

# Math behind raytracing a sphere / circle

## Introduction

Every time I'm gonna talk about raytracing it's referred to a sphere or circle if not mentioned otherwise.

When talking about raytracing we talk about having a circle and a single point in our coordinate system that's emitting lines to which we'll refer as rays.

So now we need a point in space, a direction and a circle. We're gonna calculate the scalar for the line.

Although we're gonna go through the process with a circle, tracing a sphere works exactly the same way but you have  $x, y, z$  coordinates instead of just  $x, y$  what leads to solving more parenthesis. A sphere is being mentioned down below but without in depth explanations.

## Line equation

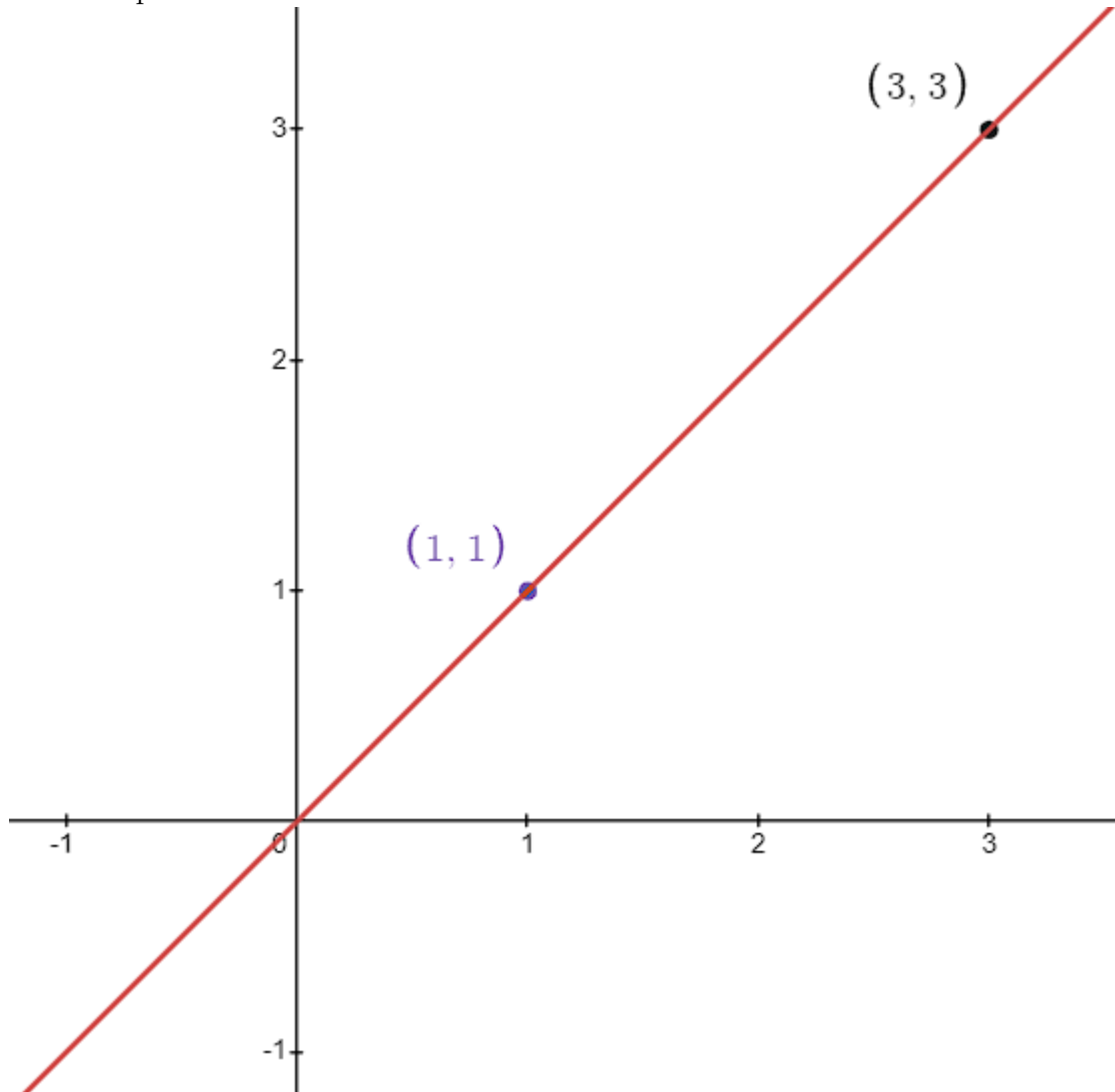
A line is defined as  $y = mx + c$ . But we don't have this equation just yet, since we only have an origin and a direction. Now we want to "emit" a line from our origin and see if it will ever intersect with our circle. If it does our intersection is defined as  $\vec{c} = \vec{a} + \vec{b}t$  ( $a = origin, b = direction, t = scalar, c = intersection$ )

We can even split this equation into solving  $x$  and  $y$  separately.

$$\vec{c} = \vec{a} + \vec{b}t \begin{cases} c_x = a_x + b_x t \\ c_y = a_y + b_y t \end{cases}$$

The scalar describes how many times we have to go in our direction from our origin.

Example:



Here the origin is (1, 1) and the destination is (3, 3). That means the direction is (1, 1) and the scalar  $t = 2$ .

$$\vec{c} = \vec{a} + \vec{b}t \begin{cases} c_x = a_x + b_x t \\ c_y = a_y + b_y t \end{cases}$$

$$c_x : 1 + 1 * 2 = 3$$

$$c_y : 1 + 1 * 2 = 3$$

$$\Rightarrow c(3, 3)$$

## Circular equation

$$(x - c_x)^2 + (y - c_y)^2 - r^2 = 0$$

$c$  = center of circle,  $r$  = radius,  $(x, y)$  is the intersection with a line.

## Find the intersection

To find the intersection of a point with a direction and a circle we insert the equation for the intersection point of the line into the circular equation.

$$(a_x + b_x t - c_x)^2 + (a_y + b_y t - c_y)^2 - r^2 = 0$$

Next step is to bring it in the form of a quadratic equation

$$ax^2 + bx + c = 0$$

Let's use  $t$  instead of  $x$  just make clear that we want to find the scalar

$$at^2 + bt + c = 0$$

1. Step: Solve the parentheses

$$\begin{aligned}(a_x + b_x t - c_x)^2 &= (a_x + b_x t - c_x) * (a_x + b_x t - c_x) \\ &= (a_x + b_x t - c_x) * (a_x + b_x t - c_x) \\ &= a_x^2 + a_x b_x t - a_x c_x + a_x b_x t + b_x^2 t^2 - b_x t c_x - a_x c_x - b_x t c_x + c_x^2 \\ &= a_x^2 + 2a_x b_x t - 2a_x c_x + b_x^2 t^2 - 2b_x t c_x + c_x^2\end{aligned}$$

Solving for  $y$  works the exact same way:

$$a_y^2 + 2a_y b_y t - 2a_y c_y + b_y^2 t^2 - 2b_y t c_y + c_y^2$$

Insert it back into the quadratic equation

$$a_x^2 + 2a_x b_x t - 2a_x c_x + b_x^2 t^2 - 2b_x t c_x + c_x^2 + a_y^2 + 2a_y b_y t - 2a_y c_y + b_y^2 t^2 - 2b_y t c_y + c_y^2 - r^2 = 0$$

2. Step: Simplify it by bringing same terms together

$$(b_x^2 + b_y^2)t^2 + (2a_x b_x - 2b_x c_x + 2a_y b_y - 2b_y c_y)t + (a_x^2 - 2a_x c_x + c_x^2 + a_y^2 - 2a_y c_y + c_y^2 - r^2) = 0$$

Now it's in the proper form to use the quadratic formula

3. Step: Solve the equation using the quadratic formula  
 Quadratic formula:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

First of all check the discriminant  $D$

$$D = b^2 - 4ac$$

If  $D < 0$  no intersections

If  $D = 0$  one intersection

If  $D > 0$  two intersections

If there are no intersections you can stop the calculations right there.

4. Step: Calculate t values

$D > 0$  (two intersections)

$$t_{1,2} = \frac{-b \pm \sqrt{D}}{2a} \left\{ \begin{array}{l} t_1 = \frac{-b + \sqrt{D}}{2a} \\ t_2 = \frac{-b - \sqrt{D}}{2a} \end{array} \right.$$

$D = 0$  (one intersection)

$$t = \frac{-b}{2a}$$

5. Step: Get correct t value (the closer one, only if you have 2 intersections)

Get both intersections:

$$\vec{c}_1 = \vec{a} + \vec{b}t_1$$

$$\vec{c}_2 = \vec{a} + \vec{b}t_2$$

Calculate the vectors:

$$\vec{v}_1 = \vec{c}_1 - \vec{a}$$

$$\vec{v}_2 = \vec{c}_2 - \vec{a}$$

Calculate the distance origin - intersection (pythagorean theorem):

$$d_1 = v_{1x}^2 + v_{1y}^2$$

$$d_2 = v_{2x}^2 + v_{2y}^2$$

Check which one is closer to the origin:

If  $d_1 < d_2$ :  $t_1$  is the correct t value

If  $d_2 < d_1$ :  $t_2$  is the correct t value

Now the intersection point can be calculated:

$$\vec{c} = \vec{a} + \vec{b}t$$

The vector from the origin to the intersection is:

$$\vec{v} = \vec{c} - \vec{a}$$

Example:

Origin  $a = (-300, 0)$

Direction  $b = (1, 0)$

Radius  $r = 100$

Circle  $c = (0, 0)$

1. Step: insert it into the quadratic equation

$$(b_x^2 + b_y^2)t^2 + (2a_xb_x - 2b_xc_x + 2a_yb_y - 2b_yc_y)t + (a_x^2 - 2a_xc_x + c_x^2 + a_y^2 - 2a_yc_y + c_y^2 - r^2) = 0$$

$$\Rightarrow (1^2 + 0^2)t^2 +$$

$$(2 * (-300) * 1 - 2 * 1 * 0 + 2 * 0 * 0 - 2 * 0 * 0)t +$$

$$((-300)^2 - 2 * (-300) * 0 + 0^2 + 0^2 - 2 * 0 * 0 + 0^2 - 100^2) = 0$$

$$\Rightarrow t^2 - 600t + 80000 = 0$$

2. Step: Solve the quadratic equation

$$\text{Discriminant } D = b^2 - 4ac \Rightarrow D = (-600)^2 - 4 * 1 * 80000 = 40000$$

$\Rightarrow D > 0$ : two intersections

Insert it into the quadratic formula:

$$t_1 = \frac{-(-600) + \sqrt{40000}}{2 * 1} = 400$$

$$t_2 = \frac{-(-600) - \sqrt{40000}}{2 * 1} = 200$$

Get the proper t value by calculating both intersections:

$$\vec{c}_1 = \vec{a} + \vec{b}t_1 = \begin{pmatrix} -300 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} * 400 = \begin{pmatrix} 100 \\ 0 \end{pmatrix}$$

$$\vec{c}_2 = \vec{a} + \vec{b}t_2 = \begin{pmatrix} -300 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} * 200 = \begin{pmatrix} -100 \\ 0 \end{pmatrix}$$

Calculate the origin - intersection vector:

$$\vec{v}_1 = \vec{c}_1 - \vec{a} = \begin{pmatrix} 100 \\ 0 \end{pmatrix} - \begin{pmatrix} -300 \\ 0 \end{pmatrix} = \begin{pmatrix} 400 \\ 0 \end{pmatrix}$$

$$\vec{v}_2 = \vec{c}_2 - \vec{a} = \begin{pmatrix} -100 \\ 0 \end{pmatrix} - \begin{pmatrix} -300 \\ 0 \end{pmatrix} = \begin{pmatrix} 200 \\ 0 \end{pmatrix}$$

Calculate the distance between the origin and the intersection using the pythagorean theorem

$$d_1 = v_{1x}^2 + v_{1y}^2 = 400^2 + 0^2 = 160000$$

$$d_2 = v_{2x}^2 + v_{2y}^2 = 200^2 + 0^2 = 40000$$

Check which one is shorter, in this case  $d_2 < d_1 \Rightarrow t_2$  is the correct  $t$  value  
Now you know  $c_2$  is your intersection and  $\vec{v}_2$  is the proper vector.

## Line equation 3D

$$\vec{c} = \vec{a} + \vec{b}t \begin{cases} c_x = a_x + b_x t \\ c_y = a_y + b_y t \\ c_z = a_z + b_z t \end{cases}$$

## Sphere equation

$$(x - c_x)^2 + (y - c_y)^2 + (z - c_z)^2 - r^2 = 0$$

c = center of sphere, r = radius,  $(x, y, z)$  is the intersection with a line.

## Find the intersection 3D

$$(a_x + b_x t - c_x)^2 + (a_y + b_y t - c_y)^2 + (a_z + b_z t - c_z)^2 - r^2 = 0$$

Same as before, just bring it in the form of a quadratic equation

$$at^2 + bt + c = 0$$

Solved parentheses:

$$a_x^2 + 2a_x b_x t - 2a_x c_x + b_x^2 t^2 - 2b_x t c_x + c_x^2$$

$$a_y^2 + 2a_y b_y t - 2a_y c_y + b_y^2 t^2 - 2b_y t c_y + c_y^2$$

$$a_z^2 + 2a_z b_z t - 2a_z c_z + b_z^2 t^2 - 2b_z t c_z + c_z^2$$

Insert it into the quadratic equation:

$$a_x^2 + 2a_x b_x t - 2a_x c_x + b_x^2 t^2 - 2b_x t c_x + c_x^2 +$$

$$a_y^2 + 2a_y b_y t - 2a_y c_y + b_y^2 t^2 - 2b_y t c_y + c_y^2 +$$

$$a_z^2 + 2a_z b_z t - 2a_z c_z + b_z^2 t^2 - 2b_z t c_z + c_z^2 - r^2 = 0$$

Simplify:

$$\begin{aligned} & (b_x^2 + b_y^2 + b_z^2)t^2 + \\ & (2a_xb_x - 2b_xc_x + 2a_yb_y - 2b_yc_y + 2a_zb_z - 2b_zc_z)t + \\ & (a_x^2 - 2a_xc_x + c_x^2 + a_y^2 - 2a_yc_y + c_y^2 + a_z^2 - 2a_zc_z + c_z^2 - r^2) = 0 \end{aligned}$$

From now on it works the exact same way as if you'd use a circle. You can simply continue from step 3. in "Find the intersection". The only noticeable difference is calculating the distance from origin - intersection

$$d_1 = v_{1x}^2 + v_{1y}^2 + v_{1z}^2$$

$$d_2 = v_{2x}^2 + v_{2y}^2 + v_{2z}^2$$