing it's referred to a sphere or circle if not mentioned otherwise.
When talking about ray tracing we talk about having a circle and a single point in our coordinate system that's emitting lines to which we'll refer as rays.
So now we need a point in space, a direction and the circle. What we're gonna calculate is the scalar for our line.
Although we're gonna go through the process with a circle, tracing a sphere works exactly the same way but you have x, y, z coordinates instead of just x, y what leads to solving more parenthesis.

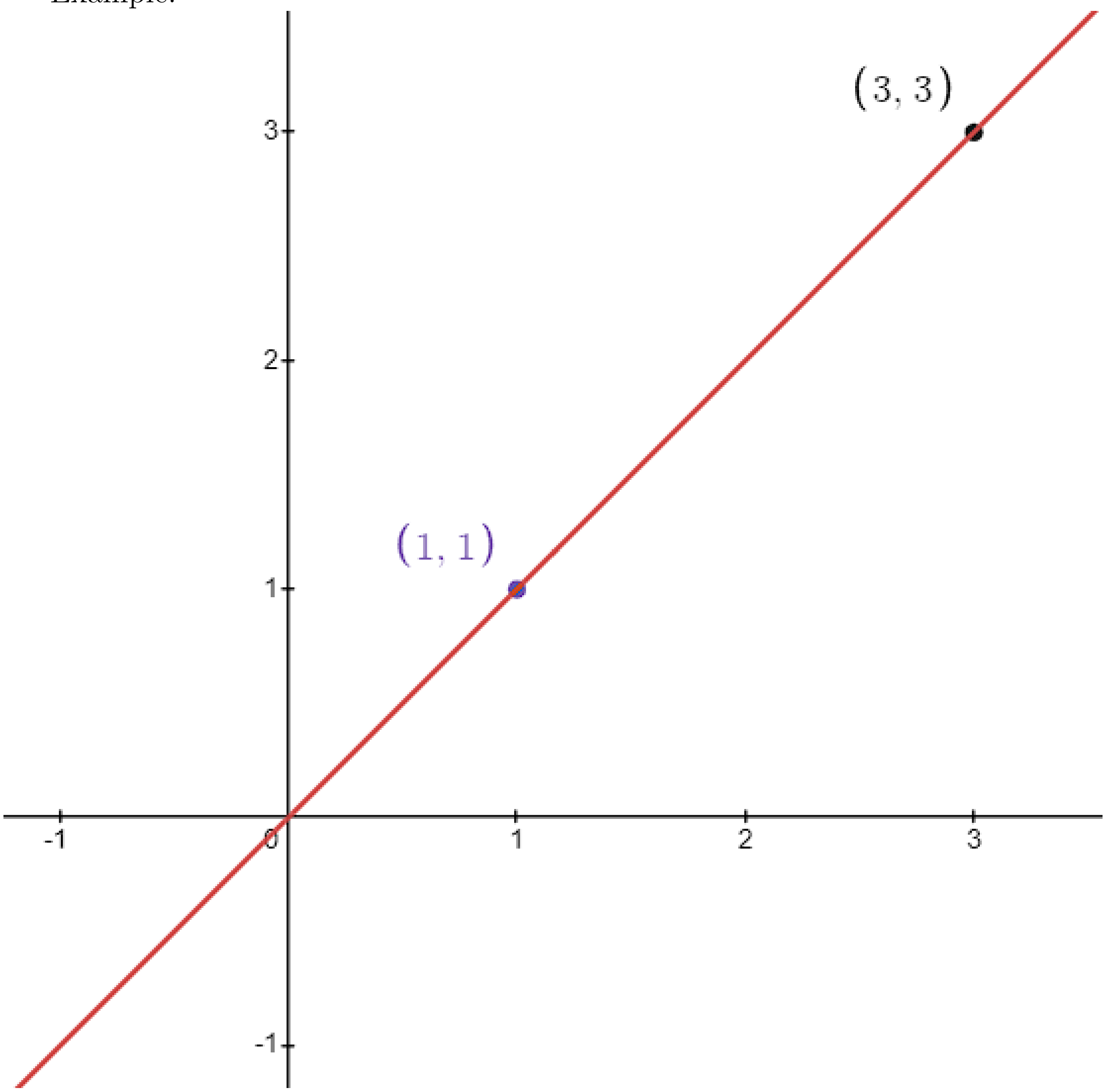
Line equation

We have a line that's described with $y = mx + c$. But we don't have this equation just yet since we only have an origin and a direction. Now we want to "emit" a line from our origin and see if it will ever intersect with our circle. If it does our intersection is described with $\vec{c} = \vec{a} + \vec{b} * t$ $(a = origin, b = direction, t = scalar, c = intersection)$
We can even split this equation into solving x and y separately.

$$\vec{c} = \vec{a} + \vec{b} * t \begin{cases} c_x = a_x + b_x * t \\ c_y = a_y + b_y * t \end{cases}$$

The scalar describes how many times we have to go in our direction from our origin.

Example:



Here the origin is $(1, 1)$ and the destination is $(3, 3)$. That means the direction is $(1, 1)$ and the scalar $t = 2$.

$$\vec{c} = \vec{a} + \vec{b} * t \begin{cases} c_x = a_x + b_x * t \\ c_y = a_y + b_y * t \end{cases}$$

$$c_x : 1 + 1 * 2 = 3$$

$$c_y : 1 + 1 * 2 = 3$$

$$=> c(3, 3)$$

Circular equation

$$(x - c_x)^2 + (y - c_y)^2 - r^2 = 0$$

c = center of circle, r = radius and (x, y) is the intersection point with a line.

Find the intersection

To find the intersection of a point with a direction and a circle we insert the equation for the intersection point of the line into the circular equation.

$$(a_x + b_x t - c_x)^2 + (a_y + b_y t - c_y)^2 - r^2 = 0$$

Next step is to bring it in the form of a quadratic equation

$$ax^2 + bx + c = 0$$

But for us t will be x since we want to find the scalar for our line.

$$at^2 + bt + c = 0$$

1. Step: Solve the parentheses

$$(a_x + b_x t - c_x)^2 = (a_x + b_x t - c_x) * (a_x + b_x t - c_x)$$
$$(a_x + b_x t - c_x) * (a_x + b_x t - c_x)$$
$$= a_x^2 + a_x b_x t - a_x c_x + a_x b_x t + b_x^2 t^2 - b_x t c_x - a_x c_x - b_x t c_x + c_x^2$$
$$= a_x^2 + 2a_x b_x t - 2a_x c_x + b_x^2 t^2 - 2b_x t c_x + c_x^2$$

Solving for y works the exact same way it's:

$$a_y^2 + 2a_y b_y t - 2a_y c_y + b_y^2 t^2 - 2b_y t c_y + c_y^2$$

Insert it back into the quadratic equation

$$a_x^2 + 2a_x b_x t - 2a_x c_x + b_x^2 t^2 - 2b_x t c_x + c_x^2 + a_y^2 + 2a_y b_y t - 2a_y c_y + b_y^2 t^2 - 2b_y t c_y + c_y^2 - r^2 = 0$$

2. Step Simplify it by bringing same terms together

$$(b_x^2 + b_y^2)t^2 + (2a_x b_x - 2b_x c_x + 2a_y b_y - 2b_y c_y)t + (a_x^2 - 2a_x c_x + c_x^2 + a_y^2 - 2a_y c_y + c_y^2 - r^2) = 0$$

Now it's in the form of the quadratic formula.

3. Step: Solve the equation using the quadratic formula
Quadratic formula:

$$\frac{-b+-\sqrt{b^2-4ac}}{2a}$$

First of all check the discriminant D

$$D = b^2 - 4ac$$

If $D < 0$ no intersections
If $D = 0$ one intersection
If $D > 0$ two intersections

If there are no intersections you can stop the calculations right here.

4. Step: Calculate the t values
$D > 0$ (two intersections)

$$t_{1,2} = \frac{-b+-\sqrt{D}}{2a} \begin{cases} t_1 = \frac{-b+\sqrt{D}}{2a} \\ t_2 = \frac{-b-\sqrt{D}}{2a} \end{cases}$$

$D = 0$ (one intersection)

$$t = \frac{-b}{2a}$$

5. Step: Get correct t value (the closer one, only if you have 2 intersections)
Get both intersection points:

$$\vec{c_1} = \vec{a} + \vec{b}t_1$$

$$\vec{c_2} = \vec{a} + \vec{b}t_2$$

Calculate the vectors

$$\vec{v_1} = \vec{c_1} - \vec{a}$$

$$\vec{v_2} = \vec{c_2} - \vec{a}$$

Calculate the distance origin, intersection (pythagorean theorem)

$$d_1 = {v_1}_x^2 + {v_1}_y^2$$
$$d_2 = {v_2}_x^2 + {v_2}_y^2$$

Check which one is closer to the origin:
If $d_1 < d_2$ t_1 is the correct t value
If $d_2 < d_1$ t_2 is the correct t value

Now the intersection point can be calculated:

$$\vec{c} = \vec{a} + \vec{b}t$$

The vector from the origin to the intersection is:

$$\vec{v} = \vec{c} - \vec{a}$$